MATEMATIKA, 2004, Jilid 20, Bil. 2, hlm. 125–132

A Glimpse on the Relationship between Feynman Integral and Integrable Systems

Zainal Abdul Aziz
Ibnu Sina Institute for Fundamental Science Studies, Faculty of Science
Universiti Teknologi Malaysia, 81310 UTM Skudai, Johor, Malaysia
E-mail: zaa@mel.fs.utm.my

Abstract We briefly review recent attempts to relate the concept of Feynman integral and integrable systems. This constitutes an endeavour on our part in making the Feynman path integral into a mathematically meaningful entity.

Keywords Feynman Integral, Integrable System, Virasoro Conjecture.

Abstrak *Kami mengimbau secara ringkas kajian-kajian terkini yang mengaitkan konsep kamiran Feynman dan sistem-sistem terkamir. Hal ini merangkumi usaha kami untuk menjadikan kamiran lintasan Feynman sebagai suatu entiti matematik yang bermakna.*

Katakunci *Kamiran Feynman, Sistem Terkamir, Konjektur Virasoro.*

1 Introduction

In the previous century, there exist two fundamental theories in theoretical physics – general relativity theory and quantum field theory. These two grand theories describe the same world at different scales, which are much related to the celebrated constants of nature: speed of light $c \approx 3 \times 10^8$ ms^{-1}, Plancks constant $\hbar \approx 6.63 \times 10^{-34}$ m^2 kgs^{-1} and gravitational constant $G \approx 6.67 \times 10^{-11}$ m^3 kg^{-1}s^{-2}. The general relativity theory describes gravitational forces at the astronomical scale, whilst quantum field theory relates the interaction of fundamental particles, electromagnetic force, the weak and strong forces. There exists inconsistency between the two theories, i.e., formal quantisation of general relativity theory would generate infinite formulae. This inconsistency between these two fundamental theories of theoretical physics consequently becomes a very important problem that attracts many researchers including Einstein to propose and formulate a grand unified theory (GUT). Einstein developed general relativity theory in order to resolve the inconsistency between special relativity theory and Newtonian gravity. Quantum field theory was developed to suit Maxwells electromagnetic theory and the special relativity theory together with the nonrelativistic quantum mechanics. Intrinsically both theories differ markedly in their approaches. In Einstein's discovery of general relativity, the logical framework had been established by researchers like Lorentz and Poincare, whilst the Riemannian geometry was found to be the exact mathematical framework. In the development of quantum field theory, there does not exist as either *a priori* basic conceptual or an appropriate exact mathematical model. The rays of understanding largely came only from the experimental evidences that play such significant roles. Influential works by Witten [18, 19, 20] *via* string theory, Jones [12] with respect to loop theory and Drinfeld [4] *via* quantum groups have changed completely this scenario.

Both the classical and quantum mechanics are characterized by two basic concepts: *states* and *observables.* The measurements are operations on the physical systems. In classical mechanics, the state is a point in the manifold (symplectic) M (phase space) and observable is the function on M. In quantum mechanics, the state of a system probably corresponds to a vector unit in a Hilbert space H while the observable quantity corresponds to the operator (self-adjoint and noncommutative) on H. Integrating special relativity to both the above-mentioned theories would direct classical mechanics towards general relativity, and quantum mechanics to quantum field theory. The path from classical mechanics to quantum mechanics is known as quantisation. Drinfeld and Witten studied the relationship between these theories through different perspectives. Following Drinfeld [4], the connection between classical mechanics and quantum mechanics can be incorporated in terms of observables. In both cases, the observables form an associative algebra that is commutative in the classical sense and noncommutative in quantum perspective. Consequently, quantisation illustrates the transformation of commutative algebra to noncommutative algebra. States are described by Drinfeld [4] in terms of Hopf algebra. This algebraic approach of Drinfeld would bear the quantum group concept that eventually link to the completely integrable system in statistical mechanics, Yang-Baxter equation and deformations of Lie algebra. Witten's approach is topological. Quantisation is expressed in terms of states, using the Feynman path integral method. Observables are represented by topological invariants. Subsequently, Witten's approach encompasses the string theory, topological quantum field theory, conformal field theory and Chern-Simons action function. The works by Jones are related to that of Drinfeld via Jones polynomial and associative representation of braid groups, their connection with the integrable system in statistical mechanics, and the combinatorical relationship between Yang-Baxter equation and Jones polynomial for loops. Similarly, the relationship between Jones' works with that of Witten's is exhibited through an interpretation of Jones polynomial in terms of topological quantum field theory. In addition, Chern-Simons Lagrangian in the Feynman integral can be used to generate Jones polynomial, or otherwise can be used to retrace the formal functional integrals as explicit mathematical quantities. The afore-mentioned scenario has opened up application of methods of field, string and integrable system theories to important progress, opening entirely new points of view in the context of Gromov-Witten invariants, Donaldson invariants and quantum-group invariants for knots and links. These would undoubtedly advance further our understanding of the bigger picture of string theory, i.e., the remarkable M-theory.

