DECISION ANALYSIS IN COMPETITIVE AND COOPERATIVE ENVIRONMENTS

by

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ABSTRACT

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This thesis contains three chapters in which we axiomatically analyze individual and collective decision problems in competitive and cooperative environments. In Chapter 1, we give a general introduction. In Chapter 2, we propose a theory of revealed preferences that allows both the status-quo bias and indecisiveness between any two alternatives. We extend a standard choice problem by adding a status-quo alternative and we incorporate standard choice theory as a special case. We characterize choice rules that satisfy two rationality requirements, status-quo bias, and strong SQ-irrelevance. In Chapter 3, we analyze bargaining situations where the agents' payoffs from disagreement depend on who among them breaks down the negotiations. We model such problems as a superset of the standard domain of Nash. On our extended domain, we analyze the implications of two central properties which, on the Nash domain, are known to be incompatible: strong monotonicity and scale invariance. We characterize bargaining rules that satisfy strong monotonicity, scale invariance, weak Pareto optimality, and continuity. In Chapter 4, we analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are possibly unequal. The agents have single peaked preferences on their consumption and production choices. For such markets, we analyze the implications of population changes as formalized by consistency and population monotonicity properties. We characterize trade rules that satisfy Pareto optimality, no-envy, and consistency as well as population monotonicity together with Pareto optimality, no-envy, and strategy-proofness.

Keywords: decision analysis, axiomatic, bargaining, status-quo, market disequilibrium.

ÖZET

DECISION ANALYSIS IN COMPETITIVE AND COOPERATIVE ENVIRONMENTS

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Bu tez, rekabetçi ve işbirlikçi ortamlardaki kişisel ve grup karar problemlerinin aksiyomatik bir şekilde incelendiği üç bölümden oluşmaktadır. İlk bölümde, genel bir giriş yapılmaktadır. İkinci bölümde, statükoya sapma ve herhangi iki alternatif arasında kararsız kalma durumlarına izin veren bir teori önerilmektedir. Standart seçim problemlerini, probleme bir statüko alternatifi eklenerek genelleştirmek yolu ile standart seçim teorisi modele dahil edilmektedir. Statükoya sapma, statükodan güçlü bağımsızlık ve diğer iki rasyonalite özelliklerini sağlayan seçim kuralları karakterize edilmektedir. İkinci bölümde anlaşmazlık sonucunun anlaşmazlığa yol açan bireyin kimliğine bağlı olduğu pazarlık problemleri analiz edilmektedir. Bu problemler, Nash modelinin bir üst uzayı olarak modellenmektedir. Nash'in uzayında tutarlı olmayan, güçlü monotonluk ve ölçekten bağımsızlık özelliklerinin bu uzaydaki sonuçları incelenmektedir. Güçlü monotonluk, ölçekten bağımsızlık, zayıf Pareto verimliliği, ve süreklilik özelliklerini sağlayan pazarlık kuralları karakterize edilmektedir. Dördüncü bölümde, fiyatın, arz ve talebin eşit olmadığı bir değerde sabitlendiği piyasalar incelenmektedir. Bireylerin tüketim ve üretim miktarları üzerine tek doruklu terichlerinin olduğu varsayılmaktadır. Bu tip piyasalarda, nüfus değişiminin etkileri tutarlılık ve nüfusta monotonluk özellikleri ile incelenmektedir. Pareto verimliliği, haset doğurmama, ve tutarlılık özelliklerini sağlayan ticaret kuralları ile Pareto verimliliği, haset doğurmama, stratejiden korunaklılık, ve nüfusta monotonluk özelliklerini sağlayan ticaret kuralları karakterize edilmektedir.

Anahtar kelimeler: karar analizi, aksiyomatik, pazarlık, statüko, temizlenmemiş piyasa.

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To my husband and my son...

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CHAPTER 1
INTRODUCTION

In the last 60 years, axiomatic analysis has been one of the primary methods to investigate economic problems in many branches of literature. An *economic problem* is given by specifying the available alternatives and the agents' characteristics (such as, preferences, endowments, etc). Given a class of problems, a *rule* associates a set of alternatives to each problem. The aim is to identify well-behaved rules. According to the axiomatic method, the desirability of a rule is evaluated in terms of its properties. A *property* is a mathematical formulation of a desirable requirement that we would like to impose on rules. Therefore, the objective of this analysis is to understand and to describe the implications of lists of properties of interest. It usually results in *characterization theorems* that identify a particular rule or possibly a family of rules as the only rule or family of rules, satisfying a given list of properties. Quoting Thomson (2010),

> "... *Characterization theorems are extremely valuable, being on the boundary between the realm of the possible and the realm of the impossible. Tracing out this boundary is the ultimate goal of the axiomatic program.* "

Axiomatic analysis is widely used in both positive and normative economics. However, the interpretation of *properties* as well as findings differ in positive and normative economics. Positive economics concerns the description and explanation of an economic phenomenon. Therefore, in positive economics, axiomatic analysis is used to explain the observed phenomenon. The introduced axioms are aimed to be on the common properties of this phenomenon. As an example, consider *Property* α in decision theory. This property says that if an alternative is chosen from a set of alternatives, then it should also be chosen from any subset that contains it. In a positive approach to describe choices made by individuals, a researcher who observes this type of behavior in experiments introduces *Property* α to restrict the description to this type of economic environments. Normative economics incorporates normative judgements about what the economy ought to be like or what particular policy actions ought to be recommended to achieve a desirable goal. Therefore, in normative economics, axiomatic analysis is used to identify rules that satisfy a list of desirable *properties.* The introduced axioms are aimed to be recommended for the economy. As an example, consider *Pareto optimality* axiom. This axiom is used in many different branches of economics and it says that it is impossible to make one person better off without making someone else worse off. In a normative approach, a researcher usually imposes *Pareto optimality* because an economic system that is Pareto inefficient implies that a certain change (for example, in allocation of goods) may result in some individuals being made better off with no individual being made worse off.

The first examples of the axiomatic method in positive economics are Samuelson (1938) and von-Neumann and Morgenstern (1944). Samuelson (1938) incorporated the axiomatic method to decision theory. He introduced the well-known "weak axiom of revealed preferences" on consumer demand functions and by this axiom he laid the foundations of revealed preference theory. Von Neumann and Morgenstern's (1944) book, *Theory of Games and Economic Behavior,* is widely considered the groundbreaking text that created game theory. They also developed a concept, known as the "Von Neumann-Morgenstern Utility" that represents preferences in situations of uncertainty. They provided the set of necessary and sufficient axioms for preferences to be representable by a Von Neumann-Morgenstern utility.

The first examples of the axiomatic method in normative economics are Arrow (1951) and Nash (1950).[1] With an application of the axiomatic method, Arrow (1951) put the discipline of social choice theory in a structured and axiomatic framework and this led to the birth of social choice theory in its modern form. This methodology helped him to prove the celebrated Arrovian impossibility theorem which says that there is no social welfare function satisfying a set of desirable conditions. Nash's (1950) work is one of the early and important application of axiomatic study. He introduced the axiomatic method to bargaining theory. He imposed a set of general assumptions that the bargaining outcome should satisfy and and showed that these assumptions actually determine the outcome uniquely.

Recently, axiomatic analysis has been used in many economic contexts. A noncomprehensive list of examples is as follows:

(**i**) A bankruptcy problem consists of a liquidation value of a bankrupt firm and creditors' claims on this value. A rule associates with each such problem a recommendation that specifes how the liquidation value is divided among its creditors. The axioms are about what the division ought to be like (see O'Neill (1982) and Aumann and Maschler (1985)).

(**ii**) A fair allocation problem consists of a social endowment and agents' preferences over this endowment. A rule associates with each such problem an allocation that

[1] Nash wrote in his paper that his approach is positive. Later, however, it is also regarded as a normative exercise.

specifies how the social endowment is allocated among the agents. The axioms are about how the endowment should be allocated fairly (see Moulin (1995) and Thomson (2010) for the review fo this literature).

(**iii**) A cost allocation problem consists of a list of quantities demanded by a set of agents for a good and the cost of producing the good at various levels. A rule associates with each such problem a recommendation that specifies how the cost of satisfying aggregate demand is divided among the agents. The axioms are used to be recommended for the division of the good (see Taumann (1988) and Kolpin (1996)).

(**iv**) A matching problem consists of two sets of agents and preference relation of each agent over the members of the other set. A rule associates with each problem a matching that specifies how the agents are paired. The axioms are about how they should be paired (see Sasaki and Toda (1992) and Kara and Sönmez (1996, 1997)).

In this thesis, we use the axiomatic method to analyze both individual and collective decision problems in different economic contexts. We use the axiomatic analysis as both a positive and a normative tool. Chapter 2 is about describing choices made by individuals. Our axioms here are designed to describe observed behavior and thus our approach is positive. Chapter 3 is about describing bargaining stituations where the disagreement outcome depends on who breaks down the negotiation. Our axioms in this chapter are again designed to describe the observed phenomenon and thus our approach is positive. Chapter 4 is about designing trade mechanisms in nonclearing markets. Our axioms in this chapter are designed to be recommended for the designed mechanisms and thus our approach is normative.

This thesis is organized as follows.

In Chapter 2, we axiomatically analyze decision problems of individuals. Standard revealed preference theory has been criticized for not being able to address two phenomena: (*i*) incomplete preferences and (*ii*) status-quo bias. Our aim in this chapter is to propose a theory of revealed preferences that allows both of these features.

One of the most widely discussed axioms of utility theory is the completeness axiom which does not leave room for an individual to remain indecisive on any occasion. In daily life there are many situations in which an individual has incomplete preferences and thereby exhibits indecisiveness. This casual observation is supported by experimental studies. Danan and Ziegelmeyer (2004) experimentally test the descriptive validity of the completeness axiom and they show that a significant number of subjects (around two-thirds) violate it. In an experiment, Brady and Ansolabehere (1989) find that approximately 20 percent of their subjects have incomplete preferences over the candidates in the 1976 and 1984 Democratic Presidential primaries.

The second feature, status-quo bias, refers to an agent whose choice behaviour is affected by the existence of an alternative he holds at the time of choice (called the status-quo). This phenomenon is documented not only by experimental studies but also by empirical work in the case of actual markets (see Kahneman, Knetsch and Thaler (1991) for surveys). Particularly, Samuelson and Zeckhauser (1988) report on several decision-making experiments where a significant number of their subjects exhibit a status-quo bias.

Motivated by these observations, we propose a theory that encompasses both (i) incomplete preferences and (ii) status-quo bias. In our model, a choice problem is (i) a feasible set S of alternatives and (ii) a status-quo point x in S (allowed to be null when there is no status-quo). Our main result is that if an agent's choice behavior satisfies a set of basic properties, then it is *rationalizable*[2] by a pair of incomplete preference relations (one "more incomplete" than the other): when there is a status-quo, the agent first compares the non-status-quo alternatives to the status-quo by using the more incomplete preference relation. He chooses the status-quo if no alternative is strictly preferred to it. However, if there are some alternatives that are strictly preferred to the status-quo, then *among them* the agent chooses alternatives that maximize the second (less incomplete) preference relation.[3] This is related to Masatlioglu and Ok (2005) that models the status-quo bias as an agent having an incomplete preference relation that he uses to compare the status-quo to another alternative (and whenever indecisive, to choose the status-quo).

In Chapter 3, we analyze bargaining situations where the agents' payoffs from disagreement depend on who among them breaks down the negotiations. A typical bargaining problem, as modeled by Nash (1950) and the vast literature that follows, is made up of two elements. The first is a set of alternative agreements on which the agents negotiate. The second element is an alternative realized in case of disagreement. This "disagreement outcome" does not however contain detailed information about the nature of disagreement. Particularly, it is assumed in the existing literature that the realized disagreement alternative is independent of who among the agents disagree(s).

In real life examples of bargaining, however, the identity of the agent who terminates the negotiations turns out to have a significant effect on the agents' "disagreement payoffs". The reunion negotiations between the northern and the southern parts of Cyprus that took place at the beginning of 2004 constitute a good example. Due to

[2] An agent's choice behaviour is *rationalizable* if there exists a preference relation such that for any choice problem, its maximizers coincide with the agent's choices.

[3] Note that, this is different than simply maximizing the second preference relation on the whole set.

a very strong support from the international community towards the island's reunion, neither party preferred to be the one to disagree. Also, each preferred the other's disagreement to some agreements which they in turn preferred to leaving the negotiation table themselves.[4] Wage negotiations between firms and labor unions constitute another example to the dependence of the disagreement payoffs on the identity of the disagreer. There, the disagreement action of the union, a strike, and that of the firm, a lockout, can be significantly different in terms of their payoff implications. Finally, note that the bargaining framework is frequently used in economic models of family (see *e.g.* Becker (1981), Manser and Brown (1980), Sen (1983) and the following literature): the married couple bargains on alternative joint-decisions and divorce is their disagreement alternative. In the current models, payoffs from divorce do not depend on who in the couple leaves the marriage. However, it seems to us that this is hardly the case in reality.

We therefore extend Nash's (1950) standard model to a nonanonymous-disagreement model of bargaining by allowing the agents' payoffs from disagreement to depend on who among them disagrees. For this, we replace the disagreement *payoff vector* in the Nash (1950) model with a disagreement *payoff matrix*. The i^{th} row of this matrix is the payoff vector that results from agent i terminating the negotiations.

On our extended domain, we analyze the implications of two central properties which, on the Nash domain, are known to be incompatible (Thomson (2010)). The first property, called *strong monotonicity* (Kalai, 1977) states that an expansion of the set of possible agreements should not make any agent worse-off. The second property, called *scale invariance* (Nash, 1950) ensures the invariance of the physical bargaining outcome with respect to utility-representation changes that leave the underlying von Neumann-Morgenstern (1944) preferences intact.

Our first result establishes the existence of nonanonymous-disagreement bargaining rules that are both *strongly monotonic* and *scale invariant*. Next, we show that *strong monotonicity*, *scale invariance*, *weak Pareto optimality*, and *continuity* characterize the class of *monotone path rules* which assign each disagreement matrix to a monotone increasing path in the payoff space and for a given problem, picks the maximal feasible point of this monotone path as the solution. Then, we add *scale invariance* to this list

[4] There is a vast number of articles that discuss the issue. For example, see the *Economist* articles dated April 17, 2004 (volume 371, issue 8371), *Cyprus: A Greek Wrecker* (page 11) and *Cyprus: A Derailment Coming* (page 25); also see *Greece's Election: Sprinting Start?* dated March 13, 2004 (volume 370, issue 8366, page 31). Also see the special issue on Cyprus of *International Debates* (2005, 3:3). Finally, an interview (in Turkish) with a former Turkish minister of external affairs, which appeared in the daily newspaper Radikal on February 16, 2004, presents a detailed discussion of the implications of disagreement.

characterize a subclass of *monotone path rules*.

We also analyze two-agent problems. We show that a *scale invariant monotone path rule* for two-agent problems can be fully defined by the specification of at most eight monotone paths.

Finally, we introduce a *symmetric monotone path rule* that we call the *Cardinal Egalitarian rule*. This is a nondecomposable rule and it is a *scale invariant* version of the well-known *Egalitarian rule* (Kalai, 1977). We show that the *Cardinal Egalitarian rule* is *weakly Pareto optimal*, *strongly monotonic*, *scale invariant*, *symmetric* and that it is the only rule to satisfy these properties on a class of two-agent problems where the agents disagree on their strict ranking of the disagreement alternatives (as, for example, was the case for the 2004 Cyprus negotiations).

In Chapter 4, we analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are possibly unequal. This stickiness of prices is observed in many markets, either because the price adjustment process is slow or because the price is controlled from the outside of the market.

The main question is the following: in such markets, how should a central authority design a mechanism (hereafter, a trade rule) that determines the trade? In this paper, we axiomatically analyze trade rules on the basis of some well-known properties in the literature.

In our model, buyers and sellers constitute two exogenously differentiated sets. There is only one traded commodity and sellers face demand from buyers. Buyers might be individuals or producers that use the commodity as input. We assume that the buyers have strictly convex preferences on consumption bundles. Thus, they have single-peaked preferences on the boundary of their budget sets, and therefore, on their consumption of the commodity. Similarly, we assume that the sellers have strictly convex production sets. Thus, their profits are single-peaked in their output.

A *trade rule* maps each economy to a feasible trade. In our model, it is made up of two components: a *trade-volume rule* and an *allocation rule*. The *trade-volume rule* determines the trade-volume that will be carried out in the economy and thus, the total consumption and the total production. Then, the *allocation rule* allocates the total consumption among the buyers and the total production among the sellers.

The following papers study the design of a mechanism that determines the trade in nonclearing markets. Barberà and Jackson (1995) analyze a pure exchange economy with a arbitrary number of agents and commodities. Each agent has a positive endowment of the commodities and a continuous, strictly convex, and monotonic preference relation on his consumption. The authors look for *strategy proof* rules that facilitate trade in this exchange economy.

Our model is closely related to Kıbrıs and Küçükşenel (2009) and Bochet, İlkılıç, and Moulin (2009). Kıbrıs and Küçükşenel (2009) analyze a class of *trade rules* each of which is a composition of the Uniform rule with a *trade-volume rule* that picks the median of total demand, total supply and an exogenous constant. They show that this class uniquely satisfies *Pareto optimality, strategy proofness, no-envy,* and an informational simplicity axiom called *independence of trade-volume*. Bochet, İlkılıç, and Moulin (2009) introduces a graph structure to this setting and they assume that a trade between a buyer and a seller is possible only if there is a link between them. They characterize *the egalitarian transfer* mechanism by the combination of *Pareto optimality, strategy proofness, voluntary trade*, and *equal treatment of equals*.

In all these papers, the authors analyze markets with a fixed population. In this thesis, we allow the population to be variable and analyze the implications of these population changes. We introduce a class of *Uniform trade rules* each of which is a composition of the *Uniform rule* and a *trade-volume rule*. We axiomatically analyze *Uniform trade rules* on the basis of some central properties concerning variations of the population, namely, *consistency* and *population monotonicity*. We also analyze the implications of standard properties such as *Pareto optimality, strategy-proofness,* and *no-envy*, and an informational simplicity property, *strong independence of trade volume*.

We first show that a particular subclass of *Uniform trade rules* uniquely satisfies *consistency* together with *Pareto optimality, no-envy*, and *strong independence of trade volume*. Next, we add *strong independence of trade volume* to the list and characterize a smaller subclass that satisfies those properties.

Next, we note that there are trade rules that simultaneously satisfy three properties, which are incompatible on standart single peaked domain: *Pareto optimality, no-envy*, and *population monotonicity*. We characterize the subclass that additionally satisfies *strategy-proofness*. Finally, we also add *strong independence of trade volume* to the list.

To sum up, in this thesis, we axiomatically analyze both individual and collective decision problems in three different contexts. In Chapter 2, we use this analysis to introduce a revealed preference theory that allows both status-quo bias and indecisiviness between any two alternatives. In Chapter 3, we introduce bargaining problems in which the disagrement outcome depends on who causes the disagreement and we axiomatically analyze bargaining rules on these problems. In Chapter 4, we axiomatically analyze markets in which the price is fixed at a level where the supply and the demand are possibly unequal.

CHAPTER 2
REVEALED INCOMPLETE PREFERENCES UNDER STATUS-QUO BIAS

2.1 Introduction

Recently, (standard) revealed preference theory has been criticized for not being able to address two phenomena: (*i*) incomplete preferences and (*ii*) status-quo bias. The aim of this chapter is to propose a theory of revealed preferences that allows both of these features.

Quoting Aumann (1962), "... Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable". Aumann argues that in daily life there are many situations in which an individual has incomplete preferences and thereby exhibits indecisiveness. His arguments are supported by experimental studies. Danan and Ziegelmeyer (2004) experimentally test the descriptive validity of the completeness axiom and they show that a significant number of subjects (around two-thirds) violate completeness. In an experiment, Brady and Ansolabehere (1989) find that approximately 20 percent of their subjects have incomplete preferences over the candidates in the 1976 and 1984 Democratic Presidential primaries. Similar results are obtained by Eliaz and Ok (2006) who show that completeness of the revealed preferences is closely related to a *Property* β (Sen, 1971) of choice and that this property is violated by a significant number of subjects. (Therefore, they weaken *Property* β to represent incomplete preferences.)

The second feature, status-quo bias, refers to an agent whose choice behaviour is affected by the existence of an alternative he holds at the time of choice (called the status-quo). This phenomenon has been repeatedly demonstrated in experiments (see Kahneman, Knetsch and Thaler (1991) for surveys). Particularly, Samuelson and Zeckhauser (1988) report on several decision-making experiments where a significant number of their subjects exhibit a status-quo bias. Masatlioglu and Ok (2005) model the status-quo bias as an agent having an incomplete preference relation that he uses to compare the status-quo to another alternative (and whenever indecisive, to choose the status-quo).

There is no experimental study that demonstrates both incomplete preferences and a status-quo bias. However, we believe that there are choice situations in which a decision maker can exhibit both features. As an example, consider a professor who has job offers. He may be evaluating these offers with respect to several criteria and thus, may be indecisive between some of them. In addition, his current job (if it exists) may bias his choices. Models that exhibit both features are already used in political theory. For example, Ashworth (2005) considers voters whose preferences are incomplete. He also assumes that there is a status-quo action that the voters take unless some alternative dominates it.

Motivated by these observations, we propose a theory that encompasses both Masatlioglu and Ok (2005) and Eliaz and Ok (2006). In our model, a choice problem is (i) a feasible set S of alternatives and (ii) a status-quo point x in S (allowed to be null when there is no status-quo). Our main result is that if an agent's choice behavior satisfies a set of basic properties, then it is *rationalizable*[1] by a pair of incomplete preference relations (one "more incomplete" than the other): when there is a status-quo, the agent first compares the non-status-quo alternatives to the status-quo by using the more incomplete preference relation. He chooses the status-quo if no alternative is strictly preferred to it. However, if there are some alternatives that are strictly preferred to the status-quo, then *among them* the agent chooses alternatives that maximize the second (less incomplete) preference relation.[2, 3]

Existence of two distinct preference relations is essential in capturing certain characteristics of the choice behaviour that we observe. We show that agents whose choice behaviour can be rationalized by a single (however incomplete) preference relation satisfy a property that significantly limits the implications of status-quo bias (see Corollaries 1 and 2).

Our model is rich enough to make a distinction between an agent being indecisive or indifferent between two alternatives. There is an observational distinction between these two cases (e.g. see Eliaz and Ok (2006)). In both of them, the agent's choices switch between the two alternatives in repetitions of the same choice problem. However, an agent being indecisive between two alternatives also implies that in terms of comparison to some third alternatives, these two alternatives differ. This feature of indecisiveness leads to certain "inconsistencies" in the choice behavior (which do not exist in the case of indifference)[4].

[1] An agent's choice behaviour is *rationalizable* if there exists a preference relation such that for any choice problem, its maximizers coincide with the agent's choices.

[2] This process is similar to Masatlioglu and Ok (2005). However, they require one of the preference relations to be complete. As a result, the agent in their model is never indecisive between two non-status-quo alternatives.

[3] Note that, this is different than simply maximizing the second preference relation on the whole set.

