J. of Inequal. & Appl., 1999, Vol. 3, pp. 313–330
Reprints available directly from the publisher
Photocopying permitted by license only

Qualitative Properties of Solutions to Elliptic Singular Problems*

S. BERHANU [a], F. GLADIALI [b] and G. PORRU [b,†]

[a] *Mathematics Department, Temple University, Philadelphia, PA 19122, USA;*
[b] *Dipartimento di Matematica, Via Ospedale 72, 09124, Cagliari, Italy*

(Received 23 February 1998; Revised 7 August 1998)

We investigate the singular boundary value problem $\Delta u + u^{-\gamma} = 0$ in D, $u = 0$ on ∂D, where $\gamma > 0$. For $\gamma > 1$, we find the estimate

$$|u(x) - b_0 \delta^{2/(\gamma+1)}(x)| < \beta \delta^{(\gamma-1)/(\gamma+1)}(x),$$

where b_0 depends on γ only, $\delta(x)$ denotes the distance from x to ∂D and β is a suitable constant. For $\gamma > 0$, we prove that the function $u^{(1+\gamma)/2}$ is concave whenever D is convex. A similar result is well known for the equation $\Delta u + u^p = 0$, with $0 \le p \le 1$. For $p = 0$, $p = 1$ and $\gamma \ge 1$ we prove convexity sharpness results.

Keywords: Singular equations; Boundary behaviour; Convexity

1991 Mathematics Subject Classification: 35B45, 35E10

1. INTRODUCTION

Let $N > 1$ and let $D \subset \mathbb{R}^N$ be a bounded smooth domain. In [1–3,5], the problem $\Delta u = u^p$ in D, $u(x) \to +\infty$ as $x \to \partial D$ is investigated. It is proved that such a problem, for $p > 1$, has a unique positive solution $u(x)$. Moreover, for $p > 3$ there exists a constant $\beta > 0$ such that

$$\left| \frac{u(x)}{\delta^{2/(1-p)}(x)} - a_0 \right| < \beta \delta(x) \quad \text{in } D, \tag{1.1}$$

* Partially supported by MURST 40% and by a University of Cagliari Research Project.
† Corresponding author. E-mail: porru@vaxca1.unica.it.

where a_0 is a constant depending on p only and $\delta(x)$ denotes the distance from x to ∂D [1,3,5].

In [6,13] it is proved that the problem

$$\Delta u + u^p = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{1.2}$$

for $p < 0$ has a unique positive solution $u(x)$ continuous up to the boundary ∂D. In the same papers it is also proved that, for $p < -1$, there exist positive constants λ, Λ such that

$$\lambda\delta^{2/(1-p)}(x) \leq u(x) \leq \Lambda\delta^{2/(1-p)}(x).$$

In Section 2 of the present paper we shall prove that, for $p < -1$ there exists $\beta > 0$ such that

$$\left|\frac{u(x)}{\delta^{2/(1-p)}(x)} - b_0\right| < \beta\delta^{(p+1)/(p-1)}(x) \quad \text{in } D, \tag{1.3}$$

where b_0 is a constant depending on p only. We emphasize that the constants a_0 in (1.1) and b_0 in (1.3) are independent of the geometry of the domain D and even of the dimension N. We also find a boundary estimate for the case $p = -1$.

Now, consider problem (1.2) with $0 < p < 1$. It is well known that this problem has a positive solution $u(x)$. Such a solution is not concave even in the radial case. Indeed, if $u = u(r)$ then the corresponding equation reads as

$$(r^{N-1}u')' + r^{N-1}u^p = 0,$$

which implies that $(r^{N-1}u')' < 0$ and $u'(r) < 0$ in $(0, R]$. Since $u(R) = 0$, we have

$$u''(R) + \frac{N-1}{r}u'(R) = 0, \tag{1.4}$$

whence, $u''(R) > 0$. This shows that $u(r)$ is not concave near $r = R$.

It is known [9,10] that if $u(x)$ is a solution of problem (1.2) with $0 \leq p < 1$ in a convex domain D then, the function $v = u^{(1-p)/2}$ is concave in D.

If $p = 1$, instead of problem (1.2) we consider the following

$$\Delta u + \lambda_1 u = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

where λ_1 is the first eigenvalue of D. If u is the (positive) corresponding eigenfunction then, $v = \log u$ is concave whenever D is convex (see [9–11]).

