A Revealed Preference Analysis of Solutions to Simple Allocation Problems*

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Abstract

We interpret solution rules on a class of simple allocation problems as data on the choices of a policy-maker. We analyze conditions under which the policy maker's choices are (i) *rational* (ii) *transitive-rational*, and (iii) *representable*; that is, they coincide with maximization of a (i) binary relation, (ii) transitive binary relation, and (iii) numerical function on the allocation space. Our main results are as follows: (i) a well known property, *contraction independence* (*a.k.a. IIA*) is equivalent to *rationality*; (ii) every *contraction independent* and *other-c monotonic* rule is *transitive-rational*; and (iii) every *contraction independent* and *other-c monotonic* rule, if additionally *continuous*, can be represented by a numerical function.

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1 Introduction

Revealed preference theory studies conditions under which by observing the choice behavior of an agent, one can discover the underlying preferences that govern it. Most of the earlier work on revealed preference theory analyzes consumers' demand choices from budget sets (e.g. see Samuelson, 1938, 1948). The underlying premise that choices reveal information about preferences, however, is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash, 1950) characterize bargaining rules which can be "rationalized" as maximizing the underlying preferences of an impartial arbitrator (or, depending on the interpretation, a social welfare function of the bargainers) (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 1999; Sánchez, 2000).[1]

In this paper, we carry out a revealed preference analysis on a class of solutions to *simple allocation problems*. A simple allocation problem for a society N is an $|N|+1$ dimensional nonnegative real vector $\left(c_1, ...c_{|N|}, E\right) \in \mathbb{R}_+^N$ satisfying $\sum_N c_i \geqq E$ where E, the **endowment** has to be allocated among agents in N who are characterized by c, the **characteristic vector**. Simple allocation problems have a wide range of applications. We discuss them in detail in *Subsection 1.1.*

We interpret an allocation rule on simple allocation problems as representing the choices of a decision-maker (*e.g.* a public-policy maker, a tax codifier or a bankruptcy judge). We then examine the conditions under which such a rule can be "rationalized" as maximizing a (possibly transitive) binary relation or a "utility" function (attributed to the decision-maker) on the allocation space. (Alternatively, an allocation rule can be interpreted as representing "social choices" of a group of agents and one can then analyze if these choices can be rationalized by a "joint-rationale" of the society.)

Note that our interpretation of a rule is different than that of the existing axiomatic literature on fair allocation (for a survey of which see Thomson, 1998). There, an allocation rule is typically interpreted as a proposal to the society evaluated by the researcher on the basis of a set of "desirable" properties. A rule is thus a normative construct. Alternatively, we interpret a rule to be a positive construct that describes the choices of a decision-maker.

An allocation rule is *rational* if there is a binary relation whose maximizers coincide with

[1] Revealed preference theory has also been applied to discuss behavioral phenomena such as the framing effect (e.g. Rubinstein and Salant, 2006, 2007) or reference dependence (e.g. Masatlioglu and Ok, 2005; Tapki, 2007; or Bossert and Sprumont, 2007).

the rule's choices for every problem. In **Section 3**, we analyze properties of *rational rules*. *Theorem 1* states that a rule is *rational* if and only if it satisfies a standard property called *contraction independence* (also called *independence of irrelevant alternatives* in the context of bargaining by Nash (1950) and *Property* α in the context of consumer choice by Sen (1971)). Peters and Wakker (1991) obtain a similar result in the context of bargaining.[2] In the context of consumer choice, however, *rationality* is stronger than *contraction independence*.

An allocation rule is *transitive-rational* if it can be rationalized by a transitive preference relation. In **Section 4**, we analyze properties of *transitive-rational* rules for two agents. *Theorem 2* states that for two agents, every *contraction independent* (that is, *rational*) rule is also *transitive-rational*. A similar result exists in consumer theory (Rose (1958)). In bargaining theory, however, *contraction independence* is weaker than *transitive-rationality* (Peters and Wakker, 1991) even for the two-agent case.

In **Section 5**, we analyze *transitive-rational* rules for an arbitrary number of agents. We first observe existence of *rational* rules that are not *transitive-rational*. (This is in line with Gale (1960), Kihlstrom, Mas-Colell, and Sonnenschein (1976), and Peters and Wakker (1994) for consumer choice and with Peters and Wakker (1991) for bargaining, who show that counterparts of Theorem 2 do not generalize.) We then observe that all such rules violate a standard property called *other-c monotonicity* which requires that a change in the characteristic value of an agent should not affect other two agents in opposite ways. *Other-c monotonicity* is satisfied by most of the known allocation rules in the literature (Thomson, 2003 and 2007). (In the context of single-peaked preferences, a similar idea is formulated and analyzed by Thomson (1993 and 1997) and Barberà, Jackson, and Neme (1997).) The main result of this section, *Theorem 3,* states that every *contraction independent* rule that is *other-c monotonic* is also *transitive-rational*. To the best of our knowledge, this result is unique to this domain, that is, we are not aware of similar results on other domains that link *rationality* (*i.e. contraction independence*) to *transitive-rationality* via auxiliary properties.

In **Section 6**, we identify conditions under which a rule F is *representable* by a numerical function (in the sense that its choices always coincide with the maximizers of the function).

[2] The critical common features of the two domains that facilitate this result are as follows. First, rules considered are singleton-valued. Second, both domains are closed under intersection. Note that this second property does not hold in the context of consumer choice: intersections of budget sets are not always budget sets.

Theorem 4 states that every *contraction independent* and *other-c monotonic* rule that is also *continuous* is *representable* by a numerical function. It is interesting to note that, unlike Debreu (1954) and the following literature (such as Peters and Wakker, 1991; Richter, 1966; or Sondermann, 1982), the proof of *Theorem 4* does not make use of a countable order-dense subset of $\mathbb{R}^N_+$. In fact, we show, in *Proposition 6*, that for simple allocation problems, it is not possible to construct such a countable collection of sets. For complete binary relations, existence of a countable order-dense set is both necessary and sufficient for the existence of a numerical representation. However, since the revealed preference relation is typically incomplete on our class, we are able to construct a numerical representation.

For a *rational rule,* the underlying preference relation is independent of the problem in consideration, particularly of the agents' characteristics. However, there might be cases where it is more natural to allow the underlying preferences to depend on this data. In **Section 7**, we explore the implications of this alternative. Particularly, we formulate a *weak rationality* notion where the representing preference relation is allowed to depend on the characteristic vector c. *Theorem 5* states that every allocation rule satisfies *weak rationality*: allowing the objective to depend on the characteristic values, every rule can be rationalized. A related result is by Young (1987) who characterizes a class of allocation rules which, for each characteristic vector, maximize an additively separable and strictly concave function. In this section, we also note that allowing the representing preference relation to depend on the social endowment E does not effect our findings.

Formally, the class of simple allocation problems is a subclass of the class of bargaining problems. (However, our results are not mere corollaries of those on bargaining problems. The more restricted structure of simple allocation problems makes new and interesting relationships between properties possible, such as noted in *Theorem 3.* At the other hand, as discussed in *Section 6*, it makes construction of a numerical representation a harder problem.) The classes of simple allocation problems and consumer choice problems have nonempty intersection but one class does not contain the other. (We note in *Subsection 1.1* that one application of our analysis is to consumer demand in fixed-price environments).

In the next subsection, we discuss the various applications of our analysis. In *Section 2*, we present our model and further discuss rational rules. In the following sections, we present our results as summarized above.

1.1 Examples and Applications

A *simple allocation problem* for a society N is an $|N|+1$ dimensional nonnegative real vector $\left(c_1, ...c_{|N|}, E\right)$ which, with the exception of the last application below, is interpreted as follows. A social endowment E of a perfectly divisible commodity is to be allocated among members of N. Each agent $i \in N$ is characterized by an amount c_i of the commodity. Next, we discuss the alternative interpretations of c and E at various applications.

1. **Taxation:** A public authority is to collect an amount E of *tax* from a society N. Each agent i has *income* c_i. This is a central and very old problem in public finance. For example, see Edgeworth (1898) and the following literature. Young (1987) proposes a class of "parametric solutions" to this problem.

2. **Bankruptcy:** A bankruptcy judge is to allocate the remaining *assets* E of a bankrupt firm among its creditors, N. Each agent i has *credited* c_i to the bankrupt firm and now, claims this amount. For example, see O'Neill (1982) and the following literature. For a detailed review of the extensive literature on taxation and bankruptcy problems, see Thomson (2003 and 2007).

3. **Permit Allocation**: The Environmental Protection Agency is to allocate an amount E of *pollution permits* among firms in N (such as CO_2 emission permits allocated among energy producers). Each firm i, depending on its location, is imposed by the local authority an *emission constraint* c_i on its pollution level. For more on this application, see Kıbrıs (2003) and the literature cited therein.

4. **Single-peaked or Saturated Preferences**: A social planner is to allocate E units of a perfectly divisible commodity among members of N. Each agent i is known to have preferences with *peak (saturation point)* c_i. The rest of the preference information is disregarded as typical in several well-known solutions to this problem, such as the Uniform rule or the Proportional rule. For example, see Sprumont (1991) and the following literature.

5. **Demand Rationing**: A supplier is to allocate its *production* E among demanders in N. Each demander i *demands* c_i units of the commodity. The supply-chain management literature contains detailed analysis of this problem.[3] For example, see Cachon

[3] We would like to thank Rakesh Vohra for bringing this application to our attention.

and Lariviere (1999) and the literature cited therein.

