The Bounded Memory Folk Theorem*

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Abstract

We show that the Folk Theorem holds for n-player discounted repeated game with bounded-memory pure strategies. Our result requires each player's payoff to be strictly above the pure minmax payoff but requires neither time-dependent strategies, nor public randomization, nor communication. The type of strategies we employ to establish our result turn out to have new features that may be important in understanding repeated interactions.

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1 Introduction

The extensive multiplicity of subgame perfect equilibrium (SPE) payoffs in repeated games, described by Folk Theorems of Fudenberg and Maskin (1986), is due to players' ability to condition their behavior arbitrarily on the past. Therefore, it is reasonable to expect, as suggested by Aumann (1981), that this multiplicity may be reduced by restricting players to use limited memory strategies.

In Barlo, Carmona, and Sabourian (2009) we show that this intuition, however, does not hold when the set of actions in the stage game of the repeated game is sufficiently "large" so that each payoff profile is not isolated. In such games we prove that the Folk Theorem with SPE as the solution concept (henceforth, we shall refer to such Folk Theorems by FT) continues to hold with one period memory strategies where at each date players' behavior depend only on the outcome of the game in the previous period. The large action space assumption is critical in establishing this results because it allows players to encode the entire history of the past into the previous period's actions.[1]

In the same study we show that when the action spaces are not "large", it is possible that no efficient payoff vector can be supported by a one period memory SPE strategy profile even if the discount factor is near one, validating the argument of Aumann (1981) with one period memory strategies and finite actions. Thence, the question is whether or not the multiplicity of equilibrium payoffs prevails with finite actions and limited memory (not necessarily restricted to be one period). More specifically, does the FT depend critically on being able to recall the history of play all the way back to the beginning?

In the current paper, we prove that the FT for discounted repeated game continues to hold with *time-independent* bounded memory pure strategies, even when the action sets are finite. Specifically we show that, when players are sufficiently patient, any strictly individually rational payoff vector can be approximately sustained by a pure subgame perfect equilibrium

[1] More formally, with rich action sets any equilibrium strategy vector in which each player strictly prefers not to deviate at every history, can be perturbed so that each player chooses different actions at different histories. With such distinct plays of the game, at each date the players can use the outcome of the previous period to coordinate their actions appropriately. Thus, the original equilibrium can be approximated by another that has one period recall.

strategy profile that at each stage recalls the outcomes of finite number of previous periods.[2] Moreover, we show that the bound on the number of periods that the player need to recall to establish the result is uniform in terms of the set of individually rational payoffs, and depends only on the desired degree of payoff approximation.

One issue in repeated game literature concerns the multiplicity of equilibrium payoffs. Another is about understanding the precise behavior that satisfy intertemporal incentives in repeated contexts. Our result is important not only because it shows that the FT does not depend on being able to recall the history of play all the way back to the beginning, but also because the kind of strategies/behaviour needed to ensure intertemporal incentives with limited memory turn out to have new features that may be significant in understanding repeated interactions.

There are many reasons why one might be interested in results with limited memory. First, there is the bounded rationality aspect in which players can only recall a finite amount of public information concerning the past. The results from psychological literature also indicate that people do not act on the entire history they observe and pay special attention to recent history. Second, in many institutional set-ups it is the convention to remove all the records after a certain number of years. Third, information that is not formally recorded is often conveyed by word of mouth or by short-lived players representing overlapping generations. Fourth, having access to past information can be costly and in equilibrium players may choose to recall a finite past. Finally, memory size may have implications for robustness of equilibria. For example, Mailath and Morris (2002) and Mailath and Morris (2006) show that private monitoring perturbations of public monitoring equilibria are robust if the equilibria have bounded recall.

To appreciate the difficulties and the novel behavioral features needed in establishing a FT with bounded memory, consider a typical "simple" strategy SPE profile used in proving the standard FT in n-player repeated game. Such a strategy profile is described by $n+1$ infinite paths $\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)}$ consisting of the equilibrium path of play $\pi^{(0)}$ and a punishment path $\pi^{(i)}$ for each player i. The strategies are such that game begins with $\pi^{(0)}$ until some player

[2] We obtain the result without introducing any randomization or any external communication device that allows the players to communicate.

deviates singly from $\pi^{(0)}$. At any stage, a single deviation by a player from any ongoing path triggers the punishment path for that player; otherwise the game continues with the ongoing path.

In the first instance, it may seem that the problem of implementing such a simple profile with bounded memory is trivial if the memory size M is sufficiently large. In particular, if each of the $n+1$ paths has a finite cycle then each can be distinguished and implemented as long as M is sufficiently large. Even when the paths are not finite one can approximate the payoff corresponding to each path by a cyclical path. Therefore finite memory should be sufficient to implement the paths approximately. But this is not enough. Strategies must also be such that after observing the outcomes of the previous M periods the following two critical properties hold: First, single player deviations can be detected and second, the identity of the deviator is revealed. If either of the above two properties were not to hold, there may be incentives for some player to deviate and manipulate the path of future play.[3]

With 1-period memory it is easy to see how such simple strategies may violate the above properties. For example consider any two action profiles a and b respectively belonging to two paths $\pi^{(i)}$ and $\pi^{(j)}$, for some i and j. Then the first property is violated if for some player k, $a_k \neq b_k$ and $a_{-k} = b_{-k}$. This is because when $(b_k, a_{-k}) = b$ is observed it is not clear if k has just deviated from $\pi^{(i)}$ and the punishment for k needs to be triggered or if the path $\pi^{(j)}$ is being followed and no deviation has occurred. Similarly, the second property is violated if, for a pair of players k and l, $a_l \neq b_l$, $a_k \neq b_k$ and $a_{-l,k} = b_{-l,k}$. This is because in this case when $(b_k, a_l, a_{-k,l}) = (b_k, a_l, b_{-k,l})$ is observed, it is not clear which of the two players k or l has deviated.

Does increasing the memory size help with ensuring that the above two properties hold? The next two examples show that these difficulties cannot be solved so easily even with large, but finite, memory.

Example 1: Consider a repeated Prisoners' Dilemma in which at every date each player can either cooperate C or defect D. Suppose that the players are sufficiently patient and we want to implement a cycle path $\pi^{(0)} = \{\pi^t\}_{t=1}^{\infty}$ consisting of playing $((C,D),(D,C))$

[3] In Barlo, Carmona, and Sabourian (2009) we refer to simple strategies that satisfy the above two properties as "confusion-proof".

repeatedly. Assume that such a path yields for each player an average payoff strictly higher than the minmax payoff generated from playing (D, D).[4] The simple strategy that plays $\pi^{(0)}$ on the equilibrium path and plays (D, D) forever for any history inconsistent with the equilibrium path, is subgame perfect with unbounded memory. However, this strategy is not subgame perfect if players can remember at most an arbitrary but finite number M of past periods. To see this, consider any history with its last M entries (henceforth called the M-tail) equal to $(a^1, \pi^2, \dots, \pi^M)$, for any $a^1 \neq \pi^1$. Then the simple strategy prescribes playing D for both players forever in the continuation game. But if $\pi^M = (D, C)$ then player 1 has the incentive to deviate. This is because if player 1 plays C instead of D at this history, the play returns to the equilibrium path in the next period, as $(\pi^2, \dots, \pi^M, \pi^{M+1})$ would be recalled. In the case when $\pi^M = (C, D)$, by an analogous reasoning, player 2 has an incentive to deviate.

One way to overcome this difficulty may be to allow the play to continue along the equilibrium path even at some histories that are inconsistent with the equilibrium path. However, this alone is not sufficient. For example, consider a strategy profile that is otherwise identical to the above simple strategy profile except that it plays π^{M+1} at any history whose M-tail equals $(a^1, \pi^2, \dots, \pi^M)$ for any a^1. In this case, if $\pi^M = (D, C)$ then player 2 will find it profitable to deviate from D to C at any history with its M-tail equal to $(a^2, a^1, \pi^2, \dots, \pi^{M-1})$, for any $a^1 \neq \pi^1$ and any a_2. By doing so, he produces a history with its M-tail equal to $(a_1, \pi_2, \dots, \pi_M)$ and brings the play back to the equilibrium path.[5] Thus, if we continue to change the strategy by allowing the play to return to the equilibrium path at these problematic histories, an inductive argument would imply that the play must be the equilibrium path after any possible history, a requirement clearly incompatible with subgame perfection.

The above example shows that increasing the memory size by itself does not guarantee that the players can identify if there has been a deviation. The next example shows that the problem of detecting the identify of the deviator can also not be easily resolved by having large but finite memory.

[4] Barlo and Carmona (2007), a predecessor to the current paper, consider the repeated Prisoners' Dilemma with bounded memory. Example 1 is from this paper, which in turn attributes it to an anonymous referee.

[5] If $\pi^M = (C, D)$, player 1 has an incentive to deviate when $(a^2, a^1, \pi^2, \dots, \pi^{M-1})$ is recalled.

Example 2: In this example there are three players, each player $i = 1, 2, 3$ has three actions α_i, β_i and γ_i in the stage game and the players discount the future by an arbitrarily small amount. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and suppose that the stage payoff u_i for each i is such that the action profile that minmaxes i is $m^i = (\beta_i, \alpha_{-i})$. Also, suppose that, for each $i = 1, 2, 3$, m^i is a Nash equilibrium of the stage game and $u_i(m^i) < u_i(m^j)$ for all $j = 0, \ldots, 3$, $j \neq i$, where $m^0 = (\gamma_1, \gamma_2, \gamma_3)$.[6] Then with no memory restriction the simple strategy profile defined by an equilibrium path $\pi^{(0)} = \{m^0, m^0, \ldots\}$ and a punishment path $\pi^{(i)} = \{m^i, m^i, \ldots\}$ for each $i = 1, 2, 3$, implements m^0 as a SPE.

Such simple strategy profile has the two features that when a deviator is identified the punishment path for that player is implemented and that after any history the continuation path corresponds to one of the four paths $\pi^{(0)}, \ldots, \pi^{(3)}$. With finite memory, irrespective of how large the memory is, implementing m^0 as a SPE with strategies that have these two features is no longer feasible. To see this fix the memory to be M and any strategy profile f with these features. By the second feature, at any history with its M-tail equal to $(\alpha, \alpha, \ldots \alpha)$ the continuation strategy prescribes playing a path $\pi^{(j)}$, for some $j = 0, \ldots, 3$. Consider any player $i \neq j$. Since f must play m^0 initially, by the first feature, if i deviates at date 1 by playing $a_i \neq m_i^0$ then f induces m^i at date 2. Also, if player i deviates again from m^i at date 2 by playing α_i instead of $m_i^i = \beta_i$, α will be observed and f would prescribe playing m^i again. Further such deviations by i induces α again and thus, by induction, f also specifies playing m^i after a history consisting of (a_i, m_{-i}^0) followed by α played $(M-1)$ times. But then at such a history player i can profitably deviate by playing α_i and inducing a history consisting of M consecutive α's. This is because his average continuation payoff from the deviation would be almost $u_i(m^j)$, whereas by not deviating he obtains $u_i(m^i)$.

The problem in the above example is that α could be the result of single deviation by 1 from m^1, 2 from m^2 or 3 from m^3. Therefore, the history consisting of α played M times can be induced by any player through a sequence of deviations and cannot be attributed to deviations by any particular player. Hence, given that $u_i(m^i) < u_i(m^j)$ for all $j = 0, \ldots, 3$, $j \neq i$, there must be some profitable opportunities for some player.

[6]It is easy to construct an example with payoffs satisfying these properties.

The problems of detecting the latest deviation and the identity of the deviator clearly do not arise with unbounded memory because, for any history, one can use induction starting from the first period of the history to find the latest deviation. With bounded memory, such inductive reasoning, by definition, is not feasible. Therefore, to deal with these problems with limited memory one needs to ensure that each of the paths that the candidate strategy profile prescribes at each history are sufficiently distinct. This can be done if each action profile in each path is distinct from those in other paths by at least *three* components (e.g. Sabourian (1998)).[7] In fact, the richness assumption in Barlo, Carmona, and Sabourian (2009) allows one to prove a Folk Theorem with bounded memory precisely because with rich action spaces, one can construct such paths at the cost of perturbing all the payoffs by a small amount. With finite action spaces, such an approach to making each path sufficiently distinct is clearly not possible.

Nevertheless, in this paper we show that the objective of making each paths sufficiently distinct, so that deviations and identity of deviators can be detected, can be achieved by ensuring that each path contains specific sequences of actions, henceforth referred to as signal sequences. Each of these signal sequences is carefully designed and appears infinitely often along its respective path so that once any of them is observed the paths or deviations are identified and the players know how to play the continuation game without the need to know the entire past history. Effectively, signal sequences can be thought of as a set of rituals that have to be played every so often so that the players can coordinate their future play in an appropriate way to preserve the incentives (punishment or reward).

Introduction of such signal sequences, however, generates a new problem: we need to ensure that it is in the interest of the players to play these sequences, i.e. the strategies with such signals must still constitute a SPE. This makes the construction of signal sequences and punishment paths needed to induce them rather intricate and complicated (particularly for games with more than 2 players). Nevertheless, we show that this is a feasible task and, thereby, demonstrate that the FT remains valid with bounded memory.

Independently and at the same time Mailath and Olszewski (2009) have considered the

[7] Assuming such distinctness, Sabourian (1998) provides a characterization for the set of SPE outcomes of repeated games for the case of no discounting and finite number of pure actions.

problem of establishing the FT with bounded memory. Their result is however a special case of ours. Specifically, they show that the FT holds with *time-dependent* bounded memory in games with *more than two players*. Our result is more general than theirs because we do not require players to condition their strategies on calendar time and because our FT also holds for two player.[8] The former is important because calendar time is unbounded and one of the reasons for limiting the analysis to bounded memory is to bound the set of objects on which the players can condition their behavior.

The main motivation of Mailath and Olszewski (2009) is however different from us as they are primary interested in demonstrating that the perfect monitoring FT is behaviorally robust to almost-perfect almost-public private monitoring. As shown by Mailath and Morris (2002) and Mailath and Morris (2006), time-dependent bounded memory, however, is all that is required for this. Therefore, their result is sufficient to establish that the above robustness exercise is valid for games with more than two players.

We are on the other hand interested to the robustness of the FT to bounds on the set of objects on which the players can condition their behavior (a bounded rationality exercise). With this in mind, we did not want take the time-dependence route as it allows for conditioning on an object that is unbounded (infinite "complexity").

In contrast to our results, in some related literature bounds on the memory do result in significant reduction in the set of equilibria in repeated set-ups. However, these results require additional assumption(s) beyond bounded memory. For example, Liu and Skrzypacz (2010) show that in a dynamic model with one long-lived player facing a sequence of short-lived players and complete information, bounds on the memory can have a dramatical impact on the equilibrium set (only Nash equilibria of the stage game are consistent with limited memory).[9] Their results, however, is critically dependent on the players being able to condition their behavior *only* on past actions of the other players (strategies are reactive).

Cole and Kocherlakota (2005) consider the repeated Prisoners' Dilemma with imperfect

[8]The proof of the FT with two players in our first version of the paper were rather cumbersome. We have simplified the proof as a result of conversations with George Mailath and Wojciech Olszwski. We would like to thank them for these very useful conversations.

[9]They also show that, with bounded memory, equilibria in games with complete and incomplete information are strikingly different.

public monitoring and finite memory. They show that for some set of parameters defection every period is the only strongly symmetric public perfect equilibrium with bounded memory (regardless of the discount factor), whereas the set strongly symmetric public perfect strategies with unbounded recall is strictly larger. The example considered by Cole and Kocherlakota (2005) does not satisfy the identifiability condition used in Fudenberg, Levine, and Maskin (1994) to establish their Folk Theorems for repeated games with imperfect monitoring. By strengthening those identifiability conditions and by allowing asymmetric strategies, Hörner and Olszewski (2009) obtain a perfect Folk Theorem with bounded memory strategies for games with (public or private but almost public) imperfect monitoring and finite action and outcome spaces. Their result, however, requires a rich set of public signals and displays a trade-off between the discount factor and the length of the memory.[10]

2 Notation and Definitions

The stage game: A *normal form game* G is defined by $G = \left(N, (A_i)_{i\in N}, (u_i)_{i\in N}\right)$, where $N = \{1, \ldots, n\}$ is a finite set of players, A_i is the set of player i's actions and $u_i : \prod_{i\in N} A_i \to \mathbb{R}$ is player i's payoff function. We assume that A_i is finite and $|A_i| \geq 2$ for all $i \in N$.

Let $A = \prod_{i\in N} A_i$ and $A_{-i} = \prod_{j\neq i} A_i$. We enumerate the set of action profiles by $A = \{a^1, \ldots, a^r\}$ with $r = |A|$.

For any $i \in N$ denote respectively the *minmax payoff* and a *minmax profile* for player i by $v_i = \min_{a_{-i}\in A_{-i}} \max_{a_i\in A_i} u_i(a_i, a_{-i})$ and $m^i \in \arg\min_{a_{-i}\in A_{-i}} \max_{a_i\in A_i} u_i(a_i, a_{-i})$. If G is a 2-player game, a *mutual minmax profile* is $\bar{m} = (m_1^2, m_2^1) \in A$. We shall denote the maximum payoff in absolute value some player can obtain by $B = \max_{i\in N} \max_{a\in A} |u_i(a)|$.

Let $\mathcal{U} = \{u \in \text{co}\,(u(A)) : u_i \geq v_i \text{ for all } i \in N\}$ denote the set of *individually rational payoffs* and $\mathcal{U}^0 = \{u \in \text{co}(u(A) : u_i > v_i \text{ for all } i \in N\}$ denote the set of *strictly individually rational payoffs*. The game G is *full-dimensional* if the interior of $\mathcal{U}$ in $\mathbb{R}^n$ is nonempty.

The repeated game: The infinitely repeated game consists of an infinite sequence of repetitions of G. We denote the action of any player i in the repeated game at any date

[10] Other works on repeated games with bounded (recall) memory include Kalai and Stanford (1988), Lehrer (1988), Aumann and Sorin (1989), Lehrer (1994), Neyman and Okada (1999), Bhaskar and Vega-Redondo (2002), and Dutta and Siconolfi (2010).

$t = 1, 2, 3, \ldots$ by $a_i^t \in A_i$. Also, let $a^t = (a_1^t, \ldots, a_n^t)$ be the profile of choices at t.

For $t \geq 1$, a t-stage history is a sequence $h = (a^1, \ldots, a^t) \in A^t$ (the t-fold Cartesian product of A). The set of all t-stage histories is denoted by $H_t = A^t$. We represent the initial (empty) history by H_0. The set of all histories is defined by $H = \bigcup_{t \in \mathbb{N}_0} H_t$.[11] We also denote the length of any history $h \in H$ by $\ell(h)$.

Let $\Pi = A \times A \times \cdots = A^\infty$ be the set of (infinite) outcome paths in the repeated game. For any $a \in A$ and $k \in \mathbb{N}$, we denote a finite path consisting of a being played k times consecutively by $(a; k)$. Also, for two positive length histories $h = (a^1, \ldots, a^{\ell(h)})$ and $\bar{h} = (\bar{a}^1, \ldots, \bar{a}^{\ell(\bar{h})})$ in H we define the *concatenation of* h *and* $\bar{h}$ by $h \cdot \bar{h} = (a^1, \ldots, a^{\ell(h)}, \bar{a}^1, \ldots, \bar{a}^{\ell(\bar{h})})$.

For any non-empty history $h = (a^1, \ldots, a^{\ell(h)}) \in H$ and any integer $0 < m \leq \ell(h)$, define the *m-tail of* h by $T^m(h) = (a^{\ell(h)-m+1}, \ldots, a^{\ell(h)})$. We also adopt the convention that $T^0(h)$ is the empty history. For all $h \in H$ and all $k' \in \mathbb{N}$ with $k' \leq \ell(h)$, let $B^{k'}(h) = (a^1, \ldots, a^{\ell(h)-k'})$ denote the history obtained from h by removing the last k' actions.

For all $i \in N$, a *strategy* for player i is a function $f_i : H \to A_i$ mapping histories into actions. The set of player i's strategies is denoted by F_i, and $F = \prod_{i \in N} F_i$ with a typical element $f = (f_1, \ldots, f_n)$. Given a strategy $f_i \in F_i$ and a history $h \in H$ we denote the *strategy induced by* f_i *at* h by $f_i|h$. Thus, $(f_i|h)(\bar{h}) = f_i(h \cdot \bar{h})$ for every $\bar{h} \in H$. We will use $(f|h)$ to denote $(f_1|h, \ldots, f_n|h)$ for every $f = (f_1, \ldots, f_n) \in F$ and $h \in H$.

Any strategy profile $f \in F$ induces an outcome path $\pi(f) = \{\pi^1(f), \pi^2(f), \ldots\} \in \Pi$ where $\pi^1(f) = f(H_0)$ and $\pi^t(f) = f(\pi^1(f), \ldots, \pi^{t-1}(f))$ for any $t > 1$.

We assume that all players discount the future returns by a common discount factor $\delta \in (0, 1)$. Thus, the payoff in the repeated game is given by $U_i(f, \delta) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(\pi^t(f))$. For any $\pi \in \Pi$, $t \in \mathbb{N}$, and $i \in N$, let $V_i^t(\pi, \delta) = (1-\delta) \sum_{r=t}^{\infty} \delta^{r-t} u^i(\pi^r)$ be the continuation payoff of player i at date t if the outcome path π is played. For simplicity, we write $V_i(\pi, \delta)$ instead of $V_i^1(\pi, \delta)$. Also, when the meaning is clear we shall not explicitly mention δ and refer to $U_i(f, \delta)$, $V_i^t(\pi, \delta)$ and $V_i(\pi, \delta)$ by $U_i(f)$, $V_i^t(\pi)$ and $V_i(\pi)$ respectively.

