RELATION BETWEEN WEAK ENTWINING STRUCTURES AND WEAK CORINGS

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Abstract. Given a commutative ring R with unit, R-algebra A and R-coalgebra C. Triple (A,C,ψ) is called (weak) entwining structure if there is R-linear map $\psi : C\otimes_R A \to A\otimes_R C$ that fulfil some axioms. In the other hand, from algebra A and coalgebra C we can consider $A\otimes_R C$ as a left A-module canonically such that (A,C,ψ) is entwined structure if only if $A\otimes_R C$ is a A-coring. In particular, we obtain that (A,C,ψ) is a weak entwined structure if only if $A\otimes_R C$ is a weak A-coring.
Keywords : algebra, coalgebra, coring, entwining structure.

1. Introduction

In this paper we assume that R is a commutative ring with unit. In Brzeziński and Wisbauer [3] R-algebra (A,μ,ι) and R-coalgebra (C,Δ,ε) is called entwined and (A,C,ψ) is said to be an entwining structure if there exists a R-linear map $\psi : C\otimes_R A \to A\otimes_R C$ such that fulfil some axioms. It is described by Brzeziński [2] on a *bow-tie* diagram.

R-algebra A dan R-coalgebra C can be considered as R-module. From A and C as R-module, we can construct tensor product $A\otimes_R C$. Moreover, from right A-action $\alpha : (A\otimes_R C)\otimes_R A \to A\otimes_R C$, $\alpha((a\otimes b)\otimes c) = a\psi(c\otimes b)$, $A\otimes_R C$ is a (A,A)-bimodule and we obtain $A\otimes_R C$ is a weak coring. From Brzeziński [4] we have relation between weak coring and weak entwining structure , i. e. (A,C,ψ) is an entwining structure if only if $A\otimes_R C$ has an A-coring structure given by the comultiplication

$$\underline{\Delta} := I_A \otimes \Delta : A\otimes_R C \to A\otimes_R C\otimes_R C \simeq (A\otimes_R C)\otimes_A (A\otimes_R C),$$

and counit $\underline{\varepsilon} := I_A \otimes \varepsilon : A \otimes_R C \to A.$ Weak coring is a structure like coring but weak coring is obtained from non-unital bimodule (see Puspita [7], Wisbauer [9]). We will see relation between coring and entwining structures can be used on weak coring $A \otimes_R C.$

In section 2 we give definitions of corings and weak corings. Those are generalization from coalgebra (see Brzeziński [3], Puspita [7] and Wisbauer [9]). In the next section from Brzeziński [4] given definitions of entwining structures and weak entwining structures. In section 4 finally we have relation between weak entwining structures and weak corings, i.e $A \otimes_R C$ is a weak coring if only if $A \otimes_R C$ is an entwining structure.

2. Corings and Weak Corings

In 1960 Sweedler Introduced the study of coalgebras and comodules over field. A vector space C over field F with comultiplication $\Delta : C \to C \otimes_F C$ and counit $\varepsilon : C \to F$ is called F-coalgebra. The study of coalgebras over commutative rings and noncommutative rings are presented in Brzeziński and Wisbauer [3]. In this section, we are given basic information of corings and weak corings (see Brzeziński [3], Puspita [7] and Wisbauer [9]). Throughout A will be an assosiative ring with unit.

Definition 2.1. *Let C be an (A,A) non-unital bimodule.*

(i). An (A,A)-bilinear map $\underline{\Delta} : C \to C \otimes_A A \otimes_A C,$ i.e $(\forall c \in C)\underline{\Delta}(c) = \sum c_{\underline{1}} \otimes 1 \otimes c_{\underline{2}}$ is called a weak comultiplication.

(ii). An (A,A)-bilinear map $\underline{\varepsilon} : C \to A$ is called weak counit for $\underline{\Delta}$ provided we have a commutative diagram on figure 1.