The purpose of this article is to trace out briefly recent attempts by researchers in relating the concept of Feynman integral to the notion of integrable systems. We believe that this would constitute the current course of action on our part to making the Feynman path integral into a mathematically meaningful entity (Zainal [21, 22]). Our main source of motivation is derived from a statement made by the eminent mathematician Prof. Michael Atiyah [1]: "*The Feynman integrals will have been given precise meanings, not by analysis, but by a mixture of combinatorial and algebraic techniques*".

2 The Form of the Relationship

Briefly, we think that Witten's [19] conjecture can be used to formulate the interesting connection between the two entities, the 'elusive' Feynman integral and the 'predictable' integrable systems (Zainal [21, 22]), that originally seems disparate in nature. Technically, this refers to a generator function of intersection numbers on moduli space for stable curves and the τ-function of the KdV (Korteweg-de Vries) hierarchy. This conjecture is derived fundamentally from two approaches with respect to two-dimensional quantum gravity. Essentially, the correlator of the two-dimensional quantum gravity is the Feynman integral with respect to the 'metric space' of the two-dimensional topological real space. One of the methods of evaluation of the path integral involves the topological field theoretic technique and finally reduces to the integration with respect to the moduli space of curves. Another method considers an approximation to the metric space with piecewise flat metric and subsequently taking the appropriate continuous limit. In the former approach, the free energy becomes a function that is defined geometrically as

$$\tau^{\,pt}\,(t_0, t_1,\, ...\,) = \exp\left[\sum_{g=0}^{\infty} \hbar^{g-1}\, F_g^{pt}\,(t_0, t_1,\, ...\,)\right],$$

where $F_g^{pt}\,(t)$ is the generator function of the intersection numbers on the moduli space of stable curves of genus g with n marked points, $\bar{M}_{g,n}$, or the genus g descendent potential of a compact symplectic manifold of $X \equiv pt$:

$$F_g\,(t_0, t_1,\, ...) := \sum_n \frac{1}{n\,!} \int_{\bar{M}_{g,n}} \prod_{i=1}^{n} \left(\sum_k t_k\, \psi_{\,i}^{\,k}\right),$$

where ψ_i^k are formal variables (summation convention on k).

In the latter approach, the generator function in the double scaling limit would result in a τ-function of the KdV hierarchy. We observe the statement of Witten's hypothesis that τ^{pt} is the τ-function (Virasoro's invariant solution) of the KdV hierarchy is justified on the fact that, there must necessarily exist only one quantum gravity. In addition, with reference to the moduli space geometry, we can deduce that τ^{pt} satifies the following string equation (summation convention on ν)

$$\frac{\partial F_0\,(t)}{\partial t_1^1} = \frac{1}{2}\,(t_0,\, t_0) + \sum_{n=0}^{\infty} \sum_{\upsilon} t_{n+1}^{\upsilon}\, \frac{\partial F_0\,(t)}{\partial t_n^{\upsilon}}$$

It is a basic fact in KdV hierarchy theory (or generally Kamdotsev-Patviashvili (KP)) that uniquely the string theory would determine one of the τ-functions of the KdV hierarchy from all the τ-functions that are being parametrized by Sato grassmanian (Date et al [3]). Finally Witten [20] formulated a generalised conjecture: analogously the generator function $\tau^{\,r-spin}$ of the intersection numbers on the moduli space of r-spin curves should be identified as the τ-function of the Gelfand-Dikii hierarchy (r-hierarchy or generalized KdV). When $r = 2$, the conjecture is reduced to the original conjecture, i.e., 2-KdV or KdV equation. This special case has been proven by Kontsevich [13] and recently there is a new proof proposed by Okounkov and Pandharipande [17]. Nevertheless, we are of the opinion that up to now, the generalised conjecture is still an open problem. In the following section, we briefly discuss recent attempts in bringing to light certain aspects of this open problem.