In [9, p. 122] it is written that if $u(x)$ is a solution to (1.2) in a convex domain D then the function

$$v(x) = \int^{u(x)} s^{-(p+1)/2}\,\mathrm{d}s$$

is a good candidate to be concave. As recalled above, this fact is known to be true for $0 \le p \le 1$. In Section 3 of the present paper we shall prove that the statement in above is also true for $p < 0$. Furthermore, we shall find the following sharpness results. Let $u(x)$ be a solution of (1.2) and let $\epsilon > 0$. Then:

(1) if $p = 0$, the function $v = u^{1/2+\epsilon}$ is not concave in some convex domain;
(2) if $p = 1$, the function $v = u^{\epsilon}$ is not concave in some convex domain;
(3) if $p \le -1$, the function $v = u^{(1-p)/2+\epsilon}$ is not concave even in the radial case.

2. BOUNDARY BEHAVIOUR

In this section the domain D is assumed to be bounded, smooth and satisfying a uniform interior and exterior sphere condition. For $\gamma > 1$, we consider the following problem

$$\Delta u + u^{-\gamma} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{2.1}$$

LEMMA 2.1 *If $u(x)$ is a positive solution to problem* (2.1) *with $\gamma > 1$, then*

$$\lim_{x \to \partial D} \frac{u(x)}{\delta^{2/(\gamma+1)}(x)} = \left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)},$$

where $\delta(x)$ denotes the distance from x to ∂D.

Proof Let $R>0$. Consider first the case of $D=B(R)$, a ball with radius R. The corresponding (radial) solution $z=z(r)$ satisfies, for $0<r<R$,

$$z'' + \frac{N-1}{r}z' + z^{-\gamma} = 0, \quad z'(0) = 0, \ z(R) = 0. \tag{2.2}$$

Multiplying (2.2) by z' we get

$$z''z' + z^{-\gamma}z' < 0.$$

After integration over $(0,r)$ we find

$$\frac{(z'(r))^2}{2} < \frac{z^{1-\gamma}(r) - z^{1-\gamma}(0)}{\gamma - 1} < \frac{z^{1-\gamma}(r)}{\gamma - 1}.$$

The latter inequality implies

$$z' > -\left(\frac{2}{\gamma - 1}\right)^{1/2} z^{(1-\gamma)/2}. \tag{2.3}$$

Insertion of (2.3) into Eq. (2.2) yields

$$z'' - \frac{N-1}{r}\left(\frac{2}{\gamma - 1}\right)^{1/2} z^{(1-\gamma)/2} + z^{-\gamma} < 0,$$

whence

$$z'' + z^{-\gamma}\left[-\frac{N-1}{r}\left(\frac{2}{\gamma - 1}\right)^{1/2} z^{(1+\gamma)/2} + 1\right] < 0.$$

Let $\epsilon > 0$. Since $z(r) \to 0$ as $r \to R$, there exists $r_\epsilon < R$ such that

$$z'' + z^{-\gamma}(1 - \epsilon/2) < 0 \quad \text{in } (r_\epsilon, R).$$

Since $z'(r) < 0$, from this inequality we find

$$z''z' + z^{-\gamma}z'(1 - \epsilon/2) > 0$$

and

$$\frac{(z'(r))^2}{2} - \frac{(z'(r_\epsilon))^2}{2} + \frac{1-\epsilon/2}{1-\gamma}[z^{1-\gamma}(r) - z^{1-\gamma}(r_\epsilon)] > 0.$$

For some $r_0 \geq r_\epsilon$, we also have

$$\frac{(z'(r))^2}{2} > \frac{1-\epsilon}{\gamma-1} z^{1-\gamma}(r) \quad \forall r \in (r_0, R)$$

or

$$z^{(\gamma-1)/2}(r) z'(r) < -(1-\epsilon)^{1/2}\left(\frac{2}{\gamma-1}\right)^{1/2}.$$

Integration over (r, R) yields

$$z(r) > (1-\epsilon)^{1/(\gamma+1)}\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)} (R-r)^{2/(\gamma+1)} \quad \forall r \in (r_0, R). \tag{2.4}$$

Now consider the annulus $D = B(R, \bar{R})$. Let $w = w(r)$ be a solution to problem (2.1) in $B(R, \bar{R})$. We have, for $R < r < \bar{R}$,

$$w'' + \frac{N-1}{r} w' + w^{-\gamma} = 0, \quad w(R) = 0, \;\; w(\bar{R}) = 0. \tag{2.5}$$

