Parabolic Stein Manifolds

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Abstract. An open Riemann surface is called parabolic in case every bounded subharmonic function on it reduces to a constant. Several authors introduced seemingly different analogs of this notion for Stein manifolds of arbitrary dimension. In the first part of this note we compile these notions of parabolicity and give some immediate relations among these different definitions. In section 3 we relate some of these notions to the linear topological type of the Fréchet space of analytic functions on the given manifold. In sections 4 and 5 we look at some examples and show, for example, that the complement of the zero set of a Weierstrass polynomial possesses a continuous plurisubharmonic exhaustion function that is maximal off a compact subset.

1. Introduction

In the theory of Riemann surfaces, simply connected manifolds which equal to complex plane are usually called parabolic and the ones which equal to the unit disk are called hyperbolic. Several authors introduced analogs of these notions for general complex manifolds of arbitrary dimension in different ways; in terms of triviality (parabolic type) and non-triviality (hyperbolic type) of the Kobayashi or Caratheodory metrics, in terms of plurisubharmonic (psh) functions etc. In some of these considerations existence of rich family of bounded holomorphic functions plays a significant role.

On the other hand attempts to generalize Nevanlinna's value distribution theory to several variables by Stoll, Griffiths, King et al. produced notions of "paraboliticity" in several complex variables defined by requiring the existence of certain special plurisubharmonic functions. The common feature of these special plurisubharmonic functions ρ defined, say on a complex manifold X of dimension n, were:

a) $\{z \in X : \rho(z) \leqslant C\} \subset\subset X$, $\forall C \varepsilon \boldsymbol{R}^+$ i.e. ρ is exhaustive, and

b) the Monge - Ampère operator $(dd^c\rho)^n$ is zero off a compact K $\subset\subset X$. That is ρ is maximal plurisubharmonic outside K.

Following Stoll, we will call a complex manifold X, $S - Parabolic$ in case there is a plurisubharmonic function ρ on X that satisfies the conditions à) and b)

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above. If a *continuous* plurisubharmonic function ρ that satisfies the conditions à) and b) above exits on X , we will call X, S^*-Parabolic.

Special exhaustion functions with certain regularity properties plays a key role in the Nevanlinna's' value distribution theory of holomorphic maps $f : X \to \mathrm{P}^m$, where $\mathrm{P}^m - m$ dimensional projective manifold (see.[**17**],[**24**],[**28**],[**29**]).

We note that without the maximality condition b), an exhaustion function $\sigma(z) \in psh(X) \cap C^\infty(X)$ always exist for any Stein manifold X, because such manifolds can be properly embedded in $\mathbb{C}_w^{2n+1}$ and one can take for σ the restriction of $\ln|w|$ to X.

The special exhaustion function $\rho(z)$ is a key object in the Nevanlinna's' value distribution theory of holomorphic maps $f : X \to \mathrm{P}^m$, where $\mathrm{P}^m - m$ dimensional projective manifold (see.[**17**],[**24**],[**28**],[**29**]).

On $S - Parabolic$ manifolds any bounded above plurisubharmonic function is constant. In particular, there are no nonconstant bounded holomorphic functions on such manifolds.

The complex manifolds, on which every bounded above plurisubharmonic function reduces to a constant, a characteristic shared by affine-algebraic manifolds , play an important role in the structure theory of Fréchet spaces of analytic functions on Stein manifolds and in finding continuous extension operators for analytic functions from complex submanifolds (see, papers of first author [**3**],[**4**],[**5**],[**6**], [**7**]). Such spaces will be called "*parabolic*" in this paper.

The parabolic manifolds (also the parabolic Stein spaces) and the structure of certain plurisubharmonic functions and currents on them here studied in detail by J.P.Demailly[**10**] and A.Zeriahi [**40**],[**41**]. Moreover for manifolds which have a special exhaustion function one can define extremal Green functions as in the classical case and apply it to the pluripotential theory on such manifolds. In the special case of an affine algebraic manifolds such a program was successfully carried out in [**41**]

The aim of this paper is to study and compare the different definitions of parabolicity and bring to attention a problem in complex potential theory that arise in this context. This problem could be looked at from a functional analysis point of view. In section 2, we compile different definitions of paraboliticity that exits in the literature, try to collate them and state some problems. In the third section we relate the notion of paraboliticity of a Stein manifold to the linear topological type of the Fréchet space of analytic functions on it. We introduce the notion of tame isomorphism to the space of entire functions and show that a Stein manifold of dimension d is S^*-parabolic if and only if it is tamely isomorphic to the entire functions in d variables. In section four we look at some examples and ways of generating parabolic manifolds. In the last section we look at complements of analytic multifunctions and show that, the Stein manifold $C^n \backslash A$, where $A \subset C^n$ is the zero variety of a Weierstrass polynomial (algebroidal function), is S^*-parabolic.

2. Different definitions of parabolicity

Definition 1. *A Stein manifold is called parabolic, in case it does not possess a non-constant bounded above plurisubharmonic function.*

Thus, parabolicity of is equal to following; if $u(z) \in psh(X)$ and $u(z) \leqslant C$, then $u(z) \equiv const$ on X. It is convenient to describe of parabolicity in term of P-measures, which is the fundamental notion of pluripotential theory [**18**]. Without loss a generality in the discussion below we will assume that our Stein manifold X is properly imbedded in $\mathbb{C}_w^{2n+1}, n = \dim X$, and $\sigma(z)$ is the restriction of $\ln |w|$ to X. Then $\sigma(z) \in psh(X) \cap C^\infty(X)$, $\{\sigma(z) \leqslant C\} \subset\subset X \ \forall C \in R$. We further assume that $0 \notin X$ and $\min \sigma(z) < 0$. We consider (σ) balls $B_R = \{z \in X : \sigma(z) < \ln R\}$ and as usual, define the class $\aleph\left(\overline{B_1}, B_R\right)$, $R > 1$, of functions $u(z) \in psh(B_R)$ such, that $u|_{B_R} \leqslant 0$, $u|_{\overline{B}_1} \leqslant -1$. We put

$$\omega\left(z, \overline{B}_1, B_R\right) = \sup\left\{u(z) : u \in \aleph\left(\overline{B_1}, B_R\right)\right\}.$$

Then regularization $\omega^*\left(z, \overline{B}_1, B_R\right)$ is called the P- measure of $\overline{B_1}$ to relation B_R. (Definition and properties of the P- measure see: [**20**],[**18**],[**36**],[**37**]).

The P- measure $\omega^*\left(z, \overline{B}_1, B_R\right)$ is plurisubharmonic on B_R, is equal to -1 on $\overline{B_1}$ and tends to 0 in $z \to \partial B_R$. It is maximal, that is $(dd^c\omega^*)^n = 0$ in $B_R \backslash \overline{B_1}$ and decreasing in R. We put $\omega^*\left(z, \overline{B}_1\right) = \lim_{R\to\infty} \omega^*\left(z, \overline{B_1}, B_R\right)$.

It follows that

$$\omega^*\left(z, \overline{B}_1\right) \in psh(X), \quad -1 \leqslant \omega^*\left(z, \overline{B_1}\right) \leqslant 0$$

and is maximal, i.e. $\left(dd^c\omega^*\left(z, \overline{B}_1\right)\right)^n = 0$, off $\overline{B_1}$.

In the construction of $\omega^*\left(z, \overline{B}_1\right)$ we have used the exhaustion function $\sigma(z)$, however it is not difficult to see that $\omega^*\left(z, \overline{B}_1\right)$ depends only on X and $\overline{B}_1$and not on the choice of exhaustion function. Moreover, defining the P-measure for any nonpluripolar compact $K \subset X$ by selecting a sequence of domains $D_j \subset\subset D_{j+1} \subset\subset X$, $X = \bigcup_{j=1}^{\infty} D_j$ and considering the limit $\omega^*(z, K) \circeq \lim_{j\to\infty} \omega^*(z, K, D_j)$, it is clear, that $\omega^*(z, K) \equiv -1$ if and only if $\omega^*\left(z, \overline{B}_1\right) \equiv -1$. Hence the later property is an inner property of X.

