ANALYTICAL AND NUMERICAL METHODS FOR THE RIEMANN PROBLEM

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ABSTRACT

In this survey we consider the classical and new methods for computing the analytical and numerical solutions to the Riemann problem, a class of boundary value problems for analytic functions, in a simply connected region Ω^+ with smooth boundary $\Gamma = \partial\Omega^+$ in the complex plane. The classical methods for solving this problem based on reducing the Riemann problem to the Dirichlet problem or to the Hilbert problem where it is required the availability of a suitable conformal mapping from Ω^+ onto the unit disk $\mathbb{D}$. Recently, the authors introduce a new method for solving the Riemann problem by transforming its boundary condition to a Fredholm integral equation of the second kind with the generalized Neumann kernel. This method has several advantages in terms of numerical operations as well as ease in programming. This paper sketches these classical and new methods and shows the advantages of our method for solving the Riemann problem using Fredholm integral equations.

Keywords: *Riemann problem, Hilbert problem, Dirichlet problem, Fredholm integral equation, eigenvalue, index of functions.*
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1. INTRODUCTION

Riemann problem is one of the basic subjects in the theory of the analytic functions. It has a rich theory, interrelations to functional analysis, several fields of complex analysis of one and several variables, and a number of applications to problems in mathematical physics.

Riemann problem has its origin in Bernhard Riemann's thesis. Since a first full solution of the problem is given by Hilbert, some references call the Riemann problem as the Riemann-Hilbert problem [1, p. 45], [10, p. 99-112] or the Hilbert problem [2, p. 207], [5, p. 42], [12, p. 154].

The Riemann problem arises in different branches of mathematics as well as in applications. For example, the complex Dirichlet problem which is one of the well-known classical boundary value problems is a particular case of the Riemann problem. Some problems from fluid dynamics or electrostatics lead to boundary problems for harmonic functions which can be reduced to Riemann problems. In the last few years, Wegmann proposed iterative methods for computing conformal maps which make use of the easy construction of the solutions to the Riemann problem [11].

Let Ω^+ be a simply connected region in the complex plane containing the origin of the coordinate system. Suppose also that $\Gamma := \partial\Omega$ is a smooth Jordan curve with the parameterization $\Gamma : t = t(\tau)$, $0 \le \tau \le \beta$, where the parameter τ need not be arc length. The unit tangent to Γ at the point t will be denoted by $T(t) = t'(s)/|t'(s)|$. Let Ω^- denotes the exterior of Γ and assume that $\infty \in \Omega^-$. The limiting values of the function $f(z)$ when the point z tends to the point $t \in \Gamma$ from inside and outside of Γ will denoted by $f^+(t)$ and $f^-(t)$, respectively. Assume that a, b and γ be real functions on the boundary Γ all satisfying the Hölder condition with $a^2(t) + b^2(t) \ne 0$ for all $t \in \Gamma$. It is required to find the functions $f = u + \mathrm{i}\,v$ analytic in Ω, continuous in the closure $\overline{\Omega} = \Omega \cup \partial\Omega$, and satisfying

$$\mathrm{Re}\left[c(t) f^+(t)\right] = \gamma(t), \qquad t \in \Gamma. \tag{1}$$

When $\gamma(t) = 0$, then we are facing with the homogeneous Riemann problem

$$\mathrm{Re}\left[c(t) f^+(t)\right] = 0, \qquad t \in \Gamma. \tag{2}$$

With $c(t)=a(t)+\mathrm{i}\,b(t)$ and $f^{+}(t)=u(t)+\mathrm{i}\,v(t)$, the boundary condition (1) may be written as

$$a(t)\,u(t)-b(t)\,v(t)=\gamma(t), \qquad t\in\Gamma . \tag{3}$$

Similarly, the Riemann problem for exterior domain Ω^{-} consists of finding all functions $f=u+\mathrm{i}\,v$ that are analytic in Ω^{-} (including at ∞), continuous on $\overline{\Omega^{-}}=\Omega^{-}\cup\Gamma$, and satisfy the boundary condition

$$\mathrm{Re}\left[c(t)f^{-}(t)\right]=\gamma(t), \quad t\in\Gamma . \tag{4}$$

It is clear that the difference of any two solutions of the non-homogeneous problem (1) is a solution to the homogeneous problem (2). Furthermore, if f is any solution of (2), then so is kf for any real k. Hence, the general solution of (1) is obtained by merely adding the general solution of (2) to a particular solution of (1). In this paper, we shall assume that $|c(t)|=1$ on Γ which is no loss of generality as can be seen by dividing the boundary condition (1) by $|c(t)|$.