We denote the repeated game described above for discount factor $\delta \in (0, 1)$ by $G^\infty(\delta)$. A strategy vector $f \in F$ is a *Nash equilibrium* of $G^\infty(\delta)$ if $U_i(f) \geq U_i(\hat{f}_i, f_{-i})$ for all $i \in N$ and $\hat{f}_i \in F_i$. Also, $f \in F$ is a *SPE* of $G^\infty(\delta)$ if $f|h$ is a Nash equilibrium for all $h \in H$.

[11] We use $\mathbb{N}_0$ and $\mathbb{N}$ to denote, respectively, the set of non-negative and positive integers.

For all $M \in \mathbb{N}$, we say that $f \in F$ is a M*-memory strategy* if $f(h) = f(\bar{h})$ for all $h, \bar{h} \in H$ such that $T^M(h) = T^M(\bar{h})$. A strategy profile f is a M*-memory SPE* if f is a M-memory strategy and a SPE.

3 The bounded memory Folk Theorem

Our main result is the following.

Theorem 1 *Let G be a n-player game and suppose that either G is full-dimensional or $n = 2$ and $\mathcal{U}^0 \neq \emptyset$. Then, for all $\varepsilon > 0$, there exists $M \in \mathbb{N}$ and $\delta^* \in (0, 1)$ such that for all $u \in \mathcal{U}$ and $\delta \geq \delta^*$, there exists a M-memory SPE f of $G^\infty(\delta)$ such that $\|U(f, \delta) - u\| < \varepsilon$.*

Restricting strategies to have bounded memory immediately implies that, after every history, the path induced by any bounded memory strategy must eventually enter a cycle. Thus, with bounded memory the set of individually rational payoffs can at best be implemented approximately. As a result, in the above FT the size of the memory M needed depends on the degree of approximation ε. However, note that M is independent of the individual rational payoff u that is being implemented and the discount factor δ.

We next provide an intuition for the proof of Theorem 1. The proof itself can be found in the Appendix.

3.1 Intuition for the 2-player case

In 2-player games, the standard FT construction for sustaining an individually rational payoff vector u as a SPE is a simple strategy profile that has the following structure: (i) it has an equilibrium path π that induces u and (ii) a common punishment path that starts with a punishment phase consisting of playing the mutual minmax $\bar{m}$ for some finite number of time T and then plays the equilibrium path π.

Our FT construction with bounded memory involves modifying the above standard construction to deal with the issues that bounded memory raises. First, as explained above, with bounded memory the set of individually rational payoffs can at best be implemented approximately.

Second, as illustrated by the examples in the Introduction, the identification of the on-going path and whether or not there has been a single player deviation can be difficult with bounded memory. This implies that the equilibrium path and the punishment path need to be chosen carefully so that the above problems can be overcome when players observe only a fixed window of past outcomes. This issue will be dealt with by designing the equilibrium cycle appropriately. The key idea is to insert a signalling sequence of actions *regularly* in the equilibrium path. The purpose of this signalling sequence is that, once players have observed it, they can infer that the play is in the equilibrium path and can, therefore, ignore the part of the history that has occurred before. For such identification to be both possible and immune to single player deviations, the following must hold: the signalling sequence of actions must appear infinitely often on the equilibrium path, it should not appear anywhere else and no single player deviation, either from the equilibrium path or from the punishment path, should be able to escape the punishment phase.

Specifically, our construction of the bounded memory equilibrium strategy is as follows. Since the discount factor is close to 1, for any path changing the order by which actions are played has an insignificant impact on the payoffs the players receive. Therefore, to approximately implement the desired payoff profile u, all that matters is that each action profile is played a fraction of times sufficiently close to its coefficient in the convex combination of stage game payoffs yielding u. This irrelevance of the order allows us to define the equilibrium path $\pi = \{\pi^1, \pi^2, \ldots\}$ as the repetition of the cycle $((a^1; p^1), \ldots, (a^r; p^r))$,[12] where, (i) a^1 is chosen to be such that it differs from the mutual minmax profile $\bar{m}$ in every coordinate (i.e. $a_i^1 \neq \bar{m}_i$ for all i), a^2 is set to equal $\bar{m}$ and all remaining actions are ordered arbitrarily; (ii) $p^1 \geq 2$ and $p^2 \geq 1$; and (iii) $p^j / \sum_{l=1}^r p^l$ is close to the coefficient of $u(a^j)$ in the convex combination yielding u, for all $j = 1, \ldots, r$.

The above equilibrium path is then implemented by the following strategy profile. It first begins at any date $t < p^1$ by playing the equilibrium action a^1 if no deviation from a^1 has occurred. It continues with playing the equilibrium path after any history of length greater or equal to p^1 if the M-tail of the history either contains p^1 consecutive occurrences of a^1 followed by the subsequent actions of the equilibrium path (if any) or, for some $t < p^1$, consists of $M - t$ consecutive occurrences of $\bar{m}$ followed by the first t actions of the equilibrium

[12] Recall that $A = \{a^1, \ldots, a^r\}$ and $(a; k)$ denotes the history consisting of the play of action profile a for k consecutive periods.

cycle. At any other history, the strategy profile prescribes playing $\bar{m}$.[13,14]

In this construction the sequence $(a^1; p^1)$ at the beginning of the equilibrium cycle is the required signalling phase described above. It trivially appears infinitely often on the equilibrium path and it differs from the punishment phase of playing $\bar{m}$ consecutively.

Furthermore, no single player deviation, either from the equilibrium path or from the punishment path, can escape the punishment phase. To see this note first that because $a_i^1 \neq \bar{m}_i$, for all $i = 1, 2$, no single player can deviate from the mutual minmaxing phase and induce the signalling phase that is necessary to escape punishment. The same holds also regarding deviations from histories whose M-tail consists of consists of $M - t$ consecutive occurrences of $\bar{m}$ followed by the first t actions of the equilibrium cycle, for some $t < p^1$, because a player deviating singly from a^1 will lead to an action different from both a^1 and $\bar{m}$. Last, consider any single-player deviation from the equilibrium path. Such a deviation does not result in a punishment phase only if the M-tail of the history after the deviation either contains p^1 consecutive occurrences of a^1 followed by the subsequent actions of the equilibrium path (if any) or, for some $t < p^1$, consists of $M - t$ consecutive occurrences of $\bar{m}$ followed by the first t actions of the equilibrium cycle. The latter cannot happen because the M-tail does not contain p^1 consecutive a^1s and hence the deviation could not be from the equilibrium path. Consider then the former case. In this case such a deviation is feasible only if the p^1-tail is $(a^1; p^1)$. Since $p^1 \geq 2$, both the action profile induced by the deviation and the action profile just before the deviation must be a^1. But, on the equilibrium path, only a^1 or $\bar{m}$ follow a^1. Since the deviation induces a^1, then it must be that the deviation is from $\bar{m}$. But $\bar{m}$ differs from a^1 in every coordinate, which implies that single-player deviation cannot produce such a history.

In the above construction of the equilibrium cycle, we have assumed that $p^1 \geq 2$ and $p^2 \geq 1$. To illustrate why these conditions cannot be weakened, consider the repeated

[13]To complete the result, we need to set M to be large enough so that with M memory (i) all individually rational payoffs can be approximately obtained by average payoff of a finite cycle paths and (ii) to distinguish between the different paths and phases.

[14]Note that the above strategy profile is not simple. This is because the punishment path is not unique: the number of times the mutual minmax action is to be played in response to a deviation depends on the number of times the mutual minmax appears before the punishment starts.

Prisoners' Dilemma described in the introduction. Since, in this example, $\bar{m} = (D, D)$ and the profile that differs from $\bar{m}$ in every component is (C, C), it follows that, if we order the set of action profiles as above, then $a^1 = (C, C)$, $a^2 = \bar{m}$, $a^3 = (D, C)$, $a^4 = (C, D)$.

To see why $p^2 \geq 1$, suppose that the equilibrium cycle is such that $p^1 = 2$, $p^2 = 0$, and $p^3 \geq 1$. If the M-tail of a history is given by $(a_1, \ldots, a_{M-2}, (C, C), (C, C))$ for some sequence of action profiles $a_1, \ldots, a_{M-2}$, then the signalling phase $(C, C), (C, C)$ is observed and the players should play (D, C). But if player 1 deviates and plays C at this history, the next period M-tail of the resulting history would be $(a_2, \ldots, a_{M-1}, (C, C), (C, C))$. Since the signaling phase is observed again such deviation does not trigger the punishment path.

To see why we need $p^1 \geq 2$, suppose that the equilibrium cycle is such that $p^1 = p^2 = 1$ and $p^3 \geq 1$. Then the strategy recommends (D, C) at any history whose M-tail equals $(a_1, \ldots, a_{M-2}, (C, C), (D, D))$, for some $a_1, \ldots, a_{M-2}$. But if player 1 deviates at this history and plays C instead, the next period M-tail of the resulting history would be $(a_2, \ldots, a_{M-2}, (C, C), (D, D), (C, C))$. Since this history induces the signalling phase (C, C), such a deviation does not trigger the punishment path.

3.2 Intuition for the $n > 2$ case

With no bounds on memory and more than two players, to implement $u \in \mathcal{U}$ the standard FT calls for the use of a simple strategy consisting of an equilibrium path $\pi^{(0)}$ and n punishment paths $\pi^{(1)}, \ldots, \pi^{(n)}$ with the following property. The punishment path $\pi^{(i)}$ for player i consists of playing the minmax profile m^i for T periods followed by a path $\hat{\pi}^{(i)}$, referred to as the reward path corresponding to $\pi^{(i)}$; thus

$$\pi^{(i),t} = \begin{cases} m^i & \text{if } t \leq T \\ \hat{\pi}^{(i),t-T} & \text{otherwise.} \end{cases}$$

Therefore, the typical FT construction consists of three sets of sequences of action profiles: (i) the equilibrium path $\pi^{(0)}$, (ii) the minmax phase for each player i consisting of playing m^i a finite number of times T, and (iii) the reward paths $\hat{\pi}^{(i)}$ for each i.

As in the above standard construction, the bounded memory strategy profiles we use to prove our FT is such that the incentives to play the equilibrium and reward paths are given by the threat of punishments, consisting of a sequence of the deviator's minmax action profile followed by the appropriate reward path. However, to identify each of the sequences described in (i)-(iii) and the appropriate action profile that has to be played, we add to the beginning of each of the above sequences a distinct signalling phase. As with the 2-player case, once players observe one of these signalling phases, they can identify what needs to be played and therefore can forget all that has happened before.

For example, each signalling phase could consist of a sequence $(s; l)$ where $s \in A$ is some fixed action profile and l is some number that is different for the different signalling phases. The idea is that when players observe a sequence of the form $(s; l)$ then, by counting the number of consecutive s's, which equals l in this sequence, players can identify which path to play.

The above, however, may not work as the players need to identify when the signalling phase starts and when it ends. Specifically, if $(s; l)$ is observed then the history is consistent with any signalling phase $(s; l')$ for all $l' \leq l$. To overcome this, we modify each signalling phase $(s; l)$ so that it is preceded and followed by another action, $s' \neq s$.

The addition of s' to the signalling phases is also not enough. First, we need to ensure that each signalling phase cannot be induced by single player deviations from another signalling phase. We deal with this problem by choosing s' to be such that it differs from s in every coordinate (i.e. $s_i \neq s'_i$ for all $i \in N$). Second, for reasons that will become clear later, we also need to assume that each signalling phase starts with two s''s and has at least two consecutive s's. Specifically, each signalling phase in our construction is described by $(s', s', (s; l), s')$ and we set l in each phase as follows: $l = i + 1$ for the minmax path of player i, $l = n + 2$ for the equilibrium path and $l = n + 2 + i$ for the reward path of player i.

As we discussed before, with δ close to 1, to approximately implement $u \in \mathcal{U}$, all that matters is that on the equilibrium path each action profile is played an appropriate fraction of times. The same holds for approximately implementing the payoffs corresponding to the reward paths. It may then seem that the simple strategy profile that we need is as follows:

(i) the equilibrium path $\pi^{(0)} = (\pi^{(0),1}, \pi^{(0),2}, \ldots)$ consists of the repetition of the following type of cycle path

$$\left(s', s', (s; n + 2), s', (a^1; p^{(0),1}), \ldots, (a^r; p^{(0),r})\right);$$

where $p^{(0),j}$ is chosen appropriately so that $\pi^{(0)}$ induce approximately u.

(ii) the reward path $\hat{\pi}^{(i)} = (\hat{\pi}^{(i),1}, \hat{\pi}^{(i),2}, \ldots)$, $i \in \{1, \ldots, n\}$, is the repetition of the cycle

$$\left(s', s', (s; n+i+2), s', (a^1; p^{(i),1}), \ldots, (a^r; p^{(i),r})\right);$$

where $p^{(i),j}$ is chosen appropriately so that $\pi^{(i)}$ induce approximately the appropriate reward payoff.

(iii) the punishment path $\pi^{(i)}$, $i \in \{1, \ldots, n\}$, is given by

$$\pi^{(i)} = \left(s', s', (s; i+1), s', (m^i; T), \hat{\pi}^{(i),1}, \hat{\pi}^{(i),2}, \ldots\right),$$

where T is chosen appropriately to deter single period deviations.

Unfortunately, the problem is great deal more complicated. An immediate issue is that we must ensure that the introduction of the signalling phases do not affect the incentives adversely. On all paths other than one's own punishment path, we can ensure that the players play the appropriate continuation path by standard construction that invokes the punishment for the deviator after any single player deviation from such phases. The same is however not the case regarding the play of one's own punishment path.

First, once we introduce a signalling phase at the beginning of each punishment path, some player may have a profitable deviation in the minmax phase of his own punishment, if such deviation restarts the punishment path. For example, deviation by i at the beginning of the minmax phase of his own punishment path induces the outcome $(s', s', (s; i+1), s', (m^i; T), \hat{\pi}^{(i),1}, \hat{\pi}^{(i),2}, \ldots)$, whereas no deviation induces $((m^i; T-1), \hat{\pi}^{(i),1}, \hat{\pi}^{(i),2}, \ldots)$. If $(s', s', (s; i+1), s')$ generates a sufficiently high average payoff, then the deviation will be profitable. To deal with this problem, we modify the above simple strategy construction by assuming that deviations by a player from his own minmax action in his punishment path are ignored and punishment path is not restarted. Such a change in the construction does not affect the incentives because there are no one-period gains to deviations during the minmax phase.

Second, some player may profitably deviate in the signalling phase of his own punishment path if such deviation restarts the signalling phase. For instance, if some player i obtains a high payoff by deviating from s' to some action a_i, he could perpetually deviate in the first period of the punishment path and obtain a path consisting in the repetition of (a_i, s'_{-i})

delivering him a higher payoff. Similarly, if some player i obtains a high payoff by deviating from s by playing some action a_i, then he could perpetually deviate in the third period of the punishment path and obtain a path consisting in the repetition of $(s', s', (a_i, s_{-i}))$ which could yield him a higher payoff.

We deal with this problem by specifying that when there is a deviation by a player in the signalling phase of his punishment path, the strategy prescribes the continuation of that particular signalling phase. But this by itself is not enough as we need to ensure that there is punishment to deter deviations during this phase (if s or s' were Nash equilibria of the stage game this would of course be unnecessary). We establish such deterrence by appropriately increasing the length of the minmax phase of the punishment path for each such deviation. Specifically, denoting the number of times that player i has deviated during the signalling phase of his punishment path by $\theta \in \{0, 1, \ldots, i+4\}$, the strategy profile requires that once the current signalling phase is over, the continuation path consists of playing $((m^i; (\theta+1)T), \hat{\pi}^{(i),1}, \hat{\pi}^{(i),2}, \ldots)$. Such construction implies that for every deviation during the signalling phase the length of the minmax phase increases by T.[15]

The above modification involving delayed punishments of deviations during the signalling phases of the punishment paths has two implications that are worth noting. First, each player i effectively has $i+5$ punishment paths indexed by $\theta \in \{0, 1, \ldots, i+4\}$.[16] We denote each of these by $\pi^{(i)}(\theta) = (s', s', (s; i+1), s', (m^i; (\theta+1)T), \hat{\pi}^{(i),1}, \hat{\pi}^{(i),2}, \ldots)$ and define the path $\pi^{(i)}(\theta)$ without its first $t-1$ elements by $\pi^{(i)}(\theta, t)$.

Second, ignoring one-period deviations by any player i during the signalling phases of i's punishment path, as proposed above, means that the minmax phase starts after any sequences

$$((a_i^1, s'_{-i}), (a_i^2, s'_{-i}), (a_i^3, s_{-i}), \ldots, (a_i^{i+3}, s_{-i}), (a_i^{i+4}, s'_{-i})), \tag{1}$$

with $a_i^l \in S_i$ for all $l = 1, \ldots, i+4$, has been observed. Therefore, it follows that the signal

[15] The reason for having $\theta+1$ instead of just θ is that player i needs to be punished even if he does not deviate in the signalling phase of his punishment path.

[16] By the number of punishment paths, we mean the number of distinct paths that a player can induce by a deviation, excluding the continuation path that occurs when the player does not deviate.

Note also that our construction does not constitute a simple strategy profile because it will have, in addition to the equilibrium path, $\sum_{i=1}^{n}(i+5)$ punishments paths.

for the punishment of player i are effectively all sequences satisfying (1) rather than just $(s', s', (s; i+1), s')$. To differentiate between any sequence described (1) from the signalling phase $(s', s', (s; i+1), s')$, we shall call the former a generalised signalling phase for player i's punishment path.

Given the above, after any history $h = (a^1, \ldots, a^\tau)$, our M period memory strategy profile f would satisfy the following conditions:

(a) *(Equilibrium and reward path histories)* Suppose the t-tail of h is $(\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),t})$, for some $i = 0, \ldots, n$ and $t \leq M$, and it includes the signalling phase $(s', s', (s; n+i+2), s')$ of $\hat{\pi}^{(i)}$, i.e. $n+i+5 \leq t$. Then f prescribes players to continue with $\hat{\pi}^{(i)}$.

(b) *(Punishment path histories)* Suppose for some $i = 1, \ldots, n$ and t such that $i+4 \leq t \leq M$, the t-tail of h has the following properties:

(i) the first $i+4$ elements of the t-tail is a generalised signalling phase of i as described in (1);

(ii) if $t \leq (\theta+1)T + i + 4$, where θ refers to the number of times that player i has deviated during the signalling phase (1), the remaining elements of the t-tail are such that the players other than i minmax i by playing m^i_{-i};

(iii) if $t > (\theta+1)T + i + 4$, in every period $i + 4 < r \leq (\theta+1)T + i + 4$ of the t-tail all players other than i minmax i by playing m^i_{-i}, and the remaining elements of the t-tail correspond to the first $t - ((\theta+1)T + i + 4)$ elements of the path $\hat{\pi}^{(i)}$.

Then f requires the players to continue with $\pi^{(i)}(\theta, t+1)$.

(c) *(Histories involving deviations from (a)–(b))* Suppose case (b) does not apply and, for some $r \in \{\tau - M, \ldots, \tau\}$, $(a^1, \ldots, a^{r-1})$ satisfies the properties described in either (a) or (b) above, a^r involves a deviation by some player i from f as described in (a) and (b), and $(a^{r+1}, \ldots, a^\tau)$ is consistent with a generalised signalling phase for player i's punishment path. Then f prescribes $\pi^{(i)}(\theta, \tau - r + 1)$, where θ refers to the number of times that player i has deviated during $(a^{r+1}, \ldots, a^\tau)$.

Conditions (a)–(c) describe the behaviour after histories that have the following feature: For some $t \leq M$, its t-tail contains the entire signalling phase of one of the equilibrium or reward path, or an entire generalised signalling phase for a punishment path. In particular, (a)–(c) specify the appropriate path to be played once these signalling phases are observed and are followed by a sequence of actions in which there are either no deviations or only single-player deviations from the path corresponding to the signalling phase.

What if a complete generalised signalling phase does not appear in the M-tail of the history? The specification of what should be played at such histories cannot be arbitrary as the equilibrium should be such that it is not in the interest of any player to deviate *during* a generalised signalling phase of another player's punishment path. To deal with this case, we assume that if a complete generalised signalling phase does not appear in the M-tail of the history as in (a)–(c) and if, for some $t \leq M$, the t-tail of the history consists of a single-player deviation from s or s' by player i followed by an incomplete generalised signalling phase for the punishment of player i, then the strategy recommends players to continue with such signalling phase. For any other history, our construction prescribes playing the equilibrium path.[17] More formally, in addition to (a)–(c) above, we assume that the equilibrium strategies satisfy the following two conditions at every history $h = (a^1, \ldots, a^\tau)$:

(d) *(Histories that involve deviations from incomplete signalling phases)* If none of the conditions (a)–(c) are satisfied and if for some $r \in \{\tau - M, \ldots, \tau\}$, a^r involves a deviation by some player i from s or s' and $(a^{r+1}, \ldots, a^\tau)$ is consistent with a generalised signalling phase of player i's punishment path, then f prescribes $\pi^{(i)}(\theta, \tau - r + 1)$, where θ refers to the number of times that player i has deviated during $(a^{r+1}, \ldots, a^\tau)$.[18]

(e) *(Other histories)* If none of conditions (a)–(d) are satisfied and the last $0 \leq t < M$ periods corresponds to the first t periods of the equilibrium path $\pi^{(0)}$, then the strategy prescribes players to continue with $\pi^{(0)}$ (when $t = 0$ the strategy recommends the first action on the equilibrium path).[19]

[17] The specification of the continuation path here is somewhat arbitrary; all that is needed is that the play results in any of the equilibrium, reward or punishment paths.