[4] As an example, consider the following voter. He has two favorite parties, A and B. If he has to vote between one of these two, he could vote for either. First, let him face the problem of voting among A, X, and Y. Suppose that he votes for A. Now, in this choice problem, replace A with B. Being

In addition to Masatlioglu and Ok (2005) and Eliaz and Ok (2006), our model is similar to Zhou (1997) and Bossert and Sprumont (2003). However, these authors do not consider incomplete preferences and they only analyze problems with a status-quo (thus unlike them, we can also discuss properties that link the choice behaviour in problems with and without a status-quo). Our model is also similar to Tversky-Kahneman (1991) and Sagi (2003) who analyze cases where an agent's preferences are dependent on a reference state (which, in our case, is a status-quo alternative). However, these authors focus on properties of preferences (rather than choices as we do).

2.2 Properties of a Choice Correspondence

Let X be a nonempty metric space of alternatives and $\mathcal{X}$ be the set of all nonempty closed subsets of X. A **choice problem** is a pair (S, x) where $S \in \mathcal{X}$ and $x \in S$ or $x = \diamond$.[5] The set of all choice problems is $\mathcal{C}(X)$. If $x \in S$, then (S, x) is a choice problem with a status-quo and we denote the set of such choice problems by $\mathcal{C}_{sq}(X)$. If $x = \diamond$, then $(S, \diamond)$ is a choice problem without a status-quo. A **choice correspondence** is a map $c : \mathcal{C}(X) \to \mathcal{X}$ such that for all $(S, x) \in \mathcal{C}(X)$, $c(S, x) \subseteq S$.

A binary relation $\succeq$ on a nonempty set X is called a **preorder** if it is reflexive ($x \succeq x$ for all $x \in X$) and transitive ($x \succeq y$ and $y \succeq z$ imply $x \succeq z$ for all $x, y, z \in X$). An antisymmetric ($x \succeq y$ and $y \succeq x$ imply $x = y$ for all $x, y \in X$) preorder is called a **partial order** and a complete ($x \succeq y$ or $y \succeq x$ for all $x, y \in X$) partial order is called a **linear order**. Let $\succeq$ be any binary relation on X. Let $x, y \in X$. Then, $x \succ y$ if and only if $x \succeq y$ and $y \not\succeq x$ and $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$. Let $\succeq$ and $\succeq'$ be two binary relations on X and $x, y \in X$. Then, $\succeq'$ is an extension of $\succeq$ if and only if $x \succ y$ implies $x \succ' y$ and $x \sim y$ implies $x \sim' y$.

Let $x \in X$ and $S \in \mathcal{X}$. Following Masatlioglu and Ok (2005), we let $\mathcal{U}_{\succ}(S, x) = \{y \in S|\ y \succ x\}$ be the **strict upper contour set of $\boldsymbol{x}$ in $\boldsymbol{S}$ with respect to $\succeq$** and $\mathcal{M}(S, \succeq) = \{x \in S|\ \mathcal{U}_{\succ}(S, x) = \emptyset\}$ be the set of all **maximal elements in $\boldsymbol{S}$ with respect to $\succeq$**. For any positive integer n and any function $\boldsymbol{u} : X \to \mathbb{R}^n$, $\mathcal{U}_{\boldsymbol{u}}(S, x) = \{y \in S|\ \boldsymbol{u}(y) > \boldsymbol{u}(x)\}$ is the **upper contour set of $\boldsymbol{x}$ in $\boldsymbol{S}$ with respect to u**[6] and $\mathcal{M}(S, \boldsymbol{u}) = \{y \in S|\ \mathcal{U}_{\boldsymbol{u}}(S, y) = \emptyset\}$ is the set of all **maximal elements in $\boldsymbol{S}$ with respect to $\boldsymbol{u}$**.

indifferent between A and B means that he chooses B in this problem. However, being indecisive between A and B refers to the case in which he chooses an alternative different than B. For further discussion, please see Eliaz and Ok (2006).

[5] $\diamond$ denotes a null alternative and is used to represent cases when there is no status-quo.

[6] For vectors in $\mathbb{R}^n$, the inequalities are defined as follows: $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, ..., n$ and $x > y$ if and only if $x \geq y$ and $x \neq y$.

Now, we define some properties. The first two are borrowed from Masatlioglu and Ok (2005). *Property* α is a straightforward extension of the "standard *Property* α" in the revealed preference theory.

Property $\boldsymbol{\alpha}$: For any $(S,x),(T,x) \in \mathcal{C}(X)$ if $y \in T \subseteq S$ and $y \in c(S,x)$, then $y \in c(T,x)$.

For the second property, suppose y is not worse than any other alternative in a feasible set S, including the status-quo alternative x (if there is one). Then, *status-quo bias* requires that when y becomes the status-quo, it will be revealed strictly preferred to every other alternative in S. (For a detailed discussion, see Masatlioglu and Ok (2005)).

Status-quo Bias: For any $(S,x) \in \mathcal{C}(X)$, if $y \in c(S,x)$, then $c(S,y) = \{y\}$.

Now, we introduce a new property which is a weakening of the counterpart of Sen's (1971) *Property* β for choice problems with status-quo (see Masatlioglu and Ok (2005) for a stronger formulation). To see the main difference between properties β and β', take any alternative y from a feasible set of alternatives S and suppose there is a chosen alternative z in S such that the following condition holds: there is a subset T of S containing both y and z such that both are chosen in T. *Property* β then says that y must also be chosen from S. Our weaker *Property* β' on the other hand requires the above condition to hold for every chosen z in S for y also to be chosen.

Property $\boldsymbol{\beta'}$: For any $(S,x) \in \mathcal{C}(X)$ and $y \in S$, if for all $z \in c(S,x)$, there exists $T \subseteq S$ such that $x,y,z \in T$[7] and $y,z \in c(T,x)$, then $y \in c(S,x)$.

Properties β' and α are together equivalent to a *"revealed non-inferiority"* property (introduced by Eliaz and Ok (2006)) which is weaker than the *weak axiom of revealed preferences.*

The following three properties relate the choice behavior of a decision maker across problems with and without a status-quo. The first two of them are borrowed form Masatlioglu and Ok (2005). (For a detailed discussion of these properties, see their paper.)

[7] If $x = \diamond$, then consider only $y,z \in T$.

Dominance: For any $(T, x) \in \mathcal{C}(X)$, if $c(T, x) = \{y\}$ for some $T \subseteq S$, and $y \in c(S, \diamond)$, then $y \in c(S, x)$.

SQ-irrelevance: For any $(S, x) \in \mathcal{C}_{sq}(X)$, if $y \in c(S, x)$ and $x \notin c(T, x)$ for any nonempty $T \subseteq S$ with $T \neq \{x\}$, then $y \in c(S, \diamond)$.

For the third property, take any alternative x from a set S. Suppose that x is never chosen from a subset $T \neq \{x\}$ of S despite the fact that it is the status-quo. Thus, x does not play a significant role in the choice problem (S, x). In such cases, *strong SQ-irrelevance* requires that dropping out the status-quo alternative does not affect the agent's choices.

Strong SQ-irrelevance: For all $(S, x) \in \mathcal{C}_{sq}(X)$, if $x \notin c(T, x)$ for any nonempty $T \subseteq S$ such that $T \neq \{x\}$, then $c(S, \diamond) = c(S, x)$.

Strong SQ-irrelavence is weaker than the combination of Masatlioglu and Ok (2005)'s *"dominance"* and *"SQ-irrelevance"*. It implies *"status-quo irrelevance"*, but not *"dominance"* as noted in the following example: let $X = \{x, y, z\}$ and

$c(\{x, y, z\}, \diamond) = \{y\}$, $c(\{x, y\}, \diamond) = \{y\}$, $c(\{x, z\}, \diamond) = \{z\}$, $c(\{y, z\}, \diamond) = \{y\}$.

$c(\{x, y, x\}, x) = \{x, z\}$, $c(\{x, y\}, x) = \{y\}$, $c(\{x, z\}, x) = \{x\}$.

However, together with *Property* α and β', *strong SQ-irrelevance* implies both properties.

Lemma 1 *(i) If a choice correspondence c satisfies SQ-irrelevance and dominance, then it satisfies strong SQ-irrelevance.*

(ii) If a choice correspondence c satisfies Property α, Property β', and strong SQ-irrelevance, then it satisfies SQ-irrelevance and dominance.

Proof. (i) Let c satisfy *SQ-irrelevance* and *dominance*. Let $(S, x) \in \mathcal{C}_{sq}(X)$. Suppose $x \notin c(T, x)$ for any nonempty $T \subseteq S$ such that $T \neq \{x\}$. Then, for any $z \in S \backslash \{x\}$, $c(\{x, z\}, x) = \{z\}$. Let $y \in S$. First, let $y \in c(S, \diamond)$. Since $c(\{x, y\}, x) = \{y\}$, by *dominance*, $y \in c(S, x)$. Second, let $y \in c(S, x)$. Then, by *SQ-irrelevance*, $y \in c(S, \diamond)$. Therefore, $c(S, x) = c(S, \diamond)$ and c satisfies *strong SQ-irrelevance.*

(ii) Let c satisfy the given properties. To show that c satisfies *SQ-irrelevance*, let $(S, x) \in \mathcal{C}_{sq}(X)$ and $y \in S$. Suppose that $y \in c(S, x)$ and $x \notin c(T, x)$ for any nonempty $T \subseteq S$ such that $T \neq \{x\}$. By *strong SQ-irrelevance*, $c(S, x) = c(S, \diamond)$. Thus, $y \in c(S, \diamond)$. To show that c satisfies *dominance*, let $y \in c(S, \diamond)$ and suppose there is $T \subseteq S$ such that $(T,x) \in \mathcal{C}_{sq}(X)$ and $c(T,x) = \{y\}$. Suppose for a contradiction that $y \notin c(S,x)$. Then by *Property* β', there is $z \in c(S,x)$ such that there is no $T' \subseteq S$ with $x,y,z \in T'$ and $y,z \in c(T',x)$. Note that $z \neq x$, because otherwise by *Property* α, $x \in c(T,x)$. Now, consider the problem $(\{x,y,z\},x)$. By *Property* α, $z \in c(\{x,y,z\},x)$. Then, $y \notin c(\{x,y,z\},x)$. Also, $x \notin c(\{x,y,z\},x)$, because otherwise, by *Property* α, $x \in c(\{x,y\},x)$ and this implies by *Property* β' that $x \in c(T,x)$. Thus, $c(\{x,y,z\},x) = \{z\}$. By *Property* α, $z \in c(\{x,z\},x)$. This implies by *Property* β' that $c(\{x,z\},x) = \{z\}$, because otherwise $x \in c(\{x,y,z\},x)$. Thus, $x \notin c(T',x)$ for any $T' \subseteq \{x,y,z\}$ with $T' \neq \{x\}$. Then, by *strong SQ-irrelevance*, $c(\{x,y,z\},\diamond) = c(\{x,y,z\},x)$. Thus, $c(\{x,y,z\},\diamond) = \{z\}$. But $y \in c(S,\diamond)$ implies by *Property* α that $y \in c(\{x,y,z\},\diamond)$, a contradiction. ■

Thus, the class of choice correspondences that satisfy *Property* α, *Property* β', and *strong SQ-irrelevance* is the same as the class of choice correspondences that satisfy *Property* α, *Property* β', *dominance*, and *SQ-irrelevance*. For simplicity, we use *strong SQ-irrelevance* instead of *dominance* and *SQ-irrelevance*.

2.3 Results

The following lemma discusses the implications of the properties introduced in Section 2.

Lemma 2 *If the choice correspondence c on $\mathcal{C}(X)$ satisfies Property α, Property β', status-quo bias, and strong SQ-irrelevance, then there is a partial order $\succeq$ and a preorder $\succeq'$ such that $\succeq'$ is an extension of $\succeq$ and*

$$c(S,\diamond) = \mathcal{M}(S,\succeq') \quad \textit{for all } S \in \mathcal{X},$$

and

$$c(S,x) = \begin{cases} \{x\} & \textit{if } \mathcal{U}_{\succ}(S,x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\succ}(S,x),\succeq') & \textit{otherwise} \end{cases}$$

for all $(S,x) \in \mathcal{C}_{sq}(X)$.

Proof. Assume that c satisfies the given properties. For any $S \in \mathcal{X}$, $x \in S$ and $y \notin S$, let $S_{y,-x} = (S \cup \{y\}) \setminus \{x\}$. Let

$$\mathcal{P}(c) := \{(x,y) \in X \times X : \ x \neq y \text{ and } c(\{x,y\},\diamond) = \{x,y\}\},$$

and let $\mathcal{I}(c)$ be the set of pairs of alternatives $(x, y) \in X \times X$ such that there is a finite set $S \in \mathcal{X}$ with $x \in S$ and $y \notin S$ and at least one of the following is true:

i) $x \in c(S, \diamond)$ but $y \notin c(S_{y,-x}, \diamond)$,

ii) $x \notin c(S, \diamond)$ but $y \in c(S_{y,-x}, \diamond)$,

iii) $c(S, \diamond) \setminus \{x\} \neq c(S_{y,-x}, \diamond) \setminus \{y\}$.

Now, consider the binary relations $\succeq$, $\succ'$, and $\sim'$ defined on X by

$x \succeq y$ if and only if $x \in c(\{x, y\}, y)$,

$x \succ' y$ if and only if $c(\{x, y\}, \diamond) = \{x\}$ and $x \neq y$,

$x \sim' y$ if and only if $(x, y) \in \mathcal{P}(c) \setminus \mathcal{I}(c)$ or $x = y$.

Note that, $\sim'$ is symmetric. To see this, take any $(x, y) \in \mathcal{P}(c) \setminus \mathcal{I}(c)$. Note that $(y, x) \in \mathcal{P}(c)$. Then we have to show that $(y, x) \notin \mathcal{I}(c)$. Take any finite $T \in \mathcal{X}$ with $y \in T$ and $x \notin T$. Let $S = T_{x,-y}$. Since $(x, y) \notin \mathcal{I}(c)$, we have $x \in c(S, \diamond)$ if and only if $y \in c(S_{y,-x}, \diamond)$. That is $y \in c(T, \diamond)$ if and only if $x \in c(T_{x,-y}, \diamond)$. Moreover, $c(T_{x,-y}, \diamond) \setminus \{x\} = c(S, \diamond) \setminus \{x\} = c(S_{y,-x}, \diamond) \setminus \{y\} = c(T, \diamond) \setminus \{y\}$. Then $(y, x) \notin \mathcal{I}(c)$.

Also, note that $\succ'$ is asymmetric and disjoint from $\sim'$. Then, define $\succeq' := \succ' \cup \sim'$. Thus, $\succeq'$ is a binary relation on X with symmetric and asymmetric parts $\sim'$ and $\succ'$.

To show that $\succeq'$ is an extension of $\succeq$, first let $x, y \in X$ be such that $x \sim y$. Then, $x \in c(\{x, y\}, y)$ and $y \in c(\{x, y\}, x)$. By *status-quo bias*, $x \in c(\{x, y\}, y)$ implies $c(\{x, y\}, x) = \{x\}$. Thus, $x = y$ and by definition of $\sim'$, $x \sim' y$. Now, suppose $x \succ y$. Then, $x \in c(\{x, y\}, y)$ and $y \notin c(\{x, y\}, x)$. By *status-quo bias*, $c(\{x, y\}, x) = \{x\}$ and $c(\{x, y\}, y) = \{x\}$. Thus, $x \neq y$ and by *strong SQ-irrelevance*, $c(\{x, y\}, \diamond) = \{x\}$. Thus $x \succ' y$.

Now, we want to prove that for all $S \in \mathcal{X}$, $c(S, \diamond) = \mathcal{M}(S, \succeq')$. First, let $x \in S$ be such that $x \in c(S, \diamond)$. Suppose for a contradiction that $x \notin \mathcal{M}(S, \succeq')$. Then, there is $y \in S$ such that $y \succ' x$. Then, $c(\{x, y\}, \diamond) = \{y\}$. On the other hand, since $x \in c(S, \diamond)$, *Property* α implies $x \in c(\{x, y\}, \diamond)$. Thus, $x = y$ contradicting $y \succ' x$ and so $c(S, \diamond) \subseteq \mathcal{M}(S, \succeq')$.

Second, let $x \in \mathcal{M}(S, \succeq')$ and suppose for a contradiction that $x \notin c(S, \diamond)$. Then, by *Property* β', there is $y \in S \setminus \{x\}$ such that $y \in c(S, \diamond)$ and for all $T \subseteq S$ with $x, y \in T$, $x \notin c(T, \diamond)$. Then, $c(\{x, y\}, \diamond) = \{y\}$. Thus, $y \succ' x$, contradicting $x \in \mathcal{M}(S, \succeq')$ and so $\mathcal{M}(S, \succeq') \subseteq c(S, \diamond)$.

Claim 1: For any $(S, x) \in \mathcal{C}_{sq}(X)$,

$$c(S, x) \subseteq \begin{cases} \{x\} & \text{if } \mathcal{U}_{\succ}(S, x) = \emptyset, \\ \mathcal{U}_{\succ}(S, x) & \text{otherwise.} \end{cases}$$

Proof of Claim 1: Assume $\mathcal{U}_{\succ}(S, x) = \emptyset$. Let $y \in S \setminus \{x\}$ and for a contradiction, suppose $y \in c(S, x)$. By *Property* α and *status-quo bias* , $c(\{x, y\}, x) = \{y\}$. Thus, $y \succ x$ contradicting $\mathcal{U}_{\succ}(S, x) = \emptyset$. Therefore, $c(S, x) = \{x\}$.

Now, let $\mathcal{U}_{\succ}(S, x) \neq \emptyset$ and first suppose $x \in c(S, x)$. Thus, by *Property* α and *status-quo bias*, for all $z \in S$, $c(\{x, z\}, x) = \{x\}$. Then, there is no $z \in S$ such that $z \succ x$, contradicting $\mathcal{U}_{\succ}(S, x) \neq \emptyset$. Thus, $x \notin c(S, x)$. Then, let $y \in S \setminus \{x\}$ be such that $y \in c(S, x)$. By *Property* α and *status-quo bias*, $c(\{x, y\}, x) = \{y\}$. Thus, $y \succ x$ and so $y \in \mathcal{U}_{\succ}(S, x)$.

Claim 2: For any $(S, x) \in \mathcal{C}_{sq}(X)$, if $\mathcal{U}_{\succ}(S, x) \neq \emptyset$, $c(S, x) = c(\mathcal{U}_{\succ}(S, x), \diamond)$.
Proof of Claim 2: We first show that $c(S, x) \subseteq c(\mathcal{U}_{\succ}(S, x), \diamond)$. Let $y \in c(S, x)$. By Claim 1, $y \in \mathcal{U}_{\succ}(S, x)$. Thus, by *Property* α, $y \in c(S, x)$ implies $y \in c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, x)$. Also by Claim 1, for any nonempty $T \subseteq \mathcal{U}_{\succ}(S, x)$, $c(T \cup \{x\}, x) \subseteq \mathcal{U}_{\succ}(T \cup \{x\}, x)$. Thus, $x \notin c(T \cup \{x\}, x)$. Then, by *strong SQ-irrelevance*, $y \in c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, \diamond)$. Then, by *Property* α, $y \in c(\mathcal{U}_{\succ}(S, x), \diamond)$. Thus, $c(S, x) \subseteq c(\mathcal{U}_{\succ}(S, x), \diamond)$.

Now, we want to show $c(\mathcal{U}_{\succ}(S, x), \diamond) \subseteq c(S, x)$. Let $y \in c(\mathcal{U}_{\succ}(S, x), \diamond)$. Since (i) $\succeq \subseteq \succeq'$, and (ii) $c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, \diamond) = \mathcal{M}(\mathcal{U}_{\succ}(S, x) \cup \{x\}, \succeq')$, we have $x \notin c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, \diamond)$. Then, by *Property* α, for any $z \in c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, \diamond)$, we have $y, z \in c(\mathcal{U}_{\succ}(S, x), \diamond)$. Thus, by *Property* β', $y \in c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, \diamond)$. On the other hand, by Claim 1, for any $T \subseteq \mathcal{U}_{\succ}(S, x)$ with $T \neq \emptyset$, $x \notin c(T \cup \{x\}, x)$. Then, by *strong SQ-irrelevance*, $y \in c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, x)$. Since $\mathcal{U}_{\succ}(S, x) \neq \emptyset$, by Claim 1, $c(S, x) \subseteq \mathcal{U}_{\succ}(S, x)$. Thus, $x \notin c(S, x)$. Then, by *Property* α, for any $z \in c(S, x)$, we have $y, z \in c(\mathcal{U}_{\succ}(S, x) \cup \{x\}, x)$. Thus, by *Property* β', $y \in c(S, x)$. Thus, $c(\mathcal{U}_{\succ}(S, x), \diamond) \subseteq c(S, x)$ and so $c(S, x) = c(\mathcal{U}_{\succ}(S, x), \diamond)$.

Thus, by $c(S, \diamond) = \mathcal{M}(S, \succeq')$ and by claims 1 and 2, we prove that for all $(S, x) \in \mathcal{C}_{sq}(X)$,

$$c(S, x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\succ}(S, x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\succ}(S, x), \succeq') & \text{otherwise.} \end{cases}$$

The proofs of $\succeq$ being a partial order and $\succeq'$ being a preorder are identical to Masatlioglu and Ok (2005, page 22) and Eliaz and Ok (2006, page 82), respectively. $\blacksquare$

Note that, the agent in our model can be both indecisive and indifferent between two non-status-quo alternatives.

The following theorem shows that whenever X is finite, a choice correspondence, c satisfies our properties if and only if it is "rationalizable" by a pair of vector-valued utility functions (one aggregating the other). Vector-valued utility functions exist also in Masatlioglu and Ok (2005) who interpret them as an evaluation of the alternatives on the basis of various distinct criteria. The ith component of the vector-valued utility function represents the agent's ranking of the alternatives with respect to the ith criterion. While in Masatlioglu and Ok (2005) the agent uses a real-valued function to aggregate these criteria (so that he has complete preferences on the alternatives), the agent in our model cannot always do so.

Theorem *Let X be finite. A choice correspondence c on $\mathcal{C}(X)$ satisfies* Property α, Property β', strong SQ-irrelevance, *and* status-quo bias *if and only if there are positive integers n, m such that $n \geq m$, an injective function $\boldsymbol{u} : X \to \mathbb{R}^n$, and a strictly increasing map $\boldsymbol{f} : \boldsymbol{u}(X) \to \mathbb{R}^m$ such that for all $S \in \mathcal{X}$,*

$$c(S, \diamond) = \mathcal{M}(S, \boldsymbol{f}(\boldsymbol{u}))$$

and

$$c(S,x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\boldsymbol{u}}(S,x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\boldsymbol{u}}(S,x), \boldsymbol{f}(\boldsymbol{u})) & \text{otherwise} \end{cases}$$

for all $(S,x) \in \mathcal{C}_{sq}(X)$.

Proof. It is straightforward to show that the described choice correspondence satisfies the given properties. Conversely, let c be a choice correspondence on $\mathcal{C}(X)$. Assume that it satisfies the given properties. Consider the partial order $\succeq$ and the preorder $\succeq'$ constructed in the Lemma.

