Bilinearity Rank of the Cone of Positive Polynomials and Related Cones

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Abstract

For a proper cone $\mathcal{K} \subset \mathbb{R}^n$ and its dual cone $\mathcal{K}^*$ the complementary slackness condition $\mathbf{x}^T\mathbf{s} = 0$ defines an n-dimensional manifold $C(\mathcal{K})$ in the space $\{ (\mathbf{x}, \mathbf{s}) \mid \mathbf{x} \in \mathcal{K}, \ \mathbf{s} \in \mathcal{K}^* \}$. When $\mathcal{K}$ is a symmetric cone, this fact translates to a set of n linearly independent bilinear identities (optimality conditions) satisfied by every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$. This proves to be very useful when optimizing over such cones, therefore it is natural to look for similar optimality conditions for non-symmetric cones. In this paper we define the *bilinearity rank* of a cone, which is the number of linearly independent bilinear identities valid for the cone, and describe a linear algebraic technique to bound this quantity. We examine several well-known cones, in particular the cone of positive polynomials $\mathcal{P}_{2n+1}$ and its dual, the closure of the moment cone $\mathcal{M}_{2n+1}$, and compute their bilinearity ranks. We show that there are exactly four linearly independent bilinear identities which hold for all $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{P}_{2n+1})$, regardless of the dimension of the cones. For nonnegative polynomials over an interval or half-line there are only two linearly independent bilinear identities. These results are extended to trigonometric and exponential polynomials.

Keywords. Optimality conditions, positive polynomials, complementarity slackness, bilinearity rank, bilinear cones

Introduction

In this paper we examine the complementarity conditions for convex cones. In particular, we are interested in those cones where complementarity can be expressed using bilinear relations. Our

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main result is that the complementarity conditions for the cone of positive polynomials and its dual, the closure of the moment cone over the real line, *cannot* be represented by bilinear relations alone. A similar result holds for the cone of positive polynomials over a given closed interval.

The cone of positive polynomials is a non-symmetric cone with many practical applications such as shape-constrained regression and the approximation of nonnegative functions (see for example [1, 13]).

It is well-known that positive polynomials over the real line are precisely those polynomials that can be written as the sum of squares of other polynomials. This property directly leads to the expression of the cone of positive polynomials as a linear image or preimage of the cone of positive semidefinite matrices, see for example [11]. For instance, optimization over the cone of positive polynomials of degree $2n$ can be expressed as the dual of a semidefinite program over $n \times n$ Hankel matrices [4]. However, this approach may significantly increase the size of the problem and introduce degeneracy. This motivates us to look for solution methods and optimality conditions which directly apply to the cone of positive polynomials.

As a first step we wish to find as simple complementary slackness conditions as is possible for the positive polynomials and the moment cones. For instance, in linear programming complementary slackness conditions are given by $x_i s_i = 0$ where x_i are the primal variables and s_i are the dual slack variables. In semidefinite programming (SDP) the complementary slackness theorem is given by $XS + SX = 0$, where, again, X is the primal matrix variable and S is the dual slack matrix. Finally for second order cone programming (SOCP) we have $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ and $s_0 x_i + s_i x_0 = 0$ (see the next section for more details). All of these relations are bilinear in the primal and dual slack variables. This property turns out to be essential in the design of primal-dual interior point algorithms. Furthermore, these bilinear forms make the machinery of certain algebraic structures available to help the understanding and improvement of such algorithms; this is especially true for SDP and SOCP.

According to a result of Güler, for every closed, pointed, convex cone $\mathcal{K}$ and its dual cone $\mathcal{K}^*$, the complementarity set $C(\mathcal{K})$, that is, the set of vector pairs $(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^{2n}$, where $\mathbf{x} \in \mathcal{K}$, $\mathbf{s} \in \mathcal{K}^*$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, is an n-dimensional manifold. In many cases, this fact translates to a computationally tractable set of n equations $f_i(\mathbf{x}, \mathbf{s}) = 0$ $(i = 1, \dots, n)$, which form the basis of complementary slackness theorems in optimization problems. Thus, it is an interesting endeavor to seek the simplest and most natural expressions for such relations. In fact, if it is at all possible to represent complementarity relations with bilinear forms, then that would be ideal, because potentially primal-dual interior point algorithms can be designed for such cones. Furthermore, bilinear relations induce algebras, and properties of these algebras may shed light on the properties of these cones and optimization problems over them [14].

In this paper we develop some techniques for proving that for certain cones, bilinear relations are not sufficient to express complementary slackness. In particular, we show this for positive polynomials, positive trigonometric polynomials, and positive exponential polynomials. The method we apply relies on results allowing the parametrization of the boundaries of these cones based on the theory of Chebyshev systems [8].

The paper is structured as follows: in Section 1 we present some fundamental concepts and results related to complementarity for proper cones, and introduce the notion of bilinear cones. In Section 2 we present a simple proof template for showing that cones are not bilinear. In the process we show a few simple cones that are not bilinear. We review necessary background information about the cone of positive polynomials $\mathcal{P}_{2n+1}$ and its dual, the closure of the moment cone $\mathcal{M}_{2n+1}$ in Section 3. Section 4 contains our main results concerning bilinear optimality constraints where we show that for the cone positive polynomials there are exactly four linearly independent bilinear complementarity relations. We also show that for the cone of positive polynomials over an interval there are exactly two such relations. Finally in Section 5 we use the notion of linearly isomorphic to show that several more cones of functions are not bilinear.

Notation

For a polynomial represented by the vector of its coefficients $\mathbf{p} = (p_0, \ldots, p_n)$ the corresponding polynomial function is denoted by $p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$. For a real $t \in \mathbb{R}$ and nonnegative integer n, $\mathbf{c}_{n+1}(t)$ denotes the moment vector $(1, t, \ldots, t^n)^\top$.

Throughout the paper we adopt the following convention: if for a range of indices the lower bound is greater than the upper bound, the range is considered to be empty.

The convex hull of a set $S \subset \mathbb{R}^n$ is denoted by $\mathrm{conv}(S)$, the closure of S is denoted by $\bar{S}$. The convex conical hull of a set $S \subset \mathbb{R}^n$ is denoted by $\mathrm{cone}(S)$.

The linear space spanned by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is denoted by $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

The inner product of vectors $\mathbf{x}$ and $\mathbf{s}$ is denoted by $\langle \mathbf{x}, \mathbf{s} \rangle = \mathbf{x}^T \mathbf{s}$.

The parity of an integer m is denoted by $m \pmod 2 = \begin{cases} 0 \text{ if } m \text{ is even} \\ 1 \text{ if } m \text{ is odd} \end{cases}$.

For a matrix $A = (a_{ij})_{m \times n}$, $\mathrm{vec}(A) \stackrel{\mathrm{def}}{=} (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{mn})^\top$. For two column vectors $\mathbf{u}$ and $\mathbf{v}$, their Kronecker product is defined to be $\mathbf{u} \otimes \mathbf{v} \stackrel{\mathrm{def}}{=} \mathrm{vec}(\mathbf{u}\mathbf{v}^\top)$.

1 Bilinear Cones

Let $\mathcal{K}$ be a proper cone in $\mathbb{R}^n$ (that is, a closed, pointed, and convex cone with nonempty interior in $\mathbb{R}^n$), and let

$$\mathcal{K}^* = \{\mathbf{z} \mid \langle \mathbf{x}, \mathbf{z} \rangle \geq 0, \quad \forall\, \mathbf{x} \in \mathcal{K}\}$$

be its *dual cone*. A pair of vectors $(\mathbf{x}, \mathbf{s})$, $\mathbf{x} \in \mathcal{K}$, $\mathbf{s} \in \mathcal{K}^*$ is said to satisfy the *complementary slackness conditions* with respect to $\mathcal{K}$ if $\langle \mathbf{x}, \mathbf{s} \rangle = 0$. We are interested in the following set:

Definition 1 *Let $\mathcal{K}$ be a proper cone, and $\mathcal{K}^*$ its dual. Then the set*

$$C(\mathcal{K}) = \{(\mathbf{x}, \mathbf{s}) \mid \mathbf{x} \in \mathcal{K}, \mathbf{s} \in \mathcal{K}^*, \langle \mathbf{x}, \mathbf{s} \rangle = 0\}$$

is called the complementarity set *of $\mathcal{K}$.*

Since for every proper cone $(\mathcal{K}^*)^* = \mathcal{K}$, it is immediate from the definition that $C(\mathcal{K})$ and $C(\mathcal{K}^*)$ are congruent: one can be obtained from the other by exchanging the first and last n coordinates.

The following theorem underlies the complementary slackness theorem for all convex optimization problems.

Theorem 2 *For each proper cone $\mathcal{K}$ in $\mathbb{R}^n$, $C(\mathcal{K})$ is an n-dimensional manifold homeomorphic to $\mathbb{R}^n$.*

A simple proof of this statement due to O. Güler [7] is given in Appendix.

To see the implications of this result for optimization problems over affine images or pre-images of proper cones, consider the following pair of dual *cone-LP* problems:

$$\begin{array}{llll} & \textbf{Primal} & & \textbf{Dual} \\ \inf & \langle \mathbf{c}, \mathbf{x} \rangle & \sup & \langle \mathbf{y}, \mathbf{b} \rangle \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} & \text{s.t.} & A^\top \mathbf{y} + \mathbf{s} = \mathbf{c} \\ & \mathbf{x} \in \mathcal{K} & & \mathbf{s} \in \mathcal{K}^* \end{array} \tag{1}$$

It is easy to see that for any feasible solution $\mathbf{x}$ of the **Primal** problem and any feasible solution $(\mathbf{y}, \mathbf{s})$ of the **Dual** problem the quantities $\langle \mathbf{c}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{b} \rangle$ and $\langle \mathbf{x}, \mathbf{s} \rangle$ are equal and nonnegative. The *strong duality theorem* for cone-LP problems states the following: Under certain regularity conditions, if both the **Primal** and **Dual** problems are feasible, then inf and sup can be replaced by min and max. Moreover, the optimal objective values are equal, i.e., $\langle \mathbf{c}, \mathbf{x}^* \rangle - \langle \mathbf{y}^*, \mathbf{b} \rangle = \langle \mathbf{x}^*, \mathbf{s}^* \rangle = 0$. It follows that at the optimum we have $(\mathbf{x}^*, \mathbf{s}^*) \in C(\mathcal{K})$. Since $C(\mathcal{K}) \in \mathbb{R}^{2n}$ is n-dimensional, it is often possible to obtain a system of equations

$$\begin{array}{l} A\mathbf{x} = \mathbf{b} \\ A^\top \mathbf{y} + \mathbf{s} = \mathbf{c} \\ f_i(\mathbf{x}, \mathbf{s}) = 0 \quad \text{for } i = 1, \ldots, n \end{array} \tag{2}$$

which is a square system, where $f_i(\mathbf{x}, \mathbf{s}) = 0$ are the complementarity equations. Many primal-dual algorithms for linear, second order and semidefinite programming problems, are based on strategies for solving this system of equations.

Let us examine some familiar examples.

Example 1 (Nonnegative orthant) When $\mathcal{K}$ is the nonnegative orthant, $\mathcal{K}^* = \mathcal{K}$. In this case if $\mathbf{x}$ and $\mathbf{s}$ contain only nonnegative components, and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then we must have $x_i s_i = 0$ for $i = 1, \ldots, n$. This is the basis of the familiar complementary slackness theorem in linear programming. ∎

Example 2 (Positive semidefinite cone) If $\mathcal{K}$ is the cone of real, symmetric positive semidefinite matrices, then $\mathcal{K}^* = \mathcal{K}$. If both X and S are real symmetric positive semidefinite matrices, and $\langle X, S \rangle = \sum_{ij} X_{ij} S_{ij} = 0$, then it is easy to show that the matrix product $XS = 0$, or equivalently $XS + SX = 0$. This is the basis of the complementary slackness theorem in semidefinite programming. ∎

Example 3 (Second order cones) Let $\mathcal{K} \in \mathbb{R}^{n+1}$ be the cone defined by all vectors $\mathbf{x}$ such that $x_0 \geq \|\overline{\mathbf{x}}\|$, where $\mathbf{x} = (x_0, x_1, \ldots, x_n)$, $\overline{\mathbf{x}} = (x_1, \ldots, x_n)$, and $\|\cdot\|$ is the Euclidean norm. This cone is also self-dual. Now if $\mathbf{x}, \mathbf{s} \in \mathcal{K}$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then from Cauchy-Schwarz-Bunyakovsky inequality it follows that $x_0 s_i + x_i s_0 = 0$ for $i = 1, \ldots, n$. These relations along with $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ are the basis of the complementary slackness theorem for the *second order cone programming* problem. ■

Example 4 (L_p cones) Generalizing the previous example, suppose instead the cone $\mathcal{K}_p$ consists of vectors $\mathbf{x}$ such that $x_0 \geq \|\overline{\mathbf{x}}\|_p$, where $\|\cdot\|_p$ is the L_p norm for some real number $p > 1$. Then it is known that the dual cone is $\mathcal{K}_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. In this case one can deduce from Hölder's inequality that if $\mathbf{x} \in \mathcal{K}_p$ and $\mathbf{s} \in \mathcal{K}_q$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then $s_0^q\, |x_i|^p - x_0^p\, |s_i|^q = 0$ for $i = 1, \ldots, n$. ■

Example 5 (L_1 and L_∞ cones) A limiting case of the previous example is when $p = 1$ (and thus $q = \infty$). Here $\mathcal{K}_1$ consists of vectors $\mathbf{x}$ such that $x_0 \geq |x_1| + \cdots + |x_n|$, and $\mathcal{K}_\infty$ consists of vectors $\mathbf{s}$ where $s_0 \geq \max_i |s_i|$. In this case, if $\mathbf{x} \in \mathcal{K}_1$, $\mathbf{s} \in \mathcal{K}_\infty$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then $x_i(s_0 - |s_i|) = 0$ for $i = 1, \ldots, n$. ■

Recall that an *algebra* is a linear space with an additional multiplication operation: $\mathbf{x} \cdot \mathbf{y} = \mathbf{z}$ defined on its vectors. The main requirement is that the components of $\mathbf{z}$ be expressed as bilinear functions of $\mathbf{x}$, and $\mathbf{y}$; in algebraic terms this multiplication must satisfy the distributive law; see for example [14]. Therefore, there are matrices Q_i such that $z_i = \mathbf{x}^\top Q_i \mathbf{y}$. If for a cone the complementarity relations can be exclusively expressed by bilinear forms, then, since these bilinear forms also define an algebra with multiplication, say "$\cdot$", the complementarity relations may be characterized by $\mathbf{x} \cdot \mathbf{s} = \mathbf{0}$. The machinery of this algebra may be useful in studying optimization problems over these cones. This motivates the following definitions.