Vanishing *of* $\omega^*(z, K)+1$ on a parabolic manifold not only imply the triviality of bounded holomorphic functions but also give some information on their growth. In fact on parabolic manifolds a kind of "Hadamard three domains theorem" with controlled exponents, is true. The precise formulation of this characteristic, that will appear below, is an adaptation of the property (DN) of Vogt [**31**], which was defined for general Fréchet spaces, to the Fréchet spaces of analytic functions. As usual we will denote by $O(M)$ the Fréchet spaces of analytic functions defined on a complex manifold M with the topology of uniform convergence on its compact subsets. The proposition we will give below is due to Zaharyuta [**38**] and it has been independently rediscovered by several other authors [**4**],[**33**]. We will include a proof of this result for the convinience of the reader.

Proposition 1. . *The following are equivalent for a Stein manifold X*

a) X is parabolic,

b) P-measures are trivial on X *i.e.* $\omega^*(z, K) = -1$ *for every non polar compact* $K \subseteq X$,

c) For every nonpolar compact set $K_0 \subset X$ *and for every compact set* K *of* X *there is another compact set*

L *containing* K *such that*

$$\|f\|_K \leq \|f\|_L^{\frac{1}{2}} \|f\|_{K_0}^{\frac{1}{2}} \quad ,\forall \ f \varepsilon O(X) \qquad (DN \ condition\ of\ Vogt)$$

where $\|*\|_H$ denotes the sup norm on H.

Proof. If X is parabolic, then $\omega^*\left(z,\overline{B}_1\right)$ being bounded and plurisubharmonic on X reduces to -1.

Conversely, let $u(z)$ be an arbitrary bounded above *psh* function on X. Let $u_R = \sup_{B_R} u(z)$, $R \geqslant 1$. If $u(z) \neq const$, then $\frac{u(z)-u_R}{u_R-u_1} \in \aleph\left(\overline{B_1}, B_R\right)$ and hence $\frac{u(z)-u_R}{u_R-u_1} \leqslant \omega^*\left(z,\overline{B}_1,B_R\right)$. It follows,that

$$u(z) \leqslant -u_1\omega^*\left(z,\overline{B}_1,B_R\right)+\left(1+\omega^*\left(z,\overline{B}_1,B_R\right)\right) \,,\, z\in B_R\,, \qquad (1)$$

which in $R\to\infty$ gives

$$u(z) \leqslant -u_1\omega^*\left(z,\overline{B}_1\right)+u_\infty\left(1+\omega^*\left(z,\overline{B}_1\right)\right) \,,\, z\in X\,. \qquad (2)$$

Now, if $\omega^*(z,\overline{B}_1)\equiv -1$, then $u(z)\leq u_1,\ z\in X$, and by maximal principle we have $u(z)=u_1\equiv const$, so that a) and b) are equivalent.

Now we fix a non-polar compact set K_0 and look at the sup norms $|\,.\,|_m$ on the sublevel balls B_m. Choose an increasing sequence of norms $\|\,.\,\|_k = |\,.\,|_{m_k}$, k=0,1,..., that satisfy the condition c) with the dominating norm $\|\,.\,\|_0$:

$$\|f\|_k \leqslant \|f\|_{k_0}^{\frac{1}{2}}\,\|f\|_{k+1}^{\frac{1}{2}} \quad \forall f\in \mathrm{O}(X)$$

Iterating this inequality one gets:

$$\|f\|_1 \leqslant \|f\|_0^{\frac{2^{k-1}-1}{2^{k-1}}}\ \|f\|_k^{\frac{1}{2^{k-1}}}\ \ \forall f\in\ \mathrm{O}(X) \qquad (3).$$

Now, denoting the sequence of domains corresponding to these norms by $D_k = B_{n_k}$ we will consider the P- measures $\omega^*(z,K_0,D_{k+1})$, $k=1,2...$ Since these functions are continuous for a fixed k, we can find analytic functions $f_1,f_2,....,f_m$ on D_{k+1} and positive numbers $a_1,a_2,...,a_m$ such that

$$\omega^*(z,K_0,D_{k+1})+1-\varepsilon \leqslant \max_{1\leqslant j\leqslant m}(a_j\ln|f_j(z)|) \leqslant \omega^*(z,K_0,D_{k+1})+1$$

pointwise on $\overline{D}_k$. We note that the compact $\overline{D_k}$ is polynomially convex in $\mathbb{C}^{2n+1}\supset X$ so by Runge's theorem the functions f_j can be uniformly approximated on $\overline{D_k}$ by functions $F\in \mathrm{O}(X)$ .This in turn by (3) gives us the estimate $\omega^*(z,K_0,D_{k+1})+1\leqslant\frac{1}{2^{k-1}}+\varepsilon, z\in D_1$. Now playing the same game with D_1 replaced by a given D_j we see that $\omega^*(z,K_0,D_{k+1})$ converge uniformly to -1 on any compact subset of X, i.e. $\omega^*(z,K_0)=-1$. This in turn implies that c) $\Rightarrow$ b).

Conversely, suppose that the P measure $\omega^*\left(z,\overline{B}_0\right)=-1$. Then for a given k and $0<\epsilon<1$, we can, in view of Dini's theorem, choose l so large that $\omega^*\left(z,\overline{B_0},B_l\right)\leqslant -1+\epsilon$ *on* B_k. Since $\omega^*\left(z,\overline{B_0},B_l\right)$ is maximal on $B_l\backslash\overline{B}_0$, the inequality

$$\frac{\ln\frac{|f(z)|}{|f|_0}}{\ln\frac{|f|_l}{|f|_0}} \leqslant \omega^*\left(z,\overline{B}_0,B_l\right)+1\,,\ z\in B_l\,,\ f\in O(X)$$

is valid. This in turn implies that $|f|_k\leqslant|f|_0^{1-\epsilon}|f|_l^{\epsilon}$,for all $f\in \mathrm{O}(X)$. Now fix a nonpolar compact set K_0. We can, replacing $\overline{B}_0$with $\overline{B}_k$, k large if necessary, assume that $K_0\subseteq B_0$. For a fixed large l, there is a $\lambda\varepsilon(0,1)$ such that

$\omega^*(z, K_0, B_l) \leqslant -\lambda$ on B_0.As above this implies:

$$\frac{|f|_0}{|f|_{K_0}} \le \left(\frac{|f|_l}{|f|_{K_0}}\right)^{1-\lambda} \quad \forall \text{ non identically zero } f \in O(X).$$

Choose $\epsilon > 0$ so that $\frac{\epsilon}{\lambda} < \frac{1}{2}$. In view of the above analysis we can find an l^+ such that $|f|_l \leqslant |f|_0^{1-\epsilon} |f|_{l^+}^{\epsilon}$ $\forall f \in \mathrm{O}(X)$.We have:

$$\left(\frac{|f|_l}{|f|_{K_0}}\right)^{\lambda} \le \frac{|f|_l}{|f|_0} \le \left(\frac{|f|_{l^+}}{|f|_0}\right)^{\epsilon} \le \left(\frac{|f|_{l^+}}{|f|_{K_0}}\right)^{\epsilon} \quad \forall \text{ non identically zero } f \in O(X)$$

This finishes the proof of the proposition. □

Definition 2. *A Stein manifold X is called S −parabolic,if there exit exhaustion function $\rho(z) \in psh(X)$ that is maximal outside a compact subset of X. If in addition we can choose ρ to be continuous then we will say that X is S^*−parabolic.*

In previous papers on parabolic manifolds (see for example [**11**],[**28**]) authors usually required the condition of continuity or C^∞ - smoothness of ρ. Here we only distinguish the case when the exhaustion function $\rho(z) \in psh(X) \cap C(X)$ is continuous.