[18] Unlike in the case of condition (c), there may be several values for r such that condition (d) holds. For example, if h satisfies none of the conditions (a)–(c) and $T^3(h) = ((a_i^1, s'_{-i}), (a_i^2, s'_{-i}), (a_i^3, s'_{-i}))$ for some $i \in N$ and $a_i^1, a_i^2, a_i^3 \neq s'_i$, then condition (d) is satisfied with $r = \tau$, $r = \tau - 1$ and $r = \tau - 2$. In our proof, we take the smallest such r (in the example in this footnote, f prescribes $\pi^{(i)}(\theta, t)$ with $\theta = 2$ and $t = 3$).

[19] Note that at histories described in (e), it is possible that the path resulting from such a history fails

To ensure that the above behaviour described (a)-(e) can be implemented when M is finite, however, several issues have to be addressed.

First, we need to set M to be large enough so that it is possible to distinguish between the different paths and phases. Specifically, let K be such that all individually rational payoffs can be approximately obtained by average payoff of cycle paths of length K.[20] Also, note that the length of the longest signalling phase in the different punishment paths, the length of the longest minmax phase and the length of the longest signalling phase of the reward paths are respectively $n+4$, $T(n+5)$ and $2n+5$. Then, it follows that for the strategy profile to implement the punishment paths, the memory size has to be at least $(n+4)+T(n+5)+(2n+5)+K$. We show in the appendix it suffices to have M greater than this bound to implement our strategy profile.

Second, even though the signalling phase of the different paths, including the generalised signalling phases as described by (1), are all different, this does not necessarily imply that, once they are observed, they can be used to identify the future path of play. For example, if the signalling phase $(s', s', (s; n+i+2), s')$ of $\hat{\pi}^{(i)}$ appears on $\hat{\pi}^{(j)}$ for $j \neq i$ then the strategy described above may not be well-defined. Furthermore, for these signalling phases to have the required property that once they are observed all previous history can be ignored, it should also be the case that they cannot be induced by a single player deviation from some other path. For example, if for some $a_j \neq s'_j$, the sequence $(s', s', (s; n+i+2), (a_j, s'_{-j}))$ appears on the reward path $\hat{\pi}^{(j)}$ then there may be an incentive for j to play s'_j on the path $\hat{\pi}^{(j)}$ after $(s', s', (s; n+i+2))$, as such a deviation induces the signalling phase of $\hat{\pi}^{(i)}$.

The issue here is that we not only need the signalling phases to be distinct from each other, they also need to be appropriately distinct with respect to the equilibrium and reward paths, as well as with respect to minmax phases. We deal with these issues as follows.

By the same argument as before, the order by which the sequence of actions $\{a^1, \ldots, a^r\}$

to be the equilibrium path. For example, suppose that $T^{n+i+4}(h) = (s', s', (s; n+i+2))$ for some $i \in N$ and that h does not satisfy (a)-(d). Then, the strategy recommends the first action on the equilibrium path s'. The resulting history, denoted by h', satisfies $T^{n+i+5}(h') = (s', s', (s; n+i+2), s')$, which equals the signalling phase of player i's reward path. At this point, player i's reward path will be played henceforth. This of course does not generate any problems as the strategy profile still implements a SPE path.

[20]Notice that K, and hence M, will depend on the degree of approximation.

are played on the equilibrium path and on each of the reward paths, as well as the number of times they are played on the path, do not matter as long as each action profile is played an appropriate number of times. This freedom to choose the order of the sequence $\{a^1, \ldots, a^r\}$ allow us to construct the equilibrium and the reward paths in such a way so that they are appropriately distinct from the signalling phases.

Specifically we achieve this as follows. The first action profile a^1 is set to be equal to s and is followed by all the action profiles of the form (a_i, s_{-i}) for some $i \in N$ and $a_i \neq s_i$. These are followed by s', and then by action profiles of the form (a_i, s'_{-i}) for some $i \in N$ and $a_i \neq s'_i$. The remaining action profiles are ordered arbitrarily.[21] With this ordering, on the equilibrium and reward paths, s' and action profiles obtained by single player deviations from s' are never followed by s or by action profiles consisting of single player deviations from s, other than in the initial signalling phases. This ordering ensures that (i) for each $i = 0, \ldots, n$ the signalling phase of $\hat{\pi}^{(i)}$ appears only once on the cycle path of $\hat{\pi}^{(i)}$ and it does not appear on $\hat{\pi}^{(j)}$, for all $j \neq i$, (ii) the generalised signalling phase for each punishment path does not appear on $\hat{\pi}^{(j)}$, for all $j = 0, \ldots, n$ and (iii) no signalling phase can be induced from single player deviations from $\hat{\pi}^{(j)}$, for all $j = 0, \ldots, n$.

There is still the issue of appropriate distinctness of the signalling phases from the minmax ones. Since the signalling phases consist of two action profiles s and s' that are distinct in every component, it follows trivially that the signalling phases, including the generalised ones, cannot occur when all players are minmaxing a specific player and furthermore the former sequences cannot be induced by single player deviations from a minmax phase. However, in our construction we assume that a deviation by any player i from his minmax profile m^i are ignored and the future play is not affected by such a deviation. This means that we must also ensure that signalling phase, including the generalised ones, cannot be induced by single player deviations from sequences $((a_i^1, m_{-i}), \ldots, (a_i^\tau, m_{-i}))$ that involve single player deviations by player i from his own minmax phase. Our requirement that each signalling phase contains at least two consecutive s's and two consecutive s''s at the beginning of these

[21]For example, when $n = 3$ and $A_i = \{\alpha, \beta\}$ for all $i \in N$, a possible ordering respecting the above properties would be $a^1 = s = (\alpha, \alpha, \alpha)$, $a^2 = (\beta, \alpha, \alpha)$, $a^3 = (\alpha, \beta, \alpha)$, $a^4 = (\alpha, \alpha, \beta)$, $a^5 = s' = (\beta, \beta, \beta)$, $a^6 = (\alpha, \beta, \beta)$, $a^7 = (\beta, \alpha, \beta)$, $a^8 = (\beta, \beta, \alpha)$.

phases deals with this issue.

To see the role of at least two consecutive s's in the signalling phases, suppose that instead of assuming that the signalling phases of the punishment path of each i has $i+1$ consecutive s's, we have i consecutive s's. This means that the signalling phase of player 1's punishment is such that s appears only once and is given by (s', s, s'). Consider then a 3-player game with $m^3 = (s_3, s'_{-3})$, a history $h = ((s';2),(s;3),s',s',s',(s'_1,s_{-1}),s')$ and $M \geq 10$. Since $s' = (s'_3, , m^3_{-3})$, $(s'_1, s_{-1}) = (s_2, m^3_{-2})$ and the signalling phase for player i's punishment is $((s';2),(s;i),s')$, it follows that h consists of the signalling phase for player 3's punishment, followed by (s'_3, m^3_{-3}) being played twice, followed by (s_2, m^3_{-2}) and followed by s', the first action of the signalling phase of player 2's punishment path. Hence, by part (c) of our construction above, the strategy prescribes continuing with punishing player 2 by playing $\big((s;2), s', (m^2;T), \hat{\pi}^{(2),1}, \hat{\pi}^{(2),2}, \ldots\big)$. But $T^4(h) = ((s';2),(s'_1,s_{-1}),s')$ is a generalised signalling phase of player 1's punishment. Thus, part (b) of our construction also applies. Therefore, the strategy also recommends $\big((m^1;2T), \hat{\pi}^{(1),1}, \hat{\pi}^{(1),2}, \ldots\big)$.

The problem here arises because $s' = (s'_3, m^3_{-3})$ and $(s'_1, s_{-1}) = (s_2, m^3_{-2})$. Hence, single-player deviations from m^3 can induce both s' and single-player deviations from s, and, as a result, the continuation strategy after history h is not well-defined.

Having s played $i+1$ times in the signalling phase of i's punishment solves the above problem as follows. In this case the signalling phase of player 1 is $(s', s', (s;2), s')$. This means that if player 1 deviates from s during his signalling phase this is preceded and succeeded by s and s' or the reverse. Since it cannot be the case that *both* s and s' can be induced by a player deviating from his own minmax profile, it follows that deviations by player 1 from his own signalling phase are not consistent with phases involving another player deviating from his own minmax phase. Hence the problem described above does not arise.

Similarly, to see the role of having two consecutive s''s at the beginning of the signalling phases, suppose that instead of assuming that the signalling phases of the punishment path of each i is $(s', s', (s;i+1), s')$, we assume that it consists of $(s', (s;i+1), s')$ with only one s' at the beginning of these phases. Consider a 3-player game with $m^1 = (s'_1, s_{-1})$, a history $h = (s', (s;2), s', (s;3), (s_2, s'_{-2}))$ and $M \geq 8$. Since $s = (s_1, m^1_{-1})$, $(s_2, s'_{-2}) = (s'_3, m^1_{-3})$ and the signalling phase for player i's punishment is $(s', (s;i+1), s')$, it follows that h consists of the signalling phase for player 1's punishment, followed by (s_1, m^1_{-1}) being played three times, followed by (s'_3, m^1_{-3}). Hence, by part (c) of our construction above, the strategy prescribes $\pi^{(3)}$. But $T^5(h) = (s', (s;3), (s_2, s'_{-2}))$ is a generalised signalling phase of player 2's punishment. Thus, part (b) of our construction also applies. Therefore, the strategy also recommends $\big((m^2; 2T), \hat{\pi}^{(2),1}, \hat{\pi}^{(2),2}, \ldots\big)$.

The problem here arises because $s = (s_1, m^1_{-1})$ and $(s_2, s'_{-2}) = (s'_3, m^1_{-3})$. Hence, single-player deviations from m^1 can induce both s and single-player deviations from s', and, as a result, the continuation strategy after history h is not well-defined.

Having two s''s at the beginning of the signalling phases solves this problem as follows. In this case, the signalling phase of player 2 would be $((s';2), (s;3), s')$. But such phase is consistent with the signalling phase of player 1 followed by 1's minmax phase only if *both* s and s' could be induced by player 1 deviating from his own minmax profile.[22] Since s and s' are distinct in every component, this is not feasible and, hence, the problem described above does not arise.

A Proof of the bounded memory Folk Theorem

For all $x \in \mathbb{R}^n$, let $||x|| = \max_{i=1,\ldots,n} |x_i|$. Since $\mathcal{U}$ is compact, it suffices to show that for all $\varepsilon > 0$ and all $u \in \mathcal{U}$, there exist $M \in \mathbb{N}$ and $\delta^* \in (0,1)$ such that for all $\delta \geq \delta^*$, there exists a M-memory SPE f of $G^\infty(\delta)$ with $\|U(f,\delta) - u\| < \varepsilon$. Furthermore, since $\mathcal{U}$ equals the closure of $\mathcal{U}^0$, we only need to show that the above holds for any $u \in \mathcal{U}^0$. Therefore, in the rest of this appendix, we show that for all $\varepsilon > 0$ and $u \in \mathcal{U}^0$, there exist $M \in \mathbb{N}$ and $\delta^* \in (0,1)$ such that for all $\delta \geq \delta^*$, there exists a M-memory SPE f of $G^\infty(\delta)$ with $\|U(f,\delta) - u\| < \varepsilon$.

A.1 2-player case

In this subsection, for convenience, we normalize payoffs so that $u_i(\bar{m}) = 0$ for both $i = 1, 2$.

Fix any $\varepsilon > 0$ and $u \in \mathcal{U}^0$. Let $0 < \eta < \min_{i=1,2}(u_i - v_i)$, $0 < \gamma < \min\{\eta/3, \varepsilon/2\}$ and $\xi > 0$ be such that $2\xi < \eta - 2\gamma$.

[22] This is because in this case the number of s''s at the beginning of the signalling phase of player 2 is different from that at the end of the signalling phase of player 1.

Order $A = \{a^1, \ldots, a^r\}$ so that $a_i^1 \neq \bar{m}_i$ for all i, and $a^2 = \bar{m}$. Also, for any $k \in \mathbb{N}$, let

$$\mathcal{U}_k = \Bigg\{ w \in \mathbb{R}^N : w = \sum_{a \in A} \frac{p_a u(a)}{k} \text{ for some } (p_a)_{a \in A} \text{ such that} \\ p_a \in \mathbb{N} \text{ for all } a, p_1 \geq 2, p_2 \geq 1 \text{ and } \sum_{a \in A} p_a = k \Bigg\}.$$

Using an analogous argument to Sorin (1992, Proposition 1.3), it follows that $\mathcal{U}_k$ converges to $\mathrm{co}(u(A))$ in the Hausdorff distance. Therefore, there must exist $K \in \mathbb{N}$ such that

$$\mathrm{co}(u(A)) \subseteq \cup_{x \in \mathcal{U}_K} B_\gamma(x). \tag{2}$$

Let $p_1, \ldots, p_r$ be such that $p_k \geq 0$ for all $1 \leq k \leq r$, $p_1 \geq 2$, $p_2 \geq 1$, $\sum_{k=1}^r p_k = K$ and

$$\left\| \sum_{k=1}^r \frac{p_k u(a^k)}{K} - u \right\| < \gamma. \tag{3}$$

Note that (2) implies that such a sequence $p_1, \ldots, p_r$ exists. Let $u' = \sum_{k=1}^r p_k u(a^k)/K$ and π consist of repetitions of the cycle $((a^1; p_1), \ldots, (a^r; p_r))$.

Let $T \in \mathbb{N}$ and $M \in \mathbb{N}$ be such that

$$T > K\left(\frac{B}{\xi} + 1\right) \text{ and } M = 2T + K. \tag{4}$$

Also, let $\delta^* \in (0, 1)$ be such that for all $\delta \in [\delta^*, 1)$

$$\max\left\{\frac{\delta^K - \delta^T}{1 - \delta^K}, \delta^T \frac{1 - \delta^{T+1}}{1 - \delta}, \delta^M\right\} > T, \tag{5}$$

$$\sup_{(x^1, \ldots, x^K) \in [-B, B]^K} \left| \frac{1 - \delta}{1 - \delta^K} \sum_{k=1}^K \delta^{k-1} x^k - \frac{1}{K} \sum_{k=1}^K x^k \right| < \gamma. \tag{6}$$

Note that such $\delta^* \in (0, 1)$ exists because the limit of the left hand side (5) and (6) as $\delta \to 1$ are, respectively, $T + 1 = \max\{(T - K)/K, T + 1, 1\}$ and 0.

Fix any $\delta \geq \delta^*$. We will prove that there is a M-memory SPE f with $||U(f, \delta) - u|| < \varepsilon$.

Note that

$$||V(\pi, \delta) - u|| \leq ||V(\pi, \delta) - u'|| + ||u' - u|| < 2\gamma < \varepsilon, \tag{7}$$

where the second inequality follows from (6) and (3) and the third from the assumption that $\gamma < \varepsilon/2$. Thus, it suffices to show that there is a M-memory SPE f with $\pi(f) = \pi$.

Before defining the strategy profile f, note the following properties of u' and $V^t(\pi, \delta)$. First, for all $i = 1, 2$,

$$u'_i > u_i - \gamma > v_i + \eta - \gamma > v_i + 2\xi, \text{ and} \tag{8}$$

$$V_i^t(\pi, \delta) > u'_i - \gamma > u_i - 2\gamma > v_i + \eta - 2\gamma > v_i + 2\xi \text{ for all } t \in \mathbb{N}. \tag{9}$$

(The first inequality in (9) follows from (6), the first in (8) and the second in (9) from (3), the second in (8) and the third in (9) since $\eta < u_i - v_i$ and the last inequality in both (8) and (9) because $2\xi < \eta - 2\gamma$).

Second, the following claim must hold.

Claim 1 *For all $i = 1, 2$, $t \in \mathbb{N}$ and $\delta \geq \delta^*$, $V_i^t(\pi, \delta) \geq \delta^T V_i(\pi, \delta)$.*

Proof. Fix any $i = 1, 2$, $t \in \mathbb{N}$ and $\delta \geq \delta^*$. Then, $V_i^t(\pi) = (1 - \delta) \sum_{l=k}^{K} \delta^{l-k} u_i(\pi^l) + \delta^{K-k+1} V_i(\pi) \geq -B(1 - \delta^{K-k+1}) + \delta^{K-k+1} V_i(\pi)$ for some $1 \leq k \leq K$. Hence, since $k \geq 1$, it follows that $V_i^t(\pi) \geq -B(1 - \delta^K) + \delta^K V_i(\pi)$.

Therefore, it suffices to show that $(\delta^K - \delta^T)V_i(\pi) \geq B(1 - \delta^K)$. This inequality holds since (9) and (5) imply that $(\delta^K - \delta^T)V_i(\pi) > (\delta^K - \delta^T)\xi > B(1 - \delta^K)$. ■

A.1.1 The strategy profile

We define the desired strategy profile f as follows. For any $k \in \mathbb{N}$ such that $0 \leq k \leq M$, let

$$\begin{aligned} H_1^k &= \{h \in H : T^k(h) = (\pi^1, \ldots, \pi^k)\}, \\ H_2^k &= \{(\pi^1, \ldots, \pi^k)\} \text{ if } k > 0 \text{ and } H_2^k = \{H_0\} \text{ if } k = 0, \text{ and} \\ H_3^k &= \{h \in H : T^M(h) = ((\bar{m}; M - k), \pi^1, \ldots, \pi^k)\}. \end{aligned}$$

We additionally define

$$H^k = \begin{cases} H_1^k & \text{if } k \geq p_1 \\ H_2^k \cup H_3^k & \text{if } k < p_1, \end{cases}$$

$H^E = \cup_{k=0}^M H^k$ and $H^P = H \setminus H^E$. Then, f is defined by

$$f(h) = \begin{cases} \pi^{k+1} & \text{if } h \in H^k \text{ for some } 0 \leq k \leq M, \\ \bar{m} & \text{otherwise.} \end{cases}$$

Claim 2 *The strategy profile f is a well defined M-memory strategy.*

Proof. By the definition of H_i^k, $i = 1, 2, 3$, the following must hold: (i) If $h \in H_1^k \cap H_1^{k'}$ for some $k > k' \geq p_1$ then it must be that $k = k' + \alpha K$ for some $\alpha \in \mathbb{N}$, implying that $\pi^{k+1} = \pi^{k'+1}$. (ii) For any $k \geq p_1$ and $k' < p_1$, $H_1^k \cap H_2^{k'} = \emptyset$ and $H_1^k \cap H_3^{k'} = \emptyset$ (if the latter were not to hold we would have $\pi^1 = \bar{m}$, a contradiction). (iii) For any $k, k' < p_1$, $k \neq k'$, $H_i^k \cap H_j^{k'} = \emptyset$ for any $i, j \in \{2, 3\}$. It then follows from (i)–(iii) that f is well-defined.

Finally, note that f is a M-memory strategy because its definition is such that $f(h)$ depends only on $T^M(h)$ for all $h \in H$. ■

A.1.2 Outcome paths induced by f and by one-shot deviations from f

The next two claims establish the continuation paths f induces after any history.

Claim 3 *If $h \in H^k$ for some $0 \leq k \leq M$, then $\pi(f|h) = (\pi^{k+1}, \pi^{k+2}, \ldots)$.*

Proof. We prove this in several steps.

Step 1: If $h \in H_1^k$ and $p_1 \leq k \leq M$ then $h \cdot f(h) \in H_1^{k'+1}$ for some k' such that $p_1 \leq k' \leq M$ and $k = \alpha K + k'$ for some $\alpha \in \mathbb{N}$. Suppose that $p_1 \leq k \leq M$ and $h \in H_1^k$. Then, we must have that $T^k(h) = (\pi^1, \ldots, \pi^k)$ and $f(h) = \pi^{k+1}$. This implies that $T^{k+1}(h \cdot f(h)) = (\pi^1, \ldots, \pi^{k+1})$. If $k < M$, the claim of this step holds because $(h \cdot \pi^{k+1}) \in H_1^{k+1}$ and $p_1 \leq k+1 \leq M$. If $k = M$, then since $M \geq 2K$, we must have $T^M(h \cdot f(h)) = (\pi^2, \ldots, \pi^{k+1}) = (\pi^2, \ldots, \pi^K, \pi^1, \ldots \pi^{k-K+1})$ with $k - K + 1 = M - K + 1 > p_1$. Hence, the claim of this step also holds in this case because $h \cdot f(h) = h \cdot \pi^{k-K+1} \in H_1^{k-K+1}$ and $k - (k - K) = K$.

Step 2: If $h \in H_1^k$ and $p_1 \leq k \leq M$ then $\pi(f|h) = (\pi^{k+1}, \pi^{k+2}, \ldots)$. This follows by induction from Step 1 and by noting that $\pi^{k'+1} = \pi^{k+1}$ if $k = \alpha K + k'$ for some $\alpha \in \mathbb{N}$.

Step 3: If $h \in H_2^k \cup H_3^k$ and $0 \leq k < p_1$ then $\pi(f|h) = (\pi^{k+1}, \pi^{k+2}, \ldots)$. If $h \in H_2^k \cup H_3^k$ and $0 \leq k < p_1$, then by induction, f induces the outcome $(\pi^{k+1}, \ldots, \pi^{p_1})$ after h. But, since $h \cdot (\pi^{k+1}, \ldots, \pi^{p_1}) \in H_1^{p_1}$, the claim of this step follows from Step 2. ■

It follows trivially from Claim 3 that $\pi(f) = (\pi^1, \pi^2, \ldots)$. Hence, f implements π.

Claim 4 *If $h \in H^P$ and $k = \max\{0 \leq k' \leq M : T^{k'}(h) = (\bar{m}; k')\}$, then $k < M$ and $\pi(f|h) = ((\bar{m}; M - k), \pi^1, \pi^2, \ldots)$.*

Proof. Fix any $h \in H^P$ and let k be as defined above.

Step 1: $k < M$. Otherwise, $k = M$ and $T^M(h) = (\bar{m}; M)$ producing a contradiction because then $h \in H_3^0 \subseteq H \setminus H^P$.

