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VARIATIONAL METHOD IN THE DESIGN OF AN OPTIMUM SOLAR WATER HEATER STORAGE TANK

Suhaimi Misha*, Amer Nordin Darus

Faculty of Mechanical Engineering
Universiti Teknologi Malaysia
81310 UTM, Skudai
Johor Darul Takzim, Malaysia

ABSTRACT

Solar water heaters are now have been accepted as a reliable source of providing hot water heating in many domestic homes and are becoming more popular. Unfortunately, solar water heaters are still considered luxurious hardware in Malaysia. A cheaper and efficient solar water heater system is required to be designed. The only way to produce cheaper and efficient product is through the optimization process to obtain the minimum cost but still maintain the specification required or known as constraints. In this paper, the vertical water storage tank will be analyzed to obtain the optimum cost. Thermosyphon-flow solar water heating system is preferred for obvious economic reasons since they do not require circulation pumps and control units. Average temperature of the hot water in the storage tank is determined through temperature distribution simulation. The overall average temperature obtained is 49.3°C and is used to solve the optimization problem. The constraints in the optimization process are the tank volume and heat losses from the water in the tank. The Lagrange multiplier method, which is based on the differentiation of the objective function and the constraints is applied. The minimum cost obtained for storage tank volume of 225 litres is RM 1321.66.

Keywords: *Solar, solar water heater, thermosyphon-flow solar water heating, Lagrange multiplier method*

1.0 INTRODUCTION

Solar water heaters collect and convert incident solar radiation energy to useful thermal energy in the form of hot water. They essentially consist of solar collectors for absorbing and converting the solar energy and an insulated storage vessel or tank to store the heated water. In a passive system, the circulation of the water within the system is by a convection current caused by density differences of the water. Water which is heated in the collector tubes tends to rise to the top of the storage tank. Cold water replaces it from the bottom of the tank. A natural convection current within the system is thus set up. This passive system is also

* Corresponding author: E-mail: suhaimimisha@utem.edu.my

known by various other names such as gravity natural convection, or thermosyphon flow solar water heating system. During the day, as long as there is sufficient sunshine, water will always recirculate in this manner and hot water will accumulate at the top of the storage tank.

The details of experimental observation of temperature and flow distribution in a natural circulation solar water heating system and its comparison with the theoretical models can be found in Chuawittayawuth *et al.* [1]. Natural circulation solar water heating systems are available in varying collector geometries (and materials), storage tank capacities and specifications of individual components. The theoretical results compares well with the measured profile of absorber plate temperature around the riser tubes. A numerical and an experimental analysis of temperature inside a storage tank with natural convection was investigated by Oliveski *et al.* [2] The numerical and experimental results showed that, as time passes, there is a thermal stratification, separating the tank into two different regions: a stratified region at the bottom and another uniform region at the top. The thermal gradients along the radius also have been analysed. At the periphery, thermal gradients are maximum, encompassing the thermal boundary layer. In the center, there is no thermal radial gradient.

Oliveski *et al.* investigated and compared models for the simulation of hot water storage tank [3]. The numerical analyses were performed with two approaches: one using a two-dimensional model in cylindrical coordinates employing the finite volume method and another using a one-dimensional model. A turbulence model for low Reynolds numbers was added to the two-dimensional model in the mixed convection region. The two dimensional model was experimentally validated and then adopted as reference. Its results were compared to those obtained with one-dimensional models with a good agreement. Malaysian climatic conditions favour the use of solar energy for water heating. Solar energy is cheap, safe to use, requires no transportation cost and also pollution-free. Solar water heater system is one of the devices that use the free solar energy. The hot water storage tank will be analysed to obtain the optimum cost.

2.0 MATHEMATICAL FORMULATION

Several governing equations are involved in this optimization process. The governing equations are derived from equations of total cost Equation (1), heat loss from the hot water storage tank Equation (2) and energy conservation in the hot water storage tank Equation (3). The governing equations involved are as follows:

Cost of tank, casing and insulator materials and welding process;

$$\begin{aligned}
 M = & [\pi r_1 L + \pi r_1^2] 2t_i \rho_{ss} D + [\pi r_2 (L+H_1+H_2) + \pi r_2^2] 2t_c \rho_{ss} D \\
 & + [\pi (r_2^2 - r_1^2) L + \pi r_2^2 H_1 + \pi r_2^2 H_2] \rho_i E + 4 \pi (r_1 + r_2) F
 \end{aligned}
 \tag{1}$$

The first term in right hand side of Equation (1) is the cost of tank wall material, the second item is cost of the casing material, the third item is cost of insulator and the last one is cost of welding at the top and bottom for the tank wall and casing. The tank material selected is stainless steel and the insulator is fiberglass wool.