Definition 3 *Let $\mathcal{K} \in \mathbb{R}^n$ be a proper cone. The $n \times n$ matrix Q is* a bilinear complementarity relation for $\mathcal{K}$ *if every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ satisfies $\mathbf{x}^\top Q \mathbf{s} = 0$.*

Note that the set of all bilinear complementarity relations for $\mathcal{K}$, denoted by $\mathcal{Q}(\mathcal{K})$, is a linear subspace of $\mathbb{R}^{n\times n}$.

Definition 4 *A proper cone $\mathcal{K} \subseteq \mathbb{R}^n$ is called* bilinear *if there exist at least n linearly independent bilinear complementarity relations for $\mathcal{K}$.*

Remark 5 *A bilinear cone $\mathcal{K} \subseteq \mathbb{R}^n$ may have more than n bilinear complementarity relations, as the following example shows. Let $\mathcal{K}$ be the three-dimensional second order cone (see Example 3), and let*

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Q_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ satisfies $\mathbf{x}^\top Q_i \mathbf{s} = 0$, $i = 1, 2, 3, 4$. These four equations are linearly independent.

The first three complementarity relations in Remark 5 are the basis of the well-known complementary slackness theorem for second order cones (see, for example, [2, Lemma 15] or [3, Chapter 8]). To see that $\mathbf{x}^\top Q_4 \mathbf{s} = 0$ for every $(\mathbf{x}, \mathbf{s}) \in \mathcal{C}(\mathcal{K})$ (where $\mathcal{K}$ denotes the three-dimensional second order cone), use the fact that $(\mathbf{x}, \mathbf{s}) \in \mathcal{C}(K)$ if and only if $\mathbf{x} = 0$, or $\mathbf{s} = 0$, or $\mathbf{x} = \alpha_1(1, \cos t, \sin t)^\top$ and $\mathbf{x} = \alpha_2(1, -\cos t, -\sin t)^\top$ with some $t \in \mathbb{R}$, $\alpha_1 > 0$, and $\alpha_2 > 0$. While the existence of four linearly independent complementarity relation may seem to contradict Theorem 2, there is no contradiction: simple arithmetic shows that for every $\mathbf{x} \in \mathcal{K}$ and $\mathbf{s} \in \mathcal{K}^* = \mathcal{K}$, the complementarity relations $\mathbf{x}^\top Q_2 \mathbf{s} = \mathbf{x}^\top Q_3 \mathbf{s} = 0$ hold if and only if $\mathbf{x} = 0$ or $\mathbf{s} = 0$ or $\mathbf{x}^\top Q_4 \mathbf{s} = 0$. Hence, with the conic constraints in place, the last relation is a consequence of the second and third, similarly to show the first complementarity relation (together with the conic constraints) alone implies the other three.

Since $C(\mathcal{K})$ and $C(\mathcal{K}^*)$ are congruent for every cone K, the dual cone $\mathcal{K}^*$ is bilinear if and only if $\mathcal{K}$ is.

From the examples above we observe that the cones in Examples 1, 2, and 3 are bilinear. Note that in Example 5, even though $\mathcal{K}_1$ and $\mathcal{K}_\infty$ are polyhedral, the complementarity relations are not bilinear due to the existence of absolute values. In Corollary 12 we show that $\mathcal{K}_1$ and $\mathcal{K}_\infty$ do not have any non-trivial bilinear complementarity relations.

The largest class of cones known to be bilinear are the *symmetric cones*. These are cones that are self-dual and homogeneous (that is, for any two points in the interior of the cone, there is a linear automorphism of the cone mapping the first point to the second one [5]). The cones in Examples 1, 2, and 3 are all symmetric. In addition, the cones of positive semidefinite complex Hermitian and quaternion Hermitian matrices are also symmetric. The second order cone, and the cones of positive semidefinite symmetric, complex Hermitian and quaternion Hermitian matrices, along with an exceptional 27 dimensional cone, are essentially the only symmetric cones; any other symmetric cones can be decomposed into direct sums of these five classes of cones.

Symmetric cones are intimately related to *Euclidean Jordan algebras*, see [5] and [9]. In such algebras the binary operation "$\circ$" is the abstraction of the operation $X \circ S = \frac{XS+SX}{2}$ in matrices. The properties of these algebras have played a major role in all aspects of optimization over such cones. In particular, design and analysis of interior point algorithms, duality, complementarity, and design of numerically efficient algorithms have been greatly simplified using the machinery of Jordan algebras. This is particularly true in the design of *primal-dual* interior point algorithms [6], [3].

There is an easy way to manufacture bilinear cones from other bilinear cones.

Definition 6 *The proper cones $\mathcal{K}$ and $\mathcal{L}$ are* linearly isomorphic *if there is a nonsingular (one-to-one and onto) linear transformation A such that $A\mathcal{K} = \mathcal{L}$.*

If two cones are linearly isomorphic, then one is bilinear if and only if the other one is. In fact, in the next section we introduce the concept of *bilinearity rank* of a cone and prove that this rank is invariant among all linearly isomorphic cones.

In the next two sections we develop techniques to prove certain cones are not bilinear.

2 A simple approach for proving cones are *not* bilinear

Recall that $\mathcal{Q}(K)$ denotes the linear space of all bilinear complementarity relations for $\mathcal{K}$, and consider the linear space

$$L(\mathcal{K}) \stackrel{\text{def}}{=} \operatorname{span}\{\mathbf{s}\mathbf{x}^\top \mid (\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})\}.$$

Proposition 7 *For every proper cone $\mathcal{K}$ we have*

$$\dim(\mathcal{Q}(\mathcal{K})) = \text{co-dim}(L(\mathcal{K})).$$

Proof. Follows immediately from the identity $\mathbf{x}^\top Q\mathbf{s} = \langle \mathbf{s}\mathbf{x}^\top, Q^\top\rangle$. ■

Since by definition $X \in L(\mathcal{K})$ implies $\operatorname{trace} X = \langle X, I\rangle = 0$, the co-dimension of $L(\mathcal{K})$ as a subspace of $\mathbb{R}^{n\times n}$ is at least 1. Now if there are m linearly independent bilinear forms Q_i such that $\langle X, Q_i\rangle = 0$ for all $X \in L(\mathcal{K})$, then $\text{co-dim}(L(\mathcal{K})) \geq m$. Therefore, if we show $n^2 - k$ linearly independent matrices $X \in L(\mathcal{K})$, then this proves that there can be at most k bilinear forms in any characterization of $C(\mathcal{K})$. In particular, $\mathcal{K}$ is bilinear if and only if $\text{co-dim}(L(\mathcal{K})) \geq n$. Note that, as Remark 5 shows, it is possible that $\text{co-dim}(L(\mathcal{K})) > n$ for a bilinear cone $\mathcal{K}$.

Definition 8 *The quantity* $\dim(Q(\mathcal{K})) = \text{co-dim}(L(\mathcal{K}))$ *is called the* bilinearity rank *of K and is denoted by $\beta(\mathcal{K})$.*

The manifolds $C(\mathcal{K})$ and $C(\mathcal{K}^*)$ are congruent for every proper cone $\mathcal{K}$, implying $\beta(\mathcal{K}) = \beta(\mathcal{K}^*)$. Furthermore, we have:

Lemma 9 *If $\mathcal{K}$ and $\mathcal{L}$ are linearly isomorphic proper cones then $\beta(\mathcal{K}) = \beta(\mathcal{L})$.*

Proof. Let A be a nonsingular linear transformation such that $A\mathcal{K} = \mathcal{L}$. Then the dual cone of $A\mathcal{K}$ is the cone $A^{-\top}\mathcal{K}^*$. Furthermore, Q_i $(i = 1, \ldots, m)$ define linearly independent bilinear complementarity conditions for $\mathcal{K}$ if and only if $A^{-\top}Q_iA^\top$ $(i = 1, \ldots, m)$ define linearly independent bilinear complementarity conditions for $A\mathcal{K}$. ■

To derive our main results, we use the following simple fact.

Proposition 10 *If there are k pairs of vectors $(\mathbf{x}_i, \mathbf{s}_i) \in C(\mathcal{K})$ for $i = 1, \ldots, k$, such that the matrices $\mathbf{s}_i\mathbf{x}_i^\top$ are linearly independent, then $\beta(\mathcal{K}) \leq n^2 - k$. In particular, if $k > n^2 - n$, then $\mathcal{K}$ is not bilinear.*

These results lead to the following template for proving certain cones are not bilinear: Suppose $\mathcal{K}$ is a proper cone in $\mathbb{R}^n$.

Step 1 Select a finite set S of orthogonal pairs of vectors $(\mathbf{x}, \mathbf{s})$, where $\mathbf{x}$ is a boundary vector of $\mathcal{K}$ and $\mathbf{s}$ is a boundary vector of $\mathcal{K}^*$.

Step 2 Form the matrix T whose rows are $\mathbf{x} \otimes \mathbf{s} = \operatorname{vec}(\mathbf{s}\mathbf{x}^\top)$, $(\mathbf{x}, \mathbf{s}) \in S$.

Step 3 If $\operatorname{rank} T > n^2 - n$, then $\mathcal{K}$ is not bilinear. More generally, $\beta(\mathcal{K}) \leq n^2 - \operatorname{rank} T$.

To see how this template works, let us show that n-dimensional polyhedral cones with more than n extreme rays are not bilinear.

Theorem 11 *If a proper polyhedral cone $\mathcal{K} \subseteq \mathbb{R}^n$ has more than n extreme rays, then $\beta(\mathcal{K}) = 1$.*

Proof. Assume that $\mathcal{K}$ is generated by the extreme rays $\{v^{(1)}, \dots, v^{(n+m)}\}$ $(n \geq 3, m \geq 1)$, and for every $i = 1, \dots, n$ let $\mathbf{e}^{(i)}$ denote the vector $(0, \dots, 0, 1, 0, \dots, 0)^\top \in \mathbb{R}^n$, with the single nonzero element in the ith position. Let F_1 and F_2 be two adjacent facets, and $F_3 = F_1 \cap F_2$ be an $(n-2)$-dimensional face. Select $n-2$ linearly independent extreme vectors of F_3, an additional extreme vector of F_1, and an additional extreme vector from F_2, such that the n selected vectors are linearly independent. Apply a nonsingular linear transformation to $\mathcal{K}$ that maps the $n-2$ selected extreme vectors of F_3 to $\mathbf{e}^{(3)}, \dots, \mathbf{e}^{(n)}$, and simultaneously maps the other two selected extreme vectors to $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$. The cone thus transformed has each $\mathbf{e}^{(i)}$ among its extreme rays, and it has m further extreme rays, each pointing outside the first orthant. We can introduce the notation $\mathbf{a}^{(i)}$ $(i = 1, \dots, m)$ for these additional extreme rays, so the transformed cone is

$$\mathcal{K}' = \operatorname{cone}\left(\left\{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \dots, \mathbf{e}^{(n)}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)}\right\}\right),$$

linearly isomorphic to $\mathcal{K}$.

Observe that by the construction of $\mathcal{K}'$, the first two coordinates of every vector in $\mathcal{K}'$ are nonnegative. For example, the entire cone lies on one side of the hyperplane spanned by the image of F_1, which is the hyperplane $\{\mathbf{x} \mid x_2 = 0\}$, and $\mathbf{e}^{(2)}$ is in the cone, hence the second coordinate of every vector in $\mathcal{K}'$ is nonnegative. Furthermore, because no n extreme rays lie in the same $(n-1)$-dimensional subspace, every coordinate of each vector $\mathbf{a}^{(i)}$ is nonzero. Let $a_j^{(i)}$ denote the jth coordinate of $\mathbf{a}^{(i)}$, and P denote the set

$$P = \left\{ j \mid a_j^{(i)} > 0 \ \ \forall i \right\}.$$

Clearly, $\{1, 2\} \subseteq P$ based on the above observations, but $|P| < n$, otherwise the vectors $\mathbf{a}^{(i)}$ would be in the first orthant.

