Matematika, 1998, Jilid 14, hlm. 1–6

A New Fourth-Order Embedded Method Based on the Harmonic Mean

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Abstract In this paper we formulate an embedded method based on harmonic and arithmetic means of order 4. This method together with RK-Harmonic scheme may be used to estimate the solutions of initial value problems. The absolute stability region of this scheme is also studied and we conclude with a numerical example to justify the effectiveness of the method.

Keywords Harmonic Mean, Runge-Kutta method, stability, error control.

Abstrak *Dalam kertas ini kami rumuskan satu kaedah terbenam berperingkat 4 yang berasaskan min harmonik dan min aritmetik. Kaedah ini berserta dengan skema RK-Harmonik boleh digunakan bagi menganggarkan penyelesaian kepada masalah nilai awal. Rantau kestabilan untuk skema ini juga dikaji dan kita simpulkan dapatan ini dengan satu contoh masalah sebagai mengesahkan keberkesanan kaedah ini.*

Katakunci *Min-Harmonik, Kaedah Runge-Kutta, kestabilan, kawalan ralat.*

1 Introduction

One of the difficulties in implementing the classical Runge-Kutta method is the absence of estimation of errors procedure in computating the numerical solutions. Several methods have been developed to overcome these weaknesses, specifically this is done by introducing the procedure that can estimate the errors in the results. Amongst them are methods developed by Merson, Scraton [2] and Fehlberg [1]. Related work was carried out by Sanugi [3], who introduced a fourth order AGM method which is based on geometric mean plus a fourth order method based on arithmetic mean. Both methods used the common k's, k_i, i = 1,2,3,4 . Similarly, we develop a much simpler embedded method of fourth order accuracy which is based on RK-Harmonic method and error estimate strategies. We shall refer this method as RK-HM-AM.

2 Formulation of Fourth Order RK-HM-AM Scheme

Sanugi and Evans [4] had introduced a fourth order RK formula based on the harmonic mean with parameters

$$a_1 = 1/2, a_2 = -1/8, a_3 = 5/8, a_4 = -1/4, a_5 = 7/20, \text{and}\, a_6 = 9/10,$$

where

$$k_1 = f(x_n, y_n), \tag{1}$$
$$k_2 = f(x_n + a_1 h, y_n + a_1 h k_1), \tag{2}$$
$$k_3 = f(x_n + (a_2 + a_3)h, y)n + a_2 h k_1 + a_3 h k_2), \tag{3}$$
$$k_4 = f(x_n + (a_4 + a_5 + a_6)h, y_n + a_4 h k_1 + a_5 h k_2 + a_6 h k_3), \tag{4}$$

and

$$y_{n+1}^{HM} = y_n + \frac{2h}{3}\left(\frac{k_1 k_2}{k_1 + k_2} + \frac{k_2 k_3}{k_2 + k_3} + \frac{k_3 k_4}{k_3 + k_4}\right). \tag{5}$$

Using the common a_i we try to formulate another scheme which combines both arithmetic mean (AM) and harmonic mean (HM). We propose the following scheme:

$$y_{n+1}^{AHM} = y_n + h\Big[d_1 k_1 + d_2 k_2 + d_3 k_3 + d_4 k_4 + d_5\left(\frac{k_1 k_2}{k_1 + k_2}\right) + d_6\left(\frac{k_2 k_3}{k_2 + k_3}\right) + d_7\left(\frac{k_3 k_4}{k_3 + k_4}\right)\Big], \tag{6}$$

with constants d_j, $j = 1, 2, \ldots, 7$, that are still undetermined and $k_i + k_{i+1} \neq 0$.

By considering $y' = f(y)$, where f is a function of y only and y is either monotonically increasing or decreasing, we expand the equations (6) and $y(x_n + h)$ using the Taylor series expansion. Using a symbolic computational package, MATHEMATICA, we compare the coefficients for h, h^2, h^3, and h^4 to obtain the following seven linear equations:

