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TAJUK PROJEK : CONJUGACY AND ORDER CLASSES OF 2-GENERATOR *p*-GROUP OF NILPOTENCY CLASS 2

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Let G be a group with identity element e and $x, y \in G$. The element x is conjugate to y if there exists an element $g \in G$ such that $g^{-1}xg = x^g = y$. The relation x is conjugate to y is an equivalence relation on G. It is reflexive since $x^e = x$ and symmetric since if $x^g = y$ then $x = gyg^{-1} = y^{g^{-1}}$. Finally it is transitive since if $x^g = y$ and $y^h = z$ then substitute x^g for y, we get $(x^g)^h = z$. Set $u = gh$. Then $x^u = z$ and x is conjugate to z. This equivalence relation induces a partition of G whose elements are called conjugacy classes. The number of conjugacy classes of G is denoted by cl_G.

Let G be a finite group and $x, y \in G$. If x and y are conjugate, then x and y have the same order. If x is conjugate to y, there exist $g \in G$ such that $g^{-1}xg = y$. Now, $xg = gy$. Thus $|xg| = |gy|$. Hence $|x| = |y|$. The conjugacy class containing the identity has only an element namely the identity. Every element of the center is in its own conjugacy class containing only that element. Hence the identity.

A finite group G is nilpotent if and only if G is the direct product of its Sylow p-subgroups. If conjugacy classes of each Sylow p-subgroups can be counted, cl_G can be found. If A and B are the Sylow p-subgroups where $G =$

$A \times B$ and $|G| = |A| \times |B|$ then $\mathrm{cl}_{A\times B} = \mathrm{cl}_A \cdot \mathrm{cl}_B$. Therefore, it suffices to count conjugacy classes for p-subgroups.

There are classes of groups in which cl_G is known. The number of conjugacy classes for an abelian group A is equal to its order, $\mathrm{cl}_A = |A|$. Formulas for the number of conjugacy classes for symmetric group S_n, alternating group A_n and dihedral group D_{2n} exist. Conjugation in S_n preserves its cycle structure and thus $\mathrm{cl}_{S_n} = p(n)$, the number of unrestricted partitions of n [1]. The number of unrestricted partitions of n also plays a role in counting cl_{A_n}. The simplest formula given by Girse [2] for which $\mathrm{cl}_{A_n} = 2p(n) + 3\sum_{r=1}^{\lfloor\sqrt{n/2}\rfloor}(-1)^r p(n-2r^2)$. For dihedral group D_{2n}, the formula for $\mathrm{cl}_{D_{2n}} = \lfloor\frac{n+2}{2}\rfloor + 1$ if $2 \nmid n$ and $\mathrm{cl}_{D_{2n}} = \lfloor\frac{n+2}{2}\rfloor + 2$ if $2 \mid n$.

In this research, our goal is to find the exact number of conjugacy classes for 2-generator p-groups of nilpotency class 2 (p an odd prime).

1.2 Research Background

For nilpotent groups, researchers only estimate the number of conjugacy classes by giving the lower bound of these groups. The difficulty of providing a formula for the exact number of conjugacy classes of these groups is because the structure of these groups are not easily understood. In this thesis, 2-generator p-groups of nilpotency class 2 (p an odd prime) are considered. These groups were classified by Bacon and Kappe in 1993 and Kappe, Sarmin and Visscher did a minor correction in 1999. There are infinitely many of these groups which are partitioned into three types and parameterized by finite α, β, γ and σ.

The Groups, Algorithms, and Programming (GAP) software is used in this research to gain insight into the 2-generator p-groups of class 2, to provide examples and to check the theoretical results obtained. GAP is a powerful tool and can be used to construct large p-groups and compute their conjugacy classes.

1.3 Problem Statement

To find general formulas for the exact number of conjugacy classes of 2-generator p-groups of class 2 (p an odd prime).

1.4 Research Objectives

The objectives of this thesis are:

(i) To provide general formulas for the exact number of conjugacy classes of 2-generator p-groups of class 2 (p an odd prime).

(ii) To obtain new structure results for these groups.

(iii) To encapsulate the results obtained in Groups, Algorithms, and Programming (GAP) for others to use.

1.5 Scope of Thesis

In this thesis, the group considered will be 2-generator p-groups of class 2 (p an odd prime).

1.6 Significance of Findings

The major contribution of this thesis is new theoretical results obtained that give the exact number of conjugacy classes of 2-generator p-groups of class 2 (p an odd prime). This thesis also contributes to a greater challenge of counting conjugacy classes of groups in general. No classes of nilpotent groups have formulas for their exact number of conjugacy classes. Therefore, this thesis provides a new and original results. Some of the results have been presented in national and international conferences and thus contribute to new findings in the field of group theory.

1.7 Thesis Outline

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Studies on the bounds of the number of conjugacy classes done by a number of researchers are reviewed. The classification of 2-generator p-groups of class 2 (p an odd prime) is stated and elaborated. Then, more background on the software package GAP used in this research is explained.

