THE EXACT NUMBER OF CONJUGACY CLASSES FOR 2-GENERATOR p-GROUPS OF NILPOTENCY CLASS 2

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To

my beloved husband

son and daughter

abah and mama

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ABSTRACT

An element x is conjugate to y in a group G if there exists an element g in G such that $g^{-1}xg = x^g = y$. The relation x is conjugate to y is an equivalence relation which induces a partition of G whose elements are called conjugacy classes. The general formula for the exact number of conjugacy classes for nilpotent groups does not exist. Researchers give only the lower bounds for the number of conjugacy classes of nilpotent groups. In this thesis, 2-generator p-groups of nilpotency class 2 (p an odd prime) are considered for their exact number of conjugacy classes. These groups have been classified by Bacon and Kappe in 1993. In 1999, Kappe, Visscher and Sarmin have corrected minor errors on the groups in the classification. Groups, Algorithms, and Programming (GAP) software is used in this research to gain insight into the structure of these groups. There are infinitely many of these groups which are partitioned into three types. For each type, there are infinitely many base groups. New structural results are found such that groups other than base groups are central extensions. As a result of this research, a general formula is derived for the exact number of conjugacy classes for each type of 2-generator p-groups of nilpotency class 2 (p an odd prime).

ABSTRAK

Suatu unsur x adalah berkonjugat dengan y dalam kumpulan G jika wujud suatu unsur g dalam G dengan $g^{-1}xg = x^g = y$. Hubungan x berkonjugat dengan y adalah hubungan kesetaraan yang mengaruh petakan bagi G yang unsur-unsurnya dinamai kelas-kelas konjugat. Rumus umum bilangan tepat bagi kelas konjugat untuk kumpulan nilpoten tidak wujud. Penyelidik-penyelidik hanya menganggarkan batas bawah bagi bilangan kelas konjugat bagi kumpulan nilpoten. Dalam kajian ini, bilangan tepat bagi kelas konjugat bagi kumpulan-p berpenjana-2 dengan kelas nilpoten 2 (p nombor perdana ganjil) akan ditentukan. Kumpulan-kumpulan tersebut telah diklasifikasikan oleh Bacon dan Kappe dalam tahun 1993. Dalam tahun 1999, Kappe, Visscher dan Sarmin telah membetulkan beberapa kesilapan kecil untuk kumpulan-kumpulan dalam klasifikasi tersebut. Perisian *Groups, Algorithms, and Programming* (GAP) telah digunakan dalam kajian ini untuk mendalami struktur kumpulan-kumpulan tersebut. Bilangan kumpulan tersebut adalah tak terhingga dan terbahagi kepada tiga jenis. Bagi setiap jenis kumpulan, terdapat tak terhingga banyaknya kumpulan-kumpulan asas. Kumpulan-kumpulan selain kumpulan asas adalah perluasan pusat dan ini merupakan struktur kumpulan yang baru ditemui. Hasil daripada kajian ini, suatu rumus umum telah diperolehi bagi bilangan tepat kelas konjugat bagi setiap jenis kumpulan-p berpenjana-2 dengan nilpoten kelas 2 (p nombor perdana ganjil).

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LIST OF NOTATION

$a \in G$	-	a is an element of G
$a \notin G$	-	a is not an element of G
$a \mid b$	-	a divides b
$a \nmid b$	-	a does not divide b
a^{-1}	-	inverse of an element
a^b	-	$b^{-1}ab$, conjugate of a by b
$a \mod b$	-	a modulo b
A_n	-	Alternating group on n letters
cl_G	-	number of conjugacy classes of G
C_n	-	Cyclic group of order n
D_{2n}	-	Dihedral group of order $2n$
$\gcd(a, b)$	-	greatest common divisor of the integers a and b
G'	-	derived subgroup or commutator subgroup of G
G/H	-	factor group of G by H
$H \leq G$	-	H is a subgroup of G
$H \triangleleft G$	-	H is a normal subgroup of G
$H \cong G$	-	H is isomorphic to G
$H \equiv G$	-	H is equivalent to G
$H \times G$	-	direct product of H and G
$H \rtimes G$	-	semidirect product of H by G
$\mathbb{N}$	-	natural numbers $0, 1, 2, ...$
S_n	-	Symmetric group on n letters
$Z(G)$	-	centre of G
$\mathbb{Z}_n$	-	integers mod n
$\lvert a\rvert$	-	the order of an element