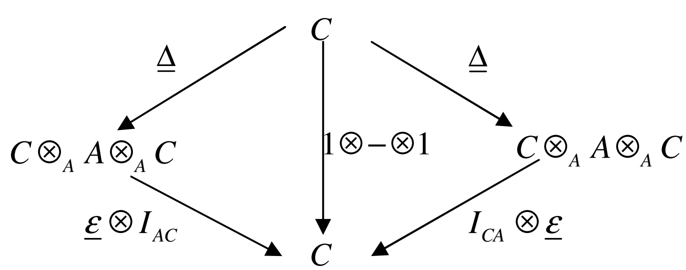

figure 1. *Weak counit diagram*

Figure 1 is commutative for $c \in C,$ $\sum \underline{\varepsilon}(c_{\underline{1}})c_{\underline{2}} = 1c1 = \sum c_{\underline{1}}\underline{\varepsilon}(c_{\underline{2}}).$

Definition 2.2. *An (A,A)-non-unital bimodule C is called weak coring provided it has weak comultiplication $\underline{\Delta}$ and weak counit $\underline{\varepsilon}$.*

Definition 2.3. *Let* $(C, \underline{\Delta}, \underline{\varepsilon})$ *be an weak* A *-coring. If* C *is an* (A, A) *-unital bimodule with left or right unital, then* C *is called* ***pre-coring.*** *If* C *is an* (A, A) *-unital, then* C *is an* A *-* ***coring***.

Based on Definition 2.3., we conclude that every A -coring are a weak A -coring. A weak A -coring is an A -coring if only if C is an (A, A) -unital bimodule.

3. Entwining Structures

Entwining structure introduced by Brzeziński and Majid [1]. Some authors have presented their observation in the same object in various text books as well see Brzeziński [4] and Brzeziński and Wisbauer[3].

Definition 3.1. *Let* (A, μ, ι) *be a* R *-algebra and* (C, Δ, ε) *be a* R *-coalgebra. Triple* (A, C, ψ) *is called entwining structure provided there exist* R *-linear map* $\psi : C \otimes_R A \to A \otimes_R C$ *such that*

(1). $\psi \circ (I_C \otimes \mu) = (\mu \otimes I_C) \circ (I_A \otimes \psi) \circ (\psi \otimes I_A)$,

(2). $(I_A \otimes \Delta) \circ \psi = (\psi \otimes I_C) \circ (I_C \otimes \psi) \circ (\Delta \otimes I_A)$,

(3). $\psi \circ (I_C \otimes \iota) = \iota \otimes I_C$,

(4). $(I_A \otimes \varepsilon) \circ \psi = \varepsilon \otimes I_A$.

The axioms in Definition 3.1. are described on **bow-tie** diagram (see Brzeziński [2], Brzeziński and Wisbauer[3]) as follow :

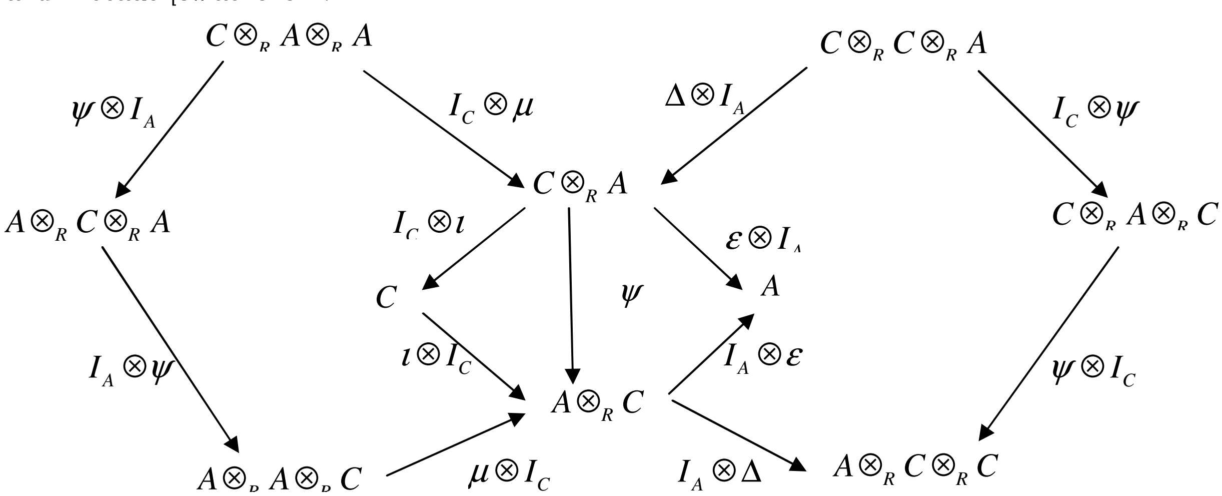

Figure 2. *bow-tie* commutative diagram

Defined a map $\psi : C \otimes_R A \to A \otimes_R C,$ $\psi(c \otimes a) = \sum a_\psi \otimes c^\psi,$ for $a_\psi \in A, c^\psi \in C.$ Figure 2 is commutative, it means that for any $c \otimes a_1 \otimes a_2 \in C \otimes_R A \otimes_R A,$ $c \otimes a \in C \otimes_R A,$