6. **Bargaining with Quasilinear Preferences and Claims:** An arbitrator is to allocate E units of a *numeriare good* among agents who have quasilinear preferences with respect to it. Each agent holds a *claim* c_i on what he should receive. For examples of bargaining problems with claims, see Chun and Thomson (1992) and the following literature. For bargaining problems with quasilinear preferences, see Moulin (1985) and the following literature.

7. **Surplus Sharing**: A social planner is to allocate the *return* E of a project among its investors in N. Each investor i has invested s_i. The project is profitable, that is, $\sum_N s_i \leqq E$. Using the principal that no agent should receive less than his investment, define the *maximal share of an agent* i as $c_i = E - \sum_{N\setminus\{i\}} s_j$. Note that $\sum_N c_i \geqq E$. The surplus sharing problem can now be analyzed as a simple allocation problem. For more on surplus-sharing problems, see Moulin (1985 and 1987) and the following literature.

8. **Consumer Choice under fixed prices and rationing:** A consumer has to allocate his *income* E among a set N of commodities. The prices of the commodities are fixed and thus, do not change from one problem to another. (With appropriate choice of consumption units, normalize the price vector so that all commodities have the same price.) As typical in the fixed-price literature, the consumer also faces "rationing constraints" on how much he can consume of each commodity. Let c_i be the agent's *consumption constraint* on commodity i. See Benassy, 1993 or Kıbrıs and Küçükşenel, 2008, for more on rationing rules.

2 Model

Let $N = \{1, ..., n\}$ be the set of agents. For $i \in N$, let e_i be the i^{th} unit vector in $\mathbb{R}^N_+$. Let $e = \sum_N e_i$. We use the vector inequalities $\leqq$, $\leq$, $<$. For $x, y \in \mathbb{R}^N_+$, let $x \vee y = (\max\{x_i, y_i\})_{i \in N}$. For each $E \in \mathbb{R}_+$, let $\Delta(E) = \left\{x \in \mathbb{R}^N_+ \mid \sum_N x_i = E\right\}$. For $c \in \mathbb{R}^N_+$, $\alpha \in \mathbb{R}_+$, and $S \subseteq N$, with an abuse of notation, we write $\left(c_S, \alpha_{N\setminus S}\right)$ to denote the vector which coincides with c on S and which chooses α for every coordinate in $N \setminus S$. For each

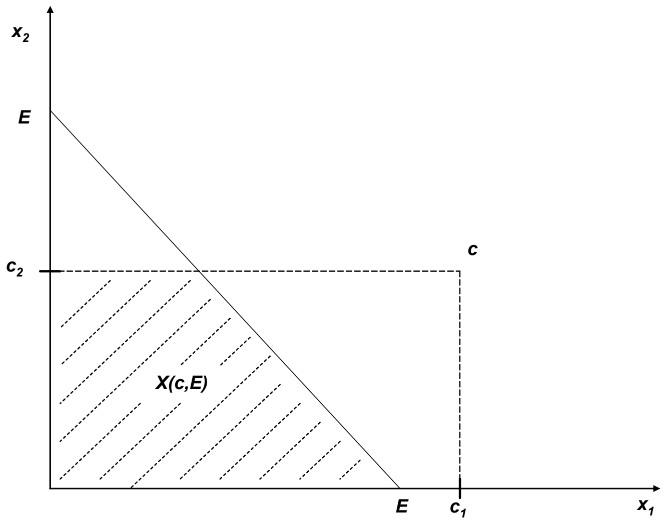

Figure 1: A two-agent simple allocation problem.

$A \subseteq \mathbb{R}^N_+$, let *comp* $\{A\} = \left\{x \in \mathbb{R}^N_+ \mid x \leqq y \text{ for some } y \in A\right\}$ be the *comprehensive hull* of A. For each $x \in \mathbb{R}^N_+$ and $\varepsilon \in \mathbb{R}_{++}$, let $V_\varepsilon(x) = \left\{y \in \mathbb{R}^N_+ \mid |x - y| < \varepsilon\right\}$.

A **simple allocation problem** for N is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geqq E$ (please see Figures 1 and 2). We call E the **endowment** and c the **characteristic vector**. As discussed in Subsection 1.1, depending on the application, E can be an asset or a liability and c can be a vector or incomes, claims, demands, preference peaks, or consumption constraints. Let $\mathcal{C}$ be the set of all simple allocation problems for N. Given a simple allocation problem $(c, E) \in \mathcal{C}$, let $X(c, E) = \left\{x \in \mathbb{R}^N_+ \mid x \leqq c \text{ and } \sum x_i \leqq E\right\}$ be the **choice set of** (c, E).

An allocation **rule** $F : \mathcal{C} \to \mathbb{R}^N_+$ assigns each simple allocation problem (c, E) to an allocation $F(c, E) \in X(c, E)$ such that $\sum_N F_i(c, E) = E$. Note that each rule F satisfies $F(c, E) \leqq c$. Depending on the application, this might be interpreted as satisfying the consumption constraints or as an efficiency requirement (as in the case of single-peaked preferences) or that no agent be taxed more than his income. Also, $\sum_N F_i(c, E) = E$ can be interpreted as an *efficiency* property or, as in taxation, a feasibility requirement or as in consumer choice, the Walras law.

The following are some well-known examples of rules. The **Proportional rule** allocates the endowment proportional to the characteristic values: for each $i \in N$, $PRO_i(c, E) =$

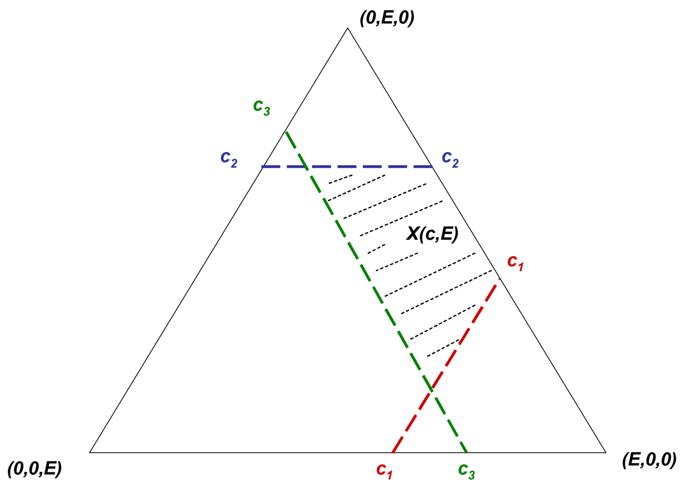

Figure 2: A three-agent simple allocation problem.

$\frac{c_i}{\sum_N c_j}E$. In the taxation literature, this rule is called a *Linear Tax.* The **Equal Gains rule** allocates the endowment equally, subject to no agent receiving more than his characteristic value: for each $i \in N$, $EG_i(c, E) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \min\{c_i, \lambda\} = E$. In the single-peaked allocation literature, this rule is called the *Uniform rule,* in the bankruptcy literature it is called the *Constrained Equal Awards rule*, and in the taxation literature, it is called the *Leveling Tax.* The **Equal Losses rule** equalizes the losses agents incur, subject to no agent receiving a negative share: for each $i \in N$, $EL_i(c, E) = \max\{0, c_i - \lambda\}$ where $\lambda \in \mathbb{R}_+$ satisfies $\sum_N \max\{0, c_i - \lambda\} = E$. In the single-peaked allocation literature, this rule is called the *Equal Distance rule,* in the bankruptcy literature it is called the *Constrained Equal Losses rule*, and in the taxation literature, it is called the *Head Tax.* The **Talmud rule** (Aumann and Maschler, 1985) assigns equal gains until each agent receives half his characteristic value and then uses the equal losses idea: $TAL(c, E) = EG\left(\frac{1}{2}c, \min\left\{E, \frac{1}{2}\sum_N c_i\right\}\right) + EL\left(\frac{1}{2}c, \max\left\{0, E - \sum_N c_i\right\}\right).$

For every rule F, we construct an induced **revealed preference relation,** $R^F \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$, as follows: for each $x, y \in \mathbb{R}^N_+$, xR^Fy if and only if there is $(c, E) \in \mathcal{C}$ such that $x = F(c, E)$ and $y \in X(c, E)$. (Note that $x = F(c, E)$ implies $E = \sum_N x_i$.) Similarly, the **strict revealed preference relation** induced by F, P^F, is defined as xP^Fy if and only if xR^Fy and $x \neq y$. The **lower contour set** of $x \in \mathbb{R}^N_+$ with respect to R^F is $L(x, R^F) =$

$\left\{y \in \mathbb{R}^N_+ \mid xR^F y\right\}$.