An analytic solution can be found for the Riemann problem for the unit disk. But for arbitrary simply connected region Ω, the standard approach in all references required the availability of suitable conformal mapping function that maps Ω onto the unit disk [2, 4, 5, 10, 12]. Recently Murid et al. [9] and Murid and Nasser [6, 7, 8], produced a new method for solving the Riemann problem without the need to use any conformal mapping. The method depends on reducing the boundary condition (1) to a Fredholm integral equation of the second kind with continuous kernel. The organization of this paper is as follows. We start with reviewing the relevant results from complex analysis in Section2. Section 3 describes, with out proof, the conformal mapping methods for solving the Riemann problem on reducing the Riemann problem to the Hilbert problem or the Dirichlet problem. The integral equation methods will be given in Section 4. Finally, short conclusions are given in Section 5.

2. PRELIMINARIES

2.1 Representations of Analytic Functions by Cauchy Type Integral

Let $\psi(t)$ be a continuous function of position on Γ. Then the integral

$$\Psi(z):=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\psi(w)}{w-z}\,dw, \quad z\notin\Gamma , \tag{5}$$

is called the Cauchy type integral and the function $\psi(w)$ is called the density.

Theorem 2.1 [2, p. 297-300] *Suppose that $c(t)$ is continuous functions on Γ such that $c(t)\neq 0$ for all $t\in\Gamma$ with $x=\mathrm{ind}_{\Gamma}(c)$. Suppose also that $f(z)$ is an analytic function in Ω^{+}. If $x>0$, then there exists a real function $\mu(t)$ of position on Γ, which depends on $2x-1$ arbitrary real constants, such that*

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\mu(w)}{c(w)\left(w-z\right)}\,dw . \tag{6}$$

If $x\le 0$, then there exists a real function $\mu(t)$ of position on Γ and a polynomial $P_{-x}(z)$, such that

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{\mu(w)}{c(w)\left(w-z\right)}\,dw+P_{-x}(z) \tag{7}$$

where $P_{-x}(z)=\sum_{k=0}^{-x}c_k z^k$, $c_0,c_1,\cdots,c_{-x-1}$ are some complex constants and c_{-x} is a purely imaginary constant. The function $\mu(w)$ and the constants $c_0,c_1,\cdots,c_{-x}$ are uniquely determined by $f(z)$.

2.2 Sokhotskyi Formulas

Let h be a complex-valued function defined on a smooth Jordan curve Γ. If $t_0\in\Gamma$, we say that h satisfies a Hölder condition at t_0 if there exist a positive constants μ and λ such that

$$\left|h(t)-h(t_0)\right|\le\mu\left|t-t_0\right|^{\lambda} \tag{1.8}$$

for all $t\in\Gamma$ sufficiently close to t_0. The constant λ is called the exponent of the Hölder condition.

Theorem 2.2 [2, p. 24] *Let* Γ *be a smooth Jordan curve and assume* h *be a complex-valued function defined and satisfied the Hölder condition on* Γ. *If a point* z *tends from inside or outside the curve* Γ, *to the point* $t \in \Gamma$, *then the Cauchy type integral*

$$C(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{h(w)}{w - z} dw, \quad z \notin \Gamma \tag{9}$$

tends to the limits

$$C^{+}(t) = \frac{1}{2} h(t) + \mathrm{PV} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{h(w)}{w - t} dw \tag{10}$$

$$C^{-}(t) = -\frac{1}{2} h(t) + \mathrm{PV} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{h(w)}{w - t} dw \tag{11}$$

where PV *stands for principal value.*

2.3 Solvability of the Riemann Problem

The solvability of the Riemann problem hinges on a topological quantity, namely, on the index of the function c with respect to the curve Γ, which it will be denoted by $\mathrm{ind}_{\Gamma}(c)$. The index of a function c is defined to be the increment of its argument under a circuit of Γ in the positive direction and it is given by

$$x = \mathrm{ind}_{\Gamma}(c) = \frac{1}{2\pi} \left[\arg c(t) \right]_{\Gamma} = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} d \ln c(t),$$

where the integral is understood in the sense of the Stieltjes integral [5, p. 30]. The following theorem from [2, pp. 222-226], with slight modifications, discusses the solvability of the Riemann problem.

Theorem 2.3 [2, p. 222] *Let* $x = \mathrm{ind}_{\Gamma}(c)$. *In the case* $x \le 0$ *the homogeneous Riemann problem* (2) *has* $-2x + 1$ *linearly independent solutions and the non-homogeneous problem* (1) *is absolutely soluble and its solution depends linearly on* $-2x + 1$ *arbitrary real constants. In the case* $x > 0$, *the homogeneous Riemann problem* (2) *has only the trivial solution and the non-homogeneous problem* (1) *is soluble only if* $2x - 1$ *conditions are satisfied. If the latter conditions are satisfied the non-homogeneous problem has a unique solution*.