Step 2: If $h \cdot (\bar{m}; l-1) \in H^P$ for some $l \in \{1, \ldots, M-k-1\}$, then $h \cdot (\bar{m}; l) \in H^P$. Suppose not; then $h \cdot (\bar{m}; l-1) \in H^P$ and $h \cdot (\bar{m}; l) \in H^{k'}$ for some $0 \leq k' \leq M$. Since, $a^1 \neq \bar{m}$ and, for any $\tau \leq p_1$, $(\pi^1, \ldots, \pi^\tau) = (a^1; \tau)$, it follows from $h \cdot (\bar{m}; l) \in H^{k'}$ that either $h \cdot (\bar{m}; l) \in H_1^{k'}$ and $k' \geq p_1$ or $T^M(h \cdot (\bar{m}; l)) = (\bar{m}; M)$. But, the latter is not possible, because we have by assumption $l < M - k$ (in fact, if $T^M(h \cdot (\bar{m}; l)) = (\bar{m}; M)$, then $T^{M-l}(h) = (\bar{m}; M-l)$ and so $k \geq M - l$); therefore, consider the former case. Then, $T^{k'-1}(h \cdot (\bar{m}; l-1)) = (\pi^1, \ldots, \pi^{k'-1})$. Since $h \cdot (\bar{m}; l-1) \in H^P$, it must be that $k' - 1 < p_1$. Hence, $k' = p_1$, $k' - 1 = p_1 - 1 \geq 1$ and $\bar{m} = \pi^{k'-1} = a^1$; but, this is a contradiction.

Step 3: $h \cdot (\bar{m}; l) \in H^P$ for all $l = 0, \ldots, M-k-1$. Since $h \in H^P$ and $f(h') = \bar{m}$ for all $h' \in H^P$, this step follows by induction from the previous step.

Step 4: $\pi(f|h) = ((\bar{m}; M-k), \pi^1, \pi^2, \ldots)$. By the previous step, f results in $(\bar{m}; M-k)$ after h. Since $T^M(h \cdot (\bar{m}; M-k)) = (\bar{m}; M) \in H_3^0$, it then follows from Claim 3 that $\pi(f|h) = ((\bar{m}; M-k), \pi^1, \pi^2, \ldots)$. ■

The following three claims characterize the consequences of a single deviation by one player from f.

Claim 5 *If $h \in H^E$, $a_i \neq f_i(h)$ and $a_{-i} = f_{-i}(h)$ for some $i \in \{1, 2\}$, then $h \cdot a \in H^P$.*

Proof. Suppose not; then $h \in H^E$, $a_i \neq f_i(h)$, $a_{-i} = f_{-i}(h)$ for some $i \in \{1, 2\}$ and $h \cdot a \in H^k$ for some $0 \leq k \leq M$. There are three different cases to consider.

Case 1: $h \cdot a = (\pi^1, \ldots, \pi^k) \in H_2^k$ for some $k < p_1$. Then we must have $a = \pi^k$, $h \in H_2^{k-1}$ and $k - 1 < p_1$. But then $f(h) = \pi^k = a$; a contradiction.

Case 2: $h \cdot a \in H_1^k$ for some $k \geq p_1$. Then $T^k(h \cdot a) = (\pi^1, \ldots, \pi^k)$, $a = \pi^k$ and $T^{k-1}(h) = (\pi^1, \ldots, \pi^{k-1})$. If $k > p_1$, then $h \in H_1^{k-1}$ and $f(h) = \pi^k = a$; a contradiction. Thus, $k = p_1$, $a = \pi^k = a^1$ and $T^{p_1 - 1}(h) = (a^1; p_1 - 1)$. Also, by construction $p_1 - 1 \geq 1$. Therefore, it follows from the construction of π (a^1 is followed by $a^2 = \bar{m}$) and the definition of f that $f(h) = a^1$ or $f(h) = \bar{m}$. Thus, either $f(h) = a$ or $f_j(h) \neq a_j$ for all $j = 1, 2$. But, both cases contradict our initial supposition that $a_i \neq f_i(h)$ and $a_{-i} = f_{-i}(h)$.

Case 3: $h \cdot a \in H_k^3$ for some $0 \leq k < p_1$. If $k = 0$, then $T^M(h \cdot a) = (\bar{m}; M)$, $a = \bar{m}$ and $T^M(h) = (a', (\bar{m}; M-1))$ for some $a' \in A$. But, since $h \in H^E$, it must also be that $a' = \bar{m}$. Thus, $T^M(h) = (\bar{m}; M)$ and $f(h) = a^1$. But, this is a contradiction, because it implies that $a_{-i} = \bar{m}_{-i} \neq a_{-i}^1 = f_{-i}(h)$. Hence, it must be that $k > 0$. Then, $a = a^1$ and

$T^M(h) = (a', (\bar{m}; M-k), (a^1; k-1))$ for some $a' \in A$. Since $k-1 < p_1$, $h \in H^E$ implies that $a' = \bar{m}$, and thus, $T^M(h) = ((\bar{m}; M-(k-1)), (a^1; k-1))$. But, this is a contradiction because it implies that $f(h) = a^1 = a$. ■

Claim 6 *If $h \in H^E$, $a_i \neq f_i(h)$ and $a_{-i} = f_{-i}(h)$ for some $i \in \{1, 2\}$, then*

$$\pi(f|h \cdot a) = \begin{cases} ((\bar{m}; M), \pi^1, \pi^2, \ldots) & \textit{if } a \neq \bar{m}, \\ ((\bar{m}; M-1), \pi^1, \pi^2, \ldots) & \textit{if } a = \bar{m} \textit{ and } T^1(h) \neq \bar{m}, \\ ((\bar{m}; M-p_2-1), \pi^1, \pi^2, \ldots) & \textit{if } a = T^1(h) = \bar{m}. \end{cases} \tag{10}$$

Proof. By Claim 5, $h \cdot a \in H^P$. Therefore, it follows from Claim 4 that $\pi(f|h \cdot a) = ((\bar{m}; M-k), \pi^1, \pi^2, \ldots)$, where $k = \max\{0 \leq k' \leq M : T^{k'}(h) = (\bar{m}; k')\}$. This means that $\pi(f|h \cdot a) = ((\bar{m}; M), \pi^1, \pi^2, \ldots)$ if $a \neq \bar{m}$ and $\pi(f|h \cdot a) = ((\bar{m}; M-1), \pi^1, \pi^2, \ldots)$ if $a = \bar{m}$ and $T^1(h) \neq \bar{m}$. Finally, consider the case $a = T^1(h) = \bar{m}$. Since $f_{-i}(h) = a_{-i} = \bar{m}_{-i} \neq a^1_{-i}$, we have $f(h) \neq a^1$. This rules out the possibility that $h \in H_2^{k'} \cup H_3^{k'}$ for some $k' < p_1$. Therefore, since $h \in H^E$, it must be that $T^{k'}(h) = (\pi^1, \ldots, \pi^{k'})$ for some $k' \geq p_1$. Also, $\pi^{k'} = T^1(h) = \bar{m}$ and $\pi^{k'+1} = f(h) \neq a = \bar{m}$; therefore, we must have $k' = p_1 + p_2$. But, this implies that $k = p_2 + 1$. Hence, we have $\pi(f|h \cdot a) = ((\bar{m}; M-p_2-1), \pi^1, \pi^2, \ldots)$. ■

Claim 7 *If $h \in H^P$, $a_i \neq f_i(h)$ and $a_{-i} = f_{-i}(h)$ for some $i \in \{1, 2\}$, then $h \cdot a \in H^P$ and $\pi(f|h \cdot a) = ((\bar{m}; M), \pi^1, \pi^2, \ldots)$.*

Proof. It follows from $h \in H^P$ that $f(h) = \bar{m}$. Thus, $a \neq \bar{m}$ and $a \neq a^1$. We will next prove that $h \cdot a \in H^P$ by showing that $h \cdot a \notin H^k$ for any $0 \leq k \leq M$: First, since $\pi^k = a^1$ for any $k < p_1$, $a \neq a^1$ implies that $h \cdot a \notin H_2^k$ for any $k < p_1$. Second, $h \cdot a \notin H_3^k$ for any $0 \leq k < p_1$ because otherwise $a = \bar{m}$ (if $k = 0$) or $a = a^1$ (if $k > 0$); a contradiction. And third, if $h \cdot a \in H_1^k$ for some $k \geq p_1$ then $\pi^k = a \neq \bar{m}$ and $\pi^k = a \neq a^1$. This implies that $k > p_1 + p_2$. Hence, $h \in H_1^{k-1}$ for some $k - 1 \geq p_1$; but, this contradicts $h \in H^P$.

It follows from above that $h \cdot a \in H^P$. Since $a \neq \bar{m}$, it follows from Claim 4 that $\pi(f|h \cdot a) = ((\bar{m}; M), \pi^1, \pi^2, \ldots)$. ■

A.1.3 Incentive conditions

Claim 8 *The strategy profile f is SPE.*

Proof. We demonstrate this result by showing that one-shot deviations are not profitable at any history.

Fix any player i, any $h \in H$ and any strategy $g_i \in F_i$ that only differs from f_i at h; thus $g_i(h) \neq f_i(h)$ and $g_i(h') = f_i(h')$ for all $h' \in H \setminus \{h\}$. We need to show that $U_i(f|h) \geq U_i(g_i, f_{-i}|h)$. To show this consider the two possible cases.

<u>Case 1</u>: *$h \in H^k$ for some* $0 \leq k \leq M$. In this case, by Claim 3 and Claim 6 respectively, $\pi(f|h) = (\pi^{k+1}, \pi^{k+2}, \ldots)$ and $\pi(g_i, f_{-i}|h) = ((a_i, \pi^{k+1}_{-i}), (\bar{m}; t), \pi^1, \pi^2, \ldots)$ for some $a_i \in A_i$ and $t \geq M - (p_2 + 1)$. Then we have

$$\begin{aligned} &U_i(f|h) - U_i(g_i, f_{-i}|h) = V_i^{k+1}(\pi) - [(1-\delta)u_i(a_i, \pi^{k+1}_{-i}) + \delta V_i((\bar{m}; t) \cdot \pi)] \geq \\ &V_i^{k+1}(\pi) - [(1-\delta)B + \delta^{2T+1} V_i(\pi)] \geq \delta^T (1 - \delta^T) V_i(\pi) - (1-\delta)B \end{aligned} \tag{11}$$

where the three inequalities in the above follow, respectively, from $u_i(a_i, \pi^{k+1}_{-i}) \leq B$, $u_i(\bar{m}) = 0$, $t \geq M - (p_2 + 1) \geq M - K = 2T$ and Claim 1. By (9), we have $V_i(\pi) > v_i + \xi \geq \xi$. By (4) and (5), we have $\delta^T(1 - \delta^T)\xi > (1-\delta)T\xi > (1-\delta)B$. Therefore, it follows from (11) that $U_i(f|h) - U_i(g_i, f_{-i}|h) > 0$.

<u>Case 2</u>: *$h \in H^P$*. In this case, by Claim 4 and Claim 7 respectively, $\pi(f|h) = ((\bar{m}; M - t), \pi^1, \pi^2, \ldots)$ for some $0 \leq t < M$ and $\pi(g_i, f_{-i}|h) = ((a_i, \bar{m}_{-i}), (\bar{m}; M), \pi^1, \pi^2, \ldots)$ for some $a_i \in A_i$. Since $u_i(\bar{m}) = 0$ and $u_i(a_i, \bar{m}_{-i}) \leq \max_{a'_i \in A_i} u_i(a'_i, \bar{m}_{-i}) = v_i$, we must then have

$$\begin{aligned} &U_i(f|h) - U_i(g_i, f_{-i}|h) \geq \delta^{M-t} V_i(\pi) - [(1-\delta)v_i + \delta^{M+1} V_i(\pi)] \geq \\ &\delta^M V_i(\pi) - [(1-\delta)v_i + \delta^{M+1} V_i(\pi)] \geq (1-\delta)(\delta^M V_i(\pi) - v_i). \end{aligned} \tag{12}$$

By (9), we have that $V_i(\pi) > v_i + 2\xi$. Also, by (5), we have $\delta^M > \frac{B}{B+\xi} \geq \frac{v_i}{v_i+\xi}$. Therefore, $\delta^M V_i(\pi) - v_i > 0$. But then, by (12), we have $U_i(f|h) - U_i(g_i, f_{-i}|h) > 0$. ■

A.2 More than 2-player case

In this subsection, for convenience, we normalize payoffs so that $v_i = 0$ for all $i \in N$.

Fix any $\varepsilon > 0$ and any $u \in \mathcal{U}^0$. Then, by Theorem 1 (Step 1) in Abreu, Dutta, and Smith (1994), for all $i \in N$ there exists $y^i \in \mathcal{U}^0$ satisfying the following property: for some $0 < \zeta' < \min_i y^i_i$, $y^i_i + \zeta' < u_i$ and $y^i_i + \zeta' < y^j_i$ for all $j \in N$ with $j \neq i$. Define $\xi > 0$ to be such that $4\xi < \zeta'$ and $\zeta = \zeta' - 4\xi$.

Fix any s and s', both in S, such that $s_i \neq s'_i$ for all $i \in N$. For all $k \in \mathbb{N}$, let $\mathcal{V}_k$ be the set of $u' \in \text{co}(u(A))$ such that $u' = \sum_{a \in S} p_a u(a)/k$ for some $\{p_a\}_{a \in A}$ satisfying $p_a \in \mathbb{N}$ and $p_a \geq 2n+2$ for all $a \in S$, $p_{s'} \geq 3$, $p_s \geq 4n+4$ and $\sum_{a \in S} p_a = k$. Using an analogous argument to Sorin (1992, Proposition 1.3), it follows that $\mathcal{V}_k$ converges to $\text{co}(u(A))$. Therefore, there must exist $K \in \mathbb{N}$ such that

$$\text{co}(u(A)) \subseteq \cup_{x \in \mathcal{V}_K} B_\xi(x). \tag{13}$$

For all $\hat{a} \in A$ and $j \in N$, let $D_j(\hat{a}) = \{a \in A : a_{-j} = \hat{a}_{-j}\}$ and $\bar{D}_j(\hat{a}) = D_j(\hat{a}) \backslash \{\hat{a}\}$. Define $D(\hat{a}) = \cup_{j \in N} D_j(\hat{a})$ and $\bar{D}(\hat{a}) = \cup_{j \in N} \bar{D}_j(\hat{a})$. Order all the actions in $A = \{a^1, \ldots, a^r\}$ as follows: $a^1 = s$, $a^2, \ldots, a^{|\bar{D}(s)|+1}$ are the different elements $\bar{D}(s)$, in any order, $a^{|\bar{D}(s)|+2} = s'$, $a^{|\bar{D}(s)|+3}, \ldots, a^{|\bar{D}(s)|+|\bar{D}(s')|+2}$ are the different elements of $\bar{D}(s')$, in any order, and all the remaining actions are then ordered arbitrarily.

To simplify notation, we also denote $y^0 = u$. For all $i \in \{0, \ldots, n\}$, let $x^i \in \mathcal{V}_K$ be such that $||x^i - y^i|| < \xi$ and $\{p_a^i\}_{a \in S}$ be such that $\frac{1}{K} \sum_{a \in S}^K p_a^i u_j(a) = x_j^i$, for all $j \in N$. For all $i \in \{0, \ldots, n\}$, define $\hat{\pi}^{(i)}$ as the repetition of the cycle

$$\left((s'; 2), (s; n+i+2), s', (a^1; p^{(i),1}), \ldots, (a^r; p^{(i),r})\right),$$

where $p^{(i),j} = p_{a^j}^i - 3$ if $a^j = s'$, $p^{(i),j} = p_{a^j}^i - (n+i+2)$ if $a^j = s$ and $p^{(i),j} = p_{a^j}^i$ otherwise. Note that the length of the cycle is K, i.e., $\sum_{j=1}^r p^{(i),j} + n + 5 + i = K$, for all $i \in \{0, \ldots, n\}$. In the construction below, $\hat{\pi}^{(i)}$ will be the equilibrium path when $i = 0$ (also sometimes denoted by $\pi^{(0)}$) and the reward path of player i when $i > 0$.

Let $T \in \mathbb{N}$ be such that

$$T > 2 \max \left\{ (K + n + 6) \frac{B}{\zeta}, K \right\}. \tag{14}$$

Also let $\pi^{(i)} = \left((s'; 2), (s; i+1), s', (m^i; T), \hat{\pi}^{(i)}\right)$ and

$$\pi^{(i)}(\theta, t) = \begin{cases} \left(\pi^{(i),t}, \ldots, \pi^{(i),i+4}, (m^i; (\theta+1)T), \hat{\pi}^{(i)}\right) & \text{if } t \leq i+4 \text{ and} \\ \left((m^i; (\theta+1)T), \hat{\pi}^{(i)}\right) & \text{if } t = i+5, \end{cases}$$

for any $i \in N$, $\theta \in \mathbb{N}_0$ and $t \in \{1, \ldots, d+5\}$. Define the size of the memory $M \in \mathbb{N}$ be such that $M \geq T(n+5) + (n+4) + (2n+5) + K$. Also, let $\delta^* \in (0, 1)$ be such that $\delta \geq \delta^*$ implies

$$\min \left\{ \frac{\delta^K - \delta^{n+6+T}}{2 - \delta^K - \delta^{n+6}}, \frac{\delta^{n+5+(n+4)T}(1 - \delta^T)}{2(1 - \delta^{n+5})}, \frac{\delta^{(n+5)(T+1)}}{2 - 2\delta^{(n+5)(T+1)} - \delta^{n+6}(1 - \delta^T)} \right\} > \frac{B}{\zeta} \tag{15}$$

$$\sup_{x \in [-B, B]^K} \left| \frac{1 - \delta}{1 - \delta^K} \sum_{k=1}^K \delta^{k-1} x^k - \frac{1}{K} \sum_{k=1}^K x^k \right| < \xi. \tag{16}$$

Note that such $\delta^* \in (0,1)$ exists because the limit of the left hand side of (15) and (16) as $\delta \to 1$ are, respectively, $\min\{(T+n+6-K)/(K+n+6), T/2(n+5)\}$ and 0, and the former limit exceeds B/ζ by (14).

Fix any $\delta \geq \delta^*$. We will now demonstrate the result by constructing a M-memory SPE strategy profile f with $||U(f)-u|| < \varepsilon$. Before constructing such a profile, note that the payoffs the different paths $\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)}$ induce satisfy the following.

Claim 9 *For all $i \in \{0, \ldots, n\}$ and $d, d' \in N$ with $d \neq d'$:*

$$-B(1-\delta^{(n+5)(T+1)}) + \delta^{(n+5)(T+1)} V_d(\hat{\pi}^{(d')}) > B(1-\delta^{n+6}) + \delta^{n+6+T} V_d(\hat{\pi}^{(d)}), \quad (17)$$

$$-B(1-\delta^K) + \delta^K V_d(\hat{\pi}^{(i)}) > (1-\delta^{n+6})B + \delta^{n+6+T} V_d(\hat{\pi}^{(d)}), \quad (18)$$

$$-(1-\delta^{n+5})B + \delta^{n+5+(n+4)T} V_d(\hat{\pi}^{(d)}) > (1-\delta^{n+5})B + \delta^{(n+5)(T+1)} V_d(\hat{\pi}^{(d)}). \quad (19)$$

Proof. First, by (16) and the definition of $\{x_j\}_{j=0}^n$, $\{y_j\}_{j=0}^n$, $\{\hat{\pi}^{(j)}\}_{j=0}^n$, ξ, ζ and ζ', we have the following two conditions:

$$V_d(\hat{\pi}^{(d)}) > x_d^d - \xi > y_d^d - 2\xi > \zeta, \quad (20)$$

$$V_d(\hat{\pi}^{(d')}) > x_d^{d'} - \xi > y_d^{d'} - 2\xi > y_d^d + \zeta' - 2\xi > V_d(\hat{\pi}^{(d)}) + \zeta' - 4\xi = V_d(\hat{\pi}^{(d)}) + \zeta. (21)$$

By (15), we have $\delta^{(n+5)(T+1)}\zeta > B(2 - 2\delta^{(n+5)(T+1)} - \delta^{n+6}(1-\delta^T))$. Then (17) follows immediately from (21).

Consider inequality (18). Since $K < T$, (17) implies (18) when $d, i \in N$ and $d \neq i$. Therefore, to demonstrate (18), it suffices to consider two cases: $i = 0$ and $d = i \in N$.

By (21) and (15) we have $(V_d(\hat{\pi}^{(0)}) - V_d(\hat{\pi}^{(d)}))\delta^K > B(2 - \delta^K - \delta^{n+6})$. This, together with $K < T$ and (20), implies that (18) holds when $i = 0$.

By (20) and (15), $V_d(\hat{\pi}^{(d)})(\delta^K - \delta^{n+6+T}) > \zeta(\delta^K - \delta^{n+6+T}) > B(2 - \delta^K - \delta^{n+6})$. This implies that (18) holds when $d = i \in N$.

