A relaxation result for energies defined on pairs set-function and applications

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Abstract

We consider, in an open subset Ω of $\mathbb{R}^N$, energies depending on the perimeter of a subset $E \subset \Omega$ (or some equivalent surface integral) and on a function u which is defined only on E. We compute the lower semicontinuous envelope of such energies. This relaxation has to take into account the fact that in the limit, the "holes" $\Omega \setminus E$ may collapse into a discontinuity of u, whose surface will be counted twice in the relaxed energy. We discuss some situations where such energies appear, and give, as an application, a new proof of convergence for an extension of Ambrosio-Tortorelli's approximation to the Mumford-Shah functional.

1 Introduction

In this paper we consider energies defined on pairs function/open subset of $\mathbb{R}^n$, of the form

$$F(u, E) = \int_E f(\nabla u)\, dx + \int_{\partial E} \varphi(\nu)\, d\mathcal{H}^{n-1},$$

with interacting bulk and surface energies. Here E is though to be smooth enough (*e.g.*, with Lipschitz boundary) so that ∂E coincides $\mathcal{H}^{n-1}$-a.e. with the essential boundary of E; *i.e.*, with the interface between the 'interior' and 'exterior' of E, and $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Energies of this type arise in physical problems for example when dealing with small drops or thin films, when bulk and surface energies can be thought to be of the same order (see, *e.g.*, [7] for a variational problem set in this framework).

Functionals as F appear also in some disguised form in many problems related to variational models in Image Segmentation, such as that by Mumford and Shah [18, 19]. A particularly successful approach to deal with such problems has proven to be the application of the direct methods of the Calculus of Variations in the framework of the special functions of bounded variation to obtain existence and regularity results. In order to apply these existence results to Image Segmentation problems a further step is necessary, of approximating free-discontinuity energies, containing competing bulk and surface integrals, by energies to which numerical methods can

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be more easily applied. This has been done in many different ways, using elliptic energies with an additional variable, finite-difference energies, non-local integrals, etc. A commom pattern can be traced in all those approximations, that start from an *ansatz* on the desired form of an 'approximate solution'. While it is easily seen, by construction of the approximating energies, that the candidate approximate solutions give the desired limit energy, it is less immediate to check that this behaviour is 'optimal'. To prove this fact (in the language of Γ-convergence we would call this the 'liminf inequality') a crucial point is, given an arbitrary sequence of minimizers, to distinguish sets in which the approximating energies computed on these functions behave as 'bulk energies', and complementary sets which we may regard as 'blurred' discontinuity sets (typically these are sets where 'gradients are high'). This point can be rephrased as comparing the candidate approximating energies with an energy as F above defined on pairs function-set, whose form is in general dependent only on the 'target' free-discontinuity energy. In the case of the Mumford and Shah functional

$$MS(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S(u) \cap \Omega)$$

($S(u)$ denotes the set of discontinuity points of u), we often end up with energies of the form

$$\widetilde{F}(u, E) = a \int_{\Omega \setminus E} |\nabla u|^2 \, dx + b\mathcal{H}^{n-1}(\Omega \cap \partial E).$$

The 'liminf inequality' is then rephrased in terms of the lower semicontinuous envelope of $\widetilde{F}$, and an optimization on the constants a and b. This remark is essentially already contained in a paper by Bourdin and Chambolle [8], but therein it is not stated explicitly in terms of a relaxation result, and is obtained by applying more elaborated approximation results.

Note that at fixed E the functional $F(\cdot, E)$ is weakly lower semicontinuous on $W^{1,p}$, provided some standard convexity and growth conditions on f are required, and that, at fixed u, $F(u, \cdot)$ can be extended to a lower semicontinuous energy on sets with finite perimeter if φ is a norm. On the contrary, the functional defined on pairs (u, E) is not lower semicontinuous. Loosely speaking, if (u_j, E_j) is a sequence with equi-bounded energy and converging to some (u, E), the limit u may be discontinuous on $\Omega \setminus E$, and its set of discontinuity points $S(u)$ may be the limit of a portion of ∂E_j. In this paper we compute the lower-semicontinuous envelope of functionals F in a direct way, and characterize it in the whole class of pairs (u, E), where E is a set of finite perimeter and u is such that $u(1 - \chi_E)$ belongs to the space $GSBV(\Omega; \mathbb{R}^m)$ of Ambrosio and De Giorgi's generalized special function of bounded variation (Theorem 2). We show that it takes the form

$$\overline{F}(u, E) = \int_{\Omega \setminus E} f(\nabla u) \, dx + \int_{\partial^* E \cap \Omega} \varphi(\nu) \, d\mathcal{H}^{n-1} + \int_{S(u) \cap \Omega \cap E_0} \widetilde{\varphi}(\nu) \, d\mathcal{H}^{n-1},$$

where E_0 denotes the points of density zero of E and $\widetilde{\varphi}(\nu) = \varphi(\nu) + \varphi(-\nu)$. The result is proved by providing separately a lower bound and an upper bound. The lower bound is obtained by reducing to the one dimensional case through Ambrosio's slicing techniques. The crucial technical point is here to check that, loosely speaking, for almost all directions, the traces of one-dimensional sections of E have density 0 on the jump set $S(u)$ precisely on the

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intersection of E_0 and $S(u)$ (Lemma 5). The upper bound is obtained by a direct construction if $S(u)$ is smooth enough, and by approximation in the general case. We show two ways to obtain such an approximation. The first one (Lemma 12) consists in applying a 'strong SBV approximation' result by Braides and Chiadò Piat [11] to a suitable modification of the function $u(1-\chi_E)$, and then construct optimal pair from this sequence of approximating functions. The second one (Remark 13) uses a mollification argument for approximating the set E first, and then the coarea formula to select approximating sets, on which then to obtain an approximation of the target u.

As applications of this result, we first give an approximation of the Mumford-Shah energy by a sequence of functions defined on pairs set-functions, by noting that $\overline{F}(u, E)$ reduces to a functional on $GSBV(\Omega)$ when $E = \emptyset$, so that $E_0 = \Omega$. Subsequently, we give a different proof of Ambrosio-Tortorelli's elliptic approximation result [5]. At the same time we provide a generalization by replacing their one-well potential by perturbed double-well potentials, which give a different smoother form of the optimal profile of solutions. This result formalizes a method that has already been used by Braides and March [13] to obtain minimizing sequences bounded in H^2 for problems in which a term penalizing the curvature of the discontinuity set is added. Finally, we outline applications to the study of crystalline films on a substrate and to water waves. As an interesting additional object for perspective work, we mention the interaction with boundary conditions and microgeomerty, that would lead to interesting problems of homogenization, as shown by Alberti and De Simone [1] already in the case when no bulk term is present. As a final bibliographical information, the results in this paper concerning the Ambrosio-Tortorelli approximation previously circulated in the form of the manuscript [14].

2 The relaxation result

Let $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition

$$c_1(|\xi|^p - 1) \leq f(\xi) \leq c_2(1 + |\xi|^p)$$

for some positive constants c_1 and c_2, and $p > 1$, and let $\varphi : \mathbb{R}^n \to [0, +\infty)$ be a convex and positively homogeneous function of degree one, with $\varphi(z) > 0$ if $z \neq 0$. For every $u \in L^1(\Omega)$ and E measurable subset of Ω, we define

$$
F(u, E) = \begin{cases} \displaystyle\int_{\Omega \setminus E} f(\nabla u) \, dx + \int_{\Omega \cap \partial E} \varphi(\nu_E) \, d\mathcal{H}^{n-1} & \text{if } u \in H^1(\Omega; \mathbb{R}^m) \text{ and } \partial E \text{ Lipschitz} \\ \\ +\infty & \text{otherwise,} \end{cases}
$$

where ν_E denotes the interior normal to E.