In order to get a uniquely given solution in the case $x \le 0$ we impose some side conditions on the solution.

Theorem 2.4 [1, p. 55] *Suppose that* $x = \mathrm{ind}_{\Gamma}(c) \le 0$, $t_k \in \Gamma$, $k = 1, 2, \cdots, -2x + 1$, *are distinct points, and* η_k *are prescribed real numbers. Then the Riemann problem* (1) *together with the side conditions*

$$\mathrm{Im}\left[c(t_k) f^{+}(t_k) \right] = \eta_k, \tag{12}$$

is uniquely solvable.

2.4 Schwarz Operator for a Simply Connected Domain

Although Dirichlet problem is a particular case of the Riemann problem, one method for solving the Riemann problem however roles on reducing the Riemann problem to a Dirichlet problem. Then the Dirichlet problem is solvable via the Schwarz operator.

Suppose that a real function $\gamma(t)$ satisfying the Hölder condition on Γ is given. The Schwarz operator S is an operator which determines the analytic function $F(z)$, the limiting value of the real part of which coincides with the function $\gamma(t)$ on Γ, and the imaginary part of which vanishes at a given points $z_0 \in \Omega$. In this paper, we assume that $z_0 = 0$.

If Γ is the unit circle the Schwarz operator is identical with the Schwarz integral [2, p. 208],

$$F(z) = \left(S\gamma \right)(z) = \frac{1}{2\pi \mathrm{i}} \int_{|t|=1} \gamma(t) \frac{t + z}{t - z} \frac{dt}{t}. \tag{13}$$

2.5 The Real Regularizing Factor

Suppose that $c(t) = a(t) + \mathrm{i}\, b(t)$ is a non-vanishing complex function prescribed on Γ which satisfies the Hölder condition on Γ. It is well known that in general an arbitrary function $c(t)$ is not a boundary value of a function analytic in the domain Ω^+. The real regularizing factor $p(t)$ is a real function of position on Γ which on multiplication by the function $c(t)$ becomes the boundary value of a function $\Phi^+(t)$ analytic in Ω^+ with exception at the origin where it may has a pole of order not exceeding $x = \mathrm{ind}_\Gamma(c)$, i.e. $\Phi^+(t) = t^x e^{\mathrm{i}\phi(t)}$ where $\phi(z)$ is an analytic functions in Ω^+. Therefore

$$p(s)c(t) = \Phi^+(t) = t^x e^{\mathrm{i}\phi(t)} = t^x e^{-\varphi_2(t)+\mathrm{i}\varphi_1(t)}. \tag{14}$$

From [2, p. 216] the real regularizing factor always exists in a unique way and is given by

$$p(t) = |t|^x e^{-\varphi_2(t)}, \tag{15}$$

where

$$\varphi_1(t) = \tan^{-1}\left(\frac{b(t)}{a(t)}\right) - x\arg t, \tag{16}$$

$$\phi(z) = \left(S\varphi_1\right)(z), \quad z \in \Omega^+, \tag{17}$$

and $\varphi_2(t) = \mathrm{Im}\left[\phi(t)\right]$.

2.6 Hilbert Problem on Simply Connected Domain

Suppose that Γ, Ω^+ and Ω^- are as describes earlier. Suppose also that $G(t)$ and $g(t)$ are two given functions of position on Γ, satisfy the Hölder condition, and $G(t)$ does not vanish on Γ. It is required to find two functions: $\Phi^+(z)$, analytic in the domain Ω^+, and $\Phi^-(z)$, analytic in Ω^-, including $z = \infty$, which satisfy on Γ the linear relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{18}$$

This problem is known as the Hilbert problem. The following Theorem gives the solution of the Hilbert problem (18).

Theorem 2.5 [2, p. 98] *Suppose that $m = \mathrm{ind}_\Gamma(G)$. In the case $m \geq 0$ the Hilbert problem* (18) *is soluble for any arbitrary g and its general solution is given by the formula*

$$\Phi(z) = \frac{X(z)}{2\pi\mathrm{i}}\int_\Gamma \frac{g(w)}{X^+(w)}\frac{dw}{w-z} + X(z)P_m(z), \tag{19}$$

where the function $X(z)$ is given by

$$X^+(z) = \mathrm{e}^{R^+(z)}, \;\; X^-(z) = \mathrm{e}^{R^-(z)}, \;\; R(z) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\ln\left(w^{-m}G(w)\right)}{w-z}dw, \tag{20}$$

and $P_m(z)$ is a polynomial of degree m with arbitrary complex coefficients. If $m = -1$ the Hilbert problem (18) *is also solvable and it has a unique solution given by* (19) *where we should set $P_m(z) = 0$.*