If r_1 is a point in $(R, \bar{R})$ where $w'(r_1) = 0$, from (2.5) we find, for $R < r < r_1$,

$$\frac{(w'(r))^2}{2} = (N-1)\int_r^{r_1} \frac{1}{t}(w'(t))^2\, \mathrm{d}t + \frac{w^{1-\gamma}(r) - w^{1-\gamma}(r_1)}{\gamma-1}. \tag{2.6}$$

By (2.5) we also find

$$w'(r) = \frac{1}{r^{N-1}}\int_r^{r_1} t^{N-1} w^{-\gamma}(t)\, \mathrm{d}t < \frac{r_1^{N-1}}{R^{N-1}}\int_r^{r_1} w^{-\gamma}(t)\, \mathrm{d}t.$$

Using the latter inequality together with de l'Ôpital rule we find

$$\lim_{r\to R}\frac{(\gamma-1)\int_r^{r_1}(1/t)(w'(t))^2\,\mathrm{d}t}{w^{1-\gamma}(r)}=\lim_{r\to R}\frac{w'(r)}{rw^{-\gamma}}\le\lim_{r\to R}\frac{r_1^{N-1}}{R^N}\frac{\int_r^{r_1}w^{-\gamma}(t)\,\mathrm{d}t}{w^{-\gamma}(r)}.$$

Using de l'Ôpital rule once more we get

$$\lim_{r\to R}\frac{(\gamma-1)\int_r^{r_1}(1/t)(w'(t))^2\,\mathrm{d}t}{w^{1-\gamma}(r)}\le\frac{r_1^{N-1}}{R^N}\lim_{r\to R}\frac{w(r)}{\gamma w'(r)}=0.$$

Recall that $w(r)\to 0$ as $r\to R$ and that, by (2.6), $w'(r)\to\infty$ as $r\to R$. If the integral of $(w')^2$ over $(0,r_1)$ is finite then we cannot apply de l'Ôpital rule, but in this case, the limit is trivial. As a consequence of this estimate, (2.6) yields, for a given $\epsilon>0$,

$$\frac{(w'(r))^2}{2}<\frac{1+\epsilon}{\gamma-1}w^{1-\gamma}(r)\quad\forall r\in(R,r_\epsilon).$$

Integrating over (R,r) with $r<r_\epsilon$, one finds

$$\frac{2}{\gamma+1}w^{(\gamma+1)/2}<\left(\frac{2}{\gamma-1}\right)^{1/2}(1+\epsilon)^{1/2}(r-R),$$

or

$$w(r)<(1+\epsilon)^{1/(\gamma+1)}\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)}(r-R)^{2/(\gamma+1)}.\qquad(2.7)$$

Recall that D has a uniform interior and exterior sphere condition. Take a point $P\in\partial D$. We may assume that $P=(R,0,\dots,0)$, that D is contained in the annulus $B(R,\bar R)$ with center in $(2R,0,\dots,0)$ and $\bar R$ large, and that D contains the ball $B(R)$ with center in $(0,\dots,0)$. Note that $B(R,\bar R)$ and $B(R)$ are tangent to ∂D in P. If $u(x)$ is the solution of problem (2.1) in D, if $w(x)$ is the solution in the annulus $B(R,\bar R)$ and if $z(x)$ is the solution in B then, by the comparison principle, we have

$$z(x)\le u(x)\le w(x)\quad\forall x\in B.$$

Let $0<r<R$. If we take a point $x=(r,0,\dots,0)$ then, by using (2.4) and (2.7) it follows that

$$(1-\epsilon)^{1/(\gamma+1)}\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)}\delta^{2/(\gamma+1)}(x)$$
$$< u(x) < (1+\epsilon)^{2/(\gamma+1)}\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)}\delta^{2/(\gamma+1)}(x).$$

Since ϵ is arbitrary, the lemma follows.

