Motion of Inertial Particles in Gaussian Fields Driven by an Infinite-Dimensional Fractional Brownian Motion

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Zusammenfassung

Die Dynamik von kleinen, trägen Partikeln, sogenannten *inertial particles*, in turbulenten Fluiden wie Luft oder Wasser bestimmt Naturphänomene wie Sandstürme oder technologische Vorgänge wie das Verbrennen von Treibstoffen und hat deshalb viele Naturwissenschaftler und Mathematiker bewogen, diese Prozesse näher zu untersuchen. Die Bewegung eines Partikels wird durch das zweite Newtonsche Gesetz beschrieben, wobei man annimmt, dass die Kraft proportional zur Differenz zwischen der Fluid- und der Partikelgeschwindigkeit ist. In dieser Arbeit studieren wir das Langzeitverhalten der Partikel unter der Hypothese, dass das Fluidgeschwindigkeitsfeld Lösung einer stochastischen partiellen Differentialgleichung vom Ornstein–Uhlenbeck–Typ ist, die durch eine unendlich–dimensionale fraktionelle Brownsche Bewegung mit beliebigem Hurst–Parameter $H \in (0, 1)$ angetrieben wird. Die Nützlichkeit dieses Ansatzes liegt darin, dass wir Geschwindigkeitszufallsfelder modellieren können, die gewünschte statistische Eigenschaften von turbulenten Fluiden, basierend auf bestimmten physikalischen Gesetzen, aufweisen und die man mit relativ geringem Rechenzeitaufwand simulieren kann. Solch ein Modell beschreibt sehr gut das sogenannte *preferential concentration*-Phänomen, welches man in numerischen und Laborexperimenten von turbulenten Fluiden beobachtet, d.h. die Partikel sammeln sich in Regionen mit schwacher Vortizität, was wir im Folgenden als *Clusterbildung* der Partikel bezeichnen. Wir beweisen die fast sichere Existenz und Eindeutigkeit von Lösungen der Partikeltransportgleichung, geben hinreichende Bedingungen an, das Modell als zufälliges dynamisches System zu beschreiben, welches einen zufälligen Attraktor hat und leiten Eigenschaften eines modifizierten Systems her, die eine Volumenkontraktion im System nahelegen. Abschließend visualisieren und untersuchen wir in numerischen Experimenten die Clusterbildung der Partikel in Abhängigkeit von den Parametern im Modell.

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Contents

1. Introduction and Notations

Motion of inertial (i.e. small heavy) particles in turbulent fluids occurs in natural phenomena as well as in technological processes and therefore has excited theoretical investigations (see e.g. [99] and the references therein). Examples for such processes are formation of raindrops, evolution of clouds and combusting of liquid fuel.
The starting point for many theoretical investigations concerning the motion of inertial particles in turbulent flows is *Stokes' law* (see e.g. [5]), which says that the force exerted by the fluid on the particle is proportional to the difference between the background fluid velocity and the particle velocity, i.e. we are concerned with the following transport equation:

$$\tau\ddot{x}(t) = v\big(x(t),t\big) - \dot{x}(t), \tag{1.1}$$

where $v(x,t)$ is the velocity of the fluid at point x in space at time t and $x(t)$ is the position of the particle at time t. Here the *response time* $\tau = \frac{m}{\nu C}$ in two dimensions or $\tau = \frac{m}{6\pi\nu r}$ in three dimensions is often called *Stokes' time*, where m is the particles' mass, r the particles' radius, ν the fluid *viscosity* and $C > 0$ a universal constant. We neglect that C actually depends on the radius r and the relative velocity making the law nonlinear ([5]). Further, an important non–dimensional parameter related to the equation (1.1) is the so–called *Stokes' number* St, which is the ratio of the particle aerodynamic time constant to an *appropriate* turbulence time scale. In turbulent fluid flows St is usually defined by $St = \tau/\tau_\eta$, where $\tau_\eta = \overline{\epsilon}^{-1/2}\nu^{1/2}$ is the *eddy turnover time* associated to the *Kolmogorov length scale* $\eta = \overline{\epsilon}^{-1/4}\nu^{3/4}$ with viscosity ν and *mean energy dissipation rate* $\overline{\epsilon}$. If $\tau \to \infty$, the equation (1.1) tends to $\ddot{x}(t) = 0$ which is the equation of motion of a particle moving with constant velocity. And if $\tau = 0$, i.e. the inertia of the particle is neglected, we get $\dot{x}(t) = v\big(x(t),t\big)$ which is the equation of motion of a *fluid particle* or a *passive tracer* which is a particle that follows the streamlines of the fluid. Various extensions of the basic model (1.1) have been considered in the literature, in particular by Maxey and collaborators ([56, 57, 97]).
Real world and numerical experiments show that the distribution of inertial particles in a turbulent fluid is highly correlated with the turbulent motion ([31, 35, 76, 90, 91]). The particles cluster in regions of low vorticity and high strain rate. This clustering phenomenon is known as *preferential concentration.* The Stokes' number St plays a central role in the effect of preferential concentration. If the particle and fluid time constant have almost the same order, i.e. $St \approx 1$, the particles concentrate in regions where straining dominates vorticity. Experiments at high or low Stokes' numbers do not show this clustering phenomenon.
In principle, the fluid velocity v should satisfy the *Navier–Stokes equations* and it is obtained through *direct numerical simulations* (DNS). DNS for a turbulent flow is computationally very expensive. Therefore, it is also important to consider simplified models for the velocity field v which simplify the rigorous mathematical analysis and careful numerical investigations, but which still include some important (statistical) properties of the turbulent flow. Hence, it is reasonable to consider v in (1.1) to be a given random field $v(x,t)$ which mimics some statistical features of the velocity field obtained from

DNS, but may not satisfy the Navier–Stokes equations, e.g. random fields whose energy spectrum is consistent with that of velocity fields obtained from DNS. Such an approach, i.e. a random velocity field which matches some statistics of turbulence, is also referred as *synthetic turbulence*. One example for such an approach is described in the book [37] by Garcia–Ojalvo and Sancho. In particular, they introduce the PDE formulation of synthetic turbulent velocity fields that is used in the model by Sigurgeirsson and Stuart ([89]), which is introduced below, together with advocating the use of the *fast Fourier transform* to simulate such velocity fields efficiently.

A model for the motion of inertial particles in two dimensions which covers the preferential concentration phenomenon was introduced by Sigurgeirsson and Stuart in [89] and analyzed in a series of papers ([42, 49, 64, 65, 66, 67, 68, 88]). This model consists of Stokes' law (1.1), where the velocity field is a Gaussian random field that is incompressible, homogeneous, isotropic and periodic in space, and stationary and Markovian in time. This gives the equations in the *non–dimensional* form

$$\tau\ddot{x}(t) = v\left(x(t),t\right) - \dot{x}(t), \quad (x(0),\dot{x}(0)) \in \mathbb{T}^2 \times \mathbb{R}^2, \tag{$S1$}$$

$$v(x,t) = \nabla^{\perp}\psi(x,t) = \Big(\frac{\partial\psi}{\partial x_2}(x,t), -\frac{\partial\psi}{\partial x_1}(x,t)\Big), \tag{$S2$}$$

$$d\psi_t = \nu A\psi_t dt + \nu^{\frac{1}{2}}Q^{\frac{1}{2}}dW_t, \quad \psi_0 \in V,\ t \geq 0, \tag{$S3$}$$

where $\tau, \nu > 0$, $\mathbb{T}^2$ is the two–dimensional torus, W is an infinite–dimensional Brownian motion in the separable Hilbert space $V := \{f \in L^{2,per}(\mathbb{T}^2)|\ \int_{\mathbb{T}^2} f(x)dx = 0\}$ and the self–adjoint operators A and $Q^{\frac{1}{2}}$ on V will be chosen to match a desired energy spectrum of the velocity field. Precise assumptions will be given later. The equation ($S3$) is interpreted as a linear stochastic evolution equation ([23]) and the stationary solution is an infinite series of *stationary Ornstein–Uhlenbeck processes*. In [89] various qualitative properties of the system (S) have been studied, such as existence and uniqueness of solutions and existence of a random attractor. In particular, numerical simulation in [89], see also [8], indicates that system (S) also covers the preferential concentration phenomenon, where the parameter τ can be now interpreted as Stokes' number, whereas ν indicates how fast the velocity field decorrelates. Therefore, the random attractor of the system (S) is highly relevant for the study of preferential concentration of inertial particles.

Similar to Stuart and Sigurgeirsson in [89], Bec simulated in [8, 9] not only the motion of inertial particles with a random velocity field as in the system (S), but also *numerically* calculated the *Lyapunov dimension* (LD) of the random attractor which is an upper bound of the *Hausdorff dimension*. It is important to note that the Hausdorff dimension of the random attractor in ($S1$) is almost sure constant due to the ergodicity of the underlying noise ($S2$)–($S3$). Bec's numerical calculation shows that there is a parameter regime $(0, \tau^*)$ with some $\tau^* > 0$ of the Stokes' number τ such that the LD is strict less than the physical phase space dimension 2. In particular, he computed that LD ≈ 2 for $\tau \approx 0$ as expected since v is *incompressible* by ($S2$) and for $\tau \to 0$ we get the limiting ODE $\dot{x}(t) = v\left(x(t),t\right)$. Then the LD decreases in τ to some minimum $1 < LD_{min}(\tau^{**}) < 2$ with $\tau^{**} \in (0, \tau^*)$ and then increases in τ with $LD(\tau) > LD(\tau^*) = 2$

for $\tau > \tau^*$. The behaviour for $\tau \to \infty$ is also reasonable since $(S1)$ tends in this case to $\ddot{x}(t) = 0$. In particular, the bounds in $(0, \tau^*)$ below 2 give us a numerical justification of preferential concentration as a *fractal clustering phenomenon*.
Further, various limits of physical interest in system (S) have been studied: rapid decorrelation in time limits ([49, 64, 65]) and diffusive scaling limits ([42, 66, 67, 68, 88]).
Also of interest are studies of system (S) for fluid particles or passive tracers, i.e. $\tau = 0$ and so $(S1)$ is replaced by $\dot{x}(t) = v\,(x(t), t)$. This problem in a similar framework was considered among others by Carmona (see [11] and references therein).

In this work we generalize system (S) to the fractional noise case, i.e. in the system (S) the stochastic evolution equation $(S3)$ will be driven by an infinite–dimensional fractional Brownian motion B^H with arbitrary Hurst parameter $H \in (0, 1)$.

$$d\psi_t = \nu A \psi_t dt + \nu^H Q^{\frac{1}{2}} dB_t^H \qquad (S'3)$$

We recover system (S) as a special case for $H = \frac{1}{2}$. The main motivation to use fractional noise, to model now a fractional Gaussian random velocity field, is that certain statistical similarities exist in the scaling behaviour of a fractional Brownian motion and a turbulent velocity field. These statistical similarities are described in Section 2. The great advantage of our generalized model for the velocity field is that it enables us to match an additional statistical property of turbulent fluids based on some statistical physical laws, which is, in general, not covered in system (S) with $H = 1/2$. Moreover, the generalized system still allows us to perform numerical experiments of relative high speed.
First we extend the results of Sigurgeirsson and Stuart in [89] to the fractional noise case with arbitrary Hurst parameter $H \in (0, 1)$, i.e. we prove the existence and uniqueness of solutions to the extended system (S')=($(S1)$,$(S2)$,$(S'3)$) and the existence of a random pullback attractor. Since a fractional Brownian motion for $H \neq \frac{1}{2}$ is not a semimartingale and not a Markov process, standard tools from Ito stochastic calculus are not available. That makes the analysis more complicated. To establish our assertions we use recent results by Maslowski and Pospisil in [54] concerning the existence of stationary ergodic mild solutions to a general class of linear stochastic evolution equations with an additive infinite–dimensional fractional Brownian motion. In Theorem 5.2.1 we prove the existence of a unique stationary ergodic mild solution ψ to $(S'3)$ with an explicit representation of this solution. The existence is due to [54], whereas we establish the uniqueness (in an appropriate sense) and the explicit representation of ψ. Then we verify in Theorem 5.2.3 and Corollary 5.2.5 regularity properties of ψ which determine the regularity of the velocity field $v = \nabla^{\perp}\psi$ and which in turn provides us sufficient conditions to prove in Corollary 5.2.8 the existence and uniqueness of solutions to $(S1)$.
To study the long–time behaviour of the particles we rewrite system (S') in Proposition 5.3.1 and Proposition 5.3.4 as a random dynamical system (RDS). In the proofs of the two propositions we work out all technical issues concerning the measurability aspects of the RDS which were left out in [89]. Subsequently, we verify in Theorem 5.3.5 the existence of a unique random pullback attractor and improve the results in [89], now in

the generalized system (S'), in such a way that we extend the universe of attracting sets from deterministic bounded sets to random *tempered* sets with an explicit representation of a random tempered the universe absorbing set. With this additional information we are also able to prove in Theorem 5.3.5 the existence of an invariant forward Markov measure for the RDS. To show the existence of the random attractor and global existence of particle paths, it is crucial to derive linear growth conditions for some functionals of ψ. We accomplish this in Lemma 5.3.2(ii). To establish the assertions of Lemma 5.3.2(ii), we adopt a method introduced by Maslowski and Schmalfuß in [53], where an analogous statement is proven for increments of an infinite–dimensional fractional Brownian motion. Further, Section 6 describes how to match desired statistical properties of the velocity field. In particular, Proposition 6.1 approves that our velocity field indeed captures the statistical property motivated in Section 2 to use fractional noise. All these results described so far, i.e. Theorem 5.2.1, Theorem 5.2.3, Corollary 5.2.5, Corollary 5.2.8, Proposition 5.3.1, Lemma 5.3.2(i)–(ii), Proposition 5.3.4, Theorem 5.3.5, Proposition 6.1 and parts of Section 2 and Section 6 were published in [84] except the assertion of the existence of an invariant forward Markov measure in Theorem 5.3.5.
Further, in view of Bec's numerical results, it would be desirable to *analytically* derive (upper) bounds of the Hausdorff dimension (HD) of the random attractor depending on the parameter τ in the generalized system (S') with fractional noise. Schmalfuß provides in [83] a general theorem to bound the HD of a random attractor. Unfortunately, his theorem is not directly applicable to our system (S') since we have a two–dimensional torus $\mathbb{T}^2$ in the position coordinates and to the best of our knowledge there are so far no results in the field of random dynamical systems on manifolds to bound the HD of a random attractor whenever the attractor is not a *random fixed point*. Therefore, we replace $\mathbb{T}^2$ by $\mathbb{R}^2$ in $(S1)$ and extend v and ψ in $(S2)$–$(S'3)$ by periodicity to $\mathbb{R}^2$. For this modified system we still can show existence and uniqueness of particle paths, but we cannot prove the existence of a random attractor. Nevertheless, we establish properties of this modified system which are very close to the assumptions of Schmalfuß' theorem in [83] to bound the HD of a random attractor. These results are on the one hand of independent interest and on the other hand suggest a volume contraction in the modified and presumably in the original system (S'), see Remark 5.3.7, Theorem 5.3.8 and Remark 5.3.9.
At last, we perform numerical experiments in system (S'). Since the unique stationary solution ψ of $(S'3)$ is an infinite series of one–dimensional stationary fractional Ornstein–Uhlenbeck (sfOU) processes, we first analytically and numerically verify in Section 4.2 the applicability of popular methods for simulating stationary Gaussian processes to the one–dimensional sfOU process. More precisely, we investigate the standard Cholesky method with complexity $\mathcal{O}(n^3)$, the Durbin–Levinson method with complexity $\mathcal{O}(n^2)$ and the circulant embedding method with complexity $\mathcal{O}(n\log(n))$. For that we establish in Proposition 4.1.2, see also Remark 4.1.3, to the best of our knowledge partly new properties of the sfOU process. It turns out that the standard Cholesky method and the Durbin–Levinson method are always applicable, whereas the circulant embedding method might only be partly of practical use. Section 7.2 then describes how the velocity field can be computed in an efficient way using the fast Fourier transform al-

gorithm. The trajectory of the particle is obtained by using the classic fourth–order Runge–Kutta scheme. Finally, in our numerical experiments in Section 7.3 we discover that the generalized model with fractional noise also captures the clustering phenomenon of preferential concentration, i.e. for $\tau \approx 1$ the clustering of the particles is distinctive, whereas for very low and large values of τ there is almost no clustering. We observe that if we increase the Hurst parameter, the clustering of the particles becomes stronger. Moreover, we heuristically deduce that, not surprisingly, the velocity field decorrelates for $\nu \to \infty$ more slowly by increasing the Hurst parameter.
To summarize, the main contributions of this work are threefold: modelling, rigorous mathematical analysis and numerical investigation. We introduce a generalized model for the velocity field with an additional statistical property motivated by some physical laws. The main analytic results are regularity properties of the unique ergodic mild solution ψ, the global existence and uniqueness of particle paths, the existence of a unique random attractor and the derivation of some properties of a modified system which suggest a volume contraction in this system. We describe how the particle motion can be simulated efficiently, numerically verify that the generalized model also captures the clustering phenomenon of preferential concentration and investigate the dependence of the clustering on the parameters τ, ν and H.

The remainder of this work is organized as follows: At the end of this section we summarize the required notations for this work. In Section 2 we present the motivation to use fractional noise to model the random velocity field. Section 3 provides preliminaries of stochastic calculus w.r.t. the (infinite–dimensional) fractional Brownian motion (Section 3.1 and Section 3.2), semigroup theory of linear operators (Section 3.3) and random dynamical systems (Section 3.4) which are needed in subsequent sections. In Section 4 we study properties of the stationary fractional Ornstein–Uhlenbeck process (Section 4.1) and describe how to simulate this process (Section 4.2). In Section 5.1 we introduce the generalized model (S') for motion of inertial particles in a fractional Gaussian random velocity field. We derive conditions for global existence and uniqueness of solutions for the system (S') in Section 5.2. Further, we verify in Section 5.3 that the system (S') defines a random dynamical system which admits a random attractor and prove properties of a modified model which suggest a volume contraction in the system. Section 6 is devoted to verifying that the random velocity field $(S2)$-$(S'3)$ captures the statistical properties of a turbulent fluid flow which were motivated in Section 2. In the last Section 7 we describe how to simulate the system (S') (Section 7.1 and Section 7.2) and study the long–time behaviour of the particles in numerical experiments (Section 7.3). Finally, the appendix is devoted to additional preliminaries for fractional calculus and tables of integrals.

In the whole thesis we use the following conventions: If E is a Banach space, then we denote by $|\cdot|_E$ the norm of E and by $\langle \cdot, \cdot \rangle_E (=: |\cdot|_E^2)$ the inner product of E if E is even a Hilbert space. In the cases $E = \mathbb{R}^n$ for some $n \in \mathbb{N}$ or $E = \mathbb{T}^2 \times \mathbb{R}^2$, where $\mathbb{T}^2$ denotes the two–dimensional torus, we just write $|\cdot| = |\cdot|_E$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_E$ for the norm and inner product, respectively. We also use $|z|$ to denote the absolute value of a scalar $z \in \mathbb{R}$ or a complex number $z \in \mathbb{C}$. Further, by $*$ we denote the complex conjugate of a complex number or a complex–valued function. For a separable Hilbert space E we denote by $\mathcal{L}(E)$ the Banach space of linear, bounded operators from E into E equipped with the operator norm $|\cdot|_{\mathcal{L}(E)}$ and $\mathcal{L}^2(E)$ equipped with the inner product $\langle T, S\rangle_{\mathcal{L}^2(E)} := \sum_{n\in\mathbb{N}} \langle Te_n, Se_n\rangle_E$, $T, S \in \mathcal{L}^2(E)$, denotes the Hilbert space of Hilbert–Schmidt operators from E into E, where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of E. For $Q \in \mathcal{L}(E)$ we denote by $Q^* \in \mathcal{L}(E)$ the adjoint of Q. We say $Q \in \mathcal{L}(E)$ is a trace-class operator on E if Q is a product of two Hilbert–Schmidt operators, i.e. $Q = T \circ S$ for some $S, T \in \mathcal{L}_2(E)$, where $\circ$ denotes the composition. If $Q \in \mathcal{L}(E)$ is a trace–class operator on E, we set $tr(Q) := \sum_{n\in\mathbb{N}} \langle Qe_n, e_n\rangle_E < \infty$ for some (and hence all) orthonormal bases $(e_n)_{n\in\mathbb{N}}$ of E. Further, we denote by $(\Omega, \mathcal{F}, \mathbb{P})$ always a probability space and $\mathbb{E}$ the expectation w.r.t. $\mathbb{P}$. For $0 < p < \infty$ and E a separable Banach space, we write $L^p(\Omega, E)$ and $L^p(\Omega) := L^p(\Omega, \mathbb{R})$ if $E = \mathbb{R}$ for the Banach space of E–valued integrable random variables $X : \Omega \to E$ (in fact, equivalence classes of random variables, where $X \sim Y$ if $X = Y$ $\mathbb{P}$–a.s.) equipped with the norm $|X|^p_{L^p} = \mathbb{E}(|X|^p_E)$. In the same way we define the spaces $L^p(\mathbb{R})$ with respect to the Lebesgue measure. We denote by $\mathcal{B}(E)$ the Borel σ–algebra of a metric space E. For a multi–index $\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{N}_0^N$ we set $|\delta| := \delta_1 + \cdots + \delta_N$ and denote the partial derivative operator by $D^\delta := \partial^{|\delta|}/\partial x_1^{\delta_1} \ldots \partial x_N^{\delta_N}$ and $D^\delta := id$ if $|\delta| = 0$. We define by $C(\mathbb{R}, E)$ the space of all E-valued continuous functions on $\mathbb{R}$ where E is a complete separable metric space with metric d. Notice that endowing $C(\mathbb{R}, E)$ with the compact open topology given by the complete metric $\widetilde{d}(f, g) = \sum_{n=1}^\infty d_n(f, g)/(2^n(1 + d_n(f, g)))$, $d_n(f, g) = \sup_{-n\le t\le n} d(f(t), g(t))$, makes $C(\mathbb{R}, E)$ a Polish space, actually a Frechet space. Let $\mathcal{O} \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, be open and bounded with closure $\overline{\mathcal{O}}$. $C^k(\overline{\mathcal{O}})$, $k \in \mathbb{N}_0$, with norm $|f|_{C^k(\overline{\mathcal{O}})} = \sum_{|\alpha|\le k} \sup_{x\in\mathcal{O}} |D^\alpha f(x)|$ denotes the separable Banach space of all real–valued k–times continuously differentiable functions on $\overline{\mathcal{O}}$. By $C^{k,\delta}(\overline{\mathcal{O}} \times \mathbb{R}^r, \mathbb{R}^s)$, $k \in \mathbb{N}_0$, $\delta \in [0, 1)$, $r, s \in \mathbb{N}$, we denote the Frechet space of functions $f : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^s$ which are k–times continuously differentiable and (for $\delta \in (0, 1)$) whose k–th derivative is locally δ-Hölder continuous with seminorms $|f|_{k,0;K} := \sum_{0\le|\alpha|\le k} \sup_{(x,y)\in\mathcal{O}\times K} |D^\alpha f(x, y)|$,

$$|f|_{k,\delta;K} = |f|_{k,0;K} + \sum_{|\alpha|=k} \sup_{\substack{(x_1,y_1),(x_2,y_2)\in\mathcal{O}\times K,\\ (x_1,y_1)\ne(x_2,y_2)}} \frac{|D^\alpha f(x_1, y_1) - D^\alpha f(x_2, y_2)|}{|(x_1, y_1) - (x_2, y_2)|^\delta},$$

where K is a compact convex subset of $\mathbb{R}^r$ and $\alpha \in \mathbb{N}_0^{m+r}$. A complete metric is given by $\rho(f, g) = \sum_{n=1}^\infty |f - g|_{k,\delta;K_n}/(2^n(1 + |f - g|_{k,\delta;K_n}))$, where $(K_n)_{n\in\mathbb{N}}$ is some increasing sequence of compact convex sets exhausting $\mathbb{R}^r$. While $C^{k,0}(\overline{\mathcal{O}} \times \mathbb{R}^r, \mathbb{R}^s)$ is separable and hence Polish, the space $C^{k,\delta}(\overline{\mathcal{O}} \times \mathbb{R}^r, \mathbb{R}^s)$ for $\delta \in (0, 1)$ is not separable (see Appendix B2 in [2] and references therein). In cases $\delta = 0$ and $k = 0$, $r = s = 1$ we set $C^{k,0}(\overline{\mathcal{O}} \times \mathbb{R}^r, \mathbb{R}^s) = C^k(\overline{\mathcal{O}} \times \mathbb{R}^r, \mathbb{R}^s)$ and $C^{0,\delta}(\overline{\mathcal{O}} \times \mathbb{R}, \mathbb{R}) = C^\delta(\overline{\mathcal{O}} \times \mathbb{R})$, respectively.

2. Motivation for the Use of Fractional Noise

Before we give the motivation to use fractional noise it is important to introduce some notations which are used frequently in this section.
Since solutions of the Navier–Stokes equations for very turbulent fluids, i.e. at large *Reynolds numbers*, are unstable in view of the sensitive dependence on the initial conditions that makes the fluid flow irregular both in space and time, a *statistical description* is needed (see e.g. [14] and [36]). Based on this and since we will model a two–dimensional fluid field, we call a measurable mapping

$$v : \Omega \times \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$$

a *(two–dimensional) random field* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that a random field is of *second–order* if $\mathbb{E}(|v(x,t)|^2) < \infty$ for all $(x,t) \in \mathbb{R}^2 \times [0,\infty)$ and *mean zero* if $\mathbb{E}(v_i(x,t)) = 0$ for all $1 \leq i \leq 2$ and $(x,t) \in \mathbb{R}^2 \times [0,\infty)$, where v_i, $1 \leq i \leq 2$, denote the components of v. Since we are only interested in velocity fields, we call a random field also a *random velocity field.* Further, we refer to a *(two–dimensional) random fluid field* v if v is a random velocity field which satisfies ω–wise, $\omega \in \Omega$, the Navier–Stokes equations in two space dimensions.
A random velocity field v is called *stationary* if for all $x \in \mathbb{R}^2$ and $s \geq 0$ $(v(x, s+t))_{t\geq 0}$ and $(v(x,t))_{t\geq 0}$ have the same finite dimensional probability distributions and analogously a random velocity field v is said to be *homogeneous* if for all $t \geq 0$ and $y \in \mathbb{R}^2$ $(v(x+y,t))_{x\in\mathbb{R}^2}$ and $(v(x,t))_{x\in\mathbb{R}^2}$ have the same finite dimensional probability distributions. If a homogeneous random velocity field is a second–order random field, the homogeneity implies that for a fixed $t \geq 0$ the *spatial covariances*

$$C_{ij}(t;x,y) := \mathbb{E}\left((v_i(x,t) - \mathbb{E}(v_i(x,t)))(v_j(y,t) - \mathbb{E}(v_j(y,t)))\right)$$

for all $1 \leq i,j \leq 2$, $x,y \in \mathbb{R}^2$ are functions of $x - y$.
We say that a random velocity field v is *isotropic* if for any orthogonal 2×2 matrix K, $t \geq 0$ and $y \in \mathbb{R}^2$ $(v(Kx+y,t))_{x\in\mathbb{R}^2}$ and $(v(x,t))_{x\in\mathbb{R}^2}$ have the same finite dimensional probability distributions, i.e. the probability distributions for the values of its components at any finite arbitrary set of points are unaffected by any translations, rotations and reflections. Notice that by our definition an isotropic random velocity field is always homogeneous. If an isotropic random velocity field is a second–order random field, the isotropy implies that for a fixed $t \geq 0$ the spatial covariances $C_{ij}(t;x,y)$ for all $1 \leq i,j \leq d$, $x,y \in \mathbb{R}^2$ are functions of $|x-y|$. In this case we set $C_{ij}(t;|x-y|) := C_{ij}(t;x,y)$. Further, we say that a second–order isotropic random velocity field v admits a *spectral density representation*, if for all $t \geq 0$ and $1 \leq i,j \leq 2$ there is a positive function $E_{ij}(t;\cdot) : [0,\infty) \to [0,\infty)$ such that

$$C_{ij}(t;|x-y|) = \int_0^\infty J_0(k|x-y|) E_{ij}(t;k)\,dk,$$

where $x, y \in \mathbb{R}^2$ and J_0 is the Bessel function of the first kind of order 0, that is,

$$J_0(r) = \sum_{m=0}^{\infty} (-1)^m \frac{(r/2)^{2m}}{(m!)^2}.$$

For a general spectral representation theorem for isotropic random fields the reader is referred to [1] page 116.
In this section we will implicitly assume that a random fluid field is a mean–zero second–order stationary and isotropic random velocity field which admits a spectral density representation. The *energy spectrum* of this random velocity field is then defined by

$$E(\cdot) := \frac{1}{2} \sum_{1 \le i \le 2} E_{ii}(t; \cdot) = \frac{1}{2} \sum_{1 \le i \le 2} E_{ii}(0; \cdot).$$

The need for the use of fractional Gaussian noise can be understood by comparing some statistical characteristics of a random fluid field and a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Due to a *phenomenological approach*, first introduced by Kolmogorov ([46]) in three dimensions and by Kraichnan ([47]), Leith ([51]) and Batchelor ([6]) in the two dimensional case, we have the following phenomenological correspondence (see Section 4b in [7]) between the relation of the *spatial second–order structure function*

$$\mathbb{E}\left(|v(x + r, t) - v(x, t)|^2\right) = C|r|^{\alpha-1} \tag{2.1}$$

and the relation of the *energy spectrum* $E(\cdot)$ of the random fluid field v

$$E(k) = \widetilde{C} k^{-\alpha}, \tag{2.2}$$

where $C, \widetilde{C} > 0$ are some constants, $1 < \alpha < 3$, $r \in \mathbb{R}^2$ and $k > 0$ in the *inertial subrange*. In particular, for $\alpha = 5/3$ we obtain the famous *Kolmogorov's two–thirds law* and *Kolmogorov's five–thirds law* (or *Kolmogorov energy spectrum*), respectively.
The connection to the fractional Brownian motion gives now *Taylor's frozen turbulence hypothesis* ([92]) which informally assumes that the spatial pattern of turbulent motion is unchanged as it is advected by a constant (in space and time) mean velocity $\overline{V}$, $|\overline{V}| := \left(\sum_i \overline{V}_i^2\right)^{\frac{1}{2}}$, let us say along the $\overline{x}$ axis. Mathematically, Taylor's hypothesis says that for any *scalar–valued fluid–mechanics variable* ξ (e.g. v_i, $i = 1, 2$) we have

$$\frac{\partial \xi}{\partial t} = -|\overline{V}| \frac{\partial \xi}{\partial \overline{x}}. \tag{2.3}$$

The frozen turbulence hypothesis enables us to express the statistical characteristics of the space differences $v(x+r, t) - v(x, t)$ in terms of the time differences $v(x, t) - v(x, t+s)$ corresponding to a fixed time t. Indeed, (2.3) implies $v_i(x, t + s) = v_i(x - \overline{V}s, t)$ and therefore by (2.1) we deduce

$$\mathbb{E}\left(|v(x, t) - v(x, t + s)|^2\right) = C|\overline{V}|^{\alpha-1} s^{\alpha-1} \tag{2.4}$$

in the inertial subrange along the time axis. For a discussion under which conditions the frozen turbulence hypothesis is valid the reader is referred to [59].
Note that due to our derivation the properties (2.2) and (2.4) are closely related to each other!
Now comparing (2.4) with the statistical property

$$\mathbb{E}\left(|\beta^H_{t+s} - \beta^H_t|^2\right) = s^{2H} \tag{2.5}$$

of the fractional Brownian motion (fBm) $(\beta^H_t)_{t\in\mathbb{R}}$ with Hurst parameter $H \in (0,1)$ indicates that it is reasonable to model the random velocity field v with noise driven by a fBm. With this motivation and since the fractional Ornstein–Uhlenbeck (OU) process (i.e. a OU process driven by a fractional Brownian motion) satisfies at least approximately local in time the property (2.5), Shao proposed in [85] a finite dimensional stationary fractional OU process to model the statistical feature (2.4) and he argued in [86] that this may remedy some inconsistencies arising in *Lagrangian stochastic models* for non–passive particle diffusion in turbulent flows.
Further, Sreenivasan and collaborators used a one–dimensional fractional Brownian motion with $H = \frac{1}{3}$ in [45] to construct a one–dimensional random velocity field as a model for turbulence by arguing that this gives some scaling behaviour that resembles Kolmogorov turbulence.
In [63] Papanicolaou and Solna use the fractional Brownian motion with special interest in $H = \frac{1}{3}$ to model turbulence and discuss the wavelet based scale spectrum that can be used for spectral estimation of such processes.
Since we are only interested in statistical properties of the random fluid field v, we introduce in Section 5 a two–dimensional, incompressible, stationary and isotropic random velocity field to capture both statistical features, (2.2) and (2.4). These assumptions (Gaussian statistics, stationarity and isotropy) are quite common in random mathematical models of turbulent fluids (see e.g. [5]). Section 6 describes in detail how to match these statistics in such a random velocity field.

3. Mathematical Background

In Section 3.1 we first introduce the one–dimensional fractional Brownian motion (fBm) and the Wiener integral w.r.t. the fBm. Further, we provide properties of these processes which are used in Section 4, where we introduce the stationary fractional Ornstein–Uhlenbeck process which is a special Wiener integral. Here we refer to [4, 58, 60, 70]. In Section 3.2 we will define the infinite–dimensional fBm as a straightforward generalization of a *cylindrical Wiener process* ([23, 72]) and recall the *Kolmogorov test* from [23] which will yield us the existence of continuous modifications of the random velocity field introduced in Section 5. Finally, we summarize some required notations and concepts of the semigroup theory of linear operators and the theory of random dynamical systems in Section 3.3 and Section 3.4, respectively, where we mainly refer to the books [2, 32, 69].

3.1. Fractional Brownian Motion on the Real Line and Wiener Integration

For the next definition recall that a stochastic process $X : \Omega \times \mathbb{R} \to \mathbb{R}$ denoted by $(X_t)_{t\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *Gaussian* if for any positive integer $n \in \mathbb{N}$ and any $t_1, \dots, t_n \in \mathbb{R}$ the joint distribution of the random variables $X_{t_1}, \dots, X_{t_n}$ is jointly normal.

Definition 3.1.1. *The (real–valued and normalized) fractional Brownian motion (fBm) on $\mathbb{R}$ with Hurst parameter $H \in (0,1)$ is a Gaussian process $\left(\beta_t^H\right)_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, having the properties*

(i) $\beta_0^H = 0$ $\mathbb{P}$-*a.s.,*

(ii) $\mathbb{E}(\beta_t^H) = 0,\ t \in \mathbb{R}$,

(iii) $\mathbb{E}(\beta_t^H \beta_s^H) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big),\ s,t \in \mathbb{R}.$

(iv) $(\beta_t^H)_{t\in\mathbb{R}}$ *has continuous sample paths* $\mathbb{P}$-*a.s.*

It is possible to consider the fBm only on $\mathbb{R}_+$ (one–sided fBm) or on an interval $[0,T]$ for some $T > 0$ with evident changes in Definition 3.1.1.
The fBm has the following *self–similarity* property: For any constant $a > 0$, the processes $\left(a^{-H}\beta_{at}^H\right)_{t\in\mathbb{R}}$, $\left(\beta_t^H\right)_{t\in\mathbb{R}}$ have the same distribution in the sense of finite–dimensional distributions. This property is an immediate consequence of the fact that the covariance $\mathbb{E}\left(\beta_t^H \beta_s^H\right)$ is homogeneous of order $2H$.
Note that

$$\mathbb{E}\big((\beta_t^H - \beta_s^H)(\beta_u^H - \beta_v^H)\big) = \frac{1}{2}\big(|s-u|^{2H} + |t-v|^{2H} - |t-u|^{2H} - |s-v|^{2H}\big) \quad (3.1.1)$$

and in particular

$$\mathbb{E}\big(|\beta_t^H - \beta_s^H|^2\big) = |t-s|^{2H}.$$

It follows from (3.1.1) that the process β^H has stationary increments, but it is not stationary itself.

One can show (see e.g. Theorem 1.6.1 and Proposition 1.7.1 in [4]) that there is a version of β^H such that the trajectories are Hölder continuous of order $H - \epsilon$ for any $\epsilon \in (0, H)$ and that the fBm does not have differentiable sample paths $\mathbb{P}$-a.s.
If $H = \frac{1}{2}$ then we have $\mathbb{E}\left(\beta_t^{\frac{1}{2}}\beta_s^{\frac{1}{2}}\right) = \min\{t, s\}$ and the increments of $\beta^{\frac{1}{2}}$ are not correlated, and consequently independent. So $\beta^{\frac{1}{2}}$ is a standard Brownian motion.
For $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and $t_1 < t_2 < t_3 < t_4$, it follows from (3.1.1) that

$$\mathbb{E}\big((\beta_{t_4}^H - \beta_{t_3}^H)(\beta_{t_2}^H - \beta_{t_1}^H)\big) = 2\left(H - \frac{1}{2}\right) H \int_{t_1}^{t_2}\int_{t_3}^{t_4} (u - v)^{2H-2} du dv. \tag{3.1.2}$$

Therefore, the increments are positively correlated for $H \in \left(\frac{1}{2}, 1\right)$ and negatively correlated for $H \in \left(0, \frac{1}{2}\right)$.

Before proceeding, we introduce an important definition.

Definition 3.1.2. *Let $(X_t)_{t\geq 0}$ be a real–valued stationary process on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite variance. We say that $(X_t)_{t\geq 0}$ is* long–range dependent *if the covariance function $\gamma(n) := Cov(X_k, X_{k+n}) = Cov(X_1, X_{1+n})$, $k, n \in \mathbb{N}$, satisfies*

$$\lim_{n\to\infty} \frac{|\gamma(n)|}{n^{-\alpha}} = C$$

for some constant $C > 0$ and $\alpha \in (0, 1)$. (In this case we have $\sum_{n=1}^{\infty} |\gamma(n)| = \infty$.) $(X_t)_{t\geq 0}$ is called short–range dependent *if $\sum_{n=1}^{\infty} |\gamma(n)| < \infty$.*

As easily seen, we have by (3.1.2)

$$r(n) := \mathbb{E}\left(\beta_1^H\left(\beta_{n+1}^H - \beta_n^H\right)\right) \sim 2\left(H - \frac{1}{2}\right) H n^{2H-2}$$

for $n \to \infty$. Therefore, the increments $X_n := \beta_{n+1}^H - \beta_n^H$, $n \in \mathbb{N}_0$, of the fBm β^H, also often called *fractional Gaussian noise*, are long–range dependent for $H > 1/2$ and short–range dependent for $H \leq 1/2$. It should be noted that the definition for long–range dependence given in Definition 3.1.2 suits the fBm very well. For alternative (and more general) definitions of long–range dependence see Definition 1.4.2 in [4].
Further, it is easy to prove (see e.g. Section 1.15 in [58]) that the fBm for $H \neq \frac{1}{2}$ is not a semimartingale and not a Markov process. As a direct consequence of this fact, one cannot use the Ito stochastic calculus developed for semimartingales in order to define the stochastic integral w.r.t. the fBm.
We have the following representations of the fBm on $\mathbb{R}$ which are due to Mandelbrot/van Ness ([52]) and Samorodnitsky/Taqqu ([80]).

Theorem 3.1.3. *Let $H \in (0, 1)$.*

(i) Let $(\tilde{\omega}_t)_{t\in\mathbb{R}}$ be a standard Brownian motion on $\mathbb{R}$ and $H \in \left(0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right)$. The process $\left(\beta_t^H\right)_{t\in\mathbb{R}}$ defined by

$$\beta_t^H := \frac{1}{C_1(H)} \int\limits_{\mathbb{R}} \left[\left((t-s)^+\right)^{H-\frac{1}{2}} - \left((-s)^+\right)^{H-\frac{1}{2}}\right] d\tilde{\omega}_s,$$

where $C_1(H) = \left(\int_{\mathbb{R}_+} \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}\right)^2 ds + \frac{1}{2H}\right)^{\frac{1}{2}} = \frac{\Gamma(H+\frac{1}{2})}{\left(2H\sin(\pi H)\Gamma(2H)\right)^{\frac{1}{2}}}$, has a continuous modification which is a fractional Brownian motion on $\mathbb{R}$ with Hurst parameter H. Here $(y)^+ := \max\{0,y\}$ for $y \in \mathbb{R}$ and $\Gamma(\cdot)$ denotes the gamma function.

(ii) Suppose that $\left(\beta_t^H\right)_{t\in\mathbb{R}}$ is a fractional Brownian motion on $\mathbb{R}$. Then

$$\left(\beta_t^H\right)_{t\in\mathbb{R}} \stackrel{d}{=} \left(\frac{1}{C_2(H)} \int\limits_{\mathbb{R}} \frac{e^{itx}-1}{ix} |x|^{\frac{1}{2}-H} d\hat{\omega}(x)\right)_{t\in\mathbb{R}},$$

where $C_2(H) = \left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}} > 0$ and $\hat{\omega} = \hat{\omega}^1 + i\hat{\omega}^2$ is a complex Gaussian measure such that $\hat{\omega}^1(A) = \hat{\omega}^1(-A)$, $\hat{\omega}^2(A) = -\hat{\omega}^2(-A)$ and $\mathbb{E}(\hat{\omega}^1(A))^2 = \mathbb{E}(\hat{\omega}^2(A))^2 = \frac{|A|}{2}$, for any Borel set A of finite Lebesgue measure $|A|$. Here $\stackrel{d}{=}$ denotes the equality in the sense of finite–dimensional distributions.

Proof. (i): Theorem 1.3.1 in [58].
(ii): Section 7.2.2 in [80]. □

Remark 3.1.4. We note that the fBm $(\beta_t^H)_{t\in[0,T]}$ on a finite interval $[0,T]$ for some $T>0$ with Hurst parameter $H \in (0,1)$ can be represented by

$$\beta_t^H = \int\limits_0^t K_H(t,s)d\tilde{\omega}_s, \ t \in [0,T], \tag{3.1.3}$$

where $K_H : [0,T]^2 \to \mathbb{R}$ is some kernel function and $(\tilde{\omega}_t)_{t\in[0,T]}$ is a standard Brownian motion on $[0,T]$ (see [60] Chapter 5 for details). In particular, the standard Brownian motion $(\tilde{\omega}_t)_{t\in[0,T]}$ that provides the integral representation (3.1.3) is unique and both processes $(\beta_t^H)_{t\in[0,T]}$, $(\tilde{\omega}_t)_{t\in[0,T]}$ generate the same filtration ([60]). In contrast to the representation (3.1.3) of the fBm $(\beta_t^H)_{t\in[0,T]}$ on a finite interval $[0,T]$ the standard Brownian motion $(\tilde{\omega}_t)_{t\in\mathbb{R}}$ and the fBm $(\beta_t^H)_{t\in\mathbb{R}}$ on $\mathbb{R}$ in Theorem 3.1.3(i) do not generate the same filtration. So, unlike the standard Brownian motion, the *two–sided* fBm $(\beta_t^H)_{t\in\mathbb{R}}$ is *not* obtained by glueing two independent copies of a *one–sided* fBm $(\beta_t^H)_{t\geq 0}$ together at $t=0$.

Now we want to introduce the Wiener integral w.r.t. the fBm on $\mathbb{R}$. Here we follow Pipiras and Taqqu [70] and use their notations.
As in the standard Brownian motion case one likes to view the integral $\int_{\mathbb{R}} f(t)d\beta^H(t)$ as approximated by $\sum_{1\le k\le n-1} a_k(\beta^H(t_{k+1}) - \beta^H(t_k))$ where a_k and $t_k < t_{k+1}$ are real numbers. So let Υ be the set of elementary (or step) functions $f : \mathbb{R} \to \mathbb{R}$ on the real line, i.e. $f(t) = \sum_{1\le k\le n-1} a_k \mathbf{1}_{[t_k,t_{k+1})}$, where $t_1 < t_2 < \cdots < t_n \in \mathbb{R}$ and $a_k \in \mathbb{R}$, $1 \le k \le n-1$. For $f \in \Upsilon$ define

$$\mathcal{I}^H(f) := \int_{\mathbb{R}} f(t)d\beta^H(t) := \sum_{1\le k\le n-1} a_k(\beta^H(t_{k+1}) - \beta^H(t_k)).$$

Then the linear space of Gaussian random variables $\{\mathcal{I}^H(f),\ f \in \Upsilon\}$ is a subset of the larger linear space

$$\overline{\mathrm{sp}}(\beta^H) := \left\{ X :\ \mathcal{I}^H(f_n) \xrightarrow[n\to\infty]{L^2(\Omega)} X,\ \text{for some } (f_n)_{n\in\mathbb{N}} \subset \Upsilon \right\}.$$

We can associate with X an equivalence class of sequences of elementary functions $(f_n)_{n\in\mathbb{N}}$ such that $\mathcal{I}^H(f_n) \to X$ for $n \to \infty$ in the $L^2(\Omega)$–sense and X is usually written in an integral form as $X = \int_{\mathbb{R}} f_X(t)d\beta_t^H$. If we endow $\overline{\mathrm{sp}}(\beta^H)$ with the inner product $\mathbb{E}(XY)$ for $X, Y \in \overline{\mathrm{sp}}(\beta^H)$, it becomes a *Gaussian Hilbert space* (see e.g. [44] for a definition). It is well–known that for the standard Brownian motion $\overline{\mathrm{sp}}(\beta^{\frac{1}{2}})$ and $L^2(\mathbb{R})$ are isometric. So we are interested in the following question:

Which classes of integrands in the definition of the Wiener integrals w.r.t. the fBm are isometric to $\overline{sp}(\beta^H)$ or at least to some of its subspaces?