Claim 1: There is a positive integer n and an injective function $\boldsymbol{u} : X \to \mathbb{R}^n$ such that for all $x, y \in X$,

$$y \succeq x \text{ if and only if } \boldsymbol{u}(y) \geq \boldsymbol{u}(x).$$

Proof of Claim 1: Let $L(\succeq)$ stand for the set of all linear orders $\succeq^*$ such that $\succeq^*$ is an extension of $\succeq$. Since X is finite, $L(\succeq)$ is a nonempty and finite set. Then, enumerate $L(\succeq) = (\succeq_i)_{i=1}^n$ and note that $\succeq = \cap_{i=1}^n \succeq_i$. Since for each $i = 1, ..., n$, $\succeq_i$ is a linear order on a finite set X, there exists a function $u_i : X \to \mathbb{R}$ such that

$$x \succeq_i y \text{ if and only if } u_i(x) \geq u_i(y).$$

Let $\boldsymbol{u} = (u_1, ..., u_n)$. Then, for all $x, y \in X$,

$$x \succeq y \text{ if and only if } \boldsymbol{u}(x) \geq \boldsymbol{u}(y).$$

Since $\succeq$ is antisymmetric, $\boldsymbol{u}$ must be injective.
Claim 2: There is a positive integer m with $m \leq n$ and a function $\boldsymbol{u'} : X \to \mathbb{R}^m$ such that for all $x, y \in X$,

$$y \succeq' x \text{ if and only if } \boldsymbol{u'}(y) \geq \boldsymbol{u'}(x).$$

Proof of Claim 2: We can show the existence of m and $\boldsymbol{u'} : X \to \mathbb{R}^m$ by using the same argument as in Claim 1. Since $\succeq'$ is an extension of $\succeq$, $L(\succeq) = (\succeq_i)_{i=1}^n$, and $L(\succeq') = (\succeq'_i)_{i=1}^m$, we have $m \leq n$.

To complete the proof, we define $\boldsymbol{f} : \boldsymbol{u}(X) \to \mathbb{R}^m$ by $\boldsymbol{f}(\boldsymbol{u}(x)) := \boldsymbol{u'}(x)$. Since $\boldsymbol{u}$ is injective, $\boldsymbol{f}$ is well-defined. Moreover, if $\boldsymbol{u}(x) > \boldsymbol{u}(y)$ for some $x, y \in X$, by Claim 1, $x \succ y$. Then, by the lemma and Claim 2, $x \succ' y$ and $\boldsymbol{f}(\boldsymbol{u}(x)) = \boldsymbol{u'}(x) > \boldsymbol{u'}(y) = \boldsymbol{f}(\boldsymbol{u}(y))$. Thus, $\boldsymbol{f}$ is strictly increasing.

Thus, by Claim 1 and 2 and by the lemma,

$$c(S, \diamond) = \mathcal{M}(S, \boldsymbol{f}(\boldsymbol{u'}))$$

and

$$c(S, x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\boldsymbol{u}}(S, x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\boldsymbol{u}}(S, x), \boldsymbol{f}(\boldsymbol{u'})) & \text{otherwise.} \end{cases}$$

■

Note that if the agent's choice behaviour satisfies our properties, then similar to Masatlioglu and Ok (2005) the status-quo alternative affects the agent's choice in the following ways: (i) it eliminates the alternatives that do not give higher utility in all evaluation criteria, (ii) it becomes the unique choice if all alternatives are eliminated, (iii) it affects the agent's choices even if it is not chosen itself (please see Masatlioglu and Ok (2005) for an example).

2.4 Independence of Unique Choice from the Status-quo

In this section, we analyze the conditions under which the two preference relations can be replaced with a single one. There are agents whose choice behaviour can satisfy all of our properties and yet cannot be rationalized by a single preference relation. In fact, if a choice behaviour can be rationalized by a single incomplete preference relation, it then has to satisfy a property that we call *independence of unique choice from the status-quo*. To understand this property , suppose x is the unique choice when there is no status-quo in the problem. Now, consider the effect of a non-status-quo alternative, y being the status-quo. *Independence of unique choice from the status-quo* then requires that x should be also chosen from the latter problem. That is if x is revealed to be superior to any alternative in the feasible set, making y the status-quo does not cause it to be revealed superior to x.

Independence of unique choice from the status-quo: For all $S \in \mathcal{X}$ and $x, y \in S$ such that $x \neq y$, if $c(S, \diamond) = \{x\}$, then $x \in c(S, y)$.

This property restricts the power of the status-quo bias significantly. Since x is chosen uniquely, it is strictly preferred to y when there is no status-quo. Then, *independence of unique choice from the status-quo* requires that y being a status-quo alternative does not create a "too"strong status-quo bias towards itself, i.e. y cannot be revealed strictly preferred to x. This contradicts with one of the well-known experimental observations, *"the preference reversal phenomenon as an endowment effect"* (Slovic and Lichtenstein (1968)).

Unfortunately, it turns out that *independence of unique choice from the status-quo* is both necessary and sufficient for a choice behaviour to be rationalized by a single incomplete preference relation.

Corollary 1 (to Lemma 2**)** *If the choice correspondence c on $\mathcal{C}(X)$ satisfies* Property α, Property β', strong SQ-irrelevance, status-quo bias, *and* independence of unique choice from the status-quo, *then there is a partial order $\succeq$ such that*

$$c(S, \diamond) = \mathcal{M}(S, \succeq) \quad \textit{forall } S \in \mathcal{X}$$

and

$$c(S, x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\succ}(S, x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\succ}(S, x), \succeq) & \text{otherwise} \end{cases}$$

for all $(S,x) \in \mathcal{C}_{sq}(X)$.

Proof. Suppose that the choice correspondence c satisfies the given properties. Then, by Lemma 2, there is a partial order $\succeq$ and a preorder $\succeq'$ such that $\succeq'$ *is an extension of* $\succeq$ *and*

$$c(S,\diamond) = \mathcal{M}(S,\succeq') \quad \textit{forall } S \in \mathcal{X},$$

and

$$c(S,x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\succ}(S,x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\succ}(S,x),\succeq') & \text{otherwise} \end{cases}$$

for all $(S,x) \in \mathcal{C}_{sq}(X)$.

Let $\succeq$ and $\succeq'$ be defined as in the proof of Lemma 2. It is sufficient to show that for any $S \in \mathcal{X}$, $\mathcal{M}(S,\succeq') = \mathcal{M}(S,\succeq)$. For this, first let $x \in S$ be such that $x \in \mathcal{M}(S,\succeq')$ and suppose for a contradiction that $x \notin \mathcal{M}(S,\succeq)$. Then, there is $y \in S$ such that $y \succ x$. Since $\succeq'$ is an extension of $\succeq$, $y \succ' x$, contradicting $x \in \mathcal{M}(S,\succeq')$. Second, let $x \in \mathcal{M}(S,\succeq)$ and suppose for a contradiction that $x \notin \mathcal{M}(S,\succeq')$. Then, there is $y \in S$ such that $y \succ' x$. Thus, $c(\{x,y\},\diamond) = \{y\}$. Then, by *independence of unique choice from the status-quo*, $y \in c(\{x,y\},x)$ and by *status-quo bias*, $c(\{x,y\},x) = \{y\}$. Thus, $y \succ x$, contradicting $x \in \mathcal{M}(S,\succeq)$. Thus, we have the desired conclusion. ■

The implications of the *independence of unique choice from the status-quo* on the representation of the revealed preferences in Theorem are as follows:

Corollary 2 (to the Theorem) *Let* X *be a nonempty finite set. A choice correspondence* c *on* $\mathcal{C}(X)$ *satisfies* Property α, Property β', strong SQ-irrelevance, status-quo bias, *and* independence of unique choice from the status-quo *if and only if there is a positive integer* n *and a function* $\boldsymbol{u} : X \to \mathbb{R}^n$ *such that for all* $S \in \mathcal{X}$,

$$c(S,\diamond) = \mathcal{M}(S,\boldsymbol{u})$$

and

$$c(S,x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\boldsymbol{u}}(S,x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\boldsymbol{u}}(S,x),\boldsymbol{u}) & \text{otherwise} \end{cases}$$

for all $(S,x) \in \mathcal{C}_{sq}(X)$.

Proof. It is straightforward to show that the choice correspondence satisfies the given properties. Conversely, let c satisfy the given properties. Consider the partial order $\succeq$ constructed in the Lemma.

Claim: There exist a positive integer n and an injective function $\boldsymbol{u} : X \to \mathbb{R}^n$ such that for all $x, y \in X$,

$$y \succeq x \text{ if and only if } \boldsymbol{u}(y) \geq \boldsymbol{u}(x).$$

Proof of Claim: The proof is the same as the proof of Claim 1 in Theorem.

Thus, by the Claim and Corollary 1,

$$c(S, \diamond) = \mathcal{M}(S, \boldsymbol{u})$$

and

$$c(S, x) = \begin{cases} \{x\} & \text{if } \mathcal{U}_{\boldsymbol{u}}(S, x) = \emptyset, \\ \mathcal{M}(\mathcal{U}_{\boldsymbol{u}}(S, x), \boldsymbol{u}) & \text{otherwise} \end{cases}$$

for all $(S, x) \in \mathcal{C}_{sq}(X)$. ■

CHAPTER 3
BARGAINING WITH NONANONYMOUS DISAGREEMENT: MONOTONIC RULES

3.1 Introduction

A typical bargaining problem, as modeled by Nash (1950) and the vast literature that follows, is made up of two elements. The first is a set of alternative agreements on which the agents negotiate. The second element is an alternative realized in case of disagreement. This "disagreement outcome" does not however contain detailed information about the nature of disagreement. Particularly, it is assumed in the existing literature that the realized disagreement alternative is independent of who among the agents disagree(s).

In real life examples of bargaining, however, the identity of the agent who terminates the negotiations turns out to have a significant effect on the agents' "disagreement payoffs". The 2004 reunion negotiations between the northern and the southern parts of Cyprus constitute a good example. Due to a very strong support from the international community towards the island's reunion, neither party preferred to be the one to disagree. Also, each preferred the other's disagreement to some agreements which they in turn preferred to leaving the negotiation table themselves.[1] Wage negotiations between firms and labor unions constitute another example to the dependence of the disagreement payoffs on the identity of the disagreer. There, the disagreement action of the union, a strike, and that of the firm, a lockout, can be significantly different in terms of their payoff implications.[2]

Note that, neither of these examples can be fully represented in the confines of Nash's (1950) standard model. We therefore extend this model to a **nonanonymous-disagreement** model of bargaining by allowing the agents' payoffs from disagreement to depend on who among them disagrees. For this, we replace the disagreement *payoff*

[1] There is a vast number of articles that discuss the issue. For example, see the *Economist* articles dated April 17, 2004 (volume 371, issue 8371), *Cyprus: A Greek Wrecker* (page 11) and *Cyprus: A Derailment Coming* (page 25); also see *Greece's Election: Sprinting Start?* dated March 13, 2004 (volume 370, issue 8366, page 31). Also see the special issue on Cyprus of *International Debates* (2005, 3:3). Finally, an interview (in Turkish) with a former Turkish minister of external affairs, which appeared in the daily newspaper Radikal on February 16, 2004, presents a detailed discussion of the implications of disagreement.

[2] A similar case may arise between two countries negotiating at the brink of a war. Among the two possible disagreement outcomes, each country might prefer the one where it leaves the negotiation table first and makes an (unexpected) "preemptive strike" against the other.

vector in the Nash (1950) model with a disagreement *payoff matrix*. The i^{th} row of this matrix is the payoff vector that results from agent i terminating the negotiations. The **standard (anonymous-disagreement)** domain of Nash (1950) is a "measure-zero" subset of ours where all rows of the disagreement matrix are identical.

Our domain extension significantly increases the amount of admissible rules. Every bargaining rule on the Nash domain has counterparts on our domain (we call such rules **decomposable** since they are a composition of a rule from the Nash domain and a function that transforms disagreement matrices to disagreement vectors). But our domain also offers an abundance of rules that are **nondecomposable** (that is, they are not counterparts of rules from the Nash domain).

On our extended domain, we analyze the implications of two central properties which, on the Nash domain, are known to be incompatible (Thomson (2010)). The first property, called **strong monotonicity** (Kalai, 1977) states that an expansion of the set of possible agreements should not make any agent worse-off. Kalai (1977) motivates it as both a normative and a positive property and argues that "if additional options were made available to the individuals in a given situation, then no one of them should lose utility because of the availability of these new options". The second property, called **scale invariance** (Nash, 1950) ensures the invariance of the physical bargaining outcome with respect to utility-representation changes that leave the underlying von Neumann-Morgenstern (1944) preferences intact. Scale invariant rules use information only about the agents' preferences (and not their utility representation) to determine the bargaining outcome.

Our first result establishes the existence of nonanonymous-disagreement bargaining rules that are both *strongly monotonic* and *scale invariant*. More specifically, in *Subsection 3.3.1,* we first present a class of **monotone path rules** which assign each disagreement matrix to a monotone increasing path in the payoff space and for a given problem, picks the maximal feasible point of this monotone path as the solution.[3] In *Theorem 3*, we show that *strong monotonicity, scale invariance, weak Pareto optimality,* and "continuity "characterize the whole class of monotone path rules. Next, we show in *Theorem 4* that adding *scale invariance* to this list characterizes a class of monotone path rules.

In this subsection, we also analyze two-agent problems. We show in *Proposition 6* that a *scale invariant* monotone path rule for two-agent problems can be fully defined

[3]This monotone path can be interpreted as an agenda in which the agents jointly improve their payoffs until doing so is no more feasible. On the Nash domain, monotone path rules are introduced by Thomson and Myerson (1980) and further discussed by Peters and Tijs (1984) (also see Thomson (2010)).

by the specification of at most eight monotone paths.

Finally, in *Subsection 3.3.2*, we introduce a *symmetric* monotone path rule that we call the **Cardinal Egalitarian rule**. This is a nondecomposable rule and it is a *scale invariant* version of the well-known Egalitarian rule (Kalai, 1977). (The Egalitarian rule violates scale invariance since it makes interpersonal utility comparisons.) The Cardinal Egalitarian rule coincides with the Egalitarian rule on a class of normalized problems and solves every other problem by using scale invariance and this normalized class. In *Theorem 7*, we show that the Cardinal Egalitarian rule is *weakly Pareto optimal, strongly monotonic, scale invariant, symmetric* and that it is the only rule to satisfy these properties on a class of two-agent problems where the agents disagree on their strict ranking of the disagreement alternatives (as, for example, was the case for the 2004 Cyprus negotiations).

Kıbrıs and Tapkı (2007) show that the class of *decomposable* rules is a *nowhere dense* subset of all bargaining rules. This class, however, contains the (uncountably many) extensions of each rule that has been analyzed in the literature until now. Thus, we then enquire if the counterparts of some standard results on the Nash domain continue to hold for decomposable rules on our extended domain. We first show that an extension of the Kalai-Smorodinsky bargaining rule uniquely satisfies the Kalai-Smorodinsky (1975) properties. This uniqueness result, however, turns out to be an exception. An infinite number of decomposable rules survive the Nash (1950), Kalai (1977), Perles-Maschler (1981), and Thomson (1981) properties even though, on the Nash domain each of these results characterizes a single rule. In that paper, we also observe that extensions to our domain of a standard independence property (by Peters, 1986) imply *decomposability.*

Gupta and Livne (1988) analyze bargaining problems with an additional reference point (in the feasible set), interpreted as a past agreement. Chun and Thomson (1992) analyze an alternative model where the reference point is not feasible (and is interpreted as a vector of "incompatible" claims). Both studies characterize rules that allocate gains proportionally to the reference point. Neither of these two papers focus on disagreement. Livne (1988) and Smorodinsky (2005), on the other hand, analyze cases where the implications of disagreement are uncertain. They thus extend the Nash (1950) model to allow probabilistic disagreement points. They characterize alternative extensions of the Nash rule to their domain. Finally, Basu (1996) analyzes cases where disagreement leads to a noncooperative game with multiple equilibria and to model them, he extends the Nash model to allow for a set of disagreement points over which the players do not have probability distributions. He characterizes an extension of the Kalai-Smorodinsky (1975) rule to this domain.

Chun and Thomson (1990a and 1990b) and Peters and van Damme (1991) remain in the Nash (1950) model but they introduce axioms to represent cases where the agents are not certain about the implications of disagreement. Chun and Thomson (1990a) show that a basic set of properties characterize the weighted Egalitarian rules. Chun and Thomson (1990b) and Peters and van Damme (1991) show that the Nash rule uniquely satisfies alternative sets of properties. Some other papers that discuss disagreement-related properties on the Nash (1950) model are Dagan, Volij, and Winter (2002), Livne (1986), and Thomson (1987).

The common feature of all of the above papers (and the current cooperative bargaining literature for that matter) is that the implications of disagreement are independent of the identity of the agent who causes it. On the other hand, there are noncooperative bargaining models in which agents are allowed to leave and take an outside option. Shaked and Sutton (1984) present one of the first examples. Ponsatí and Sákovics (1998) analyze a model where both agents can leave at each period (but the resulting payoffs are independent of who leaves) and Corominas-Bosch (2000) analyzes a model where the disagreement payoffs depend on who the last agent to reject an offer was (but the agents are not allowed to leave, disagreement is randomly determined by nature). Our model can be seen as to provide a cooperative counterpart to these noncooperative models.

3.2 Model

Let $N = \{1, ..., n\}$ be the set of agents. For each $i \in N$, let $e_i \in \mathbb{R}^N$ be the vector whose i^{th} coordinate is 1 and every other coordinate is 0. Let $\mathbf{1} \in \mathbb{R}^N$ (respectively, $\mathbf{0}$) be the vector whose every coordinate is 1 (respectively, 0). For vectors in $\mathbb{R}^N$, inequalities are defined as: $x \leqq y$ if and only if $x_i \leqq y_i$ for each $i \in N$; $x \leq y$ if and only if $x \leqq y$ and $x \neq y$; $x < y$ if and only if $x_i < y_i$ for each $i \in N$. For each $S \subseteq \mathbb{R}^N$, $Int(S)$ denotes the interior of S and $Cl(S)$ denotes the closure of S. For each $S \subseteq \mathbb{R}^N$ and $s \in S$, $conv\{S\}$ denotes the convex hull of S and $s\text{-}comp\{S\} = \{x \in \mathbb{R}^N \mid s \leqq x \leqq y$ for some $y \in S\}$ denotes the s-comprehensive hull of S. The set S is *s-comprehensive* if $s\text{-}comp\{S\} \subseteq S$. The set S is *strictly s-comprehensive* if it is *s-comprehensive* and for each $x, y \in S$ such that $x \geq y \geq s$, there is $z \in S$ such that $z > y$.

Let the **Euclidean metric** be defined as $\|x - y\| = \sqrt{\sum (x_i - y_i)^2}$ for $x, y \in \mathbb{R}^N$ and let the **Hausdorff metric** be defined as $\mu^H(S^1, S^2) = \max_{i \in \{1,2\}} \max_{x \in S^i} \min_{y \in S^j} \|x - y\|$

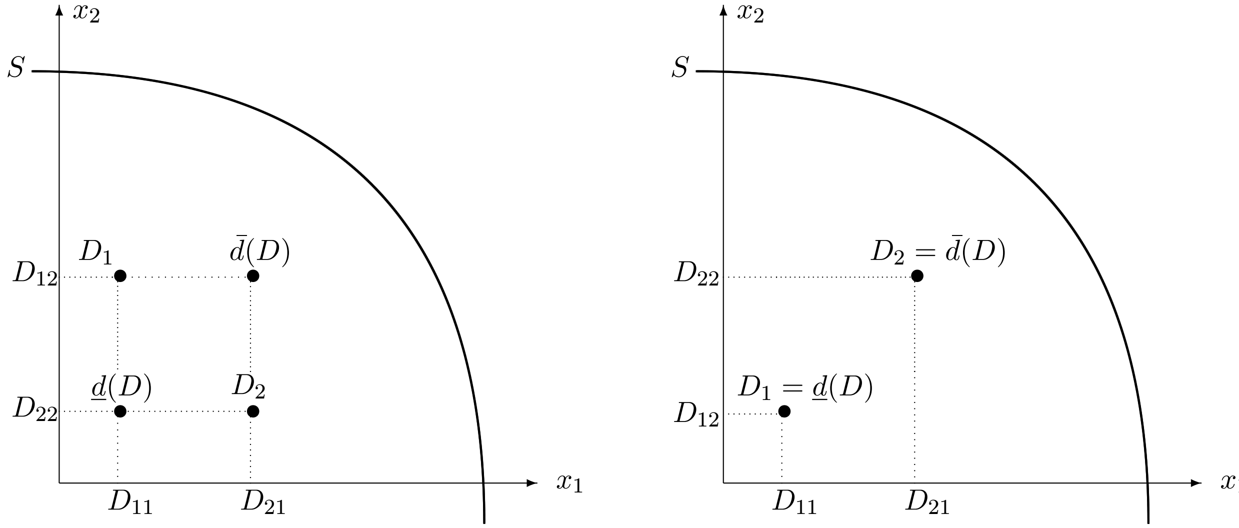

Figure 3.1: A typical bargaining problem with nonanonymous disagreement.

for compact sets $S^1, S^2 \subseteq \mathbb{R}^N$. Let

$$D = \begin{bmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{n1} & \cdots & D_{nn} \end{bmatrix} = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix} \in \mathbb{R}^{N \times N}$$

be a matrix in $\mathbb{R}^{N\times N}$. The i^{th} row vector $D_i = (D_{i1}, ..., D_{in}) \in \mathbb{R}^N$ represents the disagreement payoff profile that arises from agent i terminating the negotiations. For each $i \in N$, let $\overline{d}_i(D) = \max\{D_{ji} \mid j \in N\}$ be the maximum payoff agent i can get from disagreement and let $\underline{d}_i(D) = \min\{D_{ji} \mid j \in N\}$ be the minimal payoff. Let $\overline{d}(D) = (\overline{d}_i(D))_{i\in N}$ and $\underline{d}(D) = (\underline{d}_i(D))_{i\in N}$. Let the **metric $\boldsymbol{\mu}^{\mathbf{M}}$ on $\mathbb{R}^{\mathbf{N}\times\mathbf{N}}$** be defined as $\mu^M(D, D') = \max_{i\in N} \|D_i - D'_i\|$ for $D, D' \in \mathbb{R}^{N\times N}$.

Let Π be the set of all permutations π on N. A function $f : \mathbb{R} \to \mathbb{R}$ is *positive affine* if there is $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $f(x) = ax + b$ for each $x \in \mathbb{R}$. Let Λ be the set of all $\lambda = (\lambda_1, ..., \lambda_n)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a positive affine function.