We will prove the following relaxation theorem, in whose statement we adopt standard notation for generalized functions of bounded variations (see [4]); in particular, $S(u)$ and ν_u denote the set of essential discontinuity points of u and its measure-theoretical normal, respectively. We say that $E_j \to E$ if $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$; $M(\Omega)$ denotes the family of measurable subsets of Ω and $GSBV(\Omega; \mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued generalized special functions of bounded variation on Ω.

Theorem 1. *The lower-semicontinuous envelope of the functional F, with respect to the $L^1(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^m)$ topology, is the functional $\overline{F}: L^1(\Omega;\mathbb{R}^m) \times \mathcal{M}(\Omega) \to [0,+\infty]$ defined as:*

$$
\overline{F}(u,E) = \begin{cases} \displaystyle\int_{\Omega\setminus E} f(\nabla u)dx + \int_{\Omega\cap\partial^* E} \varphi(\nu_E)\, d\mathcal{H}^{n-1} + \int_{\Omega\cap S(u)\cap E_0} (\varphi(\nu_u) + \varphi(-\nu_u))\, d\mathcal{H}^{n-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad if\ \ u\chi_{E_0} \in GSBV(\Omega;\mathbb{R}^m) \\ \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\quad otherwise, \end{cases}
$$

where $\partial^* E$ is the reduced boundary of E and E_0 is the set of the points where E has density 0.

Furthermore, if $0 < |E| \le |\Omega|$ then for every pair (u,E) there exists a recovery sequence (u_j, E_j) such that $\lim_j F(u_j, E_j) = \overline{F}(u,E)$ and $|E_j| = |E|$.

The proof of Theorem 1 will be given in Sections 3 and 4, by showing, respectively, a lower and an upper inequality. The proof will be given in detail in the case when $m = 1$, $f(\nabla u) = a|\nabla u|^2$ and $\varphi(z) = b|z|/2$ (so that $\varphi(\nu) + \varphi(-\nu) = b$ if $|\nu| = 1$) not to overburden notation, while the extension of the proof to the general case is given at the end of each section.

3 The lower inequality

For every $u \in L^1(\Omega)$ and E measurable subset of Ω, we define

$$
F(u,E) = \begin{cases} \displaystyle a\int_{\Omega\setminus E} |\nabla u|^2 dx + \frac{b}{2}\mathcal{H}^{n-1}(\Omega\cap\partial E) & if\ \ u \in H^1(\Omega)\ and\ \partial E\ \ \text{Lipschitz} \\ \\ +\infty & \text{otherwise}, \end{cases}
$$

where a, b are positive parameters. For this choice of F Theorem 1 reads as follows.

Theorem 2. *The lower-semicontinuous envelope of the functional F with respect to the $L^1(\Omega) \times L^1(\Omega)$ topology, is the functional $\overline{F}: L^1(\Omega) \times \mathcal{M}(\Omega) \to [0,+\infty]$ defined as:*

$$
\overline{F}(u,E) = \begin{cases} \displaystyle a\int_{\Omega\setminus E} |\nabla u|^2 dx + \frac{b}{2}\mathcal{H}^{n-1}(\partial^* E \cap \Omega) + b\,\mathcal{H}^{n-1}(S(u)\cap\Omega\cap E_0) \\ \qquad\qquad\qquad\qquad\qquad\quad if\ \ u\chi_{E_0} \in GSBV(\Omega) \\ \\ +\infty \qquad\qquad\qquad\qquad\quad otherwise, \end{cases}
$$

where $\partial^* E$ is the reduced boundary of E and E_0 is the set of the points where E has density 0.

In order to prove a lower bound for the relaxation of F we will use the 'slicing' method (see [10] Chapter 15) that allows to reduce to the study of energies defined on one-dimensional sections. To this end we will need to define as customary the 'localized' versions of our energies as follows. For every $u \in L^1(\Omega)$, E measurable subset of Ω, and for every A open subset of Ω,

we define

$$
F(u, E; A) = \begin{cases} a \displaystyle\int_{A \setminus E} |\nabla u|^2 dx + \frac{b}{2} \mathcal{H}^{n-1}(\partial E \cap A) & \text{if } u \in H^1(\Omega) \text{ and } \partial E \text{ Lipschitz} \\[3em] +\infty & \text{otherwise.} \end{cases}
$$

Similarly, we define $\overline{F}(u, E; A)$.

Setting, for every $u \in L^1(\Omega)$, $E \in \mathcal{M}(\Omega)$ and A open subset of Ω :

$$
F'(u, E; A) = \inf\big\{ \liminf_{j \to \infty} F(u_j, E_j; A) \ : \ u_j \to u, \ E_j \to E \ \text{in } L^1(\Omega) \big\},
$$

we show the following inequality:

$$
F'(u, E; A) \geq \overline{F}(u, E; A). \tag{1}
$$

This corresponds to proving the following

Proposition 3. *Let* $u \in L^1(\Omega), E \in \mathcal{M}(\Omega)$ *and* A *be an open subset of* Ω. *For every sequence* $\{(u_j, E_j)\}$ *in* $L^1(\Omega) \times \mathcal{M}(\Omega)$, *such that* $u_j \to u$ *and* $E_j \to E$ *in* $L^1(\Omega)$:

$$
\liminf_{j \to +\infty} F(u_j, E_j; A) \geq \overline{F}(u, E; A).
$$

Proof. We start by stating the result concerning the one-dimensional functionals.

- The lower inequality in the 1-dimensional case

Let $I \subset \mathbb{R}$ be an interval, $u \in L^1(I)$ and let E be a measurable subset of I. Let $\{u_j\}$ be a sequence in $L^1(I)$, and let $\{E_j\}$ be a sequence of measurable subsets of I, such that

$$
u_j \to u \ \text{in} \ L^1(I) \quad \text{and} \quad E_j \to E \quad \text{as} \ j \to +\infty.
$$

We can assume $\sup_j F(u_j, E_j; I) \leq c$; then, $u_j \in H^1(I)$, the number of the connected components of E_j is uniformly bounded, and we can find a finite set $S = \{s_1, \ldots, s_L\} \subset E_0$, a finite set of intervals $\{[a_1, b_1], \ldots, [a_M, b_M]\}$ and a subsequence (not relabelled) such that

$$
\forall \, \eta > 0 \ \ \exists \overline{j} \in \mathbb{N} \ \ \forall \, j \geq \overline{j} : \ E_j \subset \bigcup_{i=1}^{L} [s_i - \eta, s_i + \eta] \cup \bigcup_{i=1}^{M} [a_i - \eta, b_i + \eta].
$$

In particular, we can assume $E = \bigcup_{i=1}^{M}[a_i, b_i]$. Then, setting $\Lambda_\eta = \bigcup_{i=1}^{L}[s_i - \eta, s_i + \eta]$ and $E_\eta = \bigcup_{i=1}^{M}[a_i - \eta, b_i + \eta]$:

$$
\int_{I \setminus (\Lambda_\eta \cup E_\eta)} |u_j'|^2 dx \leq c;
$$

by the arbitrariness of $\eta > 0$, it follows that $u \in H^1((I \setminus E) \setminus S)$, hence $u \in SBV(I \setminus E)$ and $S(u) \cap E_0 \subset S$. This allows to conclude

$$
\liminf_{j \to +\infty} F(u_j, E_j; I) \geq \overline{F}(u, E; I). \tag{2}
$$

It is easy to check that for every J open subset of I :

$$
\liminf_{j \to +\infty} F(u_j, E_j; J) \geq \overline{F}(u, E; J). \tag{3}
$$

- The lower inequality in the n-dimensional case.

We recall some definitions and properties related to the slicing procedure. For every $\xi \in S^{n-1}$ let Π_ξ be the $(n-1)$-dimensional linear subspace of $\mathbb{R}^n$ orthogonal to ξ. If $B \subseteq \mathbb{R}^n$ then B^ξ be the orthogonal projection of B on Π_ξ. For every $y \in B^\xi$ set $B^{\xi y} = \{t \in \mathbb{R} : y + t\xi \in B\}$. If $f : B \to \mathbb{R}$ let $f^{\xi y} : B^{\xi,y} \to \mathbb{R}$ be defined by $f^{\xi,y}(t) = f(y + t\xi)$.

