Strategic Behavior in Non-Atomic Games*

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Abstract

Typically, economic situations featuring a large number of agents are not modelled with a finite normal form game, rather by a non-atomic game. Consequently, the possibility of strategic interaction may be completely ignored.

In order to restore strategic interaction among agents we propose a refinement of Nash equilibrium, *strategic* equilibrium, for non-atomic games with a continuum of agents, each of whose payoff depends on what he chooses and a societal choice.

Given a non-atomic game, we consider a perturbed game in which every player believes that he alone has a small, but positive, impact on the societal choice. A strategy profile is a *strategic* equilibrium if it is a limit point of a sequence of Nash equilibria of games in which each player's belief about his impact on the societal choice goes to zero. After proving the existence of strategic equilibria, we show that every strategic equilibrium must be a Nash equilibrium of the original non-atomic game, thus, our concept of strategic equilibrium is indeed a refinement of Nash equilibrium. Next, we show that the concept of strategic equilibrium is the natural extension of Nash equilibrium in finite normal form games, to non-atomic games: That is, given any finite normal form game, we consider its non-atomic version, and prove that a strategy profile, in the non-atomic version of the given finite normal form game, is a strategic equilibrium if and only if the associated strategy profile in the finite form game is a Nash equilibrium. Finally, applications of strategic equilibrium is presented examples in which the set of strategic equilibria, in contrast with the set of Nash equilibria, does not contain any implausible Nash equilibrium strategy profiles. These examples are: a game of proportional voting, a game of allocation of public resources, and finally non-atomic Cournot oligopoly.

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1 Introduction

Typically, economic situations featuring a large number of agents are not modelled with a finite normal form game, rather by a non-atomic game. Consequently, the strategic behavior that agents are supposed to demonstrate in the economic situation modelled, may be ignored, possibly resulting in agents being indifferent between any of their actions.

In this paper, we propose the concept of *strategic equilibrium*, an equilibrium concept for non-atomic games in which the payoff of each agent depends on what he chooses and a societal choice. The equilibrium notion we present is a refinement of Nash equilibrium in non-atomic games, and is designed to rule out Nash equilibria that are due to degenerate indifference relations in agents' best response correspondences which arise because of their inability to affect the societal choice. We imagine a situation in which any player faced with a given Nash equilibrium would ask himself: Would I play my part of the Nash equilibrium if I were to have a very small, yet positive, impact on the societal choice? We argue that if the answer is no for a positive fraction of the players, then that Nash equilibrium is not plausible.

As an example consider a game where agents' utility depends only on the *societal* choice. The societal choice is, by definition, the average of each player's choice, and therefore, an agent's action does not affect the societal choice and thus cannot affect his own payoff. Thus, any agent is indifferent between any of his choices, and as a consequence any strategy profile, plausible or not, is a Nash equilibrium.

Although admittedly extreme, the previous example provides a good illustration of a situation in which the possibility of strategic interaction between players is completely absent due to fact that each player is non-atomic. In order to restore the strategic interaction in non-atomic games, for each such game, we consider a perturbed game in which every player believes that he alone has an ε, a very small yet positive, impact on the societal choice.[1] In such an ε-perturbed game, every player has an impact on the societal choice, which will be reflected in his choice. Here, it makes sense for the player to think strategically: he can change the societal choice in a neighborhood around its value implied by the choice of the others, and so his choice will depend on what the others do. Another interpretation of the strategic interaction dominant in the ε-perturbed game is that an agent would think that he alone can manipulate an ε-mass of players, i.e. that he is a dictator for a society of ε relative size.

Formally, for any $\varepsilon > 0$, we define an ε-*strategic* equilibrium to be a Nash equilibrium of the $\varepsilon-$perturbed game. The set of *strategic* equilibria will be the set of limit points of a sequence of ε-strategic equilibria, where ε, everybody's impact on the societal choice, goes to 0. After proving the existence of strategic equilibria under standard assumptions (e.g., Rath [11]), we show that every strategic equilibrium must be a Nash equilibrium of the original non-atomic game, and so our concept of strategic equilibrium is indeed a refinement of Nash equilibrium.

As in Selten [13], our equilibrium notion is designed to rule out non-robust indifference relations in best response correspondences. However, unlike his and many other refinements of Nash equilibria, our concept seeks to eliminate only those indifference relations in agents' best response correspondences that are due to the fact that in a non-atomic game each agent is atomless. For example, in a symmetric game[2] where agents' payoffs depend only on a societal

[1]It needs to be pointed out that in the ε-perturbed game agents are not rational, because an agent thinks that he alone has an ε impact on the societal choice, and does not foresee that other players have the same consideration as well.

[2]Symmetry is given in the following fashion: Agents have the same action space, and given any strategy

choice, any strategy profile is a Nash equilibrium. Obviously, in a game where agents have a common interest, every agent choosing an action which results in a least preferred strategy profile for each player, is an implausible Nash equilibrium. Moreover, such a Nash equilibrium allocation is not Pareto optimal in a game where all the agents have a common interest. On the other hand, in symmetric finite normal form games, Nash equilibria and the set of Pareto optimal strategy profiles[3] coincide. Similarly, in a symmetric non-atomic game where agents have a common interest, the set of strategic equilibria is equal to the set of Pareto optimal strategy profiles.[4]

In order to study the strategic interaction under strategic equilibrium, we consider any finite normal form game, and formulate its associated non-atomic game. Then, we prove that a strategy profile, in the non-atomic version of the given finite normal form game, is a strategic equilibrium if and only if the associated strategy profile in the finite normal form game is a Nash equilibrium.