3 Form of the Generalized Conjecture

Another generalization of Witten's conjecture [19] was proposed by Eguchi et al. [8]. In generalizing Witten's conjecture to a compact symplectic manifold of X, in particular, to any projective smooth variety X, it is obvious that τ^{pt} should be replaced by

$$\tau^X(t) = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} F_g^X(t)\right),$$

where $F_g^X(t)$ is the generating function of genus g Gromov-Witten invariants with descendents for X. Furthermore, the corresponding string equation $L_{-1}^X \tau^X$ on X is basically induced from the equation from a point. Based on that Eguchi et al. [8] managed to define $\{L_n^X\}$ for $n \geq -1$, satisfying the Virasoro relations. They conjectured that

$$L_n^X \tau^X(t) = 0, \quad n \geq -1.$$

This conjecture is normally referred to as the Virasoro conjecture. They were also able to give strong evidences for their conjecture in genus zero and a proof of $L_0^X \tau^X = 0$. Later Liu and Tian [16] proved the genus case in general. Using a different method, Dubrovin and Zhang [6] established the genus one case of Virasoro conjecture for conformal semisimple Frobenius manifolds.

The two recent major developments in Virasoro conjecture are Okounkov and Pandharipande's [17] proof for algebraic curves and that of Givental's [11] for toric Fano manifolds. In terms of formula, given a semisimple Frobenius manifold H of dimension N, he defines an operator $\hat{O}_H = \exp(\hat{o}_H)$ and a generating function

$$\tau_G^H = \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} G_g^H(t^1, \ldots, t^N)\right) = \hat{O}_H \prod_{i=1}^{N} \tau^{pt}(t^i, \hbar)$$

The essential part of $\hat{o}_H$ is a quadratic differential operator of the form

$$\frac{\hbar}{2} \sum E_{kl}^{\mu v} \frac{\partial}{\partial t_k^{\mu}} \frac{\partial}{\partial t_l^{v}},$$

with E determined by the semisimple Frobenius manifold H. The Feynman rules then dictate a formula for G_g. When the Frobenius manifold comes from geometry, i.e., $H = QH^*(X)$ (or on cohomology space $H^*(X, Q\{\{q\}\})$ over a suitable Novikov ring $Q\{\{q\}\}$), Givental conjectures that his combinatorial model is the same as the geometric model, i.e., $G_g^H = F_g^X$ when $H = QH^*(X)$. Some of the cases include toric Fano manifolds.

In order to prove the Virasoro conjecture, one defines the Virasoro operators for semisimple Frobenius manifold H by

$$L_n^H(t) = \hat{O}_H \prod_i L_n^{pt}(t_i)\, \hat{O}_H^{-1}.$$

It is obvious that L_n^H also satisfy Virasoro relations. It can be shown that this definition coincides with the definition of Eguchi et al [8] in the geometric case $H = QH^*(X)$ and coincides with Dubrovin and Zhang [6] in the conformal setting. The above Virasoro relations for H follow from Virasoro for a point (Kontsevich's theorem). It is also obvious that, when $H = QH^*(X)$, the proof of the Virasoro conjecture results in the identification of F_g^X and G_g^H.

Giventals [11] model works for any semisimple Frobenius manifolds and thus also for the Frobenius manifolds $H_{A_{r-1}}$ of the miniversal deformation space of A_{r-1} singularity. It turns out that this Frobenius manifold is isomorphic to the Frobenius manifold defined by the genus zero potential of r-spin curves and this leads to the fact that $\tau_G^{H_{A_{r-1}}}$ is a τ-function of r-KdV hierarchy. As in the case of the KdV, it can be shown that both $\tau_G^{H_{A_{r-1}}}$ and τ^{r-spin} satisfy the same string equation. Therefore, in order to prove Witten's conjecture, we have to show $G_g^{H_{A_{r-1}}} = F_g^{r-spin}$.

In order to resolve many pertinent questions relating to the afore-mentioned open problem, Lee [14] proposed the following question: Is $G_g = F_g$? That is, does the combinatorial construction coincide with the geometric one when both are available? His approach is to show that G_g satisfies enough geometric properties so that they have to be equal by some "uniqueness theorems". More specifically, the geometric properties are the tautological relations due to some functorial properties built in the Gromov-Witten theory. For simplicity, we can categorize the tautological relations as follows:

(a) genus zero tautological equations as the following genus zero equations

$$\frac{\partial F_0(t)}{\partial t_1^1} = \sum_{n=0}^{\infty}\sum_{\upsilon} t_n^{\upsilon}\frac{\partial F_0(t)}{\partial t_n^{\upsilon}} - 2F_0(t), \text{ the Dilaton Equation}$$

$$\frac{\partial F_0(t)}{\partial t_0^1} = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty}\sum_{\upsilon} t_{n+1}^{\upsilon}\frac{\partial F_0(t)}{\partial t_n^{\upsilon}}, \text{ the String Equation}$$

$$\frac{\partial^3 F_0(t)}{\partial t_{k+1}^{\alpha}\partial t_l^{\beta}\partial t_m^{\gamma}} = \sum_{\mu}\frac{\partial^2 F_0(t)}{\partial t_k^{\alpha}\partial t_0^{\mu}}\ \frac{\partial^3 F_0(t)}{\partial t_0^{\mu}\partial t_l^{\beta}\partial t_m^{\gamma}}, \text{ the Topological Recursion Relations (TRR),}$$

for all α, β, γ and all $k, l, m \geq 0$ and $F_0(t)$ is the genus zero descendent potential which satisfies many properties due to the geometry of the moduli spaces.