1. $\sum (a_1 a_2)_\psi c^\psi = \sum a_{1_\psi} a_{2_\varphi} c^{\psi\varphi}$
2. $\sum a_\psi \otimes c_{\underline{1}}^{\psi} \otimes c_{\underline{2}}^{\psi} = \sum a_{\psi\varphi} c_{\underline{1}}^{\varphi} c_{\underline{2}}^{\psi}$
3. $\sum 1_\psi \otimes c^\psi = 1 \otimes c$
4. $\sum a_\psi \varepsilon(c^\psi) = \varepsilon(c) a.$

Definition for weak entwining structures analog with Definition 3.1. The differences are caused by $A \otimes_R C$ as a non unital module so the conditions that need to be fulfiled are still involved an element unit 1. The following definition are presented in Hungerford [6] and Wisbauer [9].

Definisi 3.2. *Let* (A, μ, ι) *be a* R *- algebra and* (C, Δ, ε) *is a* R *-coalgebra. Triple* (A, C, ψ) *is called weak entwining structure provided there exist a* R *-linear maps* $\psi : C \otimes_R A \to A \otimes_R C,$ $\psi(a \otimes c) = \sum a_\psi c^\psi$ *for* $a_\psi \in A$ *and* $c^\psi \in C$ *such that :*

(1). $\sum (ab)_\psi c^\psi = \sum a_\psi b_\varphi c^{\psi\varphi}$

(2). $\sum a_\psi (c^\psi{}_{\underline{1}} \otimes 1) \otimes c^\psi{}_{\underline{2}} = \sum a_{\psi\varphi} \otimes c_{\underline{1}}^{\varphi} \otimes c_{\underline{2}}^{\psi}$

(3). $\sum a_\psi \varepsilon(c^\psi) = \sum \varepsilon(c^\psi) 1_\psi a$

(4). $\sum 1_\psi \otimes c^\psi = \sum \varepsilon(c_{\underline{1}}^{\psi}) 1_\psi \otimes c_{\underline{2}}.$

4. Weak Entwining Structures and Weak Corings

As a R-module, product tensor between R-algebra A and R-coalgebra C is denoted by $A \otimes_R C.$ In this section it will be explained the relation between (weak) entwining structures and (weak) corings) $A \otimes_R C.$ We are now proving our main theorem.

Theorem 4.1. *Let* (A, μ, ι) *be a* R *-algebra and* (C, Δ, ε) *be a* R *-coalgebra. Triple* (A, C, ψ) *is an entwining structure if only if* $A \otimes_R C$ *is an* A *-coring.*

PROOF.

$(\Leftarrow)$ Assume that $A\otimes_R C$ is an A-coring over comultiplication and counit

$$\underline{\Delta}: A\otimes_R C \xrightarrow{I_A\otimes\Delta} (A\otimes_R C)\otimes_A (A\otimes_R C) \simeq (A\otimes_R C).1\otimes_R C,$$

$$a\otimes c \;\mapsto\; \sum (a\otimes c_{\underline{1}})\otimes_A (1\otimes c_{\underline{2}}) \mapsto \sum (a\otimes c_{\underline{1}}).(1\otimes c_{\underline{2}}),$$

$$\underline{\varepsilon}: A\otimes_R C \to (A\otimes_R C).1 \xrightarrow{I_A\otimes\varepsilon} A,$$

$$a\otimes c \;\mapsto\; (a\otimes c).1 \;\mapsto\; a\varepsilon(c).$$

The following R-linear map is defined by right A-action $A\otimes_R C$.

$$\psi: C\otimes_R A \to A\otimes_R C,\; c\otimes a \mapsto (1\otimes c).a$$

$\psi(c\otimes a) = \sum a_\psi c^\psi$, for $a_\psi \in A, c^\psi \in C$. We will show that (A, C, ψ) is an entwining structure by ψ. For any $a, b\in A$ and $c\in C$