The following vectors are all boundary vectors of $(\mathcal{K}')^*$:

- $\mathbf{e}^{(k)}$ for every $k \in P$. From the definition of P, $\mathbf{e}^{(k)} \in (\mathcal{K}')^*$, and it is a boundary vector, because $\langle \mathbf{e}^{(i)}, \mathbf{e}^{(k)} \rangle = 0$ for every $i \neq k$.
- The vectors $\mathbf{b}^{(j,k)} = (b_1^{(j,k)}, \dots, b_n^{(j,k)})^\top$ for every $j \notin P$, $k \in P$, defined by

$$b_\ell^{(j,k)} = \begin{cases} -a_j^{(i_{j,k})} & \ell = k \\ \ \ a_k^{(i_{j,k})} & \ell = j \\ 0 & \text{otherwise} \end{cases},$$

  where for every given pair (j,k) $(j \notin P,\ k \in P)$ the superscript $i_{j,k}$ is chosen such that

$$\min_i \left(\frac{a_j^{(i)}}{a_k^{(i)}} \right) = \frac{a_j^{(i_{j,k})}}{a_k^{(i_{j,k})}}.$$

This definition is motivated by the facts that $\langle \mathbf{e}^{(i)}, \mathbf{b}^{(j,k)} \rangle \geq 0$ for every i and every $j \notin P$, $k \in P$ (because $\mathbf{b}^{(j,k)}$ is a coordinatewise nonnegative vector), and that $\langle \mathbf{a}^{(i)}, \mathbf{b}^{(j,k)} \rangle \geq 0$ for every i and every $j \notin P$, $k \in P$ by the choice of $i_{j,k}$. Hence $\mathbf{b}^{(j,k)}$ is indeed a vector in $(\mathcal{K}')^*$, and it is a boundary vector because $\langle \mathbf{a}^{(i_{j,k})}, \mathbf{b}^{(j,k)} \rangle = 0$.

Now we proceed according to our proof template by selecting a set S of orthogonal pairs of boundary vectors $(\mathbf{x}, \mathbf{s})$ from $\mathcal{K} \times \mathcal{K}^*$:

- $(\mathbf{e}^{(i)}, \mathbf{e}^{(k)})$, for every i, k such that $i \neq k$, $k \in P$,
- $(\mathbf{e}^{(i)}, \mathbf{b}^{(j,k)})$, for every i, j, k such that $i \neq j$, $i \neq k$, $j \notin P$, $k \in P$,
- $(\mathbf{a}^{(i_{j,k})}, \mathbf{b}^{(j,k)})$, for every j, k such that $j \notin P$, $k \in P$,

and constructing the matrix T as in Step 2 of the template.

Finally, we need to show that $\operatorname{rank} T \geq n^2 - 1$. We do this by showing $n^2 - 1$ rows of T that can be rearranged to form a triangular matrix, that is, we show a sequence of $n^2 - 1$ matrices of the form $\mathbf{x}\mathbf{s}^\top$ (with $(\mathbf{x}, \mathbf{s})$ chosen from the above list of pairs) such that each of them has a nonzero entry in a position where each of the preceding matrices has a zero entry.

- Start the sequence with the matrices $\mathbf{e}^{(i)}\mathbf{e}^{(k)\top}$ ($k \in P$, $i \neq k$) in any order. The only nonzero entry of such a matrix is the (i, k)-th entry. In particular, their nonzero entries are in off-diagonal positions in columns $k \in P$.

- For every i there is a $k_i \in P$ such that $i \neq k$, because $|P| \geq 2$. Arrange the matrices $\mathbf{e}^{(i)}\mathbf{b}^{(j,k_i)\top}$ ($i \neq j$, $j \notin P$) in any order, and put them in the sequence after the above matrices. The nonzero entries of such a matrix are the (i, j)-th and (i, k_i)-th entry. In particular, for every off-diagonal position in columns $j \notin P$ there is a matrix with a nonzero entry in that position.

- Finally, consider the matrices $\mathbf{a}^{(i_{j,k})}\mathbf{b}^{(j,k)\top}$ ($j \notin P$, $k \in P$). Their nonzero entries are in the jth and kth column. In particular, they have two diagonal nonzero entries: in positions (j, j) and (k, k). If $P = \{p_1, \dots, p_{|P|}\}$ and its complement, $\bar{P} = \{q_1, \dots, q_{|\bar{P}|}\}$, then we can arrange $n - 1$ of these matrices in the following order of superscripts (j, k):

$$(q_1, p_1), (q_1, p_2), \dots, (q_1, p_{|P|}), (q_2, p_1), (q_3, p_1), \dots, (q_{|\bar{P}|}, p_1).$$

  Append these matrices in the above order to our sequence of matrices. This way for every number $\ell \in \{1, \dots, n\} \setminus \{q_1\}$ there is a matrix in the sequence with a nonzero entry in position (ℓ, ℓ) such that all the preceding matrices have a zero entry in the same position. (Notice that in this step we use both $2 \leq |P|$ and $|P| < n$.)

The sequence of matrices constructed above satisfies all the requirements. It consists of $n^2 - 1$ matrices, one for every subscript (i, j) $(i, j = 1, \dots, n)$ except for (q_1, q_1), and we have shown that every matrix M in the sequence has a nonzero entry in a position where every matrix preceding M has a zero entry. This completes our proof. ∎

It is worth to specifically mention a special case, the dual cones $\mathcal{K}_1, \mathcal{K}_\infty$ from Example 5.

Corollary 12 *Let $n \geq 2$, and define $\mathcal{K}_1 \subseteq \mathbb{R}^{n+1}$ and $\mathcal{K}_\infty \subseteq \mathbb{R}^{n+1}$ as in Example 5. Then $\beta(\mathcal{K}_1) = \beta(\mathcal{K}_\infty) = 1$.*

Another consequence of Theorem 11 is that self-duality is not sufficient for a cone to be bilinear.

Corollary 13 *Let $m \geq 5$ be an odd number, and consider the polyhedral cone $\mathcal{K} \subseteq \mathbb{R}^3$ generated by m vectors pointing at the vertices of a regular m-gon. Let this m-gon be chosen such that each extreme ray is orthogonal to the plane of the opposite two-dimensional face of $\mathcal{K}$. Then $\mathcal{K}$ is self-dual, but not bilinear, as $\beta(\mathcal{K}) = 1$.*

Similarly, there exist homogeneous cones that are not bilinear, see [12].

The template we used to prove Theorem 11 is a special case of the following, formally more general, framework:

Step 1 Select a set S of orthogonal pairs of vectors $(\mathbf{x}, \mathbf{s})$, where $\mathbf{x}$ is a boundary vector of $\mathcal{K}$ and $\mathbf{s}$ is a boundary vector of $\mathcal{K}^*$.

Step 2 Consider the set $\mathcal{T} = \{\mathbf{x} \otimes \mathbf{s} \,|\, (\mathbf{x}, \mathbf{s}) \in S\}$.

Step 3 If $\dim(\operatorname{span}(\mathcal{T})) > n^2 - n$, then $\mathcal{K}$ is not bilinear. More generally, $\beta(\mathcal{K}) \leq n^2 - \dim(\operatorname{span}(\mathcal{T}))$.

After presenting some necessary structural results in Section 3, we shall use these steps to prove our main results in Section 4.

3 Positive Polynomials and Moment Cones

Let us first introduce the cones of positive polynomials and moment cones:

Definition 14 *The* cone of positive polynomials *(also referred to as* cone of nonnegative polynomials*) of degree $2n$*

$$\mathcal{P}_{2n+1} \stackrel{\text{def}}{=} \left\{(p_0, \ldots, p_{2n}) \in \mathbb{R}^{2n+1} \;\middle|\; p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_{2n} t^{2n} \geq 0 \quad \forall t \in \mathbb{R}\right\}$$

consists of the coefficient vectors of nonnegative polynomials of degree $2n$. Similarly, for real numbers $a < b$, the cone of positive polynomials *(or* nonnegative polynomials*)* over the interval *$[a, b]$ of degree n is the cone*

$$\mathcal{P}^{[a,b]}_{n+1} \stackrel{\text{def}}{=} \left\{(p_0, \ldots, p_n) \in \mathbb{R}^{n+1} \;\middle|\; p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n \geq 0 \quad \forall t \in [a, b]\right\}.$$

The moment cone *of dimension $2n+1$ is defined as*

$$\mathcal{M}_{2n+1} \stackrel{\text{def}}{=} \operatorname{cone}\left(\{\mathbf{c}_{2n+1}(t) \;|\; t \in \mathbb{R}\}\right), \text{ where } \mathbf{c}_{2n+1}(t) \stackrel{\text{def}}{=} (1, t, t^2, \ldots, t^{2n})^\top.$$

Similarly, the $(n+1)$-dimensional moment cone over $[a, b]$ is defined as

$$\mathcal{M}^{[a,b]}_{n+1} \stackrel{\text{def}}{=} \operatorname{cone}\left(\{\mathbf{c}_{n+1}(t) \;|\; t \in [a, b]\}\right).$$

Remark 15 *This is not the traditional definition of the moment cone. See [8] (Ch. VI) for the original definition and proof of its equivalence with the one given above.*

The cone of positive polynomials and the moment cone are closely related [8]:

Proposition 16 $\mathcal{P}^*_{2n+1} = \bar{\mathcal{M}}_{2n+1}$. *Similarly,* $(\mathcal{P}^{[a,b]}_{n+1})^* = \mathcal{M}^{[a,b]}_{n+1}$.

We will repeatedly use the following simple observation.

Proposition 17 *If* $\mathbf{p} \in \mathbb{R}^{n+1}$ *is the coefficient vector of a polynomial* p*, and* t *is real number, then* $p(t) = \langle \mathbf{p}, \mathbf{c}_{n+1}(t) \rangle$*. In particular,* $p(t) = 0$ *if and only if* $\langle \mathbf{p}, \mathbf{c}_{n+1}(t) \rangle = 0$.

In order to use the templates presented in Section 2 and prove that a cone $\mathcal{K}$ is not bilinear, it is useful to know the boundary or extreme rays of the cones $\mathcal{K}$ and $\mathcal{K}^*$. The extreme rays of $\mathcal{M}_{2n+1}$, and $\mathcal{M}^{[a,b]}_{n+1}$ are well-known:

Proposition 18 ([8, Sections 2.2 and 6.6])

1. *The nonzero extreme vectors of* $\bar{\mathcal{M}}_{2n+1}$ *are the vectors* $\alpha \mathbf{c}_{2n+1}(t)$ *for every* $\alpha > 0$ *and* $t \in \mathbb{R}$*, and the vectors* $(0, \ldots, 0, \alpha)^\top$ *for every* $\alpha > 0$.
2. *The nonzero extreme vectors of* $\mathcal{M}^{[a,b]}_{n+1}$ *are the vectors* $\alpha \mathbf{c}_{n+1}(t)$ *for every* $\alpha > 0$ *and* $t \in [a, b]$.

Finally, in the subsequent sections we will also use the following observations:

Proposition 19 ([8, Sections 2.11 and 6.6])

1. *Every real root of a nonnegative polynomial in* $\mathcal{P}_{2n+1}$ *is a multiple root with even multiplicity.*
2. *For polynomials in* $\mathcal{P}^{[a,b]}_{n+1}$ *every real root in the open interval* (a, b) *is a multiple root with even multiplicity.*

4 Main Results

In this section we show our main results, namely, that neither the cone of positive polynomials over the real line, nor the cone of positive polynomials over a closed interval are bilinear. Moreover, we give the exact bilinearity rank for these cones.

To prove our main results we need the following elementary fact from linear algebra.

Lemma 20 *Let* k *be a positive integer and let* $\mathcal{B} = \{b_1, \ldots, b_k\}$ *be a set of linearly independent vectors in a real vector space. For a set* $\{m_1, \ldots, m_k\} \subset \operatorname{span}(\mathcal{B})$ *consider the coordinates* $\alpha_{i,j} \in$

$\mathbb{R}$ $(i, j = 1, \ldots, k)$ *uniquely defined by the representations* $m_i = \sum_{j=1}^{k} \alpha_{i,j} b_j$. *(We refer to this as the* $\mathcal{B}$-representation *of* m_i.*) If the conditions*

$$\begin{aligned} \alpha_{i,i} &\neq 0 \quad \textit{for all} \quad 1 \leq i \leq k, \\ \alpha_{i,j} &= 0 \quad \textit{for all} \quad 1 \leq i < j \leq k \end{aligned}$$

hold, then the set $\{m_1, \ldots m_k\}$ *is also linearly independent.*

Proof. The claim follows immediately from the observation that the matrix $(\alpha_{i,j})_{k \times k}$ is lower triangular with a nonzero diagonal, and hence non-singular. ∎

We are going to use the following, formally more general version of the above lemma:

Corollary 21 *Let* $\mathcal{B} \subset \mathbb{R}[x_1, \ldots, x_n]$ *be a finite set of linearly independent polynomials and consider a set* $\mathcal{M} \subset \operatorname{span}(\mathcal{B})$ *with coordinates* $\alpha_{m,b}$ $(m \in \mathcal{M},\ b \in \mathcal{B})$ *defined by the representations* $m = \sum_{b \in \mathcal{B}} \alpha_{m,b} b$. *Assume that there exists an injection* $\varphi : \mathcal{B} \to \mathcal{M}$ *and a linear order* $\prec$ *on* $\varphi(\mathcal{B})$ *such that*

$$\begin{aligned} \alpha_{\varphi(b),b} &\neq 0 \quad \textit{for all} \quad b \in \mathcal{B}, \\ \alpha_{\varphi(b),d} &= 0 \quad \textit{for all} \quad b, d \in \mathcal{B} \textit{ satisfying } \varphi(b) \prec \varphi(d). \end{aligned}$$

Then $\dim(\operatorname{span}(\mathcal{M}(\mathbb{R}^n))) = |\mathcal{B}|$, *where* $\mathcal{M}(\mathbb{R}^n) \stackrel{\text{def}}{=} \{(m(\mathbf{x}))_{m \in \mathcal{M}} \mid \mathbf{x} \in \mathbb{R}^n\}$.