It is not difficult to prove, that $S - parabolic$ manifolds are parabolic. In fact, since $\rho(z)$ is maximal off some compact $K \subset\subset X$, then the balls $B_r = \{\rho(z) \leqslant \ln r\}, r \geqslant r_0$, consist K for big enough r_0 and it is not difficult to see, that

$$\omega^*(z, B_{r_0}, B_R) = \frac{\max\{-1, \rho(z) - R\}}{R - r_0}.$$

Consequently,

$$\omega^*(z, B_{r_0}) = \lim_{R\to\infty} \omega^*(z, B_{r_0}, B_R) \equiv -1 \ , \ z \in X.$$

For Stein manifolds of dimension one the notions of S −*parabolicity*, S^* − *paraboliticity*, and parabolicity coincide. This is a consequence of the existence of Evans-Selberg potentials (subharmonic exhaustion functions that are harmonic outside a given point) on a parabolic Riemann surfaces [**23**].

Problem 1. *Do the notions of S −parabolicity and S^* − paraboliticity coincide for Stein manifolds of arbitrary dimension?*

Problem 2. *Do the notions of parabolicity and S^* −paraboliticity coincide for Stein manifolds of arbitrary dimension?*

3. Spaces of Analytic Functions on Parabolic Manifolds

In this section we will relate the above discussed notions of parabolicity of a Stein manifold X to the linear topological structure of $\mathrm{O}(X)$. Next result, which is due to Aytuna-Krone-Terzioglu [**7**] and the characterizes paraboliticity of a Stein manifold X of dimension n in terms of the similarity of the linear topological structures of $\mathrm{O}(X)$ and $\mathrm{O}(\mathbb{C}^n)$

THEOREM 1. *For a Stein manifold X of dimension n the following are equivalent:*
a) X is parabolic;
b) $\mathrm{O}(X)$ is isomorphic as Fréchet spaces to $\mathrm{O}(\mathbb{C}^n)$.

The correspondence that sends an entire function to its Taylor coefficients establishes an isomorphism between $O(\mathbb{C}^d)$ and the infinite type power series space $\Lambda_\infty(\alpha_n) := \left(x = (x_n) : |x|_k := \sum |x_n| e^{k\alpha_n} < \infty \ \forall k = 1, 2, ...\right)$ with $\alpha_n = n^{\frac{1}{d}}$ $n = 1, 2, ..$ A graded Fréchet space is a tuple $(X, (|*|_k))$ where X is a Fréchet space and $(|*|_k)_{k=1}^\infty$ is a fixed system of seminorms defining the topology of X. Whenever we deal with $\Lambda_\infty(\alpha_n)$, we will tacitly assume that we are dealing with a graded space and that the grading is given by the norms defined in the above expression . We will need a definition from the structure theory of Fréchet spaces;

DEFINITION 3. *A continuous linear operator T between two graded Fréchet spaces $(X, (|*|_k))$ and $(Y, (\|*\|_k))$ is tame in case: $\exists\ A > 0\ \forall k\ \exists C > 0 : \|T(x)\|_k \leq C|x|_{k+A}, \forall x \in X$. Two graded Fréchet spaces are called tamely isomorphic in case there is a one to one tame linear operator from one onto the other whose inverse is also tame.*

The graded space $(O(\mathbb{C}), \|*\|_n)$ where $\|*\|_n$ is the sup norm on the disc with radius e^n is tamely isomorphic, under the correspondence between an entire function and the sequence of its Taylor coefficients, to the power series space $\Lambda_\infty(n)$, in view of the Cauchy's inequality. This observation motivates our next definition:

DEFINITION 4. *Let M be a Stein manifold. The space $O(M)$ is said to be tamely isomorphic to an infinite type power series space in case there is an exhaustion of M by connected holomorphically convex compact sets $(K_n)_{k=1}^\infty$ with $K_n \subset (K_{n+1})^\circ$ $, n = 1, 2, .,$ such that the graded space $\left(O(M), \left(\sup_{K_n}|*|\right)\right)$ is tamely isomorphic to a power series space $\Lambda_\infty(\alpha_n)$.*

The supremum norms are in some sense associated with the function theory whereas the power series norms are associated with the structure theory of Fréchet spaces and tameness gives one a controlled equivalence between these generating norm systems. For two nonnegative real valued functions α and β on a set T we will use the notation $\alpha(t) \prec \beta(t)$ to mean $\exists\ C > 0$ such that $\alpha(t) \leq C\beta(t)\ \forall t \in T$.

THEOREM 2. *Let M be a Stein manifold. The space of analytic functions on M, $O(M)$, is tamely isomorphic to an infinite type power series space if and only if M is $S^* - Parabolic$.*

Proof. $\Rightarrow$: Suppose that $O(M)$ is tamely isomorphic to a power series space. Fix a tame isomorphism $T:\Lambda_\infty(\alpha_n)\to O(M)$ .By assumption there is an exhaustion $\{K_n\}_n$ of M and an integer B' such that for all n large enough

$$\|T(x)\|_n \prec |x|_{n+B'} \text{ and } |x|_n \prec \|T(x)\|_{n+B'} \ \ \forall x\in O(M),$$

where $\|*\|_n$ denotes the sup norm on K_n, $n=1,2...$ Let $e_n \circeq T(\varepsilon_n)$ where ,as usual, $\varepsilon_n=(0,...,0,1,0,...)\in\Lambda_\infty(\alpha_n)$, $n=1,2....$ Set

$$\rho(z)\circeq \limsup_{\xi\to z}\limsup_n \frac{\log|e_n(\xi)|}{\alpha_n}.$$

Clearly ρ is a plurisubharmonic function on M and if we set $D_\alpha \circeq \{z:\rho(z)<\alpha\}$ for $\alpha\in\mathbb{R}$,we have:

$$K_n\subseteq D_{n+B} \text{ for large } n, \text{ where } B=B'+1.$$

Now fix an arbitrary $z_0\in D_\alpha$ choose, in view of Hartog's lemma, a small $\epsilon>0$ and a closed neighborhood η_{z_0} of z_0 such that for some $C>0$: $\sup_{w\in\eta_{z_0}}|T(\varepsilon_n)(w)|\leq Ce^{\alpha_n(\alpha-\epsilon)}$ for all n large. 'For any $x=\sum x_n\varepsilon_n\in\Lambda_\infty(\alpha_n)$ and $0<\epsilon'<<\epsilon$, we have:

$$\sup_{w\in\eta_{z_0}}|T(x)(w)| \leq C'\left(\sum_n |x_n|^2 e^{2\left(\alpha-\epsilon'\right)\alpha_n}\right)^{\frac{1}{2}} \leq C'\|T(x)\|_{L(\alpha)+B} \text{ for some } C'>0 \text{ ,where } L(\alpha)=[[\alpha]]+1.$$

Since T is onto and K_m's are holomorphically convex, we have that $\eta_{z_0}\subseteq K_{L(\alpha)+B}$. Since $z_o\in D_\alpha$ was arbitrary we conclude that $D_\alpha\subseteq K_{L(\alpha)+B}$. Combining this with our previous findings we get

$$\exists\, d>0 \text{ such that } \overline{D_\alpha}\subseteq D_{\alpha+d}\ \forall\alpha \text{ large}$$

Now fix a nice compact set K, say $K=\overline{D}$ for some domain, with the property that

$$\exists\, K'\subseteq D \text{ compact and } \beta_0>0 \text{ such that } |x|_{\beta_0}\prec \sup_{w\in K'}|T(x)| \ \ \forall x\in\Lambda_\infty(\alpha_n).$$

We wish to show that

$$\Phi(z)\circeq \limsup_{\xi\to z}\{\varphi(\xi):\varphi\in psh(M),\ \varphi|K\leq 0,\varphi\leq\rho+C \text{ for some } C=C(\varphi)\ \}$$

defines a plurisubharmonic function on M. To this end we choose a $\varphi\in psh(M)$ such that $\varphi|K\leq 0$ and $\varphi\leq\rho+C$ for some $C=C(\varphi)>0$. Choose a representation $\varphi(z)=\limsup_{\xi\to z}\limsup_n\frac{\log|f_n(\xi)|}{c_n}$ of φ on M for some $f_n\in O(M)'s$ and positive real numbers c_n , $n=1,2,3...$ In view of our assumptions we have:

$$\forall\,\epsilon>0\ \exists\, C'>0\ :\ \sup_{w\in K'}|f_n(x)|\ \leq C'e^{\epsilon c_n},\forall n.$$