2.2 Bounds On The Number Of Conjugacy Classes

The trivial group E has $\mathrm{cl}_E = 1$. Any non-trivial group G has $\mathrm{cl}_G > 1$. There are several results in the literature that attempt to give a tighter lower bound for cl_G.

In 1968, Erdos and Turan [3] proved that for any finite group G, $\mathrm{cl}_G > \log_2 \log_2 |G|$. Then Poland [4] obtained an improved bound of cl_G for nilpotent groups. If G is nilpotent, $\mathrm{cl}_G > \log_2 |G|$. A better lower bound is given by Sherman [5] in 1979 who showed that if G is nilpotent of class c, then $\mathrm{cl}_G > c(|G|^{1/c} - 1) + 1$.

The bound given by Sherman is not tight. Consider the following 2-generator 5-group of class 2:

$$G = \langle a, b | a^{5^4} = b^{5^3} = [a, b]^5 = [a, [a, b]] = [b, [a, b]] = 1\rangle.$$

Erdos and Turan gives $\mathrm{cl}_G > \log_2 \log_2 |5^8| = 4$. Using Poland's estimation, $\mathrm{cl}_G > \log_2 |5^8| = 18$. Sherman gives a better estimation for which $\mathrm{cl}_G > 2(\sqrt{5^8}-1)+1 = 1249$. We obtain $\mathrm{cl}_G = 90625$. Thus, the general lower bounds given by Erdos and Turan, Poland and Sherman are very weak.

In the next section, the classification of 2-generator p-groups of class 2 (p an odd prime) is stated and elaborated.

2.3 The Classification of 2-Generator p-Groups of Class 2 (p an odd prime)

Trebenko [7] attempted a classification of all 2-generator groups of class 2. In 1993, Bacon and Kappe [8] complete and correct Trebenko's paper to produce a classification of 2-generator p-groups of class 2, p an odd prime. In 1999, Kappe, Sarmin and Visscher [9] corrected some technical errors on parameters in original 1993 paper. The corrected classification of 2-generator p-groups of class 2 (p an odd prime) is given in the following theorem.

Theorem 2.1. [8],[9]

Let G be a finite 2-generator p-groups of class 2 (p an odd prime). Then G is isomorphic to exactly one group of the following three types:

(2.1.1) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a, b] = c$, $[a, c] = [b, c] = 1$, $|a| = p^\alpha$, $|b| = p^\beta$, $|c| = p^\gamma$, α, β, γ are integers, and $\alpha \geq \beta \geq \gamma$.

(2.1.2) $G \cong \langle a\rangle \rtimes \langle b\rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}$, $|a| = p^\alpha$, $|b| = p^\beta$, $|[a, b]| = p^\gamma$, α, β, γ are integers, $\alpha \geq 2\gamma$ and $\beta \geq \gamma$.

(2.1.3) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, where $[a,b] = a^{p^{\alpha-\gamma}}c$, $[c,b] = a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$, $|a| = p^{\alpha}$, $|b| = p^{\beta}$, $|c| = p^{\sigma}$, $|[a,b]| = p^{\gamma}$, α, β, γ, σ are integers, $\beta \geq \gamma > \sigma$ and $\alpha + \sigma \geq 2\gamma$.

Theorem 2.1 shows that 2-generator p-groups of class 2 (p an odd prime) are partitioned into three infinite families of non-isomorphic p-groups with each family having a common parameterization.

Theorem 2.1 can be written in terms of generators and relations. Magidin [10 had a characterization for 2-groups. Following [10] the next theorem gives 2-generator p-groups of class 2 (p an odd prime) in terms of generators and relations.

Theorem 2.2. Let G be a finite 2-generator p-groups of class 2 (p an odd prime). Then G is isomorphic to exactly one group of the following three types:

(2.2.1) $G \cong \langle a, b \mid a^{p^{\alpha}} = b^{p^{\beta}} = [a,b]^{p^{\gamma}} = [a,b,a], [a,b,b] = e\rangle$ where α, β, γ are integers, and $\alpha \geq \beta \geq \gamma$.

(2.2.2) $G \cong \langle a, b \mid a^{p^{\alpha}} = b^{p^{\beta}} = [a,b,a] = [a,b,b] = e, a^{p^{\alpha-\gamma}} = [a,b]\rangle$ where α, β, γ are integers, and $\alpha \geq 2\gamma$ and $\beta \geq \gamma$.

(2.2.3) $G \cong \langle a, b \mid a^{p^{\alpha}} = b^{p^{\beta}} = [a,b,a] = [a,b,b] = e, a^{p^{\alpha+\sigma-\gamma}} = [a,b]^{p^{\sigma}}\rangle$ where $\alpha, \beta, \gamma, \sigma$ are integers, and $\alpha + \sigma \geq 2\gamma$ and $\beta \geq \gamma > \sigma$.

In [10], Magidin combines (2.2.2) and (2.2.3) into one type. The groups of (2.2.2) can be obtained from (2.2.3) when $\sigma = 0$.