Proof. Let $k = |\mathcal{B}|$. It is well-known that for a vector $P = (p_1, \ldots, p_k) \in (\mathbb{R}[x_1, \ldots, x_n])^k$ consisting of linearly independent polynomials we have $\dim(\operatorname{span}(P(\mathbb{R}^n))) = k$, therefore it suffices to find a k-element linearly independent subset of $\mathcal{M}$. As φ is injective, there exists an indexing $\mathcal{B} = \{b_1, \ldots, b_k\}$ such that $\varphi(b_1) \prec \cdots \prec \varphi(b_k)$. Let $m_i = \varphi(b_i) \in \mathcal{M}$ (for all $i = 1, \ldots, k$). It is easy to verify that the sets $\{b_1, \ldots, b_k\}$ and $\{m_1, \ldots, m_k\}$ satisfy the conditions of Lemma 20. Consequently the set $\{m_1, \ldots, m_k\} \subset \mathcal{M}$ is linearly independent, which implies our claim. ∎

4.1 Positive polynomials over the real line

Theorem 22 *The cone* $\mathcal{P}_{2n+1}$ *is not bilinear, unless* $n = 1$. *More specifically, for every* n, $\beta(\mathcal{P}_{2n+1}) \leq 4$.

The second claim immediately implies the first. Note that when $n = 1$, we do have a bilinear cone linearly isomorphic to the cone of 2×2 positive semidefinite matrices.

Proof. Consider the matrix valued functions $M \colon \mathbb{R}^n \mapsto \mathbb{R}^{(2n+1)\times(2n+1)}$ defined as

$$M(t_1, \ldots, t_n) = \mathbf{c}\mathbf{p}^\top,$$

where $\mathbf{p} \in \mathcal{P}_{2n+1}$ is the coefficient vector of the polynomial $p(x) = \prod_{k=1}^{n} (x - t_k)^2$, and $\mathbf{c} = \mathbf{c}_{2n+1}(t_1) = (1, t_1, \ldots, t_1^{2n})$ is the moment vector corresponding to the first root of $\mathbf{p}$. It is easy to verify that the entries of $M = (m_{i,j})_{i,j=0}^{2n}$ satisfy the polynomial equation

$$\sum_{j=0}^{2n} m_{i,j} x^j \equiv t_1^i \prod_{k=1}^{n} (x - t_k)^2. \tag{3}$$

The polynomial $p(x)$ is clearly nonnegative everywhere, and $\mathbf{c}$ is a moment vector, furthermore, by Proposition 17, $\langle \mathbf{p}, \mathbf{c} \rangle = 0$. Therefore, following the general template of Section 2 (with $\mathbf{p}$ and $\mathbf{c}$ playing the roles of $\mathbf{x}$ and $\mathbf{s}$, and $M(\mathbb{R}^n)$ playing the role of $\mathcal{T}$), the theorem follows if $\dim(\mathrm{span}(M(\mathbb{R}^n))) = (2n+1)^2 - 4$. We show this equality using the sufficient condition presented in Corollary 21, with the set $\{m_{i,j}\}$ playing the role of set $\mathcal{M}$.

Let us define the n-variate polynomials $\Pi(k, \ell)$ by

$$\Pi(k,\ell)(t_1,\ldots,t_n) \stackrel{\text{def}}{=} \sum_{\substack{0 \le \alpha_2,\ldots,\alpha_n \le 2 \\ \alpha_2+\cdots+\alpha_n = \ell}} t_1^k \prod_{j=2}^{n} (-2)^{(\alpha_j \bmod 2)} t_j^{\alpha_j}, \tag{4}$$

whenever $0 \le k \le 2n+2$ and $0 \le \ell \le 2n-2$; for values of k and ℓ outside these ranges let us define $\Pi(k,\ell)$ to be the zero polynomial. Let $\mathcal{B}$ denote the set $\{\Pi(k,\ell) \mid 0 \le k \le 2n+2, 0 \le \ell \le 2n-2\}$. It follows from the definition that $|\mathcal{B}| = (2n+1)^2 - 4$, and that $\mathcal{B}$ is linearly independent, because no two polynomials in $\mathcal{B}$ share a common monomial. It remains to show that $\mathcal{M}$ is indeed a subset of $\mathrm{span}(\mathcal{B})$, and exhibit the injection φ and the linear order $\prec$ of Corollary 21.

The coefficient of $x^{2n-k-\ell}$ in the polynomial $\prod_{j=1}^{n}(x - t_j)^2$ is $\sum_{k=0}^{2} \Pi(k,\ell)$. From this observation and (3) it follows immediately that for every $0 \le i, j \le 2n$,

$$m_{i,j} = \Pi(i, 2n-j) + \Pi(i+1, 2n-1-j) + \Pi(i+2, 2n-2-j), \tag{5}$$

and thus $m_{i,j} \in \mathrm{span}(\mathcal{B})$, as required.

We now introduce an injection $\varphi\colon \mathcal{B} \mapsto \mathcal{M}$ by defining its inverse (where it exists): let $m_{i,j}$ be the image of the polynomial

$$\varphi^{-1}(m_{i,j}) = q_{i,j} \stackrel{\text{def}}{=} \begin{cases} \Pi(i, 2n-j) & j \ge \max\{2, i\} \\ \Pi(i+2, 2n-2-j) & j \le \min\{i-1, 2n-2\} \\ \text{not defined} & \text{otherwise} \end{cases}. \tag{6}$$

In particular, we assign a polynomial to each entry $m_{i,j}$ of $\mathcal{M}$ except for $m_{0,0}$, $m_{0,1}$, $m_{1,1}$, and $m_{2n,2n-1}$, and we assign different polynomials to different entries of M, because if $q_{i_1,j_1} = q_{i_2,j_2}$ for some $(i_1,j_1) \neq (i_2,j_2)$ and $i_1 \le j_1$, then $j_1 \ge i_1$, $i_2 - 1 \ge j_2$, $i_1 = i_2 + 2$, and $2n - j_1 = 2n - 2 - j_2$, a contradiction, as the sum of these inequalities reduces to $-1 \ge 0$. Consequently, each $\Pi(k,\ell)$ is equal to $q_{i,j}$ for precisely one pair (i,j), therefore φ is indeed an injection.

Equation (5) shows that the coefficient of $q_{i,j}$ in the $\mathcal{B}$-representation of $m_{i,j}$ is 1, so using the notation of Corollary 21, $\alpha_{\varphi(\Pi(k,\ell)),\Pi(k,\ell)} = 1$ for all $\Pi(k,\ell) \in \mathcal{B}$.

Let us define a linear order $\succ$ on $\varphi(\mathcal{B})$ in the following way: $m_{i_1,j_1} \succ m_{i_2,j_2}$ precisely when one of the following three conditions holds:

1. $i_1 - j_1 \ge 1 > i_2 - j_2$;
2. $i_1 - j_1 \ge 1$, $i_2 - j_2 \ge 1$, and either $i_1 > i_2$, or $i_1 = i_2$ but $j_1 < j_2$;
3. $i_1 - j_1 < 1$, $i_2 - j_2 < 1$, and either $j_1 < j_2$, or $j_1 = j_2$ but $i_1 > i_2$.

An easy case analysis using Equations (5) and (6) shows that if $m_{i_1,j_1} \succ m_{i_2,j_2}$, then the coefficient of q_{i_1,j_1} in the $\mathcal{B}$-representation of m_{i_2,j_2} is zero:

1. If $i_1 - j_1 \geq 1 > i_2 - j_2$, then Equations (5) and (6) show that the three terms of m_{i_1,j_1} have higher degree than those of m_{i_2,j_2}, so in particular $\Pi(i_1+2, 2n-2-j_1)$ does not appear in the $\mathcal{B}$-representation of m_{i_2,j_2}.
2. If both $i_1 - j_1, i_2 - j_2 \geq 1$, then either $i_1+2 > i_2+2$, or $i_1 = i_2$ and $2n-2-j_1 > 2n-2-j_2$. In either case, by Equation (5), $\Pi(i_1+2, 2n-2-j_1)$ does not appear in the $\mathcal{B}$-representation of m_{i_2,j_2}.
3. If both $i_1 - j_1, i_2 - j_2 \leq 0$, then either $i_1 > i_2$ and $2n - j_1 = 2n - j_2$, or $2n - j_1 > 2n - j_2$. In either case, by Equation (5), $\Pi(i_1, 2n - j_1)$ does not appear in the $\mathcal{B}$-representation of m_{i_2,j_2}.

The injection $m_{i,j} \mapsto q_{i,j}$ and the linear order $\succ$ satisfy the conditions of Corollary 21, therefore, by Equation (6),

$$\dim(\operatorname{span}(M(\mathbb{R}^n))) = |\mathcal{B}| = (2n+1)^2 - 4,$$

which completes the proof. ∎

4.2 Polynomials over a closed interval

We prove our theorem separately for polynomials of even and odd degree, since the different representations of the extreme rays would make a unified proof difficult. The main idea of the proofs is the same as in the proof of Theorem 22, however, the sets $\mathcal{M}$ and $\mathcal{B}$ are different, and the linear order $\prec$ also needs a more complicated definition.

In some cases it will be useful to restrict ourselves to the case when $[a, b] = [0, 1]$. This is without loss of generality: the same number of linearly independent bilinear complementarity relations exist for $\mathcal{P}_{n+1}^{[a,b]}$ as for $\mathcal{P}_{n+1}^{[0,1]}$, as the following proposition shows.

Proposition 23 *For every positive integer n and $a < b$, the cone $\mathcal{P}_{n+1}^{[a,b]}$ is linearly isomorphic to $\mathcal{P}_{n+1}^{[0,1]}$.*

Proof. The polynomial $p(t)$ is nonnegative over $[a, b]$ if and only if $q(t) = p\left(\frac{t-a}{b-a}\right)$ is nonnegative over $[0, 1]$. Furthermore, the coefficients of $q(t)$ can be obtained by a nonsingular linear transformation from of coefficients of p. ∎

4.2.1 Polynomials of even degree

Theorem 24 *The cone $\mathcal{P}_{2n+1}^{[a,b]}$ is not bilinear. More specifically, for every n, $\beta(\mathcal{P}_{2n+1}^{[a,b]}) \leq 2$.*

Proof. Consider the matrix valued functions $M \colon \mathbb{R}^{n+2} \mapsto \mathbb{R}^{(2n+1)\times(2n+1)}$ defined as

$$M(t_1, \ldots, t_n; \alpha, \beta) = \mathbf{c}_{2n+1}(t_1)\mathbf{p}^\top + \alpha\mathbf{c}_{2n+1}(a)\mathbf{p}_a^\top + \beta\mathbf{c}_{2n+1}(b)\mathbf{p}_b^\top,$$

where $\mathbf{p}, \mathbf{p}_a, \mathbf{p}_b \in \mathcal{P}_{2n+1}^{[a,b]}$ are the coefficient vectors of the polynomials $p(x) = \prod_{k=1}^{n}(x - t_k)^2$, $p_a(x) = x - a$, and $p_b(x) = b - x$, respectively. It is easy to verify that the entries of $M = (m_{i,j})_{i,j=0}^{2n}$ satisfy the polynomial equation

$$\sum_{j=0}^{2n} m_{i,j} x^j \equiv t_1^i \prod_{k=1}^{n}(x - t_k)^2 + \alpha a^i (x - a) + \beta b^i (b - x). \tag{7}$$

The polynomials $p(x)$, $p_a(x)$ and $p_b(x)$ are clearly nonnegative over $[a, b]$, and by Proposition 17, $\langle \mathbf{p}_a, \mathbf{c}_{2n+1}(a)\rangle = \langle \mathbf{p}_b, \mathbf{c}_{2n+1}(b)\rangle = \langle \mathbf{p}, \mathbf{c}_{2n+1}(t_1)\rangle = 0$. Consequently, following the general template of Section 2 (with $\mathbf{p}$, $\mathbf{p}_a$ and $\mathbf{p}_b$ playing the role of $\mathbf{x}$, and $\mathbf{c}_{2n+1}(t)$ playing the role of $\mathbf{s}$, and $M(\mathbb{R}^{n+2})$ playing the role of $\mathcal{T}$), the theorem follows if $\dim(\operatorname{span}(M(\mathbb{R}^{n+2}))) = (2n + 1)^2 - 2$. Finally, we will show this equality using the sufficient condition presented in Corollary 21, with the set $\{m_{i,j}\}$ playing the role of set $\mathcal{M}$.

For the rest of the proof, let us assume that $a = 0$, $b = 1$; Proposition 23 guarantees that this is without loss of generality. By convention, $a^0 = 1$ in the rest of the proof.