$$hf: \quad 1 - d_1 - d_2 - d_3 - d_4 - \frac{d_5}{2} - \frac{d_6}{2} - \frac{d_7}{2} = 0 \tag{7}$$
$$h^2 f_y: \quad \frac{1}{2} - \frac{d_2}{2} - \frac{d_3}{2} - d_4 - \frac{d_5}{8} - \frac{d_6}{8} - \frac{d_7}{8} = 0 \tag{8}$$
$$h^3 f f_y^2: \quad \frac{1}{6} - \frac{5d_3}{16} - \frac{5d_4}{8} + \frac{d_5}{32} - \frac{5d_6}{64} - \frac{13d_7}{64} = 0 \tag{9}$$
$$h^3 f^2 f_{yy}: \quad \frac{1}{6} - \frac{d_2}{8} - \frac{d_3}{8} - \frac{d_4}{2} - \frac{d_5}{32} - \frac{d_6}{16} - \frac{5d_7}{32} = 0 \tag{10}$$
$$h^4 f f_y^3: \quad \frac{1}{24} - \frac{9d_4}{32} - \frac{d_5}{128} - \frac{7d_7}{128} = 0 \tag{11}$$
$$h^4 f^2 f_y f_{yy}: \quad \frac{1}{6} - \frac{15d_3}{64} - \frac{25d_4}{32} + \frac{d_5}{64} - \frac{15d_6}{256} - \frac{53d_7}{256} = 0 \tag{12}$$
$$h^4 f^3 f_{yyy}: \quad \frac{1}{24} - \frac{d_2}{48} - \frac{d_3}{48} - \frac{d_4}{6} - \frac{d_5}{192} - \frac{d_6}{96} - \frac{3d_7}{64} = 0 \tag{13}$$

Solving (7)-(13) yields

$$d_1 = 0, d_2 = \frac{1}{6}, d_3 = \frac{1}{6}, d_4 = 0, d_5 = \frac{2}{3}, d_6 = 0, d_7 = \frac{2}{3}.$$

Thus, equation (6) becomes

$$y_{n+1}^{AHM} = y_n + h\Big[\frac{1}{6}k_2 + \frac{1}{6}k_3 + \frac{2}{3}\left(\frac{k_1 k_2}{k_1 + k_2}\right) + \frac{2}{3}\left(\frac{k_3 k_4}{k_3 + k_4}\right)\Big]. \tag{14}$$

We shall refer this formula as RK-HM-AM in the following sections.

3 Stability Analysis for RK-HM-AM Scheme

We used the test equation $y' = \lambda y$, λ is a constant, to obtain absolute stability regions for both RK-HM and RK-HM-AM methods. Substituting $\lambda_n' = \lambda y_n$ in k_i, $i = 1, 2, 3, 4$ respectively, we obtain

$$k_1 = f(y_n) = \lambda y_n \tag{15}$$

$$k_2 = f\Big(y_n + \frac{hk_1}{2}\Big) = \lambda y_n\Big(1 + \frac{\lambda h}{2}\Big) \tag{16}$$

$$k_3 = f\Big(y_n - \frac{hk_1}{8} + \frac{5hk_2}{8}\Big) = \lambda y_n\Big(1 + \frac{\lambda h}{2} + \frac{5(\lambda h)^2}{16}\Big) \tag{17}$$

$$k_4 = f\Big(y_n - \frac{hk_1}{4} + \frac{7hk_2}{20} + \frac{9hk_3}{10}\Big) = \lambda y_n\Big(1 + \lambda h + \frac{5(\lambda h)^2}{8} + \frac{9(\lambda h)^3}{32}\Big) \tag{18}$$

Substituting these values in (5) and (14) will yield

$$\frac{y_{n+1}^{HM}}{y_n} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} - \frac{3(\lambda h)^5}{128} + O(h^6) \tag{19}$$

and

$$\frac{y_{n+1}^{AHM}}{y_n} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{6} + \frac{(\lambda h)^4}{24} - \frac{47(\lambda h)^5}{3072} + O(h^6). \tag{20}$$

If $\lambda h = z$, then (19) and (20) are the stability polynomials of RK-HM and RH-HM-AM respectively,

$$Q^{HM}(z) = \frac{y_{n+1}^{HM}}{y_n} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} - \frac{3z^5}{128} + O(h^6) \tag{21}$$

and

$$Q^{AHM}(z) = \frac{y_{n+1}^{AHM}}{y_n} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} - \frac{47z^5}{3072} + O(h^6) \tag{22}$$

The absolute stability regions with respect to stabilty polynomials (21) and (22) are given in the Figure 1.

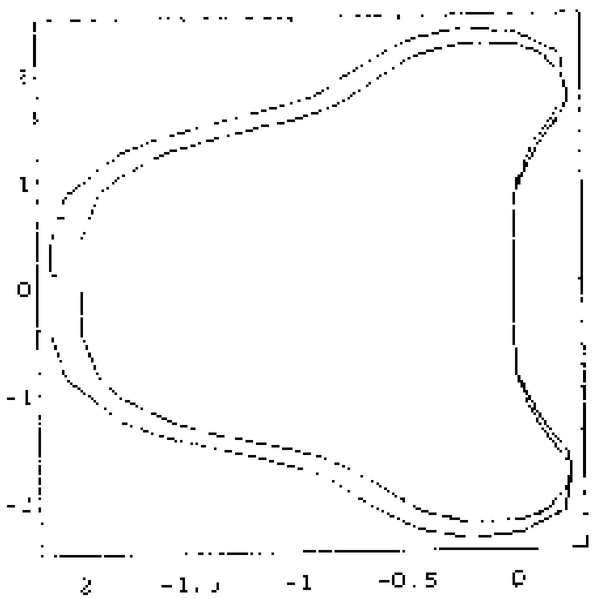

Figure 1: Absolute stability regions for the RK-HM (smaller) and RK-HM-AM methods