This result confirms that Nash equilibrium in finite normal form games and Nash equilibrium in non-atomic games are fundamentally two different equilibrium notions, in the sense that the latter does not possess the "strategic interaction property" of the former. Because of that observation, we argue that the concept of Nash equilibrium in non-atomic games is not the plausible version of the Nash equilibrium in finite normal form games, and the strategic equilibrium is the extension of Nash equilibrium in finite normal form games to non-atomic games. Consequently, we stress that strategic equilibrium does not rule out Nash equilibria which are due to indifference relations in best responses because of "the individual deviation property" of Nash equilibrium. Let us remind the reader that we rule out the Nash equilibria which are due to indifference relations in best responses because of each agent being atomless.

To see an easy example without going into the formalities, consider a finite normal form game with 2 positions. Each agent in one of those positions is allowed to choose a strategy in $\{1, 2\}$ and the payoff to each of the agent is the average choice. It is obvious that the unique Nash equilibrium strategy profile of the finite normal form game is one where each agent chooses 2. In the non-atomic version of that finite normal form game, there are two non-atomic populations from which an agent to sit in one of those positions is chosen. Since all of the agents are of measure zero, any strategy profile is a Nash equilibrium of that non-atomic game. However, the unique strategic equilibrium of the same non-atomic game is where almost every player chooses 2.

Although the formulation and definition of our concept might be similar to one of various stability and robustness refinements, the notion of strategic equilibrium is not a refinement aimed to rule out non-dynamically-stable or non-robust Nash equilibrium strategy profiles. In fact, the discussion in the previous paragraphs displays that there are dynamically non-stable Nash equilibria which are strategic. More particularly, because of our equivalence result between strategic equilibria and Nash equilibria of finite games, a mixed strategy of a finite normal form coordination game would be a strategic equilibrium in the non-atomic version of the same finite normal form coordination game. To see that consider a non-atomic coordination game.

profile, every agent gets the same payoff.

[3] A strategy profile σ is Pareto optimal if there is no σ' satisfying $u_i(\sigma') > u_i(\sigma)$ for every player i.

[4] The reason for this point is that when any agent is given some influence on the societal choice, they no longer will be indifferent among their choices. And because of the structure of symmetric games, his optimal choice will coincide with the societal optimal one. Hence, in such games the set of strategic equilibria is equal to the set of Pareto optimal strategy profiles.

Suppose that each agent in [0,1] has two possible choices, A or B. Agent's payoff when he chooses A (B respectively), is the measure of the players who have chosen A (B respectively). There are three Nash equilibria in this game: Almost every player choosing A; almost every player choosing B; and half of the agents choosing A and the other half B. Note that the last equilibrium is not dynamically-stable, and corresponds to the mixed strategy equilibrium of a finite normal form coordination game. Nonetheless, in this non-atomic coordination game the set of strategic equilibria is equal to the set of Nash equilibria. In particular, to see why the third Nash equilibrium is strategic consider the following: no matter how much weight an agent is given to affect the societal choice, still he is indifferent between A and B, as the other people are separated equally between those choices.

We apply the notion of strategic equilibrium to a game of proportional voting featuring finite number of political parties. After demonstrating that any voting profile is a Nash equilibrium of that game, we show that the unique strategic equilibrium is a strategy profile under which every agent votes for his most favored party. In the formulation of this game, it is assumed that agents' preferences are continuous in the societal choice. More specifically, each agent has a strict preference ordering on the finite set of political parties, represented by a utility function. A mixed strategy profile in the voting game induces a probability distribution on the set of parties, and the payoff of an agent is the expected utility he gets under that probability distribution. The same game can be used to the analyze the allocation of public resources on a finite set of possible projects. With that interpretation, all the implausible Nash strategy profiles are ruled out, and we are left with one in which every agent supports the project he favors the most.

Another application of the strategic equilibrium is done to explain strategic voting. A non-atomic population of agents is to determine which point in an n-dimensional simplex to choose. Given a strategy profile consisting of each agent's vote in the n-dimensional simplex, the societal choice is given by the average choice. [5] We show that under some regularity conditions to eliminate uninteresting cases, truthful voting profile is not a strategic equilibrium.

Finally, the last application we provide is done in Cournot oligopoly setup. We formulate the non-atomic Cournot oligopoly, and show that the set of strategic equilibria contains only symmetric Nash equilibria. Technically, this example is of interest as it displays the non-linearities in an agent's individual maximization problem in the perturbed game.

Section 2 gives the formal definitions of and the assumptions on non-atomic games. In Section 3 we define the concept of strategic equilibrium and prove its existence. We go on to show that any strategic equilibrium is a Nash equilibrium of the original non-atomic game. Section 4 studies the strategic interaction under strategic equilibrium. Finally, Section 5 demonstrates the applications of this refinement.