With a slight abuse of notation, let us define the polynomials $\Pi(k, \ell)$ as in the proof of Theorem 22 (see Equation (4) and the subsequent paragraph), except that now every $\Pi(k, \ell)$ has formally two additional variables, α and β, even though they do not depend on these variables. Let $\mathcal{M}$ be the set of entries of the matrix M, and let $\mathcal{B}$ be the set

$$\mathcal{B} = \{\alpha, \beta\} \cup \{\Pi(k, \ell) \mid 0 \le k \le 2n + 2, 0 \le \ell \le 2n - 2\}.$$

The elements of $\mathcal{B}$ are considered as polynomials of $n + 2$ variables, $t_1, \ldots, t_n, \alpha, \beta$. Again, it is immediate that the set $\mathcal{B}$ is linearly independent. It follows from these definitions that for every $0 \le i, j \le 2n$,

$$m_{i,j} = \Pi(i, 2n - j) + \Pi(i + 1, 2n - 1 - j) + \Pi(i + 2, 2n - 2 - j) + m'_{i,j}, \tag{8}$$

where

$$m'_{i,j} = \begin{cases} \beta & j = 0 \\ \alpha - \beta & i = 0, j = 1 \\ -\beta & i \ge 1, j = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

We now introduce an injection $\varphi\colon \mathcal{B} \mapsto \mathcal{M}$ by defining its inverse (where it exists): let $m_{i,j}$ be the image of the polynomial

$$\varphi^{-1}(m_{i,j}) = q_{i,j} \stackrel{\text{def}}{=} \begin{cases} \Pi(i, 2n - j) & j \ge \max\{2, i\} \\ \Pi(i + 2, 2n - 2 - j) & j \le \min\{i - 1, 2n - 2\} \\ \beta & i = 0, j = 0 \\ \alpha & i = 0, j = 1 \\ \text{not defined} & \text{otherwise} \end{cases}. \tag{10}$$

In particular, we assign a polynomial to each entry except for $m_{1,1}$ and $m_{2n,2n-1}$, and we assign different polynomials to different entries of M, by an argument essentially identical to that in the proof of Theorem 22. Consequently, φ is indeed an injection, and Equations (8) and (9) show that the coefficient of $q_{i,j}$ in the $\mathcal{B}$-representation of $m_{i,j}$ is 1.

Let us define a linear order $\succ$ on $\varphi(\mathcal{B})$ in the following way: $m_{i_1,j_1} \succ m_{i_2,j_2}$ precisely when one of the following four conditions holds:

1. $(i_1, j_1) = (0, 1)$;
2. $i_1 - j_1 \geq 1 > i_2 - j_2$;
3. $i_1 - j_1 \geq 1$, $i_2 - j_2 \geq 1$, and either $i_1 > i_2$, or $i_1 = i_2$ but $j_1 < j_2$;
4. $i_1 - j_1 < 1$, $i_2 - j_2 < 1$, and either $j_1 < j_2$, or $j_1 = j_2$ but $i_1 > i_2$.

An easy case analysis using Equations (8), (9), and (10) shows that if $m_{i_1,j_1} \succ m_{i_2,j_2}$, then the coefficient of q_{i_1,j_1} in the $\mathcal{B}$-representation of m_{i_2,j_2} is zero.

This case analysis is essentially identical to the one in the proof of Theorem 22, except that now we also have to take care of $q_{0,0}$ and $q_{0,1}$. Hence we examine four cases in addition to the ones in the proof of Theorem 22:

1. If $(i_1, j_1) = (0, 1)$, then $q_{i_1,j_1} = \alpha$, and this polynomial has a nonzero coefficient exclusively in the $\mathcal{B}$-representation of $m_{0,1}$.
2. The case $(i_2, j_2) = (0, 1)$ is impossible.
3. If $(i_1, j_1) = (0, 0)$, then only the fourth condition is satisfied by (i_1, j_1), so $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $i_2 - j_2 < 1$, which in the light of (10) yields $j_2 \geq 2$. Consequently, by (9), $q_{0,0} = \beta$ has zero coefficient in the $\mathcal{B}$-representation of m_{i_2,j_2}.
4. If $(i_2, j_2) = (0, 0)$, then $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $i_1 - j_1 \geq 1$, so the degree of q_{i_1,j_1} is larger than the degree of m_{i_2,j_2}. Consequently, q_{i_1,j_1} has zero coefficient in the $\mathcal{B}$-representation of $m_{0,0}$.
5. The cases in which both (i_1, j_1) and (i_2, j_2) are different from $(0, 0)$ and $(0, 1)$ are settled the same way as in the proof of Theorem 22.

The injection $m_{i,j} \mapsto q_{i,j}$ and the linear order $\succ$ satisfy the conditions of Corollary 21, therefore, by Equation (10),

$$\dim(\operatorname{span}(M(\mathbb{R}^{n+2}))) = \left|\mathcal{B}\right| = (2n+1)^2 - 2,$$

which completes the proof. ∎

4.2.2 Polynomials of odd degree

We first prove our claim for the case $n = 1$.

Lemma 25 *The cone $\mathcal{P}_4^{[a,b]}$ is not bilinear. More specifically, $\beta(\mathcal{P}_4^{[a,b]}) \leq 2$.*

Proof. Following the first version of the template given in Section 2, we present a set S of 14 pairs of vectors $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{P}_4^{[a,b]})$ such that the vectors $\operatorname{vec}(\mathbf{s}\mathbf{x}^\top)$ are linearly independent. Using Proposition 23 we can fix a and b arbitrarily; we will use $a = 1$ and $b = 6$.

For every $i = 1, \dots, 6$, let $p^{(i)}(x) = (x-1)(x-i)^2$, $q^{(i)}(x) = (6-x)(x-i)^2$, and define two additional polynomials $p^{(0)}(x) = (x-1)$ and $q^{(0)}(x) = (6-x)$. Now let S be the set consisting of the following orthogonal pairs:

- $\big(p^{(i)}, \mathbf{c}_{2n+2}(i)\big) \quad i = 1, \dots, 6,$
- $\big(q^{(i)}, \mathbf{c}_{2n+2}(i)\big) \quad i = 1, \dots, 6,$
- $\big(p^{(0)}, \mathbf{c}_{2n+2}(1)\big),$
- $\big(q^{(0)}, \mathbf{c}_{2n+2}(6)\big).$

The fact that the matrix T defined in the template using the above pairs indeed has rank 14 can be verified by direct calculation. ∎

Theorem 26 *The cone $\mathcal{P}_{2n+2}^{[a,b]}$ is not bilinear. More specifically, for every n, $\beta(\mathcal{P}_{2n+2}^{[a,b]}) \leq 2$.*

Proof. If $n = 1$, then our claim is the previous lemma. From now on, let us assume $n \geq 2$. Consider the matrix valued functions $M\colon \mathbb{R}^{2n+2} \mapsto \mathbb{R}^{(2n+2)\times(2n+2)}$, where the entries of $M(t_1, \dots, t_n; s_1, \dots, s_n; \alpha, \beta) = (m_{i,j})_{i,j=0}^{2n+1}$ are defined as

$$M(t_1, \dots, t_n; s_1, \dots, s_n; \alpha, \beta) = \mathbf{c}_{2n+2}(t_1)\mathbf{p}^\top + \mathbf{c}_{2n+2}(s_1)\mathbf{r}^\top + \alpha\mathbf{c}_{2n+2}(a)\mathbf{p}_a^\top + \beta\mathbf{c}_{2n+2}(b)\mathbf{p}_b^\top,$$

where $\mathbf{p}, \mathbf{r}, \mathbf{p}_a, \mathbf{p}_b \in \mathcal{P}_{2n+2}^{[a,b]}$ are the coefficient vectors of the polynomials $p(x) = (x-a)\prod_{k=1}^n (x-t_k)^2$, $r(x) = (b-x)\prod_{k=1}^n (x-s_k)^2$, $p_a(x) = x-a$, and $p_b(x) = b-x$, respectively. The entries of $M = (m_{i,j})_{i,j=0}^{2n+1}$ satisfy the polynomial equation

$$\sum_{j=0}^{2n+1} m_{i,j}x^j \equiv t_1^i(x-a)\prod_{k=1}^n (x-t_k)^2 + s_1^i(b-x)\prod_{k=1}^n (x-s_k)^2 + \alpha a^i(x-a) + \beta b^i(b-x). \qquad (11)$$

The polynomials $p(x)$, $p_a(x)$, $p_b(x)$, and $r(x)$ are nonnegative over $[a,b]$, and by Proposition 17,

$$\langle \mathbf{p}_a, \mathbf{c}_{2n+2}(a)\rangle = \langle \mathbf{p}_b, \mathbf{c}_{2n+2}(b)\rangle = \langle \mathbf{p}, \mathbf{c}_{2n+2}(t_1)\rangle = \langle \mathbf{r}, \mathbf{c}_{2n+2}(s_1)\rangle = 0.$$

Consequently, according to the general template of Section 2, the theorem follows if $\dim(\operatorname{span}(M(\mathbb{R}^{2n+2}))) = (2n+2)^2 - 2$. Finally, we will show this equality using the sufficient condition presented in Corollary 21.

Let us define the $(2n+2)$-variate polynomials $\Pi_1(k,\ell)$ and $\Pi_2(k,\ell)$ by

$$\begin{aligned}
&\Pi_1(k,\ell)(t_1,\dots,t_n;s_1,\dots,s_n;\alpha,\beta) \stackrel{\text{def}}{=} \\
&\qquad \Pi(k,\ell)(t_1,\dots,t_n) - a\,\Pi(k,\ell-1)(t_1,\dots,t_n), \text{ and} \\
&\Pi_2(k,\ell)(t_1,\dots,t_n;s_1,\dots,s_n;\alpha,\beta) \stackrel{\text{def}}{=} \\
&\qquad b\,\Pi(k,\ell-1)(s_1,\dots,s_n) - \Pi(k,\ell)(s_1,\dots,s_n),
\end{aligned} \tag{12}$$

where $\Pi(k,\ell)$ is defined by Equation (4) for $0 \le k \le 2n+3$, $0 \le \ell \le 2n-2$, otherwise $\Pi(k,\ell) = 0$.

For the rest of the proof, let us assume that $a = 0$, $b = 1$; Proposition 23 guarantees that this is without loss of generality.

Let $\mathcal{M}$ be the set of entries of M, and let $\mathcal{B}$ denote the set

$$\begin{aligned}
\mathcal{B} = \big\{\alpha,\beta\big\} &\cup \big\{\Pi(k,\ell)(t_1,\dots,t_n) \mid 3 \le k \le 2n+3, 0 \le \ell \le 2n-2, k+\ell \ge 2n+1\big\} \cup \\
&\cup \big\{\Pi(k,\ell)(s_1,\dots,s_n) \mid 3 \le k \le 2n+3, \ell = 2n-2\big\} \cup \\
&\cup \big\{\Pi(k,\ell)(s_1,\dots,s_n) \mid 2n \le k \le 2n+1, 0 \le \ell \le 1\big\} \cup \\
&\cup \big\{\Pi(k,\ell)(s_1,\dots,s_n) \mid 0 \le k \le 2n-1, 0 \le \ell \le 2n-2, k+\ell \le 2n\big\}.
\end{aligned}$$

Since $n \ge 2$, the last three sets in the union are disjoint. It follows from the definition that the set $\mathcal{B}$ is linearly independent, because no two polynomials in $\mathcal{B}$ share a common monomial. The coefficient of $x^{2n+1-k-\ell}$ in the polynomial $(x-a)\prod_{j=1}^{n}(x-t_j)^2$ is $\sum_{k=0}^{2}\Pi_1(k,\ell)$. Similarly, the coefficient of $x^{2n+1-k-\ell}$ in the polynomial $(b-x)\prod_{j=1}^{n}(x-s_j)^2$ is $\sum_{k=0}^{2}\Pi_2(k,\ell)$. From this observation and (11) it follows immediately that for every $0 \le i,j \le 2n+1$,

$$\begin{aligned}
m_{i,j} = {} &\Pi_1(i,2n+1-j) + \Pi_1(i+1,2n-j) + \Pi_1(i+2,2n-1-j) + \\
&+ \Pi_2(i,2n+1-j) + \Pi_2(i+1,2n-j) + \Pi_2(i+2,2n-1-j) + \\
&+ m'_{i,j},
\end{aligned} \tag{13}$$

where $m'_{i,j}$ is defined in Equation (9).

We now introduce an injection $\varphi\colon \mathcal{B} \mapsto \mathcal{M}$ by defining its inverse (where it exists): let $m_{i,j}$ be the image of the polynomial

$$\varphi^{-1}(m_{i,j}) = q_{i,j} = \begin{cases}
\alpha & i=0, j=1 \\
\Pi(i+2,2n-1-j)(t_1,\dots,t_n) & 1 \le j \le \min\{i,2n-1\} \\
\Pi(i+2,2n-2)(s_1,\dots,s_n) & i \ge 1, j=0 \\
\beta & i=0, j=0 \\
\Pi(i,2n+1-j)(s_1,\dots,s_n) & j \ge 3, j > \min\{i,2n-1\} \\
\text{not defined} & \text{otherwise}
\end{cases}. \tag{14}$$

In particular, we assign a polynomial to each entry except for $m_{0,2}$ and $m_{1,2}$, and we assign different polynomials to different entries of M, by an argument essentially identical to that in the proof of Theorem 22. Consequently, each polynomial in $\mathcal{B}$ is equal to $q_{i,j}$ for at most one pair

(i, j), and Equations (13) and (14) show that (assuming $a = 0$ and $b = 1$) the coefficient of $q_{i,j}$ in the $\mathcal{B}$-representation of $m_{i,j}$ is 1 or -1.