The following proposition from [70] gives the general answer to this question.

Proposition 3.1.5. *Let $\mathcal{C}$ be some class of integrands and let $\Upsilon \subset \mathcal{C}$ be the class of step functions. Under the assumptions*

(1) $\mathcal{C}$ is a space with inner product $(f, g)_{\mathcal{C}}$, $f, g \in \mathcal{C}$,

(2) for $f, g \in \Upsilon$, $(f, g)_{\mathcal{C}} = \mathbb{E}\big(\mathcal{I}^H(f)\mathcal{I}^H(g)\big)$,

(3) the set Υ is dense in $\mathcal{C}$,

we have the following:

(i) There is an isometry between the space $\mathcal{C}$ and a linear subspace of $\overline{sp}(\beta^H)$ which is an extension of the map $f \to \mathcal{I}^H(f)$ for $f \in \Upsilon$,

(ii) $\mathcal{C}$ is isometric to $\overline{sp}(\beta^H)$ if and only if $\mathcal{C}$ is complete.

Proof. Proposition 2.1 in [70]. $\square$

For the following, recall the definitions (A.1) and (A.2) for the fractional integral operator I_-^α and fractional derivative operator D_-^α with $\alpha > 0$.

Definition 3.1.6. *In view of Proposition 3.1.5 we define the following inner product spaces:*

(i)

$$\widetilde{\Lambda}_H = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; f \in L^2(\mathbb{R}),\ \int\limits_{\mathbb{R}} |\widehat{f}(x)|^2 |x|^{1-2H} dx < \infty \right\},$$

for $H \in (0,1)$, with inner product

$$(f,g)_{\widetilde{\Lambda}_H} = \frac{1}{C_2(H)^2} \int\limits_{\mathbb{R}} \widehat{f}(x)(\widehat{g}(x))^* |x|^{1-2H} dx,$$

where $C_2(H) > 0$ is the constant from Theorem 3.1.3(ii). Here $\widehat{f}(x)$ denotes the Fourier transform of f, i.e $\widehat{f}(x) = \int_{\mathbb{R}} e^{ixt} f(t)dt$ and $(\widehat{g}(x))^*$ the complex conjugate of $\widehat{g}(x)$.

(ii)

$$\Lambda_H = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{cases} \int_{\mathbb{R}} \left[(D_-^{\frac{1}{2}-H} f)(t)\right]^2 dt < \infty & \text{for } 0 < H < \frac{1}{2} \\ f \in L^2(\mathbb{R}) & \text{for } H = \frac{1}{2} \\ \int_{\mathbb{R}} \left[(I_-^{H-\frac{1}{2}} f)(t)\right]^2 dt < \infty & \text{for } \frac{1}{2} < H < 1 \end{cases} \right\}$$

with inner product

$$(f,g)_{\Lambda_H} = \begin{cases} \frac{\Gamma(H+\frac{1}{2})^2}{C_1(H)^2} \int_{\mathbb{R}} (D_-^{\frac{1}{2}-H} f)(t)(D_-^{\frac{1}{2}-H} g)(t)dt & \text{for } 0 < H < \frac{1}{2} \\ (f,g)_{L^2(\mathbb{R})} & \text{for } H = \frac{1}{2} \\ \frac{\Gamma(H+\frac{1}{2})^2}{C_1(H)^2} \int_{\mathbb{R}} (I_-^{H-\frac{1}{2}} f)(t)(I_-^{H-\frac{1}{2}} g)(t)dt & \text{for } \frac{1}{2} < H < 1 \end{cases},$$

where $C_1(H) > 0$ is the constant from Theorem 3.1.3(i).

(iii)

$$|\Lambda|_H = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2} du dv < \infty \right\},$$

for $H \in (1/2, 1)$, with inner product

$$(f,g)_{|\Lambda|_H} = H(2H-1) \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} f(u)g(v)|u-v|^{2H-2} du dv.$$

Remark 3.1.7. Notice that:

(i) In the Definition 3.1.6(i) the Fourier transform is defined without the normalization constant $\frac{1}{\sqrt{2\pi}}$.

(ii) Due to the definition of the fractional derivative operator D_-^{α} we can equivalently define $\Lambda_H = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \exists \phi_f \in L^2(\mathbb{R}) \text{ such that } f = I_-^{\frac{1}{2}-H} \phi_f \right\}$ for $0 < H < \frac{1}{2}$ with inner product $(f,g)_{\Lambda_H} = \frac{\Gamma(H+\frac{1}{2})^2}{C_1(H)^2} \int_{\mathbb{R}} \phi_f(t)\phi_g(t)dt$.

(iii) The definition of $\widetilde{\Lambda}_H$ is based on the spectral representation of the fBm introduced in Theorem 3.1.3(ii)

$$\left(\beta_t^H\right)_{t\in\mathbb{R}} \stackrel{d}{=} \left(\frac{1}{C_2(H)} \int_{\mathbb{R}} \frac{e^{itx}-1}{ix} |x|^{\frac{1}{2}-H} d\hat{\omega}(x) \right)_{t\in\mathbb{R}}.$$

By observing that $\frac{e^{itx}-1}{ix} = \widehat{\mathbf{1}}_{[0,t)}(x)$ and

$$\mathcal{I}^H(f) \stackrel{d}{=} \frac{1}{C_2(H)} \int_{\mathbb{R}} \widehat{f}(x)|x|^{\frac{1}{2}-H} d\hat{\omega}(x)$$

for $f \in \Upsilon$, one can deduce (see (7.2.9) in Samorodnitsky and Taqqu [80]) that

$$\mathbb{E}\big(\mathcal{I}^H(f)\mathcal{I}^H(g)\big) = \frac{1}{C_2(H)^2} \int_{\mathbb{R}} \widehat{f}(x)(\widehat{g}(x))^* |x|^{1-2H} dx$$

for $f, g \in \Upsilon$.

(iv) The introduction of the space Λ_H is motivated by the fact (see Lemma 1.1.3 in [58]) that for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ we have for all $t \in \mathbb{R}$ the equality

$$(I_-^{H-\frac{1}{2}} \mathbf{1}_{(0,t)})(x) = \frac{1}{\Gamma(H+\frac{1}{2})} \big(((t-x)^+)^{H-\frac{1}{2}} - ((-x)^+)^{H-\frac{1}{2}} \big),$$

where $I_-^{-\alpha} = D_-^{\alpha}$ for $\alpha > 0$. From this and Theorem 3.1.3(i) it follows that for $f \in \Upsilon$ we have

$$\int_{\mathbb{R}} f(u) d\beta_u^H \stackrel{d}{=} \frac{\Gamma(H+\frac{1}{2})}{C_1(H)} \int_{\mathbb{R}} (I_-^{H-\frac{1}{2}} f)(s) d\tilde{\omega}_s$$

and in particular

$$\beta_t^H \stackrel{d}{=} \frac{\Gamma(H+\frac{1}{2})}{C_1(H)} \int_{\mathbb{R}} (I_-^{H-\frac{1}{2}} \mathbf{1}_{(0,t)})(t) d\tilde{\omega}_t.$$

And hence

$$\mathbb{E}\big(\mathcal{I}^H(f)\mathcal{I}^H(g)\big) = \frac{\Gamma(H+\frac{1}{2})^2}{C_1(H)^2} \int_{\mathbb{R}} (I_-^{H-\frac{1}{2}} f)(t)(I_-^{H-\frac{1}{2}} g)(t) dt$$

for $f, g \in \Upsilon$.

(v) The definition of $|\Lambda|_H$ is based on the following observation: Let R^H be the covariance function of the fBm $(\beta_t^H)_{t\in\mathbb{R}}$ with $0 < H < 1$. Then, for $f, g \in \Upsilon$,

$$\mathbb{E}\big(\mathcal{I}^H(f)\mathcal{I}^H(g)\big) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)d^2R^H(u,v),$$

where the double integral is defined to be linear and to satisfy

$$\int_{[a,b]}\int_{[c,d]} d^2R^H(u,v) = R^H(d,b) - R^H(d,a) - (R^H(c,b) - R^H(c,a)),$$

for any real number $a < b$ and $c < d$. This may suggest that one can define the integral $\int_{\mathbb{R}} f(u)d\beta_u^H$ for functions from the space

$$|\Lambda|_H = \left\{ f:\mathbb{R}\to\mathbb{R} \;\Big|\; \int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)|d^2|R^H|(u,v) < \infty \right\},$$

where $|R^H|$ is the total variation measure of R^H. Observe however, that when $0 < H < \frac{1}{2}$ the function R^H is not of bounded variation around the diagonal $u = v$ and hence the measure $|R^H|$ is not defined. But in the case $\frac{1}{2} < H < 1$ we have

$$d^2R^H(u,v) = H(2H-1)|u-v|^{2H-2}dudv.$$

We summarize now the results of Pipiras and Taqqu from [70] in the next theorem.

Theorem 3.1.8. *We have:*

(i) The spaces $\widetilde{\Lambda}_H, \Lambda_H, |\Lambda|_H$ satisfy the assumptions (1)–(3) of Proposition 3.1.5.

(ii) The space Λ_H is complete for $H \in \big(0,\frac{1}{2}\big)$ and incomplete for $H \in \big(\frac{1}{2},1\big)$. Therefore, Λ_H is isometric to $\overline{sp}(\beta^H)$ for $H \in \big(0,\frac{1}{2}\big)$ and isometric to a subspace of $\overline{sp}(\beta^H)$ for $H \in \big(\frac{1}{2},1\big)$.

(iii) For any $H \in \big(0,\frac{1}{2}\big) \cup \big(\frac{1}{2},1\big)$ we have the strict inclusion $\widetilde{\Lambda}_H \subset \Lambda_H$ and for $H = \frac{1}{2}$ the equality $\widetilde{\Lambda}_{\frac{1}{2}} = L^2(\mathbb{R}) = \Lambda_{\frac{1}{2}}$. Further, $(f,g)_{\widetilde{\Lambda}_H} = (f,g)_{\Lambda_H}$ for $f,g \in \widetilde{\Lambda}_H$ and any $H \in (0,1)$. Since $\widetilde{\Lambda}_H$ is incomplete unless $H = \frac{1}{2}$ it is isometric to $\overline{sp}(\beta^H)$ for $H = \frac{1}{2}$ and isometric to a subspace of $\overline{sp}(\beta^H)$ for $H \neq \frac{1}{2}$.

(iv) Let $H \in \big(\frac{1}{2},1\big)$. Then $|\Lambda|_H \subset \Lambda_H$ and this inclusion is proper. Further, $(f,g)_{|\Lambda|_H} = (f,g)_{\Lambda_H}$ for $f,g \in |\Lambda|_H$ and $|\Lambda|_H$ is incomplete. So $|\Lambda|_H$ is isometric to a subspace of $\overline{sp}(\beta^H)$.

Moreover, we have the inclusions

(v) $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subset \widetilde{\Lambda}_H$ if and only if $H \in \big(\frac{1}{2},1\big)$,

(vi) $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subset L^{\frac{1}{H}} \subset |\Lambda|_H$ *for* $H \in \left(\frac{1}{2}, 1\right)$.

By Theorem 3.1.8 $\left(|\Lambda|_H, (\cdot,\cdot)_{|\Lambda|_H}\right)$ is not complete. However, by introducing a new norm on $|\Lambda|_H$,

$$\|f\| = \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} |f(u)||f(v)||u-v|^{2H-2}dudv\Big)^{\frac{1}{2}},$$

$\left(|\Lambda|_H, \|\cdot\|\right)$ becomes a Banach space (see Theorem 4.1 in [70]). The norm $\|\cdot\|$ is usually easier to work with than the norm $\|\cdot\|_{|\Lambda|_H}$.

3.2. Infinite–Dimensional Fractional Brownian Motion and the Kolmogorov Test

In the following let $(U, \langle \cdot, \cdot \rangle_U)$ be a separable Hilbert space.
The next definition provides on the one hand an infinite–dimensional analogue of the definition of a finite–dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ and on the other hand the generalization of the definition of the standard Q–Wiener process (see e.g. Definition 2.1.9 in [72]). For that purpose we recall that a U–valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *Gaussian*, if for all $u \in U$ the random variable $\langle X, u \rangle_U$ is normal. A Gaussian U–valued random variable X is said to be $\mathcal{N}(m, Q)$–distributed, if there is a vector $m \in U$ and a symmetric, non–negative linear operator $Q \in \mathcal{L}(U)$, called *covariance operator*, such that

$$\mathbb{E}(\langle X, u \rangle_U) = \langle m, u \rangle_U$$

for all $u \in U$ and

$$Cov(\langle X, u_1 \rangle_U, \langle X, u_2 \rangle_U) := \mathbb{E}\big((\langle X, u_1 \rangle_U - \langle m, u_1 \rangle_U)(\langle X, u_2 \rangle_U - \langle m, u_2 \rangle_U)\big) = \langle Q u_1, u_2 \rangle_U$$

for all $u_1, u_2 \in U$. The expectation of X is then defined by $\mathbb{E}(X) := m$. Covariance operators are trace class operators (Proposition 2.15 in [23]). Further, a U–valued stochastic process $(X(t))_{t \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is *Gaussian* if for any positive integer $n \in \mathbb{N}$, any $t_1, \dots, t_n \in \mathbb{R}$ and any $u_1, \dots, u_n \in U$, the joint distribution of the random variables $\langle X_{t_1}, u_1 \rangle_U, \dots, \langle X_{t_n}, u_n \rangle_U$ is jointly normal. In particular, for any U–valued Gaussian process $(X(t))_{t \in \mathbb{R}}$ and any $t, s \in \mathbb{R}$ there are $m(t), m(s) \in U$ and a symmetric, non–negative, trace–class operator $Q(t,s) \in \mathcal{L}(U)$ such that

$$\begin{aligned} Cov(\langle X(t), u_1 \rangle_U, \langle X(s), u_2 \rangle_U) &:= \mathbb{E}\big((\langle X(t) - m(t), u_1 \rangle_U)(\langle X(s) - m(s), u_2 \rangle_U)\big) \\ &= \langle Q(t,s) u_1, u_2 \rangle_U \end{aligned}$$

for all $u_1, u_2 \in U$. We set $Cov(X(t), X(s)) := Q(t,s)$.

Definition 3.2.1. *Let $H \in (0,1)$ and $Q \in \mathcal{L}(U)$ be a non–negative, symmetric, trace–class operator. A U–valued Gaussian process $(B_Q^H(t))_{t \in \mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a* fractional Q–Wiener process *(on $\mathbb{R}$) with Hurst parameter H if*

(i) $B_Q^H(0) = 0$ $\mathbb{P}$*-a.s.,*

(ii) $\mathbb{E}\left(B_Q^H(t)\right) = 0$ *for all* $t \in \mathbb{R}$,

(iii) $Cov\left(B_Q^H(t), B_Q^H(s)\right) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big)Q$ *for* $s, t \in \mathbb{R}$,

(iv) $(B_Q^H(t))_{t \in \mathbb{R}}$ *has U–valued, continuous sample paths $\mathbb{P}$-a.s.*

The existence of a fractional Q–Wiener process is given by the following proposition.

Proposition 3.2.2. *Let $H \in (0,1)$ and $Q \in \mathcal{L}(U)$ be a non–negative, symmetric, trace–class operator on U. Further, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. Then*

$$B_Q^H(t) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\beta_n^H(t)e_n, \quad t \in \mathbb{R}, \tag{3.2.1}$$

is a U–valued fractional Q–Wiener process $(B_Q^H(t))_{t\in\mathbb{R}}$, where β_n^H, $n \in \{k \in \mathbb{N} : \lambda_k > 0\}$, are independent fractional Brownian motions on $\mathbb{R}$ with Hurst parameter H on $(\Omega, \mathcal{F}, \mathbb{P})$. The series (3.2.1) converges in $L^2(\Omega, U)$ and has a $\mathbb{P}$-a.s. continuous modification. In particular, for any Q as above there is a fractional Q–Wiener process on U.

Proof. Proposition 1.1.1 in [71]. □

In the case then $H = \frac{1}{2}$, $(B_Q^{\frac{1}{2}}(t))_{t\in\mathbb{R}}$ is the standard Q–Wiener process. Analogous to a *(standard) cylindrical Wiener process* in U (see e.g. [72] for a definition) with $Q = id$, we can define a *cylindrical fractional Wiener process* in U with $Q = id$ by the formal series

$$B^H(t) = \sum_{n\in\mathbb{N}} \beta_n^H(t)e_n, \quad t \in \mathbb{R}, \tag{3.2.2}$$

where again $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of U and $\big((\beta_n^H(t))_{t\in\mathbb{R}}, n \in \mathbb{N}\big)$ is a sequence of independent fractional Brownian motions each with the same fixed Hurst parameter $H \in (0,1)$. The series (3.2.2) does not converge in $L^2(\Omega, U)$. So $B^H(t)$ is not a well–defined U–valued random variable. However, proceeding as in Section 2.5.1 in [72] for the standard cylindrical Wiener process, it is easy to verify that for any Hilbert space U_1 such that $U \overset{J}{\hookrightarrow} U_1$ and the linear embedding J is a Hilbert–Schmidt operator, the series $B^H(t) = \sum_{n\in\mathbb{N}} \beta_n^H(t)J(e_n)$ defines a U_1–valued random variable and $(B_Q^H(t))_{t\in\mathbb{R}}$ is a U_1–valued fractional Q_1–Wiener process, where $Q_1 := JJ^*$.
In particular, if $(B^H(t))_{t\in\mathbb{R}}$ is a cylindrical fractional Wiener process in U and $Q \in \mathcal{L}(U)$ a non–negative, symmetric, trace–class operator and $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, then

$$B_Q^H(t) := \sqrt{Q}B^H(t) := \sum_{n\in\mathbb{N}} \beta_n^H(t)\sqrt{Q}e_n = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\beta_n^H(t)e_n,$$

$t \in \mathbb{R}$, is a U–valued fractional Q–Wiener process $(B_Q^H(t))_{t\in\mathbb{R}}$, where $\sqrt{Q} \in \mathcal{L}(U)$ such that $Q = \sqrt{Q} \circ \sqrt{Q}$.

The next two lemmas will provide us in Section 5 sufficient conditions for the existence of continuous modifications of our random velocity field.

Lemma 3.2.3. *Let X be a U–valued $\mathcal{N}(0,Q)$–distributed random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Then for all $n \in \mathbb{N}$ there exists a constant $C(n) > 0$ such that*

$$\mathbb{E}(|X|_U^{2n}) \le C(n)\left(\mathbb{E}(|X|_U^2)\right)^n.$$

Proof. Corollary 2.17 in [23]. □

Lemma 3.2.4 (Kolmogorov test)**.** *Let $T > 0$ and suppose $G \subset \mathbb{R}^d$ is open and bounded. Consider a family $\{X(t,x) : t \in [0,T], x \in G\}$ of real–valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\mathbb{E}\big(|X(t_1,x_1) - X(t_2,x_2)|^\delta\big) \leq C(|t_1 - t_2|^2 + |x_1 - x_2|^2)^{(d+1+\epsilon)/2} \tag{3.2.3}$$

for $C, \epsilon > 0$, $\delta \geq 1$, all $t_1, t_2 \in [0,T]$, and all $x_1, x_2 \in G$. Then there exists a version of X (again denoted by X) such that $\mathbb{P}$-almost all sample paths

$$[0,T] \times G \to \mathbb{R}, \quad (t,x) \mapsto X(t,x)(\omega)$$

are Hölder continuous on $[0,T] \times G$ with arbitrary exponent smaller than ϵ/δ.

Proof. Theorem 3.4 in [23]. □

Remark 3.2.5. It is important to remark that if for a random field $X(t,x)_{t\in\mathbb{R},x\in G}$ (3.2.3) is even satisfied for all $t, s \in \mathbb{R}$ and the constant C is independent of t, s then by the same argumentation as in the proof of Theorem 3.4 in [23], we can deduce that for all $\gamma \in (0, \epsilon/\delta)$ there is a version of X (again denoted by X) such that $\mathbb{P}$-a.s. $X \in C^\gamma(\mathbb{R}\times G)$.

3.3. Some Semigroup Theory of Linear Operators

In this subsection we introduce the concept of an *analytic semigroup* which is *exponentially stable* and *fractional powers* of linear operators generating this semigroup. We will refer to this concept when we define our model for the particle movement in Section 5.1. Here our approach follows the books [32, 69] very closely.

In the following let X be a Banach space.
As usual, we call

$$\rho(A) := \left\{\lambda \in \mathbb{C} \middle| \ (\lambda - A) : \mathcal{D}(A) \subseteq X \to X \text{ is bijective}\right\},$$
$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

the *resolvent set* and the *spectrum* of a closed linear operator $A : \mathcal{D}(A) \subseteq X \to X$ with domain $\mathcal{D}(A)$, respectively. The *resolvent* (of A at the point $\lambda \in \rho(A)$) is denoted by

$$R(\lambda, A) := (\lambda - A)^{-1}$$

which is a bounded operator on X by the closed graph theorem, and

$$s(A) := \sup \left\{Re(\lambda) \middle| \ \lambda \in \sigma(A)\right\} \subseteq \mathbb{R} \cup \{\pm\infty\}$$

is called the *spectral bound* of A where $Re(\lambda)$ denotes the real part of $\lambda \in \mathbb{C}$ and $s(A) := -\infty$ if $\sigma(A) = \emptyset$.

Definition 3.3.1. *A one parameter family $(S(t))_{t\geq 0} \subseteq \mathcal{L}(X)$ of bounded linear operators in X is called a* strongly continuous *or C_0–*semigroup *on X if*

(i) $S(0) = id_X$*, where id_X denotes the identity operator on X.*

(ii) $S(t + s) = S(t)S(s)$ *for every $t, s \geq 0$.*

(iii) $\lim_{t\downarrow 0} S(t)x = x$ *for every $x \in X$.*

The linear operator A defined by

$$\mathcal{D}(A) := \left\{x \in X \middle| \ \lim_{t\downarrow 0} \frac{S(t)x - x}{t} \ \textit{exists}\right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{S(t)x - x}{t}, \quad x \in \mathcal{D}(A),$$

is the infinitesimal generator *of the semigroup $(S(t))_{t\geq 0}$ with domain $\mathcal{D}(A)$.*

It can be shown (see Proposition I.1.4 and Proposition V.1.22 in [32]) that for any C_0–semigroup $S(t)_{t\geq 0}$ on X there is $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$|S(t)|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad t \geq 0,$$

and

$$-\infty \leq s(A) \leq \omega_0 := \inf \Big\{ \omega \in \mathbb{R} \Big| \text{ There exists } M(\omega) \geq 1$$
$$\text{such that } |S(t)|_{\mathcal{L}(X)} \leq M(\omega)e^{\omega t},\ t \geq 0. \Big\} < \infty.$$

We call a C_0–semigroup $(S(t))_{t\geq 0}$ on X *exponentially stable* if $\omega_0 < 0$ and *bounded* if $\omega_0 \leq 0$.
In particular, a linear operator A with domain $\mathcal{D}(A)$ is the infinitesimal generator of a C_0–semigroup $(S(t))_{t\geq 0}$ on X satisfying $|S(t)|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for some $M \geq 1$, $\omega \in \mathbb{R}$, if and only if

(i) A is closed and $\mathcal{D}(A)$ is dense in X.

(ii) The resolvent set $\rho(A)$ of A contains the ray $]\omega, \infty[$ and

$$|R(\lambda, A)^n|_{\mathcal{L}(X)} \leq M/(Re(\lambda) - \omega)^n$$

for all $Re(\lambda) > \omega$ and $n \in \mathbb{N}$.

For a proof the reader is referred to Theorem 1.5.3 in [69].

We are mainly interested in analytic semigroups. In the following $arg(z)$ denotes the argument of a complex number $z \in \mathbb{C}$.

Definition 3.3.2. *A family of operators $(S(z))_{z\in\Sigma_\delta\cup\{0\}} \subseteq \mathcal{L}(X)$ with angle $\delta \in (0, \pi/2]$ and*

$$\Sigma_\delta := \{\lambda \in \mathbb{C}|\ |arg(\lambda)| < \delta\} \setminus \{0\}$$

is called analytic *if*

(i) $S(0) = id_X$ *and* $S(z_1 + z_2) = S(z_1)S(z_2)$ *for all* $z_1, z_2 \in \Sigma_\delta$.

(ii) The map $z \mapsto S(z)$ *is analytic in* Σ_δ.

(iii) $\lim_{\Sigma_{\delta'} \ni z \to 0} S(z)x = x$ *for all* $x \in X$ *and* $0 < \delta' < \delta$.

If, in addition,

(iv) $|S(z)|_{\mathcal{L}(X)}$ *is bounded in* $\Sigma_{\delta'}$ *for every* $0 < \delta' < \delta$*, we call* $(S(z))_{z\in\Sigma_\delta\cup\{0\}}$ *a* bounded *analytic semigroup.*

Obviously, the restriction of an analytic semigroup to the real axis is a C_0–semigroup. Conversely, on can prove (see Theorem II.4.6 in [32]) that a linear operator $A : \mathcal{D}(A) \subseteq X \to X$ generates a bounded analytic semigroup $(S(z))_{z\in\Sigma_\delta\cup\{0\}}$ on X, i.e. A generates a bounded C_0–semigroup $(S(t))_{t\geq 0}$ on X which is extendable to $(S(z))_{z\in\Sigma_\delta\cup\{0\}}$, if and only if A is closed, densely defined, $\Sigma_{\pi/2+\delta} \subseteq \rho(A)$ for some $\delta \in (0, \pi/2]$ and for each $\epsilon \in (0, \delta)$ there exists $M(\epsilon) \geq 1$ such that

$$|R(\lambda, A)|_{\mathcal{L}(X)} \leq \frac{M(\epsilon)}{|\lambda|}$$

for all $0 \neq \lambda \in \overline{\Sigma}_{\pi/2+\delta-\epsilon}$, where $\overline{\Sigma}_{\pi/2+\delta-\epsilon}$ denotes the closure of $\Sigma_{\pi/2+\delta-\epsilon}$. Since the multiplication of a C_0–semigroup $(S(t))_{t\geq 0}$ by $e^{\omega t}$ for some $\omega \in \mathbb{R}$ does not affect the possibility or impossibility of extending it to an analytic semigroup, the results for general C_0–semigroups follow from the corresponding results for bounded C_0–semigroups. Further, we say that an analytic semigroup is *exponentially stable* if the corresponding C_0–semigroup is exponentially stable.

The next proposition will be sufficient for our purposes in Section 5.1 to verify that our linear *drift* operator of the stochastic evolution equation generates an analytic semigroup which is exponentially stable.

Proposition 3.3.3. *Let $A : \mathcal{D}(A) \subseteq U \to U$ be a self–adjoint operator on a Hilbert space U with*

$$s(A) < 0.$$

Then A generates an analytic semigroup on U which is exponentially stable. In particular, we have

$$s(A) = \omega_0.$$

Proof. Corollary II.4.8 and Corollary V.2.10 in [32]. □

Finally, we introduce fractional powers of an operator A for which $(-A)$ generates an analytic semigroup and give some properties of such an operator. For the definition we make the following assumption:

Assumption 3.3.4. $A : \mathcal{D}(A) \subseteq X \to X$ is a densely defined closed linear operator in X such that

$$\Upsilon := \left\{ \lambda \in \mathbb{C} \middle|\ 0 < \frac{\pi}{2} - \epsilon < |arg(\lambda)| \leq \pi \right\} \cup V \subseteq \rho(A)$$

for some $\epsilon \in (0, \pi/2)$, where V is a neighbourhood of zero, and there is $M \geq 1$ such that

$$|R(\lambda, A)|_{\mathcal{L}(X)} \leq \frac{M}{1 + |\lambda|}$$

for all $\lambda \in \Upsilon$.

If A satisfies Assumption 3.3.4 then $(-A)$ is the infinitesimal generator of an analytic semigroup (see Theorem 2.5.2 in [69]).

Definition 3.3.5. *Suppose that A satisfies Assumption 3.3.4. For every $\alpha > 0$ we define*

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt, \tag{3.3.1}$$

$$A^{\alpha} := \left(A^{-\alpha}\right)^{-1}$$

and $A^0 := id_X$. Here $\Gamma(\cdot)$ denotes the gamma function and $(S(t))_{t\geq 0}$ the C_0–semigroup generated by A.

It should be noted that the integral in (3.3.1) converges in the uniform operator topology and that $A^{-\alpha}$ for $\alpha > 0$ is one–to–one (see Lemma 2.6.6 in [69]).
We conclude this subsection with some results relating A^α and the analytic semigroup generated by $(-A)$.

Theorem 3.3.6. *Suppose that A satisfies Assumption 3.3.4 and let $(S(z))_{z\in\Sigma_\delta}$ be the analytic semigroup generated by $(-A)$. Then*

(i) for all $\alpha, \beta \in \mathbb{R}$ we have
$$A^{\alpha+\beta}x = A^\alpha A^\beta x$$
for every $x \in \mathcal{D}(A^\gamma)$, where $\gamma = \max(\alpha, \beta, \alpha + \beta)$.

(ii) For every $\alpha \in \mathbb{R}$ and $x \in \mathcal{D}(A^\alpha)$ we have $S(t)A^\alpha x = A^\alpha S(t)x$ for all $t \geq 0$.

(iii) For every $\alpha \geq 0$ there is $M > 0$ such that for all $t > 0$ the operator $A^\alpha S(t)$ is bounded and
$$|A^\alpha S(t)|_{\mathcal{L}(X)} \leq Mt^{-\alpha}.$$

Proof. Theorem 2.6.8 and Theorem 2.6.13 in [69]. □

3.4. Random Dynamical Systems, Invariant Measures and the Hausdorff Dimension of the Random Attractor

We now recall some required definitions and concepts from the theory of random dynamical systems. For the general theory of random dynamical systems we refer to the excellent monograph [2].
In the following, (X, d) is a complete separable metric space and 2^X denotes the set of all subsets of X. Further, for $B \in 2^X$ we denote by $\overline{B}$ the closure of B in X and by $B^c := B \setminus X$ the complement of B in X. For $x \in X$ and $B, C \in 2^X$ we define the *semidistance* by

$$dist(x, B) := \inf_{b \in B} d(x, b) \quad \text{and} \quad dist(B, C) := \sup_{b \in B} \inf_{c \in C} d(b, c).$$

We make the convention $d(x, \emptyset) = \infty$, where $\emptyset$ denotes the empty set.

Definition 3.4.1. *A family $(\theta(t))_{t\in\mathbb{R}}$ of mappings on Ω into itself is called a* metric dynamical system *and is defined by $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ if it satisfies the following four conditions:*

(i) The mapping $(\omega, t) \mapsto \theta(t)\omega$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$–$\mathcal{F}$ measurable.

(ii) $\theta(0) = id_\Omega =$ identity map in Ω.

(iii) $(\theta(t))_{t\in\mathbb{R}}$ satisfies the flow property*, i.e. $\theta(t+s) = \theta(t) \circ \theta(s)$ for all $s, t \in \mathbb{R}$, where $\circ$ denotes the composition.*

(iv) $(\theta(t))_{t\in\mathbb{R}}$ is a family of measure preserving transformations, i.e. $\mathbb{P}(\theta(t)^{-1}(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$ and $t \in \mathbb{R}$, where $\theta(t)^{-1}(A) := \{\omega \in \Omega \mid \theta(t)\omega \in A\}$.

We say that a metric dynamical system is ergodic *if for all $A \in \mathcal{F}$, such that $\theta(t)^{-1}(A) = A$ for all $t \in \mathbb{R}$, we have $\mathbb{P}(A) \in \{0, 1\}$.*

Notice that conditions (ii) and (iii) in Definition 3.4.1 imply that all $\theta(t)$ are invertible with $\theta(t)^{-1} = \theta(-t)$.

Definition 3.4.2. *A* random dynamical system *(RDS) (on X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ with time $\mathbb{R}$) is a mapping*

$$\phi : \mathbb{R} \times \Omega \times X \to X, \ (t, \omega, x) \mapsto \phi(t, \omega, x),$$

with the following properties:

(i) Measurability*: ϕ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X)$–$\mathcal{B}(X)$ measurable.*

(ii) Cocycle property*: The mappings $\phi(t, \omega) := \phi(t, \omega, \cdot) : X \to X$ form a* cocycle over *θ, i.e. they satisfy $\phi(0, \omega) = id_X$ for all $\omega \in \Omega$ and $\phi(t+s, \omega) = \phi(t, \theta(s)\omega) \circ \phi(s, \omega)$ for all $s, t \in \mathbb{R}, \omega \in \Omega$. Here $\circ$ denotes the composition.*

We call an RDS ϕ continuous *or C^0–RDS if $(t,x) \mapsto \phi(t,\omega,x)$ is continuous for every $\omega \in \Omega$ and we say that an RDS ϕ is a C^k–RDS, where $1 \leq k \leq \infty$ if for each $(t,\omega) \in \mathbb{R}\times\Omega$ the mapping $x \mapsto \phi(t,\omega,x)$ is k–times differentiable w.r.t. $x \in X$ and the derivatives are continuous w.r.t. $(t,x) \in \mathbb{R} \times X$ for each $\omega \in \Omega$.*

We will often omit in the following the addition *'on X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ with time $\mathbb{R}$'*, speaking just of a (continuous or C^k-) RDS ϕ.

Remark 3.4.3. (i) Deterministic dynamical systems (see e.g. Temam [93] or Robinson [78]) are particular cases of RDSs. Indeed, if the RDS ϕ is independent of ω then the RDS decouples into a metric DS and *deterministic* dynamical system.

(ii) To prove the measurability of a random dynamical system ϕ it is sufficient to prove (see Lemma 1.1 in [21]) that

- $\omega \mapsto \phi(t,\omega,x)$ is $\mathcal{F}$–$\mathcal{B}(X)$ measurable for every $(t,x) \in \mathbb{R} \times X$,
- $x \mapsto \phi(t,\omega,x)$ is continuous for every $(t,\omega) \in \mathbb{R} \times \Omega$,
- $t \mapsto \phi(t,\omega,x)$ is continuous for every $(\omega,x) \in \Omega \times X$.

(iii) Condition (ii) in Definition 3.4.2 forces the cocycle to be invertible. In particular, we have
$$\phi(t,\omega)^{-1} = \phi(-t,\theta(t)\omega)$$
for all $(t,\omega) \in \mathbb{R} \times \Omega$, or equivalently,
$$\phi(-t,\omega) = \phi(t,\theta(t)^{-1}\omega)^{-1}$$
for all $(t,\omega) \in \mathbb{R} \times \Omega$. These statements are proved in [2], Theorem 1.1.6. Since ϕ is a cocycle over θ if and only if $\phi(-\cdot,\cdot)$ is a cocycle over θ^{-1}, we call
$$\psi(t,\omega) := \phi(-t,\omega)^{-1} = \phi(t,\theta(t)^{-1}\omega) = \phi(t,\theta(-t)\omega)$$
a *backward cocycle* over θ^{-1}. The important difference between the cocycle ϕ and the backward cocycle ψ lies in the asymptotic behaviour for $t \to \infty$: In general, in contrast to autonomous systems, it makes a big difference in the non–autonomous case between moving points from 0 to t, and moving points from $-t$ to 0. Only in the second case will the result be in the same *fiber* $\{\omega\} \times X$ for all t, hence can be studied for $t \to \infty$. That is the reason why the backward cocycle ψ will be of fundamental importance for the construction of the *random attractor* of ϕ.

Now we are going to introduce the *random $\mathcal{D}$–attractor* of an RDS ϕ.
The *global attractor* in the theory of deterministic dynamical systems, see e.g. Temam [93] or Robinson [78], has become one of the main concepts for the study of the asymptotic behaviour of evolution equations. Crauel and Flandoli [18], Schmalfuß [83] and Schenk–Hoppe [81] have introduced the corresponding generalization of this concept to the stochastic case. We will define the random attractor in the spirit of the paper [83] by Schmalfuß, because we will mainly work with his results.

For convenience, we assume in the following that $(X, |\cdot|_X)$ is a separable Banach space with metric $d(x, y) := |x - y|_X$.

Because of the non–autonomous noise dependence of an RDS, generalized concepts of absorption, attraction and invariance of (random) sets have to be introduced. For that purpose we also recall some facts from the theory of *measurable (closed) random sets*, sometimes also called *measurable multifunctions* (see [21]).

Definition 3.4.4. *(i) A set valued map $D : \Omega \to 2^X$ taking values in closed subsets of X is said to be* measurable *if for each $x \in X$ the map $\omega \mapsto dist(x, D(\omega))$ is $\mathcal{F}$–$\mathcal{B}([0,\infty))$ measurable. In this case D is called a* closed random set (of X).

(ii) A set valued map $\omega \mapsto D(\omega)$ is said to be an open random set *if its complement D^c is a closed random set.*

(iii) A non–empty closed random set D is called bounded *if $\sup_{x\in D(\omega)} |x|_X < \infty$ for all $\omega \in \Omega$.*

Remark 3.4.5. (i) Random sets have been investigated by Castaing and Valadier [12], who addressed them as measurable multifunctions, as also does Schmalfuß in [83]. But we decided to call them (closed) random sets, since this is more often used in the literature.

(ii) It can be shown (see Proposition 2.4 in [21]) that the statement that D is a closed random set is equivalent to the statement that for all open $U \subseteq X$ the set $\{\omega | D(\omega) \cap U \neq \emptyset\}$ is measurable. Consequently, measurability of closed set valued maps does not depend on the choice of the metric d, as might be suggested by the Definition 3.4.4(i). Further, if D is non–void, then the two statements are equivalent (see Theorem 2.6 in [21]) to the assertion that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of measurable maps $x_n : \Omega \to X$ such that

$$\overline{\bigcup_{n\in\mathbb{N}} x_n(\omega)} = D(\omega).$$

Definition 3.4.6. *A* universe of closed random sets $\mathcal{D}$ (of X) *is a system of non–empty closed random sets of X, such that $\mathcal{D}$ is closed under inclusion, i.e. if D and D' are non–empty closed random sets of X, such that $D'(\omega) \subseteq D(\omega)$, for all $\omega \in \Omega$, and $D \in \mathcal{D}$, then $D' \in \mathcal{D}$.*

We are mainly interested in the universe of tempered random sets.

Definition 3.4.7. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ be a metric dynamical system.*

(i) A positive random variable R is called tempered *if*

$$\lim_{t\to\pm\infty} e^{-c|t|} R(\theta(t)\omega) = 0$$

for any $c > 0$ and $\omega \in \Omega$.

(ii) A bounded closed random set D is said to be tempered *if $D(\omega)$ is contained in the closed ball with center* 0 *and tempered radius $R(\omega) := \sup_{x\in D(\omega)} |x|_X$, $\omega \in \Omega$.*

(iii) The system of closed random sets with bounded and non–empty tempered images forms the universe of tempered sets *(of X) which we denote by $\mathcal{G}$.*

Remark 3.4.8. The universe of tempered sets contains only sets which grow *sub–exponentially fast* including every compact deterministic set.

Now we are ready to introduce the concepts of invariance, absorption and attractor.

Definition 3.4.9. *Let ϕ be an RDS and $\mathcal{D}$ a universe of closed random sets.*

(i) A closed random set B is called (strictly) ϕ–forward invariant *if*

$$\phi(t,\omega)B(\omega) \subseteq B(\theta(t)\omega) \quad (\phi(t,\omega)B(\omega) = B(\theta(t)\omega))$$

for all $\omega \in \Omega$, $t \geq 0$.

*(ii) A closed random set $B \in \mathcal{D}$ is called $\mathcal{D}$–*absorbing *if for any $D \in \mathcal{D}$, $\omega \in \Omega$ there exists a time $t_D(\omega) \geq 0$, the so–called* absorption time*, such that for any $t > t_D(\omega)$*

$$\phi(t,\theta(-t)\omega)D(\theta(-t)\omega) \subseteq B(\omega). \tag{3.4.1}$$

(iii) A closed random set $A \in \mathcal{D} \supseteq \mathcal{G}$ with compact values is called random $\mathcal{D}$–attractor *of the RDS ϕ if A is strictly ϕ–forward invariant and for any $\omega \in \Omega$ we have*

$$dist\left(\overline{\phi(t,\theta(-t)\omega)D(\theta(-t)\omega)}, A(\omega)\right) \to 0 \ \text{as}\ t \to \infty \tag{3.4.2}$$

for any $D \in \mathcal{D}$.

If there exists a random $\mathcal{D}$–attractor, then the attractor is already unique in $\mathcal{D}$. Indeed, suppose we have two attractors $A_i \in \mathcal{D}$, $i = 1, 2$. It follows that for any $\omega \in \Omega$

$$dist(A_1(\omega), A_2(\omega)) = \lim_{t\to\infty} dist\left(\overline{\phi(t,\theta(-t)\omega)A_1(\theta(-t)\omega)}, A_2(\omega)\right) = 0.$$

Therefore, $A_1(\omega) \subseteq A_2(\omega)$ for any $\omega \in \Omega$. Similarly, we can find the contrary inclusion. So the $\mathcal{D}$–attractor is unique in $\mathcal{D}$.

Theorem 3.4.10. *Let ϕ be a continuous RDS and $\mathcal{D} \supseteq \mathcal{G}$ a universe of closed random sets. In addition, we assume the existence of a ϕ–forward invariant and $\mathcal{D}$–absorbing closed random set B with compact values. Then the RDS has a unique random $\mathcal{D}$–attractor given by*

$$A(\omega) = \bigcap_{t\in\mathbb{N}} \phi(t,\theta(-t)\omega)B(\theta(-t)\omega).$$

Proof. See Proposition 9.3.2 in [2] or Theorem 2.4 in [83]. □

Remark 3.4.11. (i) Crauel and Flandoli defined in [18] the random attractor as a strictly ϕ–forward invariant closed random set with compact values such that (3.4.2) holds $\mathbb{P}$-a.s. for all bounded deterministic sets of X. The existence of such a random attractor is assured if there is a closed random set B with compact values absorbing (in the sense of (3.4.1)) the universe of deterministic bounded sets. However, such an attractor is in general not an element of this universe of sets.

At the first view, the definition of the random attractor by Crauel and Flandoli suggests that it is not a topological, but a metric concept. But by a deep result of Crauel in [20] this random attractor is $\mathbb{P}$-a.s. uniquely determined already by the property of attracting (in the sense of (3.4.2)) all compact deterministic subsets of X (and of course, being strictly ϕ–forward invariant and a closed random set of X with compact values). This shows that a random attractor for bounded deterministic subsets of X does not depend on the choice of a metric on X: If there exist two attractors for two different metrics (both including the topology of X), then the two attractors must coincide already.

Therefore, notice that in our definition of the random $\mathcal{D}$–attractor $\mathcal{D}$ contains already all compact deterministic sets such that the random $\mathcal{D}$–attractor is not only unique in $\mathcal{D}$ but also $\mathbb{P}$-a.s. uniquely determined.

(ii) The generalized concepts of absorption and attraction in Definition 3.4.9 are called *pullback absorption* and *pullback attraction.* The noise ω is first *pulled back* in time by $\theta(-t)$ and then evolved forward by $\phi(t,\cdot)$, so that we consider the resulting image set at time zero. In general this limit in the *pullback sense* does not imply ω–wise or almost surely convergence forward in time (see [82]). But due to the θ–invariance of the probability measure $\mathbb{P}$, it is easy to prove that the pullback attraction implies forward convergence in probability, i.e.

$$\lim_{t\to\infty} \mathbb{P}\big(\{\omega\in\Omega|\ dist(\phi(t,\omega)D(\omega), A(\theta(t)\omega)) > \epsilon\}\big) = 0 \tag{3.4.3}$$

for all $\epsilon > 0$. This property has been used by Ochs [62] to define a *weak random attractor* as a closed random set with compact values satisfying strictly ϕ–forward invariance and property (3.4.3) for every bounded deterministic set D of X. Any pullback attractor is also a weak attractor, although the converse is false in general.

The concept of the random attractor is also closely related to the concept of an *invariant measure* of an RDS ϕ which we introduce next. For that purpose we first have to provide some additional notations.