In the case $m < -1$ the Hilbert problem (18) *is in general insoluble. In order that it be soluble it is necessary and sufficient that the function $g(t)$ satisfies $-m-1$ conditions,*

$$\int_\Gamma \frac{g(w)}{X^+(w)} w^{k-1} dw = 0, \;\; k = 1, 2, \cdots, -m-1. \tag{21}$$

If these conditions are satisfied the unique solution of the problem is given by (19) *where we should set $P_m(z) = 0$.*

3. SOLVING RIEMANN PROBLEM USING CONFORMAL MAPPING

3.1 Solving Riemann Problem by Reducing it to a Hilbert Problem

Suppose that g is the Riemann map from the simply connected region Ω to the unit disk $\mathbb{D}^+$ and g^{-1} be its inverse, i.e.

$$g : \Omega \to \mathbb{D}^+, \quad g^{-1} : \mathbb{D}^+ \to \Omega.$$

By the Osgood-Caratheodory theorem [3, P. 383], the map g can be extended to a homeomorphism of $\overline{\Omega}$ onto $\overline{\mathbb{D}^+} = \mathbb{D}^+ \bigcup \mathbb{T}$, where $\mathbb{T}$ is the unit circle. We assume that Γ is such that the extended map g preserves Hölder continuity on the boundary. The function $\widehat{f} := f \circ g$ solves the Riemann problem for the unit disk with the boundary condition

$$\operatorname{Re}\left[\widehat{c}(t)\widehat{f}^{+}(t)\right] = \widehat{\gamma}(t), \qquad t \in \mathbb{T}, \tag{22}$$

where $\widehat{c} = c \circ g$ and $\widehat{\gamma} = \gamma \circ g$. It thus suffices to solve the Riemann problem (22) for the unit disk and the solution of the Riemann problem (1) is given by

$$f(z) = \widehat{f}\left(g^{-1}(z)\right), \quad z \in \Omega. \tag{23}$$

The boundary condition (22) of the Riemann problem may obviously be written as

$$\widehat{f}^{+}(t) = -\frac{\overline{\widehat{c}(t)}}{\widehat{c}(t)}\overline{\widehat{f}^{+}(t)} + \frac{2\widehat{\gamma}(t)}{\widehat{c}(t)}, \quad t \in \mathbb{T}. \tag{24}$$

We will assume again that $|\widehat{c}(t)| = 1$ on Γ.

Denote the exterior of the unit circle $\mathbb{T}$ by $\mathbb{D}^-$. We shall assume that $\infty \in \mathbb{D}^-$. The unknown function $\widehat{f}(z)$ is extended to $\mathbb{D}^+ \bigcup \mathbb{D}^-$ by setting

$$\widehat{f}(z) := \overline{\widehat{f}\left(\frac{1}{z}\right)}, \quad z \in \mathbb{D}^-. \tag{25}$$

From [5, p. 43], $\widehat{f}(z)$ is analytic in $\mathbb{D}^+ \bigcup \mathbb{D}^-$ and

$$\widehat{f}^{+}(t) = \overline{\widehat{f}^{-}(t)}, \quad t \in \mathbb{T}, \tag{26}$$

and the condition (24) may be written as

$$\widehat{f}^{+}(t) = G(t)\widehat{f}^{-}(t) + g(t), \quad t \in \mathbb{T}. \tag{27}$$

where $G(t) = -\dfrac{\overline{\widehat{c}(t)}}{\widehat{c}(t)}$, $g(t) = \dfrac{2\widehat{\gamma}(t)}{\widehat{c}(t)}$.

In this way one comes to the Hilbert problem (27). A solution of (27) must be sought, bounded at infinity. Not every solution of the Hilbert problem (27) is a solution of the Riemann problem (22), because it may not satisfy the supplementary condition (25). However, with the help of a solution of (27), a solution of the original problem (22) can be constructed as follows. Let $\widehat{f}(z)$ be a solution of (27) and define the function $\widehat{f}_1(z)$ on $\mathbb{D}^+ \bigcup \mathbb{D}^-$ by

$$\widehat{f}_1(z) := \overline{\widehat{f}\left(\frac{1}{\overline{z}}\right)}. \tag{28}$$

From [4, pp. 127-128], the function $\widehat{f}_1(z)$ is a solution to (27), and the function

$$\tilde{f}(z) = \tfrac{1}{2}\left[\widehat{f}(z) + \widehat{f}_1(z)\right] \tag{29}$$

is a solution of the Hilbert problem (27) as well as the Riemann problem (22).