Let us define a linear order $\succ$ on $\varphi(\mathcal{B})$ in the following way. Let us say that a polynomial $q_{i,j}$ is *of type* k for some $k = 1, \ldots, 5$, if it is defined in the kth branch of the right-hand side of (14). Then, $m_{i_1,j_1} \succ m_{i_2,j_2}$ for some $(i_1, j_1) \neq (i_2, j_2)$ precisely when one of the following four conditions holds:

1. q_{i_1,j_1} is of smaller type than q_{i_2,j_2};
2. q_{i_1,j_1} and q_{i_2,j_2} are both of type 2, and either $i_1 > i_2$, or $i_1 = i_2$ but $j_1 < j_2$;
3. q_{i_1,j_1} and q_{i_2,j_2} are both of type 3, and $i_1 > i_2$;
4. q_{i_1,j_1} and q_{i_2,j_2} are both of type 5, and either $j_1 < j_2$, or $j_1 = j_2$ but $i_1 > i_2$.

Clearly this is indeed a linear order on $\varphi(\mathcal{B})$. An easy case analysis using (13), (9), and (14) shows that if $m_{i_1,j_1} \succ m_{i_2,j_2}$, then the coefficient of q_{i_1,j_1} in the $\mathcal{B}$-representation of m_{i_2,j_2} is zero:

1. If $(i_1, j_1) = (0, 1)$, then $q_{i_1,j_1} = \alpha$, and this polynomial has a nonzero coefficient exclusively in the $\mathcal{B}$-representation of $m_{0,1}$. In the remaining cases we assume $(i_1, j_1) \neq (0, 1)$.
2. The case $(i_2, j_2) = (0, 1)$ is impossible.
3. If $(i_1, j_1) = (0, 0)$, then $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $j_2 \geq 3$ (with a similar argument as in the proof of Theorem 24), so $q_{0,0} = \beta$ has zero coefficient in the $\mathcal{B}$-representation of m_{i_2,j_2}.
4. If $(i_2, j_2) = (0, 0)$, then $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $i_1 \geq 1$, so $q_{i_1,j_1} = \Pi(k, \ell)$ with some $k \geq 3$. Consequently, q_{i_1,j_1} has zero coefficient in the $\mathcal{B}$-representation of $m_{0,0}$. In the remaining cases we assume both (i_1, j_1) and (i_2, j_2) are different from $(0, 0)$ and $(0, 1)$.
5. The cases in which q_{i_1,j_1} and q_{i_2,j_2} are of the same type are settled the same way as in the last two cases of the case analysis in the proof of Theorem 22.
6. The case when q_{i_1,j_1} is of type 2 or 3, and q_{i_2,j_2} is of type 4, is settled the same way as in the proof of Theorem 22, by a simple degree argument.
7. The only remaining case is when q_{i_1,j_1} is of type 2 and q_{i_2,j_2} is of type 3. Then $i_1+2 > i_2+2$, and hence $q_{i_1,j_1} = \Pi(i_1 + 2, 2n - 1 - j)$ has coefficient zero in the $\mathcal{B}$-representation of m_{i_2,j_2}.

We conclude that the injection $m_{i,j} \mapsto q_{i,j}$ and the linear order $\succ$ satisfy the conditions of Corollary 21, therefore, by Equation (14),

$$\dim(\operatorname{span}(M(\mathbb{R}^{2n+2}))) = \left|\mathcal{B}\right| = (2n + 2)^2 - 2,$$

which completes the proof. ■

4.3 Lower bounds

To simplify the proof of the validity of bilinear complementarity relations, we will use the following lemma.

Lemma 27 *The bilinear equation $\mathbf{x}^\top Q\mathbf{s} = 0$ is satisfied by every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ if and only if it is satisfied by every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ such that $\mathbf{x}$ is an extreme vector of $\mathcal{K}$ and $\mathbf{s}$ is an extreme vector of $\mathcal{K}^*$.*

Proof. The *only if* direction is obvious. To show the converse implication, observe that every $\mathbf{x} \in \mathcal{K}$ and $\mathbf{s} \in \mathcal{K}^*$ can be expressed as a sum of finitely many extreme vectors of $\mathcal{K}$ and $\mathcal{K}^*$, respectively. Furthermore, if $\mathbf{x} = \sum_{i=1}^k \mathbf{x}_i$ and $\mathbf{s} = \sum_{j=1}^\ell \mathbf{s}_j$, then $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ if and only if $\langle \mathbf{x}_i, \mathbf{s}_j \rangle = 0$ for every $1 \le i \le k$, $1 \le j \le \ell$. Therefore, if $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, and the complementarity relation is satisfied by every orthogonal pair of extreme vectors, then $\langle \mathbf{x}_i, \mathbf{s}_j \rangle = 0$ for every $1 \le i \le k$, $1 \le j \le \ell$, and

$$\mathbf{x}^\top Q\mathbf{s} = \left(\sum_{i=1}^k \mathbf{x}_i\right)^\top Q \left(\sum_{j=1}^\ell \mathbf{s}_j\right) = \sum_{i=1}^k \sum_{j=1}^\ell \mathbf{x}_i^\top Q \mathbf{s}_j = 0.$$

∎

We are now ready to show that the upper bounds on the number of linearly independent bilinear complementarity relations given in Theorems 22, 24, and 26 are sharp.

Theorem 28 *For every integer $n \ge 1$, $\beta(\mathcal{P}_{2n+1}) = 4$.*

Proof. We have already proven $\beta(\mathcal{P}_{2n+1}) \le 4$. Now we prove that the following bilinear complementarity relations are satisfied by every $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$:

$$\sum_{i=0}^{2n} p_i c_i = 0, \tag{15}$$

$$\sum_{i=1}^{2n} i p_i c_{i-1} = 0, \tag{16}$$

$$\sum_{i=0}^{2n-1} (2n - i) p_i c_i = 0, \tag{17}$$

$$\sum_{i=0}^{2n-1} (2n - i) p_i c_{i+1} = 0. \tag{18}$$

It is easy to see that these conditions are indeed linearly independent. By Lemma 27 it is enough to show that the conditions are satisfied for pairs of vectors $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$ where $\mathbf{c}$ is an extreme vector of $\bar{\mathcal{M}}_{2n+1}$.

If $\mathbf{c} = c_{2n}\mathbf{e}_{2n} = (0, \ldots, 0, c_{2n})$ with some $c_{2n} > 0$ and $\langle \mathbf{p}, \mathbf{e}_{2n} \rangle = 0$, then (15), (16), and (17) trivially hold, since all the terms on the left-hand sides of these equations are zeros. Furthermore, the left-hand side of (18) simplifies to $p_{2n-1}c_{2n}$, which must be zero, because otherwise $p_{2n-1} \neq 0$, $p_{2n} = 0$, and p would be a polynomial of odd degree, which cannot be nonnegative over the entire real line.

If $\mathbf{c}$ is an extreme vector of $\mathcal{M}_{2n+1}$, then, by Proposition 18, $\mathbf{c} = \mathbf{c}(t_0)$ for some $t_0 \in \mathbb{R}$, and, using Proposition 17, $\mathbf{c}$ is orthogonal to $\mathbf{p}$ if and only if $p(t_0) = 0$. But this equation is equivalent to (15), since

$$p(t_0) = \sum_{i=0}^{2n} p_i t_0^i = \sum_{i=0}^{2n} p_i c_i.$$

By Proposition 19, every root of p has even multiplicity, therefore $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{K})$ implies $p'(t_0) = 0$, which is equivalent to (16), as

$$p'(t_0) = \sum_{i=1}^{2n} p_i i t_0^{i-1} = \sum_{i=1}^{2n} i p_i c_{i-1}.$$

Furthermore, if $p(t_0) = p'(t_0) = 0$, then $2np(t_0) - t_0 p'(t_0) = 0$, which translates to (17), since

$$2np(t_0) - t_0 p'(t_0) = \sum_{i=0}^{2n} 2np_i t_0^i - \sum_{i=1}^{2n} p_i i t_0^i = \sum_{i=0}^{2n} 2np_i c_i - \sum_{i=1}^{2n} i p_i c_i = \sum_{i=0}^{2n} (2n - i) p_i c_i.$$

Finally, $p(t_0) = p'(t_0) = 0$ also implies $2nt_0 p(t_0) - t_0^2 p'(t_0) = 0$, which is equivalent to (18):

$$2nt_0 p(t_0) - t_0^2 p'(t_0) = \sum_{i=0}^{2n} 2np_i t_0^{i+1} - \sum_{i=1}^{2n} p_i i t_0^{i+1} = \sum_{i=0}^{2n} 2np_i c_{i+1} - \sum_{i=1}^{2n} i p_i c_{i+1} = \sum_{i=0}^{2n-1} (2n - i) p_i c_{i+1}.$$

■

Theorem 29 *For every integer $n \geq 1$ and real numbers $a < b$, $\beta(\mathcal{P}_{n+1}^{[a,b]}) = 2$.*

Proof. We have already proven $\beta(\mathcal{P}_{n+1}^{[a,b]}) \leq 2$. Now we prove that the following bilinear complementarity relations are satisfied by every $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$:

$$\sum_{i=0}^{n} p_i c_i = 0, \tag{19a}$$

$$\sum_{i=0}^{n-1} \big((a+b)(n-i) p_i c_i - (n-i) p_i c_{i+1} + ab(i+1) p_{i+1} c_i \big) = 0. \tag{19b}$$

It is easy to see that these conditions are indeed linearly independent. By Lemma 27 it is enough to show that the conditions are satisfied for pairs of vectors $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$ where $\mathbf{c}$ is an extreme vector of $\bar{\mathcal{M}}_{n+1}^{[a,b]]}$.

Let $p \in \mathcal{P}_{n+1}^{[a,b]}$ be a polynomial, and $\mathbf{p}$ its coefficient vector. By Proposition 17, an extreme $\mathbf{c} = \mathbf{c}(t_0)$ is orthogonal to $\mathbf{p}$ if and only if $p(t_0) = 0$. Therefore, $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$ implies $p(t_0) = 0$, which is equivalent to (19a), as in the proof of the previous theorem. By Proposition 19, every root of p has even multiplicity, except possibly for a and b, and hence

$$(t_0 - a)(b - t_0)p'(t_0) = 0.$$

Finally, this last equality and $p(t_0) = 0$ together imply

$$n(a + b - t_0)p(t_0) - (t_0 - a)(b - t_0)p'(t_0) = 0,$$

equivalent to (19b), as simple calculation, similar to the ones in the previous two proofs, shows. ■

4.4 Müntz polynomials

Consider a vector $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$, where $0 < \lambda_0 < \cdots < \lambda_n$ are real numbers. Functions of the form $t \to \sum_{i=0}^{n} \alpha_i t^{\lambda_i}$ are called *Müntz polynomials of type* $\boldsymbol{\lambda}$, and they share many properties of ordinary polynomials. In particular, they form an extended Chebyshev system over $(0, \infty)$ [8, Chapter 1], so it is natural to ask whether the results of the previous sections generalize to the cone of Müntz polynomials nonnegative over $(0, \infty)$, denoted by

$$\mathcal{P}^{\boldsymbol{\lambda}} \stackrel{\text{def}}{=} \Big\{\mathbf{a} \mid \sum_{i=0}^{n} a_i t^{\lambda_i} \geq 0 \text{ for all } t \geq 0\Big\}.$$

In this section we show that the answer is at least "generically" yes. Similarly to the case of ordinary polynomials, if $t_i \geq 0$ is a root of a nonnegative Müntz polynomial $p(t)$, then it is also a root of $t\frac{d}{dt}p(t)$. This will lead to a non-trivial optimality constraint:

Proposition 30 $\beta(\mathcal{P}^{\boldsymbol{\lambda}}) \geq 2$.

Proof. Notice that the operator $t\frac{d}{dt}$ maps the set of Müntz polynomials of type $\boldsymbol{\lambda}$ onto itself. (Consider $\frac{d}{dt}$ to be the linear operator of formal differentiation to avoid problems when $t = 0$.) It follows from the general theory of Chebyshev systems that the nonzero extreme vectors of the dual cone of $\mathcal{P}^{\boldsymbol{\lambda}}$ are vectors of the form $\alpha(t^{\lambda_0}, \ldots, t^{\lambda_n})^\top$ and $\alpha(0, \ldots, 0, 1)^\top$, where $t \geq 0$ and $\alpha > 0$. Using the notation $\mathbf{c}(t) = (t^{\lambda_0}, \ldots, t^{\lambda_n})^\top$, the coefficient vector $\mathbf{p}$ of some $p \in \mathcal{P}^{\boldsymbol{\lambda}}$ is orthogonal to $\mathbf{c}(t)$ if and only if t is a root of p. Consequently $(\mathbf{p}, \mathbf{c}(t)) \in \mathcal{C}(\mathcal{P}^{\boldsymbol{\lambda}})$ implies $p(t) = 0$; in this case either $p'(t) = 0$ or $t = 0$ must also hold, implying $t\frac{d}{dt}p(t) = 0$. The equations $p(t) = 0$ and $t\frac{d}{dt}p(t) = 0$ translate to the linearly independent bilinear complementarity relations

$$\sum_{i=0}^{n} p_i c_i = 0, \qquad \sum_{i=0}^{n} \lambda_i p_i c_i = 0.$$

It is easily verified that these relations are also valid for pairs $(\mathbf{p}, \mathbf{c}) \in \mathcal{C}(\mathcal{P}^{\boldsymbol{\lambda}})$ when $\mathbf{c} = (0, \ldots, 0, 1)^\top$. ■

Let $\Lambda_{n+1} = \big\{(\boldsymbol{\lambda} \in \mathbb{R}^{n+1} \bigm| 0 < \lambda_0 < \cdots < \lambda_n\big\}$ denote the space of exponent vectors for Müntz polynomials. We say that $\boldsymbol{\lambda} \in \Lambda_{n+1}$ is *generic* if the following condition holds:

The differences $\lambda_i - \lambda_j$ are different for each pair (i, j) with $i \neq j$; furthermore, there exist indices $0 \le i_1 < i_2 \le n$ such that for all $j \in \{0, \ldots, n\} \setminus \{i_1, i_2\}$ the value $\lambda_{i_1} + \lambda_{i_2} + \lambda_j$ can be decomposed ***uniquely*** *as a sum of three different elements of the set $\{\lambda_0, \ldots, \lambda_n\}$.*

We remark that almost all vectors in Λ_{n+1} are generic, including those with algebraically independent coordinates.