In particular if $y_n\circeq T^{-1}(f_n)$ we have:

$$\limsup_n \frac{\log|y_n|_{\beta_0}}{c_n}\leq 0.$$

Moreover since

$$\sup_{w\in D_\alpha}|f_n(w)| \prec e^{(\alpha+d+C)c_n} \qquad \forall n,$$

we have for large m,

$$|y_n|_m \prec e^{(m+d+C+2B)c_n} \ ,\forall n.$$

Setting $|y|_t \mathrel{\mathring{=}} \sum |y_n| e^{t\alpha_n}$ for any non negative real number t, we define $h(t) \mathrel{\mathring{=}} \limsup_n \frac{\log|y_n|_t}{c_n}$ for $t > 0$. This function is an increasing convex function on the positive real numbers. Moreover it follows from the analysis above that

$$h(t) \leq \left(\frac{N+D}{N-\beta_0}\right)t - \left(\frac{N+D}{N-\beta_0}\right)\beta_0 \text{ on the interval } [\beta_0, N] \text{ for every } N\in\mathbb{N} \text{ large .}$$

Hence $h(t) \leq t - \beta_0$ for $t >> \beta_0$.Now going back, since $\sup_{w\in D_\alpha}|f_n(w)| \prec |y_n|_{\alpha+2+2B}$,we see that for z with $\rho(z) = \alpha$,

$$\begin{aligned}\varphi(z) &= \limsup_{\xi\to z}\limsup_n \frac{\log|f_n(\xi)|}{c_n} \leq h(\alpha+2+2B+d) \leq \alpha+2+2B+d-\beta_0 \\ &= \rho(z) + Q, \text{ where } Q = Q(B,d,\beta_0) \in \mathbb{R}^+.\end{aligned}$$

In particular indeed Φ is a plurisubharmonic function on M and satisfies

$$\exists\, C_1 > 0 \text{ and } C_2 > 0 \text{ such that } \rho(z) - C_1 \leq \Phi(z) \leq \rho(z) + C_2 \qquad \text{on } M.$$

Hence Φ is an exhaustion and being a free envelope [**8**], is maximal outside a compact set. Observe also that the sublevel sets $\Omega_r \mathrel{\mathring{=}} \{z : \Phi(z) < r\}$ satisfy :

$$\exists\, \kappa > 0 \text{ such that } \overline{\Omega_r} \subseteq \Omega_{r+\kappa} \text{ for } r \text{ large enough.}$$

Now fix a decreasing sequence$\{u_n\}$ of continuous plurisubharmonic functions on M converging to Φ. Fix a compact set $\dot{K}$ an $\epsilon > 0$. Choose an r so large that$\left(\frac{r+\kappa-\frac{\epsilon}{2}}{r} - 1\right)\max_{\xi\in\dot{K}}\Phi(\xi) \leq \frac{\epsilon}{2}$. There exits an n_0 such that for $n \geq n_0$ on Ω_r $u_n \leq r+\kappa$ and $u_n|K \leq \frac{\epsilon}{2}$. Hence on Ω_r :

$$\frac{u_n - \frac{\epsilon}{2}}{r+\kappa-\frac{\epsilon}{2}} \leq \omega(K,\Omega_r) = \frac{1}{r}\Phi.$$

It follows that on $\dot{K}$, $0 \leq u_n - \Phi \leq \left(\frac{r+\kappa-\frac{\epsilon}{2}}{r} - 1\right)\max_{\xi\in\dot{K}}\Phi(\xi) + \frac{\epsilon}{2} \leq \epsilon$ for $n \geq n_0$. Hence the convergence is uniform on $\dot{K}$. It follows that Φ is continuous.

$\Leftarrow$: Let M be a Stein manifold with a plurisubharmonic exhaustion function that is maximal outside a compact set. We will first examine a linear topological properties that a plurisubharmonic exhaustion function imposes on the space of analytic functions on M.Let M be a Stein manifold and $\Phi : M \to [-\infty,\infty)$ a plurisubharmonic function that is an exhaustion. Let $D_t = (x\,|\Phi(x) < t)$ for $t\in\mathbb{R}$. Choose an increasing function ℓ so that for each $t\in\mathbb{R}$, $\overline{D_t} \subset D_{\ell(t)}$. We Fix a volume form d$\mu$ on M and using the notation of Lemma 1 $[A2]$, we let

$$U_t = \left\{f \in O(M) : \int_{D_t} |f|^2\, d\varepsilon \leq 1\right\}.$$

Fix positive numbers s_1, s_2, s such that $\ell(0) < s_1 \leq \ell(s_1) \leq s_2 \leq \ell(s_2) \leq s$ and $L \geq 0$. Let

$$\Phi_L(z) \mathrel{\mathring{=}} \begin{matrix} 0 & \text{if } \Phi(z) \leq 0 \\ \frac{L\Phi(z)}{s} & \text{otherwise} \end{matrix} \qquad .$$

Consider an analytic function $f \in U_{s_2}$. In view of Lemma 1 of $[A2]$, choose a decomposition of f on $W_+ \cap W_-$,as $f = f_+ - f_-$, where $f_\pm \in O(W_\pm)$, $W_+ = \left(\overline{D_{s_1}}\right)^c$, $W_- = D_{s_2}$, so that the estimates

$$\int_{W_\pm} \left|f_\pm\right|^2 e^{-\Phi_L} d\varepsilon \le K \int_{W_+\cap W_-} |f|^2 e^{-\Phi_L} d\mu$$

hold with $K = K(M, s_1, s_2, s, \Phi) > 0$. On the other hand, using again the notation of Lemma 1 of $[A]$

$$\int_{W_+\cap W_-} |f|^2 e^{-\Phi_L} d\mu \le C \int_{W_+\cap W_-} |f|^2 e^{-\Phi_L} d\varepsilon \le C e^{-\frac{Ls_1}{s}} \text{ for some } C > 0.$$

Hence

$$\int_{W_\pm} \left|f_\pm\right|^2 e^{-\Phi_L} d\varepsilon \le C_1 e^{-\frac{Ls_1}{s}} \qquad \text{for some } C_1 > 0.$$

Now ,

$$\int_{D_0} |f_-|^2 d\varepsilon = \int_{D_0} |f_-|^2 e^{-\Phi_L} d\varepsilon \le \int_{W_-} |f_-|^2 e^{-\Phi_L} d\varepsilon \le C_1 e^{-\frac{Ls_1}{s}}$$

and

$$\int_{W_-} |f - f_-|^2 d\varepsilon \le C_2 e^{\frac{L(s_2-s_1)}{s}}.$$

Set

$$G = \begin{pmatrix} f_+ & \text{on } W_+ \\ f - f_- & \text{on } W_- \end{pmatrix}.$$

Clearly $G \in O(M)$, and,

$$\int_{D_s} |G|^2 d\varepsilon \le \int_{D_s\cap W_+} |G|^2 d\varepsilon + \int_{W_=} |G|^2 d\varepsilon \le C_3 \left(e^{\frac{L(s-s_1)}{s}} + e^{\frac{L(s_2-s_1)}{s}}\right) \le C_4 e^{\frac{L(s-s_1)}{s}}.$$

Moreover

$$\int_{D_0} |G - f|^2 d\varepsilon = \int_{D_0} |f_-|^2 d\varepsilon \le C_1 e^{-\frac{Ls_1}{s}}.$$

Hence we obtain:

$$U_{s_2} \subseteq C e^{-\frac{Ls_1}{s}} U_0 + C e^{\frac{L(s-s_1)}{s}} U_s$$

for some constant $C > 0$ which does not depend upon L.