Theorem 3.1 [1, p. 46] *If $\widehat{f}(z)$ is a solution of the Hilbert problem* (27) *with*

$$G(t) = -\frac{\overline{\widehat{c}(t)}}{\widehat{c}(t)}, \quad g(t) = \frac{2\widehat{\gamma}(t)}{\widehat{c}(t)},$$

then the solution of the Riemann problem (22)*, is given by*

$$f(z) = \frac{1}{2}\left[\widehat{f}(z) + \overline{\widehat{f}\left(1/\overline{z}\right)}\right] + X(z)P_m(z)$$

where $X(z)$ is given by (20) and the polynomial $P_m(z)$, $m = \operatorname{ind}_\Gamma(G) = -2x$, *is chosen to be,* $P_m(z) = 0$ *when* $x > 0$, *and for* $x \le 0$,

$$P_m(z) = c_0 z^{-x} + \sum_{k=1}^{-x}\left(c_k\, z^{-x+k} - \overline{c}_k\, z^{-x-k}\right),$$

c_0 *is arbitrary real constant and* c_k, $k = 1, 2, \cdots, x$ *are arbitrary complex constants.*

3.2 Solving Riemann Problem by Reducing it to a Dirichlet Problem

In this subsection, the Riemann problem will be studied using the Schwarz operator directly. At first the Dirichlet problem will modified by allowing the solution to have an isolated pole of order not greater than a given integer $x \geq 0$ at the origin.

Theorem 3.2 [1, p. 51] *Suppose that* x *is a non-negative integer. The general solution of the Dirichlet problem*

$$\mathrm{Re}\left[f(t)\right] = \gamma(t) \quad \text{on} \quad \Gamma = \partial\Omega^+ ,$$

in the space of functions analytic in $\Omega - \{0\}$ *having a pole at most of order* x *at* $z = 0$ *is*

$$f(z) = \left(S\gamma\right)(z) + \mathrm{i}\, c_0 + \sum\nolimits_{k=1}^{x}\left(c_k \omega^k(z) - \overline{c}_k \omega^{-k}(z)\right) \tag{30}$$

where $\omega(z)$ *is the Riemann map from* Ω^+ *to* $\mathbb{D}^+$ *such that* $\omega(0) = 0$ *and* $\omega'(0) > 0$*, and* c_0 *is arbitrary real constant and* c_k*,* $k = 1, 2, \cdots, x$ *are arbitrary complex constants.*

Suppose that $x = \mathrm{ind}_\Gamma(c)$. Multiplying the boundary condition (1) by the regularizing factor of $c(t)$ (formulae (14)-(17)) we reduce it to the form

$$\mathrm{Re}\left[t^x\, \mathrm{e}^{\mathrm{i}\phi(t)}\, f^+(t)\right] = p(t)\gamma(t) \, . \tag{31}$$

If we consider the Riemann problem (31) as an ordinary Dirichlet problem for the case $x \geq 0$ and as a Dirichlet problem of the type define in Theorem 3.2 for the case $x < 0$, then we have,

Theorem 3.3 [2, pp. 221-222] *Suppose that* $x = \mathrm{ind}_\Gamma(c)$*. If* $x \leq 0$*, then the Riemann problem* (1) *is solvable and its general solution is given by*

$$f(z) = z^{-x}\, \mathrm{e}^{-\mathrm{i}\phi(z)} \left\{\left(S\left(|\, t\,|^x\, \mathrm{e}^{\varphi_2(t)}\, \gamma(t)\right)\right)(z) + \mathrm{i}\, c_0 + \sum\nolimits_{k=1}^{x}\left(c_k \omega^k(z) - \overline{c}_k \omega^{-k}(z)\right)\right\} \tag{32}$$

where

$$\phi(z) = \left(S\varphi_1\right)(z) \ , \quad z \in \Omega^+ ,$$

$\varphi_2(t) = \mathrm{Im}\left[\phi(t)\right]$, $\omega(z)$ *is the Riemann map from* Ω^+ *to* $\mathbb{D}^+$ *such that* $\omega(0) = 0$ *and* $\omega'(0) > 0$*, and* c_0 *is arbitrary real constant and* c_k*,* $k = 1, 2, \cdots, x$ *are arbitrary complex constants.*

If $x > 0$*, the Riemann problem (1) is solvable if and only if the analytic function*

$$\left(S\left(|\, t\,|^x\, \mathrm{e}^{\varphi_2(t)}\, \gamma(t)\right)\right)(z)$$

has a zero of order x *at the origin of the coordinate system. The latter condition leads to* $2x - 1$ *conditions should be satisfied by* $\gamma(t)$*, in order that the solution be possible. If these conditions are satisfied, then*

$$f(z) = z^{-x}\, \mathrm{e}^{-\mathrm{i}\phi(z)} \left(S\left(|\, t\,|^x\, \mathrm{e}^{\varphi_2(t)}\, \gamma(t)\right)\right)(z) \tag{33}$$

is the unique solution to (1).