$\|G\|$	-	the order of a group G
$[a, b]$	-	$a^{-1}b^{-1}ab$, the commutator of a and b
$\langle x \rangle$	-	subgroup generated by an element x of G
$\langle X \mid R \rangle$	-	group presented by generators X and relators R
$\prod_{i=1}^{n} G_i$	-	product of $G_{i=1,\ldots,n}$
$\sum_{i=1}^{n} G_i$	-	summation of $G_{i=1,\ldots,n}$
$\lfloor \ \rfloor$	-	floor values of integer
$\cup$	-	union

CHAPTER 1

INTRODUCTION

1.1 Introduction

Let G be a group with the identity element e and $x, y \in G$. The element x is conjugate to y in G if there exists an element $g \in G$ such that $g^{-1}xg = x^g = y$. The relation x is conjugate to y is an equivalence relation on G. This equivalence relation induces a partition of G whose elements are called conjugacy classes. The number of conjugacy classes of G is denoted by $\mathbf{cl_G}$.

Let G be a finite group and $x, y \in G$. If x and y are conjugate, then x and y have the same order. Suppose $x^n = 1$. Then $y^n = (x^g)^n = (x^n)^g = 1$. Thus $|x| = |y|$. The conjugacy class containing the identity has only one element namely the identity itself. Every element of the centre is in its own conjugacy class containing only that element.

A finite group G is nilpotent if and only if G is the direct product of its Sylow p-subgroups. If conjugacy classes of each Sylow p-subgroups can be counted, cl_G can be found. By the following lemma, if A and B are the Sylow p-subgroups where $G = A \times B$ and $|G| = |A| \times |B|$ then $\mathrm{cl}_{A\times B} = \mathrm{cl}_A \cdot \mathrm{cl}_B$. Therefore, it suffices to count the conjugacy classes for p-groups.

Lemma 1.1.
Let $G = A \times B$. Suppose the number of conjugacy classes of A and B are cl_A and cl_B respectively. Then $\mathrm{cl}_G = \mathrm{cl}_A \cdot \mathrm{cl}_B$.

Proof. Let $\text{cl}_A = n$ and $\text{cl}_B = m$. Set $a_1, \ldots, a_n$ to be representatives of the conjugacy classes of A and $b_1, \ldots, b_m$ to be representatives of the conjugacy classes of B. We claim that $(a_i b_j) \in G$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ are representatives of all the conjugacy classes of G. Let (g, h) be an element of G. Then $(a_i b_j)^{(g,h)} = (a_i^g, b_j^h)$. But (a_i^g, b_j^h) is in conjugacy classes of $(a_i b_j)$. On the other hand if some $(a, b) \in G$ have a in some conjugacy classes a_i of A and b is in some conjugacy classes b_j of B. Hence (a, b) is in the conjugacy class $(a_i b_j)$ of G. There are $\text{cl}_A \cdot \text{cl}_B$ elements of the form $(a_i b_j)$ and the result follows. □

There are classes of groups in which the number of conjugacy classes are known. The number of conjugacy classes for an abelian group A is equal to its order, $\text{cl}_A = |Z(A)| = |A|$. Formulas for the number of conjugacy classes for symmetric group S_n, alternating group A_n and dihedral group D_{2n} are already found. For symmetric group, $\text{cl}_{S_n} = p(n)$, the number of partitions of n [1]. The number of partitions of n also plays a role in counting cl_{A_n}. The simplest formula given by Girse in [2] where

$$\text{cl}_{A_n} = 2p(n) + 3 \sum_{r=1}^{\lfloor \sqrt{n/2} \rfloor} (-1)^r p(n - 2r^2).$$