Set $t \circeq 1 - \frac{s_1}{s}$, and $r = e^{L(1-t)-\log C}$. Varying the parameter L, a short computation yields

$$\exists\, C > 0 \text{ such that:} \qquad U_{s_2} \subseteq \frac{1}{r} U_0 + C r^{\frac{t}{1-t}} U_s \text{ for all } r \in [1, \infty).$$

Since the above inclusion obviously holds for $0 < r \le 1$, and writing the value of t we have:

$$\exists\, D > 0 \text{ such that:} \qquad U_{s_2} \subseteq \frac{D}{r} U_0 + \frac{r^{\frac{s}{s_1}}}{r} U_s \text{ for all } r \in (0, \infty).$$

This is the conditon Ω of Vogt and Wagner [**32**] In terms of the "dual norms" this condition can we expressed as :

$$\exists\, C > 0 \text{ such that } \|x^*\|_{-s_2} \le C \left(\|x^*\|_{-0}\right)^{1-\frac{s_1}{s_2}} \left(\|x^*\|_{-s}\right)^{\frac{s_1}{s_2}} , \ \forall\, x^* \in O(M)^*,$$

where $\|x^*\|_{-t} \circeq \sup_{y\in U_t} |x^*(y)|$ (See [**32**]). We collect our findings in:

Proposition: Let M be a Stein manifold and Φ a plurisubharmonic function on M that is proper, i.e. $D_t \doteq \{z \mid \Phi(z) < t\} \subset\subset M \ ,\forall t \in \mathbb{R}$. If we have

$$\overline{D_{s_0}} \subseteq D_{s_1} \subseteq \overline{D_{s_1}} \subseteq D_{s_2} \subseteq \overline{D_{s_2}} \subseteq D_s$$

for some indexes $s_0 < s_1 < s_2 < s$, then the Fréchet space $O(M)$ has the following $\Omega-$ condition:

$$\exists C > 0 \ : \|x^*\|_{-s_2} \leq C\left(\|x^*\|_{-s_0}\right)^{\frac{s-s_1}{s-s_0}} \left(\|x^*\|_{-s}\right)^{\frac{s_1-s_0}{s-s_0}}, \quad \forall x^* \in O(M)^*.$$

Now we return to the proof of our theorem. Lets fix a continuous proper plurisubharmonic function Φ on M that is maximal outside a compact set. We can arrange things so that Φ is maximal outside a compact subset of D_0, where as usual $D_t = \{x \mid \Phi(x) < t\}$. Let us put on $O(M)$ the grading $\|f\|_n = \left(\int_{D_{n-\frac{1}{n}}} |f|^2 \, d\varepsilon\right)^{\frac{1}{2}}$, $n = 1, 2, ...$ In view of the proposition above we have an $\Omega-$ condition of type:

$$\exists C_n > 0 \ : \frac{\|x^*\|_{-n}}{\|x^*\|_{-(n+1)}} \leq C_n \left(\frac{\|x^*\|_{-(n-1)}}{\|x^*\|_{-(n+1)}}\right)^{\frac{s-s_1}{s-s_0}}, \quad \forall x^* \in O(M)^* \ \ \forall n = 2, 3....$$

where $s = n + 1 - \frac{1}{n+1}$, $s_0 = n - 1 - \frac{1}{n-1}$ and $n - 1 - \frac{1}{n+1} < s_1 < n - \frac{1}{n}$ is chosen so that $\frac{s-s_1}{s-s_0} \leq \frac{1}{2}$. With this choose of s_1 we obtain

$$\forall n \ \exists C_n > 0 \ : \|x^*\|_{-n} \leq C_n \left(\|x^*\|_{-(n+1)}\right)^{\frac{1}{2}} \left(\|x^*\|_{-(n-1)}\right)^{\frac{1}{2}}, \quad \forall x^* \in O(M)^*$$

In the terminology of [**33**], $O(M)$ with the grading $\|f\|_n = \left(\int_{D_{n-\frac{1}{n}}} |f|^2 \, d\varepsilon\right)^{\frac{1}{2}}$, $n = 1, 2....$ is an $\Omega-$space in *standard* form. On the other hand $O(M)$ with the grading $|f|_n = \sup_{D_n} |f|$, $n = 0, 1, 2...$,satisfies

$$\forall n = 1, 2, \quad \exists C_n > 0 \quad |f|_n^2 \leq C_n \, |f|_{n+1} \, |f|_n$$

in view of the maximality of Φ. In the terminology of [**33**] ,$O(M)$ with the grading $|f|_n = \sup_{D_n} |f|$, $n = 0, 1, 2..$ is a $DN-$ space in standard form. Moreover for every $n = 1, 2...$, there is a $K_n > 0$,such that $\|f\|_n \leq |f|_n$ and $|f|_n \leq K_n \|f\|_{n+2}$. Now all the requirements of 2.3 Theorem of [**33**] are satisfied with $A = I$, so $O(M)$ is tamely isomorphic to an infinite type power space. This finishes the proof of the theorem. □

Now let X be a Stein manifold with a continuous plurisubharmonic exhaustion function Φ that is maximal off $K_0 = (z : \Phi(z) \leq 0)$. We will choose a grading $(||*||_n)_n$ of $O(X)$ so that the Hilbert spaces $H_n \doteq \overline{\left(O(X), ^{||*||_n}\right)}\ _{n=0}^{\infty}$ satisfy:

a) The tuple (H_0, H_k) is admissible for the pair (K_0, D_k)in the sense of Zaharyuta [**38**], where $D_k = (z : \Phi(z) < k)$, $k \epsilon \mathbb{N}$

b) The theorem above is valid i.e. there is an infinite type power series space $\Lambda_\infty(\alpha)$ so that $\left(O(X),^{||*||_n}\right)$ is tamely isomorphic to $\Lambda_\infty(\alpha)$.

We will only use a special property of admissible pairs, so we will just refer the reader to [**38**] for the definition and a detailed discussion of this notion. However we should mention that for a given Stein manifold with a continuous exhaustion function there is a canonical way of getting admissible hilbertian norms [**38**],[**39**] and in the case of a special exhaustion function, the existence of an infinite type power series space satisfying the required property for this choice of generating norms follows from the proof the theorem given above. In what follows, we will denote the corresponding graded space by $\left(O(X),^{\Phi}\right)$.

Hence the theorem above associates to every special plurisubharmonic continuous exhaustion function Φ on a $S^*-parabolic$ Stein manifold X, an exponent sequence $(\alpha_m)_m$ such that the spaces $\left(O(X),^{\Phi}\right)$ and $\Lambda_\infty(\alpha_m)$ are tamely isomorphic. It might be of interest to examine the exponent sequences $(\alpha_m)_{m=0}^{\infty}$ obtained in this way and see how they depend upon the special exhaustion function Φ . Since $O(X)$, for a parabolic Stein manifold X of dimension n, is isomorphic to $\Lambda_\infty\left(m^{\frac{1}{n}}\right)$, regardless of the special exhaustion function we have:

$$\exists C>0 : \frac{1}{C} \leq \liminf_m \frac{\alpha_m}{m^{\frac{1}{n}}} \leq \limsup_m \frac{\alpha_m}{m^{\frac{1}{n}}} \leq C$$

for all such obtained exponent sequences $(\alpha_m)_{m=0}^{\infty}$. To proceed further we need the notion of a Kolmogorov diameter. For a vector space X, let us denote the collection of all subspaces of $Y \subset X$ with $\dim Y \leq m$, by X_m.