THEOREM 2.2 *If $u(x)$ is a positive solution to problem* (2.1) *with $\gamma>1$, then there exists $\beta>0$ such that*

$$\left|\frac{u(x)}{\delta^{2/(\gamma+1)}(x)}-\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)}\right| < \beta\delta^{(\gamma-1)/(\gamma+1)}(x) \quad \forall x\in D,$$

where δ(x) *denotes the distance from x to ∂D.*

Proof Following [5], put

$$W(x)=b_0\delta^{2/(\gamma+1)}(x)+\beta\delta(x),$$

with

$$b_0=\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]^{1/(\gamma+1)},$$

and $0<\beta$ will be fixed later. Writing δ instead of $\delta(x)$, one finds

$$W_i=b_0\frac{2}{\gamma+1}\delta^{2/(\gamma+1)-1}\delta_i+\beta\delta_i.$$

Since $\delta_i\delta_i=1$ and $-\Delta\delta=(N-1)K=H$, K being the mean curvature of the level surfaces of $\delta(x)$, we find

$$-\Delta W=b_0^{-\gamma}\delta^{-2\gamma/(\gamma+1)}+b_0\frac{2}{\gamma+1}H\delta^{(1-\gamma)/(\gamma+1)}+\beta H. \tag{2.8}$$

On the other hand, by applying Taylor expansion to the function

$$f(t) = (b_0\delta^{2/(\gamma+1)} + t)^{-\gamma}$$

one finds

$$W^{-\gamma} < b_0^{-\gamma}\delta^{-2\gamma/(\gamma+1)} - \gamma b_0^{-\gamma-1}\beta\delta^{-1} + \frac{\gamma(\gamma+1)}{2} b_0^{-\gamma-2}\beta^2\delta^{-2/(\gamma+1)}. \tag{2.9}$$

We claim that, for $\delta(x) \leq \delta_0$ small and β large, we have

$$\begin{aligned} &-\gamma b_0^{-\gamma-1}\beta\delta^{-1} + \frac{\gamma(\gamma+1)}{2} b_0^{-\gamma-2}\beta^2\delta^{-2/(\gamma+1)} \\ &\quad < b_0\frac{2}{\gamma+1}H\delta^{(1-\gamma)/(\gamma+1)} + \beta H. \end{aligned} \tag{2.10}$$

Indeed, let us rewrite (2.10) as

$$\begin{aligned} &b_0^{-\gamma-2}\beta\frac{\gamma}{2}\left[-b_0 + \beta(\gamma+1)\delta^{(\gamma-1)/(\gamma+1)}\right] \\ &\quad < \beta\frac{\gamma}{2}b_0^{-\gamma-1} + b_0\frac{2}{\gamma+1}H\delta^{2/(\gamma+1)} + \beta H\delta. \end{aligned}$$

The left hand side can be made negative for $\delta(x) < \delta_0$ by choosing

$$\beta\delta_0^{(\gamma-1)/(\gamma+1)} = \frac{b_0}{\gamma+1}. \tag{2.11}$$

Now, decrease δ_0 and increase β until the right hand side is positive. This is possible because, by the smoothness of D, H is bounded near ∂D. The claim is proved.

By (2.8)–(2.10) it follows that

$$\Delta W + W^{-\gamma} < 0 \quad \text{on } \{x \in D\colon \delta(x) < \delta_0\}.$$

Obviously, $W(x) = u(x)$ on ∂D. Increase β and decrease δ_0 according to (2.11) until we have $W(x) > u(x)$ for $\delta(x) = \delta_0$. Observe that this is possible because of Lemma 2.1.

Hence, by the comparison principle for elliptic equations [7, Theorem 9.2] we find

$$u(x) < b_0\delta^{2/(\gamma+1)}(x) + \beta\delta(x), \tag{2.12}$$

for all $x \in D$ with $\delta(x) \le \delta_0$.

To complete the proof of the theorem, let

$$W(x) = b_0\delta^{2/(\gamma+1)}(x) - \beta\delta(x),$$

with b_0 and β as before. We find

$$-\Delta W = b_0^{-\gamma}\delta^{-2\gamma/(\gamma+1)} + b_0\frac{2}{\gamma+1}H\delta^{(1-\gamma)/(\gamma+1)} - \beta H. \tag{2.13}$$

By applying Taylor expansion to the function

$$f(t) = (b_0\delta^{2/(\gamma+1)} - t)^{-\gamma}$$

one finds

$$W^{-\gamma} > b_0^{-\gamma}\delta^{-2\gamma/(\gamma+1)} + \gamma b_0^{-\gamma-1}\beta\delta^{-1}. \tag{2.14}$$

We claim that, for $\delta(x) \le \delta_0$ small and β large we have

$$\gamma b_0^{-\gamma-1}\beta\delta^{-1} > b_0\frac{2}{\gamma+1}H\delta^{(1-\gamma)/(\gamma+1)} - \beta H. \tag{2.15}$$

Indeed, inequality (2.15) can be written as

$$\left(\gamma b_0^{-\gamma-1} + H\delta\right)\beta > b_0\frac{2}{\gamma+1}H\delta^{2/(\gamma+1)}.$$

The claim follows easily.