Heat loss from the tank;

$$\begin{aligned}
 Q = & (T_{avg} - T_a) 2 \pi L \left/ \left(\frac{1}{h_1 r_1} + \frac{\ln(r_2/r_1)}{k_1} + \frac{1}{h_2 r_2} \right) \right. \\
 & + (T_t - T_a) \pi r_1^2 \left/ \left(\frac{1}{h_1} + \frac{H_1}{k_1} + \frac{1}{h_2} \right) \right. \\
 & + (T_{avg} - T_a) \pi r_1^2 \left/ \left(\frac{1}{h_1} + \frac{H_2}{k_1} + \frac{1}{h_2} \right) \right. \quad (2)
 \end{aligned}$$

The heat loss Equation (2) includes heat losses from side wall, top and bottom surfaces of the tank.

Energy equation [4]:

$$\rho_w C_p A \left[\frac{\partial T}{\partial \tau} + w \frac{\partial T}{\partial z} \right] = k_w A \frac{\partial^2 T}{\partial z^2} - h P (T - T_a) \quad (3)$$

with $h = \left(\frac{1}{h_1} + \frac{r_1}{k_1} \ln \left(\frac{r_2}{r_1} \right) + \frac{r_1}{h_2 r_2} \right)^{-1}$

A simplification in the hot water storage tank problem is obtained by assuming that the temperature distribution across any horizontal cross section in the tank is uniform, the variation of water properties may be taken as negligible and vertical velocity of water in the tank is also taken as uniform across each cross section. The energy equation for thermal transport in the water tank simplifies to one dimensional and transient equation. This equation has been validated by Oliveski et al. which shows a good agreement with two dimensional model and experiment [3].

According to Ong, about 80% of the hot water in storage tank can be assumed to be practically utilized and generally a person consumes about 45 l of hot water per day [5]. Assume the storage tank is designed for household of four peoples, therefore the storage tank capacity of about 225 l is required. The first constraint is the tank volume which equals to 225 l.

$$V = 225 \text{ l} = \pi r_1^2 L = 0.225 \text{ m}^3 \quad (4)$$

Conduction in the tank wall can be neglected due to the thickness of the material compared to the radius of the tank and the thickness of the insulator. The temperatures distributions in vertical axis of the tank need to be considered first to find the temperature value that really represents the average temperature for 10

hours duration. In the morning at around 8.00 a.m. to evening around 6.00 p.m, assume the solar energy is hot enough to maintain the temperature at the top of the tank at maximum temperature, which is about 75°C and the bottom of the tank is perfectly insulated. The appropriate initial condition is the water in the storage tank is at ambient temperature, 27°C because the warm water in the tank has been drawn out by the house occupant for showering purpose at early morning. As described earlier, the thermal stratification will be developed in the storage tank and the temperature profiles involve transient condition. Therefore, in order to obtain the average temperature within the 10 hours duration, simulation must be done. The average temperature obtained is based on the daytime. At night the average temperature will be lower than the daytime since at evening after 6.00 p.m. the hot water at the top of the tank will be drawn out for showering purpose. The top temperature will decrease and solar energy is not available at night, therefore it cannot maintain the temperature at maximum value and the average temperature at night is lower.

An efficient hot water storage tank can be produced by making the tank perfectly insulated (adiabatic), but practically it requires a thick insulator. The heat loss can be calculated by using average water temperature for the wall of the tank and bottom of the tank as well. It is more practical to use maximum temperature at the top of the tank since most of the times, the water temperature at the top is higher than average temperature. Based on the existing design of solar collector, the area 2.34 m² of solar collector is able to supply hot water at 75°C and the cold water inlet at the bottom of the tank is equal to ambient air temperature, 27°C. Initially the average temperature is assumed between the top temperature, 75°C and bottom temperature, 27°C which gives an average of 51°C. The maximum heat loss is assumed about 3% of the total internal energy and the height of the storage tank is 1 m. Generally the heat loss depends on the surface area, therefore the heat loss from Equation (2) can be written in inequality constraints, [6].