Definition 5. *Let $\left(\mathbb{X},^{|*|_k}\right)$ be a graded Fréchet space with an increasing sequence of seminorms. Let $U_i = (x\epsilon\mathbb{X} : |x|_i \leq 1), i = 1, 2....$ The m^{th} diameter of U_i with respect to U_j, $i < j$, is defined by*

$$d_m(U_i, U_j) \doteq \inf\left(\lambda > 0 : \exists\ Y\epsilon\ \mathbb{X}_k \textit{ such that } U_i \subseteq \lambda U_j + Y\right).$$

Now fix a $S^*-parabolic$ Stein manifold X and suppose that $\left(O(X),^{\Phi}\right)$ and $\Lambda_\infty(\alpha_m)$ are tamely isomorphic under an isomorphism T. So there exits an $A>0$ such that,

$$\forall k\ \exists C>0 : ||T(x)||_k \leq C\,|x|_{k+A} \quad \text{and } C\,||T(x)||_{k+A} \geq |x|_k\ \ , \forall x\epsilon\Lambda_\infty(\alpha_n).$$

We will denote by U_i and V_i the unit balls corresponding to the i^{th} norms of $\left(O(X),^{\Phi}\right)$ and $\Lambda_\infty(\alpha_n)$ respectively.

Fix a $k >> l$ large and suppose

$$U_k \subseteq \lambda U_l + L,$$

for some $\lambda > 0$ some m-dimensional subspace of $O(X)$. Applying T^{-1} to both sides and using the tame continuity estimates we have:

$$\frac{1}{c}V_{k+A} \subseteq T^{-1}(U_k) \subseteq \lambda T^{-1}(U_l) + L' \subseteq \lambda C V_{l-A} + L', \qquad L' \doteq T^{-1}(L).$$

Hence

$$d_m(V_{k+A}, V_{l-A}) \leq C d_m(U_k, U_l)$$

for all m, where the constant depends only on indices k and l.

On the other hand that, arguing in a similar fashion, we also have

$$d_m(U_{k+A}, U_{l-A}) \leq C d_m(V_k, V_l)$$

for all m and for some constant $C > 0$ that depends only on indices k and l.

It is a standard fact that $d_m(V_k, V_l) = e^{(l-k)\alpha_m}$ for $k >> l$. On the other hand our requirement of admissibility of the norms $(||*||_k)_k$ gives , in view of a result of Nivoch-Poletsky-Zaharyuta (see, [**14**], [**39**]) the asymptotics

$$\lim_m \frac{-\ln d_m(U_k, U_l)}{m^{\frac{1}{n}}} = \frac{2\pi (n!)^{\frac{1}{n}}}{C\left(\overline{D_l}, D_k\right)}$$

where $D_s = (z : \Phi(z) < s)$ as above and $\forall s$, $k >> l$ and $C\left(\overline{D_l}, D_l\right)$ is the Bedford-Taylor capacity of the condenser $\left(\overline{D_l}, D_k\right)$.([**8**])

Putting all these things together we have:

$$\liminf_m \frac{\alpha_m}{m^{\frac{1}{n}}} \geq \lim_m \left[\frac{-\ln d_m(U_{k+A}, U_{l-A})}{m^{\frac{1}{n}}}\left(\frac{\ln C}{(k-l+2A) - \ln d_m(U_K, U_L)} + \frac{1}{(k-l+2A)}\right)\right] =$$

$$\frac{2\pi (n!)^{\frac{1}{n}}}{\left(C\left(\overline{D_{l-A}}, D_{k+A}\right)\right)^{\frac{1}{n}}} \cdot \frac{1}{(k-l+2A)}.$$

$$\limsup_m \frac{\alpha_m}{m^{\frac{1}{n}}} \leq \lim_m \left[\frac{-\ln d_m(U_{k+A}, U_{l-A})}{m^{\frac{1}{n}}}\left(\frac{\ln C}{(k-l) - \ln d_m(U_k, U_l)} + \frac{1}{(k-l+)}\right)\right] =$$

$$\frac{2\pi (n!)^{\frac{1}{n}}}{\left(C\left(\overline{D_{l-A}}, D_{k+A}\right)\right)^{\frac{1}{n}}} \cdot \frac{1}{(k-l)}.$$

On the other hand, since Φ is maximal off a compact set we can use the function

$$\rho(z) = \frac{\Phi - (l-A)}{(k+A) - (l-A)}$$

to compute the capacity of the condenser $\left(\overline{D_{l-A}}, D_{k+A}\right)$ for k and l large enough . To be precise, in our case we get [**8**]:

$$C\left(\overline{D_{l-A}}, D_{k+A}\right) = \frac{1}{(k-l+2A)^n} \int_X (dd^c\Phi)^n.$$

Taking the limit as k and l goes to infinity we get:

$$\lim_m \frac{\alpha_m}{m^{\frac{1}{n}}} = \int_X (dd^c\Phi)^n.$$

We collect our findings in the proposition below. As usual $||*||_K$ denote the sup norm on a given compact set K.

Proposition 2. *Let X be a S^*-parabolic Stein manifold of dimension n. Fix a plurisubharmonic exhaustion function Φ on X that is maximal outside a compact set. Then the exponent sequence $(\alpha_m)_n$ of the infinite type power series space associated to X by Theorem 2 above satisfies:*

$$\lim_m \frac{\alpha_m}{m^{\frac{1}{n}}} = \int_X (dd^c\Phi)^n.$$

COROLLARY 1. *A Stein manifold X of dimension n is S^*-parabolic if and only if there exits an exhaustion of X by compact holomorphically convex sets $(K_m)_m$ such that $\left(O\left(M\right),^{||*||_{K_m}}\right)$ and $\left(O\left(\mathbb{C}^n\right),^{||*||_{\Delta_m}}\right)$ are tamely isomorphic where Δ_m is the polydisc in $\mathbb{C}^n$ with radius $m = 1, 2, ...$*

.

4. Some Examples

In this section we will look at some ways of generating parabolic manifolds and give some examples.

An immediate class of parabolic manifolds can be obtained by considering Stein manifolds that admit a proper analytic surjections onto some $\mathbb{C}^n$. Affine algebraic manifolds belong to this class. Moreover such manifolds are S^*-parabolic [**29**].

Demailly [**10**] considered the manifolds X which admit a continuous plurisubharmonic exhaustion function with the property that,

$$\lim_{r\to\infty}\frac{\int_{B_r}\left(dd^c\varphi\right)^n}{\ln r} = 0, \tag{4}$$

where $B_r = \{\varphi\left(z\right) < \ln r\}$.

We note , that S^* –parabolic manifolds satisfy the condition (4) . In fact , if $\rho\left(z\right)$ is special exhaustion function ,then $\left(dd^c\rho\right)^n = 0$ off a compact $K \subset\subset X$ so $\int_{B_r}\left(dd^c\rho\right)^n = \int_K\left(dd^c\rho\right)^n = const\,,\ \ r \geqslant r_0$. Hence, (4) holds.

If the X has a continuous plurisubharmonic exhaustion function satisfying the condition (4), then every bounded above plurisubharmonic function on X is constant [**10**], so that this kind of manifolds are parabolic . In fact a more general result is also true.

THEOREM 3. **.** *If on a Stein manifold X there exist a plurisubharmonic (not necessary continuous) exhaustion function that satisfies the following condition:*

$$\lim_{r\to\infty}\inf\frac{\int_{B_r}\left(dd^c\varphi\right)^n}{[\ln r]^n} = 0, \tag{5}$$

then X is parabolic.