4 Error Estimation

In RK-Fehlberg scheme [1], estimated errors are obtained by taking the difference of fifth order RK and fourth order RK solutions. While in Merson scheme [2], estimated errors are obtained by taking difference of solutions of two fourth order methods with different number of stages. In our scheme, we used two methods (5) and (14) sharing the same a's and k's (equal number of stages) but having different form of y_{n+1}.

Local truncation errors (LTE) for RK-HM method is given as,

$$\mathrm{LTE(HM)} = y(x_n+h) - y_{n+1}^{HM} = \Big(\frac{61}{1920} f f_y^4 - \frac{61}{7680} f^2 f_y^2 f_{yy} - \frac{17}{960} f^3 f_{yy}^2 - \frac{1}{640} f^3 f_y f_{yyy} - \frac{1}{2880} f^4 f_{yyyy} \Big) h^5 + O(h^6) \tag{23}$$

while LTE for RK-HM-AM is given as

$$\mathrm{LTE(AHM)} = y(x_n+h) - y_{n+1}^{AHM} = \Big(\frac{121}{5120} f f_y^4 - \frac{61}{7680} f^2 f_y^2 f_{yy} - \frac{17}{960} f^3 f_{yy}^2 - \frac{1}{640} f^3 f_y f_{yyy} - \frac{1}{2880} f^4 f_{yyyy} \Big) h^5 + O(h^6) \tag{24}$$

Now, subtracting (23) from (24) yields,

$$y_{n+1}^{HM} - y_{n+1}^{AHM} = \frac{25}{3072} f f_y^4 h^5,$$

or

$$f f_y^4 h^5 = \frac{3072}{25}\left(y_{n+1}^{HM} - y_{n+1}^{AHM}\right). \tag{25}$$

If $f_{yy} = 0$, then $f_{yyy} = f_{yyyy} = 0$, i.e. f is a linear function in y. Thus using (25) we obtain

$$\mathrm{LTE(HM)} = 3.904\left(y_{n+1}^{HM} - y_{n+1}^{AHM}\right) \tag{26}$$

and

$$\mathrm{LTE(AHM)} = 2.904\left(y_{n+1}^{HM} - y_{n+1}^{AHM}\right) \tag{27}$$

This implies that the estimated errors of RK-HM and RK-HM-AM are approximately four and three times the difference of both methods.

5 Numerical Results

The folowing example shows how error estimation and error control are used.

$$y' = \frac{1}{y}, \quad y(0) = 1, \quad 0 \leq x \leq 1.25.$$

The exact solution is $y(x) = \sqrt{2x+1}$.

Using error estimation $= 2.904\left(y^{HM} - y^{AHM}\right)$, with error tolerance $= 10^{-6}$ we obtain the following results.

COMPUTER OUTPUT

ERROR TOLERANCE =.1D-05

X0 = 0.00000

Y0 = 0.1000000D+01

Table 1: H = 0.1250000

X	EXACT SOLN	YAHM	|ERROR|	|EST. ERROR|
0.1250	1.1180340	1.1180337	0.3344949E-06	0.5463125E-06
0.2500	1.2247449	1.2247443	0.6111894E-06	0.2171449E-06
0.3750	1.3228757	1.3228750	0.6969985E-06	0.1036227E-06
0.5000	1.4142136	1.4142128	0.7167391E-06	0.5678877E-07
0.6250	1.5000000	1.4999993	0.7110623E-06	0.3479989E-07

Table 2: H= 0.2500000

X	EXACT SOLN	YAHM	|ERROR|	|EST. ERROR|
0.8750	1.6583124	1.6583114	0.9471451E-06	0.4742222E-06
1.1250	1.8027756	1.8027745	0.1148519E-05	0.2080428E-06

6 Conclusion

In this paper we have developed an embedded method of order 4 based on arithmetic and harmonic means. By having equal number of stages and different form of y_{n+1}, we are able to reduce the cost of computation for complicated function. From Table 1, obviously, for the first few steps, the absolute errors are approximately closed to the estimated errors. Consequently, the former become larger due to the effect of global errors. These are expected. Further research is recommended especially for problem of the type

$$y' = f(x,y), \quad y(a) = \eta, \quad a \leq x \leq b.$$

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