By (2.13)–(2.15) it follows that

$$\Delta W + W^{-\gamma} > 0 \quad \text{on } \{x \in D\colon \delta(x) < \delta_0\}.$$

If necessary, increase β and decrease δ_0 until we have $W(x) < u(x)$ for $\delta(x) = \delta_0$.

Again by the comparison principle we find

$$u(x) > b_0 \delta^{2/(\gamma+1)}(x) - \beta\delta(x), \tag{2.16}$$

for all $x \in D$ with $\delta(x) \leq \delta_0$.

The constants β and δ_0 in (2.12) and (2.16) can be taken with the same values. Therefore,

$$\left| \frac{u(x)}{\delta^{2/(\gamma+1)}(x)} - b_0 \right| < \beta\delta^{(\gamma-1)/(\gamma+1)}(x),$$

for all $x \in D$ such that $\delta(x) < \delta_0$. Increasing β again, the theorem follows.

By using Theorem 2.2 one finds easily that the gradient of $u(x)$ is unbounded near the boundary of D. For $\gamma = 1$, Theorem 2.2 fails. Now we give a direct estimate of the gradient in this case, improving a result of [13].

Consider the following problem:

$$\Delta u + p(x)u^{-1} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{2.17}$$

where D is a domain satisfying an interior sphere condition and $p(x)$ is a smooth function satisfying

$$0 < p_1 \leq p(x). \tag{2.18}$$

PROPOSITION 2.3 *If $u(x)$ is a solution to problem* (2.17) *with $p(x)$ satisfying* (2.18) *then the gradient of $u(x)$ is unbounded in each point of ∂D.*

Proof Let $z = z(r)$ be the solution of the following problem:

$$z'' + \frac{N-1}{r}z' + p_1 z^{-1} = 0, \quad 0 < r < R, \;\; z'(0) = 0, \;\; z(R) = 0. \tag{2.19}$$

By (2.19) we find $(r^{N-1}z')' < 0$, whence $z'(r) < 0$ in $(0, R)$.

Multiplying by z', Eq. (2.19) implies

$$z''z' + p_1 z^{-1} z' < 0.$$

Integrating on $(0, r)$ we find

$$\frac{(z')^2}{2} + p_1 \log \frac{z(r)}{z(0)} < 0,$$

whence,

$$(zz')^2 < 2p_1 z^2(r) \log \frac{z(0)}{z(r)}.$$

The latter inequality implies that

$$\lim_{r \to R} zz' = 0.$$

Let $r_0 \in (R/2, R)$ such that

$$-zz' < \frac{p_1 R}{4(N-1)} \quad \text{for } r_0 < r < R.$$

Using Eq. (2.19) once more we find, on (r_0, R)

$$z'' = \frac{1}{z}\left(-\frac{N-1}{r} zz' - p_1\right) < \frac{1}{z}\left(\frac{p_1 R}{4r} - p_1\right) < -\frac{p_1}{2z}.$$

Since $z' < 0$, from the latter inequality we find

$$z'' z' > -\frac{p_1}{2}\frac{z'}{z}.$$

Integration over (r_0, r) yields

$$(z'(r))^2 - (z'(r_0))^2 > p_1 \log \frac{z(r_0)}{z(r)}.$$

Finally, we find that

$$-z'(r) > \sqrt{p_1 \log \frac{z(r_0)}{z(r)}} \quad \text{in } (r_0, R). \tag{2.20}$$

Now consider problem (2.17) in D. We claim that the interior derivative of the solution u approaches $+\infty$ as $x \to \partial D$. Indeed, for $P_0 \in \partial D$, consider a ball $B \subset D$ and tangent to ∂D at P_0. Let z be the solution of problem (2.19) in such a ball. Since $p_1 \leq p(x)$, one finds

$$\Delta z + p(x) z^{-1} \geq 0 \quad \text{in } B.$$

By the comparison principle [7, Theorem 9.2] between the last inequality and Eq. (2.17) one finds that $u(x) \geq z(x)$ in B. As a consequence, if P is a point in B close to P_0 we have

$$\frac{u(P) - u(P_0)}{|P - P_0|} \geq \frac{z(P) - z(P_0)}{|P - P_0|}.$$

Using the last inequality together with (2.20), the proposition follows.