2 Non-Atomic Games

Let A be a non-empty, compact metric space of *actions* and $\mathcal{M}$ be the set of Borel probability measures on A endowed with the weak convergence topology. By Parthasarathy [10, Theorem II.6.4], it follows that $\mathcal{M}$ is a compact metric space. We use the following notation: we write $\mu_n \Rightarrow \mu$ whenever $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ converges to μ and ρ denote the Prohorov metric on $\mathcal{M}$, which is known to metricize the weak convergence topology. We let d_A denote the metric on A.

[5] Again in this setup, any strategy profile is a Nash equilibrium, and that is not true for strategic equilibrium.

Let $\mathcal{U}$ denote the space of continuous utility functions $u : A \times \mathcal{M} \to \mathbb{R}$ endowed with the supremum norm. The set $\mathcal{U}$ represents the space of *players' characteristics*; it is a complete, separable metric space.

A *game with a continuum of players* is characterized by a measurable function $U : [0,1] \to \mathcal{U}$, where the unit interval $[0,1]$ is endowed with the Lebesgue measure λ on the Lebesgue measurable sets and represents the set of *players*. We represent such game by $G = (([0,1],\lambda), U, A)$.

A strategy is a measurable function $f : [0,1] \to A$. A pair (U, f), where f is a strategy, induces a probability measure on $\mathcal{U} \times A$ denoted by $\lambda \circ (U, f)^{-1}$.

Given a Borel probability measure τ on $\mathcal{U} \times A$, we denote by $\tau_{\mathcal{U}}$ and τ_A the marginals of τ on $\mathcal{U}$ and A respectively. The expression $u(a,\tau) \geq u(A,\tau)$ means $u(a,\tau) \geq u(a',\tau)$ for all $a' \in A$.

Given a game $\mu = \lambda \circ U^{-1}$, a Borel probability measure τ on $\mathcal{U} \times A$ is an *equilibrium distribution for* μ if

1. $\tau_{\mathcal{U}} = \mu$, and
2. $\tau(\{(u,a) \in \mathcal{U} \times A : u(a,\tau_A) \geq u(A,\tau_A)\}) = 1$.

We will use the following notation: $B_\tau = \text{supp}(\tau) \cap \{(u,a) \in \mathcal{U} \times A : u(a,\tau_A) \geq u(A,\tau_A)\}$. Note that B_τ is closed, and so a Borel set; hence $\tau(B_\tau)$ is well defined. Also, if (u,a) belong to B_τ, then a maximizes the function $\tilde{a} \mapsto u(\tilde{a},\tau_A)$. Thus, we are implicitly assuming that the choice of any player does not affect the distribution of actions. It is in this sense that the notions of this section describe a game with a continuum of players.

3 Strategic equilibria

As we stressed in the introduction, we wish to consider those Nash equilibria that can be seen as a limit of equilibria in games in which players have a small, yet positive, impact in the societal choice. Clearly, the reason why each player t does not have any impact on the societal choice is because $\lambda(\{t\}) = 0$. The way we give players weight is by considering the following measures: For each $\varepsilon > 0$, and $t \in [0,1]$, we define a measure $\lambda_{t,\varepsilon}$ in the following way: for all Borel-measurable set $B \subseteq A$,

$$\lambda_{t,\varepsilon}(B) = \begin{cases} \varepsilon + (1-\varepsilon)\lambda(B) & \text{if } t \in B \\ (1-\varepsilon)\lambda(B) & \text{otherwise} \end{cases}$$

Thus, in $\lambda_{t,\varepsilon}$ player t is an atom: In the game described by $\lambda_{t,\varepsilon}$, agent t believes that his choices can have an ε impact on the societal choice. The following Lemma makes this precise.

Lemma 1 *For any measurable* $f : [0,1] \to A$*,*

$$\int f \mathrm{d}\lambda_{t,\varepsilon} = \varepsilon f(t) + (1-\varepsilon)\int f \mathrm{d}\lambda.$$

Proof. If f is simple, $f = \sum_{j=1}^{J} a_j \chi_{A_j}$, with $t \in A_1$, then

$$\begin{aligned} \int f \mathrm{d}\lambda_{t,\varepsilon} &= \varepsilon a_1 + (1-\varepsilon)a_1\lambda(A_1) + (1-\varepsilon)\sum_{j=2}^{J} a_j \lambda(A_j) \\ &= \varepsilon f(t) + (1-\varepsilon)\int f \mathrm{d}\lambda. \end{aligned}$$

The general case follows from this by a limit argument. ∎

Given a strategy f, $y \in A$, and $t \in T$, let $f \setminus_t y$ denote the strategy g defined by $g(t) = y$, and $g(\tilde{t}) = f(\tilde{t})$, for all $\tilde{t} \neq t$.