The following results hold [4]:

(i) Let $u \in GSBV(\Omega)$ and let $D^k u$ stand for any of D^a, D^j or D^c (the absolutely continuous, jump or Cantor part of the derivative). Then, for every $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Omega^\xi$ we have $u^{\xi y} \in GSBV(\Omega^{\xi y})$; moreover, denoting by $\langle D^k u, \xi \rangle$ the component of $D^k u$ along ξ, the following representation holds:

$$\int_B \langle D^k u, \xi \rangle = \int_{B^\xi} D^k u^{\xi y}(B^{\xi y}) d\mathcal{H}^{n-1}(y).$$

(ii) Let $u \in L^1(\Omega)$; assume that $u^{\xi y} \in SBV(\Omega^{\xi y})$ for every ξ in a basis of $\mathbb{R}^n$ and for a.e. $y \in \Omega^\xi$, and that

$$\int_{\Omega^\xi} |Du^{\xi y}|(\Omega^{\xi y}) d\mathcal{H}^{n-1}(y) < +\infty.$$

Then $u \in SBV(\Omega)$.

Before proceeding in the proof we recall a result concerning the supremum of a family of measures (see [10] Lemma 15.2)

Proposition 4. *Let Ω be an open subset of $\mathbb{R}^n$ and μ a finite, positive set function defined on the family of open subsets of Ω. Let λ be a positive Borel measure on Ω, and $\{g_i\}_{i \in \mathbb{N}}$ a family of positive Borel functions on Ω. Assume that $\mu(A) \geq \int_A g_i\, d\lambda$ for every A and i, and that $\mu(A \cup B) \geq \mu(A) + \mu(B)$ whenever $A, B \subset\subset \Omega$ and $\overline{A} \cap \overline{B} = \emptyset$ (superadditivity). Then $\mu(A) \geq \int_A \left(\sup_i g_i \right) d\lambda$ for every A.*

Moreover, we will use the following property.

Lemma 5. *Let $\Gamma \subset E_0$ be a $(n-1)$-rectifiable subset, $\xi \in S^{n-1}$ such that ξ is not orthogonal to the normal ν_Γ to Γ at any point of Γ; then, for almost every $y \in \Pi_\xi$, for every $t \in \Gamma^{\xi y}$, $E^{\xi y}$ has density 0 in t.*

Proof of Lemma 5. Since Γ is contained in a countable union of C^1-hypersurfaces, up to localization on one of those surfaces and a deformation argument, we can assume $\Gamma \subset \Pi_\xi$.

We set:

$$\Gamma_k = \left\{ y \in \Gamma \ : \ \Theta(0, E^{\xi y}) \geq \frac{1}{k} \right\}, \quad \Gamma^+ = \bigcup_{k \in \mathbb{N}} \Gamma_k,$$

where $\Theta(0, E^{\xi y})$ denotes the (one-dimensional) density of the set $E^{\xi y}$ in 0.

Let us assume by contradiction, that $\mathcal{H}^{n-1}(\Gamma^+) > 0$. Then, there exists k such that $\mathcal{H}^{n-1}(\Gamma^k) > 0$. Since $\Gamma_k \subset E_0$, Lebesgue's dominated convergence theorem gives (up to subsequences):

$$\lim_{\varrho \to 0} \int_{\Gamma_k} \frac{|E \cap Q_\varrho^\xi(x)|}{\varrho^n} d\mathcal{H}^{n-1}(x) = 0.$$

Then

$$
\begin{aligned}
0 &= c(\xi) \liminf_{\varrho \to 0} \frac{1}{\varrho^{n-1}} \int_{\Gamma_k} \int_{\Gamma_k \cap Q_\varrho^\xi(x)} \frac{|E^{\xi y} \cap (-\varrho/2, \varrho/2)|}{\varrho} \, dy \, dx \\
&= c(\xi) \liminf_{\varrho \to 0} \frac{1}{\varrho^{n-1}} \int_{\Gamma_k} \int_{\Gamma_k \cap Q_\varrho^\xi(y)} \frac{|E^{\xi y} \cap (-\varrho/2, \varrho/2)|}{\varrho} \, dx \, dy \\
&= c(\xi) \liminf_{\varrho \to 0} \int_{\Gamma_k} \Big(\frac{1}{\varrho^{n-1}} |\Gamma_k \cap Q_\varrho^\xi(y)| \Big) \frac{|E^{\xi y} \cap (-\varrho/2, \varrho/2)|}{\varrho} \, dy \\
&\geq c(\xi) \int_{\Gamma_k} \liminf_{\varrho \to 0} \frac{|E^{\xi y} \cap (-\varrho/2, \varrho/2)|}{\varrho} \, dy \geq c(\xi) \frac{\mathcal{H}^{n-1}(\Gamma_k)}{k},
\end{aligned}
$$

where $c(\xi)$ is a positive constant depending on the angle between ξ and ν_Γ; this gives the contradiction. $\qquad\square$

Now we apply the slicing method to complete the proof of Proposition 3.

Let ϕ and $\overline{\phi}$ denote the one dimensional versions of the functionals F and $\overline{F}$, respectively. For every $\xi \in S^{n-1}$ we define $F^\xi : L^1(\Omega) \times \mathcal{M}(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ as

$$
F^\xi(v, B; A) = \int_{\Pi_\xi} \phi(v^{\xi, y}, B^{\xi, y}; A^{\xi, y}) \, d\mathcal{H}^{n-1}(y).
$$

Note that $F \geq F^\xi$ for every ξ.

An application of the Fatou Lemma and the one dimensional inequality (3) give

$$
F'(u, E; A) \geq \overline{F}^\xi(u, E; A),
$$

where, for $v \in L^1(\Omega)$ and $B \in \mathcal{P}(\Omega)$:

$$
\overline{F}^\xi(v, B; A) = \int_{\Pi_\xi} \overline{\phi}(v^{\xi y}, B^{\xi y}; A^{\xi y}) \, d\mathcal{H}^{n-1}(y).
$$

Thus, if $F'(u, E; A)$ is finite, it follows that $E^{\xi y}$ is a finite union of intervals which we can suppose closed, and that $u^{\xi y} \in SBV(A^{\xi y} \setminus E^{\xi y})$ for a.a. y in Π_ξ; moreover

$$
\int_{\Pi_\xi} \Big(\int_{A^{\xi y} \setminus E^{\xi y}} |(u^{\xi y})'|^2 dt + \mathcal{H}^0((S(u^{\xi y}) \cap (E^{\xi y})_0 \cap A^{\xi y})) \Big) d\mathcal{H}^{n-1}(y) < +\infty.
$$

Then, if in addition $u \in L^\infty(\Omega)$, we get

$$
\int_{\Pi_\xi} |D(u^{\xi y})| (A^{\xi, y} \setminus E^{\xi y}) d\mathcal{H}^{n-1}(y) < +\infty.
$$

Recalling (ii), we deduce that, assuming $u \in L^\infty(\Omega)$, if $F'(u, E; A)$ is finite, then $u\chi_{E_0} \in SBV(A)$. A truncation argument allows to conclude that $F'(u, E; A)$ is finite only if $u\chi_{E_0} \in GSBV(A)$; in order to conclude applying (i), we need to prove that $S(u^{\xi y}) \cap (E_0)^{\xi y} \subset (S(u^{\xi y}) \cap E^{\xi y})_0$; this follows from Lemma 5, with $\Gamma = S(u) \cap E_0 \cap \{\langle \xi, \nu_u \rangle \neq 0\}$ Now, recalling (i), we get:

$$
\overline{F}^\xi(u, E; A) = a \int_A |\langle \nabla u, \xi \rangle|^2 dx + b \int_{A \cap S(u) \cap E_0} |\langle \nu_u, \xi \rangle| d\mathcal{H}^{n-1} + \frac{b}{2} \int_{A \cap \partial^* E} |\langle \nu_E, \xi \rangle| d\mathcal{H}^{n-1}.
$$