(i). by associative properties from right action

$$\sum (ab)_\psi \otimes c^\psi = (1\otimes c).a.b = ((1\otimes c).a).b = \left(\sum a_\psi c^\psi\right).b = \left(1\otimes \sum a_\psi c^\psi\right).b = \sum a_\psi b_\varphi c^{\psi\varphi}.$$

(ii). By comultiplication in $A\otimes_R C$ we have

$$\begin{aligned}
\underline{\Delta}(1\otimes c).a &= \underline{\Delta}\left(\sum a_\psi c^\psi\right)\\
&= \sum a_\psi \underline{\Delta}(c^\psi)\\
&= \sum a_\psi \left(c^\psi{}_{\underline{1}}\otimes c^\psi{}_{\underline{2}}\right)\\
&= \sum a_\psi \otimes c^\psi{}_{\underline{1}}\otimes c^\psi{}_{\underline{2}}
\end{aligned}$$

$$\begin{aligned}
\underline{\Delta}((1\otimes c).a) &= \underline{\Delta}(1\otimes c).a\\
&= \left(\sum (1\otimes c_{\underline{1}})\otimes_A (1\otimes c_{\underline{2}})\right).a\\
&= \sum 1\otimes c_{\underline{1}}\left(\sum a_\psi (1\otimes c_{\underline{2}})^\psi\right)\\
&= \sum 1\otimes c_{\underline{1}}\left(\sum (a_\psi 1) c_{\underline{2}}{}^\psi\right)\\
&= \sum (1\otimes c_{\underline{1}}).a_\psi \otimes c_{\underline{2}}{}^\psi\\
&= \sum a_{\psi\varphi}(1\otimes c_{\underline{1}})^\varphi \otimes c_{\underline{2}}{}^\psi\\
&= \sum a_{\psi\varphi}\otimes c_{\underline{1}}{}^\varphi \otimes c_{\underline{2}}{}^\psi
\end{aligned}$$

(iii). R -linear map $\underline{\varepsilon}$ is a module homorphism, so that

$$\begin{aligned}\sum a_{\psi}\varepsilon\left(c^{\psi}\right) &= \left(I_A \otimes \varepsilon\right)\sum a_{\psi}c^{\psi} \\ &= \left(I_A \otimes \varepsilon\right)\circ\psi\left(c\otimes a\right) \\ &= \left(I_A \otimes \varepsilon\right)\left(\left(1\otimes c\right).a\right) \\ &= \varepsilon\left(c\right)a \\ &= \varepsilon\otimes I_A\left(c\otimes a\right)\end{aligned}$$

(iv). As an A -coring, $A\otimes_R C$ is a right unital A -module, so from unital properties we have $1\otimes c = \left(1\otimes c\right).1 = \sum 1_{\psi}c^{\psi}$.

By (i) – (iv) $\left(A,C,\psi\right)$ is an entwining structure.

$\left(\Rightarrow\right)$ Suppose that $\left(A,C,\psi\right)$ is an entwining structure, to show $A\otimes_R C$ is an A -coring, so in the first step we have to show that $A\otimes_R C$ is an $\left(A,A\right)$-unital bimodule. Left A -action for $A\otimes_R C$ is trivial. By R -linear map $\psi: A\otimes_R C \to C\otimes_R A,$ defined right A -action in $A\otimes_R C$

$$\left(A\otimes_R C\right)\otimes_R A \to A\otimes_R C,\ \left(a\otimes b\right)\otimes c \mapsto a\psi\left(c\otimes b\right)$$

by right A -action above, $A\otimes_R C$ is an $\left(A,A\right)$-unital bimodule.