Definition 3.4.12. *Let ϕ be an RDS on X over $(\Omega,\mathcal{F},\mathbb{P},(\theta(t))_{t\in\mathbb{R}})$.*

(i) We call the $\mathcal{F}\otimes\mathcal{B}(X)$–measurable mapping

$$(\omega,x)\mapsto \big(\theta(t)\omega,\phi(t,\omega)x\big) =: \Theta(t)(\omega,x),\quad t\in\mathbb{R},$$

skew product *of the RDS ϕ.*

(ii) *Let $\pi_\Omega : \Omega \times X \to \Omega$, $(\omega, x) \mapsto \pi_\Omega(\omega, x) = \omega$, be here and in the following the projection onto Ω and μ a probability measure on $\big(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X)\big)$ with* marginal $\mathbb{P}$, *i.e. $\pi_\Omega\mu = \mathbb{P}$, where $\pi_\Omega\mu$ is the image measure of μ w.r.t. π_Ω. A function*

$$\mu_\cdot(\cdot) : \Omega \times \mathcal{B}(X) \to [0, 1]$$

is said to be the factorization of μ *(w.r.t. $\mathbb{P}$) if*

1. *for all $B \in \mathcal{B}(X)$, $\omega \mapsto \mu_\omega(B)$ is $\mathcal{F}$-$\mathcal{B}([0, 1])$ measurable,*
2. *for $\mathbb{P}$-a.a. $\omega \in \Omega$, $\mu_\omega(\cdot)$ is a probability measure on $(X,\mathcal{B}(X))$,*
3. *for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$*

$$\mu(A) = \int_\Omega \int_X \mathbf{1}_A(\omega, x) d\mu_\omega(x) d\mathbb{P}(\omega),$$

where $\mathbf{1}$ denotes the indicator function.

(iii) *We call the σ–algebra*

$$\mathcal{F}^- := \sigma\big\{\omega \mapsto \phi(s, \theta(-t)\omega)x \mid x \in X, 0 \le s \le t\big\}$$

the past *and*

$$\mathcal{F}^+ := \sigma\big\{\omega \mapsto \phi(t, \theta(s)\omega)x \mid x \in X, 0 \le s, t\big\}$$

the future *generated by ϕ.*

Notice that the skew product Θ satisfies the flow property, i.e. $\Theta(0) = id_{\Omega\times X}$ and $\Theta(t + s) = \Theta(t) \circ \Theta(s)$ for all $t, s \in \mathbb{R}$, and since (X, d) is a complete separable metric space, the factorization of μ exists and is $\mathbb{P}$-a.s. unique (see Proposition 1.4.3 in [2]).

Definition 3.4.13. *Let ϕ be an RDS on X over* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$.

(i) *A probability measure μ on $\big(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X)\big)$ is said to be a ϕ–*invariant measure *if it satisfies $\pi_\Omega\mu = \mathbb{P}$ and $\Theta(t)\mu = \mu$ for all $t \in \mathbb{R}$, where $\pi_\Omega\mu$ and $\Theta(t)\mu$ denote the image measure of μ w.r.t. π_Ω and $\Theta(t)$, respectively.*

(ii) *We call a ϕ–invariant measure μ a ϕ–*invariant forward Markov measure *or ϕ–*invariant backward Markov measure *if the factorization $\omega \mapsto \mu_\omega$ is $\mathcal{F}^-$–measurable or $\mathcal{F}^+$–measurable, respectively.*

(iii) *Let μ be a ϕ–invariant measure and K a closed random set. μ is said to be* supported on K *if $\mu_\omega(K(\omega)) = 1$ $\mathbb{P}$-a.s.*

Remark 3.4.14. (i) By Theorem 1.4.5(ii) in [2] μ is a ϕ–invariant measure if and only if for all $t \in \mathbb{R}$

$$\phi(t, \omega)\mu_\omega = \mu_{\theta(t)\omega} \tag{3.4.4}$$

$\mathbb{P}$-a.s. The property (3.4.4) is also called *equivariant* (w.r.t. ϕ).

(ii) For a discussion when a ϕ–invariant measure is a product measure on $\big(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X)\big)$, i.e. $\mu = \mathbb{P} \otimes \rho$, where ρ is a probability measure on $(X, \mathcal{B}(X))$, the reader is referred to Example 1.4.7 in [2].

(iii) The notion of ϕ–invariant forward/backward Markov measures was introduced by Crauel in [16] and further studied in [17] and [21]. Such Markov measures are of particular interest since they provide the connection between the notion of ϕ–invariant measures of a *white noise RDS* ϕ or sometimes called *RDS with independent increments*, i.e. where $\mathcal{F}^-$ and $\mathcal{F}^+$ are independent and the notion of invariant measures of the associated *Markov semigroup* generated by ϕ. That is also the reason for naming those measures Markov measures. For a detailed description of this relation, actually a *one-to-one correspondence*, the reader is referred to [16] Section 5.2.2, [2] Chapter 1 Section 7 or [22] Section 4. Typical examples of white noise RDSs are RDSs generated by homogeneous Markov chains (Section 2.1.3 in [2]) and stochastic differential equations driven by a standard Brownian motion (Section 2.3.8 in [2]).
Moreover, the *non–Markov property* of ϕ–invariant measures is linked to the positiveness/negativeness of the *Lyapunov exponents* w.r.t. the associated ϕ–invariant measure (see Corollary 5.4 and Remark 5.5 in [17]).

The following theorem shows the relation between the random $\mathcal{D}$–attractor and ϕ–invariant measures.

Theorem 3.4.15. *Let ϕ be a continuous RDS with a unique random $\mathcal{D}$–attractor A.*

(i) Then there exists a ϕ–invariant measure. In particular, all ϕ–invariant measures are supported on A.

(ii) If A is $\mathcal{F}^-$–measurable, then there exists a ϕ–invariant forward Markov measure.

Proof. (i): Theorem 1.6.13 in [2] and Corollary 4.4 in [20]. It should be noted that Corollary 4.4 in [20] is applicable since by our definition of the random $\mathcal{D}$–attractor $\mathcal{D}$ contains already all compact deterministic subsets of X.
(ii): Theorem 1.7.5 in [2]. □

Remark 3.4.16. A ϕ–invariant measure supported on a random $\mathcal{D}$–attractor which is $\mathcal{F}^-$–measurable must not be a ϕ–invariant forward Markov measure (see Remark 6.18 in [21] and references therein).

Next we introduce a general theorem by Schmalfuß [83] to bound the Hausdorff dimension of the random $\mathcal{D}$–attractor. For the following definition of the Hausdorff dimension we refer to Temam [93] and Robinson [78].
In the following let $(X, \langle\cdot,\cdot\rangle_X)$ be a separable Hilbert space with corresponding norm $|\cdot|_X$.
For $d \geq 0$, $\epsilon > 0$ and $Y \subseteq X$ we set

$$\mu_X(Y, d, \epsilon) := \inf \sum_{i \in I} r_i^d,$$

where the infimum is taken over all coverings of Y by a family $(B_i)_{i\in I}$ of balls of radii $0 < r_i \le \epsilon$, where the index set I is countable.

Definition 3.4.17. *Let $d \ge 0$ and $Y \subseteq X$.*

(i) The number $\mu_X(Y,d) \in [0,\infty]$ defined by

$$\mu_X(Y,d) := \lim_{\epsilon \searrow 0} \mu_X(Y,d,\epsilon) = \sup_{\epsilon>0} \mu_X(Y,d,\epsilon)$$

*is called the d-*dimensional Hausdorff measure *of Y.*

(ii) The Hausdorff dimension $d_X^H(Y) \in [0,\infty]$ *of Y is defined by*

$$d_X^H(Y) := \inf\{d \ge 0 |\ \mu_X(Y,d) = 0\},$$

where by convention $\inf \emptyset = \infty$ *for the empty set $\emptyset$.*

Remark 3.4.18. It can be shown that the d–dimensional Hausdorff measure $\mu_X(\cdot,d)$ is indeed a measure on $(X,\mathcal{B}(X))$ and a *natural* generalization of the Lebesgue measure in finite dimensions (see Section 2.10 in Federer [34]).

Now let $L \in \mathcal{L}(X)$ be a compact operator. Then we know that (L^*L) is a compact self–adjoint non–negative operator in X and we can define its *square root* $(L^*L)^{\frac{1}{2}}$ which enjoys the same properties. In particular, there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of X consisting of eigenvectors of $(L^*L)^{\frac{1}{2}}$ with corresponding eigenvalues $(\alpha_n)_{n\in\mathbb{N}}$ (in decreasing order)

$$\alpha_1 \ge \alpha_2 \ge \cdots \ge 0,\ (L^*L)^{\frac{1}{2}} e_n = \alpha_n e_n.$$

The image $L(B)$ of the unit ball B is an *ellipsoid* with *semiaxes* $\{\alpha_n e_n |\ n \in \mathbb{N}, \alpha_n > 0\}$ and (corresponding) lengths of the semiaxes $\{\alpha_n |\ n \in \mathbb{N}, \alpha_n > 0\}$, i.e.

$$L(B) = \left\{ x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle_X e_n |\ \sum_{n\in\mathbb{N},\alpha_n>0} \frac{\langle x, e_n\rangle_X^2}{\alpha_n} \le 1 \right\} \subseteq X.$$

We define for $n \in \mathbb{N}_0$, $s \in [0,1]$, $d = n + s$

$$V_0(L(B)) := 1, \quad V_n(L(B)) := \prod_{i=1}^{n} \alpha_i \text{ for } n \ge 1$$

and

$$V_d(L(B)) := (V_n(L(B)))^{1-s}(V_{n+1}(L(B)))^s.$$

Further, we set $V_d(L) := V_d(L(B))$.

Remark 3.4.19. (i) The number $V_n(L)$ for an $n \in \mathbb{N}$ can be interpreted as an $n-$ *dimensional volume* of the *parallelepiped* generated by $\alpha_1 e_1, \dots, \alpha_n e_n$, i.e. of the set of points

$$\lambda_1 \alpha_1 e_1 + \cdots + \lambda_n \alpha_n e_n,\ 0 \leq \lambda_i \leq 1,\ 1 \leq i \leq n,$$

see Chapter V Section 1.3 in Temam [93].

(ii) We motivate the introduction of the numbers $V_d(L)$ for convenience in the deterministic autonomous setting: Consider the abstract initial–value problem

$$\frac{d\phi}{dt}(t) = f(\phi(t)), \quad t > 0, \quad \phi(0) = x \in X, \tag{3.4.5}$$

with some given function $f : X \to X$. We assume that (3.4.5) has for all $x \in X$ a unique solution $\phi(t, x) = \phi(t)$, $t \geq 0$, and that f is Frechet differentiable in X with differential f' such that for all $h \in X$ the linear initial–value problem

$$\frac{d\Phi}{dt}(t) = f'(\phi(t, x))(\Phi(t)), \quad t > 0, \quad \Phi(0) = h \in X, \tag{3.4.6}$$

has a unique solution $\phi'(t, x)(h) = \Phi(t)$, $t \geq 0$, where $\phi'(t, x)$ is the compact Frechet derivative of $x \mapsto \phi(t, x)$. We can think of $(\phi'(t, x)(h))_{t \geq 0}$ as the *evolution of an infinitesimal displacement along the trajectory* $(\phi(t, x))_{t \geq 0}$ and of the numbers $V_d(\phi'(t, x))$ as the *largest distortion of an infinitesimal d–dimensional volume (element) produced by* $\phi(t, x)$ (see Section 13.2 in [78] or Chapter V Section 2.1 and Section 2.3 in [93]). Now let $A \subseteq X$ be a (compact), strictly ϕ–invariant set, i.e. $\phi(t)A = A$. The idea to bound the Hausdorff dimension of A from above is to study the evolution of $V_d(\phi'(t, x))$, $x \in A$, and to find the smallest d such that all $V_d(\phi'(t, x))$, $x \in A$, contract uniformly for $t \to \infty$, i.e. we have to estimate $\sup_{x \in A} V_d(\phi'(t, x))$. To establish

$$\sup_{x \in A} V_d(\phi'(t, x)) \leq k \tag{3.4.7}$$

for any $k \in (0, 1)$ and $t \geq \bar{t} > 0$ with some fixed $\bar{t}, d > 0$, one usually uses the *trace formula* due to Temam ([93] pp. 362–364). The trace formula asserts that for $n \in \mathbb{N}$, $s \in [0, 1)$ the *uniform expansion factors* at time $t \geq 0$ are given by

$$\overline{q}_n(t) := \sup_{x \in A} \sup_{\substack{h_i \in X, \\ |h_i|_X \leq 1, \\ i=1,\dots,n}} \Big(\int_0^t tr_n(f'(\phi(s, x)) \circ Q_{n,h_1,\dots,h_n}(s, x)) ds \Big),$$

$$\overline{q}_{n+s}(t) := s\overline{q}_{n+1}(t) + (1 - s)\overline{q}_n(t),$$

and

$$\sup_{x \in A} V_{n+s}(\phi'(t, x)) \leq \exp(\overline{q}_{n+s}(t)),$$

where $Q_{n,h_1,\dots,h_n}(s, x)$ is the orthonormal projector in X spanned by $\phi'(t, x)h_1, \dots, \phi'(t, x)h_n$ and tr_n is the trace w.r.t. this subspace. Therefore, to show (3.4.7), it is sufficient to prove that there is $\epsilon > 0$ and $\bar{t} > 0$ such that $\overline{q}_d(t) \leq -\epsilon t < 0$ for all $t \geq \bar{t}$.

The next theorem now gives an estimate of the Hausdorff dimension of the random $\mathcal{D}$–attractor A.

Theorem 3.4.20. *Let X be a separable Hilbert space and ϕ a C^1–RDS over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$. We assume that ϕ has a unique random $\mathcal{D}$–attractor A. Suppose that $x \mapsto \phi(t, \omega, x)$ has the compact Frechet derivative $\phi'(t, \omega, x)$ in x for all $(t, \omega) \in [0, \infty) \times \Omega$ and there is a $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$–$\mathcal{B}([0, \infty))$ measurable function $\nu : [0, \infty) \times \Omega \to [0, \infty)$ such that for all $\epsilon > 0$ and $(t, \omega) \in [0, \infty) \times \Omega$*

$$\sup_{u,v\in A(\omega), |u-v|_X \le \epsilon} \frac{|\phi(t, \omega, v) - \phi(t, \omega, u) - \phi'(t, \omega, u)(v - u)|_X}{|u - v|_X} \le \nu(t, \omega)\epsilon. \tag{3.4.8}$$

In addition, we assume the existence of

- *$d > 0$ and $k \in (0, 1)$ such that $d = n + s$ for some $n \in \mathbb{N}_0$, $s \in (0, 1)$ and*

$$(d + 1)^{\frac{1}{2}} k^{\frac{1}{d}} < \frac{1}{4}, \quad \beta_d k < \left(\frac{1}{4}\right)^{d+1}, \quad \beta_d := 2^n (n + 1)^{\frac{d}{2}}, \tag{3.4.9}$$

- *a positive random variable $\bar{t} : \Omega \to [0, \infty)$ such that*

$$\sup_{x\in A(\omega)} V_d(\phi'(\bar{t}(\omega), \omega, x)) \le k < 1, \ \omega \in \Omega, \tag{3.4.10}$$

- *and a positive random variable $m : \Omega \to [0, \infty)$ satisfying the conditions*

$$m(\omega)^d \ge k, \quad \sup_{x\in A(\omega)} |\phi'(\bar{t}(\omega), \omega, x))|_{\mathcal{L}(X)} \le m(\omega), \ \omega \in \Omega. \tag{3.4.11}$$

Further, we define the positive random variable

$$Z : \Omega \to [0, \infty), \ \omega \mapsto Z(\omega) = \left(\frac{m(\omega)^n}{k}\right)^{\frac{1}{s}} \nu(\bar{t}(\omega), \omega)$$

and the $\mathcal{F}$–$\mathcal{F}$ measurable mapping

$$\tilde{\theta} : \Omega \to \Omega, \ \omega \mapsto \tilde{\theta}\omega := \theta(\bar{t}(\omega))\omega.$$

We assume that $\tilde{\theta}$ preserves $\mathbb{P}$ and

$$\lim_{i\to\infty} \frac{\ln(\max\{1, Z(\tilde{\theta}_i\omega)\})}{i} = 0 \ \ \mathbb{P}\text{-a.s.}, \quad \tilde{\theta}_i := \begin{cases} id & i = 0 \\ \underbrace{\tilde{\theta} \circ \cdots \circ \tilde{\theta}}_{i} & i \in \mathbb{N}. \end{cases} \tag{3.4.12}$$

Then the Hausdorff dimension of A is less than or equal to d $\mathbb{P}$-a.s.
If the underlying metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ is ergodic then the Hausdorff dimension of A is constant $\mathbb{P}$-a.s.

Proof. Theorem 3.2 and Remark 3.3 in [83]. □

Remark 3.4.21. (i) Crauel and Flandoli [19] first developed a method to bound the Hausdorff dimension of random attractors. But their assumptions are very restrictive since they require the noise to be bounded.

Schmalfuß generalized in Theorem 3.4.20 the deterministic case in form of Theorem 3.1 in Temam [93] and, in particular, by including assumption (3.4.12), he overcomes the main difficulty that the Hausdorff measure of $A(\theta(t)\omega)$ is time–noise dependent and not uniformly bounded in t.

Similar to the results of Schmalfuß in Theorem 3.4.20 Debussche obtained in [25] bounds of the Hausdorff dimension of the random attractor with weaker assumptions, but with an ergodic underlying metric dynamical system. In particular, a closer look on the proof of Theorem 3.4.20 in [83] indicates that (3.4.8) can be replaced by the assumption that there is a $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$–$\mathcal{B}([0,\infty))$ measurable function $\nu : [0,\infty) \times \Omega \to [0,\infty)$ and $\alpha > 0$ such that for all $(t,\omega) \in [0,\infty) \times \Omega$ and $u, v \in A(\omega)$

$$|\phi(t,\omega,v) - \phi(t,\omega,u) - \phi'(t,\omega,u)(v-u)|_X \le \nu(t,\omega)|u-v|_X^{1+\alpha} \tag{3.4.13}$$

and $\nu(t,\omega) \ge 1$, and this assumption (among others) was used by Debussche in [25]. But Debussche only derived results for $d \in \mathbb{N}$ and mentioned in Remark 2.6 in [25] that analogue assertions for $d \in \mathbb{R}_+$ might be established with the concept of the *Lyapunov dimension* (see Remark V.3.5 in [93] for a definition). Since we do not have a proof for Debussche's theorem for $d \in \mathbb{R}_+$, we refer to Schmalfuß instead of Debussche.

Finally, Langa and Robinson generalized in [50] the results of Debussche in [25] to bound the *fractal dimension* of a random attractor.

(ii) The technical assumptions (3.4.8) and (3.4.13) are called *uniform differentiability (near trajectories on the attractor A)*. Such assumptions arise from the fact that in order to study the evolution of the infinitesimal volume elements $V_d(\phi'(t,\omega,x))$, $x \in A(\omega)$, we have to study the evolution of the infinitesimal displacements along the trajectory $(\phi(t,\omega,x))_{t\ge 0}$ as described in Remark 3.4.19(ii).

4. Stationary Fractional Ornstein–Uhlenbeck Process

As will be seen in Section 5.2, the stationary solution of our stochastic evolution equation will be given by an infinite series of one–dimensional stationary fractional Ornstein–Uhlenbeck (sfOU) processes. For that purpose we recall in Section 4.1 from [13] basic characteristics and in addition establish to the best of our knowledge new properties of the sfOU process. Section 4.2 is then devoted to introducing *exact* methods to simulate stationary (centered) Gaussian processes and to investigate their applicability to the sfOU process.

4.1. Properties of the Stationary Fractional Ornstein–Uhlenbeck Process

We consider the fractional Langevin equation

$$X_t = X_0 - \nu\alpha \int\limits_0^t X_s ds + \nu^H \sqrt{\lambda} \beta_t^H, \quad t \geq 0, \ X_0 \in \mathbb{R}, \tag{4.1.1}$$

where $(\beta_t^H)_{t\in\mathbb{R}}$ is a fractional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (0,1)$ and $\alpha, \lambda, \nu > 0$. The unique stationary solution of (4.1.1) is given by the stationary fractional Ornstein–Uhlenbeck process

$$Y_t := \nu^H \sqrt{\lambda} \int\limits_{-\infty}^t e^{-(t-u)\nu\alpha} d\beta_u^H, \quad t \in \mathbb{R},$$

see [13]. The uniqueness has to be understood as uniqueness in law in the class of stationary solutions adapted to the natural filtration generated by the two–sided fBm β^H. Notice also that we have $f(\cdot) = \exp(-\nu\alpha(t-\cdot))\mathbf{1}_{(-\infty,t]}(\cdot) \in \widetilde{\Lambda}_H$, where $t \in \mathbb{R}$, $\mathbf{1}$ denotes the indicator function and $\widetilde{\Lambda}_H$ is defined in Definition 3.1.6(i). So we will mainly work with the space $\widetilde{\Lambda}_H$.
First we start with an auxiliary lemma from [13].

Lemma 4.1.1. *Let $H \in (0,1/2) \cup (1/2,1)$, $\delta > 0$, $\gamma < 0$, $N \in \mathbb{N}_0$ and $-\infty \leq a < b \leq c < d < \infty$.*

(i) Then

$$\mathbb{E}\left(\int\limits_a^b e^{\delta u} d\beta_u^H \int\limits_c^d e^{\delta v} d\beta_u^H \right) = H(2H-1) \int\limits_a^b e^{\delta u} \int\limits_c^d e^{\delta v} (v-u)^{2H-2} dv du.$$

(ii) Then

$$e^x \int\limits_x^\infty e^{-y} y^\gamma dy = x^\gamma + \sum_{n=1}^N \left(\prod_{k=0}^{n-1} (\gamma - k) \right) x^{\gamma - n} + \mathcal{O}(x^{\gamma-N-1}) \tag{4.1.2}$$

and

$$e^{-x}\int_1^x e^y y^\gamma dy = x^\gamma + \sum_{n=1}^{N}(-1)^n\left(\prod_{k=0}^{n-1}(\gamma-k)\right)x^{\gamma-n} + \mathcal{O}(x^{\gamma-N-1}) \qquad (4.1.3)$$

as $x\to\infty$, *where* $\sum_{n=1}^{0}=0$.

Proof. (i): Lemma 2.1 in [13].
(ii): Lemma 2.2 in [13]. □

The following proposition will be used several times in this work. For that recall the definition of a short–/long–range dependent process from Definition 3.1.2 and the definition of the gamma function $\Gamma(\cdot)$, hyperbolic cosine $\cosh(\cdot)$ and the generalized hypergeometric function ${}_1F_2$ in Appendix B. Further, we say that a function $g:\mathbb{R}\to\mathbb{R}$ is of *bounded variation on* $\mathbb{R}$ if

$$\sup_{-\infty<a<b<\infty}\sup\left\{\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})|\middle|a=x_0<x_1<\cdots<x_n=b,\ n\in\mathbb{N}\right\}<\infty.$$

Proposition 4.1.2. *Let* Y_t, $t\in\mathbb{R}$, *be the unique stationary solution to (4.1.1) and set* $C(H):=\Gamma(2H+1)\sin(\pi H)/\pi>0$.

(i) Then for all $t,s\in\mathbb{R}$ *we have*

$$\begin{aligned}Cov\left(Y_t,Y_s\right) &:= \mathbb{E}\left(Y_tY_s\right) = C(H)\frac{\nu^{2H}\lambda}{2}\int_{-\infty}^{\infty}e^{i(t-s)x}\frac{|x|^{1-2H}}{(\nu\alpha)^2+x^2}dx\\ &= C(H)\frac{\lambda}{\alpha^{2H}}\int_0^{\infty}\cos((t-s)\nu\alpha x)\frac{x^{1-2H}}{1+x^2}dx\\ &= \begin{cases}\frac{\lambda}{2\alpha}e^{-\nu\alpha|t-s|} & \text{if } H=\frac{1}{2}\\ \frac{\lambda}{2\alpha^{2H}}\cosh(\nu\alpha(t-s))\Gamma(1+2H) \\ \qquad -\frac{\lambda(\nu|t-s|)^{2H}}{2}{}_1F_2\left(1;H+\frac{1}{2},H+1;\frac{(\nu\alpha(t-s))^2}{4}\right) & \text{if } H\neq\frac{1}{2}\end{cases}.\end{aligned} \qquad (4.1.4)$$

In particular, for all $t\in\mathbb{R}$ *we have*

$$Var(Y_t):=\mathbb{E}\left(Y_t^2\right)=\frac{\lambda}{\alpha^{2H}}\Gamma(2H)H.$$

(ii) Fix $T>0$. *Then there is a constant* $C_1(H,\lambda,\nu,\alpha,T)>0$ *such that for any* $t,s\in[-T,T]$ *we have*

$$C_1(H,\lambda,\nu,\alpha,T)|t-s|^{2H}\le\mathbb{E}\left(|Y_t-Y_s|^2\right).$$

(iii) *Then for any $\gamma \in (0, H]$ there is a constant $C_2(H, \nu, \gamma) > 0$ such that for any $t, s \in \mathbb{R}$ we have*

$$\mathbb{E}\left(|Y_t - Y_s|^2\right) \leq C_2(H, \nu, \gamma)\lambda\alpha^{2\gamma - 2H}|t - s|^{2\gamma}. \tag{4.1.5}$$

(iv) *Let $H \in (0, 1/2) \cup (1/2, 1)$ and $N \in \mathbb{N}$. Then for fixed $t \in \mathbb{R}$ and $s \to \infty$ we have*

$$Cov(Y_t, Y_{t+s}) = \frac{1}{2}\lambda\nu^{2H}\sum_{n=1}^{N}(\alpha\nu)^{-2n}\left(\prod_{k=0}^{2n-1}(2H - k)\right)s^{2H-2n} + \mathcal{O}(s^{2H-2N-2}). \tag{4.1.6}$$

In particular, the following assertions are valid:

- *$(Y_t)_{t\in\mathbb{R}}$ is short–range dependent for $H \in (0, 1/2]$.*
- *$(Y_t)_{t\in\mathbb{R}}$ is long–range dependent for $H \in (1/2, 1)$.*
- *$Cov(Y_0, Y_t)$, $r \in \mathbb{R}$, is absolutely integrable for $H \in (0, 1/2]$, i.e. $\int_{-\infty}^{\infty} |Cov(Y_0, Y_t)|dt < \infty$.*

Moreover, if $H \in (1/2, 1)$ then $Cov(Y_0, Y_t) > 0$ for all $t \in \mathbb{R}$.

(v) *If $H \in (0, 1/2]$ then there is $T > 0$ such that $Cov(Y_0, Y_t)$ is decreasing and convex on $[0, T]$.*
If $H \in (1/2, 1)$ then there is $T > 0$ such that $Cov(Y_0, Y_t)$ is decreasing and concave on $[0, T]$.

(vi) *Let $H \in (0, 1/2) \cup (1/2, 1)$. Then for fixed $t \in \mathbb{R}$ and $s \to \infty$ we have*

$$\frac{d}{ds}Cov(Y_t, Y_{t+s}) = -\frac{\nu^{2H-1}\lambda}{\alpha}H(2H - 1)s^{2H-2} + \mathcal{O}(s^{2H-4}).$$

In particular, $Cov(Y_0, Y_t)$, $t \in \mathbb{R}$, is of bounded variation on $\mathbb{R}$.

Proof. For the proof we set $C(H) := \frac{\Gamma(2H+1)\sin(\pi H)}{\pi}$ and fix without loss of generality $-\infty < s < t$.

(i): Notice first that for all $x \in \mathbb{R}$ we have the Fourier transformation

$$e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,t]}}(\cdot)(x) = \int_{\mathbb{R}} e^{u\nu\alpha}\mathbf{1}_{(-\infty,t]}(u)e^{ixu}du = \frac{e^{(\nu\alpha+ix)t}}{(\nu\alpha + ix)}$$

and

$$e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,t]}}(\cdot)(x)\left(e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,s]}}(\cdot)(x)\right)^* = \frac{e^{\nu\alpha(t+s)+ix(t-s)}}{(\nu\alpha)^2 + x^2}$$

where $\left(e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,s]}}(\cdot)(x)\right)^*$ is the complex conjugate of $e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,s]}}(\cdot)(x)$.
To calculate the (co)variance the space $\widetilde{\Lambda}_H$ (Definition 3.1.6(i)) is more convenient than

Λ_H (Definition 3.1.6(ii)). We obtain

$$
\begin{aligned}
&Cov\,(Y_t, Y_s) = \lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^{t} e^{-(t-u)\nu\alpha} d\beta_u^H \int_{-\infty}^{s} e^{-(s-u)\nu\alpha} d\beta_u^H\Big)\\
&= \lambda\nu^{2H} e^{-\nu\alpha(t+s)}\mathbb{E}\Big(\int_{-\infty}^{t} e^{u\nu\alpha} d\beta_u^H \int_{-\infty}^{s} e^{u\nu\alpha} d\beta_u^H\Big)\\
&= \lambda\nu^{2H} e^{-\nu\alpha(t+s)}(e^{\cdot\nu\alpha}\mathbf{1}_{(-\infty,t]}(\cdot), e^{\cdot\nu\alpha}\mathbf{1}_{(-\infty,s]}(\cdot))_{\widetilde{\Lambda}_H}\\
&= \lambda\nu^{2H} e^{-\nu\alpha(t+s)}\frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}\int_{\mathbb{R}} e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,t]}}(\cdot)(x)\big(e^{\cdot\nu\alpha}\widehat{\mathbf{1}_{(-\infty,s]}}(\cdot)(x)\big)^*|x|^{1-2H}dx\\
&= C(H)\frac{\lambda\nu^{2H}}{2}\int_{-\infty}^{\infty} e^{i(t-s)x}\frac{|x|^{1-2H}}{(\nu\alpha)^2+x^2}dx\\
&\overset{y=\frac{x}{\nu\alpha}}{=} C(H)\frac{\lambda}{2\alpha^{2H}}\int_{-\infty}^{\infty} e^{i\nu\alpha(t-s)y}\frac{|y|^{1-2H}}{1+y^2}dy\\
&= C(H)\frac{\lambda}{\alpha^{2H}}\int_{0}^{\infty}\cos(\nu\alpha(t-s)y)\frac{y^{1-2H}}{1+y^2}dy,
\end{aligned}
$$

where we used that $\sin(\cdot)$ and $\cos(\cdot)$ are odd and even functions, respectively. Therefore, we proved the second and third equality in (4.1.4). We only verify the fourth equality in (4.1.4) for $H \neq \frac{1}{2}$ since it is well–known for $H = \frac{1}{2}$. By using Lemma B.1(iii) with $b = |t-s|\nu\alpha$, $z = 1$, $\beta = 2-2H$ and $H \neq \frac{1}{2}$ we get

$$
\begin{aligned}
&C(H)\frac{\lambda}{\alpha^{2H}}\int_{0}^{\infty}\cos(\nu\alpha(t-s)x)\frac{x^{1-2H}}{1+x^2}dx\\
&= \frac{\lambda}{\alpha^{2H}}\frac{\Gamma(2H+1)\sin(\pi H)}{\pi}\Big[\frac{\pi}{2}\frac{\cosh(\nu\alpha(t-s))}{\sin((1-H)\pi)}\\
&\qquad - \Gamma(-2H)\cos((1-H)\pi)(\nu\alpha|t-s|)^{2H}{}_1F_2\left(1; H+\frac{1}{2}, H+1; \frac{(\nu\alpha(t-s))^2}{4}\right)\Big]\\
&=: I_1 - I_2.
\end{aligned}
$$

We remark here that $\Gamma(-2H)$ is well–defined for $H \neq \frac{1}{2}$. In the following we simplify I_1 and I_2. First we start with I_1.

$$
\begin{aligned}
I_1 &= \frac{\lambda}{\alpha^{2H}}\frac{\Gamma(2H+1)\sin(\pi H)}{\pi}\frac{\pi}{2}\frac{\cosh(\nu\alpha(t-s))}{\sin((1-H)\pi)}\\
&= \frac{\lambda}{2\alpha^{2H}}\frac{\Gamma(2H+1)}{\pi}\frac{\pi}{\Gamma(1-H)\Gamma(H)}\frac{\Gamma(H)\Gamma(1-H)}{\pi}\cosh(\nu\alpha(t-s))\pi\\
&= \frac{\lambda}{2\alpha^{2H}}\Gamma(2H+1)\cosh(\nu\alpha(t-s)),
\end{aligned}
$$

where we used in the second equality the relation $\frac{\Gamma(1-x)\Gamma(x)}{\pi} = \frac{1}{\sin(\pi x)}$ (see e.g. [40], 3.241.2, p. 319).

$$
\begin{aligned}
I_2 &= \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H+1)\sin(\pi H)}{\pi} \Gamma(-2H)\cos((1-H)\pi)(\nu\alpha|t-s|)^{2H} \\
&\qquad\qquad\qquad\qquad\qquad \times {}_1F_2\left(1; H+\frac{1}{2}, H+1; \frac{(\nu\alpha(t-s))^2}{4}\right) \\
&= -\frac{\lambda(\nu|t-s|)^{2H}}{2} \frac{\Gamma(2H+1)}{\pi} \Gamma(-2H)\sin(2H\pi) {}_1F_2\left(1; H+\frac{1}{2}, H+1; \frac{(\nu\alpha(t-s))^2}{4}\right) \\
&= -\frac{\lambda(\nu|t-s|)^{2H}}{2} \frac{\Gamma(2H+1)}{\pi} \Gamma(-2H) \frac{\pi}{\Gamma(1-2H)\Gamma(2H)} \\
&\qquad\qquad\qquad\qquad\qquad \times {}_1F_2\left(1; H+\frac{1}{2}, H+1; \frac{(\nu\alpha(t-s))^2}{4}\right) \\
&= -\frac{\lambda(\nu|t-s|)^{2H}}{2} \frac{\Gamma(2H)(2H)\Gamma(-2H)}{\Gamma(-2H)(-2H)\Gamma(2H)} {}_1F_2\left(1; H+\frac{1}{2}, H+1; \frac{(\nu\alpha(t-s))^2}{4}\right) \\
&= \frac{\lambda(\nu|t-s|)^{2H}}{2} {}_1F_2\left(1; H+\frac{1}{2}, H+1; \frac{(\nu\alpha(t-s))^2}{4}\right),
\end{aligned}
$$

where in the second equality we used the relation $\sin(\pi x)\cos((1-x)\pi) = -\sin(2\pi x)/2$, in the third equality $\frac{\Gamma(1-x)\Gamma(x)}{\pi} = \frac{1}{\sin(\pi x)}$ and finally in the fourth equality $\Gamma(1-x) = (-x)\Gamma(-x)$. Therefore, (4.1.4) is proved.
In particular, for $t = s$ we have

$$
\begin{aligned}
Var(Y_t^2) = \mathbb{E}(Y_t^2) &= \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H+1)\sin(\pi H)}{\pi} \int_0^\infty \frac{x^{1-2H}}{1+x^2} dx \\
&= \frac{\lambda}{\alpha^{2H}} \frac{B(1-H, H)\Gamma(2H+1)\sin(\pi H)}{2\pi} \\
&= \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(1-H)\Gamma(H)\Gamma(2H+1)\sin(\pi H)}{2\pi\Gamma(1)} \\
&= \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H+1)}{2} = \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H)2H}{2} = \frac{\lambda}{\alpha^{2H}} \Gamma(2H)H,
\end{aligned}
$$

where we used in the third equality Lemma B.1(i) with $\nu = 2 > \mu = 2-2H > 0$, in the fourth equality the relation $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ for the beta function $B(\cdot,\cdot)$, in the fifth equality the relation $\frac{\Gamma(1-x)\Gamma(x)}{\pi} = \frac{1}{\sin(\pi x)}$ and in the sixth equality the relation $\Gamma(x+1) = \Gamma(x)x$. Hence (i) is proved.
In addition, we note that in the case $H > \frac{1}{2}$ we can calculate the variance also using the

space $|\Lambda|_H$ instead of $\widetilde{\Lambda}_H$: In virtue of Theorem 3.1.8 we have

$$
\begin{aligned}
Var\big(Y_t^2\big) &= \mathbb{E}\big(Y_t^2\big) \\
&= \lambda\nu^{2H}e^{-2\nu\alpha t}(e^{\cdot\nu\alpha}\mathbf{1}_{(-\infty,t]}(\cdot), e^{\cdot\nu\alpha}\mathbf{1}_{(-\infty,t]}(\cdot))_{\widetilde{\Lambda}_H} \\
&= \lambda\nu^{2H}e^{-2\nu\alpha t}(e^{\cdot\nu\alpha}\mathbf{1}_{(-\infty,t]}(\cdot), e^{\cdot\nu\alpha}\mathbf{1}_{(-\infty,t]}(\cdot))_{|\Lambda|_H} \\
&= \lambda\nu^{2H}H(2H-1)\int\limits_{-\infty}^{t}\int\limits_{-\infty}^{t} e^{-(t-u)\nu\alpha}e^{-(t-v)\nu\alpha}|u-v|^{2H-2}dudv \\
&= \frac{\lambda}{\alpha^{2H}}H(2H-1)\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} e^{-x}e^{-y}|x-y|^{2H-2}dxdy \\
&= \frac{\lambda}{2\alpha^{2H}}H(2H-1)\Big[\int\limits_{0}^{\infty} e^{-y}y^{2H-2}dy + \int\limits_{0}^{\infty} e^{-x}x^{2H-2}dx\Big] \\
&= \frac{\lambda}{\alpha^{2H}}H(2H-1)\Gamma(2H-1) = \frac{\lambda}{\alpha^{2H}}\Gamma(2H)H,
\end{aligned}
$$

where we used in the sixth equality Lemma B.1(ii) (with $p, q = 1$ and $f(x) = x^{2H-2}$) and in the last equality the relation $\Gamma(x+1) = \Gamma(x)x$.

(ii)+(iii): By (i) and the change of variables $z = (t-s)\nu\alpha x$ we have

$$
\begin{aligned}
\mathbb{E}\left(|Y_t - Y_s|^2\right) &= 2C(H)\frac{\lambda}{\alpha^{2H}}\int\limits_{0}^{\infty}(1-\cos((t-s)\nu\alpha x))\frac{x^{1-2H}}{1+x^2}dx \\
&= 2C(H)\lambda(\nu(t-s))^{2H}\int\limits_{0}^{\infty}(1-\cos(z))\frac{z^{1-2H}}{((t-s)\nu\alpha)^2+z^2}dz.
\end{aligned} \tag{4.1.7}
$$

Lower bound: For any $-T \leq s < t \leq T$, where $T > 0$ is fixed, we obtain by (4.1.7)

$$
\begin{aligned}
\mathbb{E}\left(|Y_t - Y_s|^2\right) &\geq 2C(H)\lambda(\nu(t-s))^{2H}\inf_{-T\leq s<t\leq T}\int\limits_{0}^{\infty}(1-\cos(z))\frac{z^{1-2H}}{((t-s)\nu\alpha)^2+z^2}dz \\
&= 2C(H)\lambda(\nu(t-s))^{2H}\int\limits_{0}^{\infty}(1-\cos(z))\frac{z^{1-2H}}{(2T\nu\alpha)^2+z^2}dz.
\end{aligned}
$$

Upper bound: By (4.1.7) we have for all $-\infty < s < t$

$$
\begin{aligned}
\mathbb{E}\left(|Y_t - Y_s|^2\right) &\leq 2C(H)\lambda(\nu(t-s))^{2H}\sup_{-\infty<s<t}\int\limits_{0}^{\infty}(1-\cos(z))\frac{z^{1-2H}}{((t-s)\nu\alpha)^2+z^2}dz \\
&= 2C(H)\lambda(\nu(t-s))^{2H}\int\limits_{0}^{\infty}(1-\cos(z))z^{-1-2H}dz.
\end{aligned}
$$

Notice that $\int_0^\infty (1-\cos(z))z^{-1-2H}dz = 2\int_0^\infty \sin^2(z/2)z^{-1-2H}dz$ and this indefinite integral is finite. Indeed, this follows from the fact that we have $0 < H < 1$ and $\sin^2(z) = |\sin(z)-\sin(0)|^2 \le C(\epsilon)z^{2\epsilon}$ for any $\epsilon \in [0,1]$ and a constant $C(\epsilon) > 0$.
It is left to prove the relation (4.1.5) for $\gamma \in (0,H)$. By (4.1.4) and again using the Hölder continuity of $\sin(\cdot)$, we get

$$\begin{aligned}\mathbb{E}\left(|Y_t - Y_s|^2\right) &= 2C(H)\frac{\lambda}{\alpha^{2H}}\int_0^\infty (1-\cos((t-s)\nu\alpha x))\frac{x^{1-2H}}{1+x^2}dx \\ &= 4C(H)\frac{\lambda}{\alpha^{2H}}\int_0^\infty \sin^2((t-s)\nu\alpha x/2)\frac{x^{1-2H}}{1+x^2}dx \\ &\le \widetilde{C}(H,\gamma)\lambda\alpha^{2\gamma-2H}(\nu(t-s))^{2\gamma}\int_0^\infty \frac{x^{1+2\gamma-2H}}{1+x^2}dx < \infty,\end{aligned}$$

for any $\gamma \in (0,H)$ with a constant $\widetilde{C}(H,\gamma) > 0$.

(iv): Statement (4.1.6) is proved in [13], Theorem 2.3. The additional assertions are just a simple consequence of (4.1.6) and the proof of Theorem 2.3 in [13]. In particular, we have by Lemma 4.1.1(i) for $s > 0$

$$\begin{aligned}Cov(Y_0, Y_s) &= \lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^0 e^{\nu\alpha u}d\beta_u^H \int_{-\infty}^s e^{-(s-v)\nu\alpha}d\beta_v^H\Big) \\ &= e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^0 e^{\nu\alpha u}d\beta_u^H \int_{-\infty}^0 e^{v\nu\alpha}d\beta_v^H\Big) \\ &\quad + e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^0 e^{\nu\alpha u}d\beta_u^H \int_0^s e^{\nu\alpha v}d\beta_v^H\Big) \\ &= e^{-\nu\alpha s}Var(Y_0) + \lambda\nu^{2H}e^{-\nu\alpha s}H(2H-1)\int_{-\infty}^0 e^{\nu\alpha u}\int_0^s e^{\nu\alpha v}(v-u)^{2H-2}dvdu\end{aligned}$$

such that $Cov(Y_0, Y_s) > 0$ for all $s \in \mathbb{R}$ if $H \in (1/2, 1)$.
(v): The case $H = \frac{1}{2}$ is obvious.
Since the covariance function is continuous, $Var(Y_0) \ge Cov(Y_0, Y_t)$ for all $t \in \mathbb{R}$ and $Cov(Y_0, Y_t) \to 0$ as $t \to \infty$ by (4.1.6), it is sufficient to prove that there is $T > 0$ such that $\frac{d^2}{dt^2}Cov(Y_0, Y_t)$ is positive on $(0, T]$ for $H \in (0, 1/2)$ and negative for $H \in (1/2, 1)$.
Further, we remark that

$$\frac{d}{dz}{}_1F_2(a_1; b_1, b_2; z) = \frac{a_1}{b_1 b_2}{}_1F_2(a_1+1; b_1+1, b_2+1; z).$$

We have

$$\begin{aligned}\frac{d}{dt}Cov(Y_0,Y_t) = &\frac{1}{2}\Gamma(1+2H)\lambda\alpha^{1-2H}\nu\sinh(\nu\alpha t)\\
&- H\lambda\nu^{2H}t^{2H-1}{}_1F_2(1;H+1/2,H+1;(\nu\alpha t)^2/4)\\
&- \frac{1}{4(H+1/2)(H+1)}\lambda\alpha^2\nu^{2H+2}t^{2H+1}{}_1F_2(2;H+3/2,H+2;(\nu\alpha t)^2/4)\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}Cov(Y_0,Y_t) = &\frac{1}{2}\Gamma(1+2H)\lambda\alpha^{2-2H}\nu^2\cosh(\nu\alpha t)\\
&- H(2H-1)\lambda\nu^{2H}t^{2H-2}{}_1F_2(1;H+1/2,H+1;(\nu\alpha t)^2/4)\\
&- \frac{H}{2(H+1/2)(H+1)}\lambda\alpha^2\nu^{2H+2}t^{2H}{}_1F_2(2;H+3/2,H+2;(\nu\alpha t)^2/4)\\
&- \frac{(2H+1)}{4(H+1/2)(H+1)}\lambda\alpha^2\nu^{2H+2}t^{2H}{}_1F_2(2;H+3/2,H+2;(\nu\alpha t)^2/4)\\
&- \frac{1}{4(H+1/2)(H+1)(H+3/2)(H+2)}\lambda\alpha^4\nu^{2H+4}t^{2H+2}\\
&\qquad {}_1F_2(3;H+5/2,H+3;(\nu\alpha t)^2/4).\end{aligned} \tag{4.1.8}$$

The only important term in (4.1.8) as $t \to 0$ is $-H(2H-1)t^{2H-2}$. This term is positive if $H \in (0,1/2)$ and negative if $H \in (1/2,1)$.