3. CONVEXITY

In this section, $D \subset \mathbb{R}^N$ is assumed to be bounded, smooth and convex. Let $u(x)$ be a positive solution to the problem

$$\Delta u + u^{-\gamma} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{3.1}$$

with $0 < \gamma$. We want to prove that the function $v = u^{(1+\gamma)/2}$ is concave.

Using Eq. (3.1), we find

$$\Delta v = -\frac{1}{v}\left[\frac{1-\gamma}{1+\gamma}|\nabla v|^2 + \frac{1+\gamma}{2}\right] \quad \text{in } D, \qquad v(x) = 0 \quad \text{on } \partial D. \tag{3.2}$$

We first show that the function

$$Q(x) = \frac{1-\gamma}{1+\gamma}|\nabla v|^2 + \frac{1+\gamma}{2} \tag{3.3}$$

is positive in D. This fact is trivial for $\gamma \leq 1$. Let us prove it for $1 < \gamma$.

LEMMA 3.1 *If $u(x)$ is a positive solution to problem* (3.1) *with* $\gamma > 1$, *and if* $v = u^{(1+\gamma)/2}$, *then the function* $Q(x)$ *defined as in* (3.3) *is positive in* D.

Proof We have

$$Q(x) = \frac{1-\gamma^2}{2} u^{\gamma-1} P(x),$$

with

$$P(x) = \frac{|\nabla u|^2}{2} + \frac{u^{1-\gamma}}{1-\gamma}.$$

It suffices to prove that, when $1 < \gamma$, $P(x) < 0$.

For $\epsilon > 0$, consider the solution $u(x)$ to the problem

$$\Delta u + u^{-\gamma} = 0 \quad \text{in } D, \qquad u = \epsilon \quad \text{on } \partial D.$$

The corresponding function

$$P(x) = \frac{|\nabla u|^2}{2} + \frac{u^{1-\gamma}}{1-\gamma}.$$

satisfies

$$P_i = u_k u_{ki} + u^{-\gamma} u_i.$$

Note that the summation convention over repeated indices is used. This equation together with Schwarz inequality yield

$$(P_i - u^{-\gamma} u_i)^2 = (u_k u_{ki})^2 \le |\nabla u|^2 u_{ki} u_{ki},$$

from which it follows that

$$u_{ki} u_{ki} \ge \frac{P_i - 2u^{-\gamma} u_i}{|\nabla u|^2} P_i + u^{-2\gamma}.$$

Here and in the sequel we often write u_i instead of u_{x^i}. Similarly for P_i.

Using this estimate as well as Eq. (3.1) we find

$$\Delta P + \frac{2u^{-\gamma}u_i - P_i}{|\nabla u|^2} P_i \geq 0.$$

By the classical maximum principle, P attains its maximum value either when $\nabla u = 0$ or on the boundary of D. Since D is convex, Hopf's boundary lemma prevents P from having its maximum value on ∂D (see [14]). Hence,

$$\frac{|\nabla u|^2}{2} + \frac{u^{1-\gamma}}{1-\gamma} \leq \frac{M_\epsilon^{1-\gamma}}{1-\gamma} < 0,$$

where M_ϵ denotes the maximum value of $u(x) = u_\epsilon(x)$. The lemma follows as $\epsilon \rightarrow 0$.

For discussing the concavity of solutions to Eq. (3.2), we use Korevaar function [9,11]

$$C(x,y) = 2v((x+y)/2) - v(x) - v(y), \quad x, y \in D. \tag{3.4}$$

The function $v(x)$ is concave in D if and only if $C(x,y) \geq 0$ in $\bar{D} \times \bar{D}$.

PROPOSITION 3.2 *If $v(x)$ is a positive solution to problem* (3.2) *then the function $C(x,y)$ defined as in* (3.4), *cannot have a negative minimum in D.*

Proof By Lemma 3.1, the function at the right hand side of Eq. (3.2) is negative. Moreover, such a function is increasing with respect to v. The proof follows easily by [9, Theorem 3.13, p. 116]. See also [8,11].