Since, if $u\chi_{E_0} \in GSBV(A)$, the set function $F'(u, E; \cdot)$ is superadditive on disjoint open sets, an application of Proposition 4 with $\lambda = \mathcal{L}^n + \mathcal{H}^{n-1} \llcorner (S(u) \cap E_0) + \mathcal{H}^{n-1} \llcorner \partial^* E$, and

$$g_i(x) = \begin{cases} a|\langle \nabla u, \xi_i \rangle|^2 & \text{if } x \in \Omega \setminus \big((S(u) \cap E_0) \cup \partial^* E\big) \\ b|\langle \nu_u, \xi_i \rangle| & \text{if } x \in S(u) \cap E_0 \\ \frac{b}{2}|\langle \nu_E, \xi_i \rangle| & \text{if } x \in \partial^* E, \end{cases}$$

where $\{\xi_i\}$ is a dense sequence in S^{n-1} such that $\langle \xi_i, \nu_u \rangle \neq 0$ $\mathcal{H}^{n-1}$-a.e. on $S(u) \cap E_0$, gives

$$F'(u, E; A) \geq \overline{F}(u, E; A)$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Remark 6. The same proof allows to treat the case when $m \geq 1$

$$F(u, E; A) = \begin{cases} c\displaystyle\int_{A\setminus E} \|\nabla u\|^p dx + \int_{\partial E \cap A} \varphi(\nu_E) d\mathcal{H}^{n-1} & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m)\,, \ \partial E \text{ Lipschitz} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

where $\|\nabla u\| = \sup\{\nabla u \xi : |\xi| = 1\}$ and c is a positive constant, obtaining the lower inequality with $\overline{F}$ as in the thesis of Theorem 1. The necessary modifications to the slicing procedure are standard and can be found in [9] Section 4.1.2.

Remark 7. For general quasiconvex f as in Theorem 1 we may consider the lower semicontinuous functional [4] on $GSBV(\Omega; \mathbb{R}^m)$ defined by

$$G(v; A) = \int_A f(\nabla v)\, dx + c\mathcal{H}^{n-1}(S(v)).$$

If $u_j \to u$ and $E_j \to E$ we have, setting $v_j = u_j(1 - \chi_{E_j})$ and $v = u(1 - \chi_E)$

$$\begin{aligned} \liminf_j F(u, E; A) &\geq \liminf_j \Big(G(v_j, A) - |A \cap E_j| f(0) \Big) \\ &\geq G(v, A) - |A \cap E| f(0) \\ &= \int_A f((1 - \chi_E)\nabla u)\, dx + c\mathcal{H}^{n-1}(S(v)) - |A \cap E| f(0) \\ &\geq \int_{A\setminus E} f(\nabla u)\, dx. \end{aligned}$$

Note that this identification of F with G gives a lower bound that is optimal for the bulk term but not for the surface energy.

Remark 8 (Proof of the lower bound in the general case). Let now F be defined as in Theorem 1 and let u and E be such that $u\chi_{E_0} \in GSBV(\Omega; \mathbb{R}^m)$ and $F'(u, E; A) < +\infty$. By the growth conditions on f and Remark 6 we then have

$$F'(u, E; A) \geq \int_{A \cap \partial^* E} \varphi(\nu_E)\, d\mathcal{H}^{n-1} + \int_{A \cap S(u) \cap E_0} (\varphi(\nu_u) + \varphi(-\nu_u))\, d\mathcal{H}^{n-1};$$

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by Remark 7 on the other hand we obtain

$$F'(u, E; A) \geq \int_{A \setminus E} f(\nabla u) \, dx.$$

We can define apply Proposition 4 with $\mu(A) = F'(u, E; A)$, the measure λ defined by $\lambda(A) = |A| + \mathcal{H}^{n-1}(A \cap (S(u) \cap E_0)) + \mathcal{H}^{n-1}(A \cap \partial^* E)$, $g_1(x) = f(\nabla u) \chi_{E_0 \setminus S(u)}$ and $g_2(x) = \varphi(\nu_E) \chi_{\partial^* E} + (\varphi(\nu_u) + \varphi(-\nu_u)) \chi_{S(u) \cap E_0}$, to obtain the lower inequality in the general case.

4 The upper inequality

In order to give an upper estimate, we introduce, for every $u \in L^1(\Omega)$, $E \in \mathcal{M}(\Omega)$ and A open subset of Ω :

$$F''(u, E; A) = \inf\left\{\limsup_{j \to +\infty} F(u_j, E_j; A) \;:\; u_j \to u, \; E_j \to E \;\text{ in } L^1(\Omega), \right\}.$$

The proof of Theorem 2 is complete if we show that the following inequality holds for every $u \in L^1(\Omega)$, $E \in \mathcal{M}(\Omega)$ and A open subset of Ω :

$$F''(u, E; A) \leq \overline{F}(u, E; A). \tag{4}$$

It is clearly sufficient to prove the following:

Proposition 9. *Let* $u \in L^1(\Omega), E \in \mathcal{M}(\Omega)$ *such that* $u\chi_{E_0} \in GSBV(\Omega)$. *Then, there exists a sequence* $\{(u_j, E_j)\} \in H^1(\Omega) \times \mathcal{M}(\Omega)$, *with* ∂E_j *of class* C^∞, *such that* $u_j \to u$, $E_j \to E$ *in* $L^1(\Omega)$, *and*

$$\limsup_{j \to +\infty} F(u_j, E_j) \leq \overline{F}(u, E).$$

In order to construct the recovery sequence, let us recall the definition of *strong convergence in* SBV^p, introduced in [11], and an approximation lemma for SBV^p with piecewise C^1 functions.

Definition 10. *([11]) Let* $\{u_j\}$ *be a sequence of functions in* SBV^p. *We say that* u_j *converges strongly to* u *in* SBV^p *if*

- $u_j \to u$ *in* $L^1(\Omega)$,

- $\nabla u_j \to \nabla u$ *strongly in* $L^p(\Omega; \mathbb{R}^n)$,

- $\mathcal{H}^{n-1}(S(u_j) \Delta S(u)) \to 0$,

- $\displaystyle\int_{S(u_j) \cup S(u)} (|u_j^+ - u^+| + |u_j^- - u^-|) d\mathcal{H}^{n-1} \to 0$

(we choose the orientation $\nu_{u_j} = \nu_u$ $\mathcal{H}^{n-1}$*-a.e. on* $S(u_j) \cap S(u)$; *recall that if* $v \in BV(\Omega)$ *then we set* $v^+ = v^- = \widetilde{v}$ *on* $\Omega \setminus S(v)$)

Lemma 11. *([11]) If $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$ then there exists a sequence $\{u_j\}$ in $SBV^p(\Omega) \cap L^\infty(\Omega)$ with $\|u_j\|_\infty \leq \|u\|_\infty$, strongly converging to u in SBV^p, such that for each $j \in \mathbb{N}$ there exists a closed rectifiable set R_j such that $u_j \in C^1(\Omega \setminus R_j)$. Moreover, R_j can be chosen so that its Minkowski content coincides with $\mathcal{H}^{n-1}(R_j)$; i.e.,*

$$\mathcal{H}^{n-1}(R_j) = \lim_{\rho \to 0^+} \frac{1}{2\rho} |\{x : \operatorname{dist}(x, R_j) < \rho\}|.$$

The proof of this Lemma in [11] consists in finding first a compact set $K \subset S(u)$ such that $\mathcal{H}^{n-1}(S(u) \setminus K) << 1$ (which will be the main part of R_j) and then then approximating u on $\Omega \setminus K$ by a Mumford-Shah type functional.