Lemma 2 *For any strategies f and g, $a \in A$ and $t \in T$, if $\lambda \circ f^{-1} = \lambda \circ g^{-1}$ then $\lambda_{t,\varepsilon} \circ (f \setminus_t a)^{-1} = \lambda_{t,\varepsilon} \circ (g \setminus_t a)^{-1}$.*

Proof. Let $B \subseteq A$ be measurable. Then,

$$\begin{aligned} \lambda_{t,\varepsilon} \circ (f \setminus_t a)^{-1} &= \begin{cases} \varepsilon + (1-\varepsilon)\lambda \circ f^{-1}(B) & \text{if } a \in B \\ (1-\varepsilon)\lambda \circ f^{-1}(B) & \text{otherwise.} \end{cases} \\ &= \begin{cases} \varepsilon + (1-\varepsilon)\lambda \circ g^{-1}(B) & \text{if } a \in B \\ (1-\varepsilon)\lambda \circ g^{-1}(B) & \text{otherwise.} \end{cases} \\ &= \lambda_{t,\varepsilon} \circ (g \setminus_t a)^{-1}(B). \end{aligned}$$

Thus, $\lambda_{t,\varepsilon} \circ (f \setminus_t a)^{-1} = \lambda_{t,\varepsilon} \circ (g \setminus_t a)^{-1}$. ∎

Let τ be a distribution on A. Then there exist a measurable function $f : [0,1] \to A$ such that $\tau = \lambda \circ f^{-1}$. Given $\varepsilon > 0$, let for all $t \in [0,1]$ and $a \in A$, define

$$U_\varepsilon(t)(a,\tau) = U(t)(a, \lambda_{t,\varepsilon} \circ (f \setminus_t a)^{-1}).$$

By Lemma 2, U_ε is well defined. We then define the $\varepsilon-$*perturbed game* G_ε of G as $G_\varepsilon = ([0,1], A, U_\varepsilon)$. The $\varepsilon-$perturbed game has the same players, and actions spaces as the original game G, but differs from this because in G_ε every players believes that he has an ε impact on the distribution of actions.

We say that a distribution τ^* on $\mathcal{U} \times A$ is a *strategic equilibrium distribution* of G if there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_{++}$ decreasing to zero and a sequence $\{\tau_k^*\}_{k\in\mathbb{N}}$ converging to τ^* such that τ_k^* is an equilibrium distribution of G_{ε_k}, for every $k \in \mathbb{N}$.

Conceptually, our approach is in the same spirit as Selten [13]. Given $\varepsilon \in (0,1]$ and a non-atomic game, we define its ε-perturbed game, a modified version of the original non-atomic game, in which every player thinks he alone has ε impact on the distribution of actions. Then, an ε-*strategic* equilibrium distribution is an equilibrium of the ε-game. Finally, a distribution is a strategic equilibrium distribution, if it is a limit point of a sequence of ε-strategic equilibrium distributions, where $\varepsilon > 0$ and $\varepsilon \searrow 0$.

Theorem 1 is on the existence of a strategic equilibrium distribution.

Theorem 1 *Let $G \in \mathcal{G}$ be a normal form game. Then, G has a strategic equilibrium distribution.*

Proof. Note first that if $G = ([0,1], A, U)$, $\tilde{G} = ([0,1], A, V)$ and $U = V$ a.e., then τ is an equilibrium distribution of G if and only if τ is an equilibrium equilibrium of $\tilde{G}$. Hence, under the same hypothesis, τ is a strategic equilibrium distribution of G if and only if τ is a strategic equilibrium distribution of $\tilde{G}$.

Let $V : [0,1] \to \mathcal{U}$ be Borel measurable satisfy $V = U$ a.e. By the above, it is enough to show that $\tilde{G} = ([0,1], A, V)$ has a strategic equilibrium.

Let $\varepsilon > 0$. For any $t \in [0,1]$, define $g(t) : A \times \mathcal{M} \to \mathbb{R}$ by $g(t)(a,\nu) = V(t)(a, \lambda_{\varepsilon,t} \circ f)$, with $\nu = \lambda \circ f^{-1}$ and $\bar{G} = ([0,1], A, g)$. Thus, τ is an equilibrium distribution of $\tilde{G}_\varepsilon$ if and only if τ is an equilibrium distribution of $\bar{G}$. We claim that that $g : [0,1] \to \mathcal{U}$ is Borel measurable and $g(t)$ is continuous for all $t \in [0,1]$.

Claim 1 *The function $g(t) : A \times \mathcal{M} \rightarrow$ is continuous for all $t \in [0,1]$.*

Proof. Let $a \in A$ and $\tau \in \mathcal{M}$, and let $\{a_k\} \subseteq A$ and $\{\tau_k\} \subseteq \mathcal{M}$ be such that $a_k \rightarrow a$ and $\tau_k \Rightarrow \tau$. Since A is a complete metric space, by Skorokhod's Theorem (see Hildenbrand [4, Theorem 37, p. 50]), there exist measurable functions f and f_k, $k \in \mathbb{N}$, of $[0,1]$ into A such that $\tau = \lambda \circ f^{-1}$, $\tau_k = \lambda \circ f_k^{-1}$ and $\lim_k f_k = f$ a.e. in $[0,1]$. Thus, $g(t)(a_k, \tau_k) = V(t)(a_k, \lambda_{\varepsilon,t} \circ (f_k \setminus_t a_k)^{-1})$, and so it is enough to show that $\lambda_{\varepsilon,t} \circ (f_k \setminus_t a_k)^{-1} \Rightarrow \lambda_{\varepsilon,t} \circ (f \setminus_t a)^{-1}$.