(vi): We will also use some ideas of the proof of Theorem 2.3 in [13]. Fix $t \in \mathbb{R}$. For $s > 0$ large enough we obtain by Lemma 4.1.1(i)

$$\begin{aligned}Cov(Y_t,Y_{t+s}) = Cov(Y_0,Y_s) &= \lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^{0} e^{\nu\alpha u}d\beta_u^H \int_{-\infty}^{s} e^{-(s-v)\nu\alpha}d\beta_v^H\Big)\\
&= e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^{0} e^{\nu\alpha u}d\beta_u^H \int_{-\infty}^{1/(\nu\alpha)} e^{v\nu\alpha}d\beta_v^H\Big)\\
&\quad + e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^{0} e^{\nu\alpha u}d\beta_u^H \int_{1/(\nu\alpha)}^{s} e^{\nu\alpha v}d\beta_v^H\Big)\\
&= e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int_{-\infty}^{0} e^{\nu\alpha u}d\beta_u^H \int_{-\infty}^{1/(\nu\alpha)} e^{\nu\alpha v}d\beta_v^H\Big)\\
&\quad \lambda\nu^{2H}e^{-\nu\alpha s}H(2H-1)\int_{-\infty}^{0} e^{\nu\alpha u}\int_{1/(\nu\alpha)}^{s} e^{\nu\alpha v}(v-u)^{2H-2}dvdu\end{aligned}$$

$$\overset{w=\nu\alpha u, x=\nu\alpha v}{=} e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int\limits_{-\infty}^{0} e^{\nu\alpha u} d\beta_u^H \int\limits_{-\infty}^{1/(\nu\alpha)} e^{\nu\alpha v} d\beta_v^H\Big)$$
$$+ \frac{\lambda}{\alpha^{2H}} H(2H-1)e^{-\nu\alpha s}\int\limits_{-\infty}^{0} e^{w}\int\limits_{1}^{\nu\alpha s} e^{x}(x-w)^{2H-2}dxdw$$
$$\overset{y=x-w, z=x+w}{=} e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int\limits_{-\infty}^{0} e^{\nu\alpha u} d\beta_u^H \int\limits_{-\infty}^{1/(\nu\alpha)} e^{\nu\alpha v} d\beta_v^H\Big)$$
$$+ H(2H-1)\frac{\lambda}{2\alpha^{2H}}e^{-\nu\alpha s}\Big\{\int\limits_{1}^{\nu\alpha s} y^{2H-2}\int\limits_{2-y}^{y} e^{z}dzdy$$
$$+ \int\limits_{\nu\alpha s}^{\infty} y^{2H-2}\int\limits_{2-y}^{2\nu\alpha s-y} e^{z}dzdy\Big\}$$
$$= e^{-\nu\alpha s}\lambda\nu^{2H}\mathbb{E}\Big(\int\limits_{-\infty}^{0} e^{\nu\alpha u} d\beta_u^H \int\limits_{-\infty}^{1/(\nu\alpha)} e^{\nu\alpha v} d\beta_v^H\Big)$$
$$+ H(2H-1)\frac{\lambda}{2\alpha^{2H}}e^{-\nu\alpha s}\Big\{\int\limits_{1}^{\nu\alpha s} e^{y}y^{2H-2}dy - \int\limits_{1}^{\nu\alpha s} e^{2-y}y^{2H-2}dy$$
$$+ \int\limits_{\nu\alpha s}^{\infty} e^{2\nu\alpha s-y}y^{2H-2}dy - \int\limits_{\nu\alpha s}^{\infty} e^{2-y}y^{2H-2}dy\Big\}$$

and therefore

$$\frac{d}{ds}Cov(Y_0, Y_s) = \frac{d}{ds}\Big(\frac{\lambda}{2\alpha^{2H}}H(2H-1)e^{-\nu\alpha s}\Big\{\int\limits_{1}^{\nu\alpha s} e^{y}y^{2H-2}dy$$
$$+ \int\limits_{\nu\alpha s}^{\infty} e^{2\nu\alpha s-y}y^{2H-2}dy\Big\}\Big) + \mathcal{O}(e^{-\nu\alpha s})$$
$$= -\frac{\lambda\nu}{2\alpha^{2H-1}}H(2H-1)\Big(e^{-\nu\alpha s}\int\limits_{1}^{\nu\alpha s} e^{y}y^{2H-2}dy + e^{\nu\alpha s}\int\limits_{\nu\alpha s}^{\infty} e^{-y}y^{2H-2}dy\Big) + \mathcal{O}(e^{-\nu\alpha s})$$

as $s \to \infty$. Using (4.1.2) and (4.1.3) in Lemma 4.1.1(ii) we get

$$\frac{d}{ds}Cov(Y_0, Y_s) = -H(2H-1)\frac{\lambda\nu^{2H-1}}{\alpha}s^{2H-2} + \mathcal{O}(s^{2H-4}) \tag{4.1.9}$$

as $s \to \infty$. In particular, (4.1.9) implies that $Cov(Y_0, Y_s)$ is of bounded variation on $\mathbb{R}$. Indeed, by (i) $Cov(Y_0, Y_s)$ can be represented as a difference of monotone increasing functions on $[0,\infty)$ and by (iv) and (4.1.9) $Cov(Y_0, Y_s)$ converges monotonically to 0 as $s \to \infty$. $\square$

Remark 4.1.3. (i) The second equality in (4.1.4) is stated in Remark 2.4 in [13]. We proved it here explicitly. However, to the best of our knowledge, the representation of the covariance function in (4.1.4) in terms of $\cosh(\cdot)$ and ${}_1F_2(\cdot)$ as well as the statements of Proposition 4.1.2(v) and (vi) seem to be new.

(ii) The idea for the proof of the assertions of Proposition 4.1.2(ii) and (iii) for $\gamma = H$ is based on some relations introduced in the proof of Lemma 2.5 in [55], where an analogue statement as in Proposition 4.1.2(ii) is shown. For us it is very important to derive the dependence of the right-hand side of (4.1.5) on λ and α since this relation will determine the regularity properties of our random velocity field introduced in Section 5.1.
Further, the assertion in Proposition 4.1.2(iii) with the help of Lemma 3.2.3 and Lemma 3.2.4 implies that there is a version of the sfOU process which has $\mathbb{P}$-a.s. Hölder continuous paths with Hölder exponent strict less than the corresponding Hurst parameter H.

Proposition 4.1.2(i) motivates to introduce the function $f : D_H \subseteq \mathbb{R} \to \mathbb{R}$

$$x \mapsto f(x) = C(H)\frac{\nu^{2H}\lambda}{2}\frac{|x|^{1-2H}}{(\nu\alpha)^2 + x^2} \tag{4.1.10}$$

where $H \in (0,1)$, $C(H) = \frac{\Gamma(2H+1)\sin(\pi H)}{\pi}$, $D_H = \mathbb{R}$ if $H \in (0, 1/2]$ and $D_H = \mathbb{R} \setminus \{0\}$ if $H \in (1/2, 1)$. We refer to this function as the *(not normalized) spectral density* of the sfOU process. The reason why we use the term *spectral density* is that $r(t) := Cov(Y_t, Y_0)$, $t \in \mathbb{R}$, and $f(\cdot)$ are *Fourier transform pairs.* In particular, we have

$$r(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx, \quad t \in \mathbb{R},$$

and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} r(t)dt, \quad x \in D_H. \tag{4.1.11}$$

Notice that the right–hand–side of (4.1.11) is well–defined. Indeed, if $H \in (0, 1/2]$ then $r(\cdot)$ is continuous, absolutely integrable and of bounded variation on $\mathbb{R}$ by Proposition 4.1.2(i),(iv), (vi) such that (4.1.11) is satisfied (see [74] Chapter 4.7). For $H \in (1/2, 1)$ we know by Proposition 4.1.2(i),(iv),(vi) that $r(\cdot)$ is only continuous, of bounded variation on $\mathbb{R}$ and $\lim_{|t|\to\infty} r(t) = 0$. In this case the spectral density is unbounded in 0 and (4.1.11) is still valid by Lemma 3.2 in [94]. This is typical for long–range dependent processes.
Figure 1 shows sample paths of the sfOU process with fixed parameters $\lambda = \alpha = \nu = 1$

and time increment $\triangle t = 10^{-2}$ on $[0, 100]$ associated to different Hurst parameters. To simulate the sample paths we used the standard Cholesky method described in the next subsection. If we increase the Hurst parameter, we observe smoother sample paths as expected by Remark 4.1.3(ii).
Figure 2 and Figure 3 visualize the spectral density, whereas Figure 4 and Figure 5 show the covariance function of the sfOU process for different Hurst parameters and $\lambda = \alpha = \nu = 1$. As already mentioned, the spectral density has a pole in 0 for $H \in (1/2, 1)$. Further, we discover that in the case $H \in [1/2, 1)$ the covariance function $Cov(Y_0, Y_t)$ is non–negative and monotonically decreasing to 0 for $|t| \to \infty$, whereas in the case $H \in (0, 1/2)$ $Cov(Y_0, Y_t)$ is negative for large $|t|$ and also converges monotonically to 0 for $|t| \to \infty$. These observed properties are partially proved in Proposition 4.1.2.

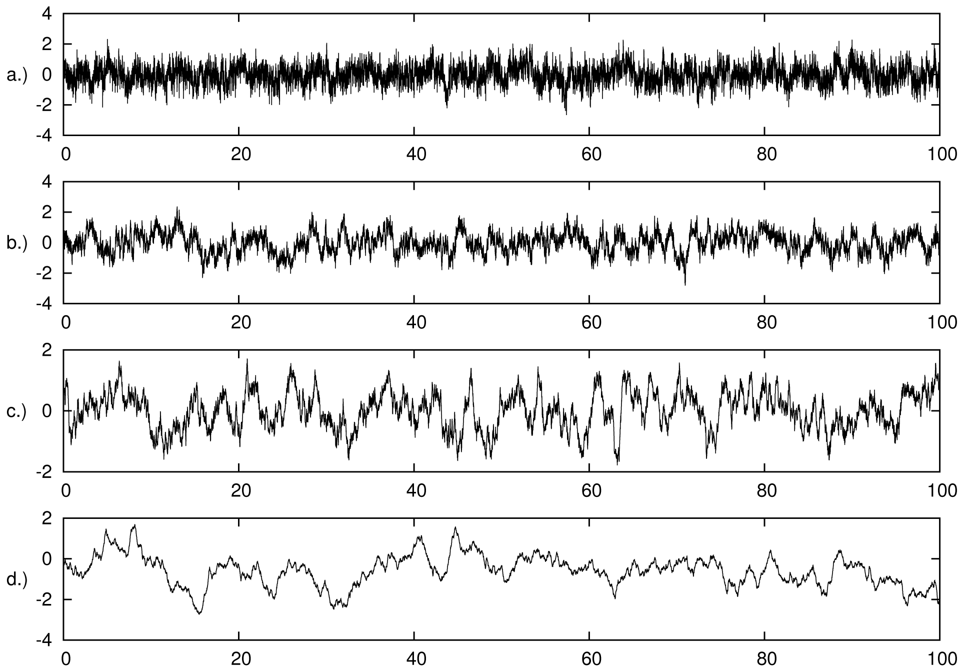

Figure 1: Sample paths of the sfOU process with $\alpha = \lambda = \nu = 1$, $\triangle t = 10^{-2}$ on $[0, 100]$ associated to different Hurst parameters H. a.) $H = 1/10$, b.) $H = 1/4$, c.) $H = 1/2$, d.) $H = 3/4$.

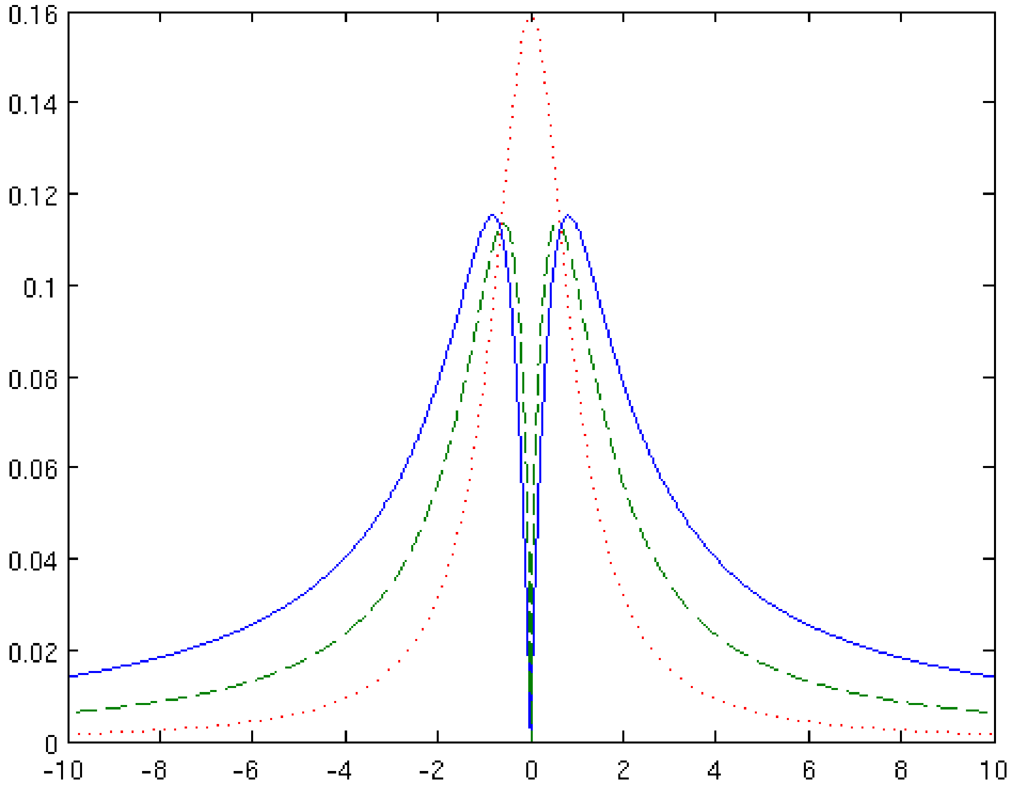

Figure 2: Spectral density of the sfOU process with $\lambda = \alpha = \nu = 1$ associated to different Hurst parameters H: $H = 1/10$ with $-$, $H = 1/4$ with $--$, $H = 1/2$ with $\cdot\cdot$.

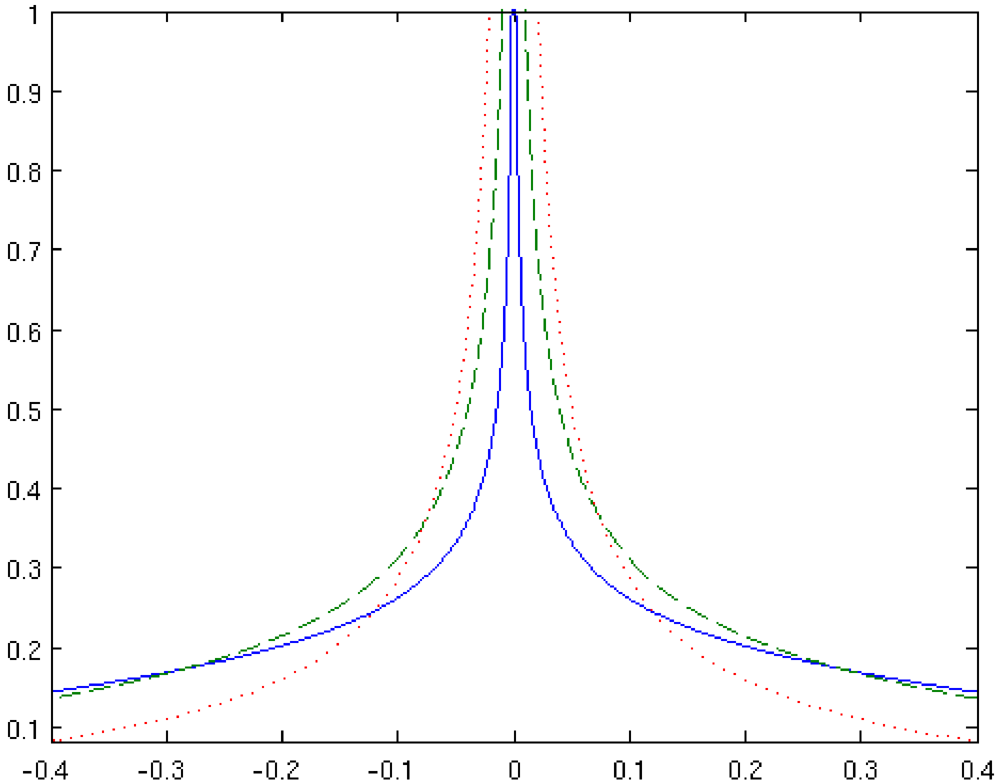

Figure 3: Spectral density of the sfOU process with $\lambda = \alpha = \nu = 1$ associated to different Hurst parameters H: $H = 2/3$ with $-$, $H = 3/4$ with $--$, $H = 9/10$ with $\cdot\cdot$.

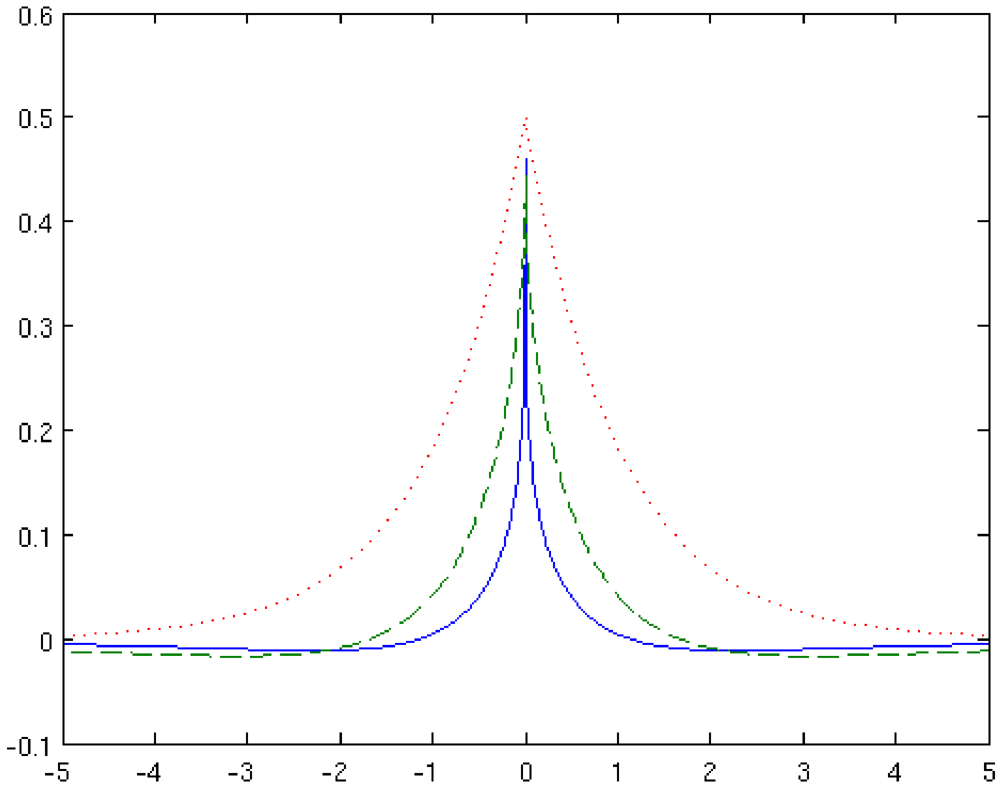

Figure 4: Covariance function of the sfOU process with $\lambda = \alpha = \nu = 1$ associated to different Hurst parameters H: $H = 1/10$ with $-$, $H = 1/4$ with $--$, $H = 1/2$ with $\cdot\cdot$.

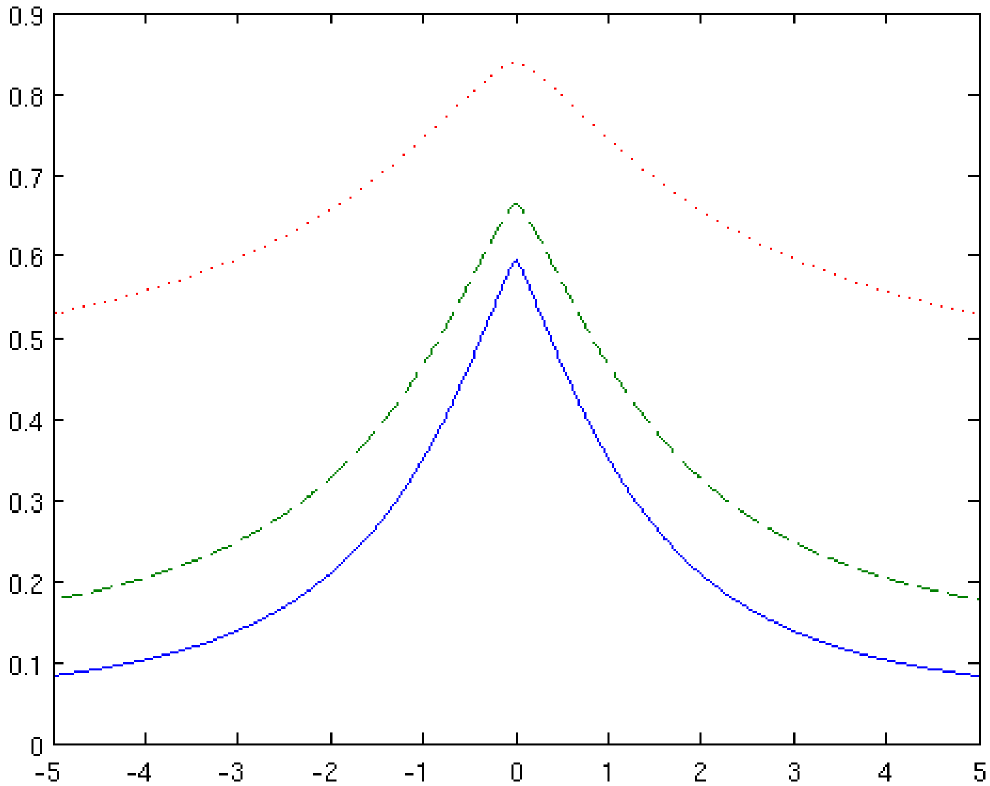

Figure 5: Covariance function of the sfOU process with $\lambda = \alpha = \nu = 1$ associated to different Hurst parameters H: $H = 2/3$ with $-$, $H = 3/4$ with $--$, $H = 9/10$ with $\cdot\cdot$.

4.2. Simulating the Stationary Fractional Ornstein–Uhlenbeck Process

In the following let $(\beta^H_t)_{t\in\mathbb{R}}$ be a real–valued and normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
To simulate the real–valued sfOU process

$$X(t) = \sqrt{\lambda}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha} d\beta^H(u) \quad , t \in \mathbb{R}, \tag{4.2.1}$$

with parameters $\lambda, \alpha, \nu > 0$ and covariance function

$$r(s) := \mathbb{E}\big(X(0)X(s)\big) \quad , s \in \mathbb{R},$$

we fix $\triangle t > 0$, $n \in \mathbb{N}_0$ and set $X_n := X(n\triangle t)$ and $r_n := r(n\triangle t)$. Further, we denote by

$$C_n := \begin{pmatrix} r_0 & r_1 & \cdots & r_n \\ r_1 & r_0 & \cdots & r_{n-1} \\ \vdots & & \ddots & \vdots \\ r_n & r_{n-1} & \cdots & r_0 \end{pmatrix} \tag{4.2.2}$$

the covariance matrix of $\mathbf{X}(n) := \big(X_0, X_1, \ldots, X_n\big)'$, where $(\cdot)'$ denotes the transpose.
We speak of an *exact method* to simulate $\mathbf{X}(n)$ if this method generates a random vector $\mathbf{Y}(n) := \big(Y_0, Y_1, \ldots, Y_n\big)'$ such that

$$\mathbf{X}(n) \overset{d}{=} \mathbf{Y}(n)$$

where $\overset{d}{=}$ denotes the equality in distribution. In particular, we are interested in the exact *standard Cholesky method*, the exact *Durbin–Levinson method* (Section 4.2.1) and the exact *circulant embedding method* (Section 4.2.2). Our main references here are [3, 10, 28].
We do not consider in this work *approximate methods*, i.e. $\mathbf{Y}(n)$ *approximates* $\mathbf{X}(n)$ in some *appropriate* sense, such as *wavelet–based* methods. Here the reader is referred to [26, 27] and references therein.

Remark 4.2.1. (i) For all $n \in \mathbb{N}$ the covariance matrix C_n is *strictly* positive definite. This follows by Proposition 5.1.1 in [10] and the fact that $\lim_{t\to\infty} r(t) = 0$ by Proposition 4.1.2(iv).

(ii) By Proposition 4.1.2(i) we have $r_0 = \frac{\lambda}{\alpha^{2H}}\Gamma(2H)H$ and

$$\begin{aligned} r_n &= r(n\triangle t) \\ &= \frac{\lambda}{\alpha^{2H}} \frac{\Gamma(2H+1)\sin(\pi H)}{\pi} \int\limits_0^\infty \cos(x\nu\alpha n\triangle t)\frac{x^{1-2H}}{1+x^2}dx \\ &= \begin{cases} \frac{\lambda}{2\alpha}e^{-\nu\alpha n\triangle t} & \text{if } H=\frac{1}{2} \\ \dfrac{\lambda}{2\alpha^{2H}}\cosh(\nu\alpha n\triangle t)\Gamma(1+2H) & \\ \qquad - \dfrac{\lambda(\nu n\triangle t)^{2H}}{2}{}_1F_2\left(1; H+\dfrac{1}{2}, H+1; \dfrac{(\nu\alpha n\triangle t)^2}{4}\right) & \text{if } H\neq\frac{1}{2} \end{cases}. \end{aligned} \tag{4.2.3}$$

Relation (4.2.3) provides us on the one hand a nice representation of r_n in terms of the functions cosh and ${}_1F_2$, on the other hand $\cosh(x)$ and ${}_1F_2(1; H+\frac{1}{2}, H+1; x)$ have an asymptotic exponential growth in $x \in [0,\infty)$ such that for large values of $\nu\alpha n\triangle t$ we have to evaluate a difference of very large numbers which leads to numerical instabilities. Therefore, it is also important to note that there are several stable integration methods to evaluate the Fourier integral

$$\int\limits_0^\infty \cos(x\nu\alpha n\triangle t)\frac{x^{1-2H}}{1+x^2}dx,$$

e.g. an adaptive integration method for Fourier integrals provided by the GNU Scientific Library ([41]) and we *strongly* recommend to use such stable methods.

(iii) For the case $H = 1/2$ the stationary Ornstein–Uhlenbeck process can be simulated more efficiently, particularly in view of memory workload by rather using methods which involve the independence of the increments of the process than the methods introduced in this subsection. Such a scheme is described in [87], Section 4.1, and we recommend to use it for this special case.

4.2.1. Standard Cholesky Method and Durbin–Levinson Method

First we introduce the standard Cholesky method and then an improved variant, the Durbin–Levinson method. The methods are based on the *Cholesky decomposition* of the covariance matrix C_n and its inverse C_n^{-1}, respectively. Since C_n and therefore C_n^{-1} are *strictly* positive definite by Remark 4.2.1(i), it turns out that both methods are applicable to simulate $\mathbf{X}(n)$.

Since the covariance matrix C_n, defined in (4.2.2), is symmetric and strictly positive definite, it admits a *Cholesky decomposition*

$$C_n = G_n G_n',$$

where $G_n := (g_{ij})_{i,j=0,1,\dots,n}$ is square lower triangular, i.e. $g_{ij} = 0$ for $j > i$. The entries of G_n are given by (see [3])

$$g_{ij} := \begin{cases} 0 & \text{if } i < j \\ \sqrt{(c_{ii} - \sum_{k=0}^{i-1} g_{ik}^2)} & \text{if } i = j \\ \frac{1}{g_{jj}}(c_{ij} - \sum_{k=0}^{j-1} g_{ik}g_{jk}) & \text{if } i > j \end{cases}$$

and therefore can be computed recursively.

The *standard Cholesky method* works now as follows: Let $Z_0, Z_1, \dots, Z_n$ be independent and identically distributed standard normal random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. We define the random variables $Y_0, Y_1, \dots, Y_n$ by

$$Y_i := \sum_{k=0}^{i} g_{ik} Z_k \quad , i = 0, 1, \dots, n.$$

Set $\mathbf{Y}(n) := \big(Y_0, Y_1, \dots, Y_n\big)'$ and $\mathbf{Z}(n) := \big(Z_0, Z_1, \dots, Z_n\big)'$. We have

$$\mathbf{Y}(n) \overset{d}{=} \mathbf{X}(n),$$

since

$$\mathbb{E}\left(\mathbf{Y}(n)\mathbf{Y}(n)'\right) = \mathbb{E}\left(G_n\mathbf{Z}(n)(G_n\mathbf{Z}(n))'\right) = G_n I_n G_n' = C_n = \mathbb{E}\left(\mathbf{X}(n)\mathbf{X}(n)'\right), \quad (4.2.4)$$

where I_n denotes the $n \times n$ identity matrix. Therefore, to sample $\mathbf{X}(n)$ we sample $\mathbf{Y}(n)$.

Remark 4.2.2. (i) In view of (4.2.4) the standard Cholesky method is *exact*, i.e. no approximation is involved.

(ii) The main drawback of this method is that it is slow because of the complexity $\mathcal{O}(n^3)$. Moreover memory could be a problem since we need to store all non–zero g_{ij}.

(iii) The advantage of this scheme is that we do not need to set the time horizon in advance. Further, the standard Cholesky method can also be used to simulate non–stationary Gaussian processes. For more details the reader is referred to [3] pp. 311–313.

As a second approach we introduce a recursive algorithm of generating X_{n+1} given $X_0, X_1, \dots, X_n$. Thus we need to know the conditional distribution of X_{n+1} given $X_0, X_1, \dots, X_n$. By a general formula for multivariate normal distributions, see Appendix A1 in [3], this distribution is normal with mean

$$\mu_{n+1} := (C_n^{-1} J_n r_{1:n+1})'(X_0, X_1, \dots, X_n)', \quad \mu_0 := 0, \quad (4.2.5)$$

and variance

$$\sigma_{n+1}^2 := r_0 - (J_n r_{1:n+1})' C_n^{-1} J_n r_{1:n+1}, \quad \sigma_0^2 := r_0, \quad (4.2.6)$$

shortly

$$X_{n+1}|(X_0, X_1, \ldots, X_n) \sim \mathcal{N}(\mu_{n+1}, \sigma_{n+1}^2). \tag{4.2.7}$$

Here we set $r_{1:n+1} := (r_1, r_2, \ldots, r_{n+1})'$ and $J_n = (j_{lk})_{l,k=0,1,\ldots,n}$ denotes the *exchange matrix* with ones on the antidiagonal, i.e. $j_{lk} = 1$ if $l = n - k$ and $j_{lk} = 0$ elsewhere, $l, k = 0, 1, \ldots, n$. Therefore, to generate $(X_0, X_1, \ldots, X_{n+1})'$ recursively, first we sample $X_0 \sim \mathcal{N}(0, r_0)$ and then $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ recursively for $i = 1, 2, \ldots, n+1$. Obviously, the main task is to find an effective recursive algorithm which avoids the matrix inversion of C_n in the calculation of μ_n and σ_n^2 such that we can effectively solve the linear system $C_n z = J_n r_{1:n+1}$.

For this purpose we introduce the *Durbin–Levinson algorithm* which computes recursively the Cholesky decomposition

$$C_n^{-1} = L_n' D_n^{-1} L_n,$$

where $L_n = (l_{ij})_{i,j=0,1,\ldots,n}$ is unit lower triangular, i.e. $l_{ii} = 1$ and $l_{ij} = 0$ for $j > i$, $i, j = 0, 1, \ldots, n$, and $D_n = (d_{ij})_{ij=0,1,\ldots,n}$ is diagonal, i.e. $d_{ij} = 0$ for $i \neq j$, $i, j = 0, 1, \ldots, n$. Recall that by Remark 4.2.1(i) C_n and C_n^{-1} are strictly positive definite such that $d_{ii} > 0$, $i = 0, 1, \ldots, n$. Further, notice that

$$C_{n+1} = \begin{pmatrix} C_n & J_n r_{1:n+1} \\ r_{1:n+1}' J_n & r_0 \end{pmatrix} = \begin{pmatrix} r_0 & r_{1:n+1}' \\ r_{1:n+1} & C_n \end{pmatrix}. \tag{4.2.8}$$

Relation (4.2.8) yields with $\psi_{n+1} := -C_n^{-1} J_n r_{1:n+1}$

$$\begin{aligned} C_{n+1}^{-1} &= \frac{1}{\sigma_{n+1}^2} \begin{pmatrix} \sigma_{n+1}^2 C_n^{-1} + \psi_{n+1}\psi_{n+1}' & \psi_{n+1} \\ \psi_{n+1}' & 1 \end{pmatrix} \\ &= \begin{pmatrix} I_n & \psi_{n+1} \\ 0_{(n+1)\times 1}' & 1 \end{pmatrix} \begin{pmatrix} C_n^{-1} & 0_{(n+1)\times 1} \\ 0_{(n+1)\times 1}' & 1/\sigma_{n+1}^2 \end{pmatrix} \begin{pmatrix} I_n & 0_{(n+1)\times 1} \\ \psi_{n+1}' & 1 \end{pmatrix} \\ &= \begin{pmatrix} L_n' & \psi_{n+1} \\ 0_{(n+1)\times 1}' & 1 \end{pmatrix} \begin{pmatrix} D_n^{-1} & 0_{(n+1)\times 1} \\ 0_{(n+1)\times 1}' & 1/\sigma_{n+1}^2 \end{pmatrix} \begin{pmatrix} L_n & 0_{(n+1)\times 1} \\ \psi_{n+1}' & 1 \end{pmatrix} \\ &= L_{n+1}' D_{n+1}^{-1} L_{n+1}, \end{aligned} \tag{4.2.9}$$

where $0_{(n+1)\times 1}$ is the column vector with zeros. In particular, we have $d_{nn} = \sigma_n^2$ and by (4.2.5)

$$\mu_0 = 0, \quad \mu_n = -\psi_n'(X_0, X_1, \ldots, X_{n-1})', \tag{4.2.10}$$

$n \geq 1$!

The Durbin–Levinson algorithm computes ψ_n and σ_n^2 (and therefore L_n and D_n^{-1}) recursively. More precisely, assume that ψ_n and σ_n^2 are known. From (4.2.9) one can deduce

$$\psi_1 = \rho_1 = -r_1/r_0, \quad \psi_{n+1} = \begin{pmatrix} 0 \\ \psi_n \end{pmatrix} + \rho_{n+1} \begin{pmatrix} 1 \\ J_{n-1}\psi_n \end{pmatrix}, \ n \geq 1, \tag{4.2.11}$$

and

$$\sigma_0^2 = r_0, \quad \sigma_{n+1}^2 = \sigma_n^2(1 - \rho_{n+1}^2), \ n \geq 0, \tag{4.2.12}$$

with

$$\rho_1 := -r_1/r_0, \quad \rho_{n+1} := -(r_{n+1} + \psi_n' r_{1:n})/\sigma_n^2, \ n \geq 1. \tag{4.2.13}$$

A detailed derivation of (4.2.11)-(4.2.13) is given in [43]. The reader is also referred to [95] and Proposition 5.2.1 in [10] for slight variations of the Durbin–Levinson algorithm.

We apply now this recursive procedure to generate X_{n+1} given $X_0, X_1, \ldots, X_n$:

We start with $X_0 \sim \mathcal{N}(\mu_0 = 0, \sigma_0^2 = r_0)$, $\psi_1 = \rho_1 = -r_1/r_0$, $\sigma_1^2 = r_0(1 - r_1^2/r_0^2)$, $\mu_1 = -\psi_1 X_0$ and $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$.

For $1 \leq i \leq n$: Given ψ_i, σ_i^2 and $(X_0, X_1, \ldots, X_i)$ we compute ρ_{i+1}, σ_{i+1}^2, ψ_{i+1}, μ_{i+1} and X_{i+1} according to (4.2.13), (4.2.12), (4.2.11), (4.2.10) and (4.2.7).

Remark 4.2.3. (i) Due to the definition of the Durbin–Levinson algorithm, the method is exact.

(ii) The complexity of the Durbin–Levinson algorithm is $\mathcal{O}(n^2)$. Therefore, it is faster than the standard Cholesky method with complexity $\mathcal{O}(n^3)$. But it is also known that the Durbin–Levinson algorithm is more sensitive to computational inaccuracies like round–off errors than the standard Cholesky method.

(iii) We have

$$\mu_{n+1} = \mathbb{E}(X_{n+1} | X_0, X_1, \ldots, X_n)$$

and

$$\sigma_{n+1}^2 = \mathbb{E}((X_{n+1} - \mu_{n+1})^2),$$

where $\mathbb{E}(X_{n+1} | X_0, X_1, \ldots, X_n)$ denotes the *conditional expectation* of X_{n+1} given $X_0, X_1, \ldots, X_n$. Therefore the Durbin–Levinson method is closely related to *prediction* in *time–series analysis*, where μ_n is the best (in mean–square sense) *linear predictor* for X_n with *prediction error* σ_n^2. In this context we should also mention the *innovation algorithm* which as well computes recursively μ_n and σ_n^2. For more details the reader is referred to [10] Chapter 5 and [3] Chapter 11.

4.2.2. Circulant Embedding Method

Analogously to the Cholesky–based methods, introduced in the last subsection, the *circulant embedding method* (CEM) tries to find a square root of a *circulant matrix* with embedded covariance matrix C_n. A circulant matrix of dimension $n \in \mathbb{N}$ is a $n \times n$

matrix of the form

$$A_n := \begin{pmatrix} a_0 & a_{n-1} & & \cdots & a_2 & a_1 \\ a_1 & a_0 & a_{n-1} & & \cdots & a_2 \\ & a_1 & a_0 & a_{n-1} & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & \vdots & & \ddots & \ddots & a_{n-1} \\ a_{n-1} & a_{n-2} & & \cdots & a_1 & a_0 \end{pmatrix},$$

$a_j \in \mathbb{C}$, $j = 0, 1, \ldots, n-1$, where the eigenvalues of A_n are given by

$$\phi_k = \sum_{j=0}^{n-1} a_j e^{-2\pi i \frac{jk}{n}}, \quad k = 0, 1, \ldots, n-1, \tag{4.2.14}$$

see e.g. Proposition XI.3.1 in [3]. Further, we recall that a one–dimensional *discrete Fourier transform* (DFT) of a vector $\mathbf{a}(n) = (a_0, a_1, \ldots, a_{n-1})' \in \mathbb{C}^n$ is defined by

$$\hat{\mathbf{a}}(n) = \mathbb{F}_n \mathbf{a}(n)/n, \qquad \hat{a}_k = \frac{1}{n} \sum_{j=0}^{n-1} a_j e^{2\pi i \frac{kj}{n}}, \quad k = 0, 1, \ldots, n-1, \tag{4.2.15}$$

with one–dimensional *inverse* DFT

$$\mathbf{a}(n) = n \mathbb{F}_n^{-1} \hat{\mathbf{a}}(n), \qquad a_k = \sum_{j=0}^{n-1} \hat{a}_j e^{-2\pi i \frac{kj}{n}}, \quad k = 0, 1, \ldots, n-1, \tag{4.2.16}$$

where $i = \sqrt{-1}$. Here $\mathbb{F}_n$ denotes the square *Fourier matrix* of order n with rows and columns indexed by $\{0, 1, \ldots, n-1\}$ and kj–th entry $e^{2\pi i \frac{kj}{n}}$. $\mathbb{F}_n^{-1}$ is its inverse given by $\mathbb{F}_n^*/n$, i.e. by the complex conjugate of $\mathbb{F}_n$ divided by n. Computing (4.2.15) and (4.2.16) via *standard* matrix multiplication has complexity $\mathcal{O}(n^2)$. But if $n = 2^m$ for some $m \in \mathbb{N}$ the *fast Fourier transform* (FFT) algorithm can be applied to calculate (4.2.15) and (4.2.16) which uses some *clever* manipulation allowing one to reduce the complexity to $\mathcal{O}(n \log(n))$ ([73]). Therefore, by (4.2.14) the FFT algorithm can be applied to calculate the eigenvalues of A_n. Moreover, we have

$$A_n = \mathbb{F}_n \Phi_n \mathbb{F}_n^*/n, \tag{4.2.17}$$

where Φ_n is the diagonal matrix with, in general, complex numbers ϕ_k, $k = 0, 1, \ldots, n-1$ on the diagonal, defined in (4.2.14).
The main idea of the CEM is to embed the covariance matrix C_n in the upper left corner

of a *symmetric* circulant matrix of order $2n$ in form of

$$\widetilde{C}_{2n} := \left(\begin{array}{ccccc|ccccc} r_0 & r_1 & \cdots & r_{n-1} & r_n & r_{n-1} & r_{n-2} & \cdots & r_2 & r_1 \\ r_1 & r_0 & \cdots & r_{n-2} & r_{n-1} & r_n & r_{n-1} & \cdots & r_3 & r_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n-1} & r_{n-2} & \cdots & r_0 & r_1 & r_2 & r_3 & \cdots & r_{n-1} & r_n \\ r_n & r_{n-1} & \cdots & r_1 & r_0 & r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} \\ \hline r_{n-1} & r_n & \cdots & r_2 & r_1 & r_0 & r_1 & \cdots & r_{n-3} & r_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_1 & r_2 & \cdots & r_n & r_{n-1} & r_{n-2} & r_{n-3} & \cdots & r_1 & r_0 \end{array}\right), \tag{4.2.18}$$

and under the assumption that $\widetilde{C}_{2n}$ is in addition positive definite and in view of (4.2.17), $\widetilde{C}_{2n}$ admits the decomposition $\widetilde{C}_{2n} = B_{2n}(B^*_{2n})'$, where

$$B_{2n} := \frac{1}{\sqrt{2n}} \mathbb{F}_{2n} \Lambda_{2n}^{\frac{1}{2}} \tag{4.2.19}$$

and $\Lambda_{2n}^{\frac{1}{2}}$ is the diagonal matrix with $\sqrt{\lambda_k}$, $k = 0, 1, \dots, 2n-1$, on the diagonal, defined by the non–negative numbers

$$\begin{aligned} \lambda_k &= \sum_{j=0}^{2n-1} s_j e^{-2\pi i \frac{jk}{2n}} = \sum_{j=0}^{n} r_j e^{-2\pi i \frac{jk}{2n}} + \sum_{j=n+1}^{2n-1} r_{2n-j} e^{-2\pi i \frac{jk}{2n}} \\ &= r_0 + r_n(-1)^k + 2\sum_{j=1}^{n-1} r_j \cos(\pi jk/n), \end{aligned} \tag{4.2.20}$$

$k = 0, 1, \dots, 2n-1$, with $s_j = r_j$, $j = 0, 1, \dots, n$, and $s_j = r_{2n-j}$, $j = n+1, n+2, \dots, 2n-1$. Further, in case that $\widetilde{C}_{2n}$ is positive definite, there is a mean zero Gaussian random vector $\widetilde{\mathbf{Y}}(2n-1) = (\widetilde{Y}_0, \widetilde{Y}_1, \dots, \widetilde{Y}_{2n-1})'$ such that $\mathbb{E}(\widetilde{\mathbf{Y}}(2n-1)\widetilde{\mathbf{Y}}(2n-1)') = \widetilde{C}_{2n}$. Since C_n is embedded in the upper left corner, the first $(n+1)$ elements of $\widetilde{\mathbf{Y}}(2n-1)$ have the desired covariance matrix C_n. To construct such a random vector it is more convenient to work with complex–valued random variables. With this approach we will build a random vector $\mathbf{Y}(2n-1) = (Y_0, Y_1, \dots, Y_{2n-1})'$ such that $\mathbf{Y}^{(0)}(n-1) = (Y_0, Y_1, \dots, Y_{n-1})'$ has the desired Gaussian distribution with mean zero and covariance matrix C_{n-1}.
For a given positive definite $\widetilde{C}_{2n}$ (and $n = 2^m$ for some $m \in \mathbb{N}$) the following CEM algorithm taken from [15] uses twice the FFT algorithm:

1.) Compute λ_k, $k = 0, 1, \dots, 2n-1$, defined in (4.2.20), via the FFT algorithm.

2.) Check that $\lambda_k \geq 0$, $k = 0, 1, \dots, 2n-1$.

3.) Generate independent standard Gaussian random variables $U_k^{(0)}$, $k = 0, 1, \dots, 2n-1$ and $U_k^{(1)}$, $k = 1, 2, \dots, n-1, n+2, \dots, 2n-1$ such that $U^{(0)} = (U_1^{(0)}, \dots, U_{2n-1}^{(0)})$ and $U^{(1)} = (U_0^{(1)}, \dots, U_{n-1}^{(1)}, U_{n+1}^{(1)}, \dots, U_{2n-1}^{(1)})$ are mutually independent, and compute the complex–valued sequence

$$w_k := \begin{cases} \sqrt{2n\lambda_0}U_0^{(0)}, & k=0 \\ \sqrt{n\lambda_k}(U_{2k-1}^{(0)} + iU_{2k}^{(1)}), & 1 \le k \le n-1 \\ \sqrt{2n\lambda_n}U_n^{(0)}, & k=n \\ \sqrt{n\lambda_k}(U_{2n-k}^{(0)} - iU_{2n-k}^{(1)}), & n+1 \le k \le 2n-1. \end{cases}$$

4.) Compute $\mathbf{Y}(2n-1) = (Y_0, Y_1, \dots, Y_{2n-1})'$ by

$$Y_k = \frac{1}{2n}\sum_{j=0}^{2n-1} w_j e^{2\pi i \frac{kj}{2n}}, \qquad k = 0, 1, \dots, 2n-1, \tag{4.2.21}$$

via FFT algorithm. $\mathbf{Y}^{(0)}(n-1) = (Y_0, Y_1, \dots, Y_{n-1})'$ has the desired Gaussian distribution.