To get the positiveness of $C(x,y)$ on the boundary of $D \times D$, we prove the following

LEMMA 3.3 *If $v(x)$ is a positive solution to problem* (3.2) *and if $y \in \partial D$, then the function*

$$\psi(x) = 2v(z) - v(x), \quad \text{with } z = (x+y)/2$$

is non-negative in $\bar{D}$.

Proof If $x \in \partial D$, we have $\psi(x) = 2v(z) \geq 0$. If $x \in D$, by computation we find

$$\nabla\psi = \nabla v(z) - \nabla v(x), \qquad \Delta\psi = \tfrac{1}{2}\Delta v(z) - \Delta v(x). \tag{3.5}$$

Using (3.2), the latter equation yields

$$\Delta\psi = -\frac{1}{2v(z)}\left(A|\nabla v(z)|^2 + B\right) + \frac{1}{v(x)}\left(A|\nabla v(x)|^2 + B\right),$$

with $A = (1-\gamma)/(1+\gamma)$ and $B = (1+\gamma)/2$. Using (3.5), this equation can be rewritten as

$$\Delta\psi + a^i\psi_i = \frac{1}{2v(z)v(x)}\left(A|\nabla v(z)|^2 + B\right)\psi, \tag{3.6}$$

with suitable smooth functions a^i. By Lemma 3.1, the coefficient of ψ in (3.6) is positive. Hence, by the classical maximum principle, we infer that $\psi(x)$ attains its minimum value on the boundary of D. The lemma is proved.

COROLLARY 3.4 *If $v(x)$ is a positive solution to problem* (3.2), *then the function $C(x, y)$ defined as in* (3.4) *is non-negative on $\bar{D} \times \bar{D}$.*

Proof It follows from Proposition 3.2 and Lemma 3.3.

THEOREM 3.5 *If $v(x)$ is a positive solution to problem* (3.2) *in a convex domain D then it is strictly concave in D.*

Proof By (3.2) and Lemma 3.1, the function $w = -v$ satisfies

$$\Delta w = \frac{1}{w}(A|\nabla w|^2 + B) > 0,$$

with $A = (1-\gamma)/(1+\gamma)$ and $B = (1+\gamma)/2$. By Corollary 3.4, w is convex. Moreover, $w \to w(A|\nabla w|^2 + B)^{-1}$ is convex. By a theorem of Korevaar and Lewis [12], we conclude that the Hessian of v has a constant rank in D. Now, v attains its maximum in $\bar{D}$ on a compact subset $K \subset D$. It is well known in this case [4] that in any neighborhood U of K, there is a point

$P \in U$ where the Hessian of v is strictly negative. It follows that v is strictly concave in D.

COROLLARY 3.6 *If $u(x)$ is a positive solution to problem* (3.1) *then it attains its maximum value in D at a single point.*

Now we prove some sharpness results on convexity.

1. Let $D \subset \mathbb{R}^N$ be a convex domain, and let $u(x)$ be a positive solution to the problem

$$\Delta u + 1 = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{3.7}$$

We recall that the function

$$v(x) = u^{1/2}$$

is concave in D [9,10]. We prove that the exponent 1/2 is sharp.

Let $N = 2$, and let D be the triangle with vertex $(-1/\sqrt{3}, 0)$, $(1/\sqrt{3}, 0)$ and (0, 1). The function

$$u(x, y) = \frac{y}{4}\left[(1 - y)^2 - 3x^2\right]$$

solves problem (3.7) in this domain. Let $\phi(y) = 4u(0, y)$. We have

$$\phi(y) = y(1 - y)^2.$$

Of course, the function $\sqrt{\phi(y)}$ is concave in (0, 1). If $\alpha > \frac{1}{2}$, $(\phi(y))^\alpha$ is not concave near $y = 1$.

If $N > 2$, one can obtain the result in above by using the function

$$u(x) = \frac{x_N}{4}\left[(1 - x_N)^2 - \frac{3}{N - 1}\sum_{i=1}^{N-1} x_i^2\right].$$

2. Let $u > 0$ be a solution of the problem

$$\Delta u + \lambda_1 u = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D.$$

Such a function is the first eigenfunction of D. We know that $v = \log u$ is concave whenever D is convex.