PROOF. Lets assume that X satisfies the condition (5), but X is not parabolic.We take a sequence $1 < r_1 < r_2 < ...., r_k \to \infty$, such, that

$$\lim_{r\to\infty}\frac{\int_{B_{r_k}}\left(dd^c\varphi\right)^n}{[\ln r_k]^n} = 0 \tag{6}$$

Without loss of generalization we can assume that the ball $B_1 = \{\varphi\left(z\right) < 0\} \neq \emptyset$. Then according the proposition 1 the P-measure $\omega^*\left(z, \overline{B_1}, B_{r_k}\right)$ decreases to $\omega^*\left(z, \overline{B_1}\right) \neq -1$ as $k \to \infty$. The function $\omega^*\left(z, \overline{B_1}\right)$ is maximal, that is $(dd^c\omega^*)^n = 0$, in $X\backslash B_1$ and is equal -1 on $\overline{B_1}$. Hence, by comparison principle of Bedford-Taylor ([**8**]) we have:

$$\int_{B_{r_k}}\left[\mathrm{dd^c}\omega^*\left(z, \overline{B_1}, B_{r_k}\right)\right]^{\mathrm{n}} = \int_{\overline{B_1}}\left[\mathrm{dd^c}\omega^*\left(z, \overline{B_1}, B_{r_k}\right)\right]^{\mathrm{n}} \geqslant \int_{\overline{B_1}}\left[\mathrm{dd^c}\omega^*\left(z, \overline{B_1}\right)\right]^{\mathrm{n}} = \alpha > 0\,.$$

However, if we apply again the comparison principle to $\omega^*\left(z,\overline{B_1},B_{r_k}\right)$ and $w(z)=\frac{\varphi(z)-\ln r_k}{\ln r_k}$, then

$$\frac{1}{(\mathrm{lnr_k})^n}\int_{B_{r_k}}[dd^c\varphi(z)]^n=\int_{B_{r_k}}[\mathrm{dd^c}w(z)]^{\mathrm{n}}\geqslant\int_{B_{r_k}}\left[\mathrm{dd^c}\omega^*\left(z,\overline{B_1},B_{r_k}\right)\right]^{\mathrm{n}}\geqslant\alpha>0\,.$$

This contradiction proves the theorem. □

Remark: Stoll [**29**] consider an analytic set, for which the solution of the equation

$$dd^c\omega_R\wedge\Psi=0\ ,\omega_R|_{\partial B_0}=-1\,,\,\omega_R|_{\partial B_R}=0\,,$$

has the parabolic property, that $\omega_R\to-1\ ,in\,R\nearrow\infty$, where Ψ is close, positive $(n-1\,,\,n-1)$ form. Atakhanov [**2**] called this kind of sets parabolic type and prove,that the sets which satisfies the (4) are this type. Moreover , he construct the Nevanlinna's equidistribution theory for holomorphic map $f:\mathrm{X}\to\mathrm{P}^m$. In particular, on this kind of sets theorems of Picard , Nevanlinna , Valiron on defect hyperplanes are true.

In the literature there exits quite a number of Liouville- type theorems for specific complex manifolds. However the property that every bounded analytic function reduces to a constant need not imply parabolicity as is well known to people working on capacity theory in Riemann surfaces. The simple example below example illustrates this point.

Example 1: We choose on complex plane $\mathbb{C}_{z_1}$ a subharmonic function u with the property that $\{u(z_1)=-\infty\}=\left\{0,1,\frac{1}{2},\frac{1}{3},...\right\}$. We let $w(z_1,z_2)=u(z_1)+\ln|z_2|$. Then $w\in psh\left(\mathbb{C}^2\right)$ and the component D of $\left\{(z_1,z_2)\in\mathbb{C}^2:w(z_1,z_2)<0\right\}$ containing the origin is pseudoconvex, and hence is a Stein manifold. Any bounded holomorphic function on it is constant by the Liouville's theorem. However, the plurisubharmonic function $w(z_1,z_2)\neq const$ and is bounded from above i.e. D is not parabolic.

Example 2: Now we consider an important class of Stein manifolds (analytic sets) with the Luoiville property, which were introduced by Sibony - Wong [**25**]. To describe these spaces we need to introduce some notation. For an n dimensional closed subvariety X of $\mathbb{C}_w^N$ let us denote by φ, the restriction of $\ln|w|$ on X. Denoting the intersection of the r ball in $\mathbb{C}^N$ with X by $B_r=\{z\in X:\varphi(z)<\ln r\}$ we can describe Sibony - Wong class as those X's so that $\sup_r\frac{vol(B_r)}{\ln r}<\infty$, where the projective volume, $vol(B_r)$, is equal to $\frac{H_{2n}(B_r)}{r^{2n}}$, H_{2n} –the Hausdorff measure ($\mathbb{R}^{2n}$ -volume) of B_r. Sibony - Wong showed that on such spaces any bounded holomorphic on function is constant. In the context when n=1, a special case of a result by Takegoshi [**30**] states that if

$$\limsup_r\frac{vol(B_r)}{g(r)}<\infty$$

where $g:\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing continuous function such that $\int_0^\infty\frac{dr}{g(r)}=\infty$, then every smooth subharmonic function on the open Riemann surface X reduces to a constant.

The proof of this proposition is based on the following estimation:

$$v(r)^2 \leq Cg(r)\frac{d}{dr}(v(r))$$

where $v(r) = \int_{B_r} du \wedge d^c u$ and $C > 0$ is a constant. We note that if u is an arbitrary subharmonic function we can approximate it by smooth subharmonic functions $u_j \downarrow u$ and since the corresponding v_j's converge to v we conclude that the above expression is also valid for arbitrary subharmonic functions and hence the proof given in [**30**] shows that such an X is parabolic.

On the other hand if $n > 1$, taking into account that $v(B_r) = \int_{B_r} (dd^c\varphi)^n$ by Wirtingers' theorem , we can deduce from Theorem 1 above that X is parabolic. Hence Sibony-Wong manifolds are parabolic. In connection with Problem 2 above it will be of interest to investigate S^* − paraboliticity of Sibony-Wong manifolds. Affine algebraic manifolds are among this class since their projective volume is finite. Moreover they are S^* − parabolic as we have already seen. On the other hand special exhaustion functions for S^* − parabolic Sibony-Wong manifolds other than the algebraic ones can not be asymptotically bigger than $\sigma(z) = \ln|z|$ restricted to X.

THEOREM 4. *Let $X \subset \mathbb{C}^N$ be a Stein manifold and $\rho(z)$ a special exhaustion function on it. If $\underline{\lim}\frac{\rho(z)}{\sigma(z)} \geqslant \alpha > 0$, then X is an affine-algebraic set in $\mathbb{C}^N$.*

PROOF. Taking $C\rho$ instead ρ, if it is necessary, we can assume that, there is some compact $K \subset\subset X$ such, that $\frac{\rho(z)}{\sigma(z)} \geqslant 1$, $z \in X \backslash K$. Let $\sup_K \rho(z) = r_0$. Then $B_r = \{z \in X : \rho(z) < \ln r\}$, $r > r_0$, is not empty and open. Hence, the closure $\overline{B_r}$ is not pluripolar. Therefore, the extremal Green function

$$V_\rho\left(z, \overline{B_r}\right) = \sup\{u(z) \in psh(X) : u|_{B_r} \leqslant 0, u(z) \leqslant C_u + \rho(z) \ \ \forall z \in X\}$$

is locally bounded on X (see [**38**]). In other side, since $\rho(z) \geqslant \sigma(z)$ outside of compact K, then

$$V\left(z, \overline{B_r}\right) \leqslant V_\rho\left(z, \overline{B_r}\right), \text{where} V\left(z, \overline{B_r}\right) = V_\sigma\left(w, \overline{B_r}\right)|_X,$$

$$V_\sigma\left(w, \overline{B_r}\right) = \sup\left\{u(w) \in psh\left(\mathbb{C}^N\right) : u|_{B_r} \leqslant 0, u(w) \leqslant C_u + \ln|w|\right\}$$

But the extremal function $V\left(z, \overline{B_r}\right)$ locally bounded on X if and only if X affine-algebraic [**18**]. This completes the proof. □