For any $\left(a\otimes c\right),\left(a'\otimes c'\right)\in A\otimes_R C$ and $r,s\in A$

(i). $\left(a\otimes c\right).x+\left(a'\otimes c'\right).x = \left(a\otimes c\right)\otimes x+\left(a'\otimes c'\right)\otimes x$

$$\begin{aligned} &= \left(\left(a\otimes c\right)+\left(a'\otimes c'\right)\right)\otimes x \\ &= \left(\left(a\otimes c\right)+\left(a'\otimes c'\right)\right).x\end{aligned}$$

(ii). $\left(a\otimes c\right).\left(x+y\right) = a\psi\left(c\otimes\left(x+y\right)\right)$

$$\begin{aligned} &= a\psi\left(c\otimes x+c\otimes y\right) \\ &= a\psi\left(c\otimes x\right)+a\psi\left(c\otimes y\right) \\ &= \left(a\otimes c\right).x+\left(a\otimes c\right).y\end{aligned}$$

(iii). $((a\otimes c).x).y=(a\psi(c\otimes x)).y$

$$=a(1\otimes c).xy$$
$$=a\psi(c\otimes xy)$$
$$=(a\otimes c)(xy)$$

(iv). $(a\otimes c).1=a\otimes\psi(c\otimes 1)$

$$=a\otimes\left(\sum 1_{\psi}c^{\psi}\right)$$
$$=a\otimes(1\otimes c)\text{ (see Definition 3.1 (4))}$$
$$=a\otimes c$$

Furthermore we define a map

$$\underline{\Delta}:A\otimes_R C\xrightarrow{I_A\otimes\Delta}(A\otimes_R C)\otimes_A(A\otimes_R C)\simeq(A\otimes_R C).1\otimes_R C,$$
$$a\otimes c\ \ \mapsto\ \ \sum(a\otimes c_{\underline{1}})\otimes_A(1\otimes c_{\underline{2}})\mapsto\sum(a\otimes c_{\underline{1}}).(1\otimes c_{\underline{2}}),$$
$$\underline{\varepsilon}:A\otimes_R C\to(A\otimes_R C).1\xrightarrow{I_A\otimes\varepsilon}A,$$
$$a\otimes c\ \mapsto\ (a\otimes c).1\ \ \mapsto\ a\varepsilon(c).$$

R -linear map $\underline{\Delta}$ and $\underline{\varepsilon}$ above sequentially are a comultiplication and counit for an (A,A)-bimodule $A\otimes_R C$

$$(I_{A\otimes_R C}\otimes\underline{\varepsilon})\circ\underline{\Delta}(a\otimes c)=(I_{A\otimes_R C}\otimes\underline{\varepsilon})\sum(a\otimes c_{\underline{1}}).1\otimes c_{\underline{2}}$$
$$=\sum a\otimes c_{\underline{1}}\varepsilon(c_{\underline{2}})$$
$$=a\sum c_{\underline{1}}\varepsilon(c_{\underline{2}})$$
$$=a\otimes c\text{ (by counital as a coalgebra).}$$ □

Theorem 4.2. *Let* (A,μ,ι) *be a* R*- algebra and* (C,Δ,ε) *be a* R*-coalgebra. Triple* (A,C,ψ) *is an entwining structure if only if* $A\otimes_R C$ *is a weak* A*-coring.*

PROOF.

$(\Leftarrow)$ We have that $A\otimes_R C$ is a weak A- coring over weak comultiplication $\underline{\Delta}$ and weak counit $\underline{\varepsilon}$.

$$\underline{\Delta}:A\otimes_R C\xrightarrow{I_A\otimes\Delta}(A\otimes_R C)\otimes_A A\otimes_A(A\otimes_R C)\simeq(A\otimes_R C)\otimes_A(A\otimes_R C)\simeq(A\otimes_R C).1\otimes_R C,$$

$$a\otimes c \;\mapsto\; \sum\left(a\otimes c_{\underline{1}}\right)\otimes_A\left(1\otimes c_{\underline{2}}\right)\mapsto\sum\left(a\otimes c_{\underline{1}}\right).\left(1\otimes c_{\underline{2}}\right),$$

$$\underline{\varepsilon}: A\otimes_R C\to\left(A\otimes_R C\right).1\xrightarrow{I_A\otimes\varepsilon}A,$$

$$a\otimes c\;\mapsto\;\left(a\otimes c\right).1\;\mapsto\; a\varepsilon\left(c\right).$$

By right A -action of $A\otimes_R C,$ we defined a R -linear map

$$\psi: C\otimes_R A\to A\otimes_R C,\; c\otimes a\mapsto\left(1\otimes c\right).a$$

$\psi\left(c\otimes a\right)=\sum a_\psi c^\psi,$ for $a_\psi\in A, c^\psi\in C.$ We must show that $\left(A,C,\psi\right)$ fulfil Definition 3.2. For 3.2 (1) analog with Theorem 4.2 (i). For any $a,b\in A$ and $c\in C$