Let $h : A \rightarrow \mathbb{R}$ be continuous. Then, by the Change-of-variable formula (see Hildenbrand [4, Theorem 36, p. 50]) and Lemma 1, one obtains

$$\begin{aligned}
&\int_A h \mathrm{d}\lambda_{\varepsilon,t} \circ (f_k \setminus_t a_k)^{-1} = \\
&\int_{[0,1]} h \circ (f_k \setminus_t a_k) \mathrm{d}\lambda_{\varepsilon,t} = \\
&\varepsilon h(a_k) + (1-\varepsilon) \int_{[0,1]} h \circ f_k \mathrm{d}\lambda = \\
&\varepsilon h(a_k) + (1-\varepsilon) \int_A h \mathrm{d}\lambda \circ f_k^{-1} \rightarrow \\
&\varepsilon h(a) + (1-\varepsilon) \int_A h \mathrm{d}\lambda \circ f^{-1} = \\
&\int_A h \mathrm{d}\lambda_{\varepsilon,t} \circ (f \setminus_t a)^{-1}.
\end{aligned}$$

Thus, indeed we have $\lambda_{\varepsilon,t} \circ (f_k \setminus_t a_k)^{-1} \Rightarrow \lambda_{\varepsilon,t} \circ (f \setminus_t a)^{-1}$. ■

Claim 2 *The function $g : [0,1] \rightarrow \mathcal{U}$ is Borel measurable.*

Proof. Note first that $V_{(a,\tau)}$ defined by $t \mapsto V(t)(a,\tau)$ is Borel measurable for all a and τ. This follows because $V_{(a,\tau)} = \pi_{(a,\tau)} \circ V$, where $\pi_{(a,\tau)}(u) = u(a,\tau)$ is continuous.

This fact implies that $g_{(a,\tau)}$ is Borel measurable for all a and τ, as follows: note that $\lambda_{\varepsilon,t} \circ (f \setminus_t a)^{-1} = \lambda_{\varepsilon,\tilde{t}} \circ (f \setminus_{\tilde{t}} a)^{-1}$, for any measurable $f : [0,1] \rightarrow A$, $a \in A$ and $t, \tilde{t} \in [0,1]$. Thus, we can write $g_{(a,\tau)} = V_{(a,\lambda_{\varepsilon,0} \circ (f \setminus_0 a)^{-1})}$ if $\tau = \lambda \circ f^{-1}$, and so $g_{(a,\tau)}$ is Borel measurable. This in turn implies that g is Borel measurable. ■

By Theorem 1 in Mas-Colell [7], it follows that $\bar{G}$, and so $\tilde{G}_\varepsilon$, has an equilibrium distribution.

To finish the proof, we let τ_n^* be an equilibrium distribution of $\tilde{G}_{1/n}$. Since the set of probability measures τ on $\mathcal{U} \times A$ with the property that $\tau_{\mathcal{U}} = \lambda \circ U^{-1}$ is compact, there exists a converging subsequence. Hence, its limit point is a strategic equilibrium distribution of $\tilde{G}$. ■

We show that any strategic equilibrium is a Nash equilibrium.

Theorem 2 *Let $G \in \mathcal{G}$. Then, every strategic equilibrium distribution of G is an equilibrium distribution of G.*

Proof. Let τ^* be a strategic equilibrium distribution, and let $\{\varepsilon_k\}$ and $\{\tau_k^*\}$ be such that $\varepsilon_k \in \mathbb{R}_{++}$, $\lim_k \varepsilon_k = 0$, τ_k^* converges to τ^*, and τ_k^* is a Nash equilibrium distribution of G_{ε_k}, for all $k \in \mathbb{N}$. Then, there exist measurable functions f and f_k, $k \in \mathbb{N}$, of $[0,1]$ into A such that $\tau_A = \lambda \circ f^{-1}$, $\tau_{A,k} = \lambda \circ f_k^{-1}$ and $\lim_k f_k = f$ a.e. in $[0,1]$.

We claim that $\lambda_{\varepsilon_k,t} \circ (f_k^{-1} \setminus_t a) \Rightarrow \tau$ for all $t \in [0,1]$ and $a \in A$. To see this, let $h : A \to \mathbb{R}$ be continuous. Then,

$$
\begin{aligned}
&\int_A h \mathrm{d}\lambda_{\varepsilon_k,t} \circ (f_k^{-1} \setminus_t a) = \\
&\int_{[0,1]} h \circ (f_k \setminus_t a) \mathrm{d}\lambda_{\varepsilon_k,t} = \\
&\varepsilon_k h(a) + (1-\varepsilon_k) \int_{[0,1]} h \circ f_k \mathrm{d}\lambda \to = \\
&\int_{[0,1]} h \circ f \mathrm{d}\lambda = \\
&\int_A h \mathrm{d}\lambda \circ f^{-1} = \\
&\int_A h \mathrm{d}\tau.
\end{aligned}
$$

Thus, indeed we have that $\lambda_{\varepsilon_k,t} \circ (f_k^{-1} \setminus_t a) \Rightarrow \tau$ for all $t \in [0,1]$ and $a \in A$.

Following the same steps as in Carmona [3], we can show that τ^* is an equilibrium distribution of G.

We include its proof for the sake of completeness.

Let $\{\tau_n\}_n$ be a sequence of ε_n−equilibrium distributions, where $\varepsilon_n \searrow 0$ and let τ be such that $\tau_n \Rightarrow \tau$. Then $\tau_{A,n} \Rightarrow \tau_A$; so, taking a subsequence if necessary, we may assume that $\rho(\tau_A, \tau_{A,n}) < 1/n$.