Remark: In connection with Problem 2 it is tempting to choose a suitable plurisubharmonic exhaustion function and try to construct a special exhaustion function using the P-harmonic measures corresponding to the balls determined by this exhaustion. For example for a Stein manifold X imbedded in $\mathbb{C}^N$ we can use $\sigma(z) = \ln|z|$ restricted to X, and consider σ - balls $B_r = \{z \in X : \sigma(z) < \ln r\}$. As above, we can assume that $0 \notin X$ and that $\sup_X \sigma(z) < 0$. Let $v_j(z) = 1 + \omega^*\left(z, \overline{B_1}, B_j\right)$, $j = 2, 3, ...$. Then $v_j|_{\overline{B_1}} = 0$, $v_j|_{\partial B_j} = 1$ and $(dd^c v_j)^n = 0$ in $B_j \backslash \overline{B_1}$. Moreover, $\overline{B}_{j-1} \subset B_j$ and $v_j(z) \geqslant \frac{\sigma(z)}{j}$, $z \in B_j \backslash \overline{B}_1$. Let $\alpha_j := \max v_j|_{\partial B_{j-1}} \geqslant \frac{j-1}{j}$. Then the quantities α_j satisfy the inequalities $\frac{j-1}{j} \leqslant \alpha_j < 1$, $j = 1, 2...$ Finally, we take $u_j(z) = \frac{v_j}{\alpha_2 \alpha_3 ... \alpha_j}$, $z \in B_j$. Then

$$u_j|_{\partial B_{j-1}} = \frac{1}{\alpha_2 \alpha_3 ... \alpha_{j-1}} \frac{v_j}{\alpha_j}|_{\partial B_{j-1}} \leq \frac{1}{\alpha_2 \alpha_3 ... \alpha_{j-1}} = \frac{1}{\alpha_2 \alpha_3 ... \alpha_{j-1}} v_{j-1}|_{\partial B_{j-1}} = u_{j-1}|_{\partial B_{j-1}}.$$

Therefore $u_j(z) \le u_{j-1}(z)$, $z \in B_{j-1}$ and for some neighborhood of any fixed point $z^0 \in X\backslash\overline{B}_1$ the sequence $\{u_j(z)\}$ is defined and is decreasing a for big enough $j > j_0(z^0)$. Since, $(dd^c u_j)^n = 0$ in $B_j\backslash\overline{B_1}$, the limit $\rho(z) = \lim_{j\to\infty} u_j(z)$ will be maximal outside the set compact $\overline{B_1}$. The question is how to manage things so that such an obtained ρ will be maximal. This depends, among other things, on speed of converge to zero of the sequence of P-measures $(v_j)_j$.

5. Complements of analytic multifunctions

In this section we will take up Problem 2 stated above in the class of parabolic manifolds obtained by looking at the complement in $\mathbb{C}^n$of a zero sets of an entire function. More generally let $A \subset \mathbb{C}^n$ be a closed pluripolar set whose complement is pseudoconvex. Such sets are sometimes called "analytic multifunctions" by some authors. These kind of sets are very important in approximation theory, in the continuation of holomorphic functions and in the description of polynomial convex hulls. and were studied by various authors ([**15**], [**13**], [**35**], [**26**], [**27**],[**9**], [**1**], [**22**] and others). These sets are removable for the class of bonded plurisubharmonic functions defined on their complements. Hence their complements are parabolic Stein manifolds. We would like to restate Problem 2 given above in this setting since we hope that it will be more tractable.

PROBLEM 3. *Is the* $M = \mathbb{C}^n\backslash A$ $S-parabolic$*?*

In classical case, n=l, every closed polar set is an analytic multifunction. As is well-known, if $K \subset\subset \overline{\mathbb{C}}$ is a closed polar set in the extended complex plane $\overline{\mathbb{C}}$, then there exist a $u(z) \in Subharmonic(\overline{\mathbb{C}}) \cap harmonic(\overline{\mathbb{C}}\backslash K)$ such that $u_K \equiv -\infty$ and $u(z) - \ln|z| \to 0$ as $z \to \infty$. One can use such functions to construct a special exhaustion function on $\mathbb{C}\backslash A$.To this end fix a $z^0 \notin K \doteq A \cup \{\infty\}$ an arbitrary but fixed point, then there exist $u(z) \in psh(\overline{\mathbb{C}}\backslash\{z^0\}) \cap h(\overline{\mathbb{C}}\backslash K) : u|_E \equiv -\infty$ and $u(z) \to +\infty$ in $z \to z^0$. Therefore, $\rho(z) = -u(z)$ is exhaustion for $M = \mathbb{C}\backslash A$, with one singular point z^0.

On the other hand if $A = \{p(z) = 0\} \subset \mathbb{C}^n$ is an algebraic set, then it is easy to see that the function $\rho(z) \doteq -\frac{1}{\deg p}\ln|p| + 2\ln|z|$ is a special exhaustion function for $\mathbb{C}^n\backslash A$ [**40**]

THEOREM 5. *Let* $A = \{F(z) = z_n^k + f_1('z)z_n^{k-1} + ... + f_k('z) = 0\}$ *- a Weierstrass polynomial (algebraiodal) set in* $\mathbb{C}^n$*, where* $f_j \in O(\mathbb{C}^{n-1})$ *- entire functions, j = 1,2,...,k,* $k \geqslant 1$*. Then* $M = \mathbb{C}^n\backslash A$ *is S*-parabolic manifold.*

PROOF. We put

$$\rho(z) = -\ln|F(z)| + \ln(|'z|^2 + |F(z) - 1|^2). \tag{14}$$

Then $\rho(z) = -\infty$ precisely on the finite set $Q = \{'z = 0, F('0, z_n) = 1\}$. ...... Moreover,$\rho$ is maximal, $(dd^c\rho)^n = 0$ and continuous outside of $\mathbb{Q}$ its finite logarithmic poles in Q. We will show , that $\rho(z)$ is exhaustion on $\mathbb{C}^n/\Gamma$, i.e.

$$\{\rho(z) < R\} \subset\subset \mathbb{C}^n\backslash\Gamma \ for\ every\ R \in \mathbb{R} \tag{15}$$

If $F(z) = 0$ then $\rho(z) = +\infty + \ln(|'z|^2 + 1) = +\infty$, so that $\rho|_A = +\infty$. (15) is clear, if all $f_j, j = 0, 1, ..., k$, are constant and we assume that, among at least one is not constant. Then $M_R = \max_{|'z| \leqslant R}\{|f_1('z)|, ..., |f_k('z)|\} \to \infty$. We have: for $|'z| = R \geqslant 1$ and $|z_n| \leqslant M_R^2$ the

$$\rho(z) = \ln \frac{|'z|^2 + |F(z) - 1|^2}{|F(z)|} \geqslant \ln \frac{|'z|^2 + |F(z) - 1|^2}{1 + |F(z) - 1|} \geqslant \ln \frac{|'z|^2 + |F(z) - 1|^2}{|'z| + |F(z) - 1|} \geqslant$$

$$\geqslant \ln \frac{|'z| + |F(z) - 1|}{2} \geqslant \ln \frac{R}{2}$$

On the other hand on $|'z| \leqslant R$ *and* $|z_n| = M_R^2$ we have:

$$\rho(z) = \ln \frac{|'z|^2 + |F(z) - 1|^2}{|F(z)|} \geq \ln \frac{(M_R^{2k} - M_R M_R^{2k-2} - ... - M_R - 1)^2}{M_R^{2k} + M_R M_R^{2k-2} + ... + M_R} =$$

$$= \ln M_R^{2k} (1 + \alpha_k),$$

where $\alpha_k \to 0$ *in* $R \to \infty$.I follows that, $\rho|_{\partial U_R} \to +\infty$ *in* $R \to \infty$, where $U_R = \{(|'z| \leqslant R, |z_n| \leqslant M_R^2)\}$. Let us now consider the level set $D_C = \{\rho(z) < C\}$, $C-$ constant. It is an open set and it contains the pole set Q. If is so big, that $U_R \supset Q$ *and* $\min\{\ln \frac{R}{2}, \ln M_R^{2k}(1 + \alpha_R)\} \geqslant C$, *then* $D_C \subset\subset U_R$, since D_C has no any component outside U_R because of maximality of ρ on $M \backslash U_R$ This completes the proof that ρ is an exhaustion function. □

Corollary 2. *The complement* , $\mathbb{C}^n/\Gamma$, *of the graph* $\Gamma = \{('z, z_n) \varepsilon \mathbb{C}^n : z_n = f('z)\}$ *of an entire function* f *is* S^**-parabolic*

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