(b) genus one tautological equations as the following genus one equations; genus one Getzler's [9] equation and genus one TRR.

(c) genus two tautological equations as the three set equations in genus two by Getzler [9], Belorousski and Pandharipande [2] and by Mumford of the form (with summation convention)

$$-\langle\partial_2^x\rangle_2 + \langle\partial_1^x\partial^\mu\rangle\langle\partial^\mu\rangle_2 + \langle\partial^x\partial^\mu\rangle\langle\partial_1^\mu\rangle_2 - \langle\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^\upsilon\rangle\langle\partial^\upsilon\rangle_2 + \tfrac{7}{10}\langle\partial^x\partial^\mu\partial^\upsilon\rangle\langle\partial^\mu\rangle_1\langle\partial^\upsilon\rangle_1$$
$$+\tfrac{1}{10}\langle\partial^x\partial^\mu\partial^\upsilon\rangle\langle\partial^\mu\partial^\upsilon\rangle_1 - \tfrac{1}{240}\langle\partial^\mu\partial^\nu\partial^\upsilon\rangle\langle\partial^x\partial^\mu\rangle_1 + \tfrac{13}{240}\langle\partial^x\partial^\mu\partial^\mu\partial^\upsilon\rangle\langle\partial^\upsilon\rangle_1$$
$$+\tfrac{1}{960}\langle\partial^x\partial^\mu\partial^\mu\partial^\upsilon\partial^\upsilon\rangle = 0$$

In genus one, the uniqueness was first shown by Dubrovin and Zhang [5], in other words, the genus one potential for a conformal Frobenius manifold is uniquely determined by genus one tautological equations. Similarly, the genus uniqueness theorem is formulated by Liu [15], which stated that the genus two descendent potentials for any conformal semisimple Frobenius manifolds are uniquely determined by genus two tautological equations. It is worth noting that whether this uniqueness theorem, or any weaker version, holds for non-conformal semisimple Frobenius manifolds remains unknown.

Another way of looking at the uniqueness theorem is given by Dubrovin and Zhang [7], by proving that the Virasoro conjecture plus $(3g-2)$ −jet property uniquely determines τ-function for any semisimple analytic Frobenius manifold. The $(3g-2)$ −jet property in the geometric Gromov-Witten theory is proved by Getzler [10] and by Givental [11] in the context of semisimple Frobenius manifolds. It is also expected to hold for the τ^{r-spin} and therefore a proof of the Virasoro conjecture for τ^X should also answer positively the open problem. Lee [14] was able to show that G_2 satisfied genus two tautological equations by Mumford abovementioned, Getzler [9] and Belorousski and Pandharipande [2]. Actually, similar statement in genus one, that is G_1 satisfies genus one TRR and Getzler's case, is shown in the conformal case by Dubrovin and Zhang [9]. The above results, combined together are expected to imply that Witten's generalized conjecture and Virasoro conjecture for manifolds with conformal semisimple quantum cohomology hold up to genus two. Further investigations are needed to ensure the validity of this conclusion (see, e.g., Zhou [23]).

4 Concluding Remarks

The discussion above relates one of the surprises of modern mathematics, that is the appearance of the phenomenal KdV equation in the organization of new invariants of quantum cohomology X (or simply the symplectic manifolds X), and in the process becomes tie up with the 'elusive' Feynman integral. In other words, certain special differential equations (a subset of those known as integrable) have surprisingly appeared predominantly in topological conformal field theory. The appearance of these equations in quantum cohomology is further reflected in the well-known "Virasoro Conjecture", which asserts that the quantum cohomological invariants are fixed points of symmetries consisting of half a Virasoro algebra. These algebras are known to act on many mathematical structures, in particular on the solutions sets of most integrable equations. As our concluding remarks, lo and behold, there is little speculation or conjecture as to the reason for this truly amazing and unlikely mating of two entirely different subjects of integrable systems and topological invariants in terms of the Feynman integral.