4. SOLVING RIEMANN PROBLEM USING INTEGRAL EQUATION METHODS

4.1 Direct Integral Equation Method

This method is a generalization of the integral equation method for the Dirichlet problem. Suppose that $f(z)$ is a solution of the Riemann problem (1) in Ω^+. According to Theorem 2.2, there exists a real function of position on Γ and a polynomial $P_{-x}(z)$ ($P_{-x}(z)$ of order $-x$ when $x \leq 0$ and $P_{-x}(z) = 0$ for $x > 0$), such that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{h(w)}{c(w)\left(w - z\right)}\, dw + P_{-x}(z) \, . \tag{34}$$

This representation formula is due to Sherman, [2, p. 401]. Taking the limit $\Omega^+ \ni z \to t \in \Gamma$ and applying Sokhotskyi formula to (34), we get

$$f^{+}(t)=\frac{1}{2}\frac{h(t)}{c(t)}+\mathrm{PV}\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\frac{h(w)}{c(w)\left(w-t\right)}dw+P_{-x}(t).$$

Multiplying both sides by $2c(t)$, and taking the real part of both sides, using the hypotheses that $f(z)$ is a solution to the Riemann problem (1) and the fact that $dw=T(w)\,|\,dw\,|$, we have the integral equation

$$h(t)+\mathrm{PV}\int_{\Gamma}N(c)(t,w)h(w)dw=2\gamma(t)-2\operatorname{Re}\left[c(t)P_{-x}(t)\right], \tag{35}$$

where

$$N(c)(t,w)=\frac{1}{\pi}\operatorname{Im}\left[\frac{c(t)}{c(w)}\frac{T(w)}{w-t}\right]. \tag{36}$$

The kernel $N(c)(t,w)$ is continuous at all points $(t,w)\in\Gamma\times\Gamma$ except for $t=w$ where it is undefined. In fact, if Γ is sufficiently smooth, $N(c)(t,w)$ is continuous even at $t=w$ where $N(c)(t,t)$ is defined by $N(c)(t,t):=\lim_{w\to t}N(c)(t,w)$ [7]. Thus we can rewrite the kernel (36) as

$$N(c)(t,w)=\begin{cases}\dfrac{1}{\pi}\operatorname{Im}\left[\dfrac{c(t)}{c(w)}\dfrac{T(w)}{w-t}\right], & w\neq t,\ w,t\in\Gamma\\[2ex] \dfrac{1}{2\pi\left|t'(s)\right|}\operatorname{Im}\left[\dfrac{t''(s)}{t'(s)}-\dfrac{2c'\left(t(s)\right)t'(s)}{c\left(t(s)\right)}\right], & w=t\in\Gamma\ ,\end{cases} \tag{37}$$

and drop the symbol PV in (35).

Hence, the solving of the Riemann problem (1) is reduced to solve the integral equation

$$h(t)+\int_{\Gamma}N(c)(t,w)h(w)dw=2\gamma(t), \tag{38}$$

for the case $x>0$, and the integral equation

$$h(t)+\int_{\Gamma}N(c)(t,w)h(w)dw=2\gamma(t)-2\operatorname{Re}\left[c(t)P_{-x}(t)\right], \tag{39}$$

for the case $x\leq 0$, where $P_{-x}(z)$ is polynomial of degree $-x$.

The kernel (37) is a generalization of the familiar Neumann kernel which arises frequently in the integral equation of potential theory and conformal mapping, [4, p.282-286], so (37) will be called the generalized Neumann kernel. Suppose $c(t)$ is defined on Γ such that $c'(t)$ exist and is continuous, and $c(t)\neq 0$ for all $t\in\Gamma$. Suppose also that $x=\operatorname{ind}_{\Gamma}(c)$ and $N(c)$ is the generalized Neumann kernel defined by (37). Then the following theorems discussed the eigenproblem of $N(c)$ [7].