The initial and boundary condition for energy Equation (3) may be taken as:

$$\left. \begin{array}{l} \text{At } \tau = 0; \quad T(z) = T_a \\ \text{For } \tau > 0 : \quad T = T_i \quad \text{at } z = 0 \end{array} \right\} \quad \left. \begin{array}{l} \\ \frac{\partial T}{\partial z} = 0 \quad \text{at } z = L \end{array} \right\} \quad (5)$$

The governing equation and boundary conditions may be nondimensionalized by defining the following dimensionless temperature θ , time τ' and vertical distance Z as:

$$\theta = \frac{T - T_a}{T_i - T_a} \quad \tau' = \frac{\alpha \tau}{L^2} \quad Z = \frac{z}{L} \quad (6)$$

The dimensionless governing equation is then obtained as:

$$\frac{\partial \theta}{\partial \tau'} + W \frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial Z^2} - H \theta \quad (7)$$

where $W = \frac{wL}{\alpha}$ $H = \frac{hPL^2}{Ak_w}$; α is the thermal diffusivity of water

The initial and boundary conditions become:

$$\left. \begin{array}{l} \text{At } \tau = 0; \quad \theta(Z) = 0 \\ \text{For } \tau > 0: \quad \theta = 1 \text{ at } Z = 0 \quad \frac{\partial \theta}{\partial Z} = 0 \text{ at } Z = 1 \end{array} \right\} \quad (8)$$

3.0 METHOD OF SOLUTION

The Lagrange multiplier method, which is based on the differentiation of the objective function and the constraints will be applied. The Lagrange multipliers represent the sensitivity coefficients which are defined as the rate of change of the objective function with the constraint at the optimum. The value of the sensitivity coefficient gives an indication of the effect of the adjustment needed in the constraint in order to employ standard sizes and readily available component. Therefore, it is important to understand this optimization method and the basic concepts introduced by this approach.

This is the most important and useful method for optimization. It can be used to optimize functions that depend on a number of independent variables and when functional constraints are involved. As such, it can be applied to a wide range of practical circumstances, provided the objective function and the constraint can be expressed as continuous and differentiable functions and only equality constraints can be considered in the optimization process. According to Jaluria, the mathematical statements of this optimization problem are as follows [4].

The objective function, M is the total cost of the tank (from Equation (1)):

$$\begin{aligned} M = & [\pi r_1 L + \pi r_1^2] 421.632 + [\pi r_2 (L+H_1+H_2) + \pi r_2^2] 421.632 \\ & + [\pi (r_2^2 - r_1^2) L + \pi r_2^2 H_1 + \pi r_2^2 H_2] 138 + 140 \pi (r_1 + r_2) \end{aligned} \quad (9)$$

subject to the constraints:

$$G_1 = \pi r_1^2 L - 0.225 = 0 \quad (10)$$

$$G_2 = (T_{avg} - 27) 2 \pi L \left(\frac{1}{165 r_1} + \frac{\ln(r_2)}{0.036} - \frac{\ln(r_1)}{0.036} + \frac{1}{17 r_2} \right)^{-1} - 30 = 0 \quad (11)$$

$$G_3 = (75 - 27) \pi r_1^2 \left(\frac{1}{165} + \frac{H_1}{0.036} + \frac{1}{17} \right)^{-1} - 4 = 0 \quad (12)$$

$$G_4 = (T_{avg} - 27) \pi r_1^2 \left(\frac{1}{165} + \frac{H_2}{0.036} + \frac{1}{17} \right)^{-1} - 4 = 0 \quad (13)$$

The objective function, Equation (9) is obtained by substituting the material properties and cost into equation (1). The tank wall and casing thickness is 2 mm. The costs of stainless steel, fiberglass wool and welding are RM 13.50/kg, RM 3.00/kg and RM 35.00/m respectively. Inequalities constraints are often converted into equations before applying optimization methods. A common approach employed to convert an inequality into an equation is to use a value smaller than the constraint if a maximum is given. It can be done by introducing, often known as slack variables (ΔC), that indicate the difference from the specified limits, normally it is about 5 % or less from the actual value. In order to estimate the insulator radius and thickness (r_2 , H_1 and H_2) assumed average temperature, T_{avg} is 51°C and top temperature, T_t equals to 75°C as mentioned earlier.