Remark 1 *Note that R^F is* reflexive *(since for each $x \in \mathbb{R}^N_+$, $x = F\left(x, \sum_N x_i\right)$).*[4] *Also, both R^F and P^F are* strictly monotonic *(since if $x \geq y$, then $x = F\left(x, \sum_N x_i\right)$ and $y \in X\left(x, \sum_N x_i\right)$). Also note that $xR^F y$ implies $\sum_N x_i \geqq \sum_N y_i$.*

A rule F is **rational** if there is a binary relation $B \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ such that for each $(c, E) \in \mathcal{C}$, $F(c, E) = \{x \in X(c, E) \mid \text{for each } y \in X(c, E),\ xBy\}$. Every rule F induces a *revealed preference relation.* But not every rule is *rational.* Consider rules that satisfy the following property. A rule F satisfies **WARP** (**the weak axiom of revealed preferences**) if P^F is *asymmetric* (equivalently if R^F is *antisymmetric*).[5]

Remark 2 *WARP can equivalently be stated as follows: for each pair $(c, E), (c', E) \in \mathcal{C}$, $F(c, E) \in X(c', E)$ and $F(c, E) \neq F(c', E)$ implies $F(c', E) \notin X(c, E)$. In the statement, using the same endowment level in both problems is without loss of generality because otherwise, WARP is trivially satisfied. That is, by the previous remark, $\sum_N x_i < \sum_N y_i$ implies not $xP^F y$.*

It is well-known in the literature that *WARP* is a necessary and sufficient condition for *rationality* (Samuelson (1938, 1948) and Houthakker (1950)). The same relationship holds on our domain (the simple and standard proof is omitted).

Theorem A. A rule F satisfies *WARP* if and only if it is *rational.*

For most economic analysis, *transitivity* of the rationalizing preference relation is an important requirement.[6] A rule F is **transitive-rational** if there is a *transitive* binary relation $B \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ such that for each $(c, E) \in \mathcal{C}$, $F(c, E) = \left\{x \in X(c, E) \mid \text{for each } y \in X(c, E),\ xR^F y\right\}$.

The following property is necessary and sufficient for rationalizability by a *transitive* preference relation (see Richter, 1966; also see Ville, 1946; Houthakker, 1950; Sondermann, 1982; and Kim, 1987). A rule F satisfies **SARP** (**the strong axiom of revealed preferences**) if P^F is *acyclic.* The same relationship holds on our domain.

[4] A binary relation B on $\mathbb{R}^N_+$ is **reflexive** if for each $x \in \mathbb{R}^N_+$, xBx. It is **strictly monotonic** if for each $x, y \in \mathbb{R}^N_+$, $x \geq y$ implies xBy.

[5] A binary relation B on $\mathbb{R}^N_+$ is **asymmetric** if for each $x, y \in \mathbb{R}^N_+$, xBy implies not yBx. It is **antisymmetric** if for each $x, y \in \mathbb{R}^N_+$, xBy and yBx imply $x = y$.

[6] A binary relation B on a space X is **transitive** if for each $x, y, z \in X$, xBy and yBz imply xBz.

Theorem B. A rule F satisfies *SARP* if and only if it is *transitive-rational.*

A rule that satisfies *SARP* is rationalizable by the following transitive relation. Given the revealed preference relation R^F, its **transitive closure** $\overline{R}^F$ is defined as follows: for each $x, y \in X$, $x\overline{R}^F y$ if and only if there are $z^1, ..., z^k \in \mathbb{R}_+^N$ such that $xR^F z^1 R^F ... R^F z^k R^F y$. Let $\overline{P}^F$ be the asymmetric part of $\overline{R}^F$, that is, $x\overline{P}^F y$ if and only if $x\overline{R}^F y$ and $x \neq y$. The **lower contour set** of $x \in \mathbb{R}_+^N$ with respect to $\overline{R}^F$ is $L\left(x, \overline{R}^F\right) = \left\{y \in \mathbb{R}_+^N \mid x\overline{R}^F y\right\}$.

A rule F is **representable** if there is a real-valued function $u : \mathbb{R}_+^N \to \mathbb{R}$ such that for each $(c, E) \in \mathcal{C}$,

$$F(c, E) = \arg \max_{x \in X(c,E)} u(x).$$

Every *representable* rule is *transitive-rational* (*i.e.* satisfies *SARP*). But the converse is not true.

3 Rationality vs Contraction Independence

In this section, we analyze properties of *rational* rules. As discussed in the previous section, rationality is equivalent to the weak axiom of revealed preferences, *WARP*, which requires that the induced revealed preference relation be antisymmetric.

Next, we introduce a well-known property from revealed preference theory (as well as bargaining theory). A rule F satisfies **contraction independence** if a chosen alternative from a set is still chosen from subsets (contractions) that contain it: for each pair $(c, E), (c', E) \in \mathcal{C}$, $F(c, E) \in X(c', E) \subseteq X(c, E)$ implies $F(c', E) = F(c, E)$. In the literature, this property is also referred to as *independence of irrelevant alternatives* (Nash, 1950) or *Sen's property* α (Sen, 1971).

Remark 3 *Defining* contraction independence *on problem pairs with the same endowment E is without loss of generality. To see this, assume the problem pair $(c, E), (c', E') \in \mathcal{C}$ satisfies $F(c, E) \in X(c', E') \subseteq X(c, E)$. Then $F(c, E) \in X(c', E')$ implies $\sum_N F_i(c, E) = E \leqq E'$. Furthermore, $X(c', E') \subseteq X(c, E)$ implies $E' \leqq E$. Thus, $E = E'$.*

Lemma 1 *A rule F satisfies* contraction independence *if and only if for each $(c, E), (c', E) \in \mathcal{C}$ it satisfies the following properties*
Property (i). *if for each $i \in N$, $\min\{c_i, E\} = \min\{c'_i, E\}$, then $F(c, E) = F(c', E)$,*
Property (ii). *if $F(c, E) \leqq c' \leqq c$, then $F(c', E) = F(c, E)$.*

Proof. $(\Rightarrow)$ Assume that F satisfies *contraction independence*. Let $(c, E), (c', E) \in \mathcal{C}$. First, assume that for each $i \in N$, $\min\{c_i, E\} = \min\{c'_i, E\}$. Let $x \in \mathbb{R}^N_+$ satisfy $\sum_N x_i \leqq E$. Then, $x \leqq c$ if and only if $x \leqq c'$. Thus $X(c, E) = X(c', E)$. This implies $F(c, E) = F(c', E)$. Next, assume that $F(c, E) \leqq c' \leqq c$. Then, $F(c, E) \in X(c', E) \subseteq X(c, E)$, which by *contraction independence*, implies $F(c, E) = F(c', E)$.

$(\Leftarrow)$ Assume that (i) and (ii) are satisfied. Let $(c, E), (c', E) \in \mathcal{C}$ be such that $F(c, E) \in X(c', E) \subseteq X(c, E)$. Then for each $i \in N$, either $c'_i \leqq c_i$ or $\min\{c'_i, E\} = \min\{c_i, E\}$. Let $S = \{i \in N \mid c'_i \leqq c_i\}$. Let $c'' = (c'_S, c_{N\setminus S})$. Then $F(c, E) \leqq c'' \leqq c$ and by (ii), $F(c, E) = F(c'', E)$. Now, for each $i \in N$, $\min\{c'_i, E\} = \min\{c''_i, E\}$. Thus, by (i), $F(c'', E) = F(c', E)$. Altogether, we obtain $F(c, E) = F(c', E)$. ■

Property (i) of *Lemma 1* is called *truncation-invariance* for rules on bankruptcy and taxation problems (Thomson, 2003 and 2007). Property (ii) says that a decrease in characteristic values does not change the initially chosen allocation as long as it remains feasible.

Contraction independent rules all satisfy the following standard property. A rule F satisfies **own-c monotonicity** if an increase in an agent's characteristic value does not decrease his share: for each $(c, E), (c', E) \in C$ and $i \in N$ such that $c_i < c'_i$ and $c_{N\setminus i} = c'_{N\setminus i}$, we have $F_i(c, E) \leqq F_i(c', E)$.

Lemma 2 *If a rule F satisfies* contraction independence*, then it satisfies* own-c monotonicity.

Proof. Let $(c, E), (c', E) \in C$ and $i \in N$ be such that $c_i < c'_i$ and $c_{N\setminus i} = c'_{N\setminus i}$. Suppose $F_i(c, E) > F_i(c', E)$. Then since $F(c', E) \leqq c \leqq c'$, *contraction independence* implies $F(c, E) = F(c', E)$, a contradiction. ■

Contraction independent rules satisfy another very useful property: if $xR^F y$, it is possible to identify a set of problems for which x is the solution.

Lemma 3 *Assume that F satisfies* contraction independence*. If $xR^F y$ and $x \leqq c \leqq x \vee y$, then, $x = F(c, \sum_N x_i)$. Particularly, $x = F(x \vee y, \sum_N x_i)$.*

Proof. If $x = y$, the statement trivially holds. If $x \neq y$, since $xP^F y$, $x = F(c', E)$ for some $(c', E) \in \mathcal{C}$ such that $y \in X(c', E)$. Then $E = \sum_N x_i$. Note that $x \leqq c'$ and $y \leqq c'$. Thus, $x \vee y \leqq c'$. Then applying *contraction independence* gives the desired result. ■

It is interesting to relate *Lemma 3* to the concept of an "inverse set" (status-quo set) in bargaining theory, introduced by Peters (1986) and further analyzed by Peters and Van Damme (1991). The inverse set of a bargaining rule F at a feasible set S and an allocation x is the set of disagreement vectors that produce x. Similarly define the inverse set of a simple allocation rule F at a social endowment level E and an allocation x as the set of c vectors that produce x. Then *Lemma 3* states that the inverse set of every allocation x is comprehensive down to x.

The main result of this section is as follows.