Finally, consider (19). By (20) and (15), $V_d(\hat{\pi}^{(d)})\delta^{n+5+(n+4)T}(1-\delta^T) > \zeta\delta^{n+5+(n+4)T}(1-\delta^T) > 2B(1-\delta^{n+5})$. This implies that (19) holds. ■

A.2.1 The strategy profile

For all $\tau \in \mathbb{N}$ and $d \in N$, define

$$\Sigma^{d,\tau} = \Big\{h \in H : h = (a^t)_{t=1}^{\tau} \text{ such that } a^t \in D_d(s) \text{ if } t = 3, \ldots, d+3$$
$$\text{and } a^t \in D_d(s') \text{ if } t = 1, 2, d+4\Big\}$$

and $\Sigma^{d,0} = \{H_0\}$ for all $d \in N$. Also, for all $\tau \geq d+4$ and all $h \in \Sigma^{d,\tau}$, let

$$\theta(h) = |\{t \in \{1, 2, d+4\} : a_d^t \neq s_d'\}| + |\{t \in \{3, \ldots, d+3\} : a_d^t \neq s_d\}|.$$

For all $d \in N$ and $\tau \in \mathbb{N}$, define

$$\Gamma^{d,\tau} = \left\{ h \in H : h = (a^t)_{t=1}^{\tau} \text{ and } a^t \in D_d(m^d) \text{ for all } 1 \leq t \leq \tau \right\}.$$

Define for all $k \in \{1, \ldots, M\}$, $i \in \{0, \ldots, n\}$, $d \in N$ and $\tau, r \in \mathbb{N}_0$ the following sets:[23]

$$\begin{aligned}
H_{1,a}^{(i),k} = \;& \left\{ h \in H : T^k(h) = \left(\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),k}\right) \right\}, \\
H_{1,b}^{(i),k} = \;& \left\{ h \in H : h = \left(\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),k}\right) \right\}, \\
H_{1}^{(i),k} = \;& H_{1,a}^{(i),k} \cup H_{1,b}^{(i),k}, \\
H_2^{k,d,\tau} = \;& \Big\{ h \in H : T^k(h) = \bar{h} \cdot a \cdot \tilde{h} \text{ such that for some } k' \leq k \text{ and } i \in \{0, \ldots, n\} \\
& (1) \quad \text{either } \bar{h} \in H_{1,a}^{(i),k'} \text{ with } k' \geq n+i+5 \text{ or } \bar{h} \in H_{1,b}^{(i),k'} \text{ with } \ell(h) = k, \\
& k' < n+5 \text{ and } i = 0, (2) \quad a \in \bar{D}_d(\hat{\pi}^{(i),k'+1}), \quad (3) \quad \tilde{h} \in \Sigma^{d,\tau} \text{ and} \\
& (4) \quad \text{if } T^{d+3}(\bar{h} \cdot a) = ((s'; 2), (s; d), a) \text{ and } a \in \bar{D}_d(s), \text{ then } \ell(\tilde{h}) = 0 \Big\}, \\
H_3^{k,d} = \;& \Big\{ h \in H : T^k(h) = \bar{h} \cdot \tilde{h} \text{ such that} \quad (1) \quad \bar{h} \in \Sigma^{d,d+4} \text{ and} \\
& (2) \quad \tilde{h} \in \Gamma^{d,l} \text{ for some } 0 \leq l < (\theta(\bar{h}) + 1)T \Big\}, \\
H_4^{k,d,r} = \;& \Big\{ h \in H : T^k(h) = \bar{h} \cdot \hat{h} \cdot \tilde{h} \text{ such that } (1) \quad \bar{h} \in \Sigma^{d,d+4}, \\
& (2) \quad \hat{h} \in \Gamma^{d,l} \text{ with } l = (\theta(\bar{h}) + 1)T \text{ and } (3) \quad \tilde{h} \in H_{1,b}^{(d),r} \Big\}, \\
H_5^{k,d,\tau} = \;& \Big\{ h \in H : T^k(h) = \bar{h} \cdot a \cdot \tilde{h} \text{ such that for some } k' \leq k \text{ and } i \in N \\
& (1) \quad \text{either } \bar{h} \in H_3^{k',i}, a \in \bar{D}_d(m^i) \text{ and } d \neq i \\
& \text{or } \bar{h} \in H_4^{k',i,r} \text{ and } a \in \bar{D}_d(\hat{\pi}^{(i),r+1}) \text{ for some } r < n+i+5, \quad (2) \quad \tilde{h} \in \Sigma^{d,\tau} \text{ and} \\
& (3) \quad \text{if } T^{d+3}(\bar{h} \cdot a) = ((s'; 2), (s; d), a) \text{ and } a \in \bar{D}_d(s), \text{ then } \ell(\tilde{h}) = 0 \Big\}.
\end{aligned}$$

We next define $H_{1,a} = \cup_{i=0}^{n} \left(\cup_{k=n+i+5}^{M} H_{1,a}^{(i),k} \right)$, $H_{1,b}^{(0),0} = \{H_0\}$, $H_{1,b} = \cup_{k=0}^{n+4} H_{1,b}^{(0),k}$, $H_1 = H_{1,a} \cup H_{1,b}$, $H_2 = \cup_{k=1}^{M} \left(\cup_{d \in N} \left(\cup_{\tau=0}^{d+3} H_2^{k,d,\tau} \right) \right)$, $H_3 = \cup_{k=1}^{M} \left(\cup_{d \in N} H_3^{k,d} \right)$,
$H_4 = \cup_{k=1}^{M} \left(\cup_{d \in N} \left(\cup_{r=0}^{n+d+4} H_4^{k,d,r} \right) \right)$, and $H_5 = \cup_{k=1}^{M} \left(\cup_{d \in N} \left(\cup_{\tau=0}^{d+3} H_5^{k,d,\tau} \right) \right)$.

[23] Note that, for some of these parameters, the sets below may be empty in some cases.

Let $\tilde{\Sigma}^{d,\tau} = \{h \in H : T^{\tau+1}(h) = a \cdot \tilde{h}, \tilde{h} \in \Sigma^{d,\tau} \text{ and } a \in \bar{D}_d(s) \cup \bar{D}_d(s')\}$ for all $d \in N$ and $\tau \in \mathbb{N}_0$. Define, for all $d \in N$ and $\tau \in \{0, \ldots, d+3\}$,

$$H_6^{d,\tau} = \left(H \setminus \cup_{l=1}^{5} H_l\right) \cap \tilde{\Sigma}^{d,\tau}.$$

Let $H_6 = \cup_{d \in N} \left(\cup_{\tau=0}^{d+3} H_6^{d,\tau}\right)$. Also, for all $t \in \{0, \ldots, n+4\}$, define

$$H_7^t = \{h \in H \setminus \cup_{l=1}^{6} H_l : T^t(h) \in H_{1,b}^{(0),t}\}.$$

The strategy f is now defined as follows: For any $h \in H$,

$$f(h) = \begin{cases} \hat{\pi}^{(0),k+1} & \text{if } h \in H_{1,b}^{(0),k} \text{ for some } k \in \{0, \ldots, n+4\}, \\ \hat{\pi}^{(i),k+1} & \text{if } h \in H_{1,a}^{(i),k} \text{ for some } i \in \{0, \ldots, n\} \text{ and } k \in \{n+i+5, \ldots, M\}, \\ s & \text{if } h \in \left(\cup_{k=1}^{M} \cup_{d \in N} \cup_{\tau=2}^{d+2} (H_2^{k,d,\tau} \cup H_5^{k,d,\tau} \cup H_6^{d,\tau})\right) \cup \left(\cup_{t=2}^{n+3} H_7^t\right), \\ m^d & \text{if } h \in \cup_{k=1}^{M} H_3^{k,d} \text{ for some } d \in N, \\ \hat{\pi}^{(d),r+1} & \text{if } h \in \cup_{k=1}^{M} H_4^{k,d,r} \text{ for some } d \in N \text{ and } r \in \{0, \ldots, n+d+4\}, \\ s' & \text{otherwise.} \end{cases}$$

A.2.2 Auxiliary results

Claim 10 *For all $i \in \{0, \ldots, n\}$, the following hold:*

1. *All actions $a \neq s'$ are played for $t \geq n+i+2$ consecutive periods in $\hat{\pi}^{(i)}$.*
2. *Suppose that, for some $t \in \mathbb{N}$, $(\hat{\pi}^{(i),t}, \ldots, \hat{\pi}^{(i),t+l}) = (s', (s; l))$ and $0 < l < 2n+3$. Then either $\hat{\pi}^{(i),t+l+1} = s$ or $\hat{\pi}^{(i),t+l+1} = s'$ and $l = n+i+2$,*
3. *Suppose that, for some $t \in \mathbb{N}$, $(\hat{\pi}^{(i),t}, \ldots, \hat{\pi}^{(i),t+l+1}) = ((s'; 2), (s; l))$ and $l > 0$. Then either $\hat{\pi}^{(i),t+l+2} = s$ or $\hat{\pi}^{(i),t+l+2} = s'$ and $l = n+i+2$, and*
4. *Suppose that, for some $t \in \mathbb{N}$, $\hat{\pi}^{(i),t} \in D(s')$ and $\hat{\pi}^{(i),t+1} \in D(s)$. Then $\hat{\pi}^{(i),t} = s'$ and $\hat{\pi}^{(i),t+1} = s$. Furthermore, either (i) $\hat{\pi}^{(i),t-1} = s'$ and $t = 2 + \beta K$ or (ii) $\hat{\pi}^{(i),t-1} = s$ and $t = n+i+5+\beta K$ for some $\beta \in \mathbb{N}_0$.*
5. *Suppose that, for some $t \in \mathbb{N}$, $\hat{\pi}^{(i),t} \in D(s)$ and $\hat{\pi}^{(i),t+1} \in D(s')$. Then $\hat{\pi}^{(i),t+1} = s'$.*

Proof. This follows immediately from the ordering of A and the definition of $\hat{\pi}^{(i)}$. ■

Claim 11 *If $h \in H_{1,a}^{(i),k}$ for some $i \in \{0, \ldots, n\}$ and $k \in \{n+i+5, \ldots, M\}$, then $T^{k'}(h) \notin \Sigma^{d,k'}$ for all $d \in N$ and $k' \in \mathbb{N}$ such that $d+4 \leq k' \leq k$.*

Proof. Suppose otherwise; then there exists $h \in H^{(i),k}_{1,a}$ such that $T^{k'}(h) \in \Sigma^{d,k'}$ for some $d \in N$ and $k' \in \mathbb{N}$ with $d+4 \le k' \le k$. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $T^{k'}(h) \in \Sigma^{d,k'}$, we have $a^{k-k'+2} \in D(s')$, $a^{k-k'+3} \in D(s)$ and $a^{k-k'+d+4} \in D(s')$. Therefore, there is an action, namely $a^{k-k'+3}$, which is different from s' and is played at most $d+1$ consecutive periods in $T^k(h)$. Since $h \in H^{(i),k}_{1,a}$, this contradicts Claim 10.1. ■

Next, for all $\tau \in \mathbb{N}$ and $i \in \{0, \ldots, n\}$, define

$$\Lambda^{i,\tau} = \left\{ h \in H : h = (a^t)_{t=1}^{\tau} \text{ such that } a^t = \hat{\pi}^{(i),t} \text{ if } t \le n+i+5 \right\}.$$

Claim 12 *If $h \in H^{(i),k}_{1,a}$ for some $i \in \{0, \ldots, n\}$ and $k \in \{n+i+5, \ldots, M\}$, and $T^{k'}(h) \in \Lambda^{i',k'}$ for some $i' \in \{0, \ldots, n\}$ and $n+i'+5 \le k' \le k$, then $i = i'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{N}_0$.*

Proof. Since $h \in H^{(i),k}_{1,a}$ and $T^{k'}(h) \in \Lambda^{i',k'}$, for some $t = k+1-k'$, $(\hat{\pi}^{(i),t}, \ldots, \hat{\pi}^{(i),t+n+i'+4}) = ((s'; 2), (s; n+i'+2), s')$. It then follows from Claim 10.3 that $n+i'+2 = n+i+2$ and so $i = i'$. Hence, by Claim 10.4, $t+1 = 2+\beta K$ for some $\beta \in \mathbb{N}_0$ and so $k = k' + \beta K$. ■

Claim 13 *Suppose $h \in H^{k,d}_3 \cup H^{k,d,0}_4$ for some $k \in \{1, \ldots, M\}$ and $d \in N$. Then $T^{k'}(h) \notin \Lambda^{i,k'} \cup \Sigma^{d',k'}$ for all $i \in \{0, \ldots, n\}$, $d' \in N$ and $k' \in \mathbb{N}$ such that $3 \le k' < k$.*

Proof. Suppose otherwise; then $T^{k'}(h) \in \Lambda^{i,k'} \cup \Sigma^{d',k'}$ for some $i \in \{0, \ldots, n\}$, $d' \in N$ and $k' \in \mathbb{N}$ such that $3 \le k' < k$. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $T^{k'}(h) \in \Lambda^{i,k'} \cup \Sigma^{d',k'}$ and $h \in H^{k,d}_3 \cup H^{k,d,0}_4$ it follows, respectively, that $a^{k-k'+1}, a^{k-k'+2} \in D_{d'}(s')$ and $a^3, \ldots, a^{d+3} \in D(s)$. Consequently, $k - k' + 1 \ge d + 4$. This means that $a^{k-k'+2}_{-d} = a^{k-k'+3}_{-d} = m^d_{-d}$. Also, by appealing again to $T^{k'}(h) \in \Lambda^{i,k'} \cup \Sigma^{d',k'}$, one has $a^{k-k'+3} \in D_{d'}(s)$. Therefore, by $a^{k-k'+2} \in D_{d'}(s')$, we have that $s_{-d,d'} = s'_{-d,d'}$; a contradiction. ■

Claim 14 *If $h \in H^{k,d,r}_4$ for some $k \in \{1, \ldots, M\}$, $d \in N$ and $r \in \{1, \ldots, n+d+4\}$ and $T^{k'}(h) \in \Lambda^{i,k'}$ for some $i \in \{0, \ldots, n\}$ and $3 \le k' < k$, then $k' = r$ and $k' < n+i+5$.*

Proof. Let $T^k(h) = (a^1, \ldots, a^k)$. By $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+1} = a^{k-k'+2} = s'$ and, by $h \in H^{k,d,r}_4$, $a^t \in D(s)$ for all $t \in \{3, \ldots, d+3\} \cup \{k-r+3, \ldots, k\}$. It then follows that $k' \ge r$ and that $k - k' + 1 \in \{2, d+4, \ldots, k-r+2\}$.

Note, however, that it cannot be that $k - k' + 1 = k - r + 2$; otherwise, $a^{k-r+3} = s'$. Also, $k - k' + 1 = k - r - 1$ is not possible because otherwise, by $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+3} = s$ and, by $h \in H_4^{k,d,r}$, $a^{k-k'+3} = a^{k-r+1} = s'$. Furthermore, $k - k' + 1 = k - r$ is not possible because otherwise, by $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+3} = s$ and, by $h \in H_4^{k,d,r}$, $a^{k-r+2} = s'$.

By Claim 13, it also cannot be that $k - k' + 1 \leq k - r - 2$. The reasoning for this is as follows. Since $h \in H_4^{k,d,r}$ and $T^{k'}(h) \in \Lambda^{i,k'}$, it follows, respectively, that $B^r(h) \in H_4^{k-r,d,0}$ and $T^{k'-r}(B^r(h)) \in \Lambda^{i,k'-r}$. But then by Claim 13 it must be that $k' - r < 3$.

Hence, it follows from all the above that $k - k' + 1 = k - r + 1$, i.e. $k' = r$.

Finally, $k' < n + i + 5$ because otherwise, by $T^{k'}(h) \in \Lambda^{i,k'}$, $a^{k-k'+n+i+5} = s'$ and, by $h \in H_4^{k,d,r}$, $a^{k-r+n+i+5} = s$. ■

Claim 15 *If $h \in H_4^{k,d,r}$ for some $k \in \{1, \ldots, M\}$, $d \in N$ and $r \in \{1, \ldots, n + d + 4\}$, then $T^{k'}(h) \notin \Sigma^{d',k'}$ for all $d' \in N$ and $k' \in \mathbb{N}$ such that $d' + 4 \leq k' < k$.*

Proof. Suppose otherwise. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $h \in H_4^{k,d,r}$ and $T^{k'}(h) \in \Sigma^{d',k'}$, it follows, respectively, that $B^r(h) \in H_4^{k-r,d,0}$ and $T^{k'-r}(B^r(h)) \in \Sigma^{d',k'-r}$. But then by Claim 13 it must be that $k' - r < 3$. Therefore, $k - k' + d' + 4 > k - r + 2$. Hence, $a^{k-k'+d'+4} = a^{k-r+t}$ for some $t > 2$. But this is a contradiction since, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{k-k'+d'+4} \in D(s')$ and, by $h \in H_4^{k,d,r}$, $a^{k-r+t} = s$ for all $t > 2$. ■

Claim 16 *If $h \in H_5^{k,d,\tau}$ for some $k \in \{1, \ldots, M\}$, $d \in N$ and $\tau \in \{0, \ldots, d + 3\}$, then $T^{k'}(h) \notin \Sigma^{d',k'}$ for all $d' \in N$ and $k' \in \mathbb{N}$ such that $d' + 4 \leq k' < k$.*

Proof. Suppose not. Then, there exist $h = (a^1, \ldots, a^t) \in H_5^{k,d,\tau}$ such that $T^{k'}(h) \in \Sigma^{d',k'}$ for the parameters given in the statement of the claim. We now derive a contradiction by considering six different possibilities. Before doing so, let $\bar{h} = B^{\tau+1}(h)$ and note that, by $h \in H_5^{k,d,\tau}$, $T^k(h) \in \Sigma^{i,k}$ for some $i \in N$.

<u>Case 1</u>: $k' - (\tau + 1) \geq d' + 4$. Then, $T^{k'-(\tau+1)}(\bar{h}) \in \Sigma^{d',k'-(\tau+1)}$ and $\bar{h} \in H_3^{k-(\tau+1),i} \cup \left(\cup_{r=0}^{n+i+4} H_4^{k-(\tau+1),i,r}\right)$. But this contradicts either Claim 13 or Claim 15.

<u>Case 2</u>: $k' - (\tau + 1) < d' + 1$. Then, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{t-k'+d'+4} \in D(s')$ and, by $h \in H_5^{k,d,\tau}$ and $t - k' + d' + 4 > t - \tau + 2$, $a^{t-k'+d'+4} \in D(s)$; a contradiction.

<u>Case 3</u>: $k' - (\tau + 1) = d' + 1$. Then, by $T^{k'}(h) \in \Sigma^{d',k'}$, $a^{t-k'+d'+3} \in D(s)$ and, by $h \in H_5^{k,d,\tau}$ and $t - k' + d' + 3 = t - \tau + 1$, $a^{t-k'+d'+3} \in D(s')$; a contradiction.

<u>Case 4</u>: *$k' - (\tau + 1)$ equals either $d' + 2$ or $d' + 3$ and $\bar{h} \in H_3^{k-(\tau+1),i} \cup H_4^{k-(\tau+1),i,0}$.* Then, $3 \leq k' - (\tau + 1) < k - (\tau + 1)$ and $T^{k'-(\tau+1)}(\bar{h}) \in \Sigma^{d',k'-(\tau+1)}$. This contradicts Claim 13.

Case 5: $k' - (\tau+1) = d'+3$ and $\bar{h} \in H_4^{k-(\tau+1),i,r}$ with $r > 0$. By $T^{k'}(h) \in \Sigma^{d',k'}$, (i) $a^{t-k'+2} \in D(s')$, (ii) $a^{t-k'+l} \in D(s)$ for all $3 \leq l \leq d'+3$ and (iii) $a^{t-k'+d'+4} \in D(s')$. By $h \in H_5^{k,d,\tau}$ and $\bar{h} \in H_4^{k-(\tau+1),i,r}$, (iv) $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(i),r+1})$ and (v) $a^{t-\tau-r'} = \hat{\pi}^{(i),r-r'+1}$ for all $1 \leq r' \leq r$.

By $r > 0$ and $t-k'+d'+4 = t-\tau$, (iii) and (iv) imply that either $r = 1$ or $r = n+i+4$. If $r = 1$, by (v), $a^{t-\tau-1} = \hat{\pi}^{(i),1} = s'$. This, together with $t-\tau-1 = t-k'+d'+3$, contradicts (ii). If $r = n+i+4$, then, by (v) and $n+i-d'+3 \geq 3$, $a^{t-\tau-d'-2} = \hat{\pi}^{(i),n+i-d'+3} = s$. This, together with $t-\tau-d'-2 = t-k'+2$, contradicts (i).

Case 6: $k' - (\tau+1) = d'+2$ and $\bar{h} \in H_4^{k-(\tau+1),i,r}$ with $r > 0$. Then, by (ii) and $t-\tau = k-k'+d'+3$, $a^{t-\tau} \in D_{d'}(s)$. Therefore, by (iv), $a^{t-\tau} \in \bar{D}_d(s)$ and $d = d'$.

Next, we show that $r = d+2$. If $r > d+2$, then, by (v), $a^{t-\tau-d-1} = \hat{\pi}^{(i),r-d} = s$. But this is a contradiction because $t-k'+2 = t-\tau-d-1$ and, by (i), $a^{t-k'+2} \in D_d(s')$. If $r < d+2$, then, by (ii), $a^{t-k'+d-r+4} \in D(s)$. But this is a contradiction because $t-\tau-r+1 = t-k'+d-r+4$ and, by (v), $a^{t-\tau-r+1} = \hat{\pi}^{(i),2} = s'$.