According to Ong, the actual volume flow rate of the water flowing in a thermosyphon flow solar water heater was about 0.00251/s ($2.5 \times 10^{-6} \text{ m}^3/\text{s}$) per square meter of collector area [5]. Since the maximum temperature selected is 75°C, the collector panel area 2.34m² is chosen. Therefore the thermosyphon flow is about $5.85 \times 10^{-6} \text{ m}^3/\text{s}$. The parameters of the water storage tank are still unknown, therefore some assumptions need to be made to simulate the water temperature distribution in the storage tank. The assumptions must be based on the constraints that have been selected. For height of tank, L equals to 1 m, it will give radius of the tank, r_t equals to 0.2676 m. Then by choosing the heat losses constraint within 10 hours is about 3%, and using average temperature, T_{avg} equal to 51°C and top water temperature, T_t equal to 75°C an estimate of the insulator thickness was obtained. The estimated radius of the insulator, r_2 is 0.3183 m, insulator thickness is 0.0949 m and 0.0463 m for the top, H_1 and bottom, H_2 respectively. Now by using these values, the temperature distribution can be simulated. As a result the dimensionless governing energy equation for thermal transport in the water storage tank becomes:

$$\frac{\partial \theta}{\partial \tau} + 167.3748 \frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial Z^2} - 8.6595 \theta \quad (14)$$

Finally the above equation will be used to simulate the temperature distribution of the water in the storage tank.

The above equation is transient equation, therefore the temperature distribution changes as time increases. In order to obtain the single temperature that represents the overall average temperature within the 10 hours duration, the graph of temperature against height of the tank for each time needs to be plotted. The temperature distribution data is calculated for time interval of 1 hour, beginning

from zero to ten hours, meaning that eleven graphs will be plotted. Then for each graph the average temperature is determined using mean value theorem. Area under the curve can be determined by using trapezoidal method. Finally from the eleven average temperatures, the overall average temperature value that represents the average temperatures of the water in the tank within 10 hours duration is obtained. This overall average temperature value will be used in the analysis for constraint Equations (11) and (13). The overall average temperature obtained from the simulation is 49.3°C. By applying the same procedure as shown in Jaluria, [4] the optimum is obtained by solving nine Equations [6].

There are a total of nine simultaneous equations obtained and the unknowns are four multiplier ($\lambda_1, \lambda_2, \lambda_3$ and λ_4), corresponding to the four constraints and the five independent variables (r_1, r_2, L, H_1 and H_2). Therefore this system may be solved by numerical method to determine the values of the independent variables, which define the optimum, as well as the multiplier. By using the numerical method which is Newton's method all the unknowns can be determined. The values of independent variables are the optimum values whereas the multipliers are the sensitivity coefficients of the constraint.

4.0 RESULTS AND DISCUSSIONS

The dimensionless energy Equation (14) can be solved numerically by using MATLAB software. The results of the numerical solution are shown in Figures 1, 2 and 3. The figures show that the flow is transient in nature. Even though the time duration of 10 hours is quite long but the thermosyphon flow is too slow that the flow does not achieve steady state condition. Even though the velocity looks too small it gave significant effect on the temperature distribution. If the velocity is neglected the temperature distribution from the top to the bottom of the tank is very poor. By neglecting the velocity, the water in the storage tank can be assumed as a solid model and the heat transfer is due to conduction in the water itself from high temperature at the top to the cold water at the bottom. The thermal conductivity of water, k_w at 51°C is about 0.6416 W/m.K. The thermal conductivity of water is too low compared with solid material such as stainless steel which is 24 W/m.K. Due to the low thermal conductivity of water, the temperature distribution is very poor if the velocity is neglected as shown in Figure 4.

The overall average temperature obtained is 49.3°C as shown in Table 1. Actually, the thermal property of the water in the storage tank varies since the temperature distribution is different from the top to the bottom, but the temperature range is relatively small. Therefore, the variation in water properties may be taken as negligible. Initially the average temperature is assumed at 51°C. The difference of the thermal properties at 49.3°C and 51°C are too small and can be neglected. Therefore, the simulation of the energy equation based on thermal properties at 51°C can be accepted. Based on side wall heat loss constraint Equation 2, the heat loss from the side wall at day time is about 30 J/s. The heat loss of 30 J/s and average temperature at daytime is applied to determine the insulator thickness, and it will be used to design storage tank insulation. If the

average temperature at night is lower, it means the heat loss is less than 30 J/s which is acceptable as described in Misha [6].