Theorem 31 *If $\boldsymbol{\lambda}$ is generic, then we have $\beta(\mathcal{P}^{\boldsymbol{\lambda}}) = 2$ for every $n \ge 4$.*

Proof. We present a proof for $n = 2k$. The case when n is odd requires separate treatment using analogous arguments, similarly to what we have seen for ordinary polynomials over the half-line. According the previous proposition it suffices to show that $\beta(\mathcal{P}^{\boldsymbol{\lambda}}) \le 2$ for generic $\boldsymbol{\lambda}$.

Along the lines of the proof of Theorem 22 let $M\colon \mathbb{R}^n \mapsto \mathbb{R}^{(2k+1)\times(2k+1)}$ be the matrix-valued function given by

$$M(t_1, \ldots, t_k) = \mathbf{c}\mathbf{p}^\top,$$

where $\mathbf{p} \in \mathcal{P}^{\boldsymbol{\lambda}}$ is the coefficient vector of the Müntz polynomial p with roots $t_1, \ldots, t_k$ defined by the determinant

$$p(t) = \begin{vmatrix} t_1^{\lambda_0} & \lambda_0 t_1^{\lambda_0 - 1} & \cdots & t_k^{\lambda_0} & \lambda_0 t_k^{\lambda_0 - 1} & t^{\lambda_0} \\ \vdots & & \ddots & & & \vdots \\ t_1^{\lambda_{2k}} & \lambda_{2k} t_1^{\lambda_{2k} - 1} & \cdots & t_k^{\lambda_{2k}} & \lambda_{2k} t_k^{\lambda_{2k} - 1} & t^{\lambda_{2k}} \end{vmatrix},$$

while $\mathbf{c} = \mathbf{c}(t_1) = (t_1^{\lambda_0}, \ldots, t_1^{\lambda_{2k}})$ is an extreme vector of the dual cone corresponding to the first root of $\mathbf{p}$.

The nonnegativity of p is a corollary of the fact that Müntz polynomials form an extended Chebyshev system over $(0, \infty)$ [8, Chapter 1]. Since t_1 is a root of p, we have $\langle \mathbf{p}, \mathbf{c} \rangle = p(t_1) = 0$, implying $(\mathbf{p}, \mathbf{c}) \in \mathcal{C}(\mathcal{P}^{\boldsymbol{\lambda}})$. Following the general template of Section 2 (with $\mathbf{p}$, $\mathbf{c}$ and $M(\mathbb{R}^k)$ playing the respective roles of $\mathbf{x}$, $\mathbf{s}$ and $\mathcal{T}$) it only remains to show that $\dim(\operatorname{span}(M(\mathbb{R}^k))) \ge (2k+1)^2 - 2$ holds if $\boldsymbol{\lambda}$ is generic.

The coefficient of t^{λ_j} in $p(t)$ is

$$\sum_{\pi \in \Pi_j} \lambda_{\pi_2} \lambda_{\pi_4} \cdots \lambda_{\pi_{2k}} t_1^{\lambda_{\pi_1} + \lambda_{\pi_2} - 1} t_2^{\lambda_{\pi_3} + \lambda_{\pi_4} - 1} \cdots t_k^{\lambda_{\pi_{2k-1}} + \lambda_{\pi_{2k}} - 1},$$

where Π_j denotes the family of all permutations of the set $\{0, \ldots, 2k\} \setminus \{j\}$. The (i, j) entry of the matrix $M(t_1, \ldots, t_k)$ can then be expressed as

$$m_{i,j} = \sum_{\pi \in \Pi_j} \lambda_{\pi_2} \lambda_{\pi_4} \cdots \lambda_{\pi_{2k}} t_1^{\lambda_i + \lambda_{\pi_1} + \lambda_{\pi_2} - 1} t_2^{\lambda_{\pi_3} + \lambda_{\pi_4} - 1} \cdots t_k^{\lambda_{\pi_{2k-1}} + \lambda_{\pi_{2k}} - 1}.$$

Since $\boldsymbol{\lambda} > 0$, all monomials in the above formula have a nonzero coefficient.

The sum of the exponents in each monomial of entry $m_{i,j}$ is $\lambda_i - \lambda_j + \sum_{\ell=0}^{2k} \lambda_\ell - k$. Let $\boldsymbol{\lambda}$ be generic. Since the pairwise differences $\lambda_i - \lambda_j$ $(i \neq j)$ are different, the non-diagonal entries $m_{i,j}$, when viewed as Müntz polynomials of variables $t_1, \dots t_k$, are all linearly independent, as they do not share a common monomial. Furthermore, since $\boldsymbol{\lambda}$ is generic, monomials containing the term $t_1^{\lambda_{i_1}+\lambda_{i_2}+\lambda_j}$ do not appear in diagonal entries $m_{i,i}$, unless $i \in \{i_1, i_2, j\}$. Hence the diagonal entries $m_{j,j}$ satisfying $j \neq i_1$ and $j \neq i_2$, and the off-diagonal entries are all linearly independent Müntz polynomials, and $\dim(\operatorname{span}(M(\mathbb{R}^k))) \geq (2k+1)^2 - 2$; that is, $\beta(\mathcal{P}^{\boldsymbol{\lambda}}) \leq 2$ in the generic case. ∎

5 More non-bilinear cones

There are several ways to construct new cones from given ones in such a way that the bilinearity rank of the new cone can be calculated from the bilinearity rank of constituent cones. For some familiar operations, such as intersection or Minkowski sum, there does not appear to be a simple formula for $\beta(\mathcal{K}_1 \cap \mathcal{K}_2)$ or $\beta(\mathcal{K}_1 + \mathcal{K}_2)$. For instance, consider two three-dimensional cones with regular pentagons as their cross section. (See Corollary 13.) It is possible to intersect them one way such that the resulting polyhedral cone has four or more extreme rays, and, by Theorem 11, it has bilinearity rank equal to one; and in another way such that the intersection has exactly three extreme rays, in which case it is linearly isomorphic to the nonnegative orthant, and thus its bilinearity rank is equal to three.

For the direct (Cartesian) product of cones there is a simple and fairly obvious formula.

Lemma 32 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two proper cones in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Define*

$$K_1 \times K_2 = \big\{(\mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 \in \mathcal{K}_1 \text{ and } \mathbf{x}_2 \in \mathcal{K}_2\big\}.$$

Then $\beta(\mathcal{K}_1 \times \mathcal{K}_2) = \beta(\mathcal{K}_1) + \beta(\mathcal{K}_2)$.

Proof. It is immediate from the definition of the dual cone that $(\mathcal{K}_1 \times \mathcal{K}_2)^* = \mathcal{K}_1^* \times \mathcal{K}_2^*$. If $\big((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{s}_1, \mathbf{s}_2)\big) \in \mathcal{C}(\mathcal{K}_1 \times \mathcal{K}_2)$, then $\langle \mathbf{x}_1, \mathbf{s}_1\rangle + \langle \mathbf{x}_2, \mathbf{s}_2\rangle = 0$ is equivalent to $\langle \mathbf{x}_1, \mathbf{s}_1\rangle = \langle \mathbf{x}_2, \mathbf{s}_2\rangle = 0$. Now let Q_1 be a bilinear complementarity relation for $\mathcal{K}_1$ and Q_2 be a bilinear complementarity relation for $\mathcal{K}_2$. Then

$$\begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix} \tag{20}$$

are bilinear complementarity relations for $\mathcal{K}_1 \times \mathcal{K}_2$.

Conversely, suppose that the matrix

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

is a bilinear complementarity relation for $\mathcal{K}_1 \times \mathcal{K}_2$. Then for every $((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{s}_1, \mathbf{s}_2)) \in \mathcal{C}(\mathcal{K}_1 \times \mathcal{K}_2)$,

$$\mathbf{x}_1^\top Q_{11}\mathbf{s}_1 + \mathbf{x}_1^\top Q_{12}\mathbf{s}_2 + \mathbf{x}_2^\top Q_{21}\mathbf{s}_1 + \mathbf{x}_2^\top Q_{22}\mathbf{s}_2 = 0. \tag{21}$$

Since this relation holds for any member of $C(\mathcal{K}_1 \times \mathcal{K}_2)$, it holds when $\mathbf{x}_1 = \mathbf{s}_2 = \mathbf{0}$. Thus, for every pair $(\mathbf{x}_2, \mathbf{s}_1) \in \mathcal{K}_2 \times \mathcal{K}_1^*$ we must have $\mathbf{x}_2^\top Q_{21}\mathbf{s}_1 = 0$. Since $\mathcal{K}_1^*$ and $\mathcal{K}_2$ are both proper cones (in particular, they are full dimensional), this implies that $Q_{21} = 0$. Similarly, we have $Q_{12} = 0$. By setting $\mathbf{s}_2 = \mathbf{0}$ in (21) we see now that Q_{11} is a bilinear complementarity relation for $C(\mathcal{K}_1)$. This shows that a complete set of linearly independent bilinear complementarity relations can be obtained from $\mathcal{K}_1$ and $\mathcal{K}_2$ in the form of (20). The number of complementarity relations thus obtained is $\beta(\mathcal{K}_1 \times \mathcal{K}_2) = \beta(\mathcal{K}_1) + \beta(\mathcal{K}_2)$. ■

5.1 Calculating bilinearity rank through linear isomorphism

As we have seen in Lemma 9, applying a nonsingular linear transformation A to a cone $\mathcal{K}$ preserves its bilinearity rank. In addition, any change of basis for the cone of polynomials will result in a cone linearly isomorphic to it. Thus, for instance, the set of vectors of coefficients of nonnegative polynomials expressed in any orthogonal polynomial basis (e.g., Laguerre, Legendre, Chebyshev, etc.), or the Bernstein polynomial basis $\binom{n}{k}t^k(1-t)^{n-k}$ for $k = 0, \dots, n$ are linearly isomorphic to the cone of nonnegative polynomials in the standard basis. This fact is useful in numerical computations since the standard basis is numerically unstable and we may need to work with a more stable basis.

We have already stated in Proposition 23 that for all $a < b$ and n, the cones $\mathcal{P}_{n+1}^{[a,b]}$ and $\mathcal{P}_{n+1}^{[0,1]}$ are linearly isomorphic. More generally:

Observation 33 *Let $f(t)$ be a function with domain $\Delta \subseteq \mathbb{R}$ and range $\Omega \subseteq \mathbb{R}$; also suppose that the set of functions $\{1, f, f^2, \dots, f^n\}$ is linearly independent. Then the cone*

$$\mathcal{P}^f = \left\{ \mathbf{a} = (a_0, \dots, a_n) \,\middle|\, \sum_{j=0}^{n} a_j f^j(t) \geq 0 \ \text{ for all } t \in \Delta \right\}$$

is linearly isomorphic to the cone of ordinary polynomials nonnegative over Ω.

From this observation and using change of basis as needed we can prove linear isomorphism of a number of cones of nonnegative functions over well-known finite dimensional bases with $\mathcal{P}_{2n+1}$ or $\mathcal{P}_{n+1}^{[0,1]}$. Below we present a partial list. Most of the techniques used below are quite simple, and they are used by Karlin and Studden [8] and Nesterov [10] for other purposes.

5.1.1 Rational functions

The basis $\{t^{-m}, t^{-m+1}, \dots, t^{n-1}, t^n\}$ for nonnegative even integers n and m spans the set of rational functions with a degree n numerator and denominator t^m. Since $\sum_{i=-m}^{n} p_i t^i = t^{-m} \sum_{i=0}^{n+m} p_i t^i$, the cone of rational functions with numerator of degree n and denominator t^m nonnegative over $\Delta = \mathbb{R} \backslash \{0\}$ is linearly isomorphic to the cone of nonnegative polynomials of degree $n + m$, and therefore its bilinearity rank is 4.