Theorem 1 *A rule F satisfies* contraction independence *if and only if it satisfies* WARP*.*

Proof. ($\Leftarrow$) Assume that F satisfies *WARP.* Let $(c,E),(c',E)\in\mathcal{C}$ be such that $F(c,E)\in X(c',E)\subseteq X(c,E)$. Suppose $F(c,E)\neq F(c',E)$. Then by *WARP* (*Remark 2*), $F(c',E)\notin X(c,E)$. This contradicts $X(c',E)\subseteq X(c,E)$.

($\Rightarrow$) Suppose F violates *WARP.* Then by *Remark 2,* there is a pair $(c,E),(c',E)\in\mathcal{C}$ such that $F(c,E)\in X(c',E)$, $F(c',E)\in X(c,E)$, and $F(c,E)\neq F(c',E)$. For each $i\in N$, let $c''_i=\min\{c_i,c'_i\}$. Since $F(c,E)\leqq c$ and $F(c,E)\leqq c'$, we have $F(c,E)\leqq c''$. Thus, $\sum_N F_i(c,E)=E\leqq\sum_N c''_i$. That is, $(c'',E)\in\mathcal{C}$. Since $F(c,E)\leqq c''\leqq c$, by *contraction independence* and *Lemma 1,* $F(c,E)=F(c'',E)$. Similarly $F(c',E)\leqq c''\leqq c'$, by *contraction independence* implies $F(c',E)=F(c'',E)$, contradicting $F(c,E)\neq F(c',E)$. ■

Theorem 1 and *Lemma 1* together provide us with a simple way of checking whether a rule satisfies *WARP.* Using them, we show that the *Proportional rule*, the *Equal Losses rule*, and the *Talmud rule* all violate *WARP* and thus are not *rational.* The following example demonstrates this point.

Example 1 *The* Proportional rule*,* Equal Losses rule*, and the* Talmud rule *all violate* contraction independence*. Let $N=\{1,2\}$, $c=(10,1)$, $c'=\left(10,\frac{10}{11}\right)$, and $E=10$. We show that, for this example, all three rules violate* Property (ii) *of* Lemma 1*. Indeed $PRO(c,E)=\left(\frac{100}{11},\frac{10}{11}\right)\leqq c'\leqq c$, $EL(c,E)=\left(9\frac{1}{2},\frac{1}{2}\right)\leqq c'\leqq c$, and $TAL(c,E)=\left(9\frac{1}{2},\frac{1}{2}\right)\leqq c'\leqq c$. However, $PRO(c',E)=\left(\frac{110}{12},\frac{10}{12}\right)\neq PRO(c,E)$, $EL(c',E)=\left(9\frac{6}{11},\frac{5}{11}\right)\neq EL(c,E)$, and $TAL(c',E)=\left(9\frac{6}{11},\frac{5}{11}\right)\neq TAL(c,E)$. It is trivial to construct another example where all three rules violate* Property (i) *of* Lemma 1*. Finally note that the* Equal Gains rule *satisfies* Property (ii) *for this example: $EG(c,E)=(9,1)$ and since $1>c'_2$ the property has no bite.*

The Equal Gains rule, on the other hand, satisfies *WARP*. Because, by contrast to the other three rules, it operates on a principle (equal division) that is independent of the agents' characteristic values; it only treats them as constraints in the application of this principle.

4 Transitive-Rationality: Two Agents

In this and the following section, we analyze properties of *transitive-rational* rules. As discussed in the previous section, *transitive-rationality* is equivalent to the strong axiom of revealed preferences, *SARP*, which requires that the induced revealed strict preference relation be *acyclic*.

The main result of this section is that, for two agents, *contraction independence* is a necessary and sufficient condition for *SARP* (i.e. *transitive-rationality*). We use an auxiliary result to prove this theorem. It states that for two agents and for allocations on the same "budget hyperplane", *contraction independence* implies *transitivity* of the revealed preference relation, a property stronger than *acyclicity*.

Proposition 4 *Let $n = 2$. Assume F satisfies* contraction independence. *Let $x, y, z \in \mathbb{R}^N_+$ be such that $\sum_N x_i = \sum_N y_i = \sum_N z_i = E$. If $xP^F y$ and $yP^F z$, then $xP^F z$.*

Proof. Note that, $x \neq y$, $y \neq z$, and since by *Theorem 1, P^F* is *asymmetric*, $x \neq z$. Without loss of generality assume $x_1 < y_1$.

Note that, $xP^F y$, by *Lemma 3* implies $x = F(x \vee y, E)$. Similarly, $yP^F z$, by *Lemma 3* implies $y = F(y \vee z, E)$.

Case 1: $z_1 < x_1$. Since $z_1 < x_1 < y_1$, $x \vee y \leq y \vee z$. Thus by *contraction independence* and $y = F(y \vee z, E)$, we have $y = F(x \vee y, E)$. This contradicts $x = F(x \vee y, E)$.

Case 2: $x_1 < z_1 < y_1$. Then, $x = F(x \vee y, E)$ and $x \leq x \vee z \leq x \vee y$, by *contraction independence* implies $x = F(x \vee z, E)$. Thus, $xP^F z$.

Case 3: $y_1 < z_1$. First suppose $F(x \vee z, E) \in (y, z]$. Then, $F(x \vee z, E) \leqq y \vee z$, by *contraction independence* implies, $F(x \vee z, E) = F(y \vee z, E)$, contradicting $y = F(y \vee z, E)$. Thus, $F(x \vee z, E) \notin (y, z]$. Next, suppose $F(x \vee z, E) \in (x, y]$. Then, $F(x \vee z, E) \leqq x \vee y$, by *contraction independence* implies $F(x \vee z, E) = F(x \vee y, E)$, contradicting $x = F(x \vee y, E)$. Thus $F(x \vee z, E) \notin (x, y]$. Combining these two observations, we conclude that $x = F(x \vee z, E)$. Thus, $xP^F z$. ■

Remark 4 Transitivity *does not necessarily hold when* $x, y,$ *and* z *violate* $\sum_N x_i = \sum_N y_i = \sum_N z_i$. *For example, let* $F = EG$. *This rule satisfies* contraction independence. *Let* $x = (6, 10)$, $y = (6, 4)$, $z = (9, 1)$. *Then,* $xP^F yP^F z$. *However, we do not have* $xP^F z$ *since the Equal Gains rule always chooses equal division from a set that contains both* x *and* z.

We now present the main result of this section.

Theorem 2 *Let* $n = 2$. *A rule* F *satisfies* contraction independence *if and only if it satisfies* SARP.

Proof. First note that *SARP* implies *WARP* which in turn implies *contraction independence*. For the converse, assume the F satisfies *contraction independence* and violates *SARP*. Then, there are $x^1, ..., x^k \in \mathbb{R}_+^N$ such that for each $l \in \{1, ..., k-1\}$, $x^l P^F x^{l+1}$, and $x^k P^F x^1$. Then, $\sum_{i \in N} x_i^1 \geqq ... \geqq \sum_{i \in N} x_i^k \geqq \sum_{i \in N} x_i^1$ implies $\sum_{i \in N} x_i^1 = ... = \sum_{i \in N} x_i^k$. Then, *contraction independence*, by *Proposition 4*, implies $x^1 P^F x^1$, contradicting the definition of P^F. ■

5 Transitive-Rationality: n Agents

The main purpose of this section is to identify, for an arbitrary number of agents, conditions under which a rule is *transitive-rational* (that is, satisfies *SARP*). The following example demonstrates that with more than two agents, *Theorem 2* of the previous section fails. A rule F that satisfies *contraction independence* can allow cycles of length three or more. Similar examples exist for both consumer theory (Gale, 1960; Shafer, 1977; Peters and Wakker, 1994) and bargaining theory (Peters and Wakker, 1991).

Example 2 *A rule that satisfies* contraction independence *but violates* SARP *(please see* Figure 3*). Let* $N = \{1, 2, 3\}$. *Let*

$$F(c, E) = \begin{cases} \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & if \quad \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) \leqq c \\ (c_1, c_1, E - 2c_1) & else\ if \quad c_1 < \frac{E}{3} \ and \ (c_1, c_1, E - 2c_1) \leqq c \\ (E - 2c_2, c_2, c_2) & else\ if \quad c_2 < \frac{E}{3} \ and \ (E - 2c_2, c_2, c_2) \leqq c \\ (c_3, E - 2c_3, c_3) & else\ if \quad c_3 < \frac{E}{3} \ and \ (c_3, E - 2c_3, c_3) \leqq c \\ (c_1, c_2, E - c_1 - c_2) & else\ if \quad E - c_1 - c_2 > c_2 \ and \ c_1 > c_2 \\ (c_1, E - c_1 - c_3, c_3) & else\ if \quad E - c_1 - c_3 > c_1 \ and \ c_3 > c_1 \\ (E - c_2 - c_3, c_2, c_3) & else\ if \quad E - c_2 - c_3 > c_3 \ and \ c_2 > c_3. \end{cases}$$

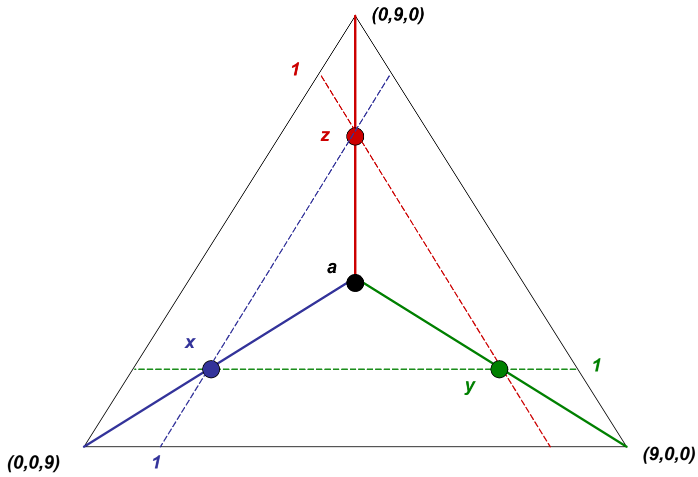

Figure 3: A three-agent rule that satisfies *contraction independence* but violates *SARP*. For $E = 9$, $x = F(1, 9, 9, E)$, $y = F(9, 1, 9, E)$, $z = F(9, 9, 1, E)$.