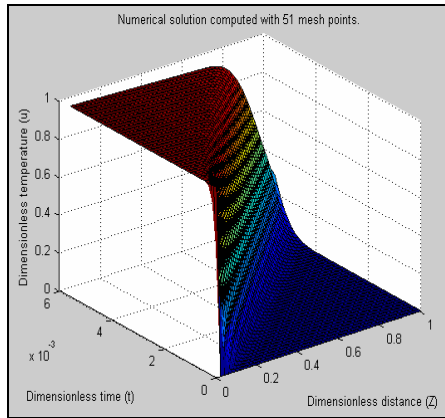


Figure 1: Numerical solution of temperature distribution in 3D

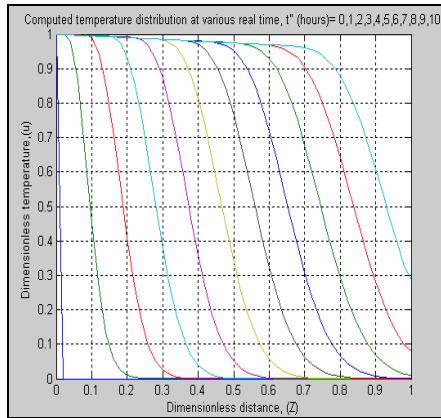


Figure 2: Variation of the temperature at various time interval (Time, $\tau = 0,1,2,3,\dots,10$ hours start from the left)

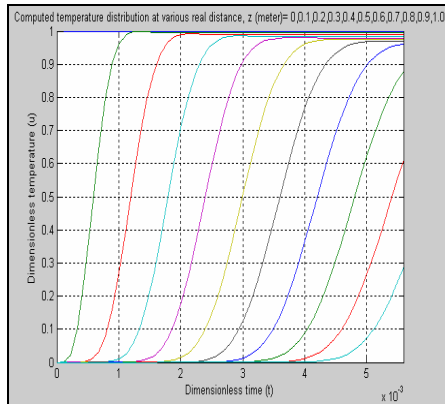


Figure 3: Variation of the temperature at various location (Distance, $z = 0, 0.1, 0.2, \dots, 1.0$ meters start from the left)

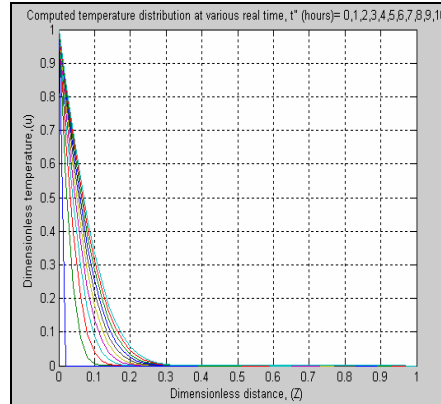


Figure 4: Variation of the temperature at various time (neglect velocity, assume $w = 0$)

Table 1: Average temperature at various time intervals

Time (hours)	Average Temperature (°C)
0	27.4
1	31.7
2	36.2
3	40.6
4	45.0
5	49.4
6	53.8
7	58.2
8	62.5
9	66.6
10	70.1
Overall Average Temperature	49.3

By assuming the height of the tank as 1 m, the estimated insulator thickness is about 5.07 cm. Since the percentage area of the side wall is larger than top and bottom areas, increasing the insulator thickness will give much effect on the cost of the insulator and casing materials. In order to have no heat loss at the side wall the overall heat transfer coefficient, h must equal to zero. This needs very thick insulation. It seems that, in order to get the overall heat transfer coefficient, h equal to zero, the radius of insulator, r_2 is almost infinity [6]. Therefore small amount of heat loss is allowed to obtain a practical and reasonable design of storage tank. If there is no heat loss at the side wall ($h=0$), the temperature distribution at certain time is about the same. However the temperature at each locations increase a bit for overall heat transfer, h equal to zero since there is no heat loss [6]. By applying the same method, the overall average temperature if no heat loss from the side wall, is 49.7°C. The difference of the overall average temperature for heat loss of 30J/s and no heat loss, is very small, about 0.4°C. Therefore, even though the heat loss at the side wall exists, it is very small and the water storage can be categorized as an efficient storage tank.