Define, for each $u \in \mathcal{U}$,

$$\beta_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a,\nu) - u(a,\tau_A)| : \rho(\nu,\tau_A) < 1/n\}.$$

Since u is continuous on $A \times \mathcal{M}$, which is compact, it follows that u is uniformly continuous. Thus, $\beta_n(u) \searrow 0$ as $n \to \infty$. We claim that β_n is continuous in $\mathcal{U}$.

Let $\eta > 0$. Define $\delta < \eta/2$. Then if $||u - v|| < \delta$, we have for any $a \in A$, and $\nu \in \mathcal{M}$ such that $\rho(\nu, \tau_A) < 1/n$

$$
\begin{aligned}
|v(a,\nu) - v(a,\tau_A)| \le |v(a,\nu) - u(a,\nu)| + |u(a,\nu) + u(a,\tau_A)| + \\
+ |v(a,\tau_A) - u(a,\tau_A)| < \delta + \beta_n(u) + \delta,
\end{aligned}
\tag{1}
$$

and so $\beta_n(v) \le 2\delta + \beta_n(u) < \eta + \beta_n(u)$. By symmetry, $\beta_n(u) < \eta + \beta_n(v)$, and so $|\beta_n(u) - \beta_n(v)| < \eta$. Hence, β_n is continuous, as claimed.

Given the definition of β_n, we have that $B_{\tau_n}^{\varepsilon_n} \subseteq D_n := \{(u,a) : u(a,\tau_A) \ge u(A,\tau_A) - \varepsilon_n - 2\beta_n(u)\}$. Since β_n is continuous, we see that D_n is closed, and so Borel measurable. Thus, $\tau_n(D_n) \ge 1 - \varepsilon_n$. Also, $D_n \searrow B_\tau$. Hence, for fixed $j \in \mathbb{N}, j \ge n$, it follows that $\tau_j(D_n) \ge \tau_j(D_j) \ge 1 - \varepsilon_j \ge 1 - \varepsilon_n$, and so $\tau(D_n) \ge \limsup_j \tau_j(D_n) \ge 1 - \varepsilon_n$. Hence, $\tau(B_\tau) = \lim_n \tau(D_n) = 1$. Therefore, τ is an equilibrium distribution of $\tau_{\mathcal{U}}$. ■

4 Strategic interaction

As we have noted in the introduction, the concept of strategic equilibrium is the correct non-atomic version of the Nash equilibrium in finite games. To show that formally, we will be using

Nash's mass action interpretation taken from his Ph.D. thesis. For any given finite normal form game, using the mass action interpretation, we will formulate its associated non-atomic version. In theorem 3 we will prove that a strategy profile, in the non-atomic version of the given finite normal form game, is a strategic equilibrium if and only if the associated strategy profile in the finite normal form game is a Nash equilibrium.

In his Ph.D. dissertation, John Nash proposed two interpretations of his equilibrium concept, with the objective of showing how equilibrium points "(...) can be connected with observable phenomenon." (Nash [8, p. 21]) One interpretation is rationalistic: if we assume that players are rational, they know the full structure of the game, the game is played just once, and there is just one Nash equilibrium, then players will play according to that equilibrium.[6]

A second interpretation, which Nash names mass-action interpretation, is less demanding on the players. In this interpretation, "[i]t is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes." (Nash [8, p. 21]) What is assumed is that there is a population of participants for each position in the game, which will be played throughout time by participants drawn at random from the different populations. If there is a stable average frequency with which each pure strategy is employed by the "average member" of the appropriate population, then this stable average frequency constitutes a Nash equilibrium.

Below we present not only a new interpretation of Nash equilibrium but also prove that the notion of strategic equilibrium is the non-atomic extension of Nash equilibrium in finite normal form games.

Consider a finite normal form game $\Gamma = (N, (\Delta(A_i), v_i)_{i\in N})$, where $N = \{1, \dots, n\}$ is the set of positions, $\Delta(A_i)$ is the set of mixed strategies over the finite action set A_i, and v_i is the usual extension to mixed strategies of the payoff function. As in Nash's mass action interpretation, imagine that this game is played in a large society divided in n groups, from each of which a participant is draw at random. For concreteness, let $T_i = [0, 1]$, and $X_i = \Delta(A_i)$, for any $i \in N$; a player $t \in T_i$ chooses an element of $\Delta(A_i)$. From each T_i a player is selected according to the Lebesgue measure, and so the probability that the player selected from the i^{th} group will play action $a_i \in A_i$ is $\int_{T_i} x_i^{a_i}$. We thus define $s_i(x_i) = \int_{T_i} x_i$, and

$$u_i(t, x) = v_i(s_1(x_1), \dots, s_n(x_n)) = \sum_{a\in A} \prod_{i\in N} s_i^{a_i}(x_i) v_i(a).$$

We denote by G the game $(T_i, P_i, u_i)_{i\in N}$

Theorem 3 *$(x_1^*, \dots, x_n^*)$ is a strategic equilibrium of G if and only if $(s_i(x_i^*))_{i=1,\dots,n}$ is a Nash equilibrium of Γ.*

Proof. (Sufficiency) Let $(x_1^*, \dots, x_n^*)$ be a strategy in G, and assume that $s^* := (s_1(x_1^*), \dots, s_n(x_n^*))$ is a Nash equilibrium of Γ. Let $i \in N$. We have that $v_i(s^*) \geq v_i(s_i, s_{-i}^*)$ for all $s_i \in \Delta(A_i)$. This implies, in particular, that $v_i(s^*) \geq v_i(\varepsilon x(t) + (1-\varepsilon)s_i^*, s_{-i}^*)$ for all $t \in T_i$, and $\varepsilon > 0$. Hence, $(x_1^*, \dots, x_n^*)$ is a Nash equilibrium of G_ε for all $\varepsilon > 0$, and so a strategic equilibrium of G.