For $\pi \in \Pi$, $S \subseteq \mathbb{R}^N$, and $D \in \mathbb{R}^{N\times N}$, let $\pi(S) = \{y \in \mathbb{R}^N \mid y = (x_{\pi(i)})_{i\in N}$ for some $x \in S\}$ and $\pi(D) = (D_{\pi(i)\pi(j)})_{i,j\in N}$. The set S (respectively, the matrix D) is **symmetric** if for every permutation $\pi \in \Pi$, $\pi(S) = S$ (respectively, $\pi(D) = D$). For $\lambda \in \Lambda$, let $\lambda(S) = \{(\lambda_1(x_1), ..., \lambda_n(x_n)) \mid x \in S\}$ and

$$\lambda(D) = \begin{bmatrix} \lambda_1(D_{11}) & \cdots & \lambda_n(D_{1n}) \\ \vdots & \ddots & \vdots \\ \lambda_1(D_{n1}) & \cdots & \lambda_n(D_{nn}) \end{bmatrix} = \begin{bmatrix} \lambda(D_1) \\ \vdots \\ \lambda(D_n) \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

A **(nonanonymous-disagreement bargaining) problem for** N is a pair (S, D) where $S \subseteq \mathbb{R}^N$ and $D \in \mathbb{R}^{N\times N}$ satisfy (see Figure 3.1): (i) for each $i \in N$, $D_i \in S$; (ii)

S is compact, convex, and $\underline{d}(D)$-comprehensive; (iii) there is $x \in S$ such that $x > \overline{d}(D)$. Assumptions (i), (ii) and a counterpart of (iii) are standard in the literature.[4] They essentially come out of problems where the agents have expected utility functions on a bounded set of lotteries.

Let $\mathcal{B}$ be the class of all problems for agents in N. Let $\mathcal{B}_= = \{(S, D) \in \mathcal{B} \mid D_1 = D_2 = ... = D_n\}$ be the subclass of **problems with anonymous disagreement**. Let $\mathcal{B}_{\neq} = \mathcal{B} \setminus \mathcal{B}_=$ be the subclass of **problems with nonanonymous disagreement**.

Let $\mathcal{B}^2_{\neq}$ be the class of **two-agent problems with nonanonymous disagreement**. Problems in $\mathcal{B}^2_{\neq}$ can be grouped into three distinct classes. In the first class of problems, the disagreement of one agent is strictly preferred by both agents to the disagreement of the other:

$$\mathcal{B}^2_{>>} = \{(S, D) \in \mathcal{B}^2_{\neq} \mid \text{there are } i, j \in \{1, 2\} \text{ such that } i \neq j \text{ and for all } k \in \{1, 2\}, D_{ik} > D_{jk}\}.$$

The second class of problems represents cases where one agent is indifferent between the two disagreement alternatives and the other has strict preferences:

$$\mathcal{B}^2_{>=} = \{(S, D) \in \mathcal{B}^2_{\neq} \mid \text{there are } i, j, k, l \in \{1, 2\} \text{ such that } i \neq j,\ k \neq l,\ D_{ik} > D_{jk} \text{ and } D_{il} = D_{jl}\}.$$

In the third class of problems, the agents disagree on their (strict) ranking of the two disagreement alternatives:

$$\mathcal{B}^2_{><} = \{(S, D) \in \mathcal{B}^2_{\neq} \mid \text{there are } i, j \in \{1, 2\} \text{ such that } i \neq j,\ D_{i1} > D_{j1}, \text{ and } D_{i2} < D_{j2}\}.$$

Let the **metric $\boldsymbol{\mu}^{\mathcal{B}}$ on $\mathcal{B}$** be defined as $\mu^{\mathcal{B}}((S, D), (S', D')) = \max\{\mu^H(S, S'), \mu^M(D, D')\}$ for $(S, D), (S', D') \in \mathcal{B}$. Given $(S, D) \in \mathcal{B}$, the set of **Pareto optimal alternatives** is $PO(S, D) = \{x \in S \mid y \geq x \text{ implies } y \notin S\}$ and the set of **weakly Pareto optimal alternatives** is $WPO(S, D) = \{x \in S \mid y > x \text{ implies } y \notin S\}$.

A **(nonanonymous-disagreement bargaining) rule** $F : \mathcal{B} \to \mathbb{R}^N$ is a function that satisfies $F(S, D) \in S$ for each $(S, D) \in \mathcal{B}$. Let $\mathcal{F}$ be the class of all rules. A rule F is **Pareto optimal** if for each $(S, D) \in \mathcal{B}$, $F(S, D) \in PO(S, D)$. It is **weakly Pareto optimal** if for each $(S, D) \in \mathcal{B}$, $F(S, D) \in WPO(S, D)$. It is **symmetric** if for each $(S, D) \in \mathcal{B}$ with *symmetric* S and D, $F(S, D)$ is also *symmetric*, that is, $F_1(S, D) = ... = F_n(S, D)$.

[4] When Assumption 3 is violated, the agents are not guaranteed to reach an agreement. Particularly, for each alternative x, there will be an agent who receives higher payoff from someone (including himself) leaving the negotiation table. It will be in the interest of this agent then to follow strategies that induce disagreement rather than to accept x.

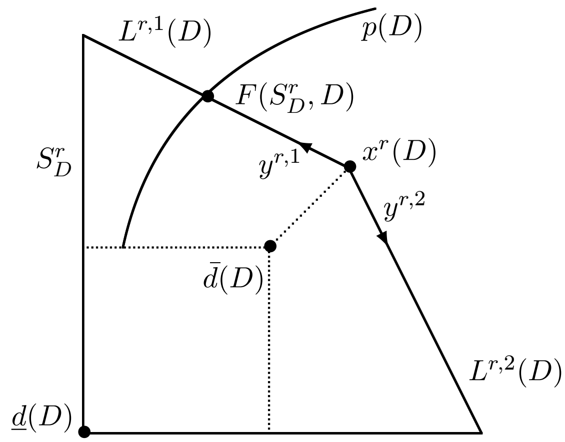

Figure 3.2: The construction of Step 1 in the proof of Theorem 3.

The following property requires small changes in the data of a problem not to have a big effect on the agreement. A rule F is **set-continuous** if for every $D \in \mathbb{R}^{N\times N}$ and for every sequence $\{(S^m, D)\}_{m\in\mathbb{N}} \subseteq \mathcal{B}$ that converges with respect to $\mu^{\mathcal{B}}$ to some $(S, D) \in \mathcal{B}$, we have $\lim_{m\to\infty} F(S^m, D) = F(S, D)$.

Next, we present two central properties in bargaining theory. The first one requires the physical bargaining outcome to be invariant under utility-representation changes as long as the underlying von Neumann-Morgenstern (1944) preference information is unchanged. A rule F is **scale invariant** if for each $(S, D) \in \mathcal{B}$ and each $\lambda \in \Lambda$, $F(\lambda(S), \lambda(D)) = \lambda(F(S, D))$. The second property requires that an expansion of the set of possible agreements make no agent worse-off. A rule F is **strongly monotonic** (Kalai, 1977) if for each $(S, D), (T, D) \in \mathcal{B}$, $T \subseteq S$ implies $F(T, D) \leqq F(S, D)$.

3.3 Results

3.3.1 Monotone Path Rules On our domain, a very large class of rules simultaneously satisfy three properties which, on the Nash domain, $\mathcal{B}_=$, are incompatible: *weak Pareto optimality, strong monotonicity,* and *scale invariance* (see Thomson (2010) for a discussion). We introduce them next.

A **monotone path** on $\mathbb{R}^N$ is the image $G \subseteq \mathbb{R}^N$ of a function $g : \mathbb{R}_+ \to \mathbb{R}^N$ which is such that for all $i \in N$, $g_i : \mathbb{R}_+ \to \mathbb{R}$ is continuous and nondecreasing and for some $j \in N$, $g_j(\mathbb{R}_+) = [g_j(0), \infty)$. Let $\mathbb{G}$ be the set of all monotone paths.

Let $p : \mathbb{R}^{N\times N} \to \mathbb{G}$ be a **path generator (function)** that maps each disagreement matrix D to a monotone path $p(D)$ such that (i) $x = \min p(D)$ is the unique member of $p(D)$ that satisfies $\underline{d}(D) \leqq x \leqq \bar{d}(D)$ and $x_i = \bar{d}_i(D)$ for some $i \in N$, and (ii) there are no $x, y \in p(D)$ such that $x \neq y$ and $x_i = y_i > \bar{d}_i(D)$ for some $i \in N$. Condition (i) requires the path $p(D)$ to start from a point x that is at the weak Pareto boundary of the rectangular set in-between $\underline{d}(D)$ and $\bar{d}(D)$. Condition (ii) strengthens the monotonicity requirement on the path $p(D)$. For example, in $\mathbb{R}^2$ it requires the path not to be vertical (respectively, horizontal) on the half-space of vectors whose first (respectively, second) coordinate is greater than that of $\overline{d}(D)$. The path generator p is **scale invariant** if for each $D \in \mathbb{R}^{N\times N}$ and $\lambda \in \Lambda$, $p(\lambda(D)) = \lambda(p(D))$.

The **monotone path rule $\mathbf{F^p}: \mathcal{B} \to \mathbb{R}^N$ with respect to the path generator p** maps each $(S, D) \in \mathcal{B}$ to the maximal point of S along $p(D)$, that is, $F^p(S, D) = WPO(S, D) \cap p(D)$.[5]

The following theorem shows that monotone path rules uniquely satisfy three basic properties.

Theorem 3 *A rule $F : \mathcal{B} \to \mathbb{R}^N$ is weakly Pareto optimal, strongly monotonic, and set-continuous if and only if it is a monotone path rule F^p.*

Proof. Monotone path rules by definition are *weakly Pareto optimal. Strong monotonicity* follows from the monotonicity of the paths $p(D)$ and *set-continuity* follows from Condition (ii) on the path generator p.

For the uniqueness part of the first statement, let $F : \mathcal{B} \to \mathbb{R}^N$ be a rule that is *weakly Pareto optimal, strongly monotonic,* and *set-continuous.*

Step 1. *(define p)* Fix arbitrary $D \in \mathbb{R}^{N\times N}$ and for each $r \in \mathbb{R}_+$, let $x^r(D) = \overline{d}(D) + r\mathbf{1}$ (see Figure 3.2). Let $f : \mathbb{R}_+ \to [0, 1]$ be an increasing continuous function such that $f(0) = 0$ and $\lim_{r\to\infty} f(r) = 1$. For each $i \in N$, let $y^{r,i} \in \mathbb{R}^N$ be such that $y^{r,i}_i = -1$ and for $j \neq i$, $y^{r,i}_j = f(r)$. Note that $y^{r,i}$ is a vector whose ith coordinate is -1 and whose jth coordinate is $f(r)$. It is used to construct the following line segment: let $L^{r,i}(D) = \{x^r(D) + ly^{r,i} \mid 0 \leqq l \leqq x^r_i(D) - \underline{d}_i(D)\}$. Now let $S^r_D = \underline{d}(D)$-*comp* $\{L^{r,1}(D), ..., L^{r,n}(D)\}$ and note that for all $r \in \mathbb{R}_{++}$, (i) S^r_D is *strictly $\underline{d}(D)$-comprehensive* and (ii) $F(S^r_D, D) \in PO(S^r_D, D)$. Finally, define $p(D) = Cl\{F(S^r_D, D) \mid r \in \mathbb{R}_{++}\}$. Note that, for each $D \in \mathbb{R}^{N\times N}$, $p(D)$ is ordered with respect to $\geq$ by *strong monotonicity* of F. Let $F^p : \mathcal{B} \to \mathbb{R}^N$ be defined as follows: for each $(S, D) \in \mathcal{B}$, $F^p(S, D) = \max\left(p(D) \cap WPO(S, D)\right)$.

Step 2. *($F = F^p$)* Now let $(S, D) \in \mathcal{B}$ and let $x^* = F^p(S, D)$ (see Figure 3.3). Then by definition of p, there is $r \in \mathbb{R}_{++}$ such that $x^* = F^p(S^r_D, D) = F(S^r_D, D)$. Let $T = S \cap S^r_D$ and note that $x^* \in T$. Since $x^* \in PO(S^r_D, D)$, we also have $x^* \in PO(T, D)$. Now, by *strong monotonicity* of F, $F(T, D) \leqq x^*$. First assume that $y \leq x^*$ implies $y \notin WPO(T, D)$. Then *weak Pareto optimality* of F implies $F(T, D) = x^*$. Alternatively if there is $y \leq x^*$ such that $y \in WPO(T, D)$, let $\{T^m\}_{m\in\mathbb{N}} \to T$ be such that for each $m \in \mathbb{N}$, $T^m \subseteq S^r_D$ is *strictly $\underline{d}(D)$-comprehensive* (this is possible since S^r_D is strictly $\underline{d}(D)$-comprehensive) and $x^* \in PO(T^m, D)$ (this is possible since $x^* \in PO(S^r_D, D)$). Then by the previous case, $F(T^m, D) = x^*$ for each $m \in \mathbb{N}$ and

[5] Continuity and monotonicity of the path $p(D)$ guarantee that this intersection is nonempty while Condition (ii) on the generator function p guarantees that it is a singleton.

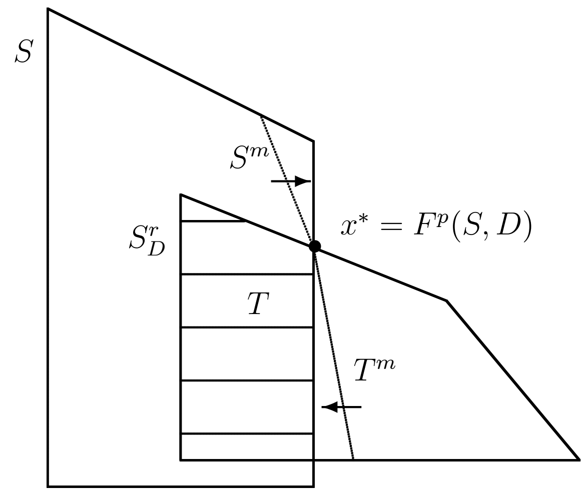

Figure 3.3: The construction of Step 2 in the proof of Theorem 3.

by *set-continuity* of F, $F(T, D) = x^*$. Finally, $T \subseteq S$, by *strong monotonicity* of F implies $x^* \leqq F(S, D)$. If $x^* \in PO(S, D)$, this implies $x^* = F(S, D)$. Alternatively if $x^* \in WPO(S, D) \setminus PO(S, D)$, let $\{S^m\}_{m \in \mathbb{N}} \to S$ be such that for each $m \in \mathbb{N}$, $T \subseteq S^m$ and $x^* \in PO(S^m, D)$ (this is possible since $x^* \in PO(T, D)$). Then by the previous case, $F(S^m, D) = x^*$ and by *set-continuity* of F, $F(S, D) = x^*$.

Step 3. *($F = F^p$ is a monotone path rule)* We show that for each $(S, D) \in \mathcal{B}$, the set $p(D) \cap WPO(S, D)$ is a singleton and thus, $F^p(S, D) = p(D) \cap WPO(S, D)$. Suppose $(S, D) \in \mathcal{B}$ is such that there is $y \leq F^p(S, D)$ satisfying $y \in p(D) \cap WPO(S, D)$. Let $\{S^m\}_{m \in \mathbb{N}} \to S$ be such that for each $m \in \mathbb{N}$, $p(D) \cap WPO(S^m, D) = \{y\}$. Then for each $m \in \mathbb{N}$, $F^p(S^m, D) = y$. This, by *set-continuity* of F^p implies $F^p(S, D) = y$, contradicting $y \neq F^p(S, D)$. ■

The following theorem characterizes *scale invariant* monotone path rules. Note that the domain reduces from $\mathcal{B}$ to $\mathcal{B}_{\neq}$ as we introduce *scale invariance*. This is because the stated properties (of Theorem 4) are not compatible on $\mathcal{B}_{=}$.

Theorem 4 *A rule on $\mathcal{B}_{\neq}$, $F : \mathcal{B}_{\neq} \to \mathbb{R}^N$ is weakly Pareto optimal, strongly monotonic, set-continuous, and scale invariant if and only if it is a monotone path rule F^p where p is scale invariant.*

Proof. The proof of Theorem 3 does not rely on the existence of a problem in $\mathcal{B}_{=}$. Therefore, its statement also holds on the subclass $\mathcal{B}_{\neq}$ of $\mathcal{B}$. That is, a rule $F : \mathcal{B}_{\neq} \to \mathbb{R}^N$ is *weakly Pareto optimal, strongly monotonic,* and *set-continuous* if and only if it is a monotone path rule F^p on $\mathcal{B}_{\neq}$. Therefore, to prove Theorem 4 it suffices to show that $F = F^p$ is *scale invariant* if and only if p is *scale invariant.* For this, let $(S, D) \in \mathcal{B}_{\neq}$ and $\lambda \in \Lambda$.

First note that $F^p(\lambda(S), \lambda(D)) = WPO(\lambda(S), \lambda(D)) \cap p(\lambda(D))$ and $WPO(\lambda(S), \lambda(D)) = \lambda(WPO(S, D))$. Then, $\lambda(F^p(S, D)) = \lambda\left(WPO\left(S, D\right) \cap p\left(D\right)\right) = \lambda\left(WPO\left(S, D\right)\right) \cap$

$\lambda(p(D)) = WPO(\lambda(S), \lambda(D)) \cap \lambda(p(D))$. Therefore, $p(\lambda(D)) = \lambda(p(D))$ implies $F^p(\lambda(S), \lambda(D)) = \lambda(F^p(S, D))$ (that is, *scale invariance* of p implies *scale invariance* of F^p).

For the other direction, suppose $p(\lambda(D)) \neq \lambda(p(D))$ for some $D \in \mathbb{R}^{N\times N}$ and $\lambda \in \Lambda$. For each $\omega \in \mathbb{R}^N_{++}$ satisfying $\sum \omega_i = 1$ and $r > \sum \omega_i \overline{d}_i(\lambda(D))$, let $T^{r,\omega} = \{x \in \mathbb{R}^N \mid \sum \omega_i x_i \leqq r \text{ and } x \geqq \underline{d}(\lambda(D))\}$ and note that $(T^{r,\omega}, \lambda(D)) \in \mathcal{B}_{\neq}$. Now, by $p(\lambda(D)) \neq \lambda(p(D))$ and the fact that both $p(\lambda(D))$ and $\lambda(p(D))$ are images of continuous functions, there is $\omega^* \in \mathbb{R}^N_{++}$ satisfying $\sum \omega^*_i = 1$ and $r^* > \sum \omega^*_i \overline{d}_i(\lambda(D))$ such that $WPO(T^{r^*,\omega^*}, \lambda(D)) \cap p(\lambda(D)) \neq WPO(T^{r^*,\omega^*}, \lambda(D)) \cap \lambda(p(D))$. But the expression on the left is $F^p(T^{r^*,\omega^*}, \lambda(D))$ and the expression on the right is $\lambda(F^p(\lambda^{-1}(T^{r^*,\omega^*}), D))$. This contradicts *scale invariance* of F^p (therefore, *scale invariance* of F^p implies *scale invariance* of p). ■

Remark 5 *On $\mathcal{B}_=$, the Nash (1950) bargaining rule uniquely satisfies weak Pareto optimality, symmetry, scale invariance, and an "independence of irrelevant alternatives" property. On $\mathcal{B}$, a large class of monotone path rules satisfy all of these properties. They are characterized by scale invariant and "symmetric" path generators.*

Two-agent problems are central in bargaining theory. We next show that for this case, *scale invariant* monotone path rules have a very simple form (for its proof, please see the Appendix).

Proposition 6 *On $\mathcal{B}^2_{\neq}$, a scale invariant monotone path rule can be completely characterized by at most eight distinct paths. On $\mathcal{B}^2_{>=}$, these paths are either vertical or horizontal.*

With more than two agents, Proposition 6 no more holds: constructing a monotone path rule potentially involves the specification of an infinite number of paths.[6]

3.3.2 Cardinal Egalitarian Rule In this subsection, we analyze the implications of *symmetry* together with *strong monotonicity* and *scale invariance*. *Symmetry* is a weakening of "anonymity" which requires that agents with identical payoff functions receive the same payoff. It thus concerns negotiations where the agents have equal "bargaining power". Implications of *symmetry* has been analyzed by many authors including Nash (1950), Kalai and Smorodinsky (1975), and Kalai (1977).

[6] This has got to do with the fact that with two agents, all disagreement matrices are divided into eight equivalence classes: two matrices in the same class are related by a positive affine transformation. In an equivalence class, it is sufficient to specify a monotone path for one matrix; scale invariance then defines the paths of the other matrices. With more agents however, the number of equivalence classes becomes infinite.

We first present a *symmetric* monotone path rule. The **Cardinal Egalitarian rule**, F^{CE} picks the maximizer, for each $D \in \mathbb{R}^{N \times N}$, of the linear monotone path that passes through $\underline{d}(D)$ and $\overline{d}(D)$: $p^{CE}(D) = \{\overline{d}(D) + r(\overline{d}(D) - \underline{d}(D)) \mid r \in \mathbb{R}_+\}$; that is, $F^{CE} = F^{p^{CE}}$.[7]

The Cardinal Egalitarian rule is well-defined for all nonanonymous-disagreement problems, $\mathcal{B}_{\neq}$, independent of the number of agents. Additional to the properties stated in the next theorem, it is *set-continuous*. Also, it is a "nondecomposable" rule. That is, it can not be written as a composition of a rule from the Nash domain and a function that transforms disagreement matrices to disagreement vectors (for more on decomposability, see Kıbrıs and Tapkı (2007)).

The Cardinal Egalitarian solution to a problem utilizes, for each agent, the difference between his maximum and minimum disagreement payoffs. Agents for whom this difference is higher receive a higher share of the surplus (over $\overline{d}(D)$) than others. As a result of this feature, the Cardinal Egalitarian rule violates "disagreement payoff monotonicity"; that is, an increase in an agent's disagreement payoff can make him worse-off (because, it can decrease the aforementioned difference). Also note that, for problems where $\underline{d}(D)$ and $\overline{d}(D)$ are much closer to each other than they are to the problem's Pareto boundary, the Cardinal Egalitarian solution can be very sensitive to small changes in D. This sensitivity increases as $\underline{d}(D)$ and $\overline{d}(D)$ get closer to each other.

The following result analyzes the properties of the Cardinal Egalitarian rule.

Theorem 7 *The Cardinal Egalitarian rule, F^{CE}, is weakly Pareto optimal, strongly monotonic, scale invariant, and symmetric on $\mathcal{B}_{\neq}$. Furthermore on $\mathcal{B}^2_{><}$, it is the unique rule that satisfies these properties.*

Proof. It is straightforward to show that F^{CE} satisfies these properties. Conversely let F be any rule on $\mathcal{B}^2_{><}$ that satisfies them. Take any $(S, D) \in \mathcal{B}^2_{><}$. We want to show that $F(S, D) = F^{CE}(S, D)$.

Consider the positive affine transformation $\lambda \in \Lambda$ such that $\lambda_i(x) = \frac{x_i - \underline{d}_i(D)}{\overline{d}_i(D) - \underline{d}_i(D)}$ for $i \in N$. Note that $\lambda(\overline{d}(D)) = \mathbf{1}$ and $\lambda(\underline{d}(D)) = \mathbf{0}$. Then, by definition $F^{CE}(\lambda(S), \lambda(D)) = (s, s)$, for some $s > 1$. Consider $S' = 0\text{-}comp\{(s, s)\}$. Note that $\lambda(D) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or

[7] We call this rule the Cardinal Egalitarian rule because it is a version of the Egalitarian rule (Kalai, 1977) that is covariant under *cardinal* (*i.e.* positive affine) transformations. To see this, let (S, D) be called a normalized problem when D is any one of the matrices $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$. For such problems, the Cardinal Egalitarian rule coincides with the Egalitarian rule. Any other problem is a cardinal transformation of a normalized problem and its Cardinal Egalitarian solution is the same cardinal transformation of the Egalitarian solution to the associated normalized problem.