The positive definiteness assumption for $\widetilde{C}_{2n}$ is crucial for the CEM algorithm and we have to investigate whether this is satisfied for the sfOU process. The next theorem gives sufficient conditions to ensure this assumption. We will discuss the applicability of these conditions to the sfOU process at the end of this subsection.

Theorem 4.2.4. *Let $n \in \mathbb{N}$, $\overline{r}_0 > 0$ and $\overline{r}_1, \dots, \overline{r}_n \in \mathbb{R}$. Define $\overline{C}_n$ and $\widetilde{\overline{C}}_{2n}$ as in (4.2.2) and (4.2.18) with $\overline{r}_0, \overline{r}_1, \dots, \overline{r}_n$.*

(i) If $\overline{r}_0, \overline{r}_1, \dots, \overline{r}_n$ form a sequence that is convex, decreasing and $\overline{r}_0 + \overline{r}_n + 2\sum_{j=1}^{n-1}\overline{r}_j \ge 0$, then $\widetilde{\overline{C}}_{2n}$ is positive definite.

(ii) If $\overline{r}_k < 0$, $k = 1, \dots, n$, then $\widetilde{\overline{C}}_{2n}$ is positive definite.

Proof. (i): Theorem 2 in [28] and the remark after that theorem.
(ii): Proposition 3.1 in [15]. □

Remark 4.2.5. (i) The CEM was first introduced by Davis et al. ([24]) and then generalized by Dietrich and Newsam ([28]). For an extension of the CEM to stationary Gaussian vector fields the reader is referred to [98].

(ii) The advantage of the CEM is that it is an exact method with complexity $\mathcal{O}(n\log(n))$ since the FFT algorithm is involved. The drawback is that one needs to predefine the simulation time horizon n.

(iii) The standard Cholesky method in mind, notice that in view of (4.2.15) and (4.2.19) we compute $\mathbf{Y}(2n-1) = \sqrt{2n}B_{2n}\mathbf{Z}(2n-1) = \frac{1}{2n}\mathbb{F}_{2n}\Lambda_{2n}^{\frac{1}{2}}\mathbf{Z}(2n-1)$ in (4.2.21) where $\mathbf{Z}(2n-1) = (Z_0, Z_1, \dots, Z_{2n-1})' = (w_0/\sqrt{\lambda_0}, w_1/\sqrt{\lambda_1}, \dots, w_{2n-1}/\sqrt{\lambda_{2n-1}})'$ and $Z_k = 0$ if $\lambda_k = 0$. But the matrix B_{2n} defined in (4.2.19) is not the matrix one would obtain via the Cholesky decomposition, since B_{2n} is not lower triangular.

(iv) The CEM can be used to simulate paths of a fractional Brownian motion (fBm) $(\beta_t^H)_{t\geq 0}$ with arbitrary Hurst parameter $H \in (0,1)$. More precisely, first one applies the CEM to get a fractional Gaussian noise (fGn) sample, i.e. $\overline{X}_k = \beta_{k+1}^H - \beta_k^H$, $k = 0,1,\ldots,n-1$ with covariance function $\mathbb{E}(\overline{X}_k\overline{X}_0) = (|k-1|^{2H} - 2|k|^{2H} + |k+1|^{2H})/2$, then computes the cumulative sums and finally uses the self–similarity property of the fBm to obtain a sample of the fBm $(\beta_{k\triangle t}^H)$, $k = 0,1,\cdots,n-1$ with a desired sample size $\triangle t > 0$. The applicability of the CEM to fGn for any $n \in \mathbb{N}$ can be proved by verifying the assumption of Theorem 4.2.4(i) for $H \in [1/2,1)$ ([39]) and Theorem 4.2.4(ii) for $H \in (0,1/2)$ ([15]).

(v) It should be noted that a strictly positive definite covariance matrix $\overline{C}_n$ of a general stationary Gaussian process can always be embedded in some circulant matrix of order $m \geq 2n$. Here the reader is referred to the discussion at the beginning of Section 3 in [28] and references therein. We leave the applicability of this approach to the sfOU process for future work.

Finally, we discuss the applicability of the CEM to the sfOU process, i.e. in particular whether the circulant matrix $\widetilde{C}_{2n}$ defined in (4.2.18) is positive definite. Notice first that by (4.2.2) and (4.2.3) the positive definiteness of the circulant matrix $\widetilde{C}_{2n}$ is independent of $\lambda > 0$. Further, if $\widetilde{C}_{2n}$ is positive/negative definite with $c = \overline{\nu}\overline{\alpha} > 0$ for some fixed $\overline{\nu},\overline{\alpha} > 0$, then $\widetilde{C}_{2n}$ is positive/negative definite for any $\nu,\alpha > 0$ such that $\nu\alpha = c$.

Case $H = \frac{1}{2}$: As easily checked, the covariance function $r(s) = \lambda e^{-\nu\alpha s}/(2\alpha)$, $s \geq 0$, is convex, decreasing and non–negative such that the sequence $r_0, r_1, \ldots, r_n$ is convex, decreasing and non–negative for any $n \in \mathbb{N}$, $\triangle t > 0$ and $\nu,\alpha,\lambda > 0$. Therefore, by Theorem 4.2.4(i) the CEM is applicable for any $n \in \mathbb{N}$, $\triangle t > 0$ and $\nu,\alpha,\lambda > 0$.
Case $H \in (1/2,1)$: Unfortunately, at least one assumption of the Statements (i)–(ii) of Theorem 4.2.4 is (for $\triangle t > 0$ small enough) violated by Proposition 4.1.2(iv) and (v). Nevertheless, we verify *numerically* that the CEM is applicable, i.e. for given $\triangle t > 0$, $n \in \mathbb{N}$, $\nu,\alpha,\lambda > 0$ and H we calculate the eigenvalues of the circulant matrix, given by (4.2.20), using the fast Fourier transform algorithm and then check those for positiveness. In the first numerical experiment we fix $\triangle t = 0.1$, $\lambda = \alpha = 1$, ν, H and test whether all eigenvalues of $\widetilde{C}_{2n}$ are positive ($= 1$) or at least one eigenvalue is strict negative ($= 0$) in dependence on $n \in \mathbb{N}$. Figure 6 with $\nu = 1$, Figure 7 with $\nu = 0.5$ and Figure 8 with $\nu = 2$ show the results associated to different values of H. We observe that there is $n_0 \in \mathbb{N}$ depending on $\triangle t, \nu, \alpha$ and H such that for all $n \geq n_0$ the circulant matrix $\widetilde{C}_{2n}$ is positive definite. Further, in Figure 6–8 we discover that n_0 increases if we increase the Hurst parameter or decrease $\alpha\nu > 0$. In the second numerical experiment we fix $\lambda = \alpha = 1$, ν, H and calculate the first $n_0 \in \mathbb{N}$, $n_0 \geq 2$, such that the circulant matrix $\widetilde{C}_{2n_0}$ is positive definite in dependence on $\triangle t = 1/m$, $m \in \mathbb{N}$ with $2 \leq m \leq 50$. Figure 9 with $\nu = 1$ and Figure 10 with $H = 0.6$ show the results associated to different values of H and ν, respectively. We notice in addition that n_0 becomes larger if we decrease $\triangle t > 0$ (i.e. increase m). Whereas Figure 9 b.), c.) and Figure 10 a.)–d.) suggest a (piecewise) linear dependence of m on n_0, Figure 9 d.) with $H = 0.9$ indi-

cates a polynomial behaviour with exponent strict larger than 1. In particular, from the numerical experiments we might conclude that the CEM for $H \in (1/2, 1)$ is on the one hand applicable, but on the other hand the CEM is probably not of practical use for large Hurst parameters and low values of $\triangle t$ and $\alpha\nu$.
Case $H \in (0, 1/2)$: Notice first that the assumptions of Theorem 4.2.4(i) are satisfied by Proposition 4.1.2(v) at least for $\triangle t > 0$ and $n \in \mathbb{N}$ small enough. Therefore, we may expect by repeating the first numerical experiment described above for the case $H \in (1/2, 1)$ that there exist $n_0 \in \mathbb{N}$ and $\triangle t > 0$ small enough such that for all $n \in \mathbb{N}$ with $1 \leq n \leq n_0$ the circulant matrix $\widetilde{C}_{2n}$ is positive definite ($= 1$) and for $n_0 + 1 \leq n$ negative definite ($= 0$). But we observe in our numerical experiments for $H \in (0, 1/2)$ that the circulant matrix $\widetilde{C}_{2n}$ is *always* positive definite, i.e. the resulting figures look like Figure 6–9 a.) for the case $H = 1/2$. We could not detect an $n \in \mathbb{N}$ for suitable choices of $\triangle t, \nu\alpha > 0$ and $H \in (0, 1/2)$ such that $\widetilde{C}_{2n}$ is negative definite.

It would be desirable to derive analytically an inequality $f(\triangle t, \nu\alpha, H) \leq n$ with some function $f : (0, \infty) \times (0, \infty) \times (0, 1) \to (0, \infty)$ such that for all $n \in \mathbb{N}$ satisfying this inequality the circulant matrix $\widetilde{C}_{2n}$ is positive definite or even to prove analytically that the CEM is always applicable for $H \in (0, 1/2)$. Such inequalities are given in [28] for different stationary Gaussian processes. Unfortunately, so far, we are not able to accomplish this for the sfOU process.

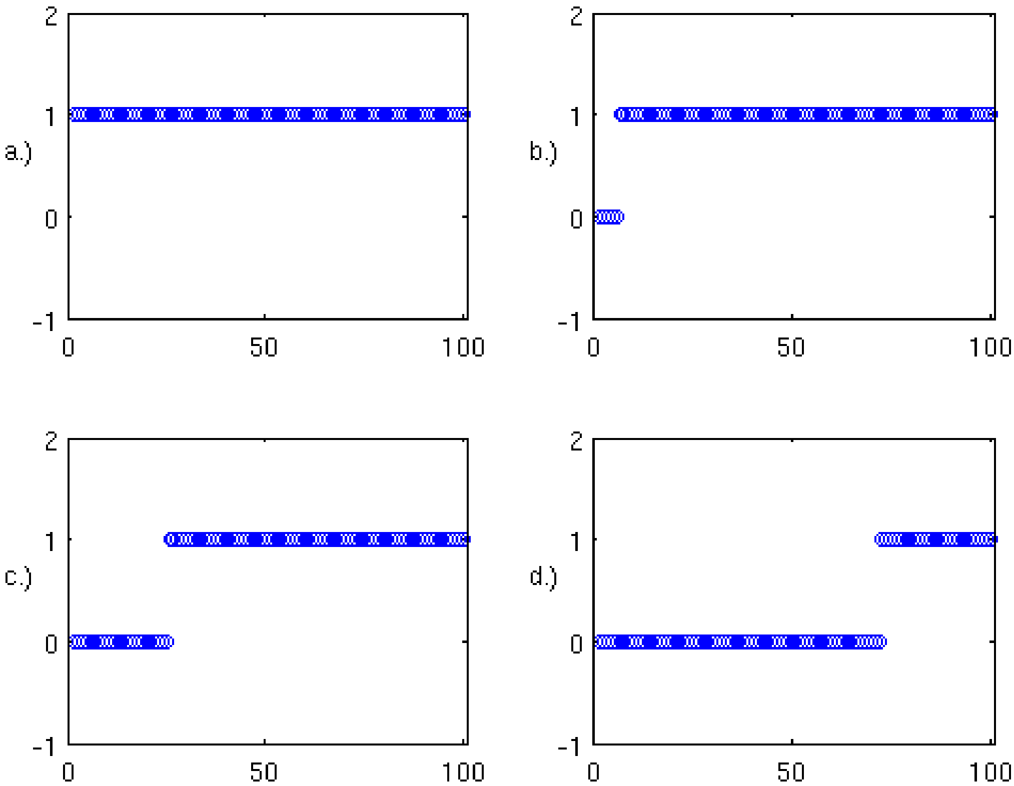

Figure 6: Positive (= 1) and negative (= 0) definiteness of $\widetilde{C}_{2n}$ with $\triangle t > 0$ and $\lambda = \alpha = \nu = 1$ in dependence on $n \in \mathbb{N}$, associated to different values of H. a.) $H = 0.5$, b.) $H = 0.6$, c.) $H = 0.75$, d.) $H = 0.9$.

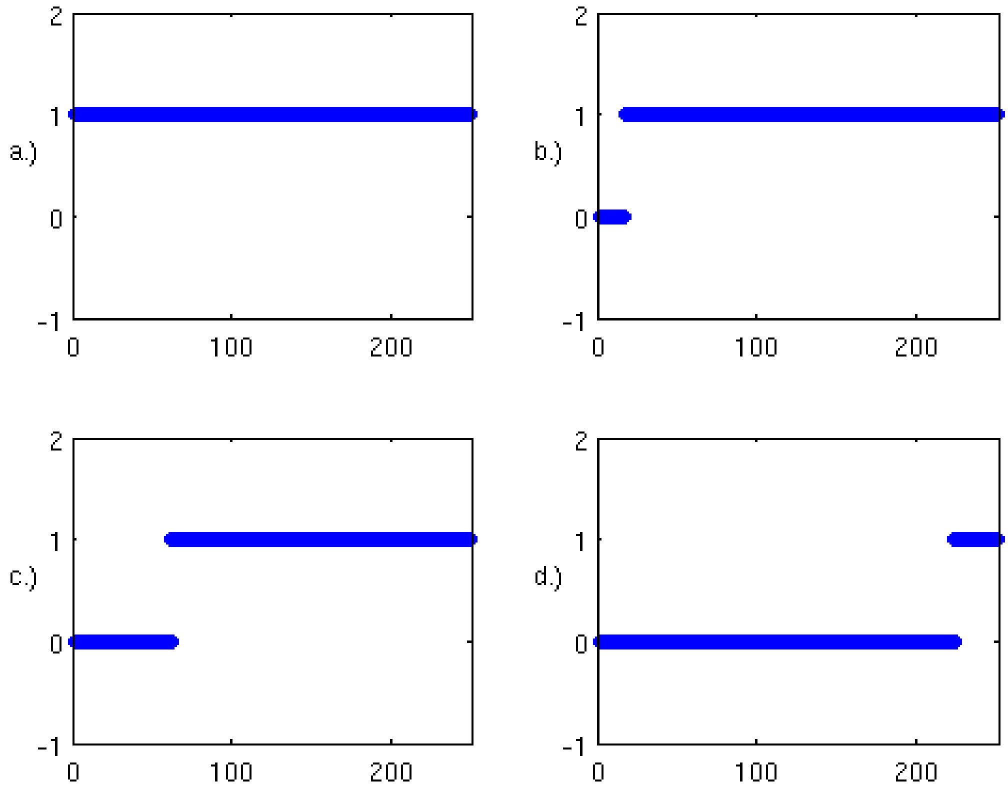

Figure 7: Positive (= 1) and negative (= 0) definiteness of $\widetilde{C}_{2n}$ with $\triangle t > 0$, $\lambda = \alpha = 1$ and $\nu = 0.5$ in dependence on $n \in \mathbb{N}$, associated to different values of H. a.) $H = 0.5$, b.) $H = 0.6$, c.) $H = 0.75$, d.) $H = 0.9$.

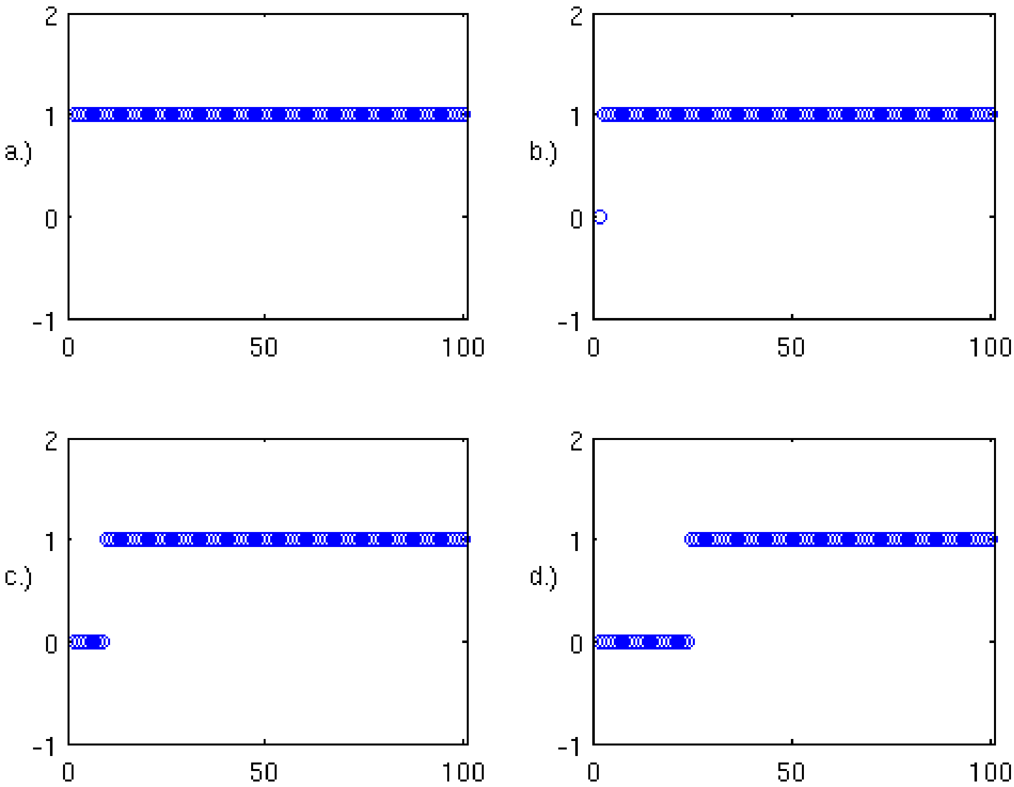

Figure 8: Positive (= 1) and negative (= 0) definiteness of $\widetilde{C}_{2n}$ with $\triangle t > 0$, $\lambda = \alpha = 1$ and $\nu = 2$ in dependence on $n \in \mathbb{N}$, associated to different values of H. a.) $H = 0.5$, b.) $H = 0.6$, c.) $H = 0.75$, d.) $H = 0.9$.

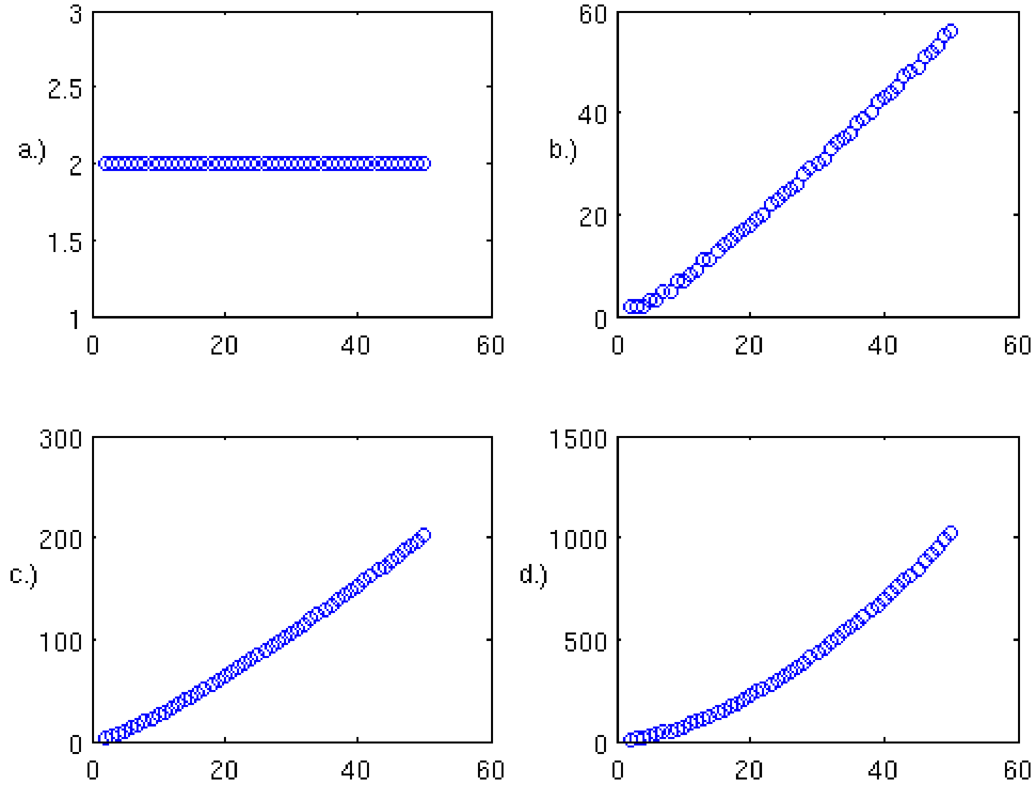

Figure 9: First $n \in \mathbb{N}$, $n \geq 2$, such that $\widetilde{C}_{2n}$ is positive definite with $\lambda = \alpha = \nu = 1$ in dependence on $\triangle t = 1/m$, $m \in \mathbb{N}$, $2 \leq m \leq 50$, associated to different values of H. a.) $H = 0.5$, b.) $H = 0.6$, c.) $H = 0.75$, d.) $H = 0.9$.

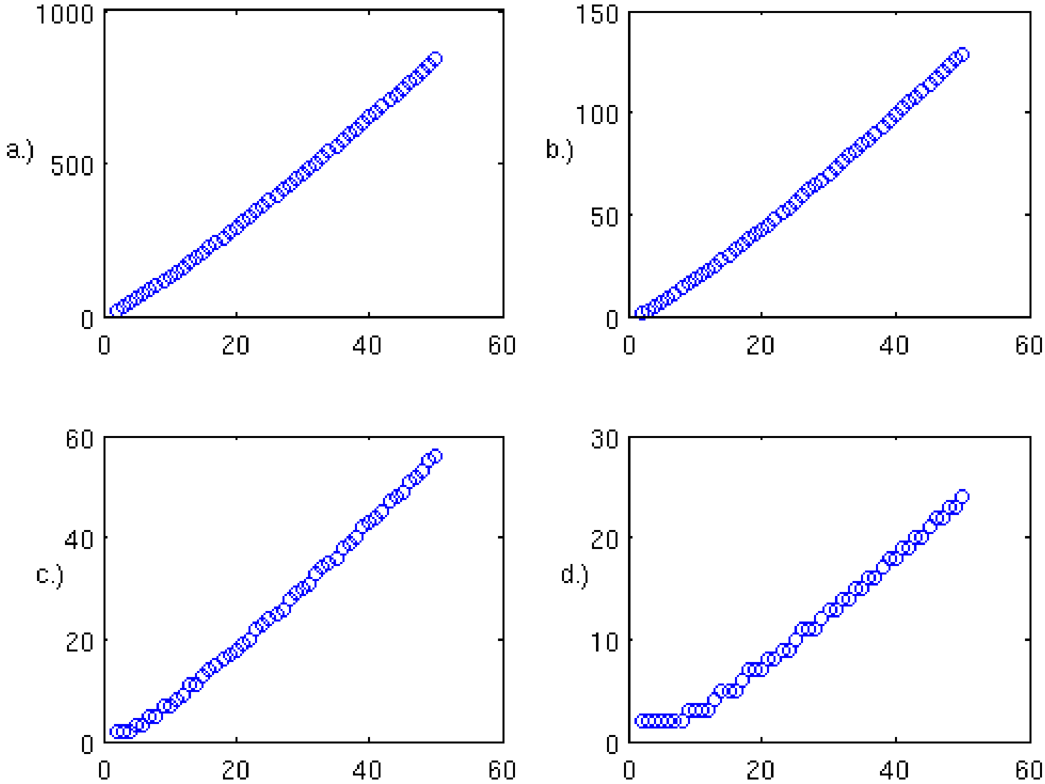

Figure 10: First $n \in \mathbb{N}$, $n \geq 2$, such that $\widetilde{C}_{2n}$ is positive definite with $\lambda = \alpha = 1$ and $H = 0.6$ in dependence on $\triangle t = 1/m$, $m \in \mathbb{N}$, $2 \leq m \leq 50$, associated to different values of ν. a.) $\nu = 0.1$, b.) $\nu = 0.5$, c.) $\nu = 1$, d.) $\nu = 2$.

5. Inertial Particles in Fractional Gaussian Fields

5.1. The Model

The concrete model we study is the following system (M) of equations in non–dimensional form:

$$\tau\ddot{x}(t) = v\big(x(t),t\big) - \dot{x}(t), \quad \big(x(0),\dot{x}(0)\big) \in \mathbb{T}^2 \times \mathbb{R}^2, \tag{M1}$$

$$v(x,t) = \nabla^{\perp}\psi(x,t) = \left(\frac{\partial\psi}{\partial x_2}(x,t), -\frac{\partial\psi}{\partial x_1}(x,t)\right), \tag{M2}$$

$$d\psi_t = \nu A\psi_t dt + \nu^H Q^{\frac{1}{2}} dB_t^H, \quad \psi_0 \in V,\ t \geq 0, \tag{M3}$$

where we assume that

Assumption 5.1.1. (i) $\tau, \nu > 0$.

(ii) $V := \{f \in L^{2,per}(\mathbb{T}^2)|\ \int_{\mathbb{T}^2} f(x)dx = 0\}$ is the separable Hilbert space with inner product $\langle f,g\rangle_V := \int_{\mathbb{T}^2} f(x)g^*(x)dx,\ f,g \in V$, and with orthonormal basis (ONB) $\big(e_k(\cdot)\big)_{k\in K} := \big(e^{i\langle k,\cdot\rangle}\big)_{k\in K}$, where $k \in K := 2\pi\mathbb{Z}^2\backslash\{(0,0)\}$ and i denotes here and in the following the imaginary unit.

(iii) $A : \mathcal{D}(A) \subset V \to V$ is a linear self–adjoint operator such that there is a strictly positive sequence $\big(\alpha_k\big)_{k\in K} \subset [c,\infty)$ with $c > 0$, $\alpha_k = \alpha_{-k}$, $Ae_k = -\alpha_k e_k$ and $\alpha_k \to \infty$ for $|k| \to \infty$.

(iv) $Q^{\frac{1}{2}} : V \to V$ is a bounded linear self–adjoint operator such that there is a positive sequence $\big(\sqrt{\lambda_k}\big)_{k\in K} \subset [0,\infty)$ with $\sqrt{\lambda_k} = \sqrt{\lambda_{-k}}$ and $Q^{\frac{1}{2}}e_k = \sqrt{\lambda_k}e_k$.

(v) $\big(B_t^H\big)_{t\geq 0}$ is a cylindrical fractional Wiener process in V with Hurst parameter $H \in (0,1)$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ by the formal series

$$B^H(t) = \sum_{k\in K} \beta_k^H(t)e_k, \quad t \in \mathbb{R}, \tag{5.1.1}$$

where $\big((\beta_k^H(t))_{t\in\mathbb{R}}, k \in K\big)$ is a sequence of complex–valued and normalized fractional Brownian motions, each with the same fixed Hurst parameter $H \in (0,1)$, i.e. $\beta_k^H = \frac{1}{\sqrt{2}}Re(\beta_k^H) + i\frac{1}{\sqrt{2}}Im(\beta_k^H)$, where $Re(\beta_k^H)$ and $Im(\beta_k^H)$ are independent real–valued and normalized fractional Brownian motions on $\mathbb{R}$, and different β_k^H are independent except $\beta_{-k}^H = (\beta_k^H)^*$.

We also will refer to the following assumption several times in this work.

Assumption 5.1.2. Suppose Assumption 5.1.1 holds and there is $m \in \mathbb{N}_0$ and $\gamma \in (0,1)$ such that

$$\sum_{k\in K} \lambda_k \alpha_k^{2\gamma-2H}|k|^{2m} < \infty \quad \text{and} \quad \sum_{k\in K} \lambda_k \alpha_k^{-2H}|k|^{2m+2\gamma} < \infty.$$

In this section we will show that the conditions on the sequences $(\beta_k^H)_{k\in K}$, $(\lambda_k)_{k\in K}$ and $(\alpha_k)_{k\in K}$ in Assumption 5.1.1, together with some growth conditions on $(\lambda_k)_{k\in K}$ and $(\alpha_k)_{k\in K}$, imply that $\psi(x,t)$ and the components of $v(x,t) = \nabla^\perp \psi(x,t)$ are real–valued. By Assumption 5.1.1 A is a strictly negative, self–adjoint operator. So by Proposition 3.3.3 (νA) generates an exponentially stable analytic semigroup on V which in the following will be denoted by $(S(t))_{t\geq 0}$. In particular, we have $|S(t)|_{\mathcal{L}(V)} \leq e^{-t\nu \inf_{k\in K} \alpha_k} \leq e^{-tc\nu}$ for all $t \geq 0$.
The domain $\mathcal{D}(A)$ of A is given by

$$\mathcal{D}(A) := \left\{ f \in V | \sum_{k\in K} \alpha_k^2 |\langle f, e_k\rangle_V|^2 < \infty \right\}.$$

Further, we define here the fractional powers $(-A)^\gamma : \mathcal{D}((-A)^\gamma) \subset V \to V$, $\gamma \geq 0$, of the strictly positive operator $(-A)$ by

$$\mathcal{D}((-A)^\gamma) := \left\{ f \in V | \sum_{k\in K} \alpha_k^{2\gamma} |\langle f, e_k\rangle_V|^2 < \infty \right\}.$$

$\mathcal{D}((-A)^\gamma)$ endowed with the inner product

$$\langle (-A)^\gamma f, (-A)^\gamma g\rangle_V = \sum_{k\in K} \alpha_k^{2\gamma} \langle f, e_k\rangle_V \langle g, e_k\rangle_V^* =: \langle f, g\rangle_{(-A)^\gamma}$$

for $f, g \in \mathcal{D}((-A)^\gamma)$, becomes a Hilbert space. Also notice that $\mathcal{D}(\nu A) = \mathcal{D}(A)$ and $\mathcal{D}((-\nu A)^\gamma) = \mathcal{D}((-A)^\gamma)$ for any $\nu > 0$ and $\gamma \geq 0$. We similarly define $\mathcal{D}((-A)^\gamma)$ for $\gamma < 0$ as the completion of V for the norm $|\cdot|_{(-A)^\gamma}$.
We are mainly interested in the special case when $A = \Delta$, where Δ denotes the Laplace operator with periodic boundary conditions. Then $\alpha_k = |k|^2$, $k \in K$, and

$$\mathcal{D}((-\Delta)) = W^{2,2}(\mathbb{T}^2) \cap V,$$

where $W^{2,2}(\mathbb{T}^2) \cap V$ denotes the Sobolev space of periodic functions on $\mathbb{T}^2$ whose weak derivatives up to order 2 are in V. In particular, we have

$$\mathcal{D}((-\Delta)^\gamma) = W^{2\gamma,2}(\mathbb{T}^2) \cap V$$

for $\gamma \geq 0$.

We will only use the following concept of solutions to equation $(M3)$.

Definition 5.1.3. *A $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$–measurable V–valued process $(\psi(t))_{t\geq 0}$ is said to be a* mild solution *of $(M3)$, if for all $t \geq 0$*

$$\psi(t) = S(t)\psi(0) + \nu^H \int_0^t S(t-s) Q^{\frac{1}{2}} dB^H(s) \tag{5.1.2}$$

$\mathbb{P}$*-a.s., where the stochastic integral on the right hand side of (5.1.2) is defined by*

$$\nu^H \int_0^t S(t-s)Q^{\frac{1}{2}}dB^H(s) := \sum_{k\in K}\sqrt{\lambda_k}\nu^H\int_0^t e^{-(t-s)\nu\alpha_k}d\beta_k^H(s)\; e_k, \tag{5.1.3}$$

provided the infinite series in (5.1.3) converges in $L^2(\Omega, V)$.

In particular, we are mainly interested in *strictly stationary solutions* of $(M3)$, which are *ergodic*.

Definition 5.1.4. *We call a mild solution* $(\psi(t))_{t\geq 0}$ strictly stationary *if for all* $k \in \mathbb{N}$ *and for all arbitrary positive numbers* $t_1, t_2, \ldots, t_k$, *the probability distribution of the* V^k*–valued random variable* $\big(\psi(t_1+r), \psi(t_2+r), \ldots, \psi(t_k+r)\big)$ *does not depend on* $r \geq 0$, *i.e.*

$$\textit{Law}\,(\psi(t_1+r), \psi(t_2+r), \ldots, \psi(t_k+r)) = \textit{Law}\,(\psi(t_1), \psi(t_2), \ldots, \psi(t_k))$$

for all $t_1, t_2, \ldots, t_k, r \geq 0$. *Here* $Law(\cdot)$ *denotes the probability distribution.*
We say that a strictly stationary mild solution $(\psi(t))_{t\geq 0}$ *of (M3) is* unique *if every strictly stationary mild solution of (M3) which is adapted to the natural filtration generated by the two–sided infinite–dimensional fractional Brownian motion (5.1.1) has the same distribution as* $(\psi(t))_{t\geq 0}$.
Further, we call a strictly stationary solution $(\psi(t))_{t\geq 0}$ ergodic *if for all measurable functionals* $\rho : V \to \mathbb{R}$ *such that* $\mathbb{E}\left(|\rho(\psi(0))|\right) < \infty$ *we have* $\mathbb{P}$*-a.s.*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \rho(\psi(t))dt = \mathbb{E}(\rho(\psi(0))). \tag{5.1.4}$$

Remark 5.1.5. (i) The definition of the ergodic mild solution is essentially taken from [54], which is also our main reference for Theorem 5.2.1 in the next subsection. The definition is motivated by Birkhoff's theorem, which says that for a V-valued strictly stationary process $(X(t))_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and any measurable functional $\rho : V \to \mathbb{R}$ with $\mathbb{E}(|\rho(X(0))|) < \infty$ there exists a measurable $\zeta : \Omega \to \mathbb{R}$ such that $\mathbb{P}$-a.s. $\lim_{T\to\infty}\frac{1}{T}\int_0^T \rho(X(t))dt = \zeta$. And $(X(t))_{t\geq 0}$ is said to be ergodic if ζ does not depend on $\omega \in \Omega$, i.e. ζ is deterministic and $\zeta = \mathbb{E}(\rho(X(0)))$. In particular, the unique ergodic mild solution $(\psi(t))_{t\in\mathbb{R}}$ of $(M3)$ realised (under suitable conditions) on $\widetilde{\Omega} = C(\mathbb{R}, C^m(\mathbb{T}^2))$, $m \in \mathbb{N}_0$, with distribution $\mathbb{P}_\psi$ and group of shifts $(\theta(t))_{t\in\mathbb{R}}$ on $\widetilde{\Omega}$ will be used in Section 5.3 as source of randomness for the *random dynamical system* generated by $(M1)$, i.e. as a *metric dynamical system* (MDS). The ergodicity of $(\psi(t))_{t\in\mathbb{R}}$ implies the ergodicity of the MDS and in view of the Birkhoff's ergodic theorem for MDSs (see e.g. [2] p. 539), we can replace $\psi(t)$ in (5.1.4) by $\theta(t)\widetilde{\psi}(0)$, $\widetilde{\psi} \in \widetilde{\Omega}$, $\mathbb{P}_\psi$-a.s.

(ii) The reader interested in more general (semi–)linear stochastic evolution equations with additive fractional noise and different concepts of solutions is referred to the articles [96, 29, 30, 61] and references therein.

5.2. Existence, Uniqueness and Regularity of Solutions of the Model

We have the following existence and uniqueness result for equation $(M3)$.

Theorem 5.2.1. *Suppose Assumption 5.1.1 holds and assume that there is $\epsilon > 0$ such that*

$$\sum_{k\in K} \lambda_k \alpha_k^{2(\epsilon-H)} < \infty. \tag{5.2.1}$$

Then there exists a unique ergodic mild solution ψ to equation $(M3)$ given by

$$\psi(t) = \sum_{k\in K} \sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u) e_k, \; t \in \mathbb{R}. \tag{5.2.2}$$

Proof. The existence of a strictly stationary mild solution to $(M3)$, which is ergodic, is ensured by Theorem 3.1 and Theorem 4.6 in [54], which have the assumptions that νA generates an exponentially stable, analytic semigroup $(S(t))_{t\geq 0}$ on V and

$$|S(t)Q^{\frac{1}{2}}|_{\mathcal{L}^2(V)} \leq Ct^{-\gamma} \qquad t \in (0,T] \tag{5.2.3}$$

for some $T > 0$, $C > 0$ and $\gamma \in [0, H)$. We have already mentioned that by Proposition 3.3.3 (νA) generates an exponentially stable, analytic semigroup $(S(t))_{t\geq 0}$ on V and (5.2.3) is also satisfied since we have

$$\begin{aligned}
|S(t)Q^{\frac{1}{2}}|_{\mathcal{L}^2(V)} &= |(-\nu A)^{\gamma}(-\nu A)^{-\gamma} S(t) Q^{\frac{1}{2}}|_{\mathcal{L}^2(V)} = |(-\nu A)^{\gamma} S(t) (-\nu A)^{-\gamma} Q^{\frac{1}{2}}|_{\mathcal{L}^2(V)} \\
&\leq |(-\nu A)^{\gamma} S(t)|_{\mathcal{L}(V)} |(-\nu A)^{-\gamma} Q^{\frac{1}{2}}|_{\mathcal{L}^2(V)} = |(-\nu A)^{\gamma} S(t)|_{\mathcal{L}(V)} \left(\sum_{k\in K} \frac{\lambda_k}{(\nu\alpha_k)^{2\gamma}} \right)^{\frac{1}{2}} \\
&\leq Ct^{-\gamma} \left(\sum_{k\in K} \frac{\lambda_k}{(\nu\alpha_k)^{2\gamma}} \right)^{\frac{1}{2}} < \infty
\end{aligned}$$

for any $\gamma \in [\max\{0, H-\epsilon\}, \infty)$, a constant $C > 0$ and any $t > 0$. Here we used (5.2.1) and Theorem 3.3.6.
Now assume that we have two strictly stationary mild solutions ψ and $\widetilde{\psi}$ to equation $(M3)$. Notice that

$$|\psi(t) - \widetilde{\psi}(t)|_V = |S(t)(\psi(0) - \widetilde{\psi}(0))|_V \leq e^{-t\nu \inf_{k\in K}\alpha_k} |\psi(0) - \widetilde{\psi}(0)|_V \to 0 \quad \text{ for } t \to \infty$$

$\mathbb{P}$-a.s. and this implies uniqueness in the sense of our definition. The representation (5.2.2) of the mild solution is just the consequence of the definition of a mild solution and stationarity. ☐

The next remark will be useful in this and the following sections.

Remark 5.2.2. Let ψ be the unique ergodic mild solution to equation $(M3)$ given by (5.2.2). For any $t \in \mathbb{R}$ and $k \in K$ we set in the following

$$\begin{aligned}\hat{\psi}_k(t) &:= \sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u) \\ &= \sqrt{\frac{\lambda_k}{2}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} dRe(\beta_k^H)(u) + i\sqrt{\frac{\lambda_k}{2}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} dIm(\beta_k^H)(u) \\ &=: \hat{\psi}_{k,Re}(t) + i\hat{\psi}_{k,Im}(t)\end{aligned}$$

and therefore $\psi(x,t) = \sum_{k\in K} \hat{\psi}_k(t)e_k(x)$, $t \in \mathbb{R}$, $x \in \mathbb{T}^2$. Further, observe that by Assumption 5.1.1 we have

$$\begin{aligned}(\hat{\psi}_k(t))^* &= \Big(\sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u)\Big)^* = \sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d(\beta_k^H)^*(u) \\ &= \sqrt{\lambda_{-k}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_{-k}} d\beta_{-k}^H(u) = \hat{\psi}_{-k}(t).\end{aligned}$$

In particular, for any $s, t \in \mathbb{R}$ and $k, k' \in K$ we obtain

$$\mathbb{E}\big(\hat{\psi}_k(t)(\hat{\psi}_{k'}(s))^*\big) = \begin{cases} 2\mathbb{E}\big(\hat{\psi}_{k,Re}(t)\hat{\psi}_{k,Re}(s)\big) = 2\mathbb{E}\big(\hat{\psi}_{k,Im}(t)\hat{\psi}_{k,Im}(s)\big) & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}$$

and

$$\mathbb{E}\big(\big|\hat{\psi}_k(t) - \hat{\psi}_k(s)\big|^2\big) = 2\mathbb{E}\big(\big|\hat{\psi}_{k,Re}(t) - \hat{\psi}_{k,Re}(s)\big|^2\big) = 2\mathbb{E}\big(\big|\hat{\psi}_{k,Im}(t) - \hat{\psi}_{k,Im}(s)\big|^2\big).$$

Therefore, to compute $\mathbb{E}\big(\hat{\psi}_k(t)(\hat{\psi}_{k'}(s))^*\big)$ or $\mathbb{E}\big(\big|\hat{\psi}_k(t)-\hat{\psi}_k(s)\big|^2\big)$, we only have to compute the associated real part and multiply it by two.

Now we turn to the path regularity of the unique ergodic mild solution of $(M3)$.

Theorem 5.2.3. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}_0$ and $\gamma \in (0,1)$. Then there is a unique ergodic mild solution ψ to equation (M3). Further, for all $\delta \in \mathbb{N}_0^2$ with $|\delta| \leq m$ there is a version of $D^\delta\psi$ (again denoted by $D^\delta\psi$) such that*

$$D^\delta\psi \in C^\epsilon\big(\mathbb{T}^2 \times \mathbb{R}\big)$$

$\mathbb{P}$-a.s. for any $\epsilon \in (0, \min\{\gamma, H\})$. In particular, $D^\delta\psi$ is real–valued.

Proof. Since Assumption 5.1.2 holds, Theorem 5.2.1 implies the existence of a unique ergodic mild solution ψ to equation $(M3)$. Further, for all $t \in \mathbb{R}$ and $x \in \mathbb{T}^2$, $\psi(x,t)$ is real–valued, since $(\hat{\psi}_k(t))^* = \hat{\psi}_{-k}(t)$ and therefore

$$\begin{aligned}(\psi(x,t))^* &= \Big(\sum_{k\in K} \hat{\psi}_k(t) e^{i<k,x>}\Big)^* = \sum_{k\in K} (\hat{\psi}_k(t))^* (e^{i<k,x>})^* \\ &= \sum_{k\in K} \hat{\psi}_{-k}(t) e^{i<-k,x>} = \sum_{k\in K} \hat{\psi}_k(t) e^{i<k,x>} = \psi(x,t).\end{aligned}$$

Now let $m \in \mathbb{N}_0$ and $\delta = (\delta_1, \delta_2) \in \mathbb{N}_0^2$ with $|\delta| = \delta_1 + \delta_2 \leq m$. By Assumption 5.1.2 it is clear that the stochastic process $(D^\delta \psi(t))_{t\in\mathbb{R}}$, defined by the formal Fourier series

$$D^\delta \psi(t) = \sum_{k\in K} \sqrt{\lambda_k} \nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u) D^\delta e_k, \; t \in \mathbb{R},$$

is a well–defined $\mathcal{D}((-A)^\zeta)$–valued stochastic process for some $\zeta \in \mathbb{R}$. But $D^\delta\psi$ is, in general, a function (and not only a *generalized function*) if $\zeta \geq 0$. However, $0 \leq \zeta \leq \gamma$ is already assured by

$$\mathbb{E}\big(|D^\delta \psi(t)|^2_{(-A)^\zeta}\big) \leq \sum_{k\in K} \alpha_k^{2\zeta} \mathbb{E}(|\hat{\psi}(t)|^2)|k|^{2m} = \Gamma(2H)H \sum_{k\in K} \lambda_k \alpha_k^{2\zeta-2H} |k|^{2m} < \infty,$$

where we used $|D^\delta e_k(x)|^2 \leq |k|^{2m}$, Remark 5.2.2, Proposition 4.1.2(i) and Assumption 5.1.2. Notice also that $D^\delta\psi(x,t)$ is real–valued by using the same argument which leads us to conclude that $\psi(x,t)$ is real–valued. Again, by $|D^\delta e_k(x)|^2 \leq |k|^{2m}$, Remark 5.2.2, Proposition 4.1.2(iii) and Assumption 5.1.2 we obtain for all $t, s \in \mathbb{R}$ and $x \in \mathbb{T}^2$

$$\begin{aligned}\mathbb{E}\big(|D^\delta\psi(x,t) - D^\delta\psi(x,s)|^2\big) &= \sum_{k\in K} \mathbb{E}\big(|\hat{\psi}_k(t) - \hat{\psi}_k(s)|^2\big) |D^\delta e_k(x)|^2 \\ &\leq \sum_{k\in K} |k|^{2m} \mathbb{E}\big(|\hat{\psi}_k(t) - \hat{\psi}_k(s)|^2\big) \\ &\leq C(H,\nu,\epsilon) \sum_{k\in K} \lambda_k \alpha_k^{2\epsilon-2H} |k|^{2m} |t-s|^{2\epsilon} < \infty\end{aligned}$$

for any $\epsilon \in (0, \min\{\gamma, H\})$ and some constant $C(H,\nu,\epsilon) > 0$. Similarly, using Assumption 5.1.2 and

$$|D^\delta e_k(x) - D^\delta e_k(y)| \leq |k|^m |e_k(x) - e_k(y)| \leq C(\eta)|k|^{m+\eta}|x-y|^\eta$$

for any $\eta \in (0,1)$ and a constant $C(\eta) > 0$, we get for all $t \in \mathbb{R}$ and $x, y \in \mathbb{T}^2$

$$\begin{aligned}\mathbb{E}\big(|D^\delta\psi(x,t) - D^\delta\psi(y,t)|^2\big) &= \sum_{k\in K} \mathbb{E}\big(|\hat{\psi}_k(t)|^2\big) |D^\delta e_k(x) - D^\delta e_k(y)|^2 \\ &\leq C(\epsilon)\Gamma(2H)H \sum_{k\in K} \lambda_k \alpha_k^{-2H} |k|^{2m+2\epsilon} |x-y|^{2\epsilon} < \infty\end{aligned}$$

for any $\epsilon \in (0, \gamma)$ and a constant $C(\epsilon) > 0$. Therefore, we obtain for all $t, s \in \mathbb{R}$ and $x, y \in \mathbb{T}^2$

$$\mathbb{E}\big(|D^\delta\psi(x,t) - D^\delta\psi(y,s)|^2\big) \leq C(H,\nu,\delta,\epsilon)\big(|t-s|^{2\epsilon} + |x-y|^{2\epsilon}\big)$$

for any $\epsilon \in (0, \min\{\gamma, H\})$ and a constant $C(H,\nu,\delta,\epsilon) > 0$. As $D^\delta\psi(x,t)$ is a normal real–valued random variable, we have by Lemma 3.2.3

$$\begin{aligned}\mathbb{E}\big(|D^\delta\psi(x,t) - D^\delta\psi(y,s)|^{2n}\big) &\leq C(H,\nu,\delta,\epsilon,n)\big(|t-s|^{2\epsilon n} + |x-y|^{2\epsilon n}\big)\\ &\leq C(H,\nu,\delta,\epsilon,n)\big(|t-s|^2 + |x-y|^2\big)^{\epsilon n}\end{aligned}$$

for all $t, s \in \mathbb{R}$, $x, y \in \mathbb{T}^2$, $n \in \mathbb{N}$, $\epsilon \in (0, \min\{\gamma, H\})$ and a constant $C(H,\nu,\delta,\epsilon,n) > 0$. Lemma 3.2.4 implies now that there is a version of $D^\delta\psi$ (again denoted by $D^\delta\psi$) such that $\mathbb{P}$-a.s.

$$D^\delta\psi \in C^\epsilon\big(\mathbb{T}^2 \times \mathbb{R}\big)$$

for any $\epsilon \in (0, \min\{\gamma, H\})$. □

Remark 5.2.4. In [61] Nualart and Viens established an analogue regularity assertion as in Theorem 5.2.3 for the mild solution of the fractional stochastic heat equation on the circle, but they did not consider the partial derivatives of the mild solution.