(Necessity) Let $(x_1^*, \dots, x_n^*)$ be a strategic equilibrium of G, and let $s^* := (s_1(x_1^*), \dots, s_n(x_n^*))$. We will show that for all $i \in N$, and $a_i \in A_i$ if $s_i^{a_i}(x_i^*) > 0$, then a_i maximizes $v_i(a_i, s_{-i}^*)$ in A_i.

Let $i \in N$, $a_i \in A_i$. If a_i does not maximize $v_i(a_i, s_{-i}^*)$ in A_i, then a_i does not maximize $v_i(a_i, s_{-i}^\varepsilon)$ in A_i for all $\varepsilon > 0$ sufficiently small, where $s^\varepsilon := (s_1(x_1^\varepsilon), \dots, s_n(x_n^\varepsilon))$, and $(x_1^\varepsilon, \dots, x_n^\varepsilon)$

[6] For a formal discussion of these ideas, see Aumann and Brandenburger [1] and the Nobel Seminar 1994 [9].

is a Nash equilibrium of G_ε, $x^\varepsilon \to x^*$, and $\varepsilon \to 0$. Hence, $x_i^{\varepsilon,a_i}(t) = 0$ a.e. $t \in T_i$, and so $x_i^{*a_i}(t) = 0$ a.e. $t \in T_i$. Thus, $s_i^{a_i}(x_i^*) = 0$. ■

5 Applications

In this section, we present examples in which the strategic equilibrium concept eliminates implausible Nash equilibrium strategy profiles.

5.1 A Game of Proportional Voting

We argue that in a model of voting, the agents should have a small but positive impact on the societal choice, because after all, the number voters in any election is finite. Another motivation for insisting voters to have small but positive impact on the societal choice is that people still vote, an observation that political economists using non-atomic games to model voting had a very hard time to explain.

We describe a game of proportional voting where any strategy profile is a Nash equilibrium. In particular, a Nash equilibrium strategy profile in which nobody votes or in which an agent favoring an extreme right (left) political party votes for an extreme left (right, respectively) political party, is an artifact of the specification of a non-atomic game, and is not due to strategic interaction among players. On the other hand, the unique strategic equilibrium of the same game is a strategy profile in which each player votes for his favorite political party.

The set of parties is given by $M = \{1, ..., \bar{m}\}$ and for each player $t \in [0,1]$ the strict preference ordering on M is characterized by $v_t(m)$, $t \in [0,1]$. That is for any m and m' in M, t strictly prefers m to m' if and only if $v_t(m) > v_t(m')$. We should note that for the simplicity of the argument we do not allow any agent be indifferent between any of the parties, m and m' in M. Let $m_t^\star$ be the favorite party of agent t, i.e. $v_t(m_t^\star) \geq v_t(m)$ for all $m \in M$.

The payoff of an agent $t \in [0,1]$ is given by an expected utility measure. More specifically, a strategy profile x will induce a probability distribution on the set of parties, which is given by $\int_0^1 x(\tau)\mathrm{d}\lambda(\tau) \in \Delta(M)$. The payoff of an agent is the weighted average of the utilities she gets from individual parties. Hence, for any $m \in M$, letting $v(t) = (v_{m_1}(t), \ldots, v_{\bar{m}}(t))$, we define the utility of agent t voting for candidate m to be

$$u^m(t,x) = \hat{u}^m\left(t, \int_{[0,1]} x\mathrm{d}\lambda\right) = v(t) \cdot \int_0^1 x(\tau)\mathrm{d}\lambda(\tau).$$

Since an agent cannot affect the societal choice, i.e. $\int_0^1 x(\tau)\mathrm{d}\lambda(\tau)$, any strategy profile is a Nash equilibrium. Yet, for any $\varepsilon > 0$, by voting to his most favorite party $m_t^\star$ instead of voting for m, $m \neq m_t^\star$, agent t would increase his expected utility by $\varepsilon(v_t(m_t^\star) - v_t(m)) > 0$. Consequently, for any $\varepsilon > 0$ agent t's best response is to vote for $m_t^\star$. Thus, the unique strategic equilibrium profile is where each player t votes only for $m_t^\star$.[7] In this unique strategic equilibrium,

[7]The assumption that no agent can be indifferent between two political parties is just to simplify the argument. If we were to allow indifference relations on M by some players, the result would essentially be the same, and would read as follows: The set of strategic equilibria will be strategy profiles in which every agent t_0 assigns zero probability to the parties $m \in M \setminus M(t_0)$, i.e. $x^m(t_0) = 0$ for all $m \in M \setminus M(t_0)$, where $M(t_0) = \{\tilde{m} \in M : v_{t_0}(\tilde{m}) \geq v_{t_0}(m), \forall m \in M\}$. That is, in any strategic equilibrium, any agent assigns zero probability to the parties that are not one of his favorites.

players vote only for the party they like the most. This result appears to be in accordance with some evidence from proportional voting systems.