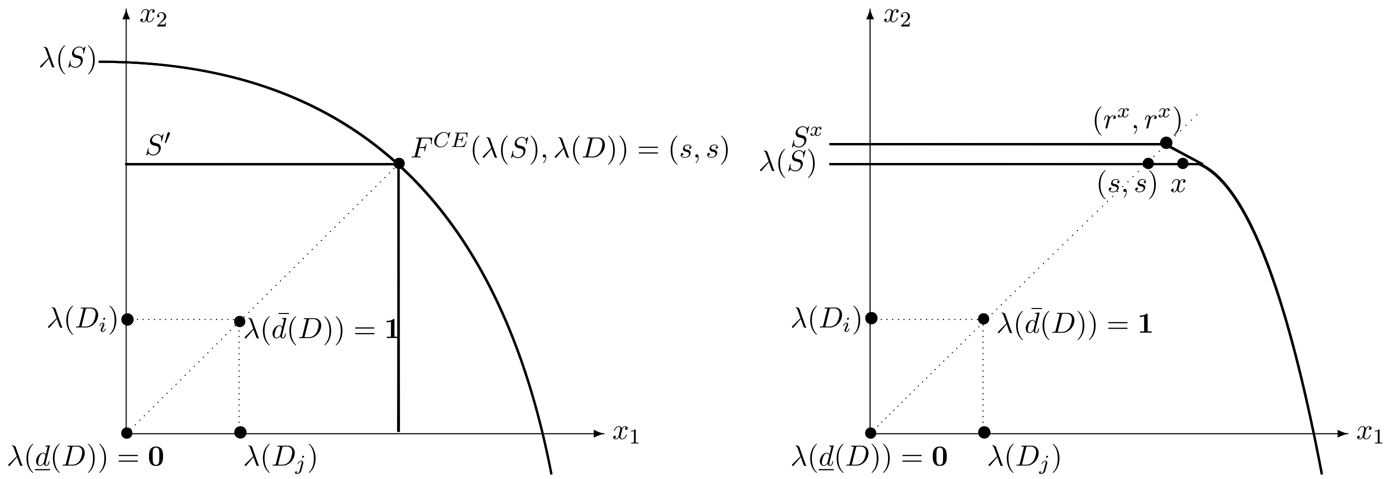

Figure 3.4: Constructing S' (on the left) and S^x (on the right) in the proof of Theorem 7.

$\lambda(D) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since S' is symmetric, $(S', \lambda(D))$ is a symmetric problem. Then, by *symmetry* and *weak Pareto optimality* of F, $F(S', \lambda(D)) = (s, s)$. Since $\lambda(S) \supseteq S'$, *strong monotonicity* of F implies $F(\lambda(S), \lambda(D)) \geqq F(S', \lambda(D))$.

Now if $(s, s) \in PO(\lambda(S), \lambda(D))$ (as in Figure 3.4, left), then $F(\lambda(S), \lambda(D)) = (s, s) = F^{CE}(\lambda(S), \lambda(D))$. Alternatively, assume that $(s, s) \in WPO(\lambda(S), \lambda(D))$ (as in Figure 3.4, right). Suppose $F(\lambda(S), \lambda(D)) = x \geq (s, s)$. Let $r^x \in \mathbb{R}$ be such that $s < r^x < \max\{x_1, x_2\}$. Let $S^x \subseteq \mathbb{R}^N$ be such that $S^x = conv\{0\text{-}comp\{r^x, r^x\}, \lambda(S)\}$. Note that $(S^x, \lambda(D)) \in \mathcal{B}_{\neq}$ and $F^{CE}(S^x, \lambda(D)) = (r^x, r^x) \in PO(S^x, \lambda(D))$. So by the previous argument, $F(S^x, \lambda(D)) = (r^x, r^x)$. Also since $s < r^x$, $S^x \supseteq \lambda(S)$. Thus by *strong monotonicity* of F, $F(S^x, \lambda(D)) = (r^x, r^x) \geqq x = F(\lambda(S), \lambda(D))$, contradicting $r^x < \max\{x_1, x_2\}$. Therefore, $F(\lambda(S), \lambda(D)) = (s, s)$.

Finally, by *scale invariance* of F and F^{CE}, $F(S, D) = \lambda^{-1}(F(\lambda(S), \lambda(D))) = \lambda^{-1}(F^{CE}(\lambda(S),$
$\lambda(D))) = F^{CE}(S, D)$. ■

Since there are no symmetric problems in $\mathcal{B}^2_{>>} \cup \mathcal{B}^2_{>=}$, any rule is *symmetric* on those classes of problems. Therefore, the properties of Theorem 7 do not pinpoint a single rule on $\mathcal{B}^2_{>>} \cup \mathcal{B}^2_{>=}$. Also, we do not state uniqueness for more than two agents. The following is an example of a rule that satisfies all the above properties and that is different from the Cardinal Egalitarian rule for problems with more than two agents. Let $\xi : \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$ be defined as

$$[\xi(D)]_{ji} = \begin{cases} D_{ji} & \text{if } D_{ji} = \underline{d}_i(D), \\ \min\{D_{ji} \mid D_{ji} \neq \underline{d}_i(D)\} & \text{otherwise.} \end{cases}$$

Then, let $F^{\xi} \in \mathcal{F}_{\neq}$ be defined as $F^{\xi}(S, D) = F^{CE}(S, \xi(D))$ for each $(S, D) \in \mathcal{B}_{\neq}$.

Remark 8 *The properties weak Pareto optimality, strong monotonicity, set-continuity,* *scale invariance, and symmetry are logically independent. For this, note that the rule* F^1 *defined as* $F^1(S,D) = \overline{d}(D)$ *satisfies all properties except weak Pareto optimality. The rule* F^2 *defined as* $F^2(S,D) = (max\{x_1 \mid \; x \in S \text{ and } x_2 = \underline{d}_2(D)\}, \underline{d}_2(D))$ *satisfies all properties except symmetry. The rule* F^3 *defined as* $F^3(S,D) = \arg\max_{x \in S} \min_{i \in N} x_i - \underline{d}_i(D)$ *satisfies all properties except scale invariance. Finally, let* $m_i^* = max\{x_i \mid x \in S \text{ and } x \geqq \underline{d}(D)\}$ *and define* F^4 *as* $F^4(S,D) = \arg\max_{x \in S} \min_{i \in N} \frac{x_i - \overline{d}_i(D)}{m_i^* - \overline{d}_i(D)}$. *This rule satisfies all properties except strong monotonicity. Finally, let* F^5 *coincide with* F^{CE} *everywhere except* $\mathcal{B}^2_{>>}$. *There, let* F^5 *be as explained in Claim 1 of the proof of Proposition 6 where* $G_i = [\mathbf{1}, \mathbf{1} + 1e_i] \cup \{\mathbf{1} + 1e_i + re_j \mid r \in \mathbb{R}_+\}$ *for* $i \neq j$. *Since each* G_i *violates Condition (ii) in the definition of the path generators,* F^5 *violates set-continuity but it satisfies all the other properties above.*

3.4 Conclusion

In this chapter, we analyze bargaining processes where the disagreement outcome depends on who terminates the negotiations. We present a cooperative bargaining model that captures this feature and we carry out an axiomatic analysis of the implications of *strong monotonicity, scale invariance*, and *symmetry* on *weakly Pareto optimal* bargaining rules.

One restriction of our model is that it does not specify the outcome of a coalition of agents jointly terminating the negotiations. Modeling coordinated disagreement by a coalition would bring in questions about the bargaining process in that coalition and move us further towards a non-transferable utility game analysis. In this model, we remain in the bargaining framework and only consider individual deviations.

In our opinion, it is essential to complement our analysis with a noncooperative approach. Studies such as Shaked and Sutton (1984), Ponsatí and Sákovics (1998), and Corominas-Bosch (2000) present an excellent starting point. The equilibria of these models, however, use only partial information on the implications of disagreement. For example, an agent's payoff from his opponent leaving has no effect on the equilibrium (except in extreme cases where the problem's individually rational region is empty).[8] Therefore, the design and analysis of noncooperative bargaining games which, in equilibrium, use full disagreement information remains an important open question.

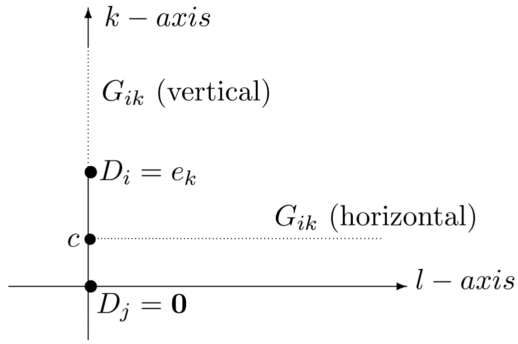

Figure 3.5: The configuration of the monotone paths in Proposition 6.

3.5 Appendix

Proof. (Proposition 6) Let F^p be a *scale invariant* monotone path rule. Note that $\mathcal{B}^2_{\neq} = \mathcal{B}^2_{>>} \cup \mathcal{B}^2_{>=} \cup \mathcal{B}^2_{><}$. Let $G_1 = p\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$, $G_2 = p\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right)$, $G_3 = p\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$, $G_4 = p\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$, $G_{11} = p\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$, $G_{12} = p\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$, $G_{21} = p\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$, and $G_{22} = p\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$.

Claim 1: On $\mathcal{B}^2_{>>}$, G_1 and G_2 suffice to describe F^p. To see this, let $(S, D) \in B^2_{>>}$ and assume $D_i > D_j$. Let $\lambda_1(x) = \frac{x - D_{j1}}{D_{i1} - D_{j1}}$ and $\lambda_2(x) = \frac{x - D_{j2}}{D_{i2} - D_{j2}}$. Note that $\lambda(D_i) = \mathbf{1}$ and $\lambda(D_j) = \mathbf{0}$. Thus $p\left(\lambda\left(D\right)\right) = G_i$. Then, by *scale invariance*, $F^p(S, D) = \lambda^{-1}(F^p(\lambda(S), \lambda(D))) = \lambda^{-1}(WPO(\lambda(S), \lambda(D)) \cap G_i)$ only uses the path G_i.

Claim 2: On $\mathcal{B}^2_{><}$, G_3 and G_4 suffice to describe F^p. The proof is similar to Claim 1.

Claim 3: On $\mathcal{B}^2_{>=}$, G_{11}, G_{12}, G_{21}, and G_{22} suffice to describe F^p. To see this, let $(S, D) \in B^2_{>=}$ and assume $D_{ik} > D_{jk}$ and for $l \neq k$, $D_{il} = D_{jl}$. Let $\lambda_k(x) = \frac{x - D_{jk}}{D_{ik} - D_{jk}}$ and for $l \neq k$, let $\lambda_l(x) = x - D_{il}$. Note that $\lambda(D_i) = e_k$ and $\lambda(D_j) = \mathbf{0}$. Therefore, $\lambda(p(D)) = G_{ik}$. Then by *scale invariance*, $F^p(S, D) = \lambda^{-1}(F^p(\lambda(S), \lambda(D))) = \lambda^{-1}(WPO(\lambda(S), \lambda(D)) \cap G_{ik})$ only uses the path G_{ik}.

By Claims 1,2 and 3, F^p can be completely characterized by at most eight distinct paths (two for $\mathcal{B}^2_{>>}$, two for $\mathcal{B}^2_{><}$, and four for $\mathcal{B}^2_{>=}$).

Claim 4: The paths G_{ik} on $\mathcal{B}^2_{>=}$ are either vertical or horizontal (see Figure 3.5). To see this, let D be as in Figure 3.5 (that is, $D_i = e_k$ and $D_j = \mathbf{0}$). Thus $p\left(D\right) = G_{ik}$. Note that, for each $r \in \mathbb{R}_+$ and $\lambda^r \in \Lambda$ defined as $\lambda^r_k\left(x\right) = x$ and $\lambda^r_l\left(x\right) = rx$, we have $\lambda^r\left(D\right) = D$. Thus $p\left(\lambda^r\left(D\right)\right) = G_{ik}$. Now, let $y \in G_{ik}$ be such that $y_l > 0$. (If there is no such y, G_{ik} is a vertical line on the k-axis and we are done.) Then $\lambda^r\left(y\right) \in \lambda^r\left(p\left(D\right)\right) = p\left(\lambda^r\left(D\right)\right) = G_{ik}$. Since $\lambda^r\left(y\right) \in G_{ik}$ holds for all r, G_{ik} is horizontal. ■

[8] Thus, our conjecture is that these noncooperative games implement *decomposable* rules.

CHAPTER 4
TRADE RULES FOR UNCLEARED MARKETS WITH A VARIABLE POPULATION

4.1 Introduction

We analyze markets in which the price of a traded commodity is fixed at a level where the supply and the demand are possibly unequal. This stickiness of prices is observed in many markets, either because the price adjustment process is slow or because the price is controlled from the outside of the market. There is a wide literature about this phenomenon, for a review, see Bènassy (1993).

The agricultural sector such as the hazelnut market provides a typical example. For political reasons, the markets in this sector are usually regulated and because of these regulations, the demand and the supply may not be equal. In fact, there is usually an excess supply. For example, in hazelnut market, the prices are determined by the government and as a result, there is usually an excess supply. For example, in Turkey, the government sets a maximum amount of production for each farmer and up to that amount, it purchases all the supply. The public health sector provides another example. The prices of public hospitals are determined by a central authority and by law, the hospitals have to attend all the patients even though there is usually an excess demand.

The main question is the following: in such markets, how should a central authority design a mechanism (hereafter, a trade rule) that determines the trade? In this paper, we axiomatically analyze trade rules on the basis of some good properties.[1]

In our model, buyers and sellers constitute two exogenously differentiated sets. There is only one traded commodity and sellers face demand from buyers. Buyers might be individuals or producers that use the commodity as input. We assume that the buyers have strictly convex preferences on consumption bundles. Thus, they have single-peaked preferences on the boundary of their budget sets, and therefore, on their consumption of the commodity. Similarly, we assume that the sellers have strictly convex production sets. Thus, their profits are single-peaked in their output.

A *trade rule* maps each economy to a feasible trade. In our model, it is made up of two components: a *trade-volume rule* and an *allocation rule*. The *trade-volume rule*

[1] Bènassy (2002) discusses some properties that a trade rule should satisfy such as *Pareto optimality, voluntary trade,* and *strategy proofness.* He also mentions the possibility of designing a good mechanism that determines the trade. However, he does not study it. He rather uses a trade rule that clears the short side of the market and uniformly rations the long side of it.

determines the trade-volume that will be carried out in the economy and thus, the total consumption and the total production. Then, the *allocation rule* allocates the total consumption among the buyers and the total production among the sellers.

Trade-volume rules are related to Moulin (1980) who analyzes the determination of a one-dimensional policy issue among agents with single-peaked preferences. Particularly, when there is only one buyer and one seller, the trade-volume is exactly like a public good for these two agents. However, this is no more true when there are multiple buyers and sellers.

The problem faced by the *allocation rule* is to allocate a social endowment (that is, the trade volume) among agents with single peaked preferences. This problem is extensively analyzed by Sprumont (1991) who proposed and analyzed a "Uniform rule" which later became a central rule of that literature (for example, see Dagan (1996), Ching (1992, 1994), Thomson (1994)). Since we analyze markets with multiple buyers and sellers, our domain is an extension of Sprumont's domain.[2] Thus, Sprumont's uniform rule will also play an important role in this paper.

Let us note that our model is not a simple conjunction of Moulin (1980) and Sprumont (1991). The interaction between the determination of the agent's shares and the trade-volume makes the model much richer. For example, the agents can manipulate their shares also by manipulating the trade-volume. Also, requirements like *Pareto optimality*, or "fairness" become much more demanding as what is to be allocated becomes endogenous. Another important difference is the existence of two types of agents (buyers and sellers) in our model. This duality limits the implications of requirements like *anonymity, no-envy*, and *population monotonicity*.

Our model is also related to Thomson (1995) and Klaus, Peters, and Storcken (1997, 1998). They analyze the reallocation of an infinitely divisible commodity among agents with single peaked preferences and individual endowments. The agents whose endowments are greater than their peaks are considered as suppliers and those whose endowments are less than their peaks as demanders. These authors also characterize a *Uniform reallocation rule*. Note that, in their model the suppliers and the demanders are not differentiated. The identities of the agents depend on the relation between their peaks and endowments. Thus, a supplier by misrepresenting his preferences can turn into a demander. In our model, however, producers and consumers are exogenously distinct identities. This difference has important implications over the properties analyzed. For example, fairness properties are much weaker in our model since they only

[2] It coincides with the *just-buyer markets* (when there is no seller in the market) and the *just-seller markets* (when there is no buyer in the market) in our model.

compare agents on the same side of the market. Also, in our model the agents do not have exogenously given endowments.

The following papers study the design of a mechanism that determines the trade in nonclearing markets. Barbera and Jackson (1995) analyze a pure exchange economy with a arbitrary number of agents and commodities. Each agent has a positive endowment of the commodities and a continuous, strictly convex, and monotonic preference relation on his consumption. The authors look for *strategy proof* rules that facilitate trade in this exchange economy.

Our model is closely related to Kıbrıs and Küçükşenel (2009) and Bochet, İlkılıç, and Moulin (2009). Kıbrıs and Küçükşenel (2009) analyze a class of *trade rules* each of which is a composition of the Uniform rule with a *trade-volume rule* that picks the median of total demand, total supply and an exogenous constant. They show that this class uniquely satisfies *Pareto optimality, strategy proofness, no-envy,* and an informational simplicity axiom called *independence of trade-volume.* Bochet, İlkılıç, and Moulin (2009) introduces a graph structure to this setting and they assume that a trade between a buyer and a seller is possible only if there is a link between them. They characterize *the egalitarian transfer* mechanism by the combination of *Pareto optimality, strategy proofness, voluntary trade*, and *equal treatment of equals.*

In all these papers, the authors analyze markets with a fixed population. In this thesis, we allow the population to be variable and analyze the implications of these population changes. We introduce a class of *Uniform trade rules* each of which is a composition of the *Uniform rule* and a *trade-volume rule.* We axiomatically analyze *Uniform trade rules* on the basis of some central properties concerning variations of the population, namely, *consistency* and *population monotonicity.* We also analyze the implications of standard properties such as *Pareto optimality, strategy-proofness*, and *no-envy*, and an informational simplicity property, *strong independence of trade volume.*

Consistency has been analyzed in many contexts such as bargaining, coalitional form games, and taxation (for a detailed discussion, see Section 4.2). Loosely speaking, a rule is *consistent* if a recommendation it makes for an economy always agrees with its recommendations for the associated reduced economies obtained by the departure of some of the agents with their promised shares. *Consistency*, however, is not well-defined for closed economies. Therefore, we analyze a specific type of an open economy by allowing possible transfers to/from outside the economy (for a detailed discussion, see Section 4.2). We show in Theorem 1 that a particular subclass of *Uniform trade rules* uniquely satisfies *consistency* together with *Pareto optimality, no-envy*, and *strong independence of trade volume.* Next, we add *strong independence of trade volume* to the list and characterize a smaller subclass that satisfies those properties. We note that each member of this subclass either clears the short side or the long side of any given market.

Population monotonicity has also been widely analyzed in many different contexts such as in classical economies, single-peaked preferences, and public goods (for a detailed discussion, see Section 4.2). Loosely speaking, it requires that for a given economy, upon the departure (equivalently, arrival) of some agents, the welfare level of the remaining agents should be affected in the same direction. Since in our model, the agents on different sides of the market are exogenously differentiated, we analyze a *population monotonicity* property which only compares agents on the same side of the market. We first note that there are trade rules that simultaneously satisfy three properties, which are incompatible on Sprumont's domain: *Pareto optimality, no-envy*, and *population monotonicity*. In Theorem 2, we characterize the subclass that additionally satisfies *strategy-proofness*. In Theorem 3, we also add *strong independence of trade volume* to the list. We note that these subclasses contain rules that do not always clear the short side[3] of the market. In fact, when we impose the *strong independence of trade volume* in Theorem 3, we find that if we only consider the markets with a transfer, then the only trade rule satisfying the desired properties is the one that always clears the long side of the market and rations the other side by Uniform rule.

This chapter is organized as follows. In Section 4.2, we introduce the model. In Section 4.3.1, we analyze the implications of *consistency* and in Section 4.3.2, the implications of *population monotonicity*. In Section 4.4, we conclude.

4.2 Model

There are countably infinite universal sets, $\mathcal{B}$ of potential buyers and $\mathcal{S}$ of potential sellers. Let $\mathcal{B} \cap \mathcal{S} = \emptyset$. There is a perfectly divisible commodity that each seller produces and each buyer consumes. Let $\mathbb{R}_{++}$ be the consumption/ production space for each agent. Let R be a preference relation over $\mathbb{R}_{++}$ and P be the strict preference relation associated with R. The preference relation R is **single-peaked** if there is $p(R) \in \mathbb{R}_{++}$ called the peak of R, such that for all $x, y \in \mathbb{R}_{++}$, $x < y \leq p(R)$ or $x > y \geq p(R)$ implies $y \; P \; x$. Each $i \in \mathcal{B} \cup \mathcal{S}$ is endowed with a continuous single-peaked preference relation R_i over $\mathbb{R}_{++}$. Let $\mathcal{R}$ denote the set of all continuous and single-peaked preference relations on $\mathbb{R}_{++}$.

[3]The *short side* of a market is where the total volume of desired transaction is smallest. It is thus the demand side if there is excess supply and the supply side if there is excess demand. The other side is called the *long side*.

Given a finite set $B \subset \mathcal{B}$ of buyers and a finite set $S \subset \mathcal{S}$ of sellers such that either $B \neq \emptyset$ or $S \neq \emptyset$, let $N = B \cup S$ be a **society**. Let $\mathcal{N}$ be the set of all societies. Let $\mathcal{N}_{\neq \emptyset}$ be the set of societies with a nonempty set of buyers and sellers. A preference profile R_N for a society N is a list $(R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Let $\mathcal{R}^N$ denote the set of all profiles for the society N. Given $N' \subset N$ and $R_N \in \mathcal{R}^N$, let $R_{N'} = (R_i)_{i \in N'}$ denote the restriction of R_N to N'.

A **market for society** $\boldsymbol{N} = B \cup S$ is a list $\boldsymbol{(R_B, R_S, T)}$ where $(R_B, R_S) \in \mathcal{R}^N$ is a profile of preferences for buyers and sellers and $T \in \mathbb{R}$ is a **transfer**. Note that T can both be positive and negative. A positive T represents a transfer made from outside. Thus, it is added to the production of the sellers and together they form the total supply. On the other hand, a negative T represents a transfer that must be made from the economy to the outside. Thus, it is considered as an addition to the total demand.