In the following $M^{n-1}(B)$ stands for the Minkowski content of a set B. As an intermediate step in the construction of the recovery sequence $\{(u_j, E_j)\}$, we apply the approximation result of Lemma 11 to prove the following:

Lemma 12. *Let E be a set of finite perimeter with $E = \Omega \setminus E_0$ and $u \in L^\infty(\Omega)$, such that $u\chi_{E_0} \in SBV(\Omega)$. Then, there exist a sequence of closed rectifiable sets $\{S_j\}$, a sequence of measurable subsets $\{F_j\}$, with ∂F_j closed rectifiable and Lipschitz in $\Omega \setminus S_j$, and a sequence of functions $\{w_j\}$ in $SBV(\Omega) \cap C^1(\Omega \setminus (S_j \cup \partial F_j))$ such that*

- $w_j \to u$ *and* $F_j \to E$ *in* $L^1(\Omega)$;

- $\mathcal{H}^{n-1}(S_j \setminus (S(w_j) \cap (F_j)_0)) = o(1)_{j \to +\infty}$;

- $\mathcal{H}^{n-1}(\partial F_j) = M^{n-1}(\partial F_j), \quad \mathcal{H}^{n-1}(S_j) = M^{n-1}(S_j)$;

- $\limsup_{j \to +\infty} \overline{F}(w_j, F_j) \leq \overline{F}(u, E)$.

Proof. We can suppose $\|u\|_\infty = 1$; we set

$$\overline{u} := \begin{cases} u & \text{in } \Omega \setminus E \\ 2 & \text{in } E. \end{cases}$$

From Lemma 11, for every j there exist a closed rectifiable set R_j and a function $\overline{u}_j \in C^1(\Omega \setminus R_j)$, such that $\overline{u}_j$ strongly converges to $\overline{u}$, and $\mathcal{H}^{n-1}(R_j \setminus S(\overline{u}_j)) \to 0$.

Setting $v_j = (\overline{u}_j \vee 1) \wedge 2$ and $v = (\overline{u} \vee 1) \wedge 2$, it follows that the sequence $\{v_j\}$ strongly converges to v; in particular,

$$|Dv_j|(\Omega) = \int_\Omega |\nabla v_j| \, dx + \int_{S(v_j)} |v_j^+ - v_j^-| \, d\mathcal{H}^{n-1} \to \int_\Omega |\nabla v| \, dx + \int_{S(v)} |v^+ - v^-| \, d\mathcal{H}^{n-1},$$

and

$$|Dv|(\Omega) = \mathcal{H}^{n-1}(S(v)) = \mathcal{H}^{n-1}(\partial^* E).$$

The coarea formula gives, for A open subset of Ω:

$$\mathcal{H}^{n-1}(\partial^* E \cap A) = \lim_{j \to +\infty} \int_1^2 \mathcal{H}^{n-1}(\partial^* E_j^t \cap A) \, dt$$

where $E_j^t = \{x \in A \; : \; \overline{u}_j \geq t\}$. Fixing $\delta > 0$, there exists $t_j \in (1, 2 - \delta)$ such that, setting $F_j = E_j^{t_j}$,

$$\mathcal{H}^{n-1}(\partial F_j \cap A) \leq \frac{1}{1 - \delta} \mathcal{H}^{n-1}(\partial E \cap A) \tag{5}$$

and ∂F_j is Lipschitz in $\Omega \setminus R_j$.

Now, we define the sequence $\{w_j\}$ as:

$$w_j := \overline{u}_j \chi_{(F_j)_0};$$

it follows that $w_j \in C^1(\Omega \setminus (R_j \cup \partial F_j))$, and setting $S_j = \overline{R_j \cap (F_j)_0}$ we get

$$\mathcal{H}^{n-1}(S_j \setminus (S(w_j) \cap (E_j)_0)) = o(1)_{j \to +\infty}.$$

Moreover:

$$\int_\Omega |\nabla u|^2 \, dx \leq \liminf_{j \to +\infty} \int_\Omega |\nabla w_j|^2 \, dx \leq \liminf_{j \to +\infty} \int_\Omega |\nabla \overline{u}_j|^2 \, dx = \int_\Omega |\nabla \overline{u}|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx,$$

and this implies the convergence

$$\nabla w_j \to \nabla u \quad \text{in} \quad L^2(\Omega). \tag{6}$$

The strong convergence $\overline{u}_j \to \overline{u}$ entails:

$$\mathcal{H}^{n-1}(\partial E) + \mathcal{H}^{n-1}(S(\overline{u}) \setminus E) = \mathcal{H}^{n-1}(S(\overline{u}))$$

$$\begin{aligned}
&= \lim_{j \to +\infty} \mathcal{H}^{n-1}(S(\overline{u}_j)) \\
&= \lim_{j \to +\infty} \left(\mathcal{H}^{n-1}(S(\overline{u}_j) \setminus F_j) + \mathcal{H}^{n-1}(S(\overline{u}_j) \cap F_j) \right) \\
&= \lim_{j \to +\infty} \left(\mathcal{H}^{n-1}(S(w_j) \setminus F_j) + \mathcal{H}^{n-1}(S(\overline{u}_j) \cap F_j) \right)
\end{aligned} \tag{7}$$

Again from the strong convergence $\overline{u}_j \to \overline{u}$ we obtain, in particular:

$$\lim_{j \to +\infty} \int_{\partial E \setminus (S(\overline{u}_j) \cap F_h)} (|1 - (\overline{u}_j)^-| + |2 - (\overline{u}_j)^+|) \, d\mathcal{H}^{n-1} = 0;$$

since $|1 - (\overline{u}_j)^-| + |2 - (\overline{u}_j)^+| \geq \delta$ in $\partial E \setminus (S(\overline{u}_j) \cap F_j)$, it follows that

$$\lim_{j \to +\infty} \mathcal{H}^{n-1}(\partial E \setminus (S(\overline{u}_j) \cap F_j)) = 0.$$

Then, we obtain

$$\begin{aligned}
\mathcal{H}^{n-1}(\partial E) &= \lim_{j \to +\infty} \mathcal{H}^{n-1}(\partial E \cap (S(\overline{u}_j) \cap F_j)) \\
&\leq \liminf_{j \to +\infty} \mathcal{H}^{n-1}(S(\overline{u}_j) \cap F_j).
\end{aligned}$$

This inequality, taking into account (7), allows to deduce that:

$$\limsup_{j \to +\infty} \mathcal{H}^{n-1}(S(w_j) \setminus F_j) \leq \mathcal{H}^{n-1}(S(\overline{u}) \setminus E) = \mathcal{H}^{n-1}(S(u) \setminus E), \tag{8}$$

concluding the proof of the Lemma. $\qquad\square$

11

Proof of Proposition 9. To prove Proposition 9, we consider a function u and a set E such that $\overline{F}(u, E)$ is finite. A truncation argument allows us to suppose $u \in L^\infty(\Omega)$. Now, we consider the sequences $\{w_j\}$, $\{F_j\}$ and $\{S_j\}$ given by Lemma 12. From the coarea formula, recalling that $\mathcal{H}^{n-1}(S_j) = M^{n-1}(S_j)$ and $|\nabla \text{dist}(\cdot, S_j)| = 1$ a.e., for fixed j it follows:

$$k \int_0^{1/k} \mathcal{H}^{n-1}(\partial^* \{x \in \Omega : \text{dist}(x, S_j) < r\}) \, dr = \int_{\{\text{dist}(\cdot, S_j) < 1/k\}} |\nabla \text{dist}(x, S_j)| \, dx$$

$$= k \left| \left\{ x \in \Omega : \text{dist}(x, S_j) < \frac{1}{k} \right\} \right| = 2\mathcal{H}^{n-1}(S_j) + o(1)_{k \to +\infty} \tag{9}$$

so that there exists $r_j^k \in (0, 1/k)$ with, letting $\Sigma_j^k = \{x \in \Omega \ : \ \text{dist}(x, S_j) < r_j^k\}$,

$$\mathcal{H}^{n-1}(\partial \Sigma_j^k) \leq 2\mathcal{H}^{n-1}(S_j) + o(1)_{k \to +\infty}.$$