(i). by weak comultiplication definition of $A\otimes_R C$

$$\begin{aligned}
\underline{\Delta}\left(1\otimes c\right).a &= \underline{\Delta}\left(\sum a_\psi c^\psi\right)\\
&=\sum a_\psi\underline{\Delta}\left(c^\psi\right)\\
&=\sum\left(a_\psi\otimes c^\psi{}_{\underline{1}}\right)\otimes\left(1\otimes c^\psi{}_{\underline{2}}\right)\\
&=\sum\left(a_\psi\otimes 1\otimes c^\psi{}_{\underline{1}}\right)\otimes\left(1\otimes c^\psi{}_{\underline{2}}\right)\\
&=\sum a_\psi\otimes\left(\left(1\otimes c^\psi{}_{\underline{1}}\right).1\right)\otimes c^\psi{}_{\underline{2}}\\
&=\sum a_\psi\psi\left(c^\psi{}_{\underline{1}}\otimes 1\right)\otimes c^\psi{}_{\underline{2}}
\end{aligned}$$

$$\begin{aligned}
\underline{\Delta}\left(\left(1\otimes c\right).a\right) &= \underline{\Delta}\left(1\otimes c\right).a\\
&=\sum\left(1\otimes c_{\underline{1}}\right)\otimes_A\left(1\otimes c_{\underline{2}}\right).a\\
&=\sum 1\otimes c_{\underline{1}}\left(\sum\left(a_\psi 1\right)c_{\underline{2}}{}^\psi\right)\\
&=\sum\left(1\otimes c_{\underline{1}}\right).a_\psi\otimes c_{\underline{2}}{}^\psi\\
&=\sum a_{\psi\varphi}\left(1\otimes c_{\underline{1}}\right)^\varphi\otimes c_{\underline{2}}{}^\psi\\
&=\sum a_{\psi\varphi}\otimes c_{\underline{1}}{}^\varphi\otimes c_{\underline{2}}{}^\psi
\end{aligned}$$

Jadi $\sum a_\psi\psi\left(c^\psi{}_{\underline{1}}\otimes 1\right)\otimes c^\psi{}_{\underline{2}}=\sum a_{\psi\varphi}\otimes c_{\underline{1}}{}^\varphi\otimes c_{\underline{2}}{}^\psi.$

(ii). Morphism $\underline{\varepsilon}$ is an A -module homomorphism so that

$$\sum a_\psi \varepsilon\left(c^\psi\right) = \left(I_A \otimes \varepsilon\right)\sum a_\psi c^\psi$$
$$= \left(I_A \otimes \varepsilon\right)\circ \psi\left(c \otimes a\right)$$
$$= \left(I_A \otimes \varepsilon\right)\left(1 \otimes c\right).a$$
$$= \left(I_A \otimes \varepsilon\right)\left(1 \otimes c\right).1.a$$
$$= \left(I_A \otimes \varepsilon\right)\left(\sum 1_\psi c^\psi\right).a$$
$$= \sum \varepsilon\left(c^\psi\right)1_\psi a$$

(iii). By weak counital we have

$$\sum 1_\psi c^\psi = \left(1 \otimes c\right).1$$
$$= \left(\underline{\varepsilon} \otimes I_A\right)\circ \underline{\Delta}\left(1 \otimes c\right) \text{ (by weak counital)}$$
$$= \left(\underline{\varepsilon} \otimes I_A\right)\left(\sum \left(1 \otimes c_{\underline{1}}\right).1 \otimes c_{\underline{2}}\right)$$
$$= \left(\underline{\varepsilon} \otimes I_A\right)\left(\sum 1_\psi c_{\underline{1}}^{\psi} \otimes c_{\underline{2}}\right)$$
$$= \sum 1_\psi \underline{\varepsilon}\left(c_{\underline{1}}^{\psi}\right) \otimes c_{\underline{2}}$$
$$= \sum \underline{\varepsilon}\left(c_{\underline{1}}^{\psi}\right) 1_\psi \otimes c_{\underline{2}}$$

(i) – (iii) showed that $\left(A, C, \psi\right)$ is a weak entwining structure. Conversely is analog with Theorem 4.2. □

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