Given a market (R_B, R_S, T) for a society $N = (B \cup S)$, a **(feasible) trade** is a vector $z \in \mathbb{R}^{B \cup S}_{++}$ such that $\sum_B z_b = \sum_S z_s + T$. Let $Z(R_B, R_S, T)$ denote the set of all trades for (R_B, R_S, T).

There are two special subclasses of markets. A market (R_B, R_S, T) is a **just-buyer market** if $B \neq \emptyset$ and $S = \emptyset$. For such markets, the feasible trades are as follows. If $T \geq 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}^B_{++} : \sum_B z_b = T\}$. If $T < 0$, then $Z(R_B, R_S, T) = \emptyset$. (This is trivial because if there is no seller, all the agents are demanders, and thus, the supply is zero. Thus, if the outside transfer is positive, it would be equal to the total supply and it is divided among the buyers. However, if there is a negative transfer (that is, a transfer must be made to outside), since there is no seller, the transfer cannot be realized. Thus, in that case there is no trade.) A market (R_B, R_S, T) is a **just-seller market** if $B = \emptyset$ and $S \neq \emptyset$. For such markets, the feasible trades are as follows. If $T \leq 0$, $Z(R_B, R_S, T) = \{z \in \mathbb{R}^S_{++} : \sum_S z_s + T = 0\}$. If $T > 0$, then $Z(R_B, R_S, T) = \emptyset$. (The explanation is similar to above.) Note that *just-buyer markets* and *just-seller markets* mathematically coincide with the allocation problems analyzed by Sprumont (1991). Thus, his domain is a restriction of ours.

Since the markets with no feasible trade are trivial, we restrict ourselves to the set of markets for which the set of trades is nonempty. Let $\mathcal{M}^N = \{(R_B, R_S, T) : (R_B, R_S) \in \mathcal{R}^N, T \in \mathbb{R},$ and $Z(R_B, R_S, T) \neq \emptyset\}$ be the set all markets for society $N = B \cup S$ and let

$$\mathcal{M} = \bigcup_{N \in \mathcal{N}} \mathcal{M}^N$$

be the set of all markets. Let $\mathcal{M}_{\mathcal{B}} = \{(R_B, R_S, T) \in \mathcal{M} :\ B \neq \emptyset,\ S = \emptyset, \text{ and } T \geq 0\}$ be the set of *just-buyer markets* and $\mathcal{M}_{\mathcal{S}} = \{(R_B, R_S, T) \in \mathcal{M} :\ B = \emptyset,\ S \neq \emptyset, \text{ and } T \leq 0\}$ be the set of *just-seller markets*.

For the analysis of the properties, the following subclasses of markets turn out to be important. Let $\mathcal{M}^2_{\neq} = \{(R_B, R_S, T) \in \mathcal{M} :$ there are $b, b' \in B$, and $s, s' \in S$ such that $p(R_b) \neq p(R_{b'})$ and $p(R_s) \neq p(R_{s'})\}$ be the set of markets in which there are at least two buyers and two sellers with different peaks, respectively. Also, let $\mathcal{M}_{nt} = \{(R_B, R_S, T) \in \mathcal{M} :\ T = 0\}$ be the set markets with no outside transfer. For notational simplicity, we will denote each $(R_B, R_S, T) \in \mathcal{M}_{nt}$ as (R_B, R_S).

Let $h(R_B, R_S, T)$ denote the **short side of the market $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$**, that is,

$$h(R_B, R_S, T) = \begin{cases} B & \text{if } \sum_B p(R_b) < \sum_S p(R_s) + T, \\ S & \text{if } \sum_S p(R_s) + T < \sum_B p(R_b). \end{cases}$$

Similarly, let $l(R_B, R_S, T)$ denote the **long side of the market $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$**, that is,

$$l(R_B, R_S, T) = \begin{cases} S & \text{if } \sum_B p(R_b) < \sum_S p(R_s) + T, \\ B & \text{if } \sum_S p(R_s) + T < \sum_B p(R_b). \end{cases}$$

A trade $z \in Z(R_B, R_S, T)$ is ***Pareto optimal with respect to $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$*** if there is no $z' \in Z(R_B, R_S, T)$ such that for all $i \in B \cup S$, $z'_i R_i z_i$ and for some $j \in B \cup S$, $z'_j P_j z_j$. The following lemma shows that in our framework, *Pareto optimality* is equivalent to the following three properties: (i) each agent in the short side of the market receives a share greater than or equal to his peak, (ii) each agent in the long side of the market receives a share less than or equal to his peak, and (iii) the total consumption is between the total supply and the total demand. Its proof is simple (see Kıbrıs and Küçükşenel (2009)).

Lemma 9 *For each $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$, a trade $z \in Z(R_B, R_S, T)$ is Pareto optimal with respect to (R_B, R_S, T) if and only if for $K \in \{B, S\}$, $h(R_B, R_S, T) = K$ implies **(i)** for each $k \in K$, $p(R_k) \leq z_k$, **(ii)** for each $l \in N \setminus K$, $z_l \leq p(R_l)$, and **(iii)** $\sum_B z_b \in [\sum_B p(R_b), \sum_S p(R_s) + T]$.*[4]

A trade rule first determines the volume of trade that will be carried out in the economy and therefore, the total production and the total consumption. Then, it

[4] By $\sum_B z_b \in [\sum_B p(R_b), \sum_S p(R_s) + T]$, we mean the total consumption is between the total supply and the total demand, that is if $h(R_B, R_S, T) = S$, then consider $[\sum_S p(R_S) + T, \sum_B p(R_b)]$. In the rest of the paper, for simplicity we will sometimes use an interval notation in a similar meaning.

allocates the total production among the sellers and the total consumption among the buyers. Before defining a trade rule, we will first define a *trade-volume rule*.

A **trade-volume rule** $\Omega : \mathcal{M} \rightarrow \mathbb{R}^2_{++}$ associates each market (R_B, R_S, T) with a vector $\Omega(R_B, R_S, T) = (\Omega_B(R_B, R_S, T), \Omega_S(R_B, R_S, T))$ whose first coordinate, $\Omega_B(R_B, R_S, T)$ is the total consumption of the buyers and the second coordinate, $\Omega_S(R_B, R_S, T)$ is the total production of the sellers. Note that, for each market (R_B, R_S, T) and a *trade-volume rule* Ω, $\Omega_B(R_B, R_S, T) = \Omega_S(R_B, R_S, T) + T$. Thus, the volume of Ω_B determines the volume of Ω_S. Therefore, with an abuse of notation, we will sometimes call Ω_B a *trade-volume rule*.

In a *just-buyer market,* the transfer is divided among the buyers. Thus, the total consumption is equal to the transfer. In a *just-seller market*, however, the sellers produce an amount that corresponds to the transfer. Thus, in that case, the total production is equal to the absolute value of the transfer. Therefore, each *trade-volume rule* Ω satisfies the following:

$$\Omega(R_B, R_S, T) = \begin{cases} (T, 0) & \text{if } (R_B, R_S, T) \in \mathcal{M}_{\mathcal{B}} \\ (0, -T) & \text{if } (R_B, R_S, T) \in \mathcal{M}_{\mathcal{S}} \\ (\Omega_B(R_B, R_S, T), \Omega_S(R_B, R_S, T)) & \text{otherwise} \end{cases}$$

Note that, the trade-volume is fixed for the *just-buyer* and the *just-seller markets*. Thus, for simplicity, we will define a *trade-volume rule* only by the volume it chooses for the other markets.

Let $\mathcal{V}$ be the set of all *trade-volume rules*. Let $\mathcal{V}^{[short,long]}$ be the set of *trade-volume rules*, Ω each of which chooses a trade-volume between the total demand and supply of the market, that is, for each market (R_B, R_S, T),

$$\Omega(R_B, R_S, T) \in [\sum_B p(R_b), \sum_S p(R_s) + T].$$

The following subclass of $\mathcal{V}^{[short,long]}$ will be used extensively in rest of the paper. Let $\mathcal{V}^{\{short,long\}}$ be the set of *trade-volume rules*, Ω each of which alternates between picking the total demand/supply of the short and the long side of the market, that is, for each market (R_B, R_S, T),

$$\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}.$$

Particularly, the following two members of $\mathcal{V}^{\{short,long\}}$ will be important in our analysis. The first one, Ω^{long} always chooses the desired trade level of the long side, that is, for each $(R_B, R_S, T) \in \mathcal{M}$,

$$\Omega^{long}(R_B, R_S, T) = \begin{cases} \sum_S p(R_s) + T & \text{if } h(R_B, R_S, T) = B \\ \sum_B p(R_b) & \text{if } h(R_B, R_S, T) = S \end{cases}$$

The second one, Ω^{short} always chooses the desired trade level of the short side. However, for the markets, (R_B, R_S, T) such that $h(R_B, R_S, T) = B$ and $\sum_B p(R_b) \leq T$, we have that $\Omega_S^{short} \leq 0$. Similarly, for the markets, (R_B, R_S, T) such that $h(R_B, R_S, T) = S$ and $\sum_S p(R_s) + T \leq 0$, we have that $\Omega_B^{short} \leq 0$. Thus, Ω^{short} is only defined for the markets in which $\sum_B p(R_b) > T$ and $\sum_S p(R_s) + T > 0$. Formally, for each $(R_B, R_S, T) \in \mathcal{M}$ such that $\sum_B p(R_b) > T$ and $\sum_S p(R_s) + T > 0$,

$$\Omega^{short}(R_B, R_S, T) = \begin{cases} \sum_B p(R_b) & \text{if } h(R_B, R_S, T) = B \\ \sum_S p(R_s) + T & \text{if } h(R_B, R_S, T) = S \end{cases}$$

For a given market $(R_B, R_S, T) \in \mathcal{M}$ and $K \in \{B, S\}$, we say that a trade-volume rule **$\boldsymbol{\Omega}$ favors $\boldsymbol{K}$ in $\boldsymbol{(R_B, R_S, T)}$** if

$$\Omega(R_B, R_S, T) = \begin{cases} \sum_B p(R_b) & \text{if } K = B \\ \sum_S p(R_s) + T & \text{if } K = S \end{cases}$$

An **allocation rule** $f : \mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}} \to \cup_{M \in \mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}}} Z(M)$ associates each *just-buyer* and *just-seller* market (R_K, T) for $K \in \{B, S\}$, with a trade $z \in Z(R_K, T)$. For example, Uniform rule, U, introduced by Sprumont (1991) is very central in the literature. In our paper, also, it will be used extensively. Formally, it is defined as follows: for each $K \in \{B, S\}$, $(R_K, T) \in \mathcal{M}_{\mathcal{K}}$, and $k \in K$,

$$U_k(R_K, T) = \begin{cases} \min\{p(R_k), \lambda\} & \text{if } \sum_K p(R_k) \geq T \\ \max\{p(R_k), \mu\} & \text{if } \sum_K p(R_k) \leq T \end{cases}$$

where λ and μ are uniquely determined by the equations, $\sum_K \min\{p(R_k), \lambda\} = T$ and $\sum_K \max\{p(R_k), \mu\} = T$.

A **trade rule** $F : \mathcal{M} \to \cup_{M \in \mathcal{M}} Z(M)$ is a composition of a *trade-volume rule* Ω and an allocation rule f: $F = f \circ \Omega$. More precisely, for each market (R_B, R_S, T) and $K \in \{B, S\}$, $F_K(R_B, R_S, T) = f(R_K, \Omega_K(R_B, R_S, T))$. A *trade rule*, $F = U \circ \Omega$, that is composed of the *Uniform rule* and a *trade-volume rule* Ω is called the **uniform trade**

rule with respect to $\boldsymbol{\Omega}$. In our analysis, $U \circ \Omega$ for some $\Omega \in \mathcal{V}^{\{short,long\}}$ turns out to be central. Kıbrıs and Küçükşenel (2009) characterize a particular class of *Uniform trade rules* for which Ω is the median of total demand, total supply, and an exogenous constant.

Let $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$ and F be a trade rule. Let $US^F(R_B, R_S, T) = \{i \in B \cup S : \; F_i(R_B, R_S, T) \neq p(R_i)\}$ be the **unsatisfied agents in $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$ with respect to $\boldsymbol{F}$**. Note that, if $F = f \circ \Omega^{long}$, then for each market $(R_B, R_S, T) \in \mathcal{M}$, $US^F(R_B, R_S, T) = h(R_B, R_S, T)$. Otherwise, however, there is a market $(R_B, R_S, T) \in \mathcal{M}$ such that $US^F(R_B, R_S, T) \cap B \neq \emptyset$, $US^F(R_B, R_S, T) \cap S \neq \emptyset$.

Now, we introduce properties of a trade rule. We start with efficiency. A trade rule F is **Pareto optimal** if for each $(R_B, R_S, T) \in \mathcal{M}$, the trade $F(R_B, R_S, T)$ is *Pareto optimal with respect to* (R_B, R_S, T).

Now, we present a fairness property. A trade is *envy free* if each buyer (respectively, seller) prefers his own consumption (respectively, production) to that of every other buyer (respectively, seller). A trade rule satisfies **no-envy** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $K \in \{B, S\}$, and $i, j \in K$, $F_i(R_B, R_S, T) R_i F_j(R_B, R_S, T)$. Since in our model the agents on different sides of the market are exogenously differentiated, this property only compares agents on the same side of the market.

The following is a property on nonmanipulability. It requires that regardless of the others' preferences, an agent is best-off with the trade associated with his true preferences. Formally, a trade rule F is **strategy proof** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $i \in N$, and $R'_i \in \mathcal{R}$, $F_i(R_i, R_{N\setminus i}, T) \; R_i \; F_i(R'_i, R_{N\setminus i}, T)$.

Next, we present some properties concerning possible variations in the number of agents. The first one is an adaptation of the standard *consistency* property to our domain. This property has been analyzed extensively in the context of bargaining by Lensberg (1987), single-peaked preferences by Thomson (1994), coalitional form games by Peleg (1986) and Hart and Mas-Colell (1989), taxation by Aumann and Maschler (1985) and Young (1987), cost allocation by Moulin (1985), fair allocation in classical economics by Thomson (1988), and matching by Sasaki and Toda (1992). To explain *consistency*, consider a trade rule F and a market (R_B, R_S, T). Suppose that F chooses the trade z. Imagine that some buyers and sellers leave with their shares they have been already assigned. This leads to a reduced market that the remaining agents are now facing. Consistency is about how the remaining agents' shares should be affected in this reduced market. If F is consistent, it should assign to them the same shares as in the initial market. However, without a transfer from outside, the recommendation for an economy may not be feasible for its reduced markets. This is one reason we consider open economies. This practice is similar to the analysis of *consistency* in economies with individual endowments. For example, Thomson (1992) introduced a "generalized economy" that consists of a preference profile of the agents, an endowment profile, and a trade vector that is updated in the reduced economies. The trade vector in that model corresponds in our model to the transfer. This leads to a reduced problem in which the remaining agents, $(B' \cup S')$ are now facing an updated transfer from T to $T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s$. Formally, given a trade rule F, for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, a **reduced market of $\boldsymbol{(R_B, R_S, T)}$ for $\boldsymbol{N'}$ at $\boldsymbol{z \equiv F(R_B, R_S, T)}$** is $r^z_{N'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T - \sum_{B \setminus B'} z_b + \sum_{S \setminus S'} z_s)$. A trade rule F is **consistent** if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $N' = (B' \cup S') \subseteq N$, if $z = F(R_B, R_S, T)$, then $z_{N'} = F(r^z_{N'}(R_B, R_S, T))$.

Consistency can also be defined for *trade-volume rules* in a similar way. A *trade-volume rule* Ω is *consistent* if for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N' = (B' \cup S') \subseteq N$, and $z \in Z(R_B, R_S, T)$, $\Omega(r^z_{N'}(R_B, R_S, T)) = \sum_{B'} z_{b'}$.

The second property concerning the population changes is *population monotonicity.* It has been extensively analyzed in classical economies by Chichilnisky and Thomson (1987), Thomson (1987), Chun and Thomson (1988), Moulin (1992), and Chun (1986), on domains of economies with indivisible goods by Alkan (1989), Tadenuma and Thomson (1990, 1993), Moulin (1990), Bevia (1992), and Fleurbaey (1993), on domains of economies with both private and public goods by Thomson (1987), Moulin (1990), in single-peaked preferences by Thomson (1995), and Klaus (2001). *Population monotonicity* requires that upon the arrival (equivalently, departure) of some agents, the welfare levels of all remaining buyers and sellers should be affected in the same direction. Since in our model agents on different sides of the market are exogenously differentiated, *population monotonicity* only compares agents on the same side of the market. Formally, a trade rule F is **population monotonic** if for each $(B \cup S) \in \mathcal{N}_{\neq \emptyset}$, $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$, $(B' \cup S') \supseteq (B \cup S)$, $K \in \{B', S'\}$, either (i) for each $i \in K$, $F_i(R_{B'}, R_{S'}, T)\ R_i\ F_i(R_B, R_S, T)$, or (ii) for each $i \in K$, $F_i(R_B, R_S, T)\ R_i\ F_i(R_{B'}, R_{S'}, T)$.[5]

There are two types of population expansions. For a given market, by the arrival of some agents, the short side of the market may remain the same. We call them *simple population expansions.* Alternatively, the arrival of sufficiently many sellers (buyers) may turn an economy in which the short side is the buyers (sellers) into one in which the short side is the sellers (buyers). We call them *radical population expansions.* Formally,

[5] *Population monotonicity* only analyzes societies with a nonempty set of buyers and sellers. If we include *just-buyer markets* and *just-seller markets*, by Thomson (1995) there is no trade rule that satisfies *Pareto optimality, no-envy*, and *population monotonicity.*

let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$. Then, the set of **simple population expansions of $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$** is defined as $r^{sim}(R_B, R_S, T) = \{(B' \cup S') \supseteq (B \cup S) : h(R_B, R_S, T) = K \text{ and } h(R_{B'}, R_{S'}, T) = K' \text{ for } K \in \{B, S\}\}$ and the set of **radical population expansions of $(\boldsymbol{R_B}, \boldsymbol{R_S}, \boldsymbol{T})$** is defined as $r^{rad}(R_B, R_S, T) = \{(B' \cup S') \supseteq (B \cup S) : h(R_B, R_S, T) = K \text{ and } h(R_{B'}, R_{S'}, T) = L' \text{ for } K \in \{B, S\} \text{ and } L = N \setminus K\}$. *Population monotonicity* allows both type of expansions.

Lastly, we present the following informational simplicity property. *Strong independence of trade-volume* requires the *trade-volume rule* only to depend on the total demand and supply but not on their individual components and the agents' identities. This property is a stronger version of *independence of trade volume* introduced by Kıbrıs and Küçükşenel (2009). In contrast to *strong independence of trade-volume*, *independence of trade volume* relates two problems with the same set of agents. Formally, Ω satisfies **strong independence of trade volume** if for each $N = (B \cup S) \in \mathcal{N}$, $N' = (B' \cup S') \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $(R'_{B'}, R'_{S'}, T) \in \mathcal{M}^{N'}$, $\sum_{b \in B} p(R_b) = \sum_{b' \in B'} p(R'_{b'})$ and $\sum_{s \in S} p(R_s) = \sum_{s' \in S'} p(R'_{s'})$ imply $\Omega(R_B, R_S, T) = \Omega(R'_{B'}, R'_{S'}, T)$.

4.3 Results

4.3.1 Consistency

The following theorem shows that the subclass of *Uniform trade rules* $F = U \circ \Omega$ where $\Omega \in \mathcal{V}^{[short,long]}$ is *consistent* uniquely satisfies *Pareto optimality, no-envy,* and *consistency.*

Theorem 1 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy,* and *consistency* if and only if $f = U$ and Ω satisfies the following:

(1.1) $\Omega \in \mathcal{V}^{[short,long]}$,

(1.2) Ω is *consistent.*

Next, we add *strong independence of trade volume* to the list and we show in Theorem 2 that under the assumption of *strong independence of trade volume*, the subclass of *Uniform trade rules* $F = U \circ \Omega$ where $\Omega \in \mathcal{V}^{\{short,long\}}$ is *consistent* uniquely satisfies *Pareto optimality, no-envy* and *consistency.*

Theorem 2 Let $\Omega \in \mathcal{V}$ satisfy *strong independence of trade volume.* A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy,* and *consistency* if and only if $f = U$ and Ω satisfies the following:

(1.1) $\Omega \in \mathcal{V}^{\{short,long\}}$,

(1.2) Ω is *consistent.*

4.3.2 Population Monotonicity Our first observation is that there are *trade rules* that simultaneously satisfy three properties which, on Sprumont's domain, are incompatible: *Pareto optimality, no-envy,* and *population_monotonicity* (see Thomson (1995) for a discussion).

Lemma 10 *The Uniform trade rule, $U \circ \Omega^{long}$ satisfies Pareto optimality, no-envy, and population monotonicity.*

Next, we characterize trade rules that satisfy *population monotonicity* together with *Pareto optimality, no-envy*, and *strategy proofness.* We show that each of these rules is a Uniform trade rule with respect to a trade volume rule, Ω that satisfies the following three properties:

Property (*i*): Formally, $\Omega \in \mathcal{V}^{\{short,long\}}$ on $\mathcal{M}^2_{\neq}$ and $\Omega \in \mathcal{V}^{[short,long]}$ on $\mathcal{M} \setminus \mathcal{M}^2_{\neq}$. This property requires that for each market, Ω chooses a trade volume that is between the total supply and the total demand. If, in addition, there are at least two buyers and two sellers with different peaks, Ω either chooses the desired trade level of the short side or the long side.

Property (*ii*): Formally, for each $(R_B, R_S, T) \in \mathcal{M} \setminus \{\mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}}\}$ such that there are $K \in \{B, S\}$, $i \in K \cap US^F(R_B, R_S, T)$, and $j \in K$ with $p(R_i) \neq p(R_j)$, and for each $(B' \cup S') \in r^{rad}(R_B, R_S, T)$, we have that Ω *favors* K in $(R_{B'}, R_{S'}, T)$.[6] For this property, consider a market in which i is an unsatisfied agent and j is another agent on the same side as i but with a different peak. Now, consider a radical population expansion of this market. *Property* (*ii*) requires that then Ω should favor the side that i belongs to.

Property (*iii*): Formally, for each $(R_B, R_S, T) \in \mathcal{M}$, $i \in US^F(R_B, R_S, T)$, and $R'_i \in \mathcal{R} \setminus \{R_i\}$, we have the following two conditions:

(i) if $i \in h(R_B, R_S, T)$ and $p(R'_i) < z_i$, then $\Omega(R'_i, R_{N\setminus\{i\}}, T) \in [\Omega(R_N, T), \sum_{j \in l(R_B,R_S,T)} p(R_j)]$,

(ii) if $i \in l(R_B, R_S, T)$ and $p(R'_i) > z_i$, then $\Omega(R'_i, R_{N\setminus\{i\}}, T) \in [\sum_{j \in h(R_B,R_S,T)} p(R_j), \Omega(R_N, T)]$.

For this property, consider a market in which i is an unsatisfied agent that belongs to the short (respectively, long) side of it. Suppose that i changes his preference relation in a way that the new peak is less (respectively, more) than the initial share, $F_i(R_B, R_S, T)$. Then, *Property* (*iii*) requires that the volume of trade moves towards the desired trade

[6]Note that, by the population expansion K becomes K' in the new market. By the abuse of notation, however, we use K.

level of the long (respectively, short) side.