Theorem 4.1 [7] *Suppose that Γ is a smooth Jordan curve. If $x\leq 0$, then $\lambda=1$ is an eigenvalue of $N(c)$ with $-2x+1$ corresponding eigenfunctions.*

Theorem 4.2 [7] *Suppose that Γ is a smooth Jordan curve. If $x>0$, the $\lambda=-1$ is an eigenvalue of $N(c)$ with $2x-1$ corresponding eigenfunctions.*

Theorem 4.3 [7] *Suppose that Γ is the unit circle. If $\lambda=1$ is an eigenvalue of $N(c)$, then $x\leq 0$.*

Theorem 4.4 [7] *Suppose that Γ is the unit circle. If $\lambda=-1$ is an eigenvalue of $N(c)$, then $x>0$.*

Unfortunately, the integral equations (38) and (39) is not a good idea for solving the Riemann problem (1) because of the following. When $x>0$, according to Theorem 4.2, -1 is an eigenvalue of the kernel $N(c)(t,w)$, and hence, the integral equation (38) is not always solvable, and if it is solvable the solution is not unique. However, to use the integral equation (39) to solve the Riemann problem (1), we need to solve a Fredholm integral equation with arbitrary right hand side. [2, p. 306].

4.2 Indirect Integral Equation Method

In this subsection we shall present our method for solving the Riemann problem in a simply connected domain with smooth boundary. The boundary condition (1) of the Riemann problem is reduced to a

Fredholm integral equation of the second kind with the generalized Neumann kernel. The difficulties facing us in the Sherman integral equations (38) and (39) will not appear here.

This method is based on the observation that to determine an analytic function $f(z)$ in the domain Ω, it is enough to determine the boundary values $f(t)$, $t \in \Gamma = \partial\Omega$, then $f(z)$ can be determined in Ω by using Cauchy integral formula. Thus to solve the Riemann problem (1), it is enough to determined

$$\mu(t) = \operatorname{Im}\left[c(t) f^{+}(t)\right], \quad t \in \Gamma. \tag{40}$$

Then from (1) and (40), $c(t) f^{+}(t) = \gamma(t) + \mathrm{i}\,\mu(t)$ and according to Cauchy integral formula, the function $f(z)$ is given by

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{f^{+}(w)}{w - z} dw = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{\gamma(w) + \mathrm{i}\,\mu(w)}{c(w)\left(w - z\right)} dw. \tag{41}$$

Suppose that γ is a real function defined on Γ and satisfies the Hölder condition. Suppose also that $c(t)$ is a complex valued function defined on Γ such that $c'(t)$ is continuous on Γ and $c(t) \neq 0$ for all $t \in \Gamma$ and define the function $L(z)$ in $\mathbb{C} \setminus \Gamma$ by

$$L(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{2\gamma(t)}{c(t)\left(t - z\right)} dt. \tag{42}$$

Theorem 4.5 gives the integral equation related to Riemann problem in the interior domain Ω^{+} of a smooth Jordan curve Γ. The proof can be found in [7].

Theorem 4.5 [9] *Let Γ be a smooth Jordan curve and Ω^{+} be its interior. Suppose that $f(z)$ is an analytic solution of the Riemann problem* (1) *in the interior domain Ω^{+}. Define $g(t) = c(t) f^{+}(t)$, $t \in \Gamma$. Then $g(t)$ satisfies the integral equation*

$$g(t) - \int_{\Gamma} N(c)(t, w) g(w) \left|dw\right| = -\overline{c(t)}\,\overline{L^{-}(t)}. \tag{43}$$

where $N(c)(t, w)$ is the generalized Neumann kernel.

As a result of Theorem 4.5, we have the following corollary.

Corollary 4.1 [8] *Suppose that $f(z)$ is a solution of the Riemann problem* (1)*. Define $\mu(t) := \operatorname{Im}\left[c(t) f(t)\right]$, $t \in \Gamma$. Then $\mu(t)$ satisfies the Fredholm integral equation*

$$\mu(t) - \int_{\Gamma} N(c)(t, w) \mu(w) \left|dw\right| = \operatorname{Im}\left[c(t) L^{-}(t)\right]. \tag{44}$$

When $x = \operatorname{ind}_{\Gamma}(c) > 0$ and Γ is the unit circle, according to Theorem 4.3, 1 is not an eigenvalue of the kernel $N(c)(t, w)$. Thus the integral equation has a unique solution. In this case, the Riemann problem has at most one solution. If it has a solution, then it is given by (41) [9].

If $x \leq 0$, then the Riemann problem is always solvable but the solution is not unique. The numerical solution of the non-uniquely solvable Riemann problem using Fredholm integral equation is studied extensively in [8]. The equivalence of the Riemann problem (1) and the integral equation (44) is proven in [8] for any smooth Jordan curve Γ.

Our method was extended to the exterior Riemann problem in [6]. The following theorem gives a Fredholm integral equation for the exterior Riemann problem.