By $\bar{h} \in H_4^{k-(\tau+1),i,r}$ and $r = d+2$, we have $T^{d+2}(\bar{h}) = ((s';2),(s;d))$. Since $a^{t-\tau} \in \bar{D}_d(s)$, it then follows from the definition of $H_5^{k,d,\tau}$ that $\tau = 0$. But this contradicts $k'-(\tau+1) = d'+2$ and $k' \geq d'+4$. ■

Claim 17 *If $T^k(h) \in \Sigma^{d,k} \cap \Sigma^{d',k}$ for some $d, d' \in N$ and $k \in \{\min\{d,d'\}+4, \ldots, M\}$, then $d = d'$.*

Proof. Suppose not; assume that $d > d'$. Let $T^k(h) = (a^1, \ldots, a^k)$. Since $T^k(h) \in \Sigma^{d,k}$, $a^{d'+4} \in D_d(s)$ and, since $T^k(h) \in \Sigma^{d',k}$, $a^{d'+4} \in D_{d'}(s')$. But this is a contradiction. ■

Claim 18 *Let $h \in H \setminus \cup_{l=1}^5 H_l$ and $a \in S$. Then, one of the following conditions hold: (a) $h \cdot a \notin \cup_{l=1}^5 H_l$, (b) $h \cdot a \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0,\ldots,n\}$ and (c) $T^{d+4}(h \cdot a) \in \Sigma^{d,d+4}$ for some $d \in N$. Furthermore, if $a \in D(s)$, then $h \cdot a \notin \cup_{l=1}^5 H_l$.*

Proof. Suppose that $h \cdot a$ does not satisfy (a)–(c). Then there are six cases to consider. (i) $h \cdot a \in H_{1,a}^{(i),k}$ *for some i and $k > n+i+5$*: then $h \in H_{1,a}$; a contradiction. (ii) $h \cdot a \in H_{1,b}$: then $h \in H_{1,b}$; a contradiction. (iii) $h \cdot a \in H_2$: then $h \in H_1 \cup H_2$; a contradiction. (iv) $h \cdot a \in H_3$ *and $T^{d+4}(h \cdot a) \notin \Sigma^{d,d+4}$ for all $d \in N$*: then $h \in H_3$; a contradiction. (v) $h \cdot a \in H_4$: then $h \in H_3 \cup H_4$; a contradiction. (vi) $h \cdot a \in H_5$: then $h \in H_3 \cup H_4 \cup H_5$; a contradiction.

Furthermore, if $a \in D(s)$, by the definition of $H_{1,a}^{(i),n+i+5}$ and $\Sigma^{d,d+4}$, (b) and (c) cannot hold. Therefore, (a) must hold, i.e. $h \cdot a \notin \cup_{l=1}^5 H_l$. ■

Claim 19 *If $h \in H_{1,a}^{(i),M}$ for some $i \in \{0,\ldots,n\}$, then $h \in H_{1,a}^{(i),k}$ for some $k < M$.*

Proof. Since $h \in H_{1,a}^{(i),M}$, $T^M(h) = (\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),M})$. Also, by assumption, $M > K + n + i + 5$. Therefore, $h \in H_{1,a}^{(i),M-K}$. ■

Claim 20 *If $h \in H_2^{M,d,\tau}$ for some $d \in N$ and $\tau \in \{0,\ldots,d+3\}$, then $h \in H_2^{k,d,\tau}$ for some $k < M$.*

Proof. Since $h \in H_2^{M,d,\tau}$, $T^M(h) = \bar{h} \cdot a \cdot \tilde{h}$, where $\bar{h}$, a, and $\tilde{h}$ are as in the definition of $H_2^{M,d,\tau}$. In particular, $\bar{h} \in H_{1,a}^{(i),k'} \cup H_{1,b}^{(0),k'}$ for some $i \in \{0,\ldots,n\}$ and $k' \leq M - (\tau+1)$. Therefore, $h \in H_2^{k'+\tau+1,d,\tau}$.

If $k'+\tau+1 < M$, the claim trivially follows. If $k'+\tau+1 = M$, then $\bar{h} = (\hat{\pi}^{(i),1}, \ldots, \hat{\pi}^{(i),M-(\tau+1)})$ for some $i \in \{0,\ldots,n\}$. Also, by assumption, $M \geq (n+4) + (2n+5) + K$. Therefore, $\bar{h} \in H_{1,a}^{(i),M-(\tau+1)-K}$ and hence $h \in H_2^{M-K,d,\tau}$. ■

Claim 21 *If $h \in H_3^{k,d}$ for some $k \in \{1,\ldots,M\}$ and $d \in N$, then $k < M$.*

Proof. Let $\bar{h}$ and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_3^{k,d}$. Then $k = \ell(\bar{h}) + \ell(\tilde{h}) < d + 4 + (\theta(\bar{h})+1)T \leq n + 4 + (n+5)T < M$. ■

Claim 22 *If $h \in H_4^{k,d,r}$ for some $k \in \{1,\ldots,M\}$, $d \in N$ and $r \in \{0,\ldots,n+d+4\}$, then $k < M$.*

Proof. Let $\bar{h}$, $\hat{h}$ and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot \hat{h} \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_4^{k,d,r}$. Then $k \leq \ell(\bar{h}) + \ell(\hat{h}) + \ell(\tilde{h}) = d + 4 + (\theta(\bar{h})+1)T + n + d + 4 \leq n + 4 + (n+5)T + 2n + 4 < M$. ■

Claim 23 *If $h \in H_5^{k,d,\tau}$ for some $k \in \{1,\ldots,M\}$, $d \in N$ and $\tau \in \{0,\ldots,d+3\}$, then $k < M$.*

Proof. Let $\bar{h}$, a and $\tilde{h}$ be such that $T^k(h) = \bar{h} \cdot a \cdot \tilde{h}$ and satisfy the conditions in the definition of $H_5^{k,d,\tau}$. By the proof of Claims 21 and 22, $\ell(\bar{h}) \leq n + 4 + (n+5)T + 2n + 4$. Therefore, $k = \ell(\bar{h}) + \ell(a \cdot \tilde{h}) \leq [n + 4 + (n+5)T + 2n + 4] + 1 + K < M$. ■

A.2.3 Well-definedness of the strategy profile

In this subsection we show that f is well defined.

Claim 24 *If $h \in H_{1,a}^{(i),k} \cap H_{1,a}^{(i'),k'}$ for some $i, i' \in \{0,\ldots,n\}$, $k \in \{n+i+5,\ldots,M\}$ and $k' \in \{n+i'+5,\ldots,M\}$, then $i = i'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{Z}$.*

Proof. It follows immediately by Claim 12. ■

Claim 25 *For all* $i \in \{0, \dots, n\}$*,* $k \in \{n+i+5, \dots, M\}$ *and* $k' \in \{0, \dots, n+4\}$*,* $H^{(i),k}_{1,a} \cap H^{(0),k'}_{1,b} = \emptyset$*.*

Proof. If $h \in H^{(0),k'}_{1,b}$, then $\ell(h) < n+5$, whereas if $h \in H^{(i),k}_{1,a}$, then $\ell(h) \geq n+i+5$ for some $i \in \{0, \dots, n\}$. Hence, $H^{(i),k}_{1,a} \cap H^{(0),k'}_{1,b} = \emptyset$. ■

Claim 26 *For all* $i \in \{0, \dots, n\}$*,* $k \in \{n+i+5, \dots, M\}$*,* $k' \in \{1, \dots, M\}$*,* $d \in N$ *and* $\tau \in \{0, \dots, d+3\}$*,* $H^{(i),k}_{1,a} \cap H^{k',d,\tau}_{2} = \emptyset$*.*

Proof. Suppose not; then there exist $h = (a^1, \dots, a^t) \in h \in H^{(i),k}_{1,a} \cap H^{k',d,\tau}_2$ for some i, k, k', d and τ as described in the claim. Since $h \in H^{(i),k}_{1,a}$, then $\hat{\pi}^{(i),r} = a^{t-k+r}$ for all $1 \leq r \leq k$. Also since $h \in H^{k',d,\tau}_2$, then $T^{k'}(h) = \bar{h} \cdot a^{t-\tau} \cdot \tilde{h}$, where $\tilde{h} \in \Sigma^{d,\tau}$, $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(i'),k'-\tau})$ and $\bar{h} \in H^{(i'),k'-(\tau+1)}_1$ for some $i' \in \{0, \dots, n\}$ satisfying either $k' - (\tau+1) \geq n+i'+5$ or $i' = 0$, $k' = t$ and $k' - (\tau+1) < n+5$. Next, consider each of these possibilities separately.

Case 1: $\bar{h} \in H^{(0),k'-(\tau+1)}_{1,b}$*,* $k' = t$ *and* $k' - (\tau+1) < n+5$. In this case $a^{k'-k+1} = a^{k'-k+2} = s'$, $\bar{h} = ((s';2), s, \dots, s)$ and $\ell(\bar{h}) = k' - (\tau+1)$. Thus, since $\tau \leq d+3$ and $k \geq n+i+5$, it must be that $k' - k + 1 < k' - \tau$ and so $k' = k$. But then $a^{t-\tau} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ and $a^{t-\tau} = \hat{\pi}^{(i),t-\tau} \in \{s, s'\}$, a contradiction.

Case 2: $\bar{h} \in H^{(i'),k'-(\tau+1)}_{1,a}$ *and* $k' - (\tau+1) \geq n+i'+5$. We consider two subcases.

Subcase 1: $k \geq n+i+5+\tau+1$. Let $\hat{h} = B^{\tau+1}(h)$. Since $k - (\tau+1) \geq n+i+5$, it follows that $\hat{h} \in H^{(i),k-(\tau+1)}_{1,a}$. Also, since $T^{k'-(\tau+1)}(\hat{h}) = \bar{h}$, $\hat{h} \in H^{(i'),k'-(\tau+1)}_{1,a}$. Then, by Claim 24, $i = i'$ and $k - (\tau+1) = k' - (\tau+1) + \beta K$ for some $\beta \in \mathbb{Z}$. This, together with $a = \hat{\pi}^{(i),k-\tau}$ and $a \in \bar{D}_d(\hat{\pi}^{(i'),k'-\tau})$, imply that $\hat{\pi}^{(i),k'-\tau} = \hat{\pi}^{(i),k-\tau} \in \bar{D}_d(\hat{\pi}^{(i),k'-\tau})$, a contradiction.

Subcase 2: $k = n+i+5+\tau$. By $h \in H^{(i),k}_{1,a}$, $a^{t-\tau-1} = s$ and $a^{t-\tau} = s'$. Also, by $h \in H^{k',d,\tau}_2$, $a^{t-\tau-1} = \hat{\pi}^{(i'),k'-\tau-1}$ and $a^{t-\tau} \in \bar{D}_d(\hat{\pi}^{(i'),k'-\tau})$. Hence, it follows from $a^{t-\tau-1} = s$ and $a^{t-\tau} = s'$, respectively, that $\hat{\pi}^{(i'),k'-\tau-1} = s$ and $\hat{\pi}^{(i'),k'-\tau} \in \bar{D}_d(s')$. But this contradicts Claim 10.5.

Subcase 3: $k = n+i+4+\tau$. First, we show that $k' \geq k$. Suppose otherwise. Since $\bar{h} \in H^{(i'),k'}_{1,a}$ then $T^{k'}(h) \in \Lambda^{i',k'}$. By Claim 12, this together with $h \in H^{(i),k}_{1,a}$ imply that $k = k' + \beta K$ for some $\beta \in \mathbb{N}$. Also, by the supposition that $k < n+i+5+\tau+1$ and $k' - (\tau+1) \geq n+i'+5$, we have $k - k' \leq n$. Since $n < K$, we have a contradiction.

By $h \in H^{(i),k}_{1,a}$, $(a^{t-\tau-(n+i+3)}, \dots, a^{t-\tau}) = ((s';2), (s;n+i+2))$. Also, by $h \in H^{k',d,\tau}_2$, $(a^{t-\tau-(n+i+3)}, \dots, a^{t-\tau}) = (\hat{\pi}^{(i'),k'-\tau-(n+i+3)}, \dots, \hat{\pi}^{(i'),k'-\tau-1}, a^{t-\tau})$. Hence, it follows from $a^{t-\tau} = s$ that $\hat{\pi}^{(i'),k'-\tau} \in \bar{D}_d(s)$. But this contradicts Claim 10.3.

Subcase 4: $k < n+i+4+\tau$. By $k \geq n+i+5$ and $\tau \leq d+3$, it follows that $\tau > 0$ and $k-\tau \geq 2$. Hence, by $h \in H_2^{k',d,\tau}$, $a^{t-\tau+1} \in D(s')$ and, by $h \in H_{1,a}^{(i),k}$ and $k-\tau < n+i+4$, $a^{t-\tau+1} = s$; but this is a contradiction. ■

Claim 27 *For all $k \in \{0,\dots,n+4\}$, $k' \in \{1,\dots,M\}$, $d \in N$ and $\tau \in \{0,\dots,d+3\}$, $H_{1,b}^{(0),k} \cap H_2^{k',d,\tau} = \emptyset$.*

Proof. Suppose otherwise; then there exists $h \in H_{1,b}^{(0),k} \cap H_2^{k',\tau,d}$. This means that $h = (\pi^{(0),1},\dots,\pi^{(0),k})$ and $T^{k'}(h) = \bar{h} \cdot a \cdot \tilde{h}$ satisfying the remaining conditions in the definition of $H_2^{k',d,\tau}$. Therefore, for some $\hat{k} < n+5$, $\bar{h} \cdot a \in H_{1,b}^{(0),\hat{k}+1}$. But this is a contradiction as, by $h \in H_2^{k',d,\tau}$, $a \in \bar{D}(\hat{\pi}^{(0),\hat{k}+1})$. ■

Claim 28 *If $h \in H_2^{k,d,\tau} \cap H_2^{k',d',\tau'}$ for some $k,k' \in \{1,\dots,M\}$, $d,d' \in N$, $\tau \in \{0,\dots,d+3\}$ and $\tau' \in \{0,\dots,d'+3\}$, then $\tau = \tau'$, $d = d'$ and $k = k' + \beta K$ for some $\beta \in \mathbb{Z}$.*

Proof. Let $h = (a^1,\dots,a^t)$. First, we establish that $\tau = \tau'$. Suppose, without loss of generality, that $\tau < \tau'$. Define $\hat{h} = B^{\tau+1}(h) = (a^1,\dots,a^{t-(\tau+1)})$ and note that $\hat{h} \in \left(\cup_{i=0}^n H_1^{(i),k-(\tau+1)}\right) \cap H_2^{k'-(\tau+1),d',\tau'-(\tau+1)}$. But this contradicts Claims 26 or 27.

By $\tau = \tau'$ and the definition of $H_2^{k,d,\tau}$ and $H_2^{k',d',\tau'}$, we have that $\hat{h} \in H_1^{(i),k-(\tau+1)}$ and $\hat{h} \in H_1^{(i'),k'-(\tau'+1)}$ for some $i,i' \in \{0,\dots,n\}$. It then follows from Claims 24 and 25 that $k = k' + \beta K$ for some $\beta \in \mathbb{Z}$ and $i = i'$. Also, by $h \in H_2^{k,d,\tau}$, $a^{t-\tau} = (a_d, \hat{\pi}_{-d}^{(i),k-\tau})$ and, by $h \in H_2^{k',d',\tau'}$, $a^{t-\tau'} = (a_{d'}, \hat{\pi}_{-d'}^{(i'),k'-\tau'})$. Since $i = i'$, $k = k' + \beta K$ and $\tau = \tau'$, then $(a_d, \hat{\pi}_{-d}^{(i),k-\tau}) = (a_{d'}, \hat{\pi}_{-d'}^{(i),k-\tau})$ and so $d = d'$. ■

Claim 29 *For all $i \in \{0,\dots,n\}$, $k \in \{n+i+5,\dots,M\}$, $k' \in \{1,\dots,M\}$ and $d \in N$, $H_{1,a}^{(i),k} \cap H_3^{k',d} = \emptyset$.*

Proof. Suppose that $h \in H_{1,a}^{(i),k} \cap H_3^{k',d}$. Since $H_3^{k',d} \subseteq \Sigma^{d,k'}$ and, by $h \in H_3^{k',d}$, $k' \geq d+4$, we have a contradiction to Claim 11 when $k \geq k'$. Also, since $H_{1,a}^{(i),k} \subseteq \Lambda^{i,k}$ and $k \geq n+i+5$, we have a contradiction to Claim 13 when $k < k'$. ■

Claim 30 *For all $k \in \{0,\dots,n+4\}$, $k' \in \{1,\dots,M\}$ and $d \in N$, $H_{1,b}^{(0),k} \cap H_3^{k',d} = \emptyset$.*

Proof. Suppose there exists $h = (a^1,\dots,a^t) \in H_{1,b}^{(0),k} \cap H_3^{k',d}$. By $h \in H_3^{k',d}$, $a^{t-k'+d+4} \in D(s')$ and, by $h \in H_{1,b}^{(0),k}$, $a^r = s$ for all $r > 2$. But this is a contradiction as it implies $d+4 \leq t-k'+d+4 \leq 2$. ■

Claim 31 *For all $k,k' \in \{1,\dots,M\}$, $d,d' \in N$ and $\tau \in \{0,\dots,d+3\}$, $H_2^{k,\tau,d} \cap H_3^{k',d'} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_2^{k,d,\tau} \cap H_3^{k',d'}$. There are two possibilities.

<u>Case 1</u>: $k' > k$. First, note that, for some $i \in \{0, \ldots, n\}$, $B^{\tau+1}(h) \in H_{1,a}^{(i),k-(\tau+1)}$ with $k-(\tau+1) \geq n+i+5$; otherwise, $B^{\tau+1}(h) \in H_{1,b}^{(0),k-(\tau+1)}$ and $\ell(h) = k < k'$; a contradiction.

It follows from $B^{\tau+1}(h) \in H_{1,a}^{(i),k-(\tau+1)}$, for some $i \in \{0, \ldots, n\}$, and $k \geq n+i+5$, that $T^k(h) \in \Lambda^{i,k}$. But this contradicts Claim 13 because $h \in H_3^{k',d'}$ and $k' > k \geq n+i+5$.

<u>Case 2</u>: $k \geq k'$. There are two possibilities.

<u>Subcase 1</u>: $k' - \tau > d' + 4$. In this case, $B^{\tau+1}(h)$ belongs to $H_1^{(i),k-(\tau+1)}$, for some $i \in \{0, \ldots, n\}$, and to $H_3^{k'-(\tau+1),d'}$. But this contradicts Claim 29 or 30.

<u>Subcase 2</u>: $k' - \tau \leq d' + 4$. Since $h \in H_3^{k',d'}$, we have (i) $a^{t-k'+r} \in D_{d'}(s')$ for $r = 1, 2$, (ii) $a^{t-k'+r} \in D_{d'}(s)$ for $r = 3, \ldots, d'+3$ and (iii) $a^{t-k'+d'+4} \in D_{d'}(s')$. When $k' - \tau = d' + 4$, by (i), (ii) and $B^{\tau+1}(h) \in H_1^{(i),k-(\tau+1)}$ for some $i \in \{0, \ldots, n\}$, Claim 10.4 implies that $(a^{t-\tau-d'-3}, \ldots, a^{t-\tau-1}) = ((s'; 2), (s; d'+1))$ and $a^{t-\tau} \in \bar{D}_d(s)$. But the latter contradicts (iii). Therefore, it must be that $k' - \tau < d' + 4$.

By $h \in H_2^{k,d,\tau}$, it must be that (iv) $a^{t-\tau+r} \in D_d(s')$ for all $r = 1, 2$ and (v) $a^{t-\tau+r} \in D_d(s)$ for all $r \geq 3$. But then by $t - k' + d' + 4 > t - \tau$, (iii) and (v), it must be that either $t - k' + d' + 4 = t - \tau + 1$ or $t - k' + d' + 4 = t - \tau + 2$. The latter, however, cannot hold because, by (ii), $a^{t-k'+d'+3} \in D(s)$ and, by (iv), $a^{t-\tau+1} \in D(s')$. Therefore, assume the former. This, together with (i), (ii), $B^{\tau+1}(h) \in H_1^{(i),k-(\tau+1)}$ for some $i \in \{0, \ldots, n\}$ and Claim 10.4 imply that $(a^{t-k'+1}, \ldots, a^{t-k'+d'+2}) = ((s'; 2), (s; d'))$ and $a^{t-k'+d'+3} \in \bar{D}_d(s)$. But the latter, together with (ii), implies $d = d'$. Hence, by part (4) of the definition of $H_2^{k,d,\tau}$, $\tau = 0$. Thus, $k' < d' + 4$; but this contradicts $h \in H_3^{k',d'}$. ■

Claim 32 *If $h \in H_3^{k,d} \cap H_3^{k',d'}$ for some $k, k' \in \{1, \ldots, M\}$ and $d, d' \in N$, then $k = k'$ and $d = d'$.*

Proof. First we show that $k = k'$. Suppose otherwise and assume, without loss of generality, that $k > k'$. By $h \in H_3^{k',d'}$, we have $T^{k'}(h) \in \Sigma^{d',k'}$ and $k > k' \geq d' + 4$. But this contradicts Claim 13 because $h \in H_3^{k,d}$.

To show that $d = d'$, by $h \in H_3^{k,d} \cap H_3^{k,d'}$, we have $T^k(h) \in \Sigma^{d,k} \cap \Sigma^{d',k}$, $k \geq d + 4$ and $k \geq d' + 4$. Hence, by Claim 17, $d = d'$. ■

Claim 33 *For all $i \in \{0, \ldots, n\}$, $k \in \{n+i+5, \ldots, M\}$, $k' \in \{1, \ldots, M\}$, $d \in N$ and $r \in \{0, \ldots, n+d+4\}$, $H_{1,a}^{(i),k} \cap H_4^{k',d,r} = \emptyset$.*

Proof. Suppose that $h \in H_{1,a}^{(i),k} \cap H_4^{k',d,r}$. Then, $T^k(h) \in \Lambda^{i,k}$, $k \geq n+i+5$, $T^{k'}(h) \in \Sigma^{d,k'}$ and $k' \geq d+4$. But these contradict Claim 11 if $k \geq k'$ and Claim 14 if $k < k'$. ■

Claim 34 *For all $k \in \{0, \ldots, n+4\}$, $k' \in \{1, \ldots, M\}$, $d \in N$ and $r \in \{0, \ldots, n+d+4\}$, $H_{1,b}^{(0),k} \cap H_4^{k',d,r} = \emptyset$.*

Proof. If $h \in H_4^{k',d,r}$ then $\ell(h) > T$. If $h \in H_{1,b}^{(0),k}$ then $\ell(h) < n+5 < T$ by (14). ■

Claim 35 *For all $k, k' \in \{1, \ldots, M\}$, $d, d' \in N$, $r \in \{0, \ldots, n+d'+4\}$ and $\tau \in \{0, \ldots, d+3\}$, $H_2^{k,d,\tau} \cap H_4^{k',d',r} = \emptyset$.*

Proof. Suppose that $h \in H_2^{k,d,\tau} \cap H_4^{k',d',r}$. By $h \in H_2^{k,d,\tau}$, $B^{\tau+1}(h) \in H_1^{(i),k-(\tau+1)}$. Also, since $h \in H_4^{k',d',r'}$ and $T > \tau+1$, then $B^{\tau+1}(h)$ belongs to $H_4^{k'-(\tau+1),d',r-(\tau+1)}$ if $r-(\tau+1) \geq 0$ or to $H_3^{k'-(\tau+1),d'}$ otherwise. But, by Claims 29, 30, 33 or 34, this is a contradiction. ■

Claim 36 *For all $k, k' \in \{1, \ldots, M\}$, $d, d' \in N$ and $r \in \{0, \ldots, n+d'+4\}$, $H_3^{k,d} \cap H_4^{k',d',r} = \emptyset$.*

Proof. Suppose that $h \in H_3^{k,d} \cap H_4^{k',d',r}$. Assume first that $k = k'$. By Claim 17, this implies $d = d'$. Since $h \in H_3^{k,d}$, it follows that $k < (\theta(T^k(h))+1)T + d + 4$ and, since $h \in H_4^{k',d',r}$, $k = k'$ and $d = d'$, it follows that $k \geq (\theta(T^k(h))+1)T + d + 4$. But this is a contradiction.