Theorem 3 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy-proofness*, and *population monotonicity* if and only if $f = U$ and Ω satisfies:

(3.1) *Property* (*i*),

(3.2) *Property* (*ii*),

(3.3) *Property* (*iii*).

In the following theorem we add *strong independence of trade volume* to the list and analyze the implications of it with *population monotonicity* and the other properties. We characterize the following class: on $\mathcal{M} \setminus \mathcal{M}_{nt}$, Ω is Ω^{long} whereas on $\mathcal{M}_{nt}$, it is either Ω^{short} or Ω^{long}.

Theorem 4 Let $\Omega \in \mathcal{V}$ satisfy *strong independence of trade volume.* Then, a trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy-proofness*, and *population monotonicity* if and only if $f = U$ and Ω satisfies the following:

(4.1) on $\mathcal{M} \setminus \mathcal{M}_{nt}$, $\Omega = \Omega^{long}$.

(4.2) on $\mathcal{M}_{nt}$, $\Omega = \Omega^{short}$ or $\Omega = \Omega^{long}$.

4.4 Conclusion

In this section, we discuss some of our results and we list some open questions.

We show in Theorem 1 that the Uniform trade rules with *consistent* trade volume rules, Ω that always choose a trade level between the total demand and supply of the market uniquely satisfy *Pareto optimality, no envy*, and *consistency.* However, once we also impose *strong independence of trade volume,* the associated trade volumes, Ω cannot choose any in-between trade level: the trade level must be either the desired trade level of the short side or the long side, that is $\Omega \in \mathcal{V}^{\{short,long\}}$ (Theorem 2).

In addition, we show that on $\mathcal{M} \setminus \mathcal{M}_{nt}$ as well as on $\mathcal{M}_{nt}$, there are infinitely many rules satisfying *Pareto optimality, no envy, strategy proofness*, and *population monotonicity.* In Theorem 2, we characterize them. Once we impose *strong independence of trade volume* to the list, we show in Theorem 3 that on $\mathcal{M} \setminus \mathcal{M}_{nt}$, there is only one rule, $U \circ \Omega^{long}$ and on $\mathcal{M}_{nt}$, only two rules, $U \circ \Omega^{short}$ and $U \circ \Omega^{long}$ that satisfy the desired properties. Therefore, although *strong independence of trade volume* gives informational simplicity, at the same time it limits the number of rules very much. Thus, it is a very strong property.

It is useful to note that replacing *strategy proofness* with *peak only* in Theorem 3 has similar implications. *Peak only* is an informational simplicity property and it requires the trade only to depend on agents' peaks but not on the whole preference relation. Replacing *strategy proofness* with *peak only* changes Theorem 3 as follows (The proof is similar to that of Theorem 3. Because of the space limitation, we omit it. However, it is available upon request.):

Theorem 5 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, peak only*, and *population monotonicity* if and only if $f = U$ and Ω satisfies:

(5.1) Property (i) of Theorem 3.

(5.2) Property (ii) of Theorem 3.

It is also useful to analyze the implications of a weaker version of *population monotonicity*, namely *weak population monotonicity*. This weaker property analyzes only *simple population expansions*. Replacing *weak population monotonicity* with *population monotonicity* changes Theorem 3 as follows:

Theorem 6 A trade rule $F = f \circ \Omega$ satisfies *Pareto optimality, no-envy, strategy proofness*, and *weak population monotonicity* if and only if $f = U$ and Ω satisfies:

(6.1) $\Omega \in \mathcal{V}^{[short,long]}$,

(6.2) Property (iii) of Theorem 3.

In this paper, we analyze markets in which there is only one traded commodity. We do this by picking a market in disequilibrium, isolating it from other related markets, and then producing a trade for it. Our properties focus on a trade rule at that particular market and not on its implications on related markets. Therefore, we do not analyze the implications of a trade rule on the overall economy. This analysis is an important follow up to our work.

4.5 Appendix

To prove Theorem 1, we use the following two lemmas. The first one analyzes the relationship between the properties satisfied by a trade rule $F = f \circ \Omega$, and its component f. It shows that *Pareto optimality, no-envy*, and *consistency* satisfied by F passes on to f.

Lemma 11 *If a trade rule $F = f \circ \Omega$ satisfies one of the following properties, then f also satisfies that property: Pareto optimality, no-envy, and consistency.*

Proof. First, suppose for a contradiction $F = f \circ \Omega$ satisfies *Pareto optimality* whereas f does not. Then, there is $K \in \{B, S\}$ and $(R_K, T) \in \mathcal{M}_K$ such that $f(R_K, T)$ is not *Pareto optimal* with respect to (R_K, T). Then, since $(R_K, T) \in \mathcal{M}$ and $F(R_K, T) = f(R_K, T)$, $F(R_K, T)$ is not *Pareto optimal* with respect to (R_K, T), a contradiction to F being *Pareto optimal*. The other properties can be proved similarly. ■

The second lemma is by Dagan (1996) on the allocation rule f. For its proof, see Dagan (1996).

Lemma 12 *(Dagan, 1996) If the potential number of agents is at least* 4 *and if an economy consists of at least* 2 *agents, then* f *satisfies Pareto optimality, no-envy, and bilateral-consistency if and only if* $f = U$.

Proof. (Theorem 1) The if part is straightforward and thus, omitted. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *consistency*, by Lemma 11, f also satisfies those properties. Then, by Lemma 12, $f = U$.

Now, let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$ and $(B' \cup S') \in \mathcal{N}$ be such that $(B' \cup S') \subseteq (B \cup S)$. Let $z \equiv F(R_B, R_S, T)$ and $z'^{z}_{B' \cup S'}(R_B, R_S, T))$. Since F is *consistent*, for each $i \in (B' \cup S')$, $z'_i = z_i$. Then, by the definition of Ω, $\Omega(r^z_{B' \cup S'}(R_B, R_S, T)) = \sum_{B'} z'_{b'} = \sum_{B'} z_{b'}$. Thus, Ω is *consistent*. ■

To prove Theorem 2, in addition to lemmas 11 and 12, we also use the following lemma. It shows that for *Pareto optimal* rules, a reduced market has the same short side as the original.

Lemma 13 *Let* F *be a Pareto optimal trade rule. Then, for each* $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, *and* $N' = (B' \cup S') \subseteq N$ *such that* $N' \in \mathcal{N}_{\neq \emptyset}$, *if* $z = F(R_B, R_S, T)$, *then we have*

(i) $h(R_B, R_S, T) = B$ *implies* $h(r^z_{B' \cup S'}(R_B, R_S, T)) = B'$, *and*

(ii) $h(R_B, R_S, T) = S$ *implies* $h(r^z_{B' \cup S'}(R_B, R_S, T)) = S'$.

Proof. Let $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, and $(B' \cup S') \subseteq N$ be such that $B' \neq \emptyset$ and $S' \neq \emptyset$. Let $z \equiv F(R_B, R_S, T)$. First, suppose $h(R_B, R_S, T) = B$. Since F is *Pareto optimal*, z is *Pareto optimal with respect to* (R_B, R_S, T). Then, by Lemma 9, for each $b \in B$, $p(R_b) \leq z_b$ and for each $s \in S$, $z_s \leq p(R_s)$. Then,

$$\begin{aligned}
\sum_{B\setminus B'} z_b + \sum_{B'} p(R_b) &\leq \sum_{B} z_b \\
&= \sum_{S} z_s + T \\
&\leq \sum_{S'} p(R_s) + \sum_{S\setminus S'} z_s + T.
\end{aligned}$$

That is $\sum_{B'} p(R_b) \leq \sum_{S'} p(R_s) + T - \sum_{B\setminus B'} z_b + \sum_{S\setminus S'} z_s$. Note that $r^z_{B'\cup S'}(R_B, R_S, T) = (R_{B'}, R_{S'}, T')$ for $T' = T - \sum_{B\setminus B'} z_b + \sum_{S\setminus S'} z_s$. Thus, $h(r^z_{B'\cup S'}(R_B, R_S, T)) = B'$. This proves (i). The proof of (ii) is similar. ■

Proof. (Theorem 2) The if part is straightforward and thus, omitted. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy,* and *consistency,* by Theorem 1, $F = U$ and $\Omega \in \mathcal{V}^{[short,long]}$ satisfies *consistency.* Now, by using *strong independence of trade volume*, we will show that $\Omega \in \mathcal{V}^{\{short,long\}}$.

For this, let $N = (B\cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$. First, assume that $h(R_B, R_S, T) = S$. Let $\sum_B p(R_b) = a$, $\sum_S p(R_s) + T = d$, and $\Omega(R_B, R_S, T) = c$. By Theorem 1, $c \in [d, a]$. Suppose for a contradiction $c \notin \{a, d\}$, that is $c \in (d, a)$. Let $\varepsilon \in \mathbb{R}_+$ be such that $\varepsilon < \min\{\frac{c}{n}, \frac{2(a-c)}{(n-2)}, \frac{2(n-1)(c-d)}{(m-1)(n-2)}\}$. Also let $m, n \in \mathbb{N}$ be such that $n \geq 3$ and $m > \max\{3, \frac{c-T}{d-T}\}$.

Let $(R_{B'}, R_{S'}, T) \in \mathcal{M}^{B'\cup S'}$ be such that $|B'| = n$, $|S'| = m$, and

$p(R_{b'_1}) = \frac{c}{n} - \varepsilon,\;\; p(R_{b'_2}) = \cdots = p(R_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1}$,
$p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)},\;\; p(R_{s'_2}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} + \frac{(n-2)(m-3)\varepsilon}{2(n-1)(m-2)}$,
$p(R_{s'_3}) = \cdots = p(R_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{(n-2)\varepsilon}{(n-1)(m-2)}$.

Also, let $(R'_{B'}, R'_{S'}, T) \in \mathcal{M}^{B'\cup S'}$ be such that

$p(R'_{b'_1}) = \frac{c}{n} - \frac{\varepsilon}{2},\;\; p(R'_{b'_2}) = \frac{a}{n-1} - \frac{c}{n(n-1)} - \frac{(n-3)\varepsilon}{2(n-1)},\;\; p(R'_{b'_3}) = \cdots = p(R'_{b'_n}) = \frac{a}{n-1} - \frac{c}{n(n-1)} + \frac{\varepsilon}{n-1}$,
$p(R'_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)},\;\; p(R'_{s'_2}) = \cdots = p(R'_{s'_m}) = \frac{d}{m-1} - \frac{T}{m} - \frac{c}{m(m-1)} - \frac{(n-2)\varepsilon}{(n-1)(m-2)}$.

Note that by the choice of ε and m, for each $k \in (B'\cup S')$, $p(R_{k'}) \geq 0$ and $p(R'_{k'}) \geq 0$. Also, $\sum_{B'} p(R_{b'}) = \sum_{B'} p(R'_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = \sum_{S'} p(R'_{s'}) = d - T$. Then, by *strong independence of trade volume*, $\Omega(R_{B'}, R_{S'}, T) = \Omega(R'_{B'}, R'_{S'}, T) = c$.

For each $K \in \{B', S'\}$, let $z_K \equiv F_K(R_{B'}, R_{S'}, T) = U(R_K, c)$ and $z'_K \equiv F_K(R'_{B'}, R'_{S'}, T) = U(R'_K, c)$. Since for each $i = 2, \cdots, n$, $p(R_{b'_1}) < \frac{c}{n} < p(R_{b'_i})$, $p(R'_{b'_1}) < \frac{c}{n} < p(R'_{b'_i})$, and $\frac{1}{(n-1)}(c - p(R'_{b'_1})) < p(R'_{b'_i})$, we have $z_{b'_1} = p(R_{b'_1}) = \frac{c}{n} - \varepsilon$, $z_{b'_i} = \frac{1}{n-1}(c - p(R_{b'_1})) = \frac{c}{n} + \frac{\varepsilon}{n-1}$, $z'_{b'_1} = p(R'_{b'_1}) = \frac{c}{n} - \frac{\varepsilon}{2}$, and $z'_{b'_i} = \frac{1}{n-1}(c - p(R'_{b'_1})) = \frac{c}{n} + \frac{\varepsilon}{2(n-1)}$.

Since for each $i = 2, \cdots, m$, $p(R_{s'_i}) < \frac{c-T}{m} < p(R_{b'_1})$, $p(R'_{s'_i}) < \frac{c-T}{m} < p(R'_{s'_1})$, and $\frac{1}{(m-1)}(c - T - p(R_{s'_1})) > p(R'_{s'_i})$, we have $z_{s'_1} = p(R_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{2(m-2)(n-1)}$, $z_{s'_i} = \frac{1}{m-1}(c - T - p(R_{s'_1})) = \frac{c}{m} - \frac{T}{m} - \frac{\varepsilon(n-2)}{2(m-2)(n-1)}$, $z'_{s'_1} = p(R'_{s'_1}) = \frac{c}{m} - \frac{T}{m} + \frac{\varepsilon(m-1)(n-2)}{(m-2)(n-1)}$, and $z'_{s'_i} = \frac{1}{m-1}(c - T - p(R'_{s'_1})) = \frac{c}{m} - \frac{T}{m} - \frac{\varepsilon(n-2)}{(m-2)(n-1)}$.

Now, let $T' = \frac{2T}{m} + \frac{2(m-n)c}{mn} - \frac{3(n-2)\varepsilon}{2(n-1)}$ and consider the following two reduced problems:

(i) $r^z_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T) = (R_{b'_1}, R_{b'_2}, R_{s'_1}, R_{s'_2}, T')$,

(ii) $r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}(R'_{B'}, R'_{S'}, T) = (R'_{b'_1}, R'_{b'_2}, R'_{s'_1}, R'_{s'_2}, T')$.

Note that, $p(R_{b'_1}) + p(R_{b'_2}) = p(R'_{b'_1}) + p(R'_{b'_2})$ and $p(R_{s'_1}) + p(R_{s'_2}) = p(R'_{s'_1}) + p(R'_{s'_2})$. Then, by *strong independence of trade volume*, $\Omega(r^z_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T)) = \Omega(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}(R'_{B'}, R'_{S'}, T))$. By *consistency*, for $i = 1, 2$, $F_{b'_i}(r^z_{b'_1,b'_2,s'_1,s'_2}(R_{B'}, R_{S'}, T)) = z_{b'_i}$ and $F_{b'_i}(r^{z'}_{b'_1,b'_2,s'_1,s'_2}(R'_{B'}, R'_{S'}, T)) = z'_{b'_i}$. Then,

$\Omega(r^z_{\{b'_1,b'_2,s'_1,s'_2\}}(R_{B'}, R_{S'}, T)) = z_{b'_1} + z_{b'_2} = \frac{2c}{n} + \frac{(2-n)\varepsilon}{n-1}$ and

$\Omega(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}(R'_{B'}, R'_{S'}, T)) = z'_{b'_1} + z'_{b'_2} = \frac{2c}{n} + \frac{(2-n)\varepsilon}{2(n-1)}$.

Then, $\Omega(r^z_{\{b'_1,b'_2,s'_1,s'_m\}}(R_{B'}, R_{S'}, T)) \neq \Omega(r^{z'}_{\{b'_1,b'_2,s'_1,s'_2\}}(R'_{B'}, R'_{S'}, T))$, a contradiction. Thus, $\Omega(R_B, R_S, T) \in \{\sum_B p(b), \sum_S p(s) + T\}$. ■

Proof. (Lemma 10) Let $(B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^{B \cup S}$. Without loss of generality, let $\sum_B p(R_b) \leq \sum_S p(R_s) + T$. Then, $\Omega^{long}(R_B, R_S, T) = \sum_S p(R_s) + T$. Let $z \equiv U \circ \Omega^{long}(R_B, R_S, T)$. Then, by the definition of U, for each $b \in B$, $z_b \geq p(R_b)$ and for each $s \in S$, $z_s = p(R_s)$. Let $(B' \cup S') \subseteq (B \cup S)$. Suppose first, $\sum_{B'} p(R_{b'}) \leq \sum_{S'} p(R_{s'}) + T$. Then, $\Omega^{long}(R_{B'}, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T$. Let $z'^{long}(R_{B'}, R_{S'}, T)$. Then, by the definition of U, for each $s' \in S'$, $z'_{s'} = p(R_{s'})$. Thus, for each $s' \in S'$, $z'_{s'}\ I_{s'}\ z_{s'}$.

Claim: We have either (i) for each $b' \in B'$, $p(R_{b'}) \leq z'_{b'} \leq z_{b'}$ or (ii) for each $b' \in B'$, $z'_{b'} \geq z_{b'}$.

Proof of Claim: Suppose for a contradiction there are $\tilde{b}, \bar{b} \in B'$ such that $z'_{\tilde{b}} > z_{\tilde{b}}$ and $z'_{\bar{b}} < z_{\bar{b}}$. By definition, $z'_{\tilde{b}} = \max\{\lambda', p(R_{\tilde{b}})\}$ where λ' is such that $\sum_{B'} \max\{\lambda', p(R_{b'})\} = \sum_{B'} p(R_{b'})$ and $z_{\tilde{b}} = \max\{\lambda, p(R_{\tilde{b}})\}$ where λ is such that $\sum_B \max\{\lambda, p(R_b)\} = \sum_B p(R_b)$. Since $z'_{\tilde{b}} > z_{\tilde{b}}$, $\lambda' > \lambda$. Then, $z'_{\bar{b}} = \max\{\lambda', p(R_{\bar{b}})\} \geq \max\{\lambda, p(R_{\bar{b}})\} = z_{\bar{b}}$, a contradiction to $z'_{\bar{b}} < z_{\bar{b}}$.

Thus, we have either (i) for each $b' \in B'$, $z'_{b'}\ R_{b'}\ z_{b'}$ or (ii) for each $b' \in B'$, $z_{b'}\ R_{b'}\ z'_{b'}$.

Now, suppose $\sum_{S'} p(R_{s'}) + T \leq \sum_{B'} p(R_{b'})$. Then, $\Omega^{long}(R_{B'}, R_{S'}, T) = \sum_{B'} p(R_{b'})$. Then, for each $b' \in B'$, $z'_{b'} = p(R_{b'})$. Thus, for each $b' \in B'$, $z'_{b'}\ R_{b'}\ z_{b'}$. Also, for each $s' \in S'$, $z'_{s'} \geq p(R_{s'})$, that is $z_{s'}\ R_{s'}\ z'_{s'}$. Therefore, $F = U \circ \Omega^{long}$ is *population monotonic*. ■

We prove Theorem 3 with the help of the following three lemmas. The first one states that if a Uniform trade rule, $U \circ \Omega$ satisfies *Pareto optimality* and *strategy proof-*

ness, then it satisfies the following property: the trade remains the same when unsatisfied agents change their preference relations without changing their peaks.

Lemma 14 *If $F = U \circ \Omega$ satisfies Pareto optimality and strategy proofness, then for each $N = (B \cup S) \in \mathcal{N}$, $(R_B, R_S, T) \in \mathcal{M}^N$, $N'^F(R_B, R_S, T)$, and $R'_{N'} \in \mathcal{R}^{N'}$ such that $p(R'_{N'}) = p(R_{N'})$, we have $F(R_B, R_S, T) = F(R'_{N'}, R_{N \setminus N'}, T)$.*

Proof. Let $U \circ \Omega$ satisfy *Pareto optimality* and *strategy proofness*. Let $N = (B \cup S) \in \mathcal{N}$ and $(R_B, R_S, T) \in \mathcal{M}^N$. Let $N'^F(R_B, R_S, T)$ and $R'_{N'} \in \mathcal{R}^{N'}$ be such that $p(R'_{N'}) = p(R_{N'})$. Let $K \in \{B, S\}$ and let $i \in N' \cap K$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(R'_i, R_{N\setminus\{i\}}, T)$. Without loss of generality, let $z_i > p(R_i)$.

Claim 1. ($z'_i = z_i$) Suppose not. Since $p(R_i) = p(R'_i)$, $h(R'_i, R_{N\setminus\{i\}}, T) = h(R_N, T)$. Then, by Lemma 9, $z'_i \geq p(R_i) = p(R'_i)$. If $z'_i > z_i$, then when the true preference relation of i is R'_i, he manipulates z' by pretending as if his true preference relation is R_i. Similarly, if $z_i > z'_i$, then when the true preference relation of i is R_i, he manipulates z by pretending as if his true preference relation is R'_i. Thus, $z'_i = z_i$.

Claim 2. ($\Omega(R_B, R_S, T) = \Omega(R'_i, R_{B\setminus\{i\}}, R_S, T)$) Since $z_i > p(R_i)$, by the definition of U, $z_i = \max\{\lambda, p(R_i)\} = \lambda$ where λ satisfies $\sum_K \max\{\lambda, p(R_k)\} = \Omega(R_B, R_S, T)$. Similarly, $z'_i = \max\{\lambda', p(R_i)\}$ where λ' satisfies $\sum_K \max\{\lambda', p(R_k)\} = \Omega(R'_i, R_{N\setminus i}, T)$. By Claim 1, $z'_i = z_i = \lambda \neq p(R_i)$. Then, $z'_i = \lambda'$ and so $\lambda' = \lambda$. Thus, $\Omega(R_B, R_S, T) = \Omega(R'_i, R_{N\setminus\{i\}}, T)$.

Claim 3. (for each $j \in N \setminus \{i\}$, $z'_j = z_j$) It follows from $F = U \circ \Omega$ and Claim 2.

By Claim 1 and 3, $z' = z$. Now, let $j \in N' \setminus \{i\}$ and apply the same argument to $(R'_i, R_{N\setminus\{i\}}, T)$. Repeating the similar argument to each $k \in N'$ proves that $U \circ \Omega(R'_{N'}, R_{N\setminus N'}, T) = U \circ \Omega(R_B, R_S, T)$. ■

The second lemma shows that *strategy proofness* satisfied by a trade rule $F = f \circ \Omega$ passes on to f. Since its proof is similar to the proof of Lemma 11, we omit it.

Lemma 15 *If a trade rule $F = f \circ \Omega$ is strategy proof, then f is also strategy proof.*

The third lemma is by Ching (1992) on the allocation rule, f.

Lemma 16 *(Ching, 1992) An allocation rule, f satisfies Pareto optimality, no-envy, and strategy proofness if and only if $f = U$.*

Proof. (Theorem 3) The if part is easy to prove. The only if part is as follows. Since F satisfies *Pareto optimality, no-envy*, and *strategy proofness*, by the lemmas 11 and 15, f also satisfies those properties. Then, by Lemma 16, $f = U$.