Upon choosing a suitable sequence k_j and defining $\Sigma_j = \Sigma_j^{k_j}$, we then have

$$\mathcal{H}^{n-1}(\partial \Sigma_j) \leq 2\mathcal{H}^{n-1}(S_j) + o(1)_{j \to +\infty}.$$

Now, setting

$$E_j = F_j \cup \Sigma_j,$$

since $\mathcal{H}^{n-1}(S_j) = \mathcal{H}^{n-1}(S(w_j) \cap E_0) + o(1)_{j \to +\infty}$, we get:

$$\mathcal{H}^{n-1}(\partial E_j) \leq \mathcal{H}^{n-1}(\partial F_j) + 2\mathcal{H}^{n-1}(S(w_j)) + o(1)_{j \to +\infty}. \tag{10}$$

For every E_j, it is easy to show that there exists a set E_j' of class C^∞ such that $E_j \subset E_j'$ and $\mathcal{H}^{n-1}(\partial E_j') = \mathcal{H}^{n-1}(\partial E_j) + o(1)$, so we can assume E_j of class C^∞. Then we can find, for every j, a function $\tilde{u}_j \in H^1(\Omega)$ such that the restriction of u_j to the set $(E_j)_0$ coincides with the restriction of w_j. We then set $u_j = \phi_j \tilde{u}_j + (1 - \phi_j)v_j$, where v_j are smooth functions converging to u in $L^1(\Omega)$, and ϕ_j are smooth functions with $\phi_j = 1$ on $(E_j)_0$ and $\phi_j(x) = 0$ if $\text{dist}(x, (E_j)_0) > 1/(2r_j)$. Clearly, the sequence $\{u_j\}$ converges to u in $L^1(\Omega)$; the inequality (10) implies:

$$\limsup_{j \to +\infty} F(u_j, E_j) \leq \limsup_{j \to +\infty} \overline{F}(u_j, E_j),$$

and this completes the proof of the upper inequality and of the Proposition 9. $\qquad \square$

Remark 13. An alternative proof of the upper inequality would consist in first approximating the set E as follows: since $\chi_E \in BV(\Omega)$, by standard results [17] there exists a sequence $\{v_j\} \subset C^\infty(\Omega)$, converging to χ_E in $L^1(\Omega)$ and such that $\int_\Omega |\nabla v_j| \to |D\chi_E| = \mathcal{H}^{n-1}(\partial^* E)$. It is easy to show that for all $s \in (0, 1)$, $\chi_{\{v_j \geq s\}} \to \chi_E$ in $L^1(\Omega)$.

Moreover, for a.a. $s \in (0, 1)$, $\partial \{v_j \geq s\} \cap \Omega$ is C^∞ (by Sard's lemma), and using the coarea formula we find

$$\mathcal{H}^{n-1}(\partial \{v_j \geq s\} \cap \Omega) \to \mathcal{H}^{n-1}(\partial^* E).$$

From the construction in [17] (made by locally convolving χ_E with suitable mollifiers) we also can assume that

$$v_j(x) \to \begin{cases} 0 & \text{in } E_0 \\ 1 & \text{in } E_1 \\ \frac{1}{2} & \text{in } E_{1/2} \end{cases} \qquad \mathcal{H}^{n-1} \text{ - a.a. } x,$$

where $E_{1/2}$ is the set of points where E has density $\frac{1}{2}$. Hence, for $s < \frac{1}{2}$, it follows that:

$$\chi_{\{v_j \geq s\}}(x) \to \begin{cases} 0 & \text{in} \quad E_0 \\ 1 & \text{in} \quad \Omega \setminus E_0, \end{cases} \qquad \mathcal{H}^{n-1} \text{ - a.a. } x,$$

recalling that $\mathcal{H}^{n-1}(\Omega \setminus (E_0 \cup E_1 \cup E_{1/2})) = 0$; in particular, for $s < \frac{1}{2}$:

$$\int_{\partial^* E} \chi_{\{v_j \geq s\}}(x) \, d\mathcal{H}^{n-1} \to \mathcal{H}^{n-1}(\partial^* E).$$

Then, for $\varepsilon > 0$, we can choose $s < \frac{1}{2}$ such that, setting $F_j = \{v_j \geq s\}$:

$$\mathcal{H}^{n-1}(\partial^* E \setminus F_j) < \varepsilon \quad \text{for} \quad j \text{ large enough.} \tag{11}$$

Now, let $u \in SBV(\Omega)$. We let $u'_j = u|_{\Omega \setminus F_j}$, which is viewed as a function defined in the open set $\Omega \setminus F_j$. Clearly $u' \in SBV(\Omega \setminus F_j)$, and one has

$$S(u'_j) \subset (\partial^* E \setminus F_j) \cup (S(u) \cap E_0),$$

so that

$$\mathcal{H}^{n-1}(S(u'_j)) \leq \mathcal{H}^{n-1}(S(u) \cap E_0) + \varepsilon. \tag{12}$$

By standard approximation results, for every j there exists $u_j \in SBV(\Omega \setminus F_j)$ with $\|u_j - u'_j\|_{L^1} < 1/j$, and such that: $\mathcal{H}^{n-1}(\overline{S(u_j)} \setminus S(u_j)) = 0$ (so that $\mathcal{H}^{n-1}(S(u_j)) = M^{n-1}(S(u_j))$), $u_j \in C^1((\Omega \setminus F_j) \setminus \overline{S(u_j)})$, and

$$\mathcal{H}^{n-1}(S(u_j)) \leq \mathcal{H}^{n-1}(S(u'_j)) + \varepsilon.$$

Recalling (11), this inequality and (12), we find:

$$\mathcal{H}^{n-1}(S(u_j)) \leq \mathcal{H}^{n-1}(S(u) \cap E_0) + 2\varepsilon.$$

We can now select an appropriate neighborhood Σ_j of $S_j = S(u_j)$ as in the previous proof, and letting $E_j = F_j \cup \Sigma_j$, we conclude as before.

Remark 14 (Proof of the upper bound in the general case). The proof is not much modified in the general case since the main difficulty is the construction of the approximating sets, which is independent of m and of the particular energy.

Some care must be taken while following the reasoning in (9). In case φ is even, then standard results on the anisotropic Minkowski contents (where the distance function is replaced with $\varphi^\circ : x \mapsto \sup_{\varphi(\xi) \leq 1} \xi \cdot x$, the polar of φ, see [6]) allow to adapt easily the previous proof to the anisotropic case. However, the nonsymmetric case is not covered by these results (although it is very likely that they extend to this situation).

We first note that if S_j is composed of a finite number of compact subsets of C^1 hypersurfaces, by the condition $M^{n-1}(S_j) = \mathcal{H}^{n-1}(S_j)$ we have

$$k\left|\left\{x \in \Omega : \operatorname{dist}(x, S_j) < \frac{1}{k}\right\}\right| = k\left|\left\{y + t\nu(y) : y \in S_j, |t| < \frac{1}{k}\right\}\right| + o(1)_{k \to +\infty} \tag{13}$$

(but in the second representation, the same point might correspond to two or more values of y and t). We then have

$$k \int_0^{1/k} \int_{\partial^* \{x \in \Omega : \text{dist}(x, S_j) < r\}} \varphi(\nabla \text{dist}(x, S_j)) \, d\mathcal{H}^{n-1}$$

$$= \int_{\{\text{dist}(\cdot, S_j) < 1/k\}} \varphi(\nabla \text{dist}(x, S_j)) |\nabla \text{dist}(x, S_j)| \, dx$$

$$\leq k \int_{S_j} \int_{-1/k}^{1/k} \varphi(\nabla \text{dist}(y + t\nu(y), S_j)) \, dt \, d\mathcal{H}^{n-1}(y) + o(1)_{k \to +\infty}$$

$$= \int_{S_j} (\varphi(\nu) + \varphi(-\nu)) \, d\mathcal{H}^{n-1} + o(1)_{k \to +\infty}$$

since $\mathcal{H}^{n-1}$-a.e. on S_j,

$$\lim_{t \to 0^\pm} \nabla \text{dist}(y + t\nu(y), S_j) = \pm \nu(y).$$

If we define Σ_j^k as before, we find that

$$\limsup_{k \to \infty} \int_{\partial^* \Sigma_j^k} \varphi(\nabla \text{dist}(x, S_j)) \, d\mathcal{H}^{n-1}$$

$$= \limsup_{k \to \infty} \int_{\partial^* \Sigma_j^k} \varphi(\nu_{\Sigma_j^k}) \, d\mathcal{H}^{n-1} \leq \int_{S_j} (\varphi(\nu) + \varphi(-\nu)) \, d\mathcal{H}^{n-1}$$

and are able to carry on the proof of Proposition 9. In the general case we can split S_j in a part composed of a finite union of compact subsets of C^1 hypersurfaces and a remainder whose Minkowski content is arbitrarily small, and proceed likewise.