Proof. Since $|a| = p^{\alpha}$, $|b| = p^{\beta}$, $|c| = |[a,b]| = p^{\gamma}$, then $a^{p^{\alpha}} = b^{p^{\beta}} = [a,b]^{p^{\gamma}} = e$. We have $[a,c] = [a,[a,b]] = a^{-1}[a,b]^{-1}a[a,b] = e$. Note that $[a,[a,b]] = [[a,b],a] = [a,b,a]$. The other commutator, $[b,c] = [b,[a,b]] = b^{-1}[a,b]^{-1}b[a,b] = e$ and $[b,[a,b]] = [[a,b],b] = [a,b,b]$. Hence, the presentation for groups in (2.2.1) is obtained satisfying $\alpha \geq \beta \geq \gamma$.

Now, $c = [a,b]a^{-p^{\alpha-\gamma}}$ and $c^{p^\sigma} = e$. Therefore, $[a,b]^{p^\sigma} = (a^{p^{\alpha-\gamma}})^{p^\sigma}$. For $\gamma > \sigma$, $([a,b]^{p^\sigma})^{p^{\gamma-\sigma}} = e$. Thus $|[a,b]| = p^\gamma$ and satisfying $\beta \geq \gamma > \sigma$ and $\alpha + \sigma \geq 2\gamma$ we have a presentation for groups in (2.2.3). If $\sigma = 0$, $[a,b] = a^{p^{\alpha-\gamma}}$ with $\beta \geq \gamma$ and $\alpha \geq 2\gamma$. Hence we have another presentation for (2.2.2). □

From now on, we refer to Theorem 2.2 for 2-generator p-groups of class 2 (p an odd prime). We shall refer the groups in (2.2.1), (2.2.2) and (2.2.3) as groups of Type 1, Type 2 and Type 3 respectively.

This research was aided by the use of computational methods in group theory. In the next section, the background and application of the computational group theory system GAP used in this thesis is reviewed.

2.4 Groups, Algorithms, and Programming (GAP)

GAP [11] is a system for computational discrete algebra, with emphasis on computational group theory. GAP provides a programming language, a library of functions that implement algebraic algorithms written in the GAP language as well as libraries of algebraic objects such as for all non-isomorphic groups up to order 2000.

GAP was started in 1986 at Lehrstuhl D für Mathematik, RWTH Aachen, Germany. After 1997, the development of GAP was coordinated from St Andrews. Nowadays, GAP is developed and maintained by GAP centers in Aachen and Braunschweig in Germany, Fort Collins in the USA and St Andrews in Scotland.

GAP was used in this research to construct examples of 2-generator p-groups of class 2 and to gain insight into the structure of these groups. Fixing a prime p and the parameters, a finitely presented group G is obtained from Theorem 2.1. A p-quotient routine is used to obtain a power conjugate presentation for G. GAP contains effective methods for computing with power conjugate groups. In particular computing the conjugacy classes of a group G. The underlying algorithms implemented by GAP for power conjugate presentations can be found in [12]. Programmes used are presented in the Appendix.

2.5 Conclusion

The exact number of conjugacy classes for nilpotent groups are not in the literature. The research in this thesis initiates toward finding the formulas for the exact number of conjugacy classes.

The next chapter provides formal definitions, general results and preliminary results used in this thesis.

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APPENDIX A

PUBLICATION/PRESENTATION IN SEMINARS/CONFERENCES

Some results from this research have been published/presented in seminars/conferences or submitted as listed in the following:

Papers published/presented in International Proceedings

P1. Azhana Ahmad, Robert Fitzgerald Morse and Nor Haniza Sarmin, On The Conjugacy Classes of 2-Generator *p*-Groups of Class 2, *Zassenhaus 2007 Group Theory Conference*, Saint Louis University, Missouri, USA, 16-18 March 2007.

P2. Azhana Ahmad, Robert Fitzgerald Morse, Nor Haniza Sarmin and Satapah Ahmad, On Counting The Number of Conjugacy Classes of 2-Generator *p*-Groups of Nilpotency Class 2, *2nd International Conference on Mathematical Sciences (ICOMS 2007)*, Ibnu Sina Institute, Universiti Teknologi Malaysia, 28-29 May 2007.

P3. Azhana Ahmad, Robert Fitzgerald Morse, Nor Haniza Sarmin and Arturo Magidin, Counting The Number of Conjugacy Classes for 2-Generator *p*-Groups of Nilpotency Class 2, *Conference on Computational Group Theory and Cohomology*, Harlaxton College, Grantham, UK, 4-8 August 2008.

P4. Azhana Ahmad, Robert Fitzgerald Morse and Nor Haniza Sarmin, The Number of Conjugacy Classes for Groups of Type 2 of 2-Generator *p*-Groups of Nilpotency Class 2, *Proceedings of The 3rd IMT-GT Regional Conference on Mathematics, Statistics and Applications (IMT-GT 2007)*, The Gurney Hotel, Penang, Malaysia, 5-6 December 2007, pp 8-11.