Is there any $\epsilon > 0$ such that $v = u^\epsilon$ is concave for all convex domains? The answer is negative, as one can see by the following example. Let $N = 2$, and let

$$D = \{0 < r < a,\ 0 < \theta < \pi/m\}.$$

Here r and θ are polar coordinates, m is an integer and a is the first zero of the mth Bessel function. The first eigenfunction of D is

$$u(x, y) = J_m(r)\sin(m\theta),$$

$J_m(r)$ being the mth Bessel function. It is known that $J_m(r)$ behaves like r^m as $r \to 0$. Hence, we must take $\epsilon \leq 1/m$ if we want the function u^ϵ to be concave. Since m can be choosen arbitrarily large, the result follows.

3. Let $z(x)$ be a solution of the problem

$$\Delta z + z^{-1} = 0 \quad \text{in } B, \qquad z = 0 \quad \text{on } \partial B, \tag{3.8}$$

where B is a ball. By Theorem 3.5, $z(x)$ is concave. Let us show that, given $\epsilon > 0$, the function $v = z^{1/(1-\epsilon)}$ is not concave near ∂B. Indeed,

$$z = v^{1-\epsilon}, \qquad \nabla z = (1-\epsilon)v^{-\epsilon}\nabla v,$$
$$\Delta z = (1-\epsilon)v^{-\epsilon}\Delta v - \epsilon(1-\epsilon)v^{-\epsilon-1}|\nabla v|^2.$$

Substituting into Eq. (3.8), we find

$$v^{1-2\epsilon}\Delta v = \epsilon v^{-2\epsilon}|\nabla v|^2 - \frac{1}{1-\epsilon}. \tag{3.9}$$

Since

$$v^{-\epsilon}|\nabla v| = \frac{1}{1-\epsilon}|\nabla z|$$

and since, by (2.20), $|\nabla v| \to \infty$ as $x \to \partial B$, the right hand side of (3.9) becomes positive near the boudary of B. As a consequence, $\Delta v > 0$ near ∂B and therefore, v cannot be concave on B.

Similarly, one can show that, if $z(x)$ is a solution of the problem

$$\Delta z + z^{-\gamma} = 0 \quad \text{in } B, \qquad z = 0 \quad \text{on } \partial B,$$

with $\gamma > 1$ then, for any $\epsilon > 0$, the function $v = z^{(1+\gamma)/2+\epsilon}$ is not concave near ∂B.

References

[1] L. Andersson and P.T. Chruściel, Solutions of the constraint equation in general relativity satisfying "hyperbolic conditions", *Dissertationes Mathematicae* **355** (1996).

[2] C. Bandle and M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, *J. d'Anal. Math.* **58** (1992), 9–24.

[3] C. Bandle and M. Marcus, On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems, *Differential and Integral Equations* **11** (1998), 23–34.

[4] R. Basener, Nonlinear Cauchy-Riemann equations and q-pseudoconvexity, *Duke Math. J.* **43** (1976), 203–215.

[5] S. Berhanu and G. Porru, Qualitative and quantitative estimates for large solutions to semilinear equations, *Communications in Applied Analysis* (to appear).

[6] M.G. Crandall, P.H. Rabinowitz and L. Tartar, On a Dirichlet problem with a singular nonlinearity, *Comm. Part. Diff. Eq.* **2** (1977), 193–222.

[7] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, 1977.

[8] A. Greco and G. Porru, Convexity of solutions to some elliptic partial differential equations, *Siam J. Math. Anal.* **24** (1993), 833–839.

[9] B. Kawohl, Rearrangements and Convexity of Level Sets in PDE, Lectures Notes in Math. 1150, Springer-Verlag, Berlin Heidelberg, 1985.

[10] B. Kawohl, When are solutions to nonlinear elliptic boundary value problems convex?, *Comm. Part. Diff. Eq.* **10** (1985), 1213–1225.

[11] N.J. Korevaar, Convex solutions to nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.* **32** (1983), 603–614.

[12] N.J. Korevaar and J. Lewis, Convex solutions of certain elliptic equations have constant rank Hessian, *Arch. Rat. Mech. Anal.* **97** (1987), 19–32.

[13] A.C. Lazer and P.J. McKenna, On a singular nonlinear elliptic boundary value problem, *Proc. American. Math. Soc.* **111** (1991), 721–730.

[14] L.E. Payne and G.A. Philippin, Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surfaces and to surfaces of constant mean curvature, *J. Nonlinear Anal.* **3** (1979), 193–211.