Suppose next that $k > k'$. Then, by $h \in H_4^{k',d',r}$, $T^{k'}(h) \in \Sigma^{d',k'}$ with $d'+4 \leq k' < k$. But this together with $h \in H_3^{k,d}$, contradicts Claim 13.

Finally, suppose that $k' > k$. Then, by $h \in H_3^{k,d}$, $T^k(h) \in \Sigma^{d,k}$ with $d+4 \leq k < k'$. But this together with $h \in H_4^{k',d',r}$ contradicts Claim 15. ■

Claim 37 *If $h \in H_4^{k,d,r} \cap H_4^{k',d',r'}$ for some $k, k' \in \{1, \ldots, M\}$, $d, d' \in N$ and $r, r' \in \mathbb{N}_0$, then $k = k'$, $d = d'$ and $r = r'$.*

Proof. To show that $k = k'$, suppose, without loss of generality, that $k > k'$. Then, by $h \in H_4^{k',d',r'}$, $T^{k'}(h) \in \Sigma^{d',k'}$ with $d'+4 \leq k' < k$. But this together with $h \in H_4^{k,d,r}$ contradicts Claim 15. Hence, $k = k'$ and, by Claim 17, $d = d'$. Thus, $r = k - (d+4+(\theta(T^k(h))+1)T) = k' - (d'+4+(\theta(T^k(h))+1)T) = r'$. ■

Claim 38 *For all $i \in \{0, \ldots, n\}$, $k \in \{n+i+5, \ldots, M\}$, $k' \in \{1, \ldots, M\}$, $d \in N$ and $\tau \in \{0, \ldots, d+3\}$, $H_{1,a}^{(i),k} \cap H_5^{k',d,\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_{1,a}^{(i),k} \cap H_5^{k',d,\tau}$. Then $h \in H_{1,a}^{(i),k}$, $T^{k'}(h) \in \Sigma^{d,k'}$ and $k' \geq d+4$. Therefore, $k' > k$; otherwise we would contradict Claim 11.

Consider next the case $k < k'$ and let $\hat{h} = B^{\tau+1}(h)$. Then, $\hat{h} \in H_3^{k'-(\tau+1),d'} \cup H_4^{k'-(\tau+1),d',r}$ for some $d' \in N$ and $0 \leq r \leq n+d'+4$. Also, by $h \in H_{1,a}^{(i),k}$, $T^{k-(\tau+1)}(\hat{h}) \in \Lambda^{i,k-(\tau+1)}$.

Furthermore, since $0 \le \tau \le d+3 \le n+3$, we have that $k-(\tau+1) \ge 1$. Therefore, by Claims 13 and 14, one of the following must hold: (1) $k-(\tau+1)=1$, (2) $k-(\tau+1)=2$ and (3) $B^{\tau+1}(h) \in H_4^{k'-(\tau+1),d',r}$, $r>0$, $k-(\tau+1)=r$ and $3 \le k-(\tau+1) < n+i+5$.

Case (1) implies that $t-\tau+1 = t-k+3$ and case (2) implies that $t-\tau+1 = t-k+4$. Since $T^k(h) \in H_{1,a}^{(i),k}$, we have $a^{t-k+3} = a^{t-k+4} = s$, and so, $a^{t-\tau+1} = s$ in both cases. Since $h \in H_5^{k',d,\tau}$, we also have $T^\tau(h) \in \Sigma^{d,\tau}$. But this implies $a^{t-\tau+1} \in D_d(s')$; a contradiction.

In case (3), we have that $t-\tau \le t-k+n+i+5$. This, together with $h \in H_{1,a}^{(i),k}$, imply that $a^{t-\tau} \in \{s,s'\}$. Since $h \in H_5^{k',d,\tau}$, $B^{\tau+1}(h) \in H_4^{k'-(\tau+1),d',r}$ and $\hat{\pi}^{(d'),r+1} \in \{s,s'\}$, it must be that $a^{t-\tau} \in \bar{D}_d(s) \cup \bar{D}_d(s')$; a contradiction. ■

Claim 39 *For all $k \in \{0,\dots,n+4\}$, $k' \in \{1,\dots,M\}$, $d \in N$ and $\tau \in \{0,\dots,d+3\}$, $H_{1,b}^{(0),k} \cap H_5^{k',d,\tau} = \emptyset$.*

Proof. Suppose $h=(a^1,\dots,a^t) \in H_{1,b}^{(0),k} \cap H_5^{k',d,\tau}$. By $h \in H_5^{k',d,\tau}$, $a^{t-k'+1}, a^{t-k'+i+4} \in D(s')$ for some $i \in N$ and, by $h \in H_{1,b}^{(0),k}$, $h = ((s';2),(s;k-2))$; a contradiction. ■

Claim 40 *For all $k,k' \in \{1,\dots,M\}$, $d,d' \in N$, $\tau \in \{0,\dots,d+3\}$ and $\tau' \in \{0,\dots,d'+3\}$, $H_2^{k,d,\tau} \cap H_5^{k',d',\tau'} = \emptyset$.*

Proof. Suppose that $h \in H_2^{k,d,\tau} \cap H_5^{k',d',\tau'}$. Assume first that $\tau \le \tau'$. Then, $B^{\tau+1}(h) \in H_1$ and $B^{\tau+1}(h) \in H_3 \cup H_4 \cup H_5$. But, this contradicts Claims 29, 30, 33, 34, 38 or 39.

Suppose next that $\tau > \tau'$. Then, $B^{\tau'+1}(h) \in H_2$ and $B^{\tau'+1} \in H_3 \cup H_4$. But, this contradicts Claim 31 or Claim 35. ■

Claim 41 *For all $k,k' \in \{1,\dots,M\}$, $d,d' \in N$ and $\tau \in \{0,\dots,d'+3\}$, $H_3^{k,d} \cap H_5^{k',d',\tau} = \emptyset$.*

Proof. Suppose that $h=(a^1,\dots,a^t) \in H_3^{k,d} \cap H_5^{k',d',\tau}$. By $h \in H_3^{k,d}$, $T^k(h) \in \Sigma^{d,k}$ with $k \ge d+4$, and, by $h \in H_5^{k',d',\tau}$, $T^{k'}(h) \in \Sigma^{i',k'}$ with $k' \ge i'+4$ for some $i' \in N$. By appealing to Claims 13, 16 and 17, it then follows that $k=k'$ and $d=i'$.

By $h \in H_5^{k',d',\tau}$ and the last two equalities, $k-(\tau+1) \ge d+4$. But then, by $h \in H_3^{k,d}$, we have $B^{\tau+1}(h) \in H_3^{k-(\tau+1),d}$. This, together with $h \in H_5^{k',d',\tau}$, implies $a^{t-\tau} \in \bar{D}_{d'}(m^d)$ and $d' \ne d$. But, since $h \in H_3^{k,d}$ and $k-(\tau+1) \ge d+4$, $a^{t-\tau} \in D_d(m^d)$; a contradiction. ■

Claim 42 *For all $k,k' \in \{1,\dots,M\}$, $d,d' \in N$, $r \in \{0,\dots,n+d+4\}$ and $\tau \in \{0,\dots,d'+3\}$, $H_4^{k,d,r} \cap H_5^{k',d',\tau} = \emptyset$.*

Proof. Suppose that $h = (a^1, \ldots, a^t) \in H_4^{k,d,r} \cap H_5^{k',d',\tau}$. By $h \in H_4^{k,d,r}$, $T^k(h) \in \Sigma^{d,k}$ with $k \geq d+4$, and, by $h \in H_5^{k',d',\tau}$, $T^{k'}(h) \in \Sigma^{i',k'}$ with $k' \geq i'+4$ for some $i' \in N$. By appealing to Claims 15 and 16, it then follows that $k = k'$.

Suppose $r \geq \tau+1$. By $h \in H_4^{k,d,r}$, $B^{\tau+1}(h) \in H_4^{k-(\tau+1),d,r-(\tau+1)}$. This implies that $a^{t-\tau} = \hat{\pi}^{(d),r-\tau}$ and, by $h \in H_5^{k',d',\tau}$ and Claims 36 and 37, $a^{t-\tau} \in \bar{D}_{d'}(\hat{\pi}^{(d),r-\tau})$; a contradiction.

Finally, suppose $r < \tau+1$. By $h \in H_4^{k,d,r}$, $B^{\tau+1}(h) \in H_3^{k-(\tau+1),d}$. This implies that $a^{t-\tau} \in D_d(m^d)$ and, by $h \in H_5^{k',d',\tau}$ and Claims 32 and 36, $a^{t-\tau} \in \bar{D}_{d'}(m^d)$ and $d' \neq d$; a contradiction. ■

Claim 43 *If $h \in H_5^{k,d,\tau} \cap H_5^{k',d',\tau'}$ for some $k, k' \in \{1, \ldots, M\}$, $d, d' \in N$, $\tau \in \{0, \ldots, d+3\}$ and $\tau' \in \{0, \ldots, d'+3\}$, then $k = k'$, $\tau = \tau'$ and $d = d'$.*

Proof. Suppose $h \in H_5^{k,d,\tau} \cap H_5^{k',d',\tau'}$. First, note that $\tau = \tau'$. Otherwise, say $\tau < \tau'$; then, by $h \in H_5^{k,d,\tau}$, $B^{\tau+1}(h) \in H_5$ and, by $h \in H_5^{k',d',\tau'}$, $B^{\tau+1}(h) \in H_3 \cup H_4$. This contradicts Claim 41 or Claim 42. Second, note that $B^{\tau+1}(h) \in \cup_{i\in N}\left(H_3^{k-(\tau+1),i} \cup \left(\cup_r H_4^{k-(\tau+1),i,r}\right)\right)$ and $B^{\tau'+1}(h) \in \cup_{i\in N}\left(H_3^{k'-(\tau'+1),i} \cup \left(\cup_r H_4^{k'-(\tau'+1),i,r}\right)\right)$. Since $\tau = \tau'$, by Claims 32, 36 and 37, $k = k'$. Finally, it follows immediately from the definitions of $H_5^{k,d,\tau}$ and $H_5^{k',d',\tau'}$, $k = k'$ and $\tau = \tau'$ that $d = d'$. ■

Claim 44 *If $h \in H_6^{d,\tau} \cap H_6^{d',\tau'}$ for some $d, d' \in N$, $\tau \in \{0, \ldots, d+3\}$ and $\tau' \in \{0, \ldots, d'+3\}$, then $d = d'$.*

Proof. Let $h = (a^1, \ldots, a^t) \in H_6^{d,\tau} \cap H_6^{d',\tau'}$. We may assume, without loss of generality, that $\tau \geq \tau'$. Then, by $h \in H_6^{d',\tau'}$, $a^{t-\tau'} \in \bar{D}_{d'}(s) \cup \bar{D}_{d'}(s')$ and, by $\tau \geq \tau'$ and $h \in H_6^{d,\tau}$, $a^{t-\tau'} \in D_d(s) \cup D_d(s')$. Thus, $d = d'$. ■

A.2.4 Outcome paths induced by f and by one-shot deviations from f

Claim 45 *If $h \in H_{1,a}^{(i),k}$ for some $i \in \{0, \ldots, n\}$ and $k \in \{n+i+5, \ldots, M\}$, then $\pi(f|h) = (\hat{\pi}^{(i),k+1}, \hat{\pi}^{(i),k+2}, \ldots)$.*

Proof. By Claim 19, we may assume that $k < M$. Hence, $f(h) = \hat{\pi}^{(i),k+1}$ and $h \cdot f(h) \in H_{1,a}^{(i),k+1}$. Thus, by induction, $\pi(f|h) = (\hat{\pi}^{(i),k+1}, \hat{\pi}^{(i),k+2}, \ldots)$. ■

Claim 46 *If $h \in H_{1,b}^{(0),k}$ for some $k \in \{0, \ldots, n+4\}$, then $\pi(f|h) = (\pi^{(0),k+1}, \pi^{(0),k+2}, \ldots)$.*

Proof. If $k < n+4$, then $h \cdot f(h) \in H_{1,b}^{(0),k+1}$. If $k = n+4$, then $h \cdot f(h) \in H_{1,a}^{(0),n+5}$. Thus, by Claim 45, $\pi(f|h) = (\pi^{(0),k+1}, \pi^{(0),k+2}, \ldots)$. ■

Claim 47 *If $h \in H_4^{k,d,r}$ for some $k \in \{1, \ldots, M\}$, $d \in N$ and $r \in \{0, \ldots, n+d+4\}$, then $\pi(f|h) = (\hat{\pi}^{(d),r+1}, \hat{\pi}^{(d),r+2}, \ldots)$.*

Proof. If $r = n+d+4$, then $h \cdot f(h) \in H_{1,a}^{(d),n+d+5}$, thus, $\pi(f|h) = (\hat{\pi}^{(d),r+1}, \hat{\pi}^{(d),r+2}, \ldots)$ by Claim 45. If $r < n+d+4$, then $f(h) = \hat{\pi}^{(d),r+1}$ and therefore $h \cdot f(h) \in H_4^{k+1,d,r+1}$. Furthermore, by Claim 22, $k < M$. Hence, by induction, $\pi(f|h) = (\hat{\pi}^{(d),r+1}, \hat{\pi}^{(d),r+2}, \ldots)$. ■

Claim 48 *If $h \in H_3^{k,d}$ for some $k \in \{1, \ldots, M\}$ and $d \in N$, then $\pi(f|h) = ((m^d; (\theta+1)T - k + d + 4), \hat{\pi}^{(d),1}, \hat{\pi}^{(d),2}, \ldots)$ where $\theta = \theta(T^k(h))$.*

Proof. Since $f(h) = m^d$, we have $h \cdot f(h) \in H_4^{k+1,d,0}$ if $k - (d+4) = (\theta+1)T - 1$ and $h \cdot f(h) \in H_3^{k+1,d}$ if $k - (d+4) < (\theta+1)T - 1$. Also, by Claim 21, $k < M$. Therefore, by Claim 47, the claim follows by induction. ■

Claim 49 *Let $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau} \cup H_6^{d,\tau}$ for some $k \in \{1, \ldots, M\}$, $d \in N$ and $\tau \in \{0, \ldots, d'+3\}$ and $\bar{a} \in D_d(f(h))$. Then there exists $\bar{\theta}(\bar{a}) \in \{0, \ldots, d+4\}$ and $t(\bar{a}) \in \{1, \ldots, d+5\}$ such that $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$ and $\bar{\theta}(f(h)) < \bar{\theta}(\bar{a})$ for any $\bar{a} \in \bar{D}_d(f(h))$.*

Proof. We establish this claim by considering the different possible cases.

<u>Case 1</u>: One of the following conditions hold: (a) $\tau = d+3$, (b) $\tau = 0$, $T^{d+3}(h) = ((s'; 2), (s; d), a)$ and $a \in \bar{D}_d(s)$, and (c) $h \in H_6^{d,0}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$. In this case, it follows that $f(h) = s'$, $\bar{a} \in D_d(s')$ and $h \cdot \bar{a} \in H_3^{d+4,d}$. Thus, by Claim 48, $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$ where $t(\bar{a}) = d+5$ and $\bar{\theta}(\bar{a}) = \theta(T^{d+3}(h) \cdot \bar{a})$. Furthermore, if $\bar{a} \in \bar{D}_d(f(h))$, then $\theta(T^{d+3}(h) \cdot \bar{a}) = \theta(T^{d+3}(h) \cdot f(h)) + 1$. Hence, $\bar{\theta}(f(h)) < \bar{\theta}(\bar{a})$.

<u>Case 2</u>: $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau}$ and none of the conditions (a)–(c) hold. In this case, $f(h) = \pi^{(d),\tau+1}$ and $h \cdot \bar{a} \in H_2^{k+1,d,\tau+1} \cup H_5^{k+1,d,\tau+1}$. Thus, by Claims 20 and 23, $f(h \cdot \bar{a}) = \hat{\pi}^{(d),\tau+2}$. Then, by induction, it follows that $f(h \cdot \bar{a} \cdot \hat{\pi}^{(d),\tau+2}) = \hat{\pi}^{(d),\tau+3}, \ldots, f(h \cdot \bar{a} \cdot \hat{\pi}^{(d),\tau+2} \cdot \ldots \cdot \hat{\pi}^{(d),d+2}) = \hat{\pi}^{(d),d+3}$. Therefore, by appealing to Case 1, we have $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$ where $t(\bar{a}) = \tau + 2$ and $\bar{\theta}(\bar{a}) = \theta(T^\tau(h) \cdot (\bar{a}, \pi^{(d),\tau+2}, \ldots, \pi^{(d),d+4}))$. It also follows from the latter that $\bar{\theta}(\bar{a}) = \bar{\theta}(f(h)) + 1$ if $\bar{a} \in \bar{D}_d(f(h))$.

<u>Case 3</u>: $h \in H_6^{d,\tau}$ and none of the conditions (a)–(c) hold. For any history $h' \in H$, define $v(h') = \max\{t \in \{0, \ldots, d+3\} : T^t(h) \in H_6^{d,t}\}$ and let $\tau' = v(h)$. Then (i) $h \in H_6^{d,\tau'}$, (ii) $T^1(h) \in \bar{D}_d(s) \cup \bar{D}_d(s')$ if $\tau' = 0$, (iii) $T^1(h) \in D_d(s')$ if $\tau' = 1$ and (iv) $f(h) = \pi^{(d),\tau'+1}$. To complete the proof in this case, we first establish two subclaims.

Subclaim 1: $h \cdot \bar{a} \in H_6^{d,\tau'+1}$. Suppose not. By (iv), $h \cdot \bar{a} \in \tilde{\Sigma}^{d,\tau'+1}$ and hence $h \cdot \bar{a} \in \cup_{l=1}^{5} H_l$. Also, if $\tau' \geq 2$, then $\bar{a} \in D(s)$. Therefore, by Claim 18, $\tau' = 0$ or 1 and either $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$ or $T^{\hat{d}+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+4}$ for some $\hat{d} \in N$.

If $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$, then $T^1(h) = s$, contradicting (ii) and (iii). If $T^{\hat{d}+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+4}$, then $T^1(h) \in D_{\hat{d}}(s)$. Therefore, by (iii), $\tau' = 0$. Then, by (ii), it follows that $T^1(h) \in \bar{D}_{\hat{d}}(s)$ and $d = \hat{d}$. This implies that $T^{d+3}(h) \in \Sigma^{d,d+3}$. Thus, by (i) and $\tau' = 0$, condition (c) is satisfied. But this contradicts our supposition.

Subclaim 2: $v(h \cdot \bar{a}) = \tau' + 1$. Since $h \cdot \bar{a} \in H_6^{d,\tau'+1}$, $v(h \cdot \bar{a}) \geq \tau' + 1 > 0$. Therefore, $h \in \tilde{\Sigma}^{d,v(h \cdot \bar{a})-1}$. Hence, $\tau' \geq v(h \cdot \bar{a}) - 1$ and thus $v(h \cdot \bar{a}) = \tau' + 1$.

It follows from the above two subclaims that $f(h \cdot \bar{a}) = \hat{\pi}^{(d),\tau'+2}$. Then, by induction, it follows that $f(h \cdot \bar{a} \cdot \hat{\pi}^{(d),\tau'+2}) = \hat{\pi}^{(d),\tau'+3}, \ldots, f(h \cdot \bar{a} \cdot \hat{\pi}^{(d),\tau'+2} \cdot \cdots \cdot \hat{\pi}^{(d),d+2}) = \hat{\pi}^{(d),d+3}$. Therefore, by appealing to Case 1, we have $\pi(f|h \cdot \bar{a}) = \pi^{(d)}(\bar{\theta}(\bar{a}), t(\bar{a}))$, where $t(\bar{a}) = \tau' + 2$ and $\bar{\theta}(\bar{a}) = \theta(T^{\tau'}(h) \cdot (\bar{a}, \pi^{(d),\tau'+2}, \ldots, \pi^{(d),d+4}))$. Furthermore, from the latter, $\bar{\theta}(\bar{a}) = \bar{\theta}(f(h)) + 1$ for all $\bar{a} \in \bar{D}_{d'}(f(h))$. ■

Claim 50 *Let $h \in H_7^k$ for some $k \in \{0, \ldots, n+4\}$. If $k \notin \{2, \ldots, n+3\}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$ for some $d \in N$, then $\pi(f|h) = \big(s', (m^d; (\theta+1)T), \hat{\pi}^{(d),1}, \ldots\big)$, where $\theta = \theta(T^{d+3}(h) \cdot s')$. Otherwise, $\pi(f|h) = \big(\hat{\pi}^{(i),k'+1}, \hat{\pi}^{(i),k'+2}, \ldots\big)$ for some $i \in \{0, \ldots, n\}$ and $k' \in \{0, \ldots, n+i+4\}$.*

Proof. If $k \notin \{2, \ldots, n+3\}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$ for some $d \in N$, then $f(h) = s'$ and $T^{d+4}(h \cdot f(h)) \in \Sigma^{d,d+4}$. Hence, $h \cdot f(h) \in H_3^{d+4,d}$ and the conclusion follows from Claim 48.