Theorem 4.6 [6] *Suppose that $f(z)$ is a solution of the exterior Riemann problem* (4)*. Define $\hat{c}(t) = t c(t)$ and $\mu(t) = \operatorname{Im}\left[c(t) f^{-}(t)\right]$, $t \in \Gamma$. Then $\mu(t)$ satisfies the Fredholm integral equation*

$$\mu(t) + \int_{\Gamma} N(\hat{c})(t, w) \mu(w) \left|dw\right| = -\operatorname{Im}\left[\hat{c}(t) \hat{L}^{+}(t)\right]. \tag{45}$$

where

$$\widehat{L}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{2\gamma(t)}{\widehat{c}(t)\left(t-z\right)} dt \,.$$

The equivalence of the exterior Riemann problem (4) and the integral equation (45) was proven in [6] for any smooth Jordan curve in the case $x \geq 0$, and for the unit circle in the case $x < 0$.

The complicated looking expressions of the right hand side of equations (44) and (45) seem to be a disadvantage of our method, but this is not true. Since the function $c(t)$ is continuously differentiable on Γ and $\gamma(t)$ is Hölder continuous on Γ, then from [4 , p. 94], the right hand side of (44) and (45) are Hölder continuous on Γ. Moreover, the right hand side of (44) can be computed numerically as follows:

$$\begin{aligned}
\mathrm{Im}\left[c(t)L^{-}(t)\right] &= \mathrm{Im}\left[c(t)\left\{-\frac{\gamma(t)}{c(t)} + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{2\gamma(w)}{c(w)\left(w-t\right)}dw\right\}\right] \\
&= \mathrm{Im}\left[c(t)\left\{-\frac{\gamma(t)}{c(t)} + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{2\gamma(w)/c(w) - 2\gamma(t)/c(t) + 2\gamma(t)/c(t)}{w-t}dw\right\}\right] \\
&= \mathrm{Im}\left[c(t)\left\{-\frac{\gamma(t)}{c(t)} + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{2\gamma(w)/c(w) - 2\gamma(t)/c(t)}{w-t}dw + 2\gamma(t)/c(t)\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{1}{w-t}dw\right\}\right]
\end{aligned}$$

Since

$$\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{1}{w-t}dw = \frac{1}{2} \text{ for all } t \in \Gamma \,,$$

we have

$$\mathrm{Im}\left[c(t)L^{-}(t)\right] = \mathrm{Im}\left[c(t)\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{2\gamma(w)/c(w) - 2\gamma(t)/c(t)}{w-t}dw\right]. \tag{46}$$

Since $2\gamma(t)/c(t)$ satisfies the Hölder condition, the integral in (46) exists as an ordinary improper integral [4, p.92].

Similarly,

$$-\mathrm{Im}\left[\widehat{c}(t)\widehat{L}^{+}(t)\right] = -\mathrm{Im}\left[\widehat{c}(t)\frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{2\gamma(w)/\widehat{c}(w) - 2\gamma(t)/\widehat{c}(t)}{\widehat{c}(w)\left(w-t\right)}dw\right]. \tag{47}$$

Thus to solve the Riemann problem (1) using the integral equation (44), we need to calculate the improper integral (46) and then solve a Fredholm integral equation with continuous kernel. Although, in the case $x \leq 0$, 1 is an eigenvalue of the kernel $N(c)(t,w)$, the integral equation is always solvable. The side conditions (12) on the Riemann problem (1) can be used to impose additional conditions to (44) in order to get a unique solution to (44) [8]. Thus the difficulties that appeared in the integral equations (38) and (39), did not appear in our integral equations (44) and (45).

6. CONCLUSIONS

This paper presented the different approaches of solving a complex boundary value problem known as the Riemann problem. One approach is by reducing the Riemann problem to the Dirichlet problem. This approach however, requires the determination of the regularizing factor, finding a particular solution of the non-homogeneous problem, and finally the determination of the conformal mapping $\omega(z)$ from Ω^{+} to $\mathbb{D}^{+}$.

The other approach involves reducing the Riemann problem to a Hilbert problem which requires the determination of the Riemann map g from Ω^{+} to $\mathbb{D}^{+}$, the inverse of the Riemann map g^{-1} from $\mathbb{D}^{+}$ to Ω^{+}, and finally the solving of a Riemann problem on the unit disk.

However, in our approach to solve the Riemann problem, we need to calculate one improper integral and to solve one Fredholm integral equation with the generalized Neumann kernel that is continuous.

It is clear that our method has the advantages in terms of the numerical operations as well as ease in programming. However, much work is still required to discuss the solvability of the integral equations (44) for the case $x > 0$ and the integral equations (45) for the case $x < 0$ on any smooth Jordan curve and its extension to nonsmooth regions.

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