The only technical point where some extra care must be used is the truncation argument at the beginning of the proof of Proposition 9. That argument is straightforward in the scalar case while in the vector case some more elaborate but by now standard truncation lemmas must be used (for example [12] Lemma 3.5).

It remains to prove that if $0 < |E| \leq \Omega$ then we can find a recovery sequence with $|E_j| = |E|$. The case $|E| = \Omega$ is trivial. In the case $0 < |E| < \Omega$ we can simply modify E_j by inserting or removing suitable balls with suitable volume close to $||E| - |E_j||$ (and possibly smoothing the resulting sets if needed). The location of the centres of such balls must be chosen as a point of density 0 or 1 for E, respectively. Details can be found, *e.g.*, in [2] Theorem 3.3.

5 Applications

5.1 An approximation of the Mumford-Shah functional

As a byproduct of Theorem 2 we have an approximation of the Mumford-Shah functional by energies defined on pairs function-set defined as follows. Let r_ε be a family of strictly positive numbers converging to 0; for every $\varepsilon > 0$ we set

$$H_\varepsilon(u, E) = \begin{cases} F(u, E) & \text{if } |E| \leq r_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 15. *The functionals H_ε Γ-converge to the energy (equivalent to the Mumford-Shah functional)*

$$H(u, E) = \begin{cases} a \int_\Omega |\nabla u|^2 + b\, \mathcal{H}^{n-1}(S(u)) & \text{if } u \in GSBV(\Omega),\ |E| = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Note that if $\liminf_{\varepsilon \to 0} H_\varepsilon(u_\varepsilon, E_\varepsilon)$ then $|E_\varepsilon| \to 0$ and hence we can limit our analysis to $|E| = 0$. The lower inequality immediately follows by applying Theorem 2, and noticing that for such E we have $E_0 = \Omega$. The same inequality implies that $u \in GSBV(\Omega)$.

Conversely, again by Theorem 2 we have that for every $u \in GSBV(\Omega)$ we can find sets E_j with $|E_j| \to 0$ and functions u_j such that $H(u, E) = \lim_j F(u_j, E_j)$. For fixed $\{\varepsilon_j\}$, upon extracting a subsequence of E_j, we can always suppose that $|E_j| \le r_{\varepsilon_j}$, so that $H(u, E) = \lim_j H_{\varepsilon_j}(u_j, E_j)$. By the arbitrariness of $\{\varepsilon_j\}$ the upper bound is proved. $\qquad\square$

5.2 A generalization of the Ambrosio and Tortorelli approximation result

In this paragraph we use Theorem 15 to give a proof of the Ambrosio-Tortorelli result [5]. We take the chance for a slight generalization.

Let $V \colon [0,1] \to [0,+\infty)$ be a continuous function vanishing at the point 1 and strictly positive on $(0,1)$, and let $V_\varepsilon \colon [0,1] \to [0,+\infty)$ be continuous functions converging uniformly to V such that $V_\varepsilon(1) = 0$, $V_\varepsilon > 0$ on $[0,1)$ and there exist $K_\varepsilon \to 0$ such that $\varepsilon \ll K_\varepsilon$ and a neighbourhood $[0, \eta]$ of 0 independent of ε where $V_\varepsilon \ge K_\varepsilon$. The prototype situation is choosing V vanishing also in 0 (a 'double-well potential') and $V_\varepsilon(z) = V(z) + \sqrt{\varepsilon}(1 - z)$.

For every $\varepsilon > 0$, we consider the functional $G_\varepsilon \colon L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ defined by:

$$G_\varepsilon(u, v) = \begin{cases} \displaystyle\int_\Omega \left(v^2 |\nabla u|^2 + \frac{V_\varepsilon(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx & \text{if } u, v \in H^1(\Omega) \text{ and } 0 \le v \le 1 \text{ a.e.} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

For $u, v \in L^1(\Omega)$, we denote by $G'(u, v)$ the functional

$$\Gamma\text{-}\liminf_{\varepsilon \to 0} G_\varepsilon(u, v) = \inf\Big\{ \liminf_{j \to \infty} G_{\varepsilon_j}(u_j, v_j)\ :\ \varepsilon_j \to 0,\ u_j \to u,\ v_j \to v \text{ in } L^1(\Omega) \Big\}.$$

Theorem 16. *For every $(u, v) \in L^1(\Omega) \times L^1(\Omega)$:*

$$G'(u, v) \ge G(u, v),$$

where

$$G(u, v) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, dx + 4 \int_0^1 \sqrt{V(s)}\, ds\ \mathcal{H}^{n-1}(S(u)) & \text{if } \quad u \in GSBV(\Omega) \\ & \text{and } \quad v = 1 \quad \text{a.e.} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Proof of Theorem 16. We prove that, if $G'(u,v)$ is finite, then $v = 1$ a.e., $u \in GSBV(\Omega)$ and

$$G'(u,v) \geq \int_\Omega |\nabla u|^2 \, dx + 4 \int_0^1 \sqrt{V(s)} \, ds \, \mathcal{H}^{n-1}(S(u)).$$

Assume $G'(u,v) < +\infty$, and $u_\varepsilon \to u$, $v_\varepsilon \to v$ be such that $G_\varepsilon(u_\varepsilon, v_\varepsilon) \leq c < +\infty$. Then

$$\left| \left\{ v_\varepsilon \leq \eta \right\} \right| \leq \frac{\varepsilon}{K_\varepsilon} G_\varepsilon(u_\varepsilon, v_\varepsilon) = o(1)$$

implies that $v \geq \eta$ in measure. Since $V_\varepsilon \to V$ uniformly, and $V(z)$ vanishes only at the point 1 for $z \geq \eta$, we deduce that that $v = 1$ a.e.

We introduce, for every A open subset of Ω, the localized functionals:

$$G_\varepsilon(u,v;A) = \begin{cases} \displaystyle\int_A \left(v^2 |\nabla u|^2 + \frac{V(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right) dx & \text{if } u,v \in H^1(\Omega), \\ & \text{and } 0 \leq v \leq 1 \text{ a.e.} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, since $G'(u,v) < +\infty$, it follows that $G'(u,v;A) := \Gamma\text{-}\liminf_{\varepsilon \to 0} G(u,v;A) < +\infty$ for every A.