5.1.2 Nonnegative polynomials over $[0, \infty)$

Consider the basis $B = \{t^n, t^{n-1}(1-t), \dots, t(1-t)^{n-1}, (1-t)^n\}$ of polynomials of degree n. Clearly the cone

$$\left\{ (p_0, \dots, p_n) \;\middle|\; \sum_{i=0}^{n} p_i t^i (1-t)^{n-i} \geq 0 \quad \forall\, t \in [0,1] \right\},$$

which consists of coefficient vectors of polynomials nonnegative over $[0,1]$, expressed in basis B, is linearly isomorphic to $\mathcal{P}_{n+1}^{[0,1]}$. On the other hand we have:

Lemma 34 (Nesterov [10]) *A polynomial $p_0(1-t)^n + p_1 t(1-t)^{n-1} + \cdots + p_n t^n$ is nonnegative over $[0,1]$ if and only if the polynomial $p_0 + p_1 t + \cdots + p_n t^n$ is nonnegative over $[0, \infty)$.*

This follows from

$$\sum_k p_k t^k (1-t)^{n-k} = (1-t)^n \sum_k p_k \left(\tfrac{t}{1-t}\right)^k$$

and the fact that $[0,1]$ is mapped to $[0, \infty]$ under $f(t) = \frac{t}{1-t}$. We thus get the following result.

Corollary 35 *The cone $\mathcal{P}_{n+1}^{[0,\infty)}$ is linearly isomorphic to $\mathcal{P}_{n+1}^{[0,1]}$. Therefore $\beta\big(\mathcal{P}^{[0,\infty)}\big) = 2$ for every $n \in \mathbb{N}$.*

5.1.3 Cosine polynomials

Consider the cone

$$\mathcal{P}_{n+1}^{\cos} \stackrel{\text{def}}{=} \{\mathbf{c} \in \mathbb{R}^{n+1} \mid \sum_{k=0}^{n} c_k \cos(kt) \geq 0 \text{ for all } t \in \mathbb{R}\}.$$

To relate this cone to the cones we have discussed before, first observe that $\cos(kt)$ can be expressed as an ordinary polynomial of degree k of $\cos(t)$. This follows immediately from applying the binomial theorem to the identity $\big(\cos(t) + i\sin(t)\big)^k = \cos(kt) + i\sin(kt)$:

$$\cos(kt) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \cos^{k-2j}(t) \big(1 - \cos^2(t)\big)^j \tag{22}$$

$$\sin(kt) = \sin(t) \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} \cos^{k-2j-1}(t) \big(1 - \cos^2(t)\big)^j \tag{23}$$

From (22) we see that expansion of $\cos(kt)$ is a polynomial of $\cos(t)$ of degree k. Thus, every vector $\mathbf{c}$ representing the cosine polynomial $c(t) = \sum_{k=0}^{n} c_k \cos(kt)$ is mapped to a vector $\mathbf{p}$ representing the ordinary polynomial $p(s) = \sum_{k=0}^{n} p_k s^k$ through the identity $\sum_{k=0}^{n} c_k \cos(kt) = p\big(\cos(t)\big)$. Furthermore, this correspondence between $\mathbf{c}$ and $\mathbf{p}$ is one-to-one and onto, since for each k the function $\cos(kt)$ is a polynomial of degree k in $\cos(t)$ the matrix mapping $\mathbf{c}$ to $\mathbf{p}$ is lower triangular with nonzero diagonal entries. Now $c(t) = p\big(\cos(t)\big) \geq 0$ for all t if and only if $p(s) \geq 0$ for all $s \in [-1,1]$. Recalling that $\mathcal{P}_{n+1}^{[-1,1]}$ is linearly isomorphic to $\mathcal{P}_{n+1}^{[0,1]}$, we have the following result.

Proposition 36 *The cone $\mathcal{P}^{\cos}_{n+1}$ is linearly isomorphic to $\mathcal{P}^{[0,1]}_{n+1}$. Therefore $\beta\big(\mathcal{P}^{\cos}_{n+1}\big) = 2$ for every $n \in \mathbb{N}$.*

5.1.4 Trigonometric polynomials

Consider the cone

$$\begin{aligned}\mathcal{P}^{\text{trig}}_{2n+1} &= \Big\{\mathbf{r} \in \mathbb{R}^{2n+1} \mid r_0 + \sum_{k=1}^{n}\big(r_{2k-1}\cos(kt) + r_{2k}\sin(kt)\big) \geq 0 \text{ for all } t \in \mathbb{R}\Big\} \\ &= \Big\{\mathbf{r} \in \mathbb{R}^{2n+1} \mid r_0 + \sum_{k=1}^{n}\big(r_{2k-1}\cos(kt) + r_{2k}\sin(kt)\big) \geq 0 \text{ for all } t \in (-\pi,\pi)\Big\}\end{aligned}$$

To transform a trigonometric polynomial $r(t) = r_0 + \sum_{k=1}^{n}\big(r_{2k-1}\cos(kt) + r_{2k}\sin(kt)\big)$ into one of the classes of polynomials already discussed we make a change of variables $t = 2\arctan(s)$. With this transformation we have

$$\begin{aligned}\sin(t) &= \frac{2s}{1+s^2} \\ \cos(t) &= \frac{1-s^2}{1+s^2}\end{aligned}$$

Using (22-23) we can write

$$r(t) = p_1\big(\tfrac{1-s^2}{1+s^2}\big) + \tfrac{2s}{1+s^2}p_2\big(\tfrac{1-s^2}{1+s^2}\big) \tag{24}$$

where p_1 and p_2 are ordinary polynomials of degree n, and $n-1$, respectively; $p_1(\cdot)$ is obtained from (22) and $p_2(\cdot)$ is obtained from (23). Multiplying by $(1+s^2)^n$ we see that

$$r(t) = (1+s^2)^{-n}p(s)$$

for some ordinary polynomial p. Substituting (22) and (23), the polynomial p can be expressed in the following basis:

$$\Big\{(1+s^2)^n, (1+s^2)^{n-1}(1-s^2), \ldots, (1-s^2)^n\Big\} \cup \Big\{s(1+s^2)^{n-1}, s(1-s^2)^{n-2}(1-s^2), \ldots, s(1-s^2)^{n-1}\Big\}$$

It is straightforward to see that this is indeed a basis. We need to simply observe that those terms that are not multiplied by s form a basis of polynomials with even degree terms, and those that involve s form a basis of polynomials with odd degree terms. Therefore, the correspondence between vector of coefficients $\mathbf{r}$ of the trigonometric polynomial $r(t)$ and the vector of coefficients $\mathbf{p}$ of the ordinary polynomial $p(s)$ in the above basis is one-to-one and onto. Furthermore, since the function $\tan(t/2)$ maps $(-\pi,\pi)$ to $\mathbb{R}$, it follows that trigonometric and ordinary polynomials of the same degree are linearly isomorphic:

Proposition 37 *The cone $\mathcal{P}^{trig}_{2n+1}$ is linearly isomorphic to $\mathcal{P}_{n+1}$. Therefore, $\beta\big(\mathcal{P}^{trig}_{2n+1}\big) = 4$ for every $n \in \mathbb{N}$.*

5.1.5 Exponential polynomials

One could ask if the results of trigonometric polynomials extend to hyperbolic functions $\sinh(\cdot)$ and $\cosh(\cdot)$. The situation is actually simpler here. Consider the cone

$$\mathcal{P}^{\exp} \stackrel{\text{def}}{=} \Big\{\mathbf{e} \mid \sum_{k=-m}^{n} e_k \exp(kt) \geq 0 \text{ for all } t \geq 0\Big\}.$$

First, there is no loss of generality if we assume $m = 0$ since every such polynomial can be mutltiplied by $\exp(mt)$. Now clearly $e(t) = e_0 + e_1 \exp(t) + \cdots + e_n \exp(nt) \geq 0$ for all $t \in \mathbb{R}$ if and only if the ordinary polynomial $e_0 + e_1 s + \cdots + e_n s^n$ is nonnegative over $[0, \infty]$. Recalling that $\mathcal{P}^{[0,\infty)}$ is linearly isomorphic to $\mathcal{P}^{[0,1]}$ we have shown the following:

Proposition 38 *The cone $\mathcal{P}^{\exp}$ is linearly isomorphic to $\mathcal{P}^{[0,1]}$. Therefore, $\beta\big(\mathcal{P}^{\exp}\big) = 2$.*

6 Conclusion

Our main motivation for this research came from our work on solving statistical nonparametric estimation problems using polynomials and polynomial splines where the estimated functions themselves required to be nonnegative, see [1] and [13]. Our goal was to see if there is an easier way than formulating these problems as semidefinite programs. In particular, are there efficient algorithms for cone-LP problems over positive polynomials? This questions led us to consider the simplest form of complementarity relations for positive polynomials, and we have found that bilinear complementarity relations alone are not sufficient.

The central question remaining open is whether there are bilinear cones other than symmetric cones and cones linearly isomorphic to them?

Another direction is to investigate more sets of cones and estimate their bilinearity rank. For example one can examine all cones of positive functions over Chebyshev systems, and cones of functions of several variables which can be expressed as sums of squares of functions over a given finite set of functions.

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Appendix: Proof of Theorem 2, due to O. Güler [7]

Recall the following basic fact.

Proposition 39 *Let $S \subseteq R^n$ be a closed convex set and $\mathbf{a} \in \mathbb{R}^n$. Then there is a unique point $\mathbf{x} = \Pi_S(\mathbf{a})$ in S which is closest to $\mathbf{a}$, i.e., there is a unique point $\mathbf{x} \in S$ such that $\mathbf{x} = \operatorname{argmin}_{\mathbf{y}\in S} \|\mathbf{a} - \mathbf{y}\|$. Furthermore, if S is a closed convex cone, then $\langle \mathbf{x}, \mathbf{x} - \mathbf{a}\rangle = 0$.*

The unique point above is called the *projection of* $\mathbf{a}$ *to* S.

We need to show a continuous bijection between the complementarity set $C(\mathcal{K})$ of $\mathcal{K}$ and $\mathbb{R}^n$ whose inverse is also continuous.

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be defined by $\varphi(\mathbf{a}) = (\mathbf{x}, \mathbf{s})$, where $\mathbf{x} = \Pi_{\mathcal{K}}(\mathbf{a})$ and $\mathbf{s} = \mathbf{x} - \mathbf{a}$. Clearly φ is continuous; we first show that $\varphi(\mathbf{a}) \in C(\mathcal{K})$ for every $\mathbf{a}$. By definition $\Pi_{\mathcal{K}}(\mathbf{a}) \in \mathcal{K}$, and by the above proposition $\langle \mathbf{x}, \mathbf{s} \rangle = 0$. It remains to show that $\mathbf{s} \in \mathcal{K}^*$.

For an arbitrary $\mathbf{u} \in \mathcal{K} \setminus \{\mathbf{x}\}$, define the convex combination $\mathbf{u}_\alpha = \alpha\mathbf{u} + (1-\alpha)\mathbf{x}$ where $0 \le \alpha \le 1$, and let $\zeta(\alpha) = \|\mathbf{a} - \mathbf{u}_\alpha\|^2$. Then ζ is a differentiable function on the interval $[0, 1]$, and $\min_{0 \le \alpha \le 1} \zeta(\alpha)$ is attained at $\alpha = 0$. Hence $\frac{d\zeta}{d\alpha}\Big|_{\alpha=0} \ge 0$.

Now, using $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, we have

$$\frac{d\zeta}{d\alpha}\bigg|_{\alpha=0} = 2\langle \mathbf{s}, \mathbf{u} - \mathbf{x} \rangle = 2\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$$

for every $\mathbf{u} \in \mathcal{K} \setminus \{\mathbf{x}\}$. Note that the inequality $\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$ also holds for $\mathbf{u} = \mathbf{x}$, implying $\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$ for every $\mathbf{u} \in \mathcal{K}$. Therefore $\mathbf{s} \in \mathcal{K}^*$.

Consider now the continuous function $\bar{\varphi}\colon C(\mathcal{K}) \to \mathbb{R}^n$ defined by $\bar{\varphi}(\mathbf{x}, \mathbf{s}) = \mathbf{x} - \mathbf{s}$. To conclude the proof we show that $\bar{\varphi} \circ \varphi = \iota_{\mathbb{R}^n}$ and $\varphi \circ \bar{\varphi} = \iota_{C(\mathcal{K})}$, where ι_S denotes the identity function of the set S. The first one is easy:

$$(\bar{\varphi} \circ \varphi)(\mathbf{a}) = \bar{\varphi}\left(\Pi_{\mathcal{K}}(\mathbf{a}), \Pi_{\mathcal{K}}(\mathbf{a}) - \mathbf{a}\right) = \mathbf{a}.$$

To show $\varphi \circ \bar{\varphi} = \iota_{C(\mathcal{K})}$, it suffices to prove that $\Pi_{\mathcal{K}}(\mathbf{x} - \mathbf{s}) = \mathbf{x}$ for every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$.

Suppose on the contrary that there is a point $\mathbf{u} \in \mathcal{K}$ such that $\|\mathbf{a} - \mathbf{u}\| < \|\mathbf{a} - \mathbf{x}\|$, where $\mathbf{a} = \mathbf{x} - \mathbf{s}$. Then, again using $\langle \mathbf{x}, \mathbf{s} \rangle = 0$,

$$0 > \langle \mathbf{a} - \mathbf{u}, \mathbf{a} - \mathbf{u} \rangle - \langle \mathbf{a} - \mathbf{x}, \mathbf{a} - \mathbf{x} \rangle = \langle \mathbf{x} - \mathbf{s} - \mathbf{u}, \mathbf{x} - \mathbf{s} - \mathbf{u} \rangle - \langle \mathbf{s}, \mathbf{s} \rangle = \|\mathbf{x} - \mathbf{u}\|^2 + 2\langle \mathbf{s}, \mathbf{u} \rangle,$$

in contradiction with $\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$, which completes the proof. ■