We will next construct a cycle. Let $E = 9$, $c^1 = (1, 9, 9)$, $c^2 = (9, 1, 9)$, $c^3 = (9, 9, 1)$, $x = (1, 1, 7)$, $y = (7, 1, 1)$, *and* $z = (1, 7, 1)$. *Then* $F(c^1, E) = x$, $F(c^2, E) = y$, *and* $F(c^3, E) = z$. *Noting that* $x \leqq c^2$, $y \leqq c^3$, *and* $z \leqq c^1$, *we obtain a violation of* SARP *due to* $xP^F zP^F yP^F x$.

Remark 5 *The rule presented in the example is continuous. In Peters and Wakker (1991), continuity prevents cycles of size three (though not bigger cycles). This is not the case in our framework.*

The rule constructed in *Example 2* violates a weak property that is satisfied by most solutions to simple allocation problems. A rule F is **other-c monotonic** if a change in agent i's characteristic value affects any other two agents in the same way: for each $(c, E) \in \mathcal{C}$, each $i \in N$, each $c'_i \in \mathbb{R}_+$ such that $(c'_i, c_{-i}, E) \in \mathcal{C}$, and each $j, k \in N \setminus \{i\}$, $F_j(c, E) > F_j(c'_i, c_{-i}, E)$ implies $F_k(c, E) \geqq F_k(c'_i, c_{-i}, E)$.

It turns out that *contraction independent* rules that are *other-c monotonic* induce a revealed preference relation that is *transitive* on a fixed endowment level.

Proposition 5 *Assume that F satisfies* contraction independence *and* other-c monotonicity*. Let $x, y, z \in \mathbb{R}^N_+$ be such that $\sum_N x_i = \sum_N y_i = \sum_N z_i = E$. If $xP^F y$ and $yP^F z$, then $xP^F z$.*

Proof. Note that, $x \neq y$, $y \neq z$, and since by *Theorem 1, P^F* is *asymmetric*, $x \neq z$. By *Lemma 3*, $x = F(x \vee y, E)$ and $y = F(y \vee z, E)$. We claim $x = F(x \vee y \vee z, E)$. This, by *contraction independence* implies $x = F(x \vee z, E)$, the desired conclusion. Contrary to the claim, suppose $w = F(x \vee y \vee z, E) \neq x$.

If $w \leqq x \vee y$, by *contraction independence*, $w = F(x \vee y, E)$, contradicting $w \neq x$. Thus, the set $S_x = \{i \in N \mid \max\{x_i, y_i\} < w_i\}$ is nonempty.

If $w \leqq y \vee z$, by *contraction independence*, $w = F(y \vee z, E) = y$, contradicting $w \nleqq x \vee y$. Thus, the set $S_y = \{i \in N \mid \max\{y_i, z_i\} < w_i\}$ is nonempty.

If there is $i \in S_x \cap S_y$, $\max\{x_i, y_i, z_i\} < w_i$ contradicts $w \leqq x \vee y \vee z$. So $S_x \cap S_y = \emptyset$.

Let $c = \left(w_{S_y}, 0_{N \setminus S_y}\right)$. Then $w \leqq y \vee z \vee c$, by *contraction independence* implies $w = F(y \vee z \vee c, E)$. Since $y = F(y \vee z, E)$, *Lemma 2* and *other-c monotonicity* implies that for each $j \notin S_y$, $y_j \geqq w_j$.[7]

Now let $j \in S_x$. Then by definition, $\max\{x_j, y_j\} < w_j$. Since $j \notin S_y$ however, by the previous paragraph, $y_j \geqq w_j$, a contradiction. ■

We now use this proposition to prove the main result of this section.

Theorem 3 *If a rule F satisfies* contraction independence *and* other-c monotonicity*, then it satisfies* SARP*.*

Proof. Assume the F satisfies *contraction independence, other-c monotonicity* and violates *SARP*. Then, there are $x^1, ..., x^k \in \mathbb{R}^N_+$ such that for each $l \in \{1, ..., k-1\}$, $x^l P^F x^{l+1}$ and $x^k P^F x^1$. Then, $\sum_{i \in N} x^1_i \geqq ... \geqq \sum_{i \in N} x^k_i \geqq \sum_{i \in N} x^1_i$ implies $\sum_{i \in N} x^1_i = ... = \sum_{i \in N} x^k_i$. Then *contraction independence* and *other-c monotonicity*, by *Proposition 5*, imply $x^1 P^F x^1$, contradicting the definition of P^F. ■

The following example demonstrates that the converse of *Theorem 3* is not true: while *SARP* implies *WARP*, it does not imply *other-c monotonicity.*

[7] To see this, let $i \in S_y$. As c_i decreases from w_i to $\max\{y_i, z_i\}$, by *Lemma 2*, i's share does not increase. By feasibility and *other-claim monotonicity*, the share of every $j \neq i$ does not decrease. Repeating this argument for each $i \in S_y$, we conclude that the share of a $j \notin S_y$ does not decrease, that is, $y_j \geqq w_j$.

Example 3 *A rule that satisfies* SARP *but violates* other-c monotonicity*. Let* $N = \{1, 2, 3\}$*. Let*

$$F(c, E) = \begin{cases} \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & if \quad \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) \leqq c \\ (c_1, c_1, E - 2c_1) & else\ if \quad c_1 < \frac{E}{3} \text{ and } (c_1, c_1, E - 2c_1) \leqq c \\ \left(\frac{E-c_2}{2}, c_2, \frac{E-c_2}{2}\right) & else\ if \quad c_2 < \frac{E}{3} \text{ and } \left(\frac{E-c_2}{2}, c_2, \frac{E-c_2}{2}\right) \leqq c \\ \left(\frac{E-c_3}{2}, \frac{E-c_3}{2}, c_3\right) & else\ if \quad c_3 < \frac{E}{3} \text{ and } \left(\frac{E-c_3}{2}, \frac{E-c_3}{2}, c_3\right) \leqq c \\ (c_1, c_2, E - c_1 - c_2) & else\ if \quad E - c_1 - c_2 > c_1 \text{ and } c_1 > c_2 \\ (E - c_2 - c_3, c_2, c_3) & else\ if \quad E - c_2 - c_3 > c_3 \text{ and } E - c_2 - c_3 > c_2 \\ (c_1, E - c_1 - c_3, c_3) & else\ if \quad E - c_1 - c_3 > c_1. \end{cases}$$

To see that F *violates other-c monotonicity, note that* $F(E, E, E, E) = \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right)$ *and* $F\left(\frac{E}{4}, E, E, E\right) = \left(\frac{E}{4}, \frac{E}{4}, \frac{E}{2}\right)$. *F satisfies SARP because a problem from which* $\left(\frac{E-c_2}{2}, c_2, \frac{E-c_2}{2}\right)$ *is chosen never contains a point of type* $\left(\frac{E-c_3}{2}, \frac{E-c_3}{2}, c_3\right)$ *and thus, it is not possible to construct a cycle.*

6 Representability

The main purpose of this section is to identify conditions under which a rule F is representable by a numerical function. The main result of this section, *Theorem 4* states that every *contraction independent* and *other-c monotonic* rule that is also *continuous* is *representable* by a numerical function.

The best-known method of constructing such a numerical representation is finding, if possible, a countable set $A \subseteq \mathbb{R}^N_+$ which is $\overline{\mathbf{P}}^{\mathbf{F}}$**-dense**: for each $x, y \in \mathbb{R}^N_+ \setminus A$, if $x\overline{P}^F y$, then there is $a \in A$ such that $x\overline{P}^F a\overline{P}^F y$. Following Debreu (1954), Peters and Wakker (1991), Richter (1966), and Sondermann (1982) construct such sets to prove representability of choice rules. The following result demonstrates that in simple allocation problems, no such set A can be constructed.

Proposition 6 *Let* F *be an arbitrary rule. Then, there is no countable set* $A \subseteq \mathbb{R}^N_+$ *which is* $\overline{P}^F$-dense.