Since F is *Pareto optimal*, by Lemma 9, $\Omega \in \mathcal{V}^{[short,long]}$. Now, let $(R_B, R_S, T) \in \mathcal{M}^2_{\neq}$. Let $|B| = n$ and $|S| = m$. Note that, $n \geq 2$ and $m \geq 2$. First, suppose $h(R_B, R_S, T) = S$. Since $\Omega \in \mathcal{V}^{[short,long]}$, $\Omega(R_B, R_S, T) \in [\sum_S p(R_s) + T, \sum_B p(R_b)]$. Suppose for a contradiction, $\Omega(R_B, R_S, T) \in (\sum_S p(R_s) + T, \sum_B p(R_b))$. Without loss of generality, enumerate $B = \{b_1, \cdots, b_n\}$ and $S = \{s_1, \cdots, s_m\}$ such that $p(R_{b_1}) \leq p(R_{b_2}) \leq \cdots \leq p(R_{b_n})$ and $p(R_{s_1}) \leq p(R_{s_2}) \leq \cdots \leq p(R_{s_m})$. Let $z \equiv F(R_B, R_S, T) = U \circ \Omega(R_B, R_S, T)$. Then, by the definition of U, $z_{b_1} \leq p(R_{b_1})$, $z_{b_n} < p(R_{b_n})$, $z_{s_1} > p(R_{s_1})$, and $z_{s_m} \geq p(R_{s_m})$. Now, let l be the smallest integer such that $p(R_{s_m})l + \sum_S p(R_s) + T > \sum_B p(R_b)$. Then, let $S' = S \cup \{s_{m+1}, \cdots, s_{m+l}\}$ and for each $i = 1, \cdots, l$, let $R_{s_{m+i}} = R_{s_m}$. Note that $h(R_B, R_{S'}, T) = B$. Let $z' \equiv F(R_B, R_{S'}, T)$. Since $\Omega \in \mathcal{V}^{[short,long]}$, $\Omega(R_B, R_{S'}, T) \in [\sum_B p(R_b), \sum_{S'} p(R_{s'}) + T]$. First, suppose $\sum_B p(R_b) < \Omega(R_B, R_{S'}, T) \leq \sum_{S'} p(R_{s'}) + T$. Then, by the definition of U, we have one of the following cases:

Case 1: Let $z_{b_1} = p(R_{b_1})$, $z_{b_n} < p(R_{b_n})$, and $z'_{b_1} = z'_{b_n} > p(R_{b_n})$. Then, let $R'_{b_n} \in \mathcal{R}$ be such that $p(R'_{b_n}) = p(R_{b_n})$ and $z'_{b_n} \ P'_{b_n} \ z_{b_n}$. By Lemma 14, $F(R_{B \setminus \{b_n\}}, R'_{b_n}, R_S, T) = z$ and $F(R_{B \setminus \{b_n\}}, R'_{b_n}, R_{S'}, T) = z'$. Then, we have $z_{b_1} \ P_{b_1} \ z'_{b_1}$ and $z'_{b_n} \ P'_{b_n} \ z_{b_n}$, a contradiction to F satisfying *population monotonicity*.

Case 2: Let $z_{b_1} = p(R_{b_1})$, $z_{b_n} < p(R_{b_n})$, $z'_{b_1} > p(R_{b_1})$, and $z'_{b_n} = p(R_{b_n})$. Then, we have $z_{b_1} \ P_{b_1} \ z'_{b_1}$ and $z'_{b_n} \ P_{b_n} \ z_{b_n}$, a contradiction to F satisfying *population_monotonicity*.

Case 3: Let $z_{b_1} = z_{b_n} < p(R_{b_1})$, and $z'_{b_1} = z'_{b_n} > p(R_{b_n})$. Then, consider R'_{b_1} and R'_{b_n} such that $p(R'_{b_1}) = p(R_{b_1})$, $p(R'_{b_n}) = p(R_{b_n})$, $z_{b_1} \ P'_{b_1} \ z'_{b_1}$, and $z'_{b_n} \ P'_{b_n} \ z_{b_n}$. By Lemma 14, $F(R_{B \setminus \{b_1, b_n\}}, R'_{b_1}, R'_{b_n}, R_S, T) = z$ and $F(R_{B \setminus \{b_1, b_n\}}, R'_{b_1}, R'_{b_n}, R_{S'}, T) = z'$. Then, we have $z_{b_1} \ P'_{b_1} \ z'_{b_1}$ and $z'_{b_n} \ P'_{b_n} \ z_{b_n}$, a contradiction to F satisfying *population monotonicity*.

Case 4: Let $z_{b_1} = z_{b_n} < p(R_{b_1})$, $z'_{b_1} > p(R_{b_1})$, and $z'_{b_n} = p(R_{b_n})$. Then, consider R'_{b_1} such that $p(R'_{b_1}) = p(R_{b_1})$ and $z_{b_1} \ P'_{b_1} \ z'_{b_1}$. By Lemma 14, $F(R_{B \setminus \{b_1\}}, R'_{b_1}, R_S, T) = z$ and $F(R_{B \setminus \{b_1\}}, R'_{b_1}, R_{S'}, T) = z'$. Then, we have $z_{b_1} \ P'_{b_1} \ z'_{b_1}$ and $z'_{b_n} \ P_{b_n} \ z_{b_n}$, a contradiction to F satisfying *population monotonicity*.

Second, suppose $\Omega(R_B, R_{S'}, T) = \sum_B p(R_b)$. Then, similar argument proves that in each case of z_{s_1}, z'_{s_1}, z_{s_m}, and z'_{s_m}, there is a violation of *population monotonicity*. Thus, $\Omega(R_B, R_S, T) \in \{\sum_B p(R_b), \sum_S p(R_s) + T\}$. Similar argument proves the other case in which $h(R_B, R_S, T) = B$, for this just replace S with B. This proves (2.1).

For (2.2), let $(R_B, R_S, T) \in \mathcal{M} \setminus \{\mathcal{M}_{\mathcal{B}} \cup \mathcal{M}_{\mathcal{S}}\}$ and $(B' \cup S'^{rad}(R_B, R_S, T)$. Let $K \in \{B, S\}$ and $i \in K$ be such that $i \in US^F(R_B, R_S, T)$. Also, let $j \in K$ be such that $p(R_j) \neq p(R_i)$. First, let $h(R_B, R_S, T) = K$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(R_{B'}, R_{S'}, T)$. By *Pareto optimality,* Lemma 9 implies $z_i > p(R_i)$ and $z_j \geq p(R_j)$.

Since $(B' \cup S'^{rad}(R_B, R_S, T), h(R_{B'}, R_{S'}, T) = N' \setminus K' \equiv L'$. Suppose for a contradiction Ω does not favor K' in $(R_{B'}, R_{S'}, T)$. Then, by Lemma 9, for each $k' \in K'$, $z'_{k'} \leq p(R_{k'})$. Then, we have four cases:

Case 1: Let $z_i > p(R_i)$, $z_j = p(R_j)$, $z'_i = p(R_i)$, and $z'_j < p(R_j)$. Then, we have $z'_i \ P_i \ z_i$ and $z_j \ P_j \ z'_j$, a contradiction to F satisfying *population monotonicity*.

Case 2: Let $z_i > p(R_i)$, $z_j = p(R_j)$, $z'_i = z'_j < \min\{p(R_i), p(R_j)\}$. Then, consider $R'_i \in \mathcal{R}$ such that $p(R'_i) = p(R_i)$ and $z'_i \ P'_i \ z_i$. By Lemma 14, $F(R'_i, R_{(B \cup S) \setminus \{i\}}, T) = z$ and $F(R'_i, R_{(B' \cup S') \setminus \{i\}}, T) = z'$. Then, we have $z'_i \ P'_i \ z_i$ and $z_j \ P_j \ z'_j$, a contradiction to F satisfying *population monotonicity*.

Case 3: Let $z_i = z_j > \max\{p(R_i), p(R_j)\}$, $z'_i = p(R_i)$, and $z'_j < p(R_j)$. Then, consider $R'_j \in \mathcal{R}$ such that $p(R'_j) = p(R_j)$ and $z_j \ P'_j \ z'_j$. By Lemma 14, $F(R'_j, R_{(B \cup S) \setminus \{j\}}, T) = z$ and $F(R'_j, R_{(B' \cup S') \setminus \{j\}}, T) = z'$. Then, we have $z'_i \ P_i \ z_i$ and $z_j \ P'_j \ z'_j$, a contradiction to F satisfying *population monotonicity*.

Case 4: Let $z_i = z_j > \max\{p(R_i), p(R_j)\}$ and $z'_i = z'_j < \min\{p(R_i), p(R_j)\}$. Then, let $R'_i \in \mathcal{R}$ be such that $p(R'_i) = p(R_i)$ and $z_i \ P'_i \ z'_i$. Also, let $R'_j \in \mathcal{R}$ be such that $p(R'_j) = p(R_j)$ and $z'_j \ P'_j \ z_j$. By Lemma 14, $F(R'_i, R'_j, R_{(B \cup S) \setminus \{i,j\}}, T) = z$ and $F(R'_i, R'_j, R_{(B' \cup S') \setminus \{i,j\}}, T) = z'$. Then, we have $z_i \ P'_i \ z'_i$ and $z'_j \ P'_j \ z_j$, a contradiction to F satisfying *population monotonicity*.

Thus, if $h(R_B, R_S, T) = K$, Ω must favor K' in $(R_{B'}, R_{S'}, T)$. If $h(R_B, R_S, T) = N \setminus K$, the proof is very similar. Thus, Ω must favor K' in $(R_{B'}, R_{S'}, T)$. Therefore Ω satisfies *Property* (ii) and this proves (3.2).

For 3.3, let $(R_B, R_S, T) \in \mathcal{M}$ and $i \in US^F(R_B, R_S, T)$. Without loss of generality, let $i \in h(R_B, R_S, T)$. Let $R'_i \in \mathcal{R} \setminus \{R_i\}$ be such that $p(R'_i) < F_i(R_B, R_S, T)$. Let $z \equiv F(R_B, R_S, T)$ and $z' \equiv F(R'_i, R_{N \setminus \{i\}}, T)$. Note that, since $i \in US^F(R_B, R_S, T)$ and $i \in h(R_B, R_S, T)$, $z_i > p(R_i)$. By *strategy proofness* and Lemma 6, $z'_i \geq z_i$, otherwise when the true preference relation of i is R_i, he gains by declaring R'_i. Note that, since $f = U$, $z_i = \max\{p(R_i), \lambda\}$ where λ satisfies $\sum_B \max\{p(R_b), \lambda\} = \Omega(R_B, R_S, T)$ and $z'_i = \max\{p(R'_i), \lambda'\}$ where λ' satisfies $\sum_B \max\{p(R'_b), \lambda'\} = \Omega(R'_i, R_{N \setminus \{i\}}, T)$. Since $z_i > p(R_i)$, $z_i = \lambda$. Also, since $z'_i \geq z_i > p(R'_i)$, $z'_i = \lambda'$. Then, since $z'_i \geq z_i$, $\lambda' \geq \lambda$. Therefore, $\Omega(R'_i, R_{N \setminus \{i\}}, T) \geq \Omega(R_B, R_S, T)$. By *Pareto optimality*, Lemma 9 implies that $\Omega(R'_i, R_{N \setminus \{i\}}, T) \in [\Omega(R_B, R_S, T), \sum_{j \in l(R_B, R_S, T)} p(R_j)]$. The proof of the case in which $i \in l(R_B, R_S, T)$ is very similar. Thus, Ω satisfies *Property* (iii). ■

Proof. (Theorem 4) Let Ω satisfy *strong independence of trade volume*. Let $F = f \circ \Omega$ satisfy *Pareto optimality, no-envy, strategy proofness*, and *population monotonicity*. By Theorem 2, $f = U$, Ω satisfies properties (ii) and (iii). Also, Ω satisfies Property (i), that is, $\Omega \in \mathcal{V}^{\{short, long\}}$ on $\mathcal{M}^2_{\neq}$, and $\Omega \in \mathcal{V}^{[short, long]}$ on $\mathcal{M} \setminus \mathcal{M}^2_{\neq}$. Then, by *strong_independence of trade volume*, $\Omega \in \mathcal{V}^{\{short,long\}}$ on $\mathcal{M}$. We will prove (3.1) by the following claims:

Claim 1. For each $T > 0$, $(R_B, R_S, T) \in \mathcal{M}\backslash\mathcal{M}_{nt}$, $\Omega(R_B, R_S, T) = \Omega^{long}(R_B, R_S, T)$.

Proof of Claim 1. Let $T > 0$. Let $(R_B, R_S, T) \in \mathcal{M} \setminus \mathcal{M}_{nt}$. Let $\sum_B p(R_b) = a$ and $\sum_S p(R_s) = d$. First, suppose $h(R_B, R_S, T) = S$, that is $d + T < a$. Then, let $B' = \{b'_1, b'_2\}$ and $S' = \{s'_1, s'_2\}$. Let $R_{(B' \cup S)} \in \mathcal{R}^{B' \cup S'}$ be such that $p(R_{b'_1}) = T/6$, $p(R_{b'_2}) = T/3$, $p(R_{s'_1}) = d/3$, and $p(R_{s'_2}) = 2d/3$. Note that $(R_{B'}, R_{S'}, T) \in \mathcal{M} \setminus \mathcal{M}_{nt}$. Then, $\Omega(R_{B'}, R_{S'}, T) \in \{\sum_{B'} p(R_{b'}), \sum_{S'} p(R_{s'}) + T\}$. Note that $\sum_{B'} p(R_{b'}) = T/2$ and $\sum_{S'} p(R_{s'}) + T = d + T$. Thus, $h(R_{B'}, R_{S'}, T) = B'$ and $\sum_{B'} p(R_{b'}) < T$. Then, by *feasibility*, $\Omega(R_{B'}, R_{S'}, T) = \sum_{S'} p(R_{s'}) + T = \Omega^{long}(R_{B'}, R_{S'}, T)$. Then, $F_{b'_1}(R_{B'}, R_{S'}, T) > T/6$. Now, let $B'' = B' \cup \{b'_3\}$ and $R_{b'_3} \in \mathcal{R}$ be such that $p(R_{b'_3}) = a - T/2$. Note that $(R_{B''}, R_{S'}, T) \in \mathcal{M} \setminus \{(\mathcal{M}_{nt} \cup \mathcal{M}_0)\}$ and $\sum_{B''} p(R_{b'}) = a$, and $\sum_{S'} p(R_{s'}) = d$. Then, by Theorem 2, Ω should favor B'' in $(R_{B''}, R_{S'}, T)$, that is $\Omega(R_{B''}, R_{S'}, T) = \Omega^{long}(R_{B''}, R_{S'}, T)$. Then, by *strong independence of trade volume*, $\Omega(R_B, R_S, T) = \Omega^{long}(R_B, R_S, T)$. Now, suppose $h(R_B, R_S, T) = B$. By Theorem 2, $\Omega(R_B, R_S, T) \in \{\Omega^{short}(R_B, R_S, T), \Omega^{long}(R_B, R_S, T)\}$. Suppose for a contradiction $\Omega(R_B, R_S, T) = \Omega^{short}(R_B, R_S, T)$. let $B' = \{b'_1, b'_2\}$ and $S' = \{s'_1, s'_2\}$. Let $R_{(B' \cup S)} \in \mathcal{R}^{B' \cup S'}$ be such that $p(R_{b'_1}) = a/3$, $p(R_{b'_2}) = 2a/3$, $p(R_{s'_1}) = d/3$, and $p(R_{s'_2}) = 2d/3$. Note that $\sum_{B'} p(R_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = d$. Then, by *strong independence of trade volume*, $\Omega(R_{B'}, R_{S'}, T) = \Omega(R_B, R_S, T) = \Omega^{short}(R_B, R_S, T)$. Then, $s'_2 \in US^F(R_{B'}, R_{S'}, T)$. Now, let $B'' = B' \cup \{b'_3\}$ and $R_{b'_3} \in \mathcal{R}$ be such that $p(R_{b'_3}) = d + T$. Note that $(R_{B''}, R_{S'}, T) \in \mathcal{M} \setminus \mathcal{M}_{nt}$ and $\sum_{B''} p(R_{b'}) = a + d + T$, and $\sum_{S'} p(R_{s'}) = d$. Note that, $h(R_{B''}, R_{S'}, T) = S$. Then, $(B'' \cup S'') \in r^{rad}(R_{B'}, R_{S'}, T)$ and by Property (ii), Ω should favor S' in $(R_{B''}, R_{S'}, T)$. Thus, $\Omega(R_{B''}, R_{S'}, T) = \Omega^{short}(R_{B''}, R_{S'}, T)$, a contradiction to the first case.

Claim 2. For each $T < 0$, $(R_B, R_S, T) \in \mathcal{M}\backslash\mathcal{M}_{nt}$, $\Omega(R_B, R_S, T) = \Omega^{long}(R_B, R_S, T)$.

Proof of Claim 2. The proof is very similar to the proof of Claim 1.

Thus, by claims 1 and 2, on $\mathcal{M} \setminus \mathcal{M}_{nt}$, $\Omega = \Omega^{long}$.

The following claims will prove 3.2.

Claim 3. Let $(R_B, R_S), (R'_{B'}, R'_{S'}) \in \mathcal{M}_{nt}$. Let $K \in \{B, S\}$. Suppose that $h(R_B, R_S) = K$ and $h(R'_{B'}, R'_{S'}) = K'$. Then, we have either $[\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{short}(R'_{B'}, R'_{S'})]$ or $[\Omega(R_B, R_S) = \Omega^{long}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{long}(R'_{B'}, R'_{S'})]$.

Proof of Claim 3. Without loss of generality, let $K = B$. Suppose for a contradiction, $\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{long}(R'_{B'}, R'_{S'})$. Let $\sum_B p(R_b) = a$, $\sum_{B'} p(R'_{b'}) = a'$, $\sum_S p(R_s) = d$, and $\sum_{S'} p(R'_{s'}) = d'$. First, let $a > a'$

and $d > d'$. Now, let $\tilde{B} = \{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3\}$ and $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3\}$. Also, let $(R_{\tilde{B}}, R_{\tilde{S}}) \in \mathcal{M}_{nt}$ be such that $p(R_{\tilde{b}_1}) = a'/3$, $p(R_{\tilde{b}_2}) = 2a'/3$, $p(R_{\tilde{b}_3}) = a - a'$, $p(R_{\tilde{s}_1}) = d'/3$, $p(R_{\tilde{s}_2}) = 2d'/3$, and $p(R_{\tilde{s}_3}) = d - d'$. Note that $\sum_{\tilde{B}} p(R_{\tilde{b}}) = a$ and $\sum_{\tilde{S}} p(R_{\tilde{s}}) = d$. Then, by *strong independence_of trade volume*, $\Omega(R_{\tilde{B}}, R_{\tilde{S}}) = \Omega^{short}(R_{\tilde{B}}, R_{\tilde{S}}) = a$. Note that, there is $\tilde{s}_i \in \tilde{S}$ such that $\tilde{s}_i \in US^F(R_{\tilde{B}}, R_{\tilde{S}})$. Similarly, let $\tilde{B}' = \{\tilde{b}'_1, \tilde{b}'_2\}$ and $\tilde{S}' = \{\tilde{s}_1, \tilde{s}_2\}$. $(R_{\tilde{B}'}, R_{\tilde{S}'}) \in \mathcal{M}_{nt}$ be such that $p(R_{\tilde{b}'_1}) = a'/3$, $p(R_{\tilde{b}'_2}) = 2a'/3$, $p(R_{\tilde{s}'_1}) = d'/3$, $p(R_{\tilde{s}'_2}) = 2d'/3$. Note that $\sum_{\tilde{B}'} p(R_{\tilde{b}'}) = a'$ and $\sum_{\tilde{S}'} p(R_{\tilde{s}'}) = d'$. Then, by *strong independence of trade volume*, $\Omega(R_{\tilde{B}'}, R_{\tilde{S}'}) = \Omega^{long}(R_{\tilde{B}'}, R_{\tilde{S}'}) = d'$. Note that $\tilde{b}'_1 \in US^F(R_{\tilde{B}'}, R_{\tilde{S}'})$. Now, let $B'' = \tilde{B} \cup \{\tilde{b}_4\}$ and $S'' = \tilde{S}$. Let $R_{\tilde{b}_4} \in \mathcal{R}$ be such that $p(R_{\tilde{b}_4}) = d + d' - a$. Now, consider $(R_{B''}, R_{S''})$. Note that $\sum_{B''} p(R_{\tilde{b}}) = d + d'$ and $\sum_{S''} p(R_{\tilde{s}}) = d$. Thus, $h(R_{B''}, R_{S''}) = S''$. That is, $(B'' \cup S''^{rad}(R_{\tilde{B}}, R_{\tilde{S}}) \cap r^{rad}(R_{\tilde{B}'}, R_{\tilde{S}'})$. Then, by *Property* (ii), we have the following: if we consider $(R_{\tilde{B}}, R_{\tilde{S}})$, Ω should favor S''. If we consider $(R_{\tilde{B}'}, R_{\tilde{S}'})$, Ω should favor B'', a contradiction. Thus, we have either $[\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{short}(R'_{B'}, R'_{S'})]$ or $[\Omega(R_B, R_S) = \Omega^{long}(R_B, R_S)$ and $\Omega(R'_{B'}, R'_{S'}) = \Omega^{long}(R'_{B'}, R'_{S'})]$. The proofs of the other relations of a, a', d, and d' are very similar.

Claim 4. On $\mathcal{M}_{nt}$, $\Omega = \Omega^{short}$ or $\Omega = \Omega^{long}$.

Proof of Claim 4. By Claim 3, first suppose for each $(R_B, R_S) \in \mathcal{M}_{nt}$ such that $h(R_B, R_S) = B$, $\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$. Let $(R_B, R_S) \in \mathcal{M}_{nt}$ be such that $h(R_B, R_S) = B$. Let $\sum_B p(R_b) = a$ and $\sum_S p(R_s) = d$. Then, $\Omega(R_B, R_S) = a$. Now, let $B' = \{b'_1, b'_2\}$ and $S' = \{s'_1, s'_2\}$. Let $R_{B'}, R_{S'} \in \mathcal{R}$ be such that $p(R_{b'_1}) = a/3$, $p(R_{b'_2}) = 2a/3$, $p(R_{s'_1}) = d/3$, and $p(R_{s'_2}) = 2d/3$. Note that, $\sum_{B'} p(R_{b'}) = a$ and $\sum_{S'} p(R_{s'}) = d$. Then, by *strong independence of trade volume*, $\Omega(R_{B'}, R_{S'}) = a$. Then, $s'_2 \in US^F(R_{B'}, R_{S'})$. Now, let $B'' = B' \cup \{b'_3\}$ and $R_{b'_3} \in \mathcal{R}$ be such that $p(R_{b'_3}) = d$. Consider $(R_{B''}, R_{S'}) \in \mathcal{M}_{nt}$. Note that $\sum_{B''} p(R_{b''}) = a + d$. Thus, $h(R_{B''}, R_{S'}) = S'$, that is, $(B'' \cup S'^{rad}(R_{B'}, R_{S'})$. Then, by Property (ii), Ω should favor S' in $(R_{B''}, R_{S'})$, that is, $\Omega(R_{B''}, R_{S'}) = \Omega^{short}(R_{B''}, R_{S'})$. Then, by Claim 1, for each $(R_B, R_S) \in \mathcal{M}_{nt}$ such that $h(R_B, R_S) = S$, $\Omega(R_B, R_S) = \Omega^{short}(R_B, R_S)$. The proof of the other case in which for each $(R_B, R_S) \in \mathcal{M}_{nt}$ such that $h(R_B, R_S) = B$, $\Omega(R_B, R_S) = \Omega^{long}(R_B, R_S)$ is similar. Thus, on $\mathcal{M}_{nt}$, $\Omega = \Omega^{short}$ or $\Omega = \Omega^{long}$. ■

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