Corollary 5.2.5. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}_0$, $\gamma \in (0,1)$, and let ψ be the unique ergodic mild solution to equation (M3). Then there is a version of ψ (again denoted by ψ) such that $\mathbb{P}$-a.s.*

$$\psi \in C(\mathbb{R}, C^m(\mathbb{T}^2)).$$

Further, for all $-\infty < T_1 < T_2 < \infty$ and $p \geq 1$ there is a positive random variable $K = K(H,\nu,m,\gamma,T_1,T_2,p) : \Omega \to [0,\infty)$ with $\mathbb{E}\big(K^p\big) < \infty$, such that $\mathbb{P}$-a.s.

$$|\psi(\omega)|_{C([T_1,T_2],C^m(\mathbb{T}^2))} \leq K(\omega). \tag{5.2.4}$$

Proof. Recall from the proof of Theorem 5.2.3 that for any $\delta \in \mathbb{N}_0^2$, $|\delta| \leq m$, $\epsilon \in (0, \min\{\gamma, H\})$, $k \in \mathbb{N}$ with $\epsilon k \geq 1$ and $s, t \in \mathbb{R}$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{T}^2$ we have

$$\begin{aligned}\mathbb{E}\big(|D^\delta\psi(x,t) - D^\delta\psi(y,s)|^{2k}\big) &\leq C(H,\nu,m,\delta,\epsilon,k)\big(|t-s|^{\epsilon 2k} + |x-y|^{\epsilon 2k}\big)\\ &\leq \tilde{C}(H,\nu,m,\delta,\epsilon,k)\big(|t-s|^{\epsilon 2k} + |x_1-y_1|^{\epsilon 2k} + |x_2-y_2|^{\epsilon 2k}\big)\end{aligned}$$

for some constants $C(H,\nu,m,\delta,\epsilon,k), \tilde{C}(H,\nu,m,\delta,\epsilon,k) > 0$. By Theorem 1.4.1 in [48], the so–called *Kolmogorov's continuity theorem for random fields*, for $l \in \mathbb{N}$, $\epsilon 2l > 3$, $-\infty < T_1 < T_2 < \infty$ and any

$$0 < \beta < \frac{\epsilon 2l}{2l}\left(\frac{\frac{3}{3\frac{1}{\epsilon 2l}} - 3}{\frac{3}{3\frac{1}{\epsilon 2l}}}\right) = \epsilon\left(\frac{\epsilon 2l - 3}{\epsilon 2l}\right) < \min\{\gamma, H\},$$

there is a positive random variable $K = K(H, \nu, m, \delta, \epsilon, \beta, l, T_1, T_2) : \Omega \to [0, \infty)$ with

$$\mathbb{E}\big(K^{2l}\big) < \infty \tag{5.2.5}$$

and a version of ψ (again denoted by ψ) such that for all $t, s \in [T_1, T_2]$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{T}^2$ we have $\mathbb{P}$-a.s.

$$|D^\delta \psi(x,t)(\omega) - D^\delta \psi(y,s)(\omega)| \le K(\omega)\big(|t-s|^\beta + |x_1 - y_1|^\beta + |x_2 - y_2|^\beta\big).$$

In particular, for $t, t_0 \in [T_1, T_2]$ and $x = (x_1, x_2), x_0 = (x_{0,1}, x_{0,2}) \in \mathbb{T}^2$ we have $\mathbb{P}$-a.s.

$$\begin{aligned} |D^\delta \psi(x,t)(\omega)| &\le |D^\delta \psi(x,t)(\omega) - D^\delta \psi(x_0,t_0)(\omega)| + |D^\delta \psi(x_0,t_0)(\omega)| \\ &\le K(\omega)\big(|t-t_0|^\beta + |x_1 - x_{0,1}|^\beta + |x_2 - x_{0,2}|^\beta\big) + |D^\delta \psi(t_0, x_0)(\omega)| \end{aligned}$$

and therefore $\mathbb{P}$-a.s.

$$\sup_{t \in [T_1,T_2]} \sup_{x \in \mathbb{T}^2} |D^\delta \psi(x,t)(\omega)| \le K(\omega) C(\beta, T_1, T_2) + |D^\delta \psi(x_0, t_0)(\omega)| \tag{5.2.6}$$

for some constant $C(\beta, T_1, T_2) > 0$. The assertions of the corollary now follow by (5.2.5), (5.2.6) and Lemma 3.2.3 since we have

$$\begin{aligned} \mathbb{E}\big(|D^\delta \psi(x_0,t_0)|^2\big) &= \sum_{k \in K} \lambda_k \nu^{2H} \mathbb{E}\left(\Big| \int_{-\infty}^{t_0} e^{-(t_0-u)\nu\alpha_k} d\beta_k^H(u) \Big|^2 \right) |D^\delta e_k(x_0)|^2 \\ &\le \Gamma(2H) H \sum_{k \in K} \frac{\lambda_k}{\alpha_k^{2H}} |k|^{2m} < \infty \end{aligned}$$

for every $x_0 \in \mathbb{T}^2$, $t_0 \in \mathbb{R}$ and $\delta \in \mathbb{N}_0^2$, $|\delta| \le m$, where we used Assumption 5.1.2, Proposition 4.1.2(i) and Remark 5.2.2. □

Remark 5.2.6. As already mentioned in Remark 5.1.5(i), the unique ergodic mild solution ψ of $(M3)$ realized under the assumptions of Corollary 5.2.5 on $\widetilde{\Omega} = C(\mathbb{R}, C^m(\mathbb{T}^2))$, $m \in \mathbb{N}_0$, with associated Borel σ–algebra $\mathcal{F}$, distribution $\mathbb{P}_\psi$ and group of shifts $(\theta(t))_{t \in \mathbb{R}}$ is used in Section 5.3 as a metric dynamical system for the random dynamical system generated by $(M1)$. The spaces $C(\mathbb{R}, C^m(\mathbb{T}^2))$, $m \in \mathbb{N}_0$, are suitable for that purpose since (see the discussion at the end of Section 1) endowing them with compact open topology makes them to Polish spaces, whereas $C^\epsilon(\mathbb{T}^2 \times \mathbb{R})$, $\epsilon \in (0,1)$, are not separable, which could lead to measurability problems. Moreover, the proof of Lemma 5.3.2(ii) in Section 5.3 uses the fact from Corollary 5.2.5 that the right hand side of (5.2.4) is integrable w.r.t. $\mathbb{P}$. And Lemma 5.3.2(ii) is of particular importance since it provides linear growth conditions for some functionals of ψ which are crucial to prove global existence of solutions to $(M1)$ and the existence of the random attractor of the random dynamical system.

We apply Corollary 5.2.5 to state an existence and uniqueness result for the transport equation $(M1)$. Consider the differential equation $(M1)$ as a first order system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = f_{\psi(\omega),\tau}(t,(x(t),\dot{x}(t))), \quad \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} \in \mathbb{T}^2 \times \mathbb{R}^2, \tag{5.2.7}$$

where $\tau > 0$ and $f_{\psi(\omega),\tau} : \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^4$ is defined by

$$(t,x,y) \mapsto f_{\psi(\omega),\tau}(t,x,y) = \begin{pmatrix} y \\ \frac{1}{\tau}(\nabla^{\perp}\psi(x,t)(\omega) - y) \end{pmatrix}. \tag{5.2.8}$$

Here $\psi(\cdot,\cdot)(\omega)$ with $\omega \in \Omega$ denotes a realization of the ergodic mild solution of $(M3)$.

Definition 5.2.7. *We say that $(M1)$ has a* unique local C^m–solution $\mathbb{P}$-*a.s. for some $m \in \mathbb{N}$ if for all $(\overline{x},\overline{y}) \in \mathbb{T}^2 \times \mathbb{R}^2$ there is an open interval $I(\omega) \subseteq \mathbb{R}$ including 0 and a function*

$$\begin{pmatrix} x(\cdot) \\ \dot{x}(\cdot) \end{pmatrix} \in C^1\left(I(\omega), C^m\big(\mathbb{T}^2 \times \mathbb{R}^2, \mathbb{T}^2 \times \mathbb{R}^2\big)\right),$$

which uniquely satisfies the equation (5.2.7) for all $t \in I(\omega)$ $\mathbb{P}$-a.s.
We say that $(M1)$ has a unique global C^m–solution $\mathbb{P}$-*a.s. for some $m \in \mathbb{N}$ if $(M1)$ has a unique local C^m–solution with $I(\omega) = \mathbb{R}$ $\mathbb{P}$-a.s.*

The unique local/global solution $\binom{x(\cdot)}{\dot{x}(\cdot)}$ to (5.2.7) is interpreted in such a way that we extend $f_{\psi(\omega),\tau}$ in the position coordinates $x = (x_1,x_2)$ by periodicity to $\mathbb{R}^2$ and consider the unique local/global solution $\binom{\widetilde{x}(\cdot)}{\dot{\widetilde{x}}(\cdot)}$ in the extended system whenever it exists. Then we simply take $\widetilde{x}(t)$ mod 1 in each coordinate.

Corollary 5.2.8. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}$, $m \geq 2$, and $\gamma \in (0,1)$. Then $(M1)$ has a unique global C^{m-1}–solution $\mathbb{P}$-a.s.*

Proof. By Corollary 5.2.5 there is a version of the strictly stationary solution ψ to equation $(M3)$ (again denoted by ψ) such that $\psi \in C(\mathbb{R}, C^m(\mathbb{T}^2))$ $\mathbb{P}$-a.s.. This implies that $f_{\psi(\omega),\tau} \in C(\mathbb{R}, C^{m-1}(\mathbb{T}^2 \times \mathbb{R}^2, \mathbb{R}^4))$ where $f_{\psi(\omega),\tau}$ is defined in (5.2.8). Therefore (see e.g. Appendix B in [2]) $(M1)$ has a unique local C^{m-1}–solution $\mathbb{P}$-a.s. To establish $\mathbb{P}$-a.s. global solutions, we have to find locally integrable, positive functions $\alpha_\omega, \beta_\omega : \mathbb{R} \to [0,\infty)$ which may depend upon the realization $\omega \in \Omega$ such that $\mathbb{P}$-a.s.

$$|f_{\psi(\omega),\tau}(t,x,y)| \leq \alpha_\omega(t)|(x,y)| + \beta_\omega(t).$$

By Lemma 5.3.2(ii) (see Section 5.3 below) there is a constant $K(\omega) > 0$ such that $\mathbb{P}$-a.s.

$$\big|\frac{\partial}{\partial x_1}\psi(x,t)(\omega)\big|^2 + \big|\frac{\partial}{\partial x_2}\psi(x,t)(\omega)\big|^2 \leq |\psi(t)(\omega)|^2_{C^1(\mathbb{T}^2)} \leq (|t| + K(\omega))^2$$

for all $t \in \mathbb{R}$. Hence, we have for all $t \in \mathbb{R}$, $x = (x_1,x_2) \in \mathbb{T}^2$ and $y = (y_1,y_2) \in \mathbb{R}^2$

$$\begin{aligned} |f_{\psi(\omega),\tau}(t,x,y)|^2 &= y_1^2 + y_2^2 + \big|\frac{\partial}{\partial x_2}\psi(x,t)(\omega) - y_1\big|^2 + \big| -\frac{\partial}{\partial x_1}\psi(x,t)(\omega) - y_2\big|^2 \\ &\leq 3(y_1^2 + y_2^2) + 2\left(\big|\frac{\partial}{\partial x_1}\psi(x,t)(\omega)\big|^2 + \big|\frac{\partial}{\partial x_2}\psi(x,t)(\omega)\big|^2\right) \\ &\leq 3|(x_1,x_2,y_1,y_2)|^2 + 2\left(|t| + K(\omega)\right)^2 \end{aligned}$$

$\mathbb{P}$-a.s. and in particular

$$|f_{\psi(\omega),\tau}(t,x,y)| \leq \sqrt{3}|(x,y)| + \sqrt{2}\,(|t| + K(\omega))\,.$$

$\square$

5.3. The Model as a Random Dynamical System and the Existence of the Random Attractor and of Invariant Measures

In this subsection we verify under appropriate assumptions that the system (M) defines a random dynamical system. Further, we prove the existence of a unique random $\mathcal{D}$–attractor and an invariant forward Markov measure. Finally, we establish properties of a modified model which suggest a volume contraction in the system.

Proposition 5.3.1. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}_0$ and $\gamma \in (0,1)$. Then the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ defines an ergodic metric dynamical system where*

- *$\Omega = C\big(\mathbb{R}, C^m(\mathbb{T}^2)\big)$ equipped with the compact open topology given by the complete metric $d(\psi, \widetilde{\psi}) := \sum_{n=1}^{\infty} |\psi - \widetilde{\psi}|_n / \big(2^n(1 + |\psi - \widetilde{\psi}|_n)\big)$, where $|\psi - \widetilde{\psi}|_n := \sup_{-n\le t\le n} |\psi(t) - \widetilde{\psi}(t)|_{C^m(\mathbb{T}^2)}$,*
- *$\mathcal{F}$ is the associated Borel σ–algebra, which is the trace in Ω of the product σ–algebra $(\mathcal{B}(C^m(\mathbb{T}^2)))^{\otimes\mathbb{R}}$,*
- *$\mathbb{P}$ is the distribution of the ergodic mild solution of $(M3)$,*
- *$(\theta(t))_{t\in\mathbb{R}}$ is the group of shifts, i.e. $\theta(t)\psi(s) = \psi(t+s)$ for all $t, s \in \mathbb{R}$ and $\psi \in \Omega$.*

Proof. By Corollary 5.2.5 the ergodic mild solution ψ to equation $(M3)$ is realized on $\Omega := C\big(\mathbb{R}, C^m(\mathbb{T}^2)\big)$. Endowing Ω with the compact open topology makes Ω a Polish space, actually a Frechet space.
The group of shifts $(\theta(t))_{t\in\mathbb{R}}$ on Ω defined by $\theta(t)\psi(\cdot) = \psi(t+\cdot)$ for $t \in \mathbb{R}$ satisfies the flow property and is measure preserving, since $\mathbb{P}$ is the distribution of the ergodic mild solution of the equation $(M3)$.
Obviously, $t \mapsto \theta(t)\psi$ is continuous for all $\psi \in \Omega$ and $\psi \mapsto \theta(t)\psi$ is continuous for all $t \in \mathbb{R}$. Therefore, by Lemma 1.1 in [21] $(t,\psi) \mapsto \theta(t)\psi$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$–$\mathcal{F}$ measurable.
Hence, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ defines an ergodic metric dynamical system. $\square$

The next lemma will be used several times in this subsection.

Lemma 5.3.2. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}_0$, $\gamma \in (0,1)$, and let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ be the ergodic metric dynamical system introduced in Proposition 5.3.1. Then the following assertions are valid:*

(i) For all $-\infty < T_1 < t < T_2 < \infty$ the mappings $\psi \mapsto |\psi(t)|_{C^m(\mathbb{T}^2)}$ and $\psi \mapsto |\psi|_{C([T_1,T_2],C^m(\mathbb{T}^2))}$ are $\mathcal{F}$–$\mathcal{B}(\mathbb{R})$ measurable and there is a constant $C = C(m, H, \nu, \gamma, T_1, T_2) > 0$ such that

$$\mathbb{E}\left(|id_\Omega(t)|^2_{C^m(\mathbb{T}^2)}\right) \le \mathbb{E}\left(|id_\Omega|^2_{C([T_1,T_2],C^m(\mathbb{T}^2))}\right) \le C, \tag{5.3.1}$$

where $id_\Omega : \Omega \to \Omega$, $\psi \mapsto \psi$.

(ii) There is a $(\theta(t))_{t\in\mathbb{R}}$–invariant set $\mathcal{F} \ni \Omega_0 = \Omega_0(m,\gamma,H,\nu) \subset \Omega$ (i.e. $\theta(t)^{-1}\Omega_0 = \Omega_0$ for all $t \in \mathbb{R}$) with $\mathbb{P}(\Omega_0) = 1$ such that for all $\delta > 0$, $\psi \in \Omega_0$ there is a constant $C(\psi) = C(\psi, m, \gamma, H, \nu, \delta) > 0$ such that

$$|\psi(t)|_{C^m(\mathbb{T}^2)} \leq \delta|t| + C(\psi) \tag{5.3.2}$$

for all $t \in \mathbb{R}$. In particular, the mapping

$$\psi \mapsto \begin{cases} \int_{-\infty}^0 e^{\frac{s}{\tau}} |\psi(s)|^2_{C^m(\mathbb{T}^2)} ds & \text{for } \psi \in \Omega_0 \\ 0 & \text{for } \psi \notin \Omega_0 \end{cases}$$

with $\tau > 0$ is well–defined and $\mathcal{F}$–$\mathcal{B}(\mathbb{R})$ measurable.

(iii) There is a $(\theta(t))_{t\in\mathbb{R}}$–invariant set $\mathcal{F} \ni \Omega_1 = \Omega_1(m,\gamma,H,\nu) \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for all $\psi \in \Omega_1$ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |\psi(s)|^2_{C^m(\mathbb{T}^2)} ds = \mathbb{E}\left(|id_\Omega(0)|^2_{C^m(\mathbb{T}^2)}\right) < \infty.$$

Proof. (i): Since Ω is endowed with the compact open topology, the mappings $\psi \mapsto |\psi(t)|_{C^m(\mathbb{T}^2)}$ and $\psi \mapsto |\psi|_{C([T_1,T_2],C^m(\mathbb{T}^2))}$ for all $-\infty < T_1 < t < T_2 < \infty$ are $\mathcal{F}$–$\mathcal{B}(\mathbb{R})$ measurable. Further, by Corollary 5.2.5 for any $-\infty < T_1 < T_2 < \infty$ there is a positive random variable $K(T_1,T_2) = K(m,\gamma,H,\nu,T_1,T_2) \in L^1(\Omega, \mathcal{F}^{\mathbb{P}}, \mathbb{P})$, where $\mathcal{F}^{\mathbb{P}}$ denotes the completion of $\mathcal{F}$ w.r.t. $\mathbb{P}$ such that $\mathbb{P}$-a.s.

$$|\psi|_{C([T_1,T_2],C^m(\mathbb{T}^2))} \leq K(T_1,T_2,\psi), \tag{5.3.3}$$

since K may be measurable only with respect to the completed σ–algebra $\mathcal{F}^{\mathbb{P}}$. So for all $-\infty < T_1 < t < T_2 < \infty$ we have

$$\begin{aligned} \mathbb{E}\big(|id_\Omega(t)|_{C^m(\mathbb{T}^2)}\big) \leq \mathbb{E}\big(|id_\Omega|_{C([T_1,T_2],C^m(\mathbb{T}^2))}\big) &= \mathbb{E}^{\mathbb{P}}\big(|id_\Omega|_{C([T_1,T_2],C^m(\mathbb{T}^2))}\big) \\ &\leq \mathbb{E}^{\mathbb{P}}\big(K(T_1,T_2)\big) < \infty, \end{aligned} \tag{5.3.4}$$

where $\mathbb{E}^{\mathbb{P}}$ is related to the extension of $\mathbb{P}$ to $\mathcal{F}^{\mathbb{P}}$.
(ii): We have by (5.3.3)

$$\sup_{r\in[0,1]} |\theta(r)\psi|_{C([T_1,T_2],C^m(\mathbb{T}^2))} = |\psi|_{C([T_1,T_2+1],C^m(\mathbb{T}^2))} \leq K(T_1,T_2+1,\psi) \tag{5.3.5}$$

and

$$\begin{aligned} \mathbb{E}\big(\sup_{r\in[0,1]} |\theta(r) id_\Omega|_{C([T_1,T_2],C^m(\mathbb{T}^2))}\big) &= \mathbb{E}\big(|id_\Omega|_{C([T_1,T_2+1],C^m(\mathbb{T}^2))}\big) \\ &= \mathbb{E}^{\mathbb{P}}\big(|id_\Omega|_{C([T_1,T_2+1],C^m(\mathbb{T}^2))}\big) \leq \mathbb{E}^{\mathbb{P}}\big(K(T_1,T_2+1)\big) < \infty. \end{aligned} \tag{5.3.6}$$

Taking (5.3.5) and (5.3.6) into account, Proposition 4.1.3 in [2] (*the dichotomy of linear growth for stationary processes*) with the measurable mapping $\psi \mapsto |\psi|_{C([0,1],C^m(\mathbb{T}^2))}$ implies that

$$\limsup_{t\to\pm\infty} \frac{|\theta(t)\psi|_{C([0,1],C^m(\mathbb{T}^2))}}{|t|} = 0$$

on a $(\theta(t))_{t\in\mathbb{R}}$–invariant set $\mathcal{F} \ni \Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$. Therefore, for any $\delta > 0$, $\psi \in \Omega_0$ there is a constant $T(\delta,\psi) = T(\delta,m,\psi) > 0$ such that

$$|\psi(t)|_{C^m(\mathbb{T}^2)} = |\theta(t)\psi(0)|_{C^m(\mathbb{T}^2)} \leq |\theta(t)\psi|_{C([0,1],C^m(\mathbb{T}^2))} \leq \delta|t|$$

for $|t| \geq T(\delta,\psi)$. Hence, by (5.3.3) for any $\delta > 0$, $\psi \in \Omega_0$ we have

$$|\psi(t)|_{C^m(\mathbb{T}^2)} = |\theta(t)\psi(0)|_{C^m(\mathbb{T}^2)} \leq \delta|t| + K(-T(\delta,\psi), T(\delta,\psi), \psi) \tag{5.3.7}$$

for all $t \in \mathbb{R}$. Therefore, for $\tau > 0$ the mapping

$$\psi \mapsto \begin{cases} \int_{-\infty}^{0} e^{\frac{s}{\tau}} |\psi(s)|^2_{C^m(\mathbb{T}^2)} ds & \text{for } \psi \in \Omega_0 \\ 0 & \text{for } \psi \notin \Omega_0 \end{cases}$$

is well–defined and $\mathcal{F}$–$\mathcal{B}(\mathbb{R})$ measurable, since for $\tau > 0$ and $n \in \mathbb{N}$ the mappings

$$\psi \mapsto \begin{cases} \int_{-n}^{0} e^{\frac{s}{\tau}} |\psi(s)|^2_{C^m(\mathbb{T}^2)} ds & \text{for } \psi \in \Omega_0 \\ 0 & \text{for } \psi \notin \Omega_0 \end{cases}$$

are finite, $\mathcal{F}$–$\mathcal{B}(\mathbb{R})$ measurable and the ψ–wise limits for $n \to \infty$ are finite by (5.3.7). Finally, assertion (iii) follows by Birkhoff's ergodic theorem (see e.g. [2] p. 539) and the ergodicity of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$. The finiteness of $\mathbb{E}\left(|id_\Omega(0)|^2_{C^m(\mathbb{T}^2)}\right)$ is assured by (5.3.1). □

Remark 5.3.3. The idea for the proof of the assertion in Lemma 5.3.2(ii) is adopted from the proof of Lemma 2.4 in the research article [53] by Maslowski and Schmalfuß where an analogue statement was proven for increments of an infinite–dimensional fractional Brownian motion with Hurst parameter $H > 1/2$. The same approach works also for arbitrary $H \in (0,1)$, as mentioned in [38]. It was apparent for us that we can apply this procedure in our context, too. Moreover, we emphasise that the linear growth condition (5.3.2) is crucial to prove the global existence of particle paths (Corollary 5.2.8) and the existence of the random $\mathcal{D}$–attractor (Theorem 5.3.5).

Proposition 5.3.4. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}$, $m \geq 2$ and $\gamma \in (0,1)$. Then the function $\phi : \mathbb{R}\times\Omega\times\mathbb{T}^2\times\mathbb{R}^2 \longrightarrow \mathbb{T}^2\times\mathbb{R}^2$*

$$(t,\psi,(x,y)) \mapsto \phi(t,\psi,(x,y)) := \phi(t,\psi)\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix},$$

defines a C^{m-1}–RDS over the ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t\in\mathbb{R}})$ introduced in Proposition 5.3.1, where $\binom{x(t)}{\dot{x}(t)} \in \mathbb{T}^2 \times \mathbb{R}^2$ is the unique global C^{m-1}–solution for $\tau > 0$, $\psi \in \Omega$ and $(x, y) \in \mathbb{T}^2 \times \mathbb{R}^2$ at time $t \in \mathbb{R}$ to

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = f_{\psi,\tau}(t, (x(t), \dot{x}(t))), \ \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.3.8}$$

with $f_{\psi,\tau} : \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$(t, x, y) \mapsto f_{\psi,\tau}(t, (x, y)) = \begin{pmatrix} y \\ \frac{1}{\tau}(\nabla^\perp \psi(x, t) - y) \end{pmatrix}.$$

Here we change Ω to $\Omega := \Omega_0 \cap \Omega_1$, i.e. to the intersection of the $(\theta(t))_{t\in\mathbb{R}}$–invariant sets Ω_0 and Ω_1 introduced in Lemma 5.3.2(ii) and (iii).

Proof. First notice that by Corollary 5.2.8 for all $(x, y) \in \mathbb{T}^2 \times \mathbb{R}^2$, $\psi \in \Omega$ (i.e. $\psi \in \Omega_0 \cap \Omega_1$) there is a unique global C^{m-1}–solution $\binom{x(t)}{\dot{x}(t)}$ to equation (5.3.8).
Since $t \mapsto \phi(t, \psi)\binom{x}{y}$ is continuous for every $(\psi, (x, y)) \in \Omega \times \mathbb{T}^2 \times \mathbb{R}^2$ and $(x, y) \mapsto \phi(t, \psi)\binom{x}{y}$ is continuous for every $(t, \psi) \in \mathbb{R} \times \Omega$, in order to prove the measurability of ϕ, by Remark 3.4.3(ii) we only need to prove the measurability of $\psi \mapsto \phi(t, \psi)\binom{x}{y}$ for every $(t, (x, y)) \in \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$. But this measurability is obvious: Since Ω is equipped with the compact open topology, $\psi \mapsto \psi(x, t)$ is measurable for every $(x, t) \in \mathbb{T}^2 \times \mathbb{R}$ and therefore also $\psi \mapsto \phi(t, \psi)\binom{x}{y}$.
The cocycle property is just a consequence of the uniqueness of the solution to equation (5.3.8). $\square$

The next theorem ensures the existence of the random $\mathcal{D}$–attractor and a ϕ–invariant forward Markov measure.

Theorem 5.3.5. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}$, $m \geq 2$ and $\gamma \in (0, 1)$. Then the C^{m-1}–RDS ϕ defined in Proposition 5.3.4 has a unique random $\mathcal{D}$–attractor $\mathcal{A}$ where $\mathcal{D}$ is the universe of tempered sets of $\mathbb{T}^2 \times \mathbb{R}^2$. Further, for any $\delta > 0$*

$$B^\delta(\psi) := \left\{ (x, y) \in \mathbb{T}^2 \times \mathbb{R}^2 \middle| \ |y|^2 \leq \frac{(1+\delta)}{\tau} \int_{-\infty}^{0} e^{\frac{u}{\tau}} |\psi(u)|^2_{C^1(\mathbb{T}^2)} du \right\}, \ \psi \in \Omega, \tag{5.3.9}$$

is a $\mathcal{D}$–absorbing and ϕ–forward invariant closed random set such that

$$\mathcal{A}(\psi) = \bigcap_{t\in\mathbb{N}} \phi(t, \theta(-t)\psi) B^\delta(\theta(-t)\psi), \ \psi \in \Omega. \tag{5.3.10}$$

In particular, all ϕ–invariant measures are supported on the random $\mathcal{D}$–attractor and there exists a ϕ–invariant forward Markov measure.

Proof. To prove the existence of the random $\mathcal{D}$–attractor, we only have to show in view of Theorem 3.4.10 that for any $\delta > 0$ the random set $B(\psi) := B^\delta(\psi)$ defined in (5.3.9) is a $\mathcal{D}$–absorbing and ϕ–forward invariant closed random set.
Notice here that $\psi \mapsto \int_{-\infty}^0 e^{\frac{u}{\tau}} |\psi(u)|^2_{C^1(\mathbb{T}^2)} du$ is measurable and finite for every $\psi \in \Omega$. This follows by Lemma 5.3.2(ii) and recall here again the change of Ω in Proposition 5.3.4. Further, the random set–valued map B takes values in closed bounded subsets of $\mathbb{T}^2 \times \mathbb{R}^2$ and is measurable. Therefore, B is a closed random set with compact values.
In the following, for any set $C \subseteq \mathbb{T}^2 \times \mathbb{R}^2$ we set

$$\pi_{\mathbb{T}^2}(C) := \{\pi_{\mathbb{T}^2}(x,y) | (x,y) \in C\}, \quad \pi_{\mathbb{R}^2}(C) := \{\pi_{\mathbb{R}^2}(x,y) | (x,y) \in C\},$$

where

$$\pi_{\mathbb{T}^2} : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2,\ (x,y) \mapsto x, \quad \text{and} \quad \pi_{\mathbb{R}^2} : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto y.$$

Next we prove that $\psi \mapsto B(\psi)$ is ϕ–forward invariant, i.e. $\phi(t,\psi)B(\psi) \subseteq B(\theta(t)\psi)$ for all $t \geq 0$, $\psi \in \Omega$. So to prove the ϕ–forward invariance of B, we only have to show that

$$\sup_{(x,y)\in B(\psi)} \left|\pi_{\mathbb{R}^2}(\phi(t,\psi)\begin{pmatrix} x \\ y \end{pmatrix})\right|^2 = \sup_{(x,y)\in B(\psi)} \{|\dot{x}(t)|^2 \mid \dot{x}(0) = y\} \leq \sup_{(x,y)\in B(\theta(t)\psi)} |y|^2$$

for any $t \geq 0$, $\psi \in \Omega$, since $\pi_{\mathbb{T}^2}(B(\theta(t)\psi)) = \mathbb{T}^2$ for any $t \geq 0$ and $\psi \in \Omega$.
To estimate $|\dot{x}(t)|^2$ for $t > 0$, we take the inner product of the equation

$$\tau \ddot{x}(s) = \nabla^\perp \psi(x(s), s) - \dot{x}(s)$$

with $\dot{x}(s)$ and obtain

$$\begin{aligned} \tau \frac{1}{2}\frac{d}{ds}|\dot{x}(s)|^2 = \langle \dot{x}(s), \nabla^\perp \psi(x(s),s)\rangle - |\dot{x}(s)|^2 &\leq \frac{1}{2}|\nabla^\perp \psi(x(s),s)|^2 + \frac{1}{2}|\dot{x}(s)|^2 - |\dot{x}(s)|^2 \\ &= \frac{1}{2}|\nabla^\perp \psi(x(s),s)|^2 - \frac{1}{2}|\dot{x}(s)|^2, \end{aligned}$$

where we used $\langle z_1, z_2\rangle \leq \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$ for $z_1, z_2 \in \mathbb{R}^2$.
Multiplying by $\frac{2}{\tau}e^{\frac{s}{\tau}}$ on each side gives

$$\frac{d}{ds}\left(e^{\frac{s}{\tau}}|\dot{x}(s)|^2\right) \leq \frac{1}{\tau}e^{\frac{s}{\tau}}|\nabla^\perp \psi(x(s),s)|^2. \tag{5.3.11}$$

By integrating the inequality (5.3.11) from 0 to t we get

$$e^{\frac{t}{\tau}}|\dot{x}(t)|^2 - |\dot{x}(0)|^2 \leq \frac{1}{\tau}\int_0^t e^{\frac{s}{\tau}}|\nabla^\perp \psi(x(s),s)|^2 ds,$$

so that

$$
\begin{aligned}
|\dot{x}(t)|^2 &\le e^{-\frac{t}{\tau}}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_0^t e^{-\frac{(t-s)}{\tau}}|\nabla^\perp\psi(x(s),s)|^2 ds \\
&\le e^{-\frac{t}{\tau}}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_0^t e^{-\frac{(t-s)}{\tau}}|\psi(s)|^2_{C^1(\mathbb{T}^2)} ds \\
&\overset{u=s-t}{=} e^{-\frac{t}{\tau}}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_{-t}^0 e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du.
\end{aligned}
$$

Therefore

$$
\left|\pi_{\mathbb{R}^2}(\phi(t,\psi)\begin{pmatrix} x(0) \\ \dot{x}(0)\end{pmatrix})\right|^2 = |\dot{x}(t)|^2 \le e^{-\frac{t}{\tau}}|\dot{x}(0)|^2 + \frac{1}{\tau}\int_{-t}^0 e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du. \tag{5.3.12}
$$

Now by (5.3.12) and the definition of $B(\psi)$, we obtain for $t \ge 0$

$$
\begin{aligned}
\sup_{(x,y)\in B(\psi)} \left|\pi_{\mathbb{R}^2}(\phi(t,\psi)\begin{pmatrix} x \\ y\end{pmatrix})\right|^2 &\le e^{-\frac{t}{\tau}} \sup_{(x,y)\in B(\psi)} |y|^2 + \frac{1}{\tau}\int_{-t}^0 e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du \\
&\le \frac{(1+\delta)}{\tau} e^{-\frac{t}{\tau}} \int_{-\infty}^0 e^{\frac{u}{\tau}}|\psi(u)|^2_{C^1(\mathbb{T}^2)} du + \frac{(1+\delta)}{\tau}\int_{-t}^0 e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du \\
&= \frac{(1+\delta)}{\tau}\int_{-\infty}^{-t} e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du + \frac{(1+\delta)}{\tau}\int_{-t}^0 e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du \\
&= \frac{(1+\delta)}{\tau}\int_{-\infty}^0 e^{\frac{u}{\tau}}|\psi(u+t)|^2_{C^1(\mathbb{T}^2)} du \\
&= \sup_{(x,y)\in B(\theta(t)\psi)} |y|^2.
\end{aligned}
$$

So $\psi \mapsto B(\psi)$ is ϕ–forward invariant.
Finally, we prove that $\psi \mapsto B(\psi)$ is $\mathcal{D}$–absorbing. For any $D \in \mathcal{D}$ we have

$$
\pi_{\mathbb{T}^2}(\phi(t,\theta(-t)\psi)D((\theta(-t))\psi)) \subseteq \pi_{\mathbb{T}^2}(B(\psi)) = \mathbb{T}^2
$$

and by (5.3.12)

$$
\begin{aligned}
\sup_{(x,y)\in D(\theta(-t)\psi)} & \left|\pi_{\mathbb{R}^2}(\phi(t,\theta(-t)\psi)\begin{pmatrix}x\\ y\end{pmatrix})\right|^2 \\
& \le e^{-\frac{t}{\tau}} \sup_{(x,y)\in D(\theta(-t)\psi)} |y|^2 + \frac{1}{\tau}\int_{-t}^{0} e^{\frac{u}{\tau}} |\psi(u+t-t)|^2_{C^1(\mathbb{T}^2)} du \\
& = e^{-\frac{t}{\tau}} \sup_{(x,y)\in D(\theta(-t)\psi)} |y|^2 + \frac{1}{\tau}\int_{-t}^{0} e^{\frac{u}{\tau}} |\psi(u)|^2_{C^1(\mathbb{T}^2)} du
\end{aligned} \tag{5.3.13}
$$

for any $t \geq 0$ and $\psi \in \Omega$. By the definition of the set $\mathcal{D}$, the first term on the right hand side of (5.3.13) converges for $t \to \infty$ to zero. The second term tends for $t \to \infty$ to $\frac{1}{\tau}\int_{-\infty}^{0} e^{\frac{u}{\tau}} |\psi(u)|^2_{C^1(\mathbb{T}^2)} du$. Further, Lemma 5.3.2(ii) implies that

$$
e^{-tc}\frac{(1+\delta)}{\tau}\int_{-\infty}^{0} e^{\frac{u}{\tau}} |\psi(u)|^2_{C^1(\mathbb{T}^2)} du \to 0 \quad \text{as } t \to \infty
$$

for any $c, \delta > 0$ and $\psi \in \Omega$. This ensures $B \in \mathcal{D}$, i.e. B is also $\mathcal{D}$–absorbing. The existence of the random $\mathcal{D}$–attractor follows now by Theorem 3.4.10. The additional assertions are verified by Theorem 3.4.15, since $\mathcal{D}$ is the universe of tempered sets of $\mathbb{T}^2 \times \mathbb{R}^2$ and the random $\mathcal{D}$–attractor is $\mathcal{F}^-$–measurable by (5.3.9) and (5.3.10). □

Remark 5.3.6. In Theorem 5.3.5 we generalized Sigurgeirsson and Stuart's results in [89] concerning the existence and uniqueness of the random attractor to the fractional noise case with arbitrary Hurst parameter $H \in (0, 1)$ and improved it in such a way that we extended the universe of attracting sets from deterministic bounded sets to random tempered sets with an explicit representation of a random tempered the universe absorbing set. With these new information we were able to prove in addition the existence of a ϕ–invariant forward Markov measure.

The next step would be to derive (upper) bounds of the Hausdorff dimension of the random $\mathcal{D}$–attractor. But as already mentioned, we cannot directly apply Theorem 3.4.20 to bound the Hausdorff dimension of the random $\mathcal{D}$–attractor since on the one hand in $(M1)$ we have a two–dimensional torus $\mathbb{T}^2$ in the position coordinates and Theorem 3.4.20 does not cover this manifold case. On the other hand Theorem 3.4.20 would be usable for the by periodicity extended system introduced in the following Remark 5.3.7 if we could prove the existence of a random attractor in this modified system, but this is not the case. Nevertheless, in Theorem 5.3.8 below we verify properties of the modified system which are very close to the assumptions of Theorem 3.4.20 and (3.4.13) in Remark 3.4.21(i) to bound the Hausdorff dimension of the random attractor. This suggests at least that some *volume decrease* is happening in the modified and presumably in the original system (M), see Remark 5.3.9 below.

Remark 5.3.7. In the following we extend v and the stationary mild solution ψ in $(M2)$–$(M3)$ by periodicity to $\widetilde{v}$ and $\widetilde{\psi}$ on $\mathbb{R}^2$ and replace $(M1)$ by

$$\frac{d}{dt}\begin{pmatrix}\widetilde{x}(t)\\ \tilde{\dot{x}}(t)\end{pmatrix} = \widetilde{f}_{\widetilde{\psi},\tau}(t,(\widetilde{x}(t),\tilde{\dot{x}}(t))),\ \begin{pmatrix}\widetilde{x}(0)\\ \tilde{\dot{x}}(0)\end{pmatrix} \in \mathbb{R}^4,$$

with $\widetilde{f}_{\widetilde{\psi},\tau} : \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^4$ defined by

$$(t,x,y) \mapsto \widetilde{f}_{\widetilde{\psi},\tau}(t,(x,y)) = \begin{pmatrix} y \\ \frac{1}{\tau}(\nabla^\perp\widetilde{\psi}(x,t)-y)\end{pmatrix}.$$

In this modified system we can re–prove all assertions of Proposition 5.3.1, Lemma 5.3.2 and Proposition 5.3.4, whenever their assumptions are satisfied. But we are not able to prove the existence of a random attractor as stated in Theorem 5.3.5. In particular, in Proposition 5.3.1 we replace Ω by $\widetilde{\Omega} := C(\mathbb{R}, C^m(\mathbb{R}^2))$ endowed with compact open topology with associated Borel σ–algebra $\widetilde{\mathcal{F}}$, distribution $\widetilde{\mathbb{P}}$ and group of shifts $(\widetilde{\theta}(t))_{t\in\mathbb{R}}$. Further, in the same way we can show that there are $\widetilde{\mathcal{F}}$–measurable $(\widetilde{\theta}(t))_{t\in\mathbb{R}}$–invariant sets $\widetilde{\Omega}_0,\widetilde{\Omega}_1 \subseteq \widetilde{\Omega}$ satisfying the properties (ii)–(iii) of Lemma 5.3.2 and that $\widetilde{\phi} : \mathbb{R}\times\widetilde{\Omega}\times\mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^4$,

$$(t,\widetilde{\psi},(x,y)) \mapsto \widetilde{\phi}(t,\widetilde{\psi},(x,y)) := \begin{pmatrix}\widetilde{x}(t)\\ \tilde{\dot{x}}(t)\end{pmatrix},$$

defines a C^{m-1}–RDS over the ergodic metric dynamical system $\left(\widetilde{\Omega} := \widetilde{\Omega}_0\cap\widetilde{\Omega}_1, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, (\widetilde{\theta}(t))_{t\in\mathbb{R}}\right)$ analogous to Proposition 5.3.4. Also notice that $|\widetilde{\psi}|_{C([-T,T],C^m(\overline{\mathcal{O}}))} = |\widetilde{\psi}|_{C([-T,T],C^m(\mathbb{T}^2))}$ for all $T>0$, $\widetilde{\psi}\in\widetilde{\Omega}$ and any open, bounded $\mathcal{O}\subseteq\mathbb{R}^2$ with closure $\overline{\mathcal{O}}$ and $\mathbb{T}^2 = [0,1]^2 \subseteq \mathcal{O}$.

For convenience, in the rest of this subsection we will drop the tilde for denoting the modified system introduced in Remark 5.3.7.

In the following for $\psi\in\Omega$ and $t\in\mathbb{R}$ we define

$$\phi'(t,\psi,(x,y)) := \left(\left(\frac{\partial}{\partial x_j}\phi_i(t,\psi,(x,y))\right)_{1\le i\le 4, 1\le j\le 2}, \left(\frac{\partial}{\partial y_j}\phi_i(t,\psi,(x,y))\right)_{1\le i\le 4,1\le j\le 2}\right) \tag{5.3.14}$$

and

$$f'_{\psi,\tau}(t,(x,y)) := \frac{1}{\tau}\begin{pmatrix} 0 & 0 & \tau & 0\\ 0 & 0 & 0 & \tau\\ \frac{\partial^2}{\partial x_1\partial x_2}\psi(x,t) & \frac{\partial^2}{\partial x_2^2}\psi(x,t) & -1 & 0\\ -\frac{\partial^2}{\partial x_1^2}\psi(x,t) & -\frac{\partial^2}{\partial x_2\partial x_1}\psi(x,t) & 0 & -1\end{pmatrix}, \tag{5.3.15}$$

the Jacobians of the C^{m-1}–RDS $\phi(t,\psi)$ and $f_{\psi,\tau}(t,\cdot)$ at $(x,y)\in\mathbb{R}^4$ introduced in Remark 5.3.7 in the by periodicity extended system.
For the following theorem recall the definition of $V_d(L)$ from Section 3.4.