Proof. Let A be a $\overline{P}^F$*-dense* set. For each $E \in \mathbb{R}_+$, let $x^E = F(E_N, E)$ and let $y^E \in \Delta(E) \setminus \{x^E\}$. Then, $x^E\overline{P}^F y^E$ and thus, there is $a^E \in A$ such that $x^E\overline{P}^F a^E\overline{P}^F y^E$. By

Remark 1, $\sum x_i^E \geqq \sum a_i^E \geqq \sum y_i^E$. Thus, $\sum a_i^E = E$. So for $E' \neq E$, $a^{E'} \neq a^E$. Since there is uncountably many E, the set A is then also uncountable. ■

For complete binary relations, existence of a countable order-dense set is both necessary and sufficient for representability. However, since $\overline{P}^F$ is possibly incomplete, this is only a sufficient condition. So, *Proposition 6* only means that we will have to pursue a different path than Debreu (1954) and the following papers.

For each rule F, we define the associated function $u^F : \mathbb{R}_+^N \to \mathbb{R}$ so that the numerical value of an alternative $x \in \mathbb{R}_+^N$ is the Lebesque measure, μ, of an extension of its lower-contour set (please see *Figure 4*):

$$u^F(x) = \mu\left(comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\}\right).$$

That is, to obtain the value of an alternative x, we move up $L\left(x, \overline{R}^F\right)$ (its lower contour set with respect to the transitive $\overline{R}^F$) by the unit vector e and then calculate the Lebesque measure of the comprehensive hull of this set. This operation (of moving up by e and taking the comprehensive hull) guarantees that for every $x \in \mathbb{R}_+^N$ (particularly, those vectors at the boundary of $\mathbb{R}_+^N$), $comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\}$ is a set of positive measure.

Also note that, replacing $L\left(x, R^F\right)$ with $L\left(x, \overline{R}^F\right)$ in the above representation does not work. For example if F is the Equal Gains rule, $x = (11, 10)$ and $y = (10, 10)$, $\mu\left(comp\left\{L\left(x, R^F\right) + \{e\}\right\}\right) = 192 < \mu\left(comp\left\{L\left(y, R^F\right) + \{e\}\right\}\right) = 241$ even though $xP^F y$.

A rule F is **c-continuous** if for each $E \in \mathbb{R}_+$, $F(., E)$ is a continuous function. It is **E-continuous** if for each $c \in \mathbb{R}_+^N$, $F(c, .)$ is a continuous function. Finally, a rule F is **continuous** if it is both *c-continuous* and *E-continuous*. The following example demonstrates that our construction does not work for rules that violate either continuity property.

Example 4 *(A discontinuous rule that is not representable by u^F) Let $N = \{1, 2\}$. Let F be defined as*

$$F(c, E) = \begin{cases} EG(c, E) & if \quad E < 2, \\ (\min\{c_1, E\}, E - \min\{c_1, E\}) & if \quad 2 \leqq E < 3, \\ (E, 0) & if \quad 3 \leqq E,\ c_1 \geqq E, \\ EG(c, E) & if \quad 3 \leqq E,\ c_1 < E. \end{cases}$$

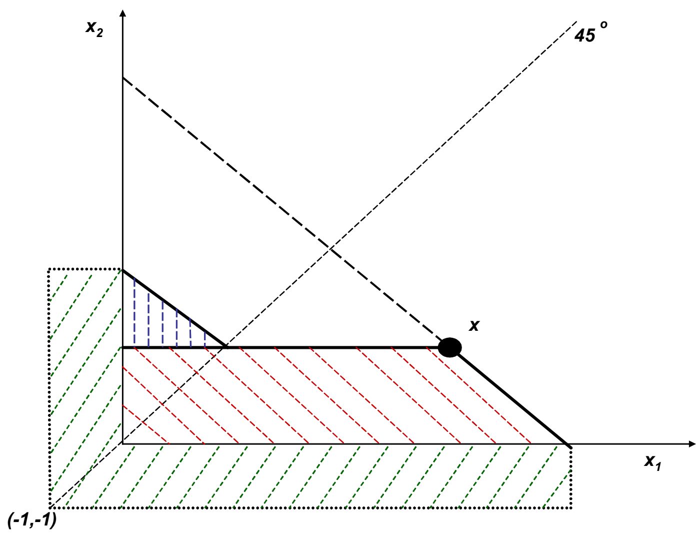

Figure 4: The sets $L\left(x, R^F\right)$ (the red-shaded trapezoid), $L\left(x, \overline{R}^F\right)$ (the trapezoid plus the blue-shaded triangle), and $comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\}$ (the trapezoid and the triangle plus the green-shaded rectangular area).

Now let $x = (2,0)$, $y = (1,1)$, $z = (4,0)$, *and* $w = (2,2)$. *Since* $x = F(2,2,2)$, $x\overline{P}^F y$. *However, since* F *violates* E-continuity *at* $E = 2$, $u^F(x) = 2 = u^F(y)$. *Similarly, since* $z = F(4,4,4)$, $z\overline{P}^F w$. *However, since* F *violates* c-continuity *at* $c_1 = E = 4$, $u^F(z) = 8 = u^F(w)$.

The main result of this section, *Theorem 4*, shows that any *contraction independent*, *other-c monotonic*, and *continuous* rule is *representable* by a numerical function of the above form. To prove it, we use a series of lemmas presented next.

The following lemma shows that the set of characteristic vectors that deliver the same alternative have a lattice structure.

Lemma 7 *Let* F *be* contraction independent *and* other-c monotonic*. If* $F(c, E) = F(c', E) = x$*, then* $F(c \vee c', E) = x$.

Proof. *Suppose* $F(c \vee c', E) = y \neq x$. *Let* $S = \{i \in N \mid y_i > c_i\}$ *and* $S' = \{i \in N \mid y_i > c'_i\}$. *If* $S = \emptyset$, *then* $y \leqq c$ *and, by* contraction independence*,* $F(c, E) = y$*, a contradiction. Thus* $S \neq \emptyset$. *Similarly,* $S' \neq \emptyset$. *Also,* $y \in X(c \vee c', E)$ *implies* $S \cap S' = \emptyset$. *Now, let* $c'' = \left(y_S, 0_{N \setminus S}\right)$

and note that, by contraction independence*,* $y = F(c \vee c'', E)$*. Since* $x = F(c, E)$*, by* Lemma 2 *and* other-c monotonicity*, for each* $i \notin S$*,* $x_i \geqq y_i$*. Now let* $j \in S'$*. Then,* $x_j \leqq c'_j < y_i$*. Since* $j \notin S$*, also* $x_j \geqq y_j$*, a contradiction.* ■

The following lemma shows that *other-c monotonic* and *c-continuous* rules choose boundary allocations under certain contractions of the feasible set.

Lemma 8 *Assume that* F *is* contraction independent *and* c-continuous*. Then, for each* $(c, E) \in \mathcal{C}$*,* $i \in N$*, and* $\delta \in \mathbb{R}_+$*,* $F_i(c + \delta e_i, E) > c_i$ *implies* $F_i(c, E) = c_i$*.*

Proof. Let $(c, E) \in \mathcal{C}$, $i \in N$, and $\delta \in \mathbb{R}_+$ satisfy $F_i(c + \delta e_i, E) > c_i$. Suppose $F_i(c, E) < c_i$. Then by *c-continuity*, there is $\varepsilon < \delta$ such that $F_i(c + \varepsilon e_i, E) = c_i$. But then, $F(c + \varepsilon e_i, E) \leqq c \leqq c + \varepsilon e_i$, by *contraction independence*, implies $F(c, E) = F(c + \varepsilon e_i, E)$, a contradiction. ■

The following lemma uses the previous to establish an even stronger boundary condition for *other-c monotonic* rules.

Lemma 9 *Assume that* F *satisfies* contraction independence, c-continuity*, and* other-c monotonicity*. Let* $(c, E) \in \mathcal{C}$ *and* $x \in \Delta(E)$ *be such that* $x \leqq c$*, there is* $i \in N$ *such that* $x_i < F_i(c, E)$*, and there is* $c^* \leqq c$ *such that* $(c^*, E) \in \mathcal{C}$ *and* $x = F(c^*, E)$*. Then* $x_i = c^*_i$*.*

Proof. Let $F\left(c^*_i, c^*_{N\setminus i}, E\right) = x$. Let $j \in N \setminus \{i\}$. Since, $c^*_j \leqq c_j$, by *Lemma 2*, we have $x_j \leqq F_j\left(c^*_i, c_j, c^*_{N\setminus ij}, E\right)$. By *other-c monotonicity* then, $x_i \geqq F_i\left(c^*_i, c_j, c^*_{N\setminus ij}, E\right)$. Replacing each c^*_j with c_j, we conclude

$$c^*_i \geqq x_i \geqq F_i\left(c^*_i, c_{N\setminus i}, E\right).$$

Since $x_i < F_i(c, E)$, we have $F\left(c^*_i, c_{N\setminus i}, E\right) \neq F(c, E)$. If $F_i(c, E) \leqq c^*_i$, *contraction independence* implies $F\left(c^*_i, c_{N\setminus i}, E\right) \neq F(c, E)$, a contradiction. Thus, $F_i(c, E) > c^*_i$. Then *contraction independence* and *c-continuity*, by *Lemma 8*, imply $F_i\left(c^*_i, c_{N\setminus i}, E\right) = c^*_i$. Thus, $x_i = c^*_i$. ■

The following lemma shows that the lower contour set of each alternative x is equal to the feasible set of a particular problem where each c_i is either x_i or $\sum_N x_i$.