Let $\{\varepsilon_j\}$ be a positive infinitesimal sequence, and $\{u_j\}, \{v_j\}$ sequences in $H^1(\Omega)$ respectively converging in $L^1(\Omega)$ to u and v, with $0 \leq v_j \leq 1$ a.e. An application of the coarea formula gives:

$$\begin{aligned} G_{\varepsilon_j}(u_j, v_j; A) &\geq \int_A |\nabla u_j|^2 \, dx + 2 \int_A \left(\sqrt{V(v_j)} + o(1) \right) |\nabla v_j| \, dx \\ &\geq \int_A |\nabla u_j|^2 \, dx + 2 \int_0^1 \left(\sqrt{V(v_j)} + o(1) \right) \mathcal{H}^{n-1}(\partial\{v_j < s\} \cap A) \, ds. \end{aligned}$$

Now, we fix $\delta \in (0,1)$. The Mean Value Theorem ensures the existence of $t_j^\delta \in (\delta, 1)$ such that

$$\int_\delta^1 \sqrt{V(s)} \mathcal{H}^{n-1}(\partial\{v_j < s\} \cap A) \, ds \geq \int_\delta^1 \sqrt{V(v_j)} \, ds \, \mathcal{H}^{n-1}(\partial E_j^\delta \cap A),$$

where $E_j^\delta = \{v_j < t_j^\delta\}$; hence

$$G_{\varepsilon_j}(u_j, v_j; A) \geq \delta^2 \int_{A \setminus E_j^\delta} |\nabla u_j|^2 \, dx + 2 \int_\delta^1 \left(\sqrt{V(v_j)} + o(1) \right) ds \, \mathcal{H}^{n-1}(\partial E_j^\delta \cap A). \tag{14}$$

Since $v = 1$ a.e. in Ω, it follows that $|E_j^\delta| \to 0$ as $j \to +\infty$, for every δ. An application of Theorem 15 with $a = \delta^2$ and $b = 4 \int_\delta^1 \sqrt{V(s)} \, ds$ allows to deduce that $u \in GSBV(\Omega)$; moreover,

$$\liminf_{j \to +\infty} \left(\delta^2 \int_{A \setminus E_j^\delta} |\nabla u_j|^2 \, dx + 2 \int_\delta^1 \sqrt{V(s)} \, ds \, \mathcal{H}^{n-1}(\partial E_j^\delta \cap A) \right)$$

$$\geq \delta^2 \int_A |\nabla u|^2 \, dx + 4 \int_\delta^1 \sqrt{V(s)} \, ds \, \mathcal{H}^{n-1}(S(u) \cap A).$$

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Therefore, recalling (14), for every $\delta \in (0,1)$ we get:

$$\liminf_{j \to +\infty} G_{\varepsilon_j}(u_j, v_j; A) \geq \delta^2 \int_A |\nabla u|^2 dx + 4 \int_\delta^1 \sqrt{V(s)} ds \ \mathcal{H}^{n-1}(S(u) \cap A). \tag{15}$$

In order to apply Proposition 4 to the set function

$$\mu(A) = \Gamma\text{-}\liminf_{\varepsilon \to 0} G_\varepsilon(u, v; A),$$

which is superadditive on disjoint open sets when $u \in GSBV(\Omega)$ and $v = 1$ a.e., we define:

$$g_i(x) = \begin{cases} \delta_i^2 |\nabla u|^2 & \text{if} \quad x \in \Omega \setminus S(u) \\ 4 \int_{\delta_i}^1 \sqrt{W(s)} ds & \text{if} \quad x \in S(u), \end{cases}$$

where $\{\delta_i\}$ is a dense sequence in $(0,1)$, and $\lambda = \mathcal{L}^n + \mathcal{H}^{n-1} \llcorner S(u)$. From (15) we have $\mu(A) \geq \sup_i \int_A g_i d\lambda$; then Proposition 4 gives

$$\mu(A) \geq \int_A |\nabla u|^2 \, dx + 4 \int_0^1 \sqrt{V(s)} ds \mathcal{H}^{n-1}(S(u) \cap A),$$

concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.3 Other applications

We quickly mention two other applications in which an energy, similar to our $F(u, E)$, is used.

5.3.1 Crystalline film on a substrate

In [7, 16], the following energy is introduced:

$$\mathcal{E}(u, h) \ = \ \int_{\{0 < x_n < h(x')\}} W(\nabla u(x)) \, dx \ + \ \int_\omega \sqrt{1 + |\nabla h(x')|^2} \, dx' \tag{16}$$

where: $\omega = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ is the $(n-1)$-dimensional torus, $h : \Omega \to [0, +\infty)$ is a smooth function, $W : \mathbb{R}^{n \times n} \to [0, +\infty)$ is a quasiconvex function with growth $p > 1$, and u is a $\mathbb{R}^n$-valued displacement. This energy is supposed to be a simplified model for a thin layer of crystal (whose reference configuration is $\Omega_h = \{x = (x', x_n) : x' \in \Omega, 0 < x_n < h(x')\}$, the subgraph of h), deposited on a crystalline substrate $\{x : x_n \leq 0\}$. Due to the mismatch between both crystalline lattices, the crystal layer is stretched at the interface $\{x_n = 0\}$. This is expressed by a Dirichlet condition on the displacement u: $u(x', 0) = (\delta \, x', 0)$ (for instance) on this interface ($\delta \neq 0$ is a parameter related to the mismatch). Moreover, it is assumed for simplicity that $\omega \ni x' \mapsto u(x', x_n) - (\delta \, x', 0)$ is periodic (for all x_n), in other words: $u - (\delta \, x', 0) \in W^{1,p}(\Omega_h; \mathbb{R}^n)$.

The effect of the surface tension of the crystal is represented by the term $\int_\omega \sqrt{1 + |\nabla h|^2}$ which penalizes its total surface. A competition occurs between the elastic term of the energy (which would like to release the stress induced by the mismatch: one easily sees that infinitely many vertical fractures can totally release this stress and make the elastic energy as small as wanted) and the surface tension (which would like, on the contrary, the surface of the crystal to be as flat as possible).

In [7], the two-dimensional case ($n = 2$) is considered. The material is supposed to be linear-elastic, that is, $W(\nabla u) = (\mathbf{A}e(u)) : e(u)$ where $e(u)$ is the symmetrized gradient of u and the tensor $\mathbf{A}$ defines the *Hooke's law* of the material. A relaxation formula is given, that extends (16) to any lower-semicontinuous h, and any $u \in SBV(\Omega_h; \mathbb{R}^2)$. An Ambrosio-Tortorelli type approximation is introduced, and the result of numerical experiments are shown.

In [16], the last two authors have tried to extend the mathematical results of [7] to higher dimension. Similar results are shown but the proofs are significantly more technical.

When computing the lower semicontinuous envelope of (16) (in some reasonable topology), as in the present paper one computes a lower estimate and show that it is also an upper estimate. Up to minor technical details, the lower estimate is easily deduced from the present paper. The main difficulties arise when showing the upper estimate. In (16), the role of the "hole" E is played by the complement of Ω_h, $\{x : x_n \geq h(x')\}$. It is thus constrained to be a supergraph. One therefore needs to show that any "generalized" supergraph (in some sense) is approximated by smoother sets *that are still supergraphs*, with almost the same surface tension.

5.3.2 Water waves

In a forthcoming study, the second author and Eric Séré introduce the following problem, which models a periodic water wave in presence of gravity and surface tension:

$$\min_{u,F} \left\{ \frac{\mu}{V - \int_F |\nabla u(x)|^2 \, dx} + \beta \int_F x_2 \, dx + \mathcal{H}^1(\partial F) \right\}. \tag{17}$$

Here, $F \subset \omega \times (0, +\infty)$, where $\omega = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus, and u is such that $u(x_1, x_2) - x_1$ is periodic (that is, in $H^1(F)$). The constant V is the (fixed) volume of F. In this setting, the set F represent a "column" of water in the ocean (of depth V), in a frame which is moving at constant speed -1, which is the speed of the wave. The speed of the water is ∇u (in the frame of the wave, hence $\nabla u - (1, 0)$ in a fixed frame). The second term in the energy is a gravity term, while the length $\mathcal{H}^1(\partial F)$ is the surface energy of the free surface of the water (more precisely, it should be written $\mathcal{H}^1(\partial(F \cup (\omega \times (-\infty, 0]))))$).

The functional in (17), and in particular its first term, is inspired by a recent work of Boris Buffoni [15]. The special form of this term rules out the trivial solution $u \equiv (x, 0)$, $F = \omega \times (0, V)$ (no wave, still water). Again, the functional in (17) is a variant of the functional $F(u, E)$ which is studied in the present paper and its relaxation can be derived from the result in Section 2.

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