Theorem 5.3.8. *Suppose Assumption 5.1.2 holds with $m \in \mathbb{N}$, $m \leq 3$, and $\gamma \in (0,1)$. Let ϕ be the C^{m-1}–RDS introduced in Remark 5.3.7 in the by periodicity extended system and ϕ' be its Jacobian defined in (5.3.14). Further, we set $\mathcal{F}_t := \sigma(id_\Omega(r)|\ r \leq t)$, $t \geq 0$, and*

$$B_{H,\nu,\lambda,\alpha} := \mathbb{E}\left(|id_\Omega(0)|_{C^2(\mathbb{T}^2)}\right).$$

Then the following assertions are valid.

(i) For all $\tau > 0$ there are constants $C_1(\tau), C_2(\tau) > 0$ only depending on τ such that the $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$–$\mathcal{B}([0,\infty))$ measurable mapping $\xi_\tau : [0,\infty) \times \Omega \to [0,\infty)$,

$$(t,\psi) \mapsto \xi_\tau(t,\psi) = C_1(\tau) \exp\left(C_2(\tau) \int_0^t (1 + |\psi(s)|^2_{C^3(\mathbb{T}^2)}) ds\right) \tag{5.3.16}$$

satisfies for all $t \geq 0$, $(x,y), h \in \mathbb{R}^4$ and $\psi \in \Omega$

$$|\phi(t,\psi,(x,y)+h) - \phi(t,\psi,(x,y)) - \phi'(t,\psi,(x,y))h| \leq \xi_\tau(t,\psi)|h|^{\frac{3}{2}}$$

and $\xi_\tau(t,\psi) \geq 1$.

(ii) For any $l \in (0,1)$, $\tau > 0$ and $\epsilon_1 \in (0, \frac{2}{\tau})$ there is a finite $(\mathcal{F}_t)_{t\geq 0}$–stopping time $\bar{t}_{1,\tau} : \Omega \to [0,\infty)$ depending on $B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_1, l$ such that

$$\sup_{(x,y)\in\mathbb{R}^4} V_{3+s_1}\left(\phi'(\bar{t}_{1,\tau}(\psi), \psi, (x,y))\right) \leq l,\ \ \psi \in \Omega,$$

where $s_1 = s_1(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_1)$ is defined by

$$0 \leq \max\left\{0, \frac{\epsilon_1 + \frac{1}{\tau}\left(B_{H,\nu,\lambda,\alpha} - 1\right) + \frac{\tau}{4}}{\frac{2}{\tau} + \frac{1}{\tau}\left(B_{H,\nu,\lambda,\alpha} - 1\right) + \frac{\tau}{4}}\right\} =: s_1 < 1.$$

If in addition

$$B_{H,\nu,\lambda,\alpha} < 1,$$

then for any $l \in (0,1)$, $0 < \tau < \tau_* := 2\sqrt{1 - B_{H,\nu,\lambda,\alpha}}$ and any $\epsilon_2 \in \big(0, \frac{1}{\tau}(1 - B_{H,\nu,\lambda,\alpha}) - \frac{\tau}{4}\big)$ there is a finite $(\mathcal{F}_t)_{t\geq 0}$–stopping time $\bar{t}_{2,\tau} : \Omega \to [0,\infty)$ depending on $B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2, l$ such that

$$\sup_{(x,y)\in\mathbb{R}^4} V_{2+s_2}\left(\phi'(\bar{t}_{2,\tau}(\psi), \psi, (x,y))\right) \leq l,\ \ \psi \in \Omega,$$

where $s_2 = s_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2)$, whenever $B_{H,\nu,\lambda,\alpha} < 1$, $0 < \tau < \tau_*$, is defined by

$$0 < \frac{\epsilon_2 + \frac{2}{\tau} B_{H,\nu,\lambda,\alpha} + \frac{\tau}{2}}{\frac{1}{\tau}\left(1 - B_{H,\nu,\lambda,\alpha}\right) - \frac{\tau}{4} + \frac{2}{\tau} B_{H,\nu,\lambda,\alpha} + \frac{\tau}{2}} =: s_2 < 1.$$

(iii) We set

$$\bar{t}_\tau := \bar{t}_{B_{H,\nu,\lambda,\alpha},\tau} := \begin{cases} \bar{t}_{1,\tau} & \text{if } B_{H,\nu,\lambda,\alpha} \geq 1 \\ \begin{cases} \bar{t}_{1,\tau} & \text{if } \tau > \tau_* \\ \bar{t}_{2,\tau} & \text{if } \tau < \tau_* \end{cases} & \text{if } B_{H,\nu,\lambda,\alpha} < 1 \end{cases},$$

$$d(\tau) := d(B_{H,\nu,\lambda,\alpha},\tau) := \begin{cases} 3 & \text{if } B_{H,\nu,\lambda,\alpha} \geq 1 \\ \begin{cases} 3 & \text{if } \tau > \tau_* \\ 2 & \text{if } \tau < \tau_* \end{cases} & \text{if } B_{H,\nu,\lambda,\alpha} < 1 \end{cases},$$

$$0 < s(\tau) := s(B_{H,\nu,\lambda,\alpha},\tau,\epsilon) := \begin{cases} s_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1) & \text{if } B_{H,\nu,\lambda,\alpha} \geq 1 \\ \begin{cases} s_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1) & \text{if } \tau > \tau_* \\ s_2(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_2) & \text{if } \tau < \tau_* \end{cases} & \text{if } B_{H,\nu,\lambda,\alpha} < 1 \end{cases},$$

where τ_*, $\bar{t}_{j,\tau}$, $s_j(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_j)$, $j = 1, 2$, *are defined in (ii). Then for all* $\tau > 0$ *the random variables* $m_\tau, Z_\tau : \Omega \to [0,\infty)$ *defined by*

$$\psi \mapsto m_\tau(\psi) := \exp\big(C_3(\tau) \int_0^{\bar{t}_\tau(\psi)} (1 + |\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\big),$$

$$\psi \mapsto Z_\tau(\psi) := \left(\frac{m_\tau(\psi)^{d(\tau)}}{l}\right)^{\frac{1}{s(\tau)}} \xi_\tau(\bar{t}_\tau(\psi),\psi)$$

with $l \in (0,1)$ *and a constant* $C_3(\tau) > 0$ *only depending on* τ *satisfy the conditions*

(a) $(m_\tau(\psi))^{d(\tau)+s(\tau)} \geq 1$, $\psi \in \Omega$,

(b) $\sup_{(x,y)\in\mathbb{R}^4} |\phi'(\bar{t}_\tau(\psi),\psi,(x,y))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)} \leq m_\tau(\psi)$, $\psi \in \Omega$,

(c)

$$\lim_{i\to\infty} \frac{\ln(\max\{1, Z_\tau(\tilde{\theta}_i(\psi)\})}{i} = 0 \;\; \mathbb{P}\text{-}a.s., \quad \tilde{\theta}_i := \begin{cases} id & i = 0 \\ \underbrace{\tilde{\theta} \circ \cdots \circ \tilde{\theta}}_{i-times} & i \in \mathbb{N} \end{cases},$$

where ξ_τ *is defined in (i) and the measure preserving random variable* $\tilde{\theta} : \Omega \to \Omega$ *is defined by*

$$\psi \mapsto \tilde{\theta}(\psi) := \theta(\bar{t}_\tau(\psi))\psi.$$

We will prove the assertions of Theorem 5.3.8 in several lemmas. Before proceeding, we interpret the results of Theorem 5.3.8 in the next remark.

Remark 5.3.9. As mentioned in Section 1, in [8, 9] Bec calculated in the system (M) with $H = 1/2$ numerically the Lyapunov dimension (LD) (see Remark V.3.5 in [93] for a definition) which is in this system at most 4 and which is an upper bound of the $\mathbb{P}$-a.s. constant Hausdorff dimension of the random attractor in dependence on the Stokes's number τ. He computed that $LD \approx 2$ for $\tau \approx 0$, LD decreases in τ to some minimum $1 < LD_{min}(\hat{\tau}) < 2$ and then increases in τ with $LD(\tau) > LD(\overline{\tau}) = 2$ for $\tau > \overline{\tau} > \hat{\tau} > 0$. In particular, these bounds are in accordance with the limiting behaviour of the ODE $(M1)$ for $\tau \to 0$ and $\tau \to \infty$, and give a numerical justification of preferential concentration as a fractal clustering phenomenon. If we could prove the existence of a random $\mathcal{D}$–attractor in the by periodicity extended system introduced in Remark 5.3.7, then the assertions of Theorem 5.3.8 would imply that the $\mathbb{P}$-a.s. constant Hausdorff dimension of the random $\mathcal{D}$–attractor is bounded from above by

$$d := \begin{cases} 3 + s_1^0(B_{H,\nu,\lambda,\alpha}, \tau) & \text{if } B_{H,\nu,\lambda,\alpha} \geq 1 \\ \begin{cases} 3 + s_1^0(B_{H,\nu,\lambda,\alpha}, \tau) & \text{if } \tau \geq \tau_* \\ 2 + s_2^0(B_{H,\nu,\lambda,\alpha}, \tau) & \text{if } \tau < \tau_* \end{cases} & \text{if } B_{H,\nu,\lambda,\alpha} < 1 \end{cases},$$

where $s_1^0(B_{H,\nu,\lambda,\alpha}, \tau)$, whenever $B_{H,\nu,\lambda,\alpha} = \mathbb{E}(|id_\Omega(0)|_{C^2(\mathbb{T}^2)}) \geq 1$ or $B_{H,\nu,\lambda,\alpha} < 1$, $\tau \geq \tau_* := 2\sqrt{1 - B_{H,\nu,\lambda,\alpha}}$, is defined by

$$0 \leq s_1^0(B_{H,\nu,\lambda,\alpha}, \tau) := \lim_{\epsilon_1 \downarrow 0} s_1(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_1) = \frac{\frac{1}{\tau}\big(B_{H,\nu,\lambda,\alpha} - 1\big) + \frac{\tau}{4}}{\frac{1}{\tau}\big(B_{H,\nu,\lambda,\alpha} + 1\big) + \frac{\tau}{4}} < 1$$

and $s_2^0(B_{H,\nu,\lambda,\alpha}, \tau)$, whenever $B_{H,\nu,\lambda,\alpha} < 1$, $0 < \tau < \tau_*$, is defined by

$$0 < s_2^0(B_{H,\nu,\lambda,\alpha}, \tau) := \lim_{\epsilon_2 \downarrow 0} s_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2) = \frac{\frac{2}{\tau} B_{H,\nu,\lambda,\alpha} + \frac{\tau}{2}}{\frac{1}{\tau}\big(1 + B_{H,\nu,\lambda,\alpha}\big) + \frac{\tau}{4}} < 1,$$

where $s_1(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_1)$ and $s_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2)$ are defined in Theorem 5.3.8(ii). Comparing these upper bounds with the numerical results by Bec, they do not look impressive. In particular, we do not have a bound strict less than 2. Also notice that in Theorem 5.3.8 only the statistical property $B_{H,\nu,\lambda,\alpha} < 1$ is involved. Probably, one could derive smaller upper bounds by using more *statistical information* of the underlying metric dynamical system, i.e. of the noise. But that are just hypothetical conjectures since we cannot prove the existence of the random $\mathcal{D}$–attractor in the modified system. However, the assertions of Theorem 5.3.8 suggest at least a volume contraction in the modified and presumably in the original system (M). To see this, recall from Remark 3.4.19(ii) that the numbers $V_d(\phi'(t, \psi, (x, y)))$, $d = 3 + s_1, 2 + s_2$ in Theorem 5.3.8(ii) can be interpreted as the largest distortion of an infinitesimal d–dimensional volume produced by $\phi(t, \psi, (x, y))$. Further, analogous to the autonomous deterministic case in Remark 3.4.19(ii) we have due to the trace formula

$$\sup_{(x,y) \in \mathbb{R}^4} V_d(\phi'(t, \psi, (x, y))) \leq \exp(\overline{q}_d(t, \psi)), \quad t \geq 0. \tag{5.3.17}$$

Here

$$
\begin{aligned}
\overline{q}_4(t,\psi) &:= \sup_{(x,y)\in\mathbb{R}^4} \int_0^t tr(f'_{\psi,\tau}(s,(x(s),\dot{x}(s))))ds,\\
\overline{q}_n(t,\psi) &:= \sup_{(x,y)\in\mathbb{R}^4} \sup_{\substack{h_i\in\mathbb{R}^4,\\ |h_i|\le 1,\\ i=1,\dots,n}} \Big(\int_0^t tr_n(f'_{\psi,\tau}(s,(x(s),\dot{x}(s)))\circ Q_{n,h_1,\dots,h_n}(s,\psi,(x,y)))ds\Big),\\
\overline{q}_{n+s}(t,\psi) &:= s\overline{q}_{n+1}(t,\psi)+(1-s)\overline{q}_n(t,\psi),
\end{aligned}
$$

where $s\in[0,1)$, $Q_{n,h_1,\dots,h_n}(s,\psi,(x,y))$, $n\in\mathbb{N}$, $1\le n\le 3$, is the orthonormal projector in $\mathbb{R}^4$ spanned by $\phi'(t,\psi,(x,y))h_1,\dots,\phi'(t,\psi,(x,y))h_n$ and tr_n is the trace w.r.t. this subspace. In Lemma 5.3.11 below we will show that for all $\tau>0$ and $\epsilon_1\in(0,2/\tau)$ there exist $s_1=s_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1)\in[0,1)$ which is defined in Theorem 5.3.8(ii) and $t_1=t_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1,\psi)\ge 0$ such that for all $t\ge t_1$ we have

$$
\overline{q}_{3+s_1}(t,\psi)\le -\epsilon_1 t<0 \tag{5.3.18}
$$

and if $B_{H,\nu,\lambda,\alpha}<1$ then for all $0<\tau<2\sqrt{1-B_{H,\nu,\lambda,\alpha}}$ and $\epsilon_2\in\big(0,\frac{1}{\tau}(1-B_{H,\nu,\lambda,\alpha})-\frac{\tau}{4}\big)$ there exist $s_2=s_2(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_2)\in(0,1)$ which is defined in Theorem 5.3.8(ii) and $t_2=t_2(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_2,\psi)\ge 0$ such that for all $t\ge t_2$ we have

$$
\overline{q}_{2+s_2}(t,\psi)\le -\epsilon_2 t<0. \tag{5.3.19}
$$

Therefore, in view of (5.3.17), (5.3.18) and (5.3.19) the volume elements $V_d(\phi'(t,\psi,(x,y)))$, $d=3+s_1,2+s_2$, are uniformly contracting as $t\to\infty$.

Lemma 5.3.10. *Suppose the assumptions of Theorem 5.3.8 hold. Then assertion (i) of Theorem 5.3.8 is valid.*

Proof. We fix $t\ge 0$, $(x,y),h\in\mathbb{R}^4$, $\psi\in\Omega$ and set

$$
\phi(t,\psi,(x,y)) =: \begin{pmatrix} x(t)\\ \dot{x}(t)\end{pmatrix},\ \phi(t,\psi,(x,y)+h) =: \begin{pmatrix} \tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix}.
$$

We obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\phi(t,\psi,(x,y))-\phi(t,\psi,(x,y)+h)|^2 = \frac{1}{2}\frac{d}{dt}\Big|\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}-\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix}\Big|^2\\
&= \langle\begin{pmatrix}x(t)-\tilde{x}(t)\\ \dot{x}(t)-\dot{\tilde{x}}(t)\end{pmatrix},\begin{pmatrix}\dot{x}(t)-\dot{\tilde{x}}(t)\\ \ddot{x}(t)-\ddot{\tilde{x}}(t)\end{pmatrix}\rangle\\
&= \langle x(t)-\tilde{x}(t),\dot{x}(t)-\dot{\tilde{x}}(t)\rangle + \langle \dot{x}(t)-\dot{\tilde{x}}(t),\ddot{x}(t)-\ddot{\tilde{x}}(t)\rangle\\
&= \langle x(t)-\tilde{x}(t),\dot{x}(t)-\dot{\tilde{x}}(t)\rangle - \frac{1}{\tau}|\dot{x}(t)-\dot{\tilde{x}}(t)|^2\\
&\qquad\qquad + \frac{1}{\tau}\langle \dot{x}(t)-\dot{\tilde{x}}(t),\nabla^{\perp}\psi(x(t),t)-\nabla^{\perp}\psi(\tilde{x}(t),t)\rangle\\
&\le \frac{\tau}{2}|x(t)-\tilde{x}(t)|^2+\frac{1}{2\tau}|\dot{x}(t)-\dot{\tilde{x}}(t)|^2-\frac{1}{\tau}|\dot{x}(t)-\dot{\tilde{x}}(t)|^2 \qquad (5.3.20)\\
&\qquad\qquad + \frac{1}{2\tau}|\dot{x}(t)-\dot{\tilde{x}}(t)|^2+\frac{1}{2\tau}|\nabla^{\perp}\psi(x(t),t)-\nabla^{\perp}\psi(\tilde{x}(t),t)|^2\\
&= \frac{\tau}{2}|x(t)-\tilde{x}(t)|^2+\frac{1}{2\tau}|\nabla^{\perp}\psi(x(t),t)-\nabla^{\perp}\psi(\tilde{x}(t),t)|^2\\
&\le \frac{\tau}{2}|x(t)-\tilde{x}(t)|^2+\frac{2}{2\tau}|\psi(t)|^2_{C^2(\mathbb{T}^2)}|x(t)-\tilde{x}(t)|^2\\
&\le \big(\frac{\tau}{2}+\frac{1}{\tau}\big)(1+|\psi(t)|^2_{C^2(\mathbb{T}^2)})|x(t)-\tilde{x}(t)|^2\\
&\le \big(\frac{\tau}{2}+\frac{1}{\tau}\big)(1+|\psi(t)|^2_{C^2(\mathbb{T}^2)})\Big|\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}-\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix}\Big|^2,
\end{aligned}
$$

where in the first inequality in (5.3.20) we used Young's inequality $|ab|\le\epsilon|a|^2+\frac{(2\epsilon)^{-1}}{2}|b|^2$ with $a,b\in\mathbb{R}$, $\epsilon=\frac{1}{2\tau}$ and $\langle z_1,z_2\rangle\le\frac{1}{2}|z_1|^2+\frac{1}{2}|z_2|^2$ for $z_1,z_2\in\mathbb{R}^2$. The second inequality in (5.3.20) follows by

$$\sum_{1\le i\le 2}\left|\frac{\partial}{\partial x_i}\psi(x(t),t)-\frac{\partial}{\partial x_i}\psi(\tilde{x}(t),t)\right|^2\le 2|\psi(t)|^2_{C^2(\mathbb{T}^2)}|x(t)-\tilde{x}(t)|^2$$

since $\psi(t)\in C^m(\mathbb{R}^2)$ for some $m\ge 3$ and $\psi(t)$ is periodic with period 1 in both variables. So by Gronwall's inequality we get

$$
\begin{aligned}
|\phi(t,\psi,(x,y))-\phi(t,\psi,(x,y)+h)|^2 &= \Big|\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}-\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix}\Big|^2\\
&\le |h|^2\exp\Big(\big(\tau+\frac{2}{\tau}\big)\int_0^t(1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\Big)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
|\phi(t,\psi,(x,y))-\phi(t,\psi,(x,y)+h)| &= \Big|\begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}-\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix}\Big|\\
&\le |h|\exp\Big(C_{0,1}(\tau)\int_0^t(1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\Big)
\end{aligned}
\qquad (5.3.21)
$$

with $C_{0,1}(\tau) := \frac{1}{2}(\tau + \frac{2}{\tau})$. Further, notice that

$$
\begin{aligned}
&|f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)} \\
&\qquad = \frac{1}{\tau} \sup_{\substack{a,b\in\mathbb{R}^4,\\ |a|,|b|\le 1}} \langle \begin{pmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \\ \frac{\partial^2}{\partial x_1 \partial x_2}\psi(x(t),t) & \frac{\partial^2}{\partial x_2^2}\psi(x(t),t) & -1 & 0 \\ -\frac{\partial^2}{\partial x_1^2}\psi(x(t),t) & -\frac{\partial^2}{\partial x_2 \partial x_1}\psi(x(t),t) & 0 & -1 \end{pmatrix} a, b\rangle \\
&\qquad \le \frac{1}{\tau}\big(2\tau + 2 + \sum_{i,j=1}^{2} |\frac{\partial^2}{\partial x_i \partial x_j}\psi(x,t)|\big) \\
&\qquad \le C_{0,2}(\tau)\big(1 + |\psi(t)|_{C^2(\mathbb{T}^2)}\big)
\end{aligned} \tag{5.3.22}
$$

with $C_{0,2}(\tau) := \frac{2\tau+3}{\tau}$ and similarly

$$
\begin{aligned}
|f''_{\psi,\tau}(t,(x(t),\dot{x}(t)))|_{\mathcal{L}(\mathbb{R}^4,\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4))} &\le \frac{C}{\tau} \sum_{i,j,k=1}^{2} |\frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\psi(x(t),t)| \\
&\le C_{0,3}(\tau)|\psi(t)|_{C^3(\mathbb{T}^2)}
\end{aligned} \tag{5.3.23}
$$

with $C_{0,3}(\tau) := \frac{C}{\tau}$ and some constant $C > 0$, where $f''_{\psi,\tau}(t,(x(t),\dot{x}(t)))$ denotes the second Frechet derivative of $f_{\psi,\tau}(t,\cdot)$ in $(x(t),\dot{x}(t)) \in \mathbb{R}^4$.
Moreover, $\phi'(t,\psi,(x,y))h = v(t)$ satisfies the equation

$$v(t) = h + \int_0^t f'_{\psi,\tau}(s,(x(s),\dot{x}(s)))v(s)ds$$

and the inequality

$$|v(t)| \le |h| + \int_0^t |f'_{\psi,\tau}(s,(x(s),\dot{x}(s)))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)}|v(s)|ds.$$

Gronwall's inequality and (5.3.22) imply then

$$
\begin{aligned}
|\phi'(t,\psi,(x,y))h| &\le |h| \exp\big(\int_0^t |f'_{\psi,\tau}(s,(x(s),\dot{x}(s)))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)}ds\big) \\
&\le |h| \exp\big(C_{0,2}(\tau)\int_0^t (1 + |\psi(s)|_{C^2(\mathbb{T}^2)})ds\big) \\
&\le |h| \exp\big(C_{0,4}(\tau)\int_0^t (1 + |\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\big)
\end{aligned} \tag{5.3.24}
$$

with $C_{0,4}(\tau) := 2C_{0,2}(\tau) = \frac{4\tau+6}{\tau}$.
Further, by Taylor's theorem (see e.g. [100] Theorem 4.C on page 242) and (5.3.23) we have

$$
\begin{aligned}
&|f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)| \\
&\leq \frac{1}{2}\sup_{0\leq\tau\leq 1}|f''_{\psi,\tau}\Big(t,(x(t),\dot{x}(t)) + \tau\big((\tilde{x}(t),\dot{\tilde{x}}(t)) - (x(t),\dot{x}(t))\big)\Big) \\
&\qquad\qquad \Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)| \\
&\leq \frac{1}{2}\sup_{0\leq\tau\leq 1}|f''_{\psi,\tau}(t,(x(t),\dot{x}(t)) + \tau((\tilde{x}(t),\dot{\tilde{x}}(t)) - (x(t),\dot{x}(t))))|_{\mathcal{L}(\mathbb{R}^4,\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4))} \\
&\qquad\qquad \times |\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}|^2 \\
&\leq \frac{C_{0,3}(\tau)}{2}|\psi(t)|_{C^3(\mathbb{T}^2)}|\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}|^2 \\
&= C_{0,5}(\tau)|\psi(t)|_{C^3(\mathbb{T}^2)}|\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}|^2
\end{aligned}
\tag{5.3.25}
$$

with $C_{0,5}(\tau) := \frac{C_{0,3}(\tau)}{2}$.
For fixed $t \geq 0$, $(x,y), h \in \mathbb{R}^4$, $\psi \in \Omega$ we define

$$
\begin{aligned}
\triangle_h(t,\psi) &:= \phi(t,\psi,(x,y)+h) - \phi(t,\psi,(x,y)) - \phi'(t,\psi,(x,y))h \\
&= \begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix} - \phi'(t,\psi,(x,y))h.
\end{aligned}
$$

Notice that $\triangle_h(t,\psi)$ satisfies the equations

$$
\begin{aligned}
&\frac{d}{dt}\triangle_h(t,\psi) = f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) + f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\triangle_h(t,\psi) \\
&\qquad\qquad - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big), \\
&\quad \triangle_h(0,\psi) = 0 \in \mathbb{R}^4
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\triangle_h(t,\psi)|^2 &= \langle f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) \\
&\qquad - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big), \triangle_h(t,\psi)\rangle \\
&\quad + \langle f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\triangle_h(t,\psi), \triangle_h(t,\psi)\rangle \\
&= I_1 + I_2 \leq |I_1| + |I_2|.
\end{aligned}
$$

In the following we estimate $|I_1|$, $|I_2|$. We start with $|I_1|$.

$$
\begin{aligned}
&|I_1| = |\langle f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) \\
&\qquad\qquad - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big), \triangle_h(t,\psi)\rangle| \\
&\le |f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) \\
&\qquad\qquad - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)||\triangle_h(t,\psi)| \\
&= |f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) \\
&\quad - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)||\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix} - \phi'(t,\psi,(x,y))h| \\
&\le |f_{\psi,\tau}(t,(\tilde{x}(t),\dot{\tilde{x}}(t))) - f_{\psi,\tau}(t,(x(t),\dot{x}(t))) \\
&\quad - f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)|\Big(|\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}| + |\phi'(t,\psi,(x,y))h|\Big) \\
&\le C_{0,5}(\tau)|\psi(t)|_{C^3(\mathbb{T}^2)}|\Big(\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}\Big)|^2\Big(|\begin{pmatrix}\tilde{x}(t)\\ \dot{\tilde{x}}(t)\end{pmatrix} - \begin{pmatrix}x(t)\\ \dot{x}(t)\end{pmatrix}| + |\phi'(t,\psi,(x,y))h|\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le C_{0,5}(\tau)|\psi(t)|_{C^3(\mathbb{T}^2)}|h|^2 \exp\Big(2C_{0,1}(\tau)\int_0^t (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\Big) \\
&\quad \times \Big(|h|\exp\Big(C_{0,1}(\tau)\int_0^t (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\Big) + |h|\exp\Big(C_{0,4}(\tau)\int_0^t (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\Big)\Big) \\
&\le 2C_{0,5}(\tau)|\psi(t)|_{C^3(\mathbb{T}^2)}|h|^3 \exp\Big(C_1(\tau)\int_0^t (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\Big)
\end{aligned}
\tag{5.3.26}
$$

with $C_1(\tau) := 4C_{0,1}(\tau) + C_{0,4}(\tau)$, where we used (5.3.25), (5.3.21) and (5.3.24).
Next we estimate $|I_2|$.

$$
\begin{aligned}
|I_2| &= |\langle f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\triangle_h(t,\psi), \triangle_h(t,\psi)\rangle| \le |f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)}|\triangle_h(t,\psi)|^2 \\
&\le C_{0,2}(\tau)(1+|\psi(t)|_{C^2(\mathbb{T}^2)})|\triangle_h(t,\psi)|^2 \\
&\le C_2(\tau)(1+|\psi(t)|^2_{C^2(\mathbb{T}^2)})|\triangle_h(t,\psi)|^2
\end{aligned}
\tag{5.3.27}
$$

with $C_2(\tau) := C_{0,2}(\tau) + 2$, where we used (5.3.22).

Therefore, by (5.3.26) and (5.3.27) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\triangle_h(t,\psi)|^2 &\le |I_1|+|I_2| \\ &\le 2C_{0,5}(\tau)|\psi(t)|_{C^3(\mathbb{T}^2)}|h|^3\exp\big(C_1(\tau)\int_0^t(1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\big) \\ &\quad + C_2(\tau)(1+|\psi(t)|^2_{C^2(\mathbb{T}^2)})|\triangle_h(t,\psi)|^2.\end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned}|\triangle_h(t,\psi)|^2 &\le 0+4C_{0,5}(\tau)|h|^3\int_0^t|\psi(s)|_{C^3(\mathbb{T}^2)}\exp\big(C_1(\tau)\int_0^s(1+|\psi(r)|^2_{C^2(\mathbb{T}^2)})dr\big) \\ &\qquad\times\exp\big(2C_2(\tau)\int_s^t(1+|\psi(r)|^2_{C^2(\mathbb{T}^2)})dr\big)ds \\ &\le 4C_{0,5}(\tau)|h|^3\int_0^t|\psi(s)|_{C^3(\mathbb{T}^2)}ds\exp\big((C_1(\tau)+2C_2(\tau))\int_0^t(1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\big) \\ &\le C_3(\tau)|h|^3\exp\big(C_4(\tau)\int_0^t(1+|\psi(s)|^2_{C^3(\mathbb{T}^2)})ds\big)\end{aligned}$$

with $C_3(\tau) := 4C_{0,5}(\tau)$ and $C_4(\tau) := C_1(\tau)+2C_2(\tau)+2$.
Finally, by Lemma 5.3.2(ii) and Remark 5.3.7 the mapping $\xi_\tau : [0,\infty)\times\Omega\to[0,\infty)$

$$(t,\psi)\mapsto\xi_\tau(t,\psi)=\sqrt{C_5(\tau)}\exp\left(\frac{C_4(\tau)}{2}\int_0^t(1+|\psi(s)|^2_{C^3(\mathbb{T}^2)})ds\right)$$

with $C_5(\tau) := C_3(\tau)+1$ is well–defined, $\mathcal{B}([0,\infty))\otimes\mathcal{F}$–$\mathcal{B}([0,\infty))$ measurable and satisfies the assertions of Theorem 5.3.8(i). □

Lemma 5.3.11. *Suppose the assumptions of Theorem 5.3.8 hold. Then assertion (ii) of Theorem 5.3.8 is valid.*

Proof. To establish

$$\sup_{(x,y)\in\mathbb{R}^4} V_d(\phi'(t,\psi,(x,y))))\le k,\ \ \psi\in\Omega,$$

for $k\in(0,1)$ and certain $t>0$, $0<d\le 4$, we use, as mentioned in Remark 5.3.9, the *trace formula* due to Temam ([93] pp. 362–364). Recall again that the trace formula asserts that the uniform volume expansion factors for $\psi\in\Omega$ and $(x,y)\in\mathbb{R}^4$ at time

$t \geq 0$ are given by

$$\overline{q}_4(t,\psi) := \sup_{(x,y)\in\mathbb{R}^4} \int_0^t tr(f'_{\psi,\tau}(s,(x(s),\dot{x}(s))))ds,$$

$$\overline{q}_n(t,\psi) := \sup_{(x,y)\in\mathbb{R}^4} \sup_{\substack{h_i\in\mathbb{R}^4,\\ |h_i|\leq 1,\\ i=1,\dots,n}} \Big(\int_0^t tr_n(f'_{\psi,\tau}(s,(x(s),\dot{x}(s))) \circ Q_{n,h_1,\dots,h_n}(s,\psi,(x,y)))ds\Big)$$

and

$$\sup_{(x,y)\in\mathbb{R}^4} V_n(\phi'(t,\psi,(x,y))) \leq \exp(\overline{q}_n(t,\psi)), \tag{5.3.28}$$

where $Q_{n,h_1,\dots,h_n}(s,\psi,(x,y))$, $n \in \mathbb{N}$, $1 \leq n \leq 3$, is the orthonormal projector in $\mathbb{R}^4$ spanned by $\phi'(t,\psi,(x,y))h_1,\dots,\phi'(t,\psi,(x,y))h_n$ and tr_n is the trace w.r.t. this subspace. $f'_{\psi,\tau}$ is defined in (5.3.15). Actually, it is more convenient to consider the quantities $\overline{q}_n(t,\psi)/t$ rather than $\overline{q}_n(t,\psi)$ as will be seen later in this proof. By (5.3.28) we have

$$\begin{aligned}
&\frac{1}{t}\ln\Big(\sup_{(x,y)\in\mathbb{R}^4} V_d(\phi'(t,\psi,(x,y)))\Big)\\
&\qquad = \frac{1}{t}\ln\Big(\sup_{(x,y)\in\mathbb{R}^4}\big(V_{n+1}(\phi'(t,\psi,(x,y)))^s V_n(\phi'(t,\psi,(x,y)))^{1-s}\big)\Big)\\
&\qquad \leq \frac{s}{t}\overline{q}_{n+1}(t,\psi) + \frac{(1-s)}{t}\overline{q}_n(t,\psi)
\end{aligned} \tag{5.3.29}$$

with $n \in \mathbb{N}$, $1 \leq n \leq 3$, $s \in [0,1)$ and $d = n+s$.
We will show that for all $\tau > 0$ there is $\epsilon_1 > 0$ such that for large $t > 0$ (depending upon the realization ψ)

$$\frac{s_1}{t}\overline{q}_4(t,\psi) + \frac{(1-s_1)}{t}\overline{q}_3(t,\psi) \leq -\epsilon_1 < 0$$

for some $s_1 \in [0,1)$ and under the additional assumption $\mathbb{E}\left(|id_\Omega(0)|_{C^2(\mathbb{T}^2)}\right) < 1$ we will show that there is $\tau_* > 0$ such that for all $0 < \tau < \tau_*$ there is $\epsilon_2 > 0$ such that for large $t > 0$ (depending upon the realization ψ)

$$\frac{s_2}{t}\overline{q}_3(t,\psi) + \frac{(1-s_2)}{t}\overline{q}_2(t,\psi) \leq -\epsilon_2 < 0$$

for some $s_2 \in (0,1)$, since then for any $k \in (0,1)$ we can prove that there are families of finite stopping times $\overline{t}_{1,\tau} : \Omega \to [0,\infty)$, $\tau > 0$, $\overline{t}_{2,\tau} : \Omega \to [0,\infty)$, $0 < \tau < \tau_*$, such that

$$\sup_{(x,y)\in\mathbb{R}^4} V_{3+s_1}(\phi'(\overline{t}_{1,\tau}(\psi),\psi,(x,y)))) \leq k,\ \ \psi \in \Omega,$$

for a fixed $\tau > 0$ and

$$\sup_{(x,y)\in\mathbb{R}^4} V_{2+s_2}(\phi'(\overline{t}_{2,\tau}(\psi),\psi,(x,y)))) \leq k,\ \ \psi \in \Omega,$$

for a fixed $0 < \tau < \tau_*$.
First we estimate $\overline{q}_4(t,\psi)$, $\overline{q}_3(t,\psi)$ and $\overline{q}_2(t,\psi)$. Notice that for $t \geq 0$, $(x(t), \dot{x}(t)) \in \mathbb{R}^4$ we have

$$\begin{aligned} tr(f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))) &= tr\big(\frac{1}{\tau}\begin{pmatrix} 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \\ \frac{\partial^2}{\partial x_1 \partial x_2}\psi(x(t),t) & \frac{\partial^2}{\partial x_2^2}\psi(x(t),t) & -1 & 0 \\ -\frac{\partial^2}{\partial x_1^2}\psi(x(t),t) & -\frac{\partial^2}{\partial x_2 \partial x_1}\psi(x(t),t) & 0 & -1 \end{pmatrix}\big) \\ &= -\frac{2}{\tau} \end{aligned} \tag{5.3.30}$$

and for any arbitrary orthonormal system $e^i = (e_1^i, e_2^i, e_3^i, e_4^i)$, $i = 1,2,3,4$, in $\mathbb{R}^4$, $j \in \{1,2,3\}$ and $t \geq 0$ we estimate

$$\begin{aligned} &\sum_{i=1}^{4-j} \langle f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))e^i, e^i\rangle \\ &\quad = tr\big(f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))\big) - \sum_{i=4-j+1}^{4} \langle f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))e^i, e^i\rangle \\ &\quad = -\frac{2}{\tau} - \sum_{i=4-j+1}^{4} \langle \begin{pmatrix} e_3^i \\ e_4^i \\ \frac{1}{\tau}\big(\frac{\partial^2}{\partial x_1 \partial x_2}\psi(x(t),t)e_1^i + \frac{\partial^2}{\partial x_2^2}\psi(x(t),t)e_2^i - e_3^i\big) \\ -\frac{1}{\tau}\big(\frac{\partial^2}{\partial x_1^2}\psi(x(t),t)e_1^i + \frac{\partial^2}{\partial x_2 \partial x_1}\psi(x(t),t)e_2^i + e_4^i\big) \end{pmatrix}, \begin{pmatrix} e_1^i \\ e_2^i \\ e_3^i \\ e_4^i \end{pmatrix}\rangle \\ &\quad = -\frac{2}{\tau} + \sum_{i=4-j+1}^{4} \Big(- e_1^i e_3^i - e_2^i e_4^i + \frac{(e_3^i)^2}{\tau} + \frac{(e_4^i)^2}{\tau} \\ &\qquad\qquad + \frac{1}{\tau}\big[- e_3^i\big(\frac{\partial^2}{\partial x_1 \partial x_2}\psi(x(t),t)e_1^i + \frac{\partial^2}{\partial x_2^2}\psi(x(t),t)e_2^i\big) \\ &\qquad\qquad\quad + e_4^i\big(\frac{\partial^2}{\partial x_1^2}\psi(x(t),t)e_1^i + \frac{\partial^2}{\partial x_2 \partial x_1}\psi(x(t),t)e_2^i\big)\big]\Big) \\ &\quad = -\frac{2}{\tau} + \sum_{i=4-j+1}^{4} (I_1^i + \frac{1}{\tau}I_2^i). \end{aligned} \tag{5.3.31}$$

Next we estimate I_1^i and I_2^i separately.

$$\begin{aligned} I_1^i &= -e_1^i e_3^i - e_2^i e_4^i + \frac{(e_3^i)^2}{\tau} + \frac{(e_4^i)^2}{\tau} \\ &= -e_1^i e_3^i - e_2^i e_4^i - \frac{(e_1^i)^2}{\tau} - \frac{(e_2^i)^2}{\tau} + \frac{1}{\tau} \\ &\le \tau\frac{(e_3^i)^2}{4} + \tau\frac{(e_4^i)^2}{4} + \frac{(e_1^i)^2}{\tau} + \frac{(e_2^i)^2}{\tau} - \frac{(e_1^i)^2}{\tau} - \frac{(e_2^i)^2}{\tau} + \frac{1}{\tau} \\ &\le \frac{\tau}{4} + \frac{1}{\tau}, \end{aligned} \tag{5.3.32}$$

where in the first inequality we used twice Young's inequality, i.e. $|ab| \le \eta|a|^p + (p\eta)^{1-q}|b|^q/q$ with $p = q = 2$, $\eta = \frac{1}{\tau}$, $a = e_1^i, e_2^i$ and $b = e_3^i, e_4^i$. Further,

$$\begin{aligned} I_2^i = -e_3^i &\big(\frac{\partial^2}{\partial x_1 \partial x_2}\psi(x(t),t)e_1^i + \frac{\partial^2}{\partial x_2^2}\psi(x(t),t)e_2^i\big) \\ &+ e_4^i\big(\frac{\partial^2}{\partial x_1^2}\psi(x(t),t)e_1^i + \frac{\partial^2}{\partial x_2 \partial x_1}\psi(x(t),t)e_2^i\big) \\ &\le \sum_{1\le l,m\le 2}\Big|\frac{\partial^2}{\partial x_l \partial x_m}\psi(x(t),t)\Big| \le |\psi(t)|_{C^2(\mathbb{T}^2)}. \end{aligned} \tag{5.3.33}$$

So by (5.3.31), (5.3.32) and (5.3.33) we obtain

$$\begin{aligned} \sum_{i=1}^{3}\langle f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))e^i, e^i\rangle &\le -\frac{2}{\tau} + \frac{\tau}{4} + \frac{1}{\tau} + \frac{1}{\tau}|\psi(t)|_{C^2(\mathbb{T}^2)} \\ &= \frac{1}{\tau}\big(|\psi(t)|_{C^2(\mathbb{T}^2)} - 1\big) + \frac{\tau}{4} \end{aligned} \tag{5.3.34}$$

and

$$\begin{aligned} \sum_{i=1}^{2}\langle f'_{\psi,\tau}(t,(x(t),\dot{x}(t)))e^i, e^i\rangle &\le -\frac{2}{\tau} + \frac{2\tau}{4} + \frac{2}{\tau} + \frac{2}{\tau}|\psi(t)|_{C^2(\mathbb{T}^2)} \\ &= \frac{2}{\tau}|\psi(t)|_{C^2(\mathbb{T}^2)} + \frac{\tau}{2}. \end{aligned} \tag{5.3.35}$$

Therefore, by (5.3.30), (5.3.34) and (5.3.35) we have for $t \ge 0$, $\psi \in \Omega$,

$$\begin{aligned} \overline{q}_4(t,\psi) &\le -\frac{2t}{\tau}, \\ \overline{q}_3(t,\psi) &\le \frac{1}{\tau}\Big(\int_0^t |\psi(s)|_{C^2(\mathbb{T}^2)}ds - t\Big) + \frac{\tau t}{4} \end{aligned}$$

and

$$\overline{q}_2(t,\psi) \le \frac{2}{\tau}\int_0^t |\psi(s)|_{C^2(\mathbb{T}^2)}ds + \frac{\tau t}{2}.$$

In particular, we obtain

$$\begin{aligned}\frac{s}{t}\overline{q}_4(t,\psi)+&\frac{(1-s)}{t}\overline{q}_3(t,\psi)\\&\le -\frac{2s}{\tau}+\frac{(1-s)}{\tau}\Big(\frac{1}{t}\int\limits_0^t|\psi(r)|_{C^2(\mathbb{T}^2)}dr-1\Big)+\frac{\tau(1-s)}{4}\end{aligned}$$

and

$$\begin{aligned}&\frac{s}{t}\overline{q}_3(t,\psi)+\frac{(1-s)}{t}\overline{q}_2(t,\psi)\\&\quad\le\frac{s}{\tau}\Big(\frac{1}{t}\int\limits_0^t|\psi(r)|_{C^2(\mathbb{T}^2)}dr-1\Big)+\frac{\tau s}{4}+\frac{2(1-s)}{\tau t}\int\limits_0^t|\psi(r)|_{C^2(\mathbb{T}^2)}dr+\frac{\tau(1-s)}{2}\end{aligned}$$

for $t\ge 0$, $\psi\in\Omega$, $s\in(0,1)$. By Lemma 5.3.2(iii) and Remark 5.3.7 for any $\mu>0$ and $\psi\in\Omega$ (recall here the change of Ω in Remark 5.3.7) there is $t_0(\mu,\psi)\ge 0$ such that

$$\frac{1}{t}\int\limits_0^t|\psi(s)|_{C^2(\mathbb{T}^2)}ds\le\mathbb{E}\big(|id_\Omega(0)|_{C^2(\mathbb{T}^2)}\big)+\mu$$

for any $t>t_0(\psi,\mu)$. In the following we set

$$B_{H,\nu,\lambda,\alpha}:=\mathbb{E}\big(|id_\Omega(0)|_{C^2(\mathbb{T}^2)}\big).$$

This implies that for all $\tau>0$ and $\epsilon_1\in(0,\frac{2}{\tau})$ there is $s_1=s_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1)\in[0,1)$ and for all $\psi\in\Omega$ there is $t_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1,\psi)\ge 0$ and such that for $t\ge t_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1,\psi)$

$$\begin{aligned}&\frac{s_1}{t}\overline{q}_4(t,\psi)+\frac{(1-s_1)}{t}\overline{q}_3(t,\psi)\\&\qquad\le-\frac{2s_1}{\tau}+\frac{(1-s_1)}{\tau}\big(B_{H,\nu,\lambda,\alpha}-1\big)+\frac{\tau(1-s_1)}{4}=-\epsilon_1<0.\end{aligned}\tag{5.3.36}$$

So $s_1=s_1(B_{H,\nu,\lambda,\alpha},\tau,\epsilon_1)\in[0,1)$ in (5.3.36) is defined by

$$\max\left\{0,\frac{\epsilon_1+\frac{1}{\tau}\big(B_{H,\nu,\lambda,\alpha}-1\big)+\frac{\tau}{4}}{\frac{2}{\tau}+\frac{1}{\tau}\big(B_{H,\nu,\lambda,\alpha}-1\big)+\frac{\tau}{4}}\right\}=:s_1.\tag{5.3.37}$$

For any $k\in(0,1)$, $\tau>0$ let $\overline{t}_{1,\tau}(\psi)$ be the *first time* such that

$$\ln(k)=-\frac{2s_1}{\tau}\overline{t}_{1,\tau}(\psi)+\frac{\tau(1-s_1)}{4}\overline{t}_{1,\tau}(\psi)+\frac{(1-s_1)}{\tau}\Big(\int\limits_0^{\overline{t}_{1,\tau}(\psi)}|\psi(r)|_{C^2(\mathbb{T}^2)}dr-\overline{t}_{1,\tau}(\psi)\Big)\tag{5.3.38}$$

with $s_1 = s_1(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_1) \in [0,1)$ given by (5.3.37).
Now we assume that in addition

$$B_{H,\nu,\lambda,\alpha} < 1. \tag{5.3.39}$$

Notice that for $\tau_1, \tau_2 > 0$ such that

$$\tau_2 < \tau_* := 2\sqrt{1 - B_{H,\nu,\lambda,\alpha}} < \tau_1$$

we have by (5.3.39)

$$\frac{1}{\tau_*}\big(B_{H,\nu,\lambda,\alpha} - 1\big) + \frac{\tau_*}{4} = 0$$

and

$$\frac{1}{\tau_2}\big(B_{H,\nu,\lambda,\alpha} - 1\big) + \frac{\tau_2}{4} < 0 < \frac{1}{\tau_1}\big(B_{H,\nu,\lambda,\alpha} - 1\big) + \frac{\tau_1}{4}.$$

This implies that for all $0 < \tau < \tau_*$ and $\epsilon_2 \in \big(0, \frac{1}{\tau}(1 - B_{H,\nu,\lambda,\alpha}) - \frac{\tau}{4}\big)$ there is $s_2 = s_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2) \in (0,1)$ and for all $\psi \in \Omega$ there is $t_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2, \psi) \geq 0$ and such that for $t \geq t_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2, \psi)$

$$\begin{aligned}
&\frac{s_2}{t}\overline{q}_3(t,\psi) + \frac{(1-s_2)}{t}\overline{q}_2(t,\psi) \\
&\qquad \leq \frac{s_2}{\tau}\big(B_{H,\nu,\lambda,\alpha} - 1\big) + \frac{\tau s_2}{4} + \frac{2(1-s_2)}{\tau}B_{H,\nu,\lambda,\alpha} + \frac{\tau(1-s_2)}{2} = -\epsilon_2 < 0.
\end{aligned} \tag{5.3.40}$$

Therefore, $s_2 = s_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2) \in (0,1)$ in (5.3.40) is defined by

$$\frac{\epsilon_2 + \frac{2}{\tau}B_{H,\nu,\lambda,\alpha} + \frac{\tau}{2}}{\frac{1}{\tau}\big(1 - B_{H,\nu,\lambda,\alpha}\big) - \frac{\tau}{4} + \frac{2}{\tau}B_{H,\nu,\lambda,\alpha} + \frac{\tau}{2}} =: s_2. \tag{5.3.41}$$

For any $k \in (0,1)$, $0 < \tau < \tau_*$ let $\overline{t}_{2,\tau}(\psi)$ be the *first time* such that

$$\begin{aligned}
\ln(k) = {} & \frac{s_2}{\tau}\Big(\int_0^{\overline{t}_{2,\tau}(\psi)} |\psi(r)|_{C^2(\mathbb{T}^2)}dr - \overline{t}_{2,\tau}(\psi)\Big) + \frac{\tau s_2}{4}\overline{t}_{2,\tau}(\psi) \\
& + \frac{2(1-s_2)}{\tau}\int_0^{\overline{t}_{2,\tau}(\psi)} |\psi(r)|_{C^2(\mathbb{T}^2)}dr + \frac{\tau(1-s_2)}{2}\overline{t}_{2,\tau}(\psi)
\end{aligned} \tag{5.3.42}$$

with $s_2 = s_2(B_{H,\nu,\lambda,\alpha}, \tau, \epsilon_2) \in (0,1)$ given by (5.3.41).
Finally, we show that $\overline{t}_{1,\tau}, \overline{t}_{2,\tau} : \Omega \to [0,\infty)$ are indeed finite $(\mathcal{F}_t)_{t\geq 0}$–stopping times, where $\mathcal{F}_t := \sigma(id_\Omega(s)|\ s \leq t)$. The above considerations ensure the finiteness of $\overline{t}_{1,\tau}$ and $\overline{t}_{2,\tau}$ for any $\psi \in \Omega$. Further, we have to prove the $\mathcal{F}_t$–measurability of the events $\{\overline{t}_{1,\tau} \leq t\}$, $\{\overline{t}_{2,\tau} \leq t\}$ for any $t \geq 0$. For that let $Y_{1,\tau}(t,\psi)$, $t \geq 0$, $\psi \in \Omega$, be the process defined by the right hand side of (5.3.38) and $Y_{2,\tau}(t,\psi)$, $t \geq 0$, $\psi \in \Omega$, be the process defined by the right hand side of (5.3.42) (with $\overline{t}_{i,\tau}(\psi)$, $i = 1, 2$, replaced

by the time index t). To prove $\{\bar{t}_{1,\tau} \le t\}, \{\bar{t}_{2,\tau} \le t\} \in \mathcal{F}_t$, $t \ge 0$, we have to show $\{\inf_{r\in[0,t]} Y_{1,\tau}(r,\cdot) \le \ln(k)\}, \{\inf_{r\in[0,t]} Y_{2,\tau}(r,\cdot) \le \ln(k)\} \in \mathcal{F}_t$, since $r \mapsto Y_{1,\tau}(r,\psi)$, $r \mapsto Y_{2,\tau}(r,\psi)$ are continuous, $Y_{1,\tau}(0,\psi), Y_{1,\tau}(0,\psi) = 0$ for all $\psi \in \Omega$ and $\ln(k) < 0$. Therefore it is sufficient to prove that $Y_{1,\tau}(r,\cdot), Y_{2,\tau}(r,\cdot)$ are $\mathcal{F}_r$–measurable. But this measurability is already assured by the definition of $Y_{1,\tau}$ and $Y_{2,\tau}$. □

Finally, we verify the last assertion of Theorem 5.3.8.