Lemma 10 *Let F be* contraction independent*,* other-c monotonic*, and* continuous*. Then for each $x \in \mathbb{R}^N_+$, there is $c^x \in \mathbb{R}^N_+$ such that $L\left(x, R^F\right) = X\left(c^x, \sum_N x_i\right)$. Furthermore, if there is $i \in N$ such that $c^x_i < \sum_N x_i$, then $c^x_i = x_i$.*

Proof. Let $x \in \mathbb{R}^N_+$ and let $\overline{E} = \sum_N x_i$.

Step 1: There is $c^x \in \mathbb{R}^N_+$ such that $L\left(x, R^F\right) = X\left(c^x, \overline{E}\right)$.

Let $C^x = \left\{c \in \mathbb{R}^N_+ \mid \sum_N c_i \geqq \overline{E},\ c \leqq \overline{E}_N,\ F\left(c, \overline{E}\right) = x\right\}$. By *continuity* of F, C^x is compact. Let

$$c^x \in \arg\max_{c \in C^x} \sum_N c_i.$$

By continuity of $\sum_N c_i$ and compactness of C^x, such c^x exists. By *Lemma 7*, c^x is unique.

We claim that for each $c \in C^x$, $X\left(c, \overline{E}\right) \subseteq X\left(c^x, \overline{E}\right)$, that is, $c \leqq c^x$. To see this, suppose there is $c \in C^x$ such that $c \nleqq c^x$. Let $c' = c \vee c^x$ and note that $c' \geq c^x$. By *Lemma 7*, $c' \in C^x$. But then $\sum_N c'_i > \sum_N c^x_i$, a contradiction.

Now let $\left(c, \overline{E}\right) \in \mathcal{C}$ such that $F\left(c, \overline{E}\right) = x$. We claim, $X\left(c, \overline{E}\right) \subseteq X\left(c^x, \overline{E}\right)$. If $c \leqq \overline{E}_N$, by the previous paragraph, $X\left(c, \overline{E}\right) \subseteq X\left(c^x, \overline{E}\right)$. If $c \nleqq \overline{E}_N$, the set $S = \left\{i \in N \mid c_i > \overline{E}\right\}$ is nonempty. Let $c' = \left(\overline{E}_S, c_{N \setminus S}\right)$ and note that, since $c' \in C^x$, $X\left(c', \overline{E}\right) \subseteq X\left(c^x, \overline{E}\right)$. Since $X\left(c, \overline{E}\right) = X\left(c', \overline{E}\right)$, we also have $X\left(c, \overline{E}\right) \subseteq X\left(c^x, \overline{E}\right)$.

Since $c^x \in C^x$, $X\left(c^x, \overline{E}\right) \subseteq L\left(x, R^F\right)$. Conversely, let $y \in L\left(x, R^F\right)$. Then there is $\left(c, \overline{E}\right) \in \mathcal{C}$ such that $F\left(c, \overline{E}\right) = x$ and $y \in X\left(c, \overline{E}\right)$. By the previous paragraph, $X\left(c, \overline{E}\right) \subseteq X\left(c^x, \overline{E}\right)$. Thus, $y \in X\left(c^x, \overline{E}\right)$. These two set inclusions give the desired result.

Step 2: If there is $i \in N$ such that $c^x_i < \overline{E}$, then $c^x_i = x_i$.

If there is $i \in N$ such that $c^x_i < \overline{E}$, $F\left(\overline{E}_i, c^x_{N \setminus i}, \overline{E}\right) \neq x$. (Otherwise, $\left(\overline{E}_i, c^x_{N \setminus i}\right) \in C^x$ and $\left(\overline{E}_i, c^x_{N \setminus i}\right) \geq c^x$, a contradiction.) By *contraction independence*, $F_i\left(\overline{E}_i, c^x_{N \setminus i}, \overline{E}\right) > c^x_i$. Thus $x_i < F_i\left(\overline{E}_i, c^x_{N \setminus i}, \overline{E}\right)$. This, by *Lemma 9*, implies $x_i = c^x_i$. ■

The following lemma establishes that in a neighborhood of x, $L\left(x, R^F\right)$ and $L\left(x, \overline{R}^F\right)$ coincide.

Lemma 11 *Let F be* contraction independent*,* other-c monotonic*, and* continuous*. Then for each $x \in \mathbb{R}^N_+$, there is $\delta \in \mathbb{R}_{++}$ such that if $E \in \left(\sum_N x_i - \delta, \sum_N x_i\right]$ and $y \in L\left(x, R^F\right) \cap \Delta(E)$, then $L\left(y, R^F\right) \subseteq L\left(x, R^F\right)$.*

Proof. Let F satisfy the given properties. Let $x \in \mathbb{R}^N_+$ and let $\overline{E} = \sum_N x_i$. By *Lemma 10*, there is $S \subseteq N$ such that for $c^x = \left(x_S, \overline{E}_{N \setminus S}\right)$, $L\left(x, R^F\right) = X\left(c^x, \overline{E}\right)$.

If $S = \emptyset$, the statement holds for all δ. To see this, first note that then, $L\left(x, R^F\right) = X\left(\overline{E}_N, \overline{E}\right)$. Let $y \in L\left(x, R^F\right)$. By *Lemma 10*, there is $c^y \in [0, \overline{E}]^N$ such that $L\left(y, R^F\right) = X\left(c^y, \sum_N y_i\right)$. But $c^y \leqq \overline{E}_N$ and $\sum_N y_i \leqq \overline{E}$. Thus, $L\left(y, R^F\right) \subseteq L\left(x, R^F\right)$.

Alternatively assume $S \neq \emptyset$.

Step 1: If $T \subseteq N$ is such that $T \not\supseteq S$, then $x \neq F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right)$. To see this, suppose otherwise. Let $c = \left(x_T, \overline{E}_{N \setminus T}\right)$ and note that $c \vee c^x \geq c^x$. By *Lemma 7*, $x = F\left(c \vee c^x, \overline{E}\right)$. But then, $L\left(x, R^F\right) \supseteq X\left(c \vee c^x, \overline{E}\right) \supset X\left(c^x, \overline{E}\right)$, a contradiction.

Step 2: If $T \subseteq N$ is such that $T \not\supseteq S$, then $F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right) \notin X\left(c^x, \overline{E}\right)$. To see this, suppose otherwise. Note that $x \in X\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right)$. Thus, by *contraction independence* (which by *Theorem 1*, is equivalent to *WARP*) and by *Remark 2*, $F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right) = x$, contradicting Step 1.

Step 3: Defining ε and δ. Let

$$\varepsilon = \frac{1}{2} \min \left\{ \left| x - F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right) \right| \mid T \subseteq N \textit{ such that } T \not\supseteq S \right\}.$$

By Step 1, $\varepsilon > 0$. Now let $T \subseteq N$ be such that $T \not\supseteq S$. By *continuity* of F, there is a real number $\delta_T > 0$ such that $E \in \left(\overline{E} - \delta_T, \overline{E}\right]$ implies $F\left(x_T, E_{N \setminus T}, E\right) \in V_\varepsilon\left(F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right)\right)$. Let $\delta = \min\left\{\delta_T \mid T \subseteq N \textit{ such that } T \not\supseteq S\right\}$.

Step 4: If $E \in \left(\overline{E} - \delta, \overline{E}\right]$ and $T \subseteq N$ is such that $T \not\supseteq S$, then $F\left(x_T, E_{N \setminus T}, E\right) \notin X\left(c^x, \overline{E}\right)$. By Step 3, $F\left(x_T, E_{N \setminus T}, E\right) \in V_\varepsilon\left(F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right)\right)$. Noting that, by definition of ε, $V_\varepsilon\left(F\left(x_T, \overline{E}_{N \setminus T}, \overline{E}\right)\right) \cap X\left(c^x, \overline{E}\right) = \emptyset$, we have the desired conclusion.

Step 5: If $E \in \left(\overline{E} - \delta, \overline{E}\right]$, $y \in X\left(c^x, \overline{E}\right) \cap \Delta(E)$, and $T \subseteq N$ is such that $T \not\supseteq S$, then $F\left(y_T, E_{N \setminus T}, E\right) \neq y$. To see this, first note that, by Step 4, $F\left(x_{T \cap S}, E_{N \setminus (T \cap S)}, E\right) \notin X\left(c^x, \overline{E}\right)$. (Note that $T \cap S = \emptyset$ or $T \cap S = T$ are both possible.) Thus, there is $i \in S \setminus T$ such that $F_i\left(x_{T \cap S}, E_{N \setminus (T \cap S)}, E\right) > x_i$. Since $x_{T \cap S} \geqq y_{T \cap S}$ and $i \notin T$, by *Lemma 2* and *other-c monotonicity,* $F_i\left(y_T, E_{N \setminus T}, E\right) \geqq F_i\left(x_{T \cap S}, E_{N \setminus (T \cap S)}, E\right)$. Since $y \in X\left(c^x, \overline{E}\right)$ and $i \in S$, $y_i \leqq x_i$. Together, $F_i\left(y_T, E_{N \setminus T}, E\right) > y_i$. Thus, $F\left(y_T, E_{N \setminus T}, E\right) \neq y$.