If $k \notin \{2, \ldots, n+3\}$ and $T^{n+i+4}(h) = ((s'; 2), (s; n+i+2))$ for some $i \in \{0, \ldots, n\}$, then $f(h) = s'$ and $T^{n+i+5}(h) = ((s'; 2), (s; n+i+2), s')$. Hence, $h \cdot f(h) \in H_{1,a}^{(i),n+i+5}$ and the conclusion follows from Claim 45. For the remainder of the proof, therefore assume that the following holds:

$$\begin{aligned} &\text{if } k \notin \{2, \ldots, n+3\}, \text{ then } T^{n+i+4}(h) \neq ((s'; 2), (s; n+i+2)), \\ &\text{for all } i \in \{0, \ldots, n\}, \text{ and } T^{d+3}(h) \notin \Sigma^{d,d+3}, \text{ for all } d \in N. \end{aligned} \tag{22}$$

Next, for any history $h' \in H$, define $v(h') = \max\{t \in \{0, \ldots, n+4\} : T^t(h') \in H_{1,b}^{(t),0}\}$ and let $k' = v(h)$. If $k' = n+4$, then $f(h) = s'$ and $h \cdot f(h) \in H_{1,a}^{(0),n+5}$, and the conclusion follows from Claim 45.

If $k' \in \{0, \ldots, n+3\}$, then the claim follows by induction if it is the case that $h \cdot f(h) \in H_7^{k'+1}$ and $v(h \cdot f(h)) = k'+1$. Next, we show that the latter is indeed the case in several steps.

Step 1: $h \cdot f(h) \notin H_6$. Otherwise, $h \cdot f(h) \in H_6^{d,\tau}$ for some $d \in N$ and $\tau \in \{0, \ldots, d+3\}$. Then if $\tau > 0$, $T^{\tau-1}(h) \in \tilde{\Sigma}^{d,\tau-1}$; but this is a contradiction as this implies that $h \in H_6$. If $\tau = 0$, then by the definition of H_6^0, $f(h) = T^1(h \cdot f(h)) \in \bar{D}(s) \cup \bar{D}(s')$. But this is a contradiction because, by $h \in H_7^{k'}$, $f(h) \in \{s, s'\}$.

Step 2: $h \cdot f(h) \notin \cup_{l=1}^5 H_l$. When $k' \geq 2$, then $f(h) = s$. Therefore, the claim in this step follows immediately from Claim 18. Next suppose that $k' \in \{0,1\}$ and that $h \cdot f(h) \in \cup_{l=1}^5 H_l$. Then Claim 18 implies that either $h \in H_{1,a}^{(i),n+i+4}$ for some $i \in \{0, \ldots, n\}$ or $T^{d+3}(h) \in \Sigma^{d,d+3}$ for some $d \in N$. But this contradicts our supposition in (22).

Step 3: $T^{k'+1}(h \cdot f(h)) \in H_{1,b}^{(0),k'+1}$ *and* $v(h \cdot f(h)) = k'+1$. By $h \in H_7^{k'}$, $f(h) = \pi^{(0),k'+1}$ and so $T^{k'+1}(h \cdot f(h)) \in H_{1,b}^{(0),k'+1}$. Hence, $v(h \cdot f(h)) \geq k'+1$. If it were the case that $v(h \cdot f(h)) > k'+1$, then $T^{v(h \cdot f(h))-1}(h) \in H_{1,b}^{(0),v(h \cdot f(h))-1}$, implying $v(h) \geq v(h \cdot f(h)) - 1$. Since $k' = v(h)$ and $v(h \cdot f(h)) > k'+1$, this is a contradiction. ■

Claim 51 *Let $h \in H_1$ and $\bar{a} \in \bar{D}_d(f(h))$ for some $d \in N$. Then $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^d(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$.*

Proof. By assumption, $h \in H_{1,a}^{(i),k} \cup H_{1,b}^{(0),l}$ for some i, k and l, and $\bar{a} \in \bar{D}_d(f(h))$. Thus, $h \cdot \bar{a} \in H_2^{k+1,d,0}$. Also, by Claim 49, $\pi(f|h \cdot \bar{a} \cdot f(h \cdot \bar{a})) = \pi^{(d)}(\theta, t)$ for some θ and t. ■

Claim 52 *Let $h \in H_2^{k,d',\tau'} \cup H_5^{k,d',\tau'} \cup H_6^{d',\tau'}$ for some $k \in \{1, \ldots, M\}$, $d' \in N$ and $\tau' \in \{0, \ldots, d'+3\}$. Let $d \neq d'$ and $\bar{a} \in \bar{D}_d(f(h))$. Then $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$.*

Proof. We first argue that it is sufficient to show that either $h \cdot \bar{a} \in H_3^{d+4,d}$ or $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. If the former holds, then the claim follows from Claim 48. If the latter holds, then this together with $f(h) \in \{s, s'\}$ and $\bar{a} \in \bar{D}_d(f(h))$ imply that $h \cdot \bar{a} \in H_6^{d,0}$. Then the claim follows from Claim 49.

We next establish that either $h \cdot \bar{a} \in H_3^{d+4,d}$ or $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. Suppose not; then we derive a contradiction for the different possible cases as follows.

<u>*Case 1*</u>*:* $T^{\hat{d}+4}(h \cdot \bar{a}) \in \Sigma^{\hat{d},\hat{d}+4}$ *for some* $\hat{d} \in N$. Then $\bar{a} \in D_{\hat{d}}(s')$, which together with $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ implies that $\hat{d} = d$. Hence, $h \cdot \bar{a} \in H_3^{d+4,d}$; a contradiction.

Case 2: $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ *for some* $i \in \{0,\ldots,n\}$. Then $\bar{a} = s'$ and this contradicts $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$.

Case 3: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ *and, for some* $i \in \{0,\ldots,n\}$, $k' > n+i+5$, $\hat{k} \in \{1,\ldots,M\}$ *and* $\hat{d} \in N$, $h \cdot \bar{a} \in H_{1,a}^{(i),k'} \cup H_{1,b} \cup H_2^{\hat{k},\hat{d},0} \cup H_5^{\hat{k},\hat{d},0}$. Then, by the latter, $h \in H_1 \cup H_3 \cup H_4$ and, by Claims 26, 27, 31, 35, 38, 39, 41 and 42, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

Case 4: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ *and, for some* $\hat{k} \in \{1,\ldots,M\}$, $\hat{d} \in N$ *and* $\hat{\tau} \in \{1,\ldots,\hat{d}+3\}$, $h \cdot \bar{a} \in H_2^{\hat{k},\hat{d},\hat{\tau}} \cup H_5^{\hat{k},\hat{d},\hat{\tau}}$. By the latter, $\bar{a} \in D_{\hat{d}}(s) \cup D_{\hat{d}}(s')$. Since we also have $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$ it follows that $\hat{d} = d$. This, together with $h \cdot \bar{a} \in H_2^{\hat{k},\hat{d},\hat{\tau}} \cup H_5^{\hat{k},\hat{d},\hat{\tau}}$, imply that $h \in H_2^{\hat{k}-1,d,\hat{\tau}-1} \cup H_5^{\hat{k}-1,d,\hat{\tau}-1}$ which, by Claims 28, 40 and 43, contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

Case 5: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$, $h \cdot \bar{a} \in H_3$ *and* $T^{\hat{d}+4}(h \cdot \bar{a}) \notin \Sigma^{\hat{d},\hat{d}+4}$ *for all* $\hat{d} \in N$. Then $h \in H_3$ and, by Claims 31 and 41, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

Case 6: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$ *and* $h \cdot \bar{a} \in H_4$. Then $h \in H_3 \cup H_4$ and, by Claims 31, 35, 41 and 42, this contradicts the supposition that $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau}$.

Case 7: $h \in H_6^{d',\tau}$, $h \cdot \bar{a} \notin H_{1,a}^{(i),n+i+5}$ *for all* $i \in \{0,\ldots,n\}$ *and* $T^{\hat{d}+4}(h \cdot \bar{a}) \notin \Sigma^{\hat{d},\hat{d}+4}$ *for all* $\hat{d} \in N$. Then $h \notin \cup_{l=1}^{5} H_l$ and, by Claim 18, we have $h \cdot \bar{a} \notin \cup_{l=1}^{5} H_l$; a contradiction. ■

Claim 53 *Let* $h \in H_3^{k,d'}$ *for some* $k \in \{1,\ldots,M\}$ *and* $d' \in N$ *and let* $\bar{a} \in \bar{D}_d(f(h))$ *for some* $d \in N$. *If* $d = d'$, *then* $\pi(f|h \cdot \bar{a}) = \big((m^d; (\theta+1)T - [k+1-(d+4)]), \hat{\pi}^{(d),1}, \ldots\big)$, *where* $\theta = \theta(T^k(h))$. *If* $d \neq d'$, *then* $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^{(d)}(\theta, t)$ *for some* $\theta \in \{0,\ldots,d+4\}$ *and* $t \in \{1,\ldots,d+5\}$.

Proof. If $d = d'$, then $h \cdot \bar{a} \in H_4^{k+1,d,0}$ if $k+1-(d+4) = (\theta+1)T$ and $h \cdot \bar{a} \in H_3^{k+1,d}$, otherwise. Thus, in this case, the result follows from Claim 47 and Claim 48, respectively.

If $d \neq d'$, then $h \cdot \bar{a} \in H_5^{k+1,d,0}$. Thus, by Claim 49, $\pi(f|h \cdot \bar{a} \cdot f(h \cdot \bar{a})) = \pi^{(d)}(\theta, t)$ for some $\theta \in \{0,\ldots,d+4\}$ and $t \in \{1,\ldots,d+5\}$. ■

Claim 54 *Let* $h \in H_4^{k,d',r}$ *for some* $k \in \{1,\ldots,M\}$, $d' \in N$ *and* $r \in \{0,\ldots,n+d'+4\}$, *and let* $\bar{a} \in \bar{D}_d(f(h))$ *for some* $d \in N$. *Then* $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^{(d)}(\theta, t)$ *for some* $\theta \in \{0,\ldots,d+4\}$ *and* $t \in \{1,\ldots,d+5\}$.

Proof. We have that $h \cdot \bar{a} \in H_5^{k+1,0,d}$. Thus, by Claim 49, $\pi(f|h \cdot \bar{a} \cdot f(h \cdot \bar{a})) = \pi^{(d)}(\theta, t)$ for some $\theta \in \{0,\ldots,d+4\}$ and $t \in \{1,\ldots,d+5\}$. ■

Claim 55 *Let $h \in H_7^k$ for some $k \in \{0, \ldots, n+4\}$ and $\bar{a} \in \bar{D}_d(f(h))$ for some $d \in N$. If $k \notin \{2, \ldots, n+3\}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$, then $\pi(f|h \cdot \bar{a}) = \big((m^d; (\theta+1)T), \hat{\pi}^{(d),1}, \ldots\big)$, where $\theta = \theta(T^{d+3}(h) \cdot s') + 1$. Otherwise, $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$.*

Proof. If $k \notin \{2, \ldots, n+3\}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$, then $f(h) = s'$, $\bar{a} \in \bar{D}_d(s')$ and $T^{d+4}(h \cdot \bar{a}) \in \Sigma^{d,d+4}$. Hence, $h \cdot \bar{a} \in H_3^{d+4,d}$ and the conclusion follows from Claim 48.

For the remainder of the proof, therefore assume that the following holds:

$$\text{If } k \notin \{2, \ldots, n+3\}, \text{ then } T^{d+3}(h) \notin \Sigma^{d,d+3}. \tag{23}$$

Then we will show that $h \cdot \bar{a} \in H_6^{d,0}$ which, by Claim 49, establishes the conclusion of the claim. We prove the former in two steps.

Step 1: $h \cdot \bar{a} \in \tilde{\Sigma}^{d,0}$. Since $h \in H_7^k$, $f(h) \in \{s, s'\}$. Therefore, $\bar{a} \in \bar{D}_d(s) \cup \bar{D}_d(s')$.

Step 2: $h \cdot \bar{a} \notin \cup_{l=1}^5 H_l$. Suppose otherwise. Then, by Claim 18, $\bar{a} \notin D(s)$. Hence, by $h \in H_7^k$, (i) $\bar{a} \in \bar{D}_d(s')$ and (ii) $k \notin \{2, \ldots, n+3\}$. Furthermore, Claim 18 implies that either $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$ or $T^{d'+4}(h \cdot \bar{a}) \in \Sigma^{d',d'+4}$ for some $d' \in N$. If $h \cdot \bar{a} \in H_{1,a}^{(i),n+i+5}$ for some $i \in \{0, \ldots, n\}$, then $\bar{a} = s'$, a contradiction to (i). If $T^{d'+4}(h \cdot \bar{a}) \in \Sigma^{d',d'+4}$ for some $d' \in N$, then $\bar{a} \in D_{d'}(s')$, which, by (i), implies that $d = d'$. Thus, $T^{d+3}(h) \in \Sigma^{d,d+3}$. But this, together with (ii), contradict our supposition in (23). ■

A.2.5 Incentive conditions

To complete the proof of the theorem, we next establish the following for all $h \in H$:

$$V_d(\pi(f|h)) \geq (1-\delta)u_d(\bar{a}) + \delta V_d(\pi(f|h \cdot \bar{a})) \text{ for all } d \in N \text{ and } \bar{a} \in \bar{D}_d(f(h)). \tag{24}$$

<u>*Case 1*</u>*: $h \in H_1 \cup H_4$.* In this case, by Claims 45, 46 and 47, $\pi(f|h) = (\hat{\pi}^{(i),k}, \ldots)$ for some $i \in \{0, \ldots, n\}$ and $k \leq M$. Also, by Claims 51 and 54, $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$. Therefore, the left-hand side of (24) must be greater or equal to $-B(1-\delta^{K-k+1}) + \delta^{K-k+1}V_d(\hat{\pi}^{(i)}) \geq -B(1-\delta^K) + \delta^K V_d(\hat{\pi}^{(i)})$ and the right-hand side of (24) is less than or equal to $(1-\delta^{d+7-t})B + \delta^{d+7-t+(\theta+1)T}V_d(\hat{\pi}^{(d)}) \leq (1-\delta^{d+6})B + \delta^{d+6+T}V_d(\hat{\pi}^{(d)})$. Thus, by (18), (24) must hold.

<u>*Case 2*</u>*: $h \in H_3^{k,d'}$ for some $k \in \{1, \ldots, M\}$ and $d' \in N$.* Claim 48 implies that $\pi(f|h) = ((m^{d'}; (\theta+1)T - [k-(d'+4)], \hat{\pi}^{(d'),1}, \ldots)$, where $\theta = \theta(T^k(h))$. Claim 53 implies that $\pi(f|h \cdot$

$\bar{a}) = \big((m^d; (\theta+1)T - [k+1-(d+4)], \hat{\pi}^{(d),1}, \ldots\big)$ if $d = d'$ and $\pi(f|h\cdot\bar{a}) = f(h\cdot\bar{a})\cdot\pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$ if $d \neq d'$. Clearly, the deviation is not profitable if $d = d'$. When $d \neq d'$, the left-hand side of (24) must be greater or equal to $(1-\delta^{(\theta+1)T-k+d'+4})u_d(m^{d'}) + \delta^{(\theta+1)T-k+d'+4}V_d(\hat{\pi}^{(d')}) \geq -(1-\delta^{(n+5)T})B + \delta^{(n+5)T}V_d(\hat{\pi}^{(d')})$ and the right-hand side of (24) is less than or equal to $(1-\delta^{d+6})B + \delta^{d+6+T}V_d(\hat{\pi}^{(d)})$. Thus, by (17), (24) must hold.

Case 3: $h \in H_2^{k,d,\tau} \cup H_5^{k,d,\tau} \cup H_6^{d,\tau}$ for some $k \in \{1, \ldots, M\}$ and $\tau \in \{0, \ldots, d+3\}$. Claim 49 implies that $\pi(f|h) = f(h)\cdot\pi^{(d)}(\theta, t)$ and $\pi(f|h\cdot\bar{a}) = \pi^{(d)}(\theta', t')$ for some $t, t' \in \{1, \ldots, d+5\}$ and $\theta, \theta' \in \{0, \ldots, d+4\}$ such that $\theta < \theta'$. Therefore, the left-hand side of (24) must be greater or equal to $-(1-\delta^{d+6-t})B + \delta^{d+6-t+(\theta+1)T}V_d(\hat{\pi}^{(d)}) \geq -(1-\delta^{d+5})B + \delta^{d+5+(\theta+1)T}V_d(\hat{\pi}^{(d)})$ and the right-hand side of (24) is less than or equal to $(1-\delta^{d+6-t'})B + \delta^{d+6-t'+(\theta'+1)T}V_d(\hat{\pi}^{(d)}) \leq (1-\delta^{d+5})B + \delta^{d+5+(\theta'+1)T}V_d(\hat{\pi}^{(d)})$. Thus, by (19) and $\theta < \theta'$, (24) must hold.

Case 4: $h \in H_2^{k,d',\tau} \cup H_5^{k,d',\tau} \cup H_6^{d',\tau}$ for some $k \in \{1, \ldots, M\}$, $d' \in N$ and $\tau \in \{0, \ldots, d'+3\}$ such that $d' \neq d$. Claim 49 implies that $\pi(f|h) = f(h)\cdot\pi^{(d')}(\theta', t')$ for some $t' \in \{1, \ldots, d'+5\}$ and $\theta' \in \{0, \ldots, d'+4\}$. Also, Claim 52 implies that $\pi(f|h\cdot\bar{a}) = \pi^{(d)}(\theta, t)$ for some $t \in \{1, \ldots, d+5\}$ and $\theta \in \{0, \ldots, d+4\}$. Therefore, the left-hand side of (24) must be greater or equal to $-(1-\delta^{d'+6-t'+(\theta'+1)T})B + \delta^{d'+6-t'+(\theta'+1)T}V_d(\hat{\pi}^{(d')}) \geq -(1-\delta^{(n+5)(T+1)})B + \delta^{(n+5)(T+1)}V_d(\hat{\pi}^{(d')})$ and the right-hand side of (24) is less than or equal to $(1-\delta^{d+5})B + \delta^{d+5+(\theta+1)T}V_d(\hat{\pi}^{(d)}) \leq (1-\delta^{n+6})B + \delta^{n+6+T}V_d(\hat{\pi}^{(d)})$. Thus, by (17), (24) must hold.

Case 5: $h \in H_7^k$ for some $k \notin \{2, \ldots, n+3\}$ and $T^{d+3}(h) \in \Sigma^{d,d+3}$. Claims 50 and 55 imply that $\pi(f|h) = \big(s', (m^d; (\theta+1)T), \hat{\pi}^{(d),1}, \ldots\big)$ and $\pi(f|h\cdot\bar{a}) = \big((m^d; (\theta+2)T), \hat{\pi}^{(d),1}, \ldots\big)$ for some $\theta \in \{0, \ldots, d+3\}$. Therefore, the left-hand side of (24) must be greater or equal to $-(1-\delta)B + \delta^{(\theta+1)T}V_d(\hat{\pi}^{(d)})$ and the right-hand side of (24) is less than or equal to $(1-\delta)B + \delta^{(\theta+2)T}V_d(\hat{\pi}^{(d)})$. Thus, by (19), (24) must hold.

Case 6: $h \in H_7^k$ for some $k \notin \{2, \ldots, n+3\}$ and $T^{d'+3}(h) \in \Sigma^{d',d'+3}$ for some $d' \neq d$. Claims 50 and 55 imply that $\pi(f|h) = \big(s', (m^{d'}; (\theta'+1)T), \hat{\pi}^{(d'),1}, \ldots\big)$ for some $\theta' \in \{0, \ldots, d'+4\}$ and $\pi(f|h\cdot\bar{a}) = f(h\cdot\bar{a})\cdot\pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$. Therefore, the left-hand side of (24) must be greater or equal to $-(1-\delta^{1+(\theta'+1)T})B + \delta^{1+(\theta'+1)T}V_d(\hat{\pi}^{(d')}) \geq -(1-\delta^{(n+5)(T+1)})B + \delta^{(n+5)(T+1)}V_d(\hat{\pi}^{(d')})$ and the right-hand side of (24) is less than or equal to $(1-\delta^{d+6})B + \delta^{d+6+(\theta+1)T}V_d(\hat{\pi}^{(d)}) \leq (1-\delta^{n+6})B + \delta^{n+6+T}V_d(\hat{\pi}^{(d)})$.

Thus, by (17), (24) must hold.

Case 7: $h \in H_7^k$ and either $k \in \{2, \ldots, n+3\}$ or $T^{d'+3}(h) \notin \Sigma^{d',d'+3}$ for all $d' \in N$. Claim 50 implies that $\pi(f|h) = (\hat{\pi}^{(i),k}, \ldots)$ for some $i \in \{0, \ldots, n\}$ and $k \leq M$. Also, by Claim 55, $\pi(f|h \cdot \bar{a}) = f(h \cdot \bar{a}) \cdot \pi^{(d)}(\theta, t)$ for some $\theta \in \{0, \ldots, d+4\}$ and $t \in \{1, \ldots, d+5\}$. Therefore, by an identical argument as in Case 1, (24) must hold.

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