For dihedral group D_{2n}, the formula for $\text{cl}_{D_{2n}}$ is given as

$$\text{cl}_{D_{2n}} = \begin{cases} \lfloor \frac{n+2}{2} \rfloor + 1 & \textit{if } 2 \nmid 2. \\ \lfloor \frac{n+2}{2} \rfloor + 2 & \textit{if } 2 \mid n. \end{cases}$$

1.2 Research Background

For nilpotent groups, many researchers only estimate the number of conjugacy classes by giving the lower bound of these groups. The trial of classifying finite nilpotent groups is a difficult task. There is no clear description of these groups which is the reason why the researchers only give the bounds on the number of conjugacy classes. However, there are some collection of nilpotent groups that have been classified. In this research, 2-generator p-groups of nilpotency class 2 (p an odd prime) are considered. These groups were classified by Bacon and Kappe in 1993 and Kappe, Visscher and Sarmin did minor corrections in 1999. There are infinitely many of these groups which are partitioned into three types and parameterized by finite α, β, γ and σ.

The Groups, Algorithms, and Programming (GAP) software is used in this research to gain insight into the 2-generator p-groups of class 2 (p an odd prime), to provide examples and to check the theoretical results obtained. GAP is a powerful tool and can be used to construct large p-groups and compute their conjugacy classes.

1.3 Problem Statement

To find general formulas for the exact number of conjugacy classes for 2-generator p-groups of class 2 (p an odd prime).

1.4 Research Objectives

The objectives of this thesis are

(i) to provide general formulas for the exact number of conjugacy classes for 2-generator p-groups of class 2 (p an odd prime).

(ii) to obtain new structure results for these groups including:

- Conjugations of elements in a group.
- Conjugations of elements between a group and its extension.
- Characterization of a group by a central extension.

- Determination of base groups.
- Order of centre of a group.

(iii) to encapsulate the results obtained in Groups, Algorithms, and Programming (GAP) for others to use.

1.5 Scope of Thesis

In this thesis, the group considered will be 2-generator p-groups of class 2 (p an odd prime).

1.6 Significance of Findings

The major contribution of this thesis will be the new theoretical results on the exact number of conjugacy classes for 2-generator p-groups of class 2 (p an odd prime). This thesis also contributes to a greater challenge of counting conjugacy classes of groups in general. No classes of nilpotent groups have formulas for their exact number of conjugacy classes. Therefore, this thesis provides original results. Some of the results have been presented in national and international conferences and thus contribute to new findings in the field of group theory.

1.7 Thesis Outline

There are six chapters in this thesis. Chapter 1 provides the introduction of the thesis. This chapter discusses research background, problem statement, research objectives, scope and significance of findings of the thesis.

In Chapter 2, the bounds on the number of conjugacy classes for nilpotent groups given by several researchers are compared. We show that the bounds exist are very weak. The original classification of 2-generator p-groups of class 2 (p an odd prime) is stated. The modified classification of 2-generator p-groups of class 2 (p an odd prime) is given in terms of generators and relations. Then, the background and application of GAP in this research are presented.

Chapter 3 presents some definitons on group theory and number theory. In this chapter also, basic results for nilpotent groups are included. The new structural results obtained for 2-generator p-groups of class 2 (p an odd prime) are given and proved.

The main results of this thesis are given in Chapter 4. The general formulas for the exact number of conjugacy classes for 2-generator p-groups of class 2 (p an odd prime) are given according to their types. From the general formulas, an immediate result of the number of conjugacy classes for certain finite nilpotent groups of class 2 can be obtained.

Chapter 5 provides GAP programmes that have been used to construct examples for 2-generator p-groups of class 2 (p an odd prime). Not only that, the properties of these groups can be computed and lead to producing several lemmas. GAP is also used to check our theoretical results by the examples generated. We illustrate some examples and explain the interpretation of the commands used.

Finally, Chapter 6 concludes this thesis by giving summarization of the thesis and suggestions for future research.

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