Step 6: If $E \in \left(\overline{E} - \delta, \overline{E}\right]$ and $y \in X\left(c^x, \overline{E}\right) \cap \Delta(E)$, then $L\left(y, R^F\right) \subseteq L\left(x, R^F\right)$. To see this, first note that, by *Lemma 10*, there is $T \subseteq N$ such that for $c^y = \left(y_T, E_{N \setminus T}\right)$, we have $L\left(y, R^F\right) = X\left(c^y, E\right)$. By Step 5, $T \not\supseteq S$ implies $F\left(c^y, E\right) \neq y$. Thus, $T \supseteq S$. Then, $c^y \leqq c^x$ and $E \leqq \overline{E}$ implies $X\left(c^y, E\right) \subseteq X\left(c^x, \overline{E}\right)$. That is, $L\left(y, R^F\right) \subseteq L\left(x, R^F\right)$. ■

We next present the main result of this section.

Theorem 4 *If a rule F is* contraction independent, other-c monotonic, *and* continuous, *then it is representable by a function u^F.*

Proof. Let F satisfy the given properties. First assume that for each $x, y \in \mathbb{R}^N_+$ such that $xP^F y$, we have $u^F(x) > u^F(y)$. Then, u^F represents F. To see this, let $(c, E) \in \mathcal{C}$. For each $y \in X(c, E) \setminus \{F(c, E)\}$, we have $F(c, E) P^F y$ and thus, $u^F(F(c, E)) > u^F(y)$. Therefore, $F(c, E) = \arg\max_{x \in X(c,E)} u^F(x)$.

We next show that for each $x, y \in \mathbb{R}^N_+$ such that $xP^F y$, we have $u^F(x) > u^F(y)$. Let $x, y \in \mathbb{R}^N_+$ satisfy $xP^F y$. Then, by *Remark 1*, $\sum_N x_i \geqq \sum_N y_i$. Also note that, by definition of u^F and transitivity of $\overline{R}^F$, $u^F(x) \geqq u^F(y)$. We next show that equality is not possible.
Case 1: $\sum_N x_i > \sum_N y_i$. Let $\varepsilon = \frac{1}{2}(\sum_N x_i - \sum_N y_i)$. Then

$$V_\varepsilon(x + e) \cap comp\{x + \{e\}\} \subseteq comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\} \setminus comp\left\{L\left(y, \overline{R}^F\right) + \{e\}\right\}$$

and

$$\mu(V_\varepsilon(x + e) \cap comp\{x + \{e\}\}) > 0.$$

Thus,

$$u^F(x) - u^F(y) \geqq \mu(V_\varepsilon(x + e) \cap comp\{x + \{e\}\}) > 0.$$

Case 2: $\sum_N x_i = \sum_N y_i$. Let $\overline{E} = \sum_N x_i$. By *Lemma 10*, there is $c^x, c^y \in \mathbb{R}^N_+$ such that $L(x, R^F) = X(c^x, \overline{E})$ and $L(y, R^F) = X(c^y, \overline{E})$.
Step 1: Defining δ_1, δ_2, and ε. Since $x \notin L(y, R^F)$, there is $i \in N$ such that $x_i > c^y_i$. Let $\delta_1 = \frac{1}{2}|x_i - c^y_i|$. Note that, $z \in V_{\delta_1}(x)$ implies $z \notin L(y, R^F)$. Next, by *Lemma 11*, there is $\delta_2 \in \mathbb{R}_{++}$ such that if $E \in (\overline{E} - \delta_2, \overline{E}]$ and $z \in L(y, R) \cap \Delta(E)$, then $L(z, R^F) \subseteq L(y, R^F)$. Let $\varepsilon = \min\{\delta_1, \delta_2\}$ and note that $\varepsilon > 0$.
Step 2: If $z \in L\left(y, \overline{R}^F\right)$, then $z \notin V_\varepsilon(x)$. If $\sum_N z_i \leqq \overline{E} - \varepsilon$, then $z \notin V_\varepsilon(x)$ trivially holds. Alternatively assume $\sum_N z_i > \overline{E} - \varepsilon$. Since $z \in L\left(y, \overline{R}^F\right)$, there is $w^1, ..., w^k \in \mathbb{R}^N_+$ such that $y = w^1 P^F w^2 P^F ... P^F w^k = z$. Let $l \in \{1, ..., k-1\}$. Then, $\sum_N w^l_i \geqq \sum_N z_i > \overline{E} - \varepsilon \geqq \overline{E} - \delta_2$. Thus, if $w^l \in L(y, R^F)$, by *Lemma 11*, $L(w^l, R^F) \subseteq L(y, R^F)$. Since $w^l P^F w^{l+1}$, this implies $w^{l+1} \in L(y, R^F)$. Overall, $w^1 \in L(y, R^F)$ and for each $l \in \{1, ..., k-1\}$, $w^{l+1} \in L(y, R^F)$. Therefore, $z \in L(y, R^F)$. This and $\varepsilon \leqq \delta_1$ implies $z \notin V_\varepsilon(x)$.

Step 3: $u^F(x) > u^F(y)$. By Step 2, $V_\varepsilon(x) \cap L\left(y, \overline{R}^F\right) = \emptyset$. Particularly, as noted in Step 1, there is $i \in N$ such that $x_i - \varepsilon > c_i^y$. We now claim

$$V_\varepsilon(x+e) \cap comp\left\{L\left(y, \overline{R}^F\right) + \{e\}\right\} = \emptyset.$$

To see this, let $z \in V_\varepsilon(x+e)$ and $w \in comp\left\{L\left(y, \overline{R}^F\right) + \{e\}\right\}$. Then, $z_i > x_i + 1 - \varepsilon > c_i^y + 1 \geqq w_i$ implies $z \neq w$.

Now note that $V_\varepsilon(x) \cap L\left(x; \overline{R}^F\right) \neq \emptyset$ implies $V_\varepsilon(x+e) \cap \left\{L\left(x, \overline{R}^F\right) + \{e\}\right\} \neq \emptyset$. Thus

$$V_\varepsilon(x+e) \cap comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\} \neq \emptyset.$$

Since $comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\}$ has dimension n and contains x,

$$\mu\left(V_\varepsilon(x+e) \cap comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\}\right) > 0.$$

Finally, since

$$V_\varepsilon(x+e) \cap \quad comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\} \subseteq comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\} \backslash comp\left\{L\left(y, \overline{R}^F\right) + \{e\}\right\},$$

we have

$$u^F(x) - u^F(y) \geqq \mu\left(V_\varepsilon(x+e) \cap comp\left\{L\left(x, \overline{R}^F\right) + \{e\}\right\}\right) > 0.$$

Thus, $u^F(x) > u^F(y)$. ■

7 Weaker Notions of Rationality

In this section, we explore the implications of natural weakenings of rationality. Rationality requires the existence of a binary relation which is independent of the data of a given problem. However, there might be cases where it is more natural to allow the underlying preferences to depend on the social endowment or the characteristic vector.

We start by noting that allowing the underlying preferences to depend on the social endowment E does not affect our results. To see this, note that our results do not utilize properties that relate solutions to problems with different social endowment levels. Thus, our whole analysis would be true under a fixed endowment level, say $E = 1$. It is interesting to note that then, by *Propositions 4* and *5*, *contraction independence* (and for $n > 2$, *other-c* *monotonicity*) imply transitivity of the revealed preference relation R^F. Also note that then, in *Theorem 4*, *continuity* can be replaced with *c-continuity*.

Allowing the underlying preferences to depend on the characteristic vector has more radical implications. We discuss them next. Formally, a rule F is **weakly rational** if for each $c \in \mathbb{R}^N_+$, there is a binary relation $B(c) \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ such that for each $(c, E) \in \mathcal{C}$, $F(c, E) = \{x \in X(c, E) \mid \text{for each } y \in X(c, E),\ xB(c)\, y\}$.

It turns out that every rule satisfies this weak property.

Theorem 5 *Every rule F is* weakly rational.

Proof. Let $c \in \mathbb{R}^N_+$. We first claim that $F(c, .)$ satisfies *WARP*. To see this, define the revealed preference relation induced by c, $R^F(c)$ as follows: for each $x, y \in \mathbb{R}^N_+$, $xR^F(c)\, y$ if and only if there is $E \in \mathbb{R}_+$ such that $(c, E) \in \mathcal{C}$, $x = F(c, E)$ and $y \in X(c, E)$. Next, we show that $R^F(c)$ is *antisymmetric*: assume $xR^F(c)\, y$ and $yR^F(c)\, x$. Let E^x be such that $x = F(c, E^x)$ and $y \in X(c, E^x)$. Similarly, let E^y be such that $y = F(c, E^y)$ and $x \in X(c, E^y)$. Then $\sum_N x_i \leqq E^y = \sum_N y_i \leqq E^x = \sum_N x_i$. Thus, $E^x = E^y$. But then, $x = F(c, E^x)$ and $y = F(c, E^y)$ together imply $x = y$.

Now let $E \in \mathbb{R}_+$ be such that $(c, E) \in \mathcal{C}$. Then by definition of $R^F(c)$, for each $y \in X(c, E)$, $F(c, E)R^F(c)\, y$ and (by *antisymmetry* of $R^F(c)$) for each $y \in X(c, E) \setminus \{F(c, E)\}$, not $yR^F(c)\, F(c, E)$. Thus, $F(c, E) = \left\{x \in X(c, E) \mid \text{for each } y \in X(c, E),\ xR^F(c)\, y\right\}$.

Since this argument is true for each $c \in \mathbb{R}^N_+$, F is *weakly rational.* ■

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