Acknowledgements
This research is partially supported by MOSTI/RMC Grant for Fundamental Sciences, Vot. No. 75086. The author would like to express his deepest gratitude to his fellow researcher, Assoc. Prof. Dr Nor Haniza Sarmin, for her unwavering resourcefulness. The author likes to thank the referee whose comments led to some improvements of this article.

References

[1] M.Atiyah, *Responses to "Theoretical Mathematics toward a cultural synthesis of Mathematics and Theoretical Physics", by A. Jaffe and F. Quinn,* Bull. Amer. Math. Soc.30 (2) (1994), 1–2.

[2] P. Belorousski & R. Pandharipande, *A descendent relation in genus 2.* Ann. Scuola Norm. Sup. Pisa Cl.. Sci. 29, (2000), 171–191.

[3] E. Date, M. Jimbo, M. Kashiwara & T. Miwa, *Transformation groups for soliton equations,* in M. Jimbo & T. Miwa (eds.), Proc.RIMS Symposium on Nonlinear Integrable Systems Classical and Quantum Theory, Singapore: World Scientific; (1983), 39–119.

[4] V.G. Drinfeld, *Quantum Groups,* in A.M. Gleason, (eds), Proc. Internat. Congress Math., Berkeley, Providence: Amer. Math Soc. Vol. 1: (1987) 798–820

[5] B. Dubrovin & Y. Zhang, *Bi-Hamiltonian hierarchies in 2D topological field theory at one-loop approximation,* Comm. Math. Phys. 198, (1998), 311–361.

[6] B. Dubrovin & Y. Zhang, *Frobenius manifolds and Virasoro constraints,* Selecta Math. (N.S.) 5, (1998), 423-466

[7] B. Dubrovin & Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants,* (2001) Preprint (math.DG/0108160).

[8] T. Eguchi, K. Hori & C-S. Xiong, *Quantum cohomology and Virasoro algebra,* Phys. Lett. B 402: (1997) 71-80.

[9] E. Getzler, *Topological recursion relations in genus 2,* in Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Scientific, New Jersey, USA, (1998), 73–106.

[10] E. Getzler, *The jet-space of a Frobenius manifold and higher-genus Gromov-Witten invariants,* (2002) Preprint (math.AG/0211338).

[11] A. Givental, *Semisimple Frobenius structures at higher genus,* Internat. Math. Res. Notices, 23, (2001), 1265-1286.

[12] V.F.R. Jones, *Von Neumann algebras in mathematics and physics,* in I. Satake (ed.), Proc. Internat. Congress Math., Kyoto. Tokyo: Springer-Verlag Vol 1: (1991), 121–138

[13] M. Kontsevich, *Integration theory on the moduli space of curves and the matrix Airy functions,* Commun. Math. Phys 147, (1992), 1–23

[14] Y.P. Lee, *Witten's conjecture, Virasoro conjecture, and invariance of tautological equations,* (2003), draft.

[15] X. Liu, *Quantum product on the big phase space and the Virasoro conjecture,* Adv. Math. 169, (2002), 313–375.

[16] X. Liu, & G. Tian, *Virasoro constraints for quantum cohomology,* J. Differential Geom. 50, (1998) 537–590.

[17] A. Okounkov & R. Pandharipande, *Gromov Witten Theory, Hurwitz numbers and Matrix models I,* (2001) Preprint . (math. AG/ 0101147)

[18] E. Witten, *Physics and geometry,* in A.M. Gleason, (eds.), Proc. Internat. Congress Math., Berkeley. Providence: Amer. Math Soc. Vol. 1: (1987), 267–303.

[19] E. Witten, *Two dimensional gravity and intersection theory on moduli space,* in Surveys in Differential Geometry, Bethlehem: Lehigh Univ.; (1991), 243–310

[20] E. Witten, *Algebraic geometry associated with matrix models of two dimensional gravity,* in Topological Methods in Modern Mathematics, Houston: Publish or Perish ; (1993), 235–269

[21] A.A. Zainal, *Functional integral solution of the complex diffusion equation, with complex quadratic potential in a classical space,* Bull. Malaysian Math. Sc. Soc.(Second Series), 24, (2001) 111–128.

[22] A.A. Zainal, *Kamiran Feynman dan Sistem Terkamir Lengkap,* in Ummul Khair Salma Din et al. (eds.), Prosiding Seminar Mengenang Prof. Dr. Shaharir Mohamad Zain, 3 June 2004, Bangi: FST UKM and PERSAMA; (2004), 30–54. (in Malay)

[23] J. Zhou, *Proof and interpretation of a string duality* Math. Res. Lett. 11, (2004), 213–229.