5.2 A Game of Allocation of Public Resources

A slightly modified version of the game presented in the previous section can be used in the analysis of allocating resources on public projects.

Suppose that the set of possible public projects is $M = \{1, ..., \bar{m}\}$. As before, for each player $t \in [0, 1]$ the strict preference ordering on M is characterized by $v_t(m)$, $t \in [0, 1]$. That is for any m and m' in M, t strictly prefers m to m' if and only if $v_t(m) > v_t(m')$. We do not allow any agent be indifferent between any of the projects, m and m' in M. Let $m_t^\star$ be the favorite project of agent t, i.e. $v_t(m_t^\star) \geq v_t(m)$ for all $m \in M$.

There is a fixed amount of perfectly divisible public resources, $B \in \mathbb{R}_{++}$, and any fraction of that can be allocated to these projects, each of which requires the same kind of resources to be operated. What fraction of the public resources a project gets is determined in the following manner: Each agent t announces the weight vector, in $\Delta(M)$, with which he wants the public resource, B, be allocated. Consequently, the resulting strategy profile x determines a distribution $p \in \Delta(M)$ given by $(p_1, \ldots, p_{\bar{m}}) = \int_0^1 x(\tau)\mathrm{d}\lambda(\tau)$, and $p_m B$ of the public resources are allocated to project $m \in M$. A public project that receives $p_m B$ resources, makes a utility contribution of $v_t(m) p_m B$ to agent t. Hence, the payoff of an agent t supporting project m is

$$u^m\left(t, \int_{[0,1]} x\mathrm{d}\lambda\right) = Bv(t) \cdot \int_0^1 x(\tau)\mathrm{d}\lambda(\tau).$$

This game is a scaled version of the voting game presented in the previous section. Therefore, as shown before, any strategy profile is a Nash equilibrium. On the other hand, due to the same arguments supplied in the previous section, the unique strategic equilibrium is a strategy profile in which any agent $t \in [0, 1]$ points to $m_t^\star \in M$, his favorite project.

5.3 Cournot Oligopoly

In this section we will formulate and analyze the Cournot oligopoly, and demonstrate that strategic equilibrium strategy profiles consist of the non-symmetric Nash equilibrium strategy profiles.

The set of agents is given by $T = [0, 1]$ and each of them can choose a quantity $x(t) \in [0, \bar{q}]$, where $\bar{q} \geq 1$, and the symmetric unit cost of production for each $t \in [0, 1]$ is 0. The inverse demand is given by $p = 1 - \int \mathbf{x}\mathrm{d}\lambda$.[8] Hence the profit function of firm t is

$$\Pi\left(x(t), \int \mathbf{x}\right) = \left(1 - \int \mathbf{x}\right) x(t).$$

The set of Nash equilibria in this game is any strategy profile $\mathbf{x}$ satisfying $\int \mathbf{x}\mathrm{d}\lambda = 1$. The reason is that as long as $\int \mathbf{x}\mathrm{d}\lambda = 1$, $p = 0$, thus, any agent $t \in [0, 1]$ would be indifferent between any of their choices in $\in [0, \bar{q}]$, since each agent is atomless. Moreover, if $\bar{q} = 1$ the Nash equilibrium is unique and is given by $\mathbf{x}(t) = 1$ for almost every player t in $[0, 1]$.

[8]Please note that for simplicity we allow for negative prices. That could be corrected by working with a symmetric and positive unit cost.

Given a profile $\mathbf{x}$ and $\varepsilon > 0$, the profit of $t \in [0,1]$ is

$$\Pi^{\varepsilon}\left(x_t, \int \mathbf{x}\right) = \left(1 - (1-\varepsilon)\int \mathbf{x} - \varepsilon x_t\right) x_t.$$

Thus, the best response of $t \in [0,1]$ is

$$x_t^{\varepsilon} = \frac{1 - (1-\varepsilon)\int \mathbf{x}}{2\varepsilon}.$$

In equilibrium,

$$\int \mathbf{x}^{\varepsilon} = \int \left(\frac{1 - (1-\varepsilon)\int \mathbf{x}^{\varepsilon}}{2\varepsilon}\right) = \frac{1 - (1-\varepsilon)\int \mathbf{x}^{\varepsilon}}{2\varepsilon}.$$

Thus, $\int \mathbf{x}^{\varepsilon} = \frac{1}{1+\varepsilon}$ which gives us (by substituting back to the best response function)

$$\mathbf{x}_t^{\varepsilon} = \frac{1}{1+\varepsilon}.$$

Obviously, this term converges to $\mathbf{x}_t^{\star} = 1$ for almost every player $t \in [0,1]$. Hence, the set of strategic equilibria is given by $\mathbf{x}^{\star} : \mathbf{x}_t^{\star} = 1$ for almost every $t \in [0,1]$. Thus, unlike for Nash equilibrium, there a unique strategic equilibrium.

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