The set of nonlinear equations is solved by numerical method. The results are shown in Table 2. The values displayed in the result is after rounding to a certain decimal points, but to solve it numerically by computer, the program use 16 digits precision, otherwise the results may not converge to a solution. The optimum cost obtained is RM 1321.66. The sensitivity coefficient for volume constraint of Equation (13) is given by $-\lambda_1$ with $S_{cl} = 3888.53$. It means at the optimal proportions an extra 1 m³ volume of the storage tank would increase the cost to RM 3888.53 which is more than the original cost. Let us assume the constraint on the volume is increased from 0.225 to 0.226. By applying the same procedure to obtain the optimum independent variables but changing the volume constraint to 0.226 instead of 0.225, the rate of change, $(\partial M / \partial V)^*$ equals 3886.36 which is close to the sensitivity coefficient, S_{cl} . Let us change the side wall heat losses from 30 J/s to 31 J/s and then for top and bottom surfaces of heat loss change from

4 J/s to 5 J/s. The comparison between sensitivity coefficient, S_c and the rate of change, $(\partial M / \partial G)^*$ when the constraint is changed, are shown in Table 3. The negative sign of the sensitivity coefficient means that by increasing heat losses, the cost will decrease. In order to increase the heat losses from the storage tank, thinner insulator is required and it will reduce the cost of the insulator and casing. According to Jaluria, the slight difference in the change in M^* from the calculated value of sensitivity coefficient, S_c is the result of the nonlinear equations which make S_c a function of the independent variables and not a constant [4]. When the height, L moves away from the optimum (0.886 m) whether increasing or decreasing, the cost is increased. It means the optimum value obtained is a minimum.

Table 2: Unknown values obtained from numerical solution

Optimum unknowns	Results
r_1	0.2843
r_2	0.3276
H_1	0.1074
H_2	0.0486
L	0.8860
λ_1	-3888.53
λ_2	4.8860
λ_3	13.1774
λ_4	6.1220

Table 3: Comparison between S_c and $(\partial M / \partial G)^*$

Constraints	Sensitivity coefficient, S_c	Rate of change, $\left(\frac{\partial M}{\partial G}\right)^*$
G_1	3888.53	3886.36
G_2	-4.89	-4.72
G_3	-13.18	-10.68
G_4	-6.12	-4.93

5.0 CONCLUSIONS

A simplification in the hot water storage tank problem is obtained by assuming that the temperature distribution across any horizontal cross section in the tank is uniform, the variation of water properties may be taken as negligible and vertical velocity of water in the tank is also taken as uniform across each cross section. The temperature distribution simulation is done by using guess storage tank dimensions which are based on the constraints and current design. The overall

average temperature obtained is 49.3 °C. The overall average temperature is used in the constraints Equations (11) and (13) to solve the optimization problem.

The optimization method used is Lagrange multiplier. The optimum cost obtained is RM 1321.66. The Lagrange multipliers also provide information, through the multipliers, on the sensitivity of the optimum with respect to changes in the constraint. This parameter is useful in adjusting the design variables to come up with the final design. The optimum values of the storage tank dimensions are shown in Table 2. The sensitivity coefficients of each constraint are shown in Table 3. In order to get optimum price for the overall system of solar water heater, further research may be done to get the optimum of other components such as the solar collector and cold water feed.

NOMENCLATURE

r_1	radius of tank, m
r_2	radius of insulator, m
L	height of tank, m
H_1	insulator thickness at the top of the tank, m
H_2	insulator thickness at the bottom of the tank, m
A	cross-sectional area of tank, m ²
C_p	specific heat of water, kJ / kg °C
D	price of stainless steel, RM/kg
E	price of fiberglass wool, RM/kg
F	welding cost, RM/m.
G_1	constraint 1, m ³
G_n	constraint n , J/s ; $n = 2,3,4$
h	overall heat transfer coefficient, W/m ² .K
h_1	convective heat transfer coefficient of the water, W/m ² .K
h_2	convective heat transfer coefficient of the ambient air, W/m ² .K
k_l	thermal conductivity of the fiberglass wool, W/m.K
k_w	thermal conductivity of water, W/m.K
P	perimeter of tank, m
S_{cl}	sensitivity coefficient constraint 1
T	water temperature in the tank, °C
T_t	top water temperature in the tank, °C
T_a	ambient temperature, °C
T_{avg}	average water temperature in the tank, °C
V	volume of tank, m ³
w	average vertical water velocity in the tank, m/s
ρ_{ss}	density of tank material (stainless steel), kg / m ³
ρ_i	density of insulator (fiberglass wool), kg / m ³
ρ_w	density of water, kg / m ³
τ	physical time, s
λ_n	Lagrange multiplier constraint n ; $n = 1,2,3,4$

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