Lemma 5.3.12. *Suppose the assumptions of Theorem 5.3.8 hold. Then assertion (iii) of Theorem 5.3.8 is valid.*

Proof. Notice that by (5.3.24) in the proof of Lemma 5.3.10 we have for $(x,y) \in \mathbb{R}^4$, $t \ge 0$, $\psi \in \Omega$,

$$|\phi'(t,\psi,(x,y))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)} \le \exp\big(\overline{C}(\tau)\int_0^t (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\big), \tag{5.3.43}$$

where $\overline{C}(\tau) := \frac{4\tau+6}{\tau}$.
For the following recall the definitions of $\bar{t}_\tau$, $d(\tau)$ and $s(\tau)$ from Theorem 5.3.8(iii). We define for a fixed $\tau > 0$ the random variable $m_\tau : \Omega \mapsto [0,\infty)$

$$\psi \mapsto m_\tau(\psi) = \exp\big(\overline{C}(\tau)\int_0^{\bar{t}_\tau(\psi)} (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds\big) \tag{5.3.44}$$

with $\overline{C}(\tau) := \frac{4\tau+6}{\tau}$. By (5.3.43) and the definitions of $d(\tau)$, $s(\tau)$, m_τ we have for $\psi \in \Omega$

$$(m_\tau(\psi))^{d(\tau)+s(\tau)} \ge 1, \quad \sup_{(x,y)\in\mathbb{R}^4} |\phi'(\bar{t}_\tau(\psi),\psi,(x,y))|_{\mathcal{L}(\mathbb{R}^4,\mathbb{R}^4)} \le m_\tau(\psi).$$

So conditions (a)-(b) in Theorem 5.3.8(iii) are fulfilled.
Now, for a fixed $\tau > 0$ we define the positive random variable

$$Z_\tau : \Omega \to [0,\infty), \psi \mapsto Z_\tau(\psi) = \left(\frac{m_\tau(\psi)^{d(\tau)}}{l}\right)^{\frac{1}{s(\tau)}} \xi_\tau(\bar{t}_\tau(\psi),\psi),$$

where $l \in (0,1)$ and $\xi_\tau(t,\psi)$ is defined in Theorem 5.3.8(i). So by (5.3.44) and (5.3.16) there are constants $C_1, C_2, C_3 > 0$ independent of ψ, but depending on τ, $d(\tau)$, $s(\tau)$ and l such that

$$\begin{aligned}\ln(Z_\tau(\psi)) &\le C_1 + C_2 \int_0^{\bar{t}_\tau(\psi)} (1+|\psi(s)|^2_{C^2(\mathbb{T}^2)})ds + C_3 \int_0^{\bar{t}_\tau(\psi)} (1+|\psi(s)|_{C^3(\mathbb{T}^2)})ds \\ &\le C_1 + C_4 \int_0^{\bar{t}_\tau(\psi)} (1+|\psi(s)|^2_{C^3(\mathbb{T}^2)})ds\end{aligned} \tag{5.3.45}$$

with $C_4 := C_2 + C_3 + 2$.
Further, we define recursively for $\psi \in \Omega$

$$\bar{t}_0(\psi) := 0, \quad \bar{t}_i(\psi) := \bar{t}_\tau(\theta(\bar{t}_{i-1}(\psi))\psi) + \bar{t}_{i-1}(\psi),\ i \in \mathbb{N},$$

and

$$\tilde{\theta}_i\psi := \theta(\bar{t}_i(\psi))\psi,\ i \in \mathbb{N}.$$

By the definition of θ and since $\bar{t}_\tau$ is a $(\mathcal{F}_t)_{t\geq 0}$–stopping time with $\mathcal{F}_t = \sigma(id_\Omega(r)|\ r \leq t)$, the $\mathcal{F}$–$\mathcal{F}$ measurable mapping

$$\tilde{\theta} := \tilde{\theta}_1 : \Omega \to \Omega, \quad \psi \mapsto \tilde{\theta}\psi = \theta(\bar{t}_\tau(\psi))\psi$$

preserves $\mathbb{P}$.
It is only left to prove

$$\begin{aligned}\lim_{i\to\infty} \frac{\ln(\max\{1, Z_\tau(\tilde{\theta}_i\psi)\})}{i} &\leq \lim_{i\to\infty} \frac{1}{i} \int_0^{\bar{t}_\tau(\tilde{\theta}_i\psi)} (1 + |\tilde{\theta}_i\psi(r)|^2_{C^3(\mathbb{T}^2)})dr \\ &= \lim_{i\to\infty} \frac{1}{i} \int_0^{\bar{t}_\tau(\theta(\bar{t}_i(\psi))\psi)} (1 + |\theta(\bar{t}_i(\psi))\psi(r)|^2_{C^3(\mathbb{T}^2)})dr = 0,\end{aligned} \tag{5.3.46}$$

$\mathbb{P}$-a.s.
By the definition of $\bar{t}_\tau$, i.e. by the definition of $\bar{t}_{1,\tau}$, $\bar{t}_{2,\tau}$ in (5.3.38), (5.3.42) in the proof of Lemma 5.3.11 and by the recursive definition of $\bar{t}_i$, $i \in \mathbb{N}$, there is a constant $C > 0$ such that

$$\lim_{i\to\infty} \frac{\bar{t}_i(\psi)}{i} = C,\ \psi \in \Omega. \tag{5.3.47}$$

Therefore $\bar{t}_i(\psi) \to \infty$ for $i \to \infty$, $\psi \in \Omega$, and

$$\lim_{i\to\infty} \frac{\bar{t}_{i+1}(\psi)}{\bar{t}_i(\psi)} = \lim_{i\to\infty} \frac{\bar{t}_i(\psi)}{\bar{t}_{i+1}(\psi)} = 1,\ \psi \in \Omega. \tag{5.3.48}$$

Indeed, this can be seen by adding up the equation (5.3.38) for
$\psi, \theta(\bar{t}_1(\psi))\psi, \ldots, \theta(\bar{t}_{i-1}(\psi))\psi$, $i \in \mathbb{N}$, $\psi \in \Omega$, and dividing this sum by $\bar{t}_i(\psi)$, i.e.

$$\begin{aligned}&\frac{1}{\bar{t}_i(\psi)} \sum_{j=0}^{i-1} \Big[\Big(-\frac{2s_1}{\tau} + \frac{\tau(1-s_1)}{4}\Big)\bar{t}_{1,\tau}(\theta(\bar{t}_j(\psi))\psi) \\ &\qquad + \frac{(1-s_1)}{\tau}\Big(\int_0^{\bar{t}_{1,\tau}(\theta(t_j(\psi))\psi)} |\theta(\bar{t}_j(\psi))\psi(r)|_{C^2(\mathbb{T}^2)}dr - \bar{t}_{1,\tau}(\theta(\bar{t}_j(\psi))\psi)\Big)\Big] \\ &= \frac{1}{\bar{t}_i(\psi)}\Big[\Big(-\frac{2s_1}{\tau} + \frac{\tau(1-s_1)}{4}\Big)\bar{t}_i(\psi) + \frac{(1-s_1)}{\tau}\Big(\int_0^{\bar{t}_i(\psi)} |\psi(r)|_{C^2(\mathbb{T}^2)}dr - \bar{t}_i(\psi)\Big)\Big] \\ &\xrightarrow{i\to\infty} -\frac{2s_1}{\tau} + \frac{\tau(1-s_1)}{4} + \frac{(1-s_1)}{\tau}\Big(\mathbb{E}\Big(|id_\Omega(0)|_{C^2(\mathbb{T}^2)}\Big) - 1\Big),\end{aligned}$$

where we used Lemma 5.3.2(iii), Remark 5.3.7 and the fact that $\sum_{j=0}^{i-1}\bar{t}_{1,\tau}(\theta(\bar{t}_j(\psi))\psi) = \bar{t}_i(\psi) = \bar{t}_{1,\tau}(\theta(\bar{t}_{i-1}(\psi))\psi) + \bar{t}_{i-1}(\psi)$. In an analogous way this can be done with equation (5.3.42). Therefore, by (5.3.46), (5.3.47), (5.3.48) and Lemma 5.3.2(iii) (recall here again the change of Ω in Remark 5.3.7) we finally deduce for $\psi \in \Omega$

$$\begin{aligned}
0 &= \lim_{i\to\infty}\Big[\frac{1}{\bar{t}_{i+1}(\psi)}\int_0^{\bar{t}_{i+1}(\psi)}(1+|\psi(r)|^2_{C^3(\mathbb{T}^2)})dr - \frac{1}{\bar{t}_i(\psi)}\int_0^{\bar{t}_i(\psi)}(1+|\psi(r)|^2_{C^3(\mathbb{T}^2)})dr\Big] \\
&= \lim_{i\to\infty}\Big[\frac{\bar{t}_i(\psi)}{\bar{t}_{i+1}(\psi)}\frac{1}{\bar{t}_i(\psi)}\int_0^{\bar{t}_{i+1}(\psi)}(1+|\psi(r)|^2_{C^3(\mathbb{T}^2)})dr - \frac{1}{\bar{t}_i(\psi)}\int_0^{\bar{t}_i(\psi)}(1+|\psi(r)|^2_{C^3(\mathbb{T}^2)})dr\Big] \\
&= \lim_{i\to\infty}\frac{1}{\bar{t}_i(\psi)}\int_{\bar{t}_i(\psi)}^{\bar{t}_{i+1}(\psi)}(1+|\psi(r)|^2_{C^3(\mathbb{T}^2)})dr \\
&= \lim_{i\to\infty}\frac{1}{\bar{t}_i(\psi)}\int_{\bar{t}_i(\psi)}^{\bar{t}_\tau(\theta(\bar{t}_i(\psi))\psi)+\bar{t}_i(\psi)}(1+|\psi(r)|^2_{C^3(\mathbb{T}^2)})dr \\
&= \lim_{i\to\infty}\frac{1}{\bar{t}_i(\psi)}\int_0^{\bar{t}_\tau(\theta(\bar{t}_i(\psi))\psi)}(1+|\theta(\bar{t}_i(\psi))\psi(r)|^2_{C^3(\mathbb{T}^2)})dr \\
&= \lim_{i\to\infty}\frac{1}{i}\int_0^{\bar{t}_\tau(\theta(\bar{t}_i(\psi))\psi)}(1+|\theta(\bar{t}_i(\psi))\psi(r)|^2_{C^3(\mathbb{T}^2)})dr \\
&\geq \lim_{i\to\infty}\frac{\ln(\max\{1, Z_\tau(\tilde{\theta}_i\psi)\})}{i} = 0.
\end{aligned}$$

$\square$

6. Matching Desired Statistical Properties of the Velocity Field

This short section is devoted to verifying that the random velocity field $v = \nabla^\perp \psi$, where ψ is the unique ergodic mild solution of $(M3)$ introduced in Section 5.2, captures the statistical properties of a turbulent fluid flow which were described in Section 2.
We suppose in this section that Assumption 5.1.1 in Section 5.1 holds and set $A = \Delta$ (and thereby $\alpha_k = |k|^2$, $k \in K = 2\pi\mathbb{Z}^2\backslash\{(0,0)\}$). Further, assume that

$$\sum_{k\in K} \lambda_k |k|^{2+4\gamma-4H} < \infty \tag{6.1}$$

for some $\gamma > 0$. So by Corollary 5.2.5 there is a unique ergodic mild solution ψ of $(M3)$ and there is a version of ψ (again denoted by ψ) such that $\mathbb{P}$-a.s. $\psi \in C\left(\mathbb{R}, C^1(\mathbb{T}^2)\right)$.
First notice that by the definition of the random velocity field $v = \nabla^\perp \psi = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right)$, the field is already *incompressible* and the components of v are real–valued by Theorem 5.2.3. In particular, we have

$$v(x,t) = \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} = \sum_{k\in K} i \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} \hat{\psi}_k(t) e_k(x) = \sum_{k\in K} \hat{v}_k(t) e_k(x), \;\; t \in \mathbb{R}, \;\; x \in \mathbb{T}^2, \tag{6.2}$$

where we set $\hat{v}_k(t) := i\binom{k_2}{-k_1}\hat{\psi}_k(t)$. Again recall here that

$$\hat{\psi}_k(t) = \sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u), \qquad k \in K, \;\; t \in \mathbb{R}.$$

Since $(\hat{\psi}_k(t))_{t\in\mathbb{R}}$, $k \in K$, are mean zero Gaussian processes, independent, except $(\hat{\psi}_k)^* = \hat{\psi}_{-k}$, v is a *mean zero Gaussian random field.*
The *autocovariance function* $R : \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ of v is given by

$$\begin{aligned} R(x,y,t,s) &= \big(\mathbb{E}\big(v_i(x,t)v_j(y,s)\big)\big)_{1\le i,j\le 2} \\ &= \sum_{k\in K} \begin{pmatrix} k_2^2 & -k_2k_1 \\ -k_1k_2 & k_1^2 \end{pmatrix} \lambda_k |k|^{-4H} \delta_k(t-s; H, \nu)\; e_k(x-y), \end{aligned}$$

where we set

$$\delta_k(t-s; H,\nu) := \frac{\Gamma(2H+1)\sin(\pi H)}{\pi} \int_0^\infty \cos((t-s)\nu|k|^2 z) \frac{|z|}{1+z^2} dz$$

and where we applied Proposition 4.1.2(i) and Remark 5.2.2. Therefore, v is *stationary* and *homogeneous.*
For $k \in K$, the energy of the Fourier mode k is defined by

$$\begin{aligned} \mathcal{E}(k) &= \frac{1}{2}\mathbb{E}\left(\hat{v}_k(t)(\hat{v}_k(t))^*\right) = \frac{1}{2}\mathbb{E}\left(|\hat{v}_k(t)|^2\right) = \frac{1}{2}|k|^2\mathbb{E}\left(|\hat{\psi}_k(t)|^2\right) = \frac{1}{2}|k|^2\mathbb{E}\left(|\hat{\psi}_k(0)|^2\right) \\ &= \frac{\Gamma(2H)H}{2}\lambda_k|k|^{2-4H}, \end{aligned}$$

where we again applied Proposition 4.1.2(i) and Remark 5.2.2.
To ensure *isotropy*, we set $\lambda_k := \zeta(|k|)$, $k \in K$, for a suitable positive function $\zeta : [0,\infty) \to [0,\infty)$, but keep in mind that (6.1) should still be satisfied. Then the energy of a Fourier mode $k \in K$ depends only on the length of k, in that, $\mathcal{E}(\kappa) := \mathcal{E}(k) = \mathcal{E}(k')$ whenever $\kappa = |k| = |k'|$ for $k, k' \in K$. In such an isotropic random field of the form (6.2) it is customary to define the *energy spectrum* in terms of total energy in all the Fourier modes of the same length $\kappa = |k|$ by $E(\kappa) := \#\{k \in K \big| \ |k| = \kappa\} \ \mathcal{E}(\kappa)$. Clearly, $E(\kappa)$ can be approximated by $E(\kappa) \approx C\kappa\mathcal{E}(\kappa)$ with some constant $C > 0$.
In general, the energy spectrum $E(\cdot)$ can be divided in three ranges:

- For small $|k|$ where the energy is injected, $E(\cdot)$ increases in $|k|$ algebraically.
- For large $|k|$ where the energy dissipates, we just set λ_k and therefore the energy spectrum to zero (*ultraviolet cut–off*).
- For intermediate $|k|$, in the so–called *inertial subrange*, $E(\cdot)$ decays in $|k|$ algebraically.

The spectrum $(\lambda_k)_{k\in K}$ of Q can be chosen so that the energy spectrum of v matches experimentally observed energy spectra of a turbulent fluid flow. We introduce three energy spectra which were used by Sigurgeirsson and Stuart in [89], see also [37] pp. 112:

- *Kolmogorov spectrum*: $E(|k|) \propto |k|^{-\frac{5}{3}}$ and therefore $\mathcal{E}(|k|) \propto \lambda_k |k|^{2-4H} \propto |k|^{-\frac{8}{3}}$, i.e. $\lambda_k \propto |k|^{-\frac{14}{3}+4H}$.
- *Kraichnan spectrum*: $\lambda_k |k|^{2-4H} \propto |k|^2 e^{-|k|^2}$, i.e. $\lambda_k \propto |k|^{4H} e^{-|k|^2}$.
- *Karman–Obukhov spectrum*, the so–called *long–tail Kolmogorov spectrum*: $\lambda_k |k|^{2-4H} \propto |k|^2 (1+|k|^2)^{-\frac{7}{3}}$, i.e. $\lambda_k \propto |k|^{4H}(1+|k|^2)^{-\frac{7}{3}}$. This spectrum was introduced to study Kolmogorov turbulence with a long $-\frac{5}{3}$ tail in the spectrum for large $|k|$.

Obviously, the decay of the spectrum $(\lambda_k)_{k\in K}$ of Q as $|k| \to \infty$ determines the regularity of the velocity field v and by this also the regularity of the transport equation $(M1)$. From the physical point of view by applying Corollary 5.2.8 it is clear that for any spectrum in the inertial subrange we have a unique global solution to $(M1)$ due to the ultraviolet cut–off. From the mathematical point of view it is interesting to ask whether there is a (unique) solution to $(M1)$ if we match the spectrum $(\lambda_k)_{k\in K}$ for all modes without the cut–off. Since the Kraichnan spectrum decays exponentially fast, we get again a unique global solution to $(M1)$. In view of the Kolmogorov spectrum, Karman–Obukhov spectrum and Assumption 5.1.2 in Corollary 5.2.5 we have

$$|k|^{-\frac{14}{3}+4H-4H+2m+4\gamma} = |k|^{-2}|k|^{-\frac{8}{3}+2m+4\gamma}$$

and $-\frac{8}{3} + 2m + 4\gamma < 0$ is satisfied for $m = 1$ and $\gamma < \frac{1}{6}$. So by Corollary 5.2.5 there is a version of ψ (again denoted by ψ) such that $\mathbb{P}$-a.s. $\psi \in C\left(\mathbb{R}, C^1(\mathbb{T}^2)\right)$ and

$v = \nabla^{\perp}\psi \in C\left(\mathbb{R}, C(\mathbb{T}^2, \mathbb{R}^2)\right)$. Hence, by the classic Peano existence theorem there is a solution to $(M1)$, but uniqueness may fail.

Further, recall that our main motivation to use fractional noise was to match the statistical property

$$\mathbb{E}\left(|v(x,t) - v(x,s)|^2\right) \sim C\ |t-s|^{2H}$$

for $t, s \geq 0$, $x \in \mathbb{T}^2$ and some constant $C > 0$. We have the following result.

Proposition 6.1. *Suppose Assumption 5.1.1 holds and that there is $m \in \mathbb{N}$ and $\epsilon > 0$ such that*

$$\sum_{k \in K} \lambda_k |k|^{2m+\epsilon} < \infty. \tag{6.3}$$

Then there is a unique ergodic mild solution ψ to equation $(M3)$ and there is a version of ψ (again denoted by ψ) such that $\mathbb{P}$-a.s. $\psi \in C\left(\mathbb{R}, C^m(\mathbb{T}^2)\right)$ and $v = \nabla^{\perp}\psi \in C\left(\mathbb{R}, C^{m-1}(\mathbb{T}^2, \mathbb{R}^2)\right)$. Further, there is a constant $C(H,\nu) > 0$ such that for any $t, s \in \mathbb{R}$ and $x \in \mathbb{T}^2$ we have

$$\mathbb{E}\left(|v(x,t) - v(x,s)|^2\right) \leq C(H,\nu)|t-s|^{2H} \tag{6.4}$$

and for a fixed $T > 0$ there is a constant $C(H,\nu,T) > 0$ such that for any $t, s \in [-T, T]$ and $x \in \mathbb{T}^2$ we have

$$C(H,\nu,T)|t-s|^{2H} \leq \mathbb{E}\left(|v(x,t) - v(x,s)|^2\right). \tag{6.5}$$

Proof. The first assertion follows by Corollary 5.2.5 in view of (6.3). Further, by (6.2) we have

$$\begin{aligned}
\mathbb{E}\left(|v(x,t) - v(x,s)|^2\right) &= \sum_{k \in K} |k|^2 \mathbb{E}\left(|\hat{\psi}_k(t) - \hat{\psi}_k(s)|^2\right) |e_k(x)|^2 \\
&= \sum_{k \in K} |k|^2 \mathbb{E}\left(|\hat{\psi}_k(t) - \hat{\psi}_k(s)|^2\right)
\end{aligned}$$

for any $t, s \in \mathbb{R}$ and $x \in \mathbb{T}^2$. Statements (6.4) and (6.5) now follow by (6.3), Proposition 4.1.2(ii),(iii) and Remark 5.2.2. $\square$

Remark 6.2. It should be noted that (6.3) in Proposition 6.1 is very restrictive and not satisfied for $H \in [\frac{1}{6}, 1)$ if $m = 1$ and if we use the Kolmogorov or Karman–Obukhov spectrum. But in view of the ultraviolet cut–off in the region where the energy dissipates, (6.3) is fulfilled for any energy spectrum in the inertial subrange.

7. Numerical Simulation of the Long–Time Behaviour of Inertial Particles

7.1. Introduction

In this section we describe how to accomplish numerical simulation of the system (M) introduced in Section 5.1. We suppose that Assumption 5.1.1 in Section 5.1 with $A = \triangle$ (and thereby $\alpha_k = |k|^2$, $k \in K = 2\pi\mathbb{Z}^2 \backslash \{(0,0)\}$) holds. Further, we assume that

$$\lambda_k = \begin{cases} \xi(|k|) & \text{if } |k| \leq 2\pi R \\ 0 & \text{else} \end{cases}, k \in K,$$

for a fixed $R \in \mathbb{N}$ and a *suitable* function $\xi : [0,\infty) \to [0,\infty)$ to obtain an isotropic random velocity field v with a desired energy spectrum (see Section 6). So by Corollary 5.2.5 there is a unique strictly stationary mild solution ψ to $(M3)$ such that there is a version of ψ (again denoted by ψ) such that $\psi \in C(\mathbb{R}, C^\infty(\mathbb{T}^2))$ $\mathbb{P}$-a.s. and

$$v(x,t) = \sum_{k\in K,\ |k|\leq 2\pi R} i \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} \hat{\psi}_k(t) e^{i<k,x>}, \qquad x \in \mathbb{T}^2,\ t \in \mathbb{R}, \tag{7.1.1}$$

with $v \in C(\mathbb{R}, C^\infty(\mathbb{T}^2, \mathbb{R}^2))$ $\mathbb{P}$-a.s.
Here recall that

$$\begin{aligned} \hat{\psi}_k(t) &= \sqrt{\lambda_k}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} d\beta_k^H(u) \\ &= \sqrt{\frac{\lambda_k}{2}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} dRe(\beta_k^H)(u) + i\sqrt{\frac{\lambda_k}{2}}\nu^H \int_{-\infty}^{t} e^{-(t-u)\nu\alpha_k} dIm(\beta_k^H)(u) \\ &= \hat{\psi}_{k,Re}(t) + i\hat{\psi}_{k,Im}(t). \end{aligned}$$

and $\big((\beta_k^H(t))_{t\in\mathbb{R}}, k \in K, |k| \leq 2\pi R\big)$ is a sequence of complex–valued and normalized fractional Brownian motions each with the same fixed Hurst parameter $H \in (0,1)$, i.e. $\beta_k^H = \frac{1}{\sqrt{2}} Re(\beta_k^H) + i\frac{1}{\sqrt{2}} Im(\beta_k^H)$, where $Re(\beta_k^H)$ and $Im(\beta_k^H)$ are independent real–valued and normalized fractional Brownian motions on $\mathbb{R}$, and different β_k^H are independent, except $\beta_{-k}^H = (\beta_k^H)^*$. This implies $\psi_{-k} = (\psi_k)^*$.
In particular, by Corollary 5.2.8 the random ordinary differential equation $(M1)$ has a unique global solution $\mathbb{P}$-a.s.

In Section 4.2 we have already introduced and discussed the applicability of several *exact* methods to simulate a real–valued stationary fractional Ornstein–Uhlenbeck process, i.e. in particular the processes $\hat{\psi}_{k,Re}$, $\hat{\psi}_{k,Im}$ and thereby $\hat{\psi}_k = \hat{\psi}_{k,Re} + i\hat{\psi}_{k,Im}$ for a fixed $k \in K$. We here use for our numerical experiments the standard Cholesky method, because we started to investigate the other methods thereafter.

Giving the realisations of the coefficients $\hat{\psi}_k$, $k \in K$, $|k| \leq 2\pi R$, we specify in Section 7.2 how to evaluate the finite series (7.1.1) using the *fast Fourier transform* (FFT) algorithm. The trajectory of the inertial particle is then obtained by using the classic fourth–order Runge–Kutta scheme.
Finally, we perform numerical experiments studying the effect of the different parameters on particle distribution in the random field in Section 7.3.

7.2. Simulating the Velocity Field and the Particle Movement

In the following we assume that the realisations of

$$(\hat{\psi}_{k,Re}(j\triangle t))_{j=0,1,\dots,n},\ (\hat{\psi}_{k,Im}(j\triangle t))_{j=0,1,\dots,n}$$

and thereby of

$$\hat{\psi}_k(j\triangle t)) = \hat{\psi}_{k,Re}(j\triangle t)) + i\hat{\psi}_{k,Im}(j\triangle t)), \quad j = 0, 1, \dots, n,$$

for a fixed $\triangle t > 0$, $n \in \mathbb{N}$ and $k \in K = 2\pi\mathbb{Z}^2 \backslash \{(0,0)\}$ with $|k| \leq 2\pi R$ for some fixed $R \in \mathbb{N}$ are given. Further we define $N := 2R + 2$.

To evaluate the random velocity field v defined by (7.1.1) at time $j\triangle t$ with fixed $j \in \{0, 1, \dots, n\}$ and at space position $(x_1 = l_1/N, x_2 = l_2/N)$ with fixed $l_m \in \{0, 1, \dots, N-1\}$, $m = 1, 2$, we have to calculate the finite series

$$\begin{aligned} v_1(x_1, x_2, j\triangle t) &:= v_1(l_1/N, l_2/N, j\triangle t) := v_1(l_1, l_2, j) \\ &:= \sum_{\bar{k}_1=-R}^{R} \sum_{\bar{k}_2=-R}^{R} \hat{v}_{1,\bar{k}}(j) e^{2\pi i(\bar{k}_1 \frac{l_1}{N} + \bar{k}_2 \frac{l_2}{N})} \\ &:= \sum_{\bar{k}_1=-R}^{R} \sum_{\bar{k}_2=-R}^{R} 2\pi i \bar{k}_2 \hat{\psi}_{(2\pi\bar{k}_1, 2\pi\bar{k}_2)}(j\triangle t) e^{2\pi i(\bar{k}_1 \frac{l_1}{N} + \bar{k}_2 \frac{l_2}{N})} \end{aligned} \tag{7.2.1}$$

and

$$\begin{aligned} v_2(x_1, x_2, j\triangle t) &:= v_2(l_1/N, l_2/N, j\triangle t) := v_2(l_1, l_2, j) \\ &:= \sum_{\bar{k}_1=-R}^{R} \sum_{\bar{k}_2=-R}^{R} \hat{v}_{2,\bar{k}}(j) e^{2\pi i(\bar{k}_1 \frac{l_1}{N} + \bar{k}_2 \frac{l_2}{N})} \\ &:= \sum_{\bar{k}_1=-R}^{R} \sum_{\bar{k}_2=-R}^{R} (-2)\pi i \bar{k}_1 \hat{\psi}_{(2\pi\bar{k}_1, 2\pi\bar{k}_2)}(j\triangle t) e^{2\pi i(\bar{k}_1 \frac{l_1}{N} + \bar{k}_2 \frac{l_2}{N})} \end{aligned} \tag{7.2.2}$$

with $\hat{\psi}_{(2\pi\bar{k}_1, 2\pi\bar{k}_1)}(j\triangle t) := 0$ if $(\bar{k}_1, \bar{k}_2) = (0,0)$ or $|(2\pi\bar{k}_1, 2\pi\bar{k}_1)| > 2\pi R$ and where we set

$$\begin{aligned} \hat{v}_{1,\bar{k}}(j) &:= \hat{v}_{1,(\bar{k}_1,\bar{k}_2)}(j) := 2\pi i \bar{k}_2 \hat{\psi}_{(2\pi\bar{k}_1, 2\pi\bar{k}_2)}(j\triangle t), \\ \hat{v}_{2,\bar{k}}(j) &:= \hat{v}_{2,(\bar{k}_1,\bar{k}_2)}(j) := (-2)\pi i \bar{k}_1 \hat{\psi}_{(2\pi\bar{k}_1, 2\pi\bar{k}_2)}(j\triangle t). \end{aligned}$$

To evaluate the series (7.2.1) and (7.2.2), we use the *fast Fourier transform* (FFT) algorithm. For that purpose recall that the *two–dimensional inverse discrete Fourier transform* DFT, $Z \in \mathbb{C}^{M \times M}$, for some fixed $M \in \mathbb{N}$, of an element $\hat{Z} \in \mathbb{C}^{M \times M}$ is usually defined as

$$Z(l_1, l_2) := \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} \hat{Z}_m e^{2\pi i(m_1 \frac{l_1}{M} + m_2 \frac{l_2}{M})} \tag{7.2.3}$$

for $l_1, l_2 = 0, 1, \ldots, M-1$.
We should be able to use (7.2.3) to compute (7.2.1) and (7.2.2). For that set $m_1 := \overline{k}_1 + R + 1$ and $m_2 := \overline{k}_2 + R + 1$. We obtain for $p \in \{1, 2\}$

$$\begin{aligned}
v_p(l_1, l_2, j) &= \sum_{\overline{k}_1=-R}^{R} \sum_{\overline{k}_2=-R}^{R} \hat{v}_{p,\overline{k}}(j) e^{2\pi i(\overline{k}_1 \frac{l_1}{N} + \overline{k}_2 \frac{l_2}{N})} \\
&= \sum_{m_1=1}^{2R+1} \sum_{m_1=1}^{2R+1} \hat{v}_{p,(m_1-R-1,m_2-R-1)}(j) e^{2\pi i\left((m_1-R-1)\frac{l_1}{N} + (m_2-R-1)\frac{l_2}{N}\right)} \\
&= e^{-\pi i(2R+2)(\frac{l_1}{N} + \frac{l_2}{N})} \sum_{m_1=1}^{2R+1} \sum_{m_1=1}^{2R+1} \hat{v}_{p,(m_1-R-1,m_2-R-1)}(j) e^{2\pi i(m_1 \frac{l_1}{N} + m_2 \frac{l_2}{N})} \\
&= e^{-\pi i N(\frac{l_1}{N} + \frac{l_2}{N})} \sum_{m_1=0}^{N-1} \sum_{m_1=0}^{N-1} \hat{Z}_{p,(m_1,m_2)}(j) e^{2\pi i(m_1 \frac{l_1}{N} + m_2 \frac{l_2}{N})} \\
&= (-1)^{l_1+l_2} \sum_{m_1=0}^{N-1} \sum_{m_1=0}^{N-1} \hat{Z}_{p,(m_1,m_2)}(j) e^{2\pi i(m_1 \frac{l_1}{N} + m_2 \frac{l_2}{N})},
\end{aligned}$$

where we set

$$\hat{Z}_{p,(m_1,m_2)}(j) := \begin{cases} 0 & \text{if } m_1 = 0 \text{ or } m_2 = 0 \\ \hat{v}_{p,(m_1-R-1,m_2-R-1)}(j) & \text{if } 1 \leq m_1, m_2 \leq 2R+1 \end{cases}$$

and where we used that

$$e^{-\pi i(l_1+l_2)} = (e^{-\pi i})^{l_1+l_2} = (-1)^{l_1+l_2}.$$

Notice carefully that after taking the inverse DFT of $\hat{Z}_p(j)$, we have to change the sign of the elements for which $l_1 + l_2$ is odd, to obtain $v_p(j)$. For $R \in \mathbb{N}$ such that $N = 2R + 2 = 2^p$ for some $p \in \mathbb{N}$ we obtain a complexity of $\mathcal{O}(N \log(N))$ with the FFT algorithm. Further, recall again that because of the symmetry $(\hat{\psi}_k)^* = \hat{\psi}_{-k}$ the components v_1 and v_2 of the velocity field are real–valued. This symmetry can be used to reduce the amount of work required by FFT almost by half. For our computation we use a *complex–to–real* FFT provided by [33].

Now giving the velocity field $(v(l_1/N, l_2/N, j\triangle t))_{l_1,l_2 \in \{0,1,\ldots,N-1\}, j \in \{0,1,\ldots,n\}}$, we integrate the trajectory of the inertial particle using the classic *fourth–order Runge–Kutta* scheme with time step double of that used for the velocity field, where we apply *bilinear interpolation* of the velocity field at particle locations.

7.3. Numerical Experiments

We now visualize the long–time behaviour of particle motions varying different parameters.

Varying the time scale ratio τ

In the first experiment we simulate the motion of 10^4 particles uniformly distributed on the unit square at time zero with zero velocities according to system (M) with the Kolmogorov spectrum. We fix $\nu = 10^{-2}$, $\triangle t = 10^{-1}$, $R = 5$ and the Hurst parameter, but varying the values of τ. Figure 11, Figure 12 and Figure 13 with different fixed Hurst parameters show the final positions of the particles in the phase space at time $T = 500$ associated to four different Stokes' numbers in form of τ. In all three figures the clustering is distinctive for $\tau = 10^{-1}$ and $\tau = 1$, but there is almost no clustering for very low and large values of τ, i.e. for $\tau = 10^{-4}$ and $\tau = 10^2$ in the experiment. Further, given a realisation of the velocity field with $\nu = 10^{-2}$, $H = 3/4$ Figure 14b.) and d.) show the final positions of 5000 particles at time $T = 200$ initially uniformly distributed on the left rectangle (Figure 14a.)) and on the right rectangle (Figure 14c.)) on the unit square with zero velocities, respectively. Despite the very different initial data the clustering in Figure 14b.) and Figure 14d.) is almost indistinguishable. This indicates that the particles have converged to a subset of the random attractor. Therefore, we conclude that also the generalized model with fractional noise captures the clustering phenomenon of preferential concentration. In addition, if we compare the pictures in Figure 11–13, we discover that if we increase the Hurst parameter, the clustering seems to become stronger. Unfortunately, we do not have an explanation for that. In Theorem 5.3.8 we proved assertions which suggest a volume decrease in the (modified) system (M) and the volume decrease might be stronger if the functional $B_{H,\nu,\lambda,\alpha} = \mathbb{E}(|\psi(0)|_{C^2(\mathbb{T}^2)})$ in Theorem 5.3.8 is small. But we do not know how the Hurst parameter is involved in this functional.
Stuart and Sigurgeirsson used in [88, 89] for their simulation the Karman–Obukhov spectrum introduced in Section 6. Repeating the experiments with that spectrum under same conditions produces analogues results for $H = 1/2$.

Varying the fluid correlation time ν

In the next experiment we fix τ and H, but vary now the values of ν. For that we first *derive* limit equations of $(M1)$ for $\nu \to 0$ and $\nu \to \infty$ using similar *heuristic* arguments as Sigurgeirsson and Stuart in [88] and leave the rigorous derivation for future study. In the following let $X_H(t) = \sqrt{\lambda}\nu^H \int_{-\infty}^{t} e^{-(t-s)\nu\alpha} d\beta_t^H$, $t \in \mathbb{R}$, with $\lambda, \nu, \alpha > 0$ be the stationary fractional Ornstein–Uhlenbeck process. By Proposition 4.1.2(i) with $C(H) = \Gamma(2H+1)\sin(\pi H)/\pi$ we have

$$\mathbb{E}(X_H(t)X_H(s)) = C(H)\frac{\lambda}{\alpha^{2H}}\int_0^\infty \cos((t-s)\nu\alpha x)\frac{x^{1-2H}}{1+x^2}dx \approx Var(X_H(0)),$$

as $\nu \approx 0$. Therefore, we assume that for $\nu = 0$, i.e. at *infinite correlation time*, the velocity field v is frozen at its initial time value, i.e.

$$\bar{v}(x) = \nabla^{\perp}\psi(x,0) = \sum_{k\in K} \hat{\psi}_k(0)\nabla^{\perp}e_k(x), \quad x \in \mathbb{T}^2,$$

such that $(M1)$ reduces to the autonomous system

$$\tau\ddot{x}(t) = \bar{v}(x(t)) - \dot{x}(t), \quad (x(0),\dot{x}(0)) \in \mathbb{T}^2 \times \mathbb{R}^2. \tag{7.3.1}$$

Since (7.3.1) is dissipative, $\mathbb{T}^2$ compact and using some aspects of the proof of Theorem 5.3.5, it is clear that (7.3.1) admits a global attractor $\mathbb{P}$-a.s.
Now we turn to the limit $\nu \to \infty$. Sigurgeirsson and Stuart argued for $H = 1/2$ in [88] as follows: Since $e^{-\gamma|t|} \approx 2\delta(t)/\gamma$ for $\gamma \gg 1$ and therefore $\mathbb{E}(X_{\frac{1}{2}}(t)X_{\frac{1}{2}}(0)) = \frac{\lambda}{2\alpha}e^{-\nu\alpha|t|} \approx \frac{\lambda}{\nu\alpha^2}\delta(t)$ for $\nu \gg 1$, where $\delta(\cdot)$ denotes the Dirac delta function, Sigurgeirsson and Stuart supposed that $X_{\frac{1}{2}}(t)$ approximates *(standard) Gaussian white noise with constant* $\sqrt{\frac{\lambda}{\nu\alpha^2}}$, i.e. $X_{\frac{1}{2}}(t) \approx \sqrt{\frac{\lambda}{\nu\alpha^2}}\frac{d\beta_t^{\frac{1}{2}}}{dt}$. From this and

$$\sum_{k\in K} \frac{\lambda_k}{\alpha_k^2}\nabla^{\perp}e_k(x)((\nabla^{\perp}e_k(x))^*)' = \sum_{k\in K}\frac{\lambda_k}{\alpha_k^2}\begin{pmatrix} k_2^2 & -k_1k_2 \\ -k_1k_2 & k_1^2\end{pmatrix}$$

they assumed that

$$v(x,t) \approx \widetilde{v}(x,t) := \frac{1}{\sqrt{\nu}}\sum_{k\in K}\frac{\sqrt{\lambda_k}}{\alpha_k}\frac{d\beta_k^{\frac{1}{2}}(t)}{dt}\nabla^{\perp}e_k(x)$$

for $\nu \gg 1$ such that

$$\tau\ddot{x}(t) \approx -\dot{x}(t) \tag{7.3.2}$$

as $\nu \to \infty$. And for (7.3.2) they noted that a *Liouville equation argument* shows that initially uniform particle distribution in position space will be preserved, provided the velocities are chosen independently of positions. For the general case $H \in (0,1)$ we argue similarly, but in spectral domain. From Section 4.1 we know that the not normalized spectral density of $(X_H(t))_{t\in\mathbb{R}}$ is given by

$$f(x) = \widetilde{C}(H)\nu^{2H}\lambda\frac{|x|^{1-2H}}{(\nu\alpha)^2 + x^2}, \quad x \in D_H \subseteq \mathbb{R},$$

where $\widetilde{C}(H) = C(H)/2$, $D_H = \mathbb{R}$ if $H \in (0,1/2]$ and $D_H = \mathbb{R}\setminus\{0\}$ if $H \in (1/2,1)$. We have

$$f(x) \approx \widetilde{f}(x) = \widetilde{C}(H)\frac{\nu^{2H-2}\lambda}{\alpha^2}|x|^{1-2H}$$

for $\nu \gg 1$. $\widetilde{f}(\cdot)$ is (up to some positive constant depending on H) the spectral density of *fractional Gaussian white noise* $\sqrt{\frac{\nu^{2H-2}\lambda}{\alpha^2}}\frac{d\beta_t^H}{dt}$ *with constant* $\sqrt{\frac{\nu^{2H-2}\lambda}{\alpha^2}}$ (see Corollary 1 in [77]). It should be noted that the term *fractional Gaussian noise* is often used (as in [77]) to denote $\frac{d\beta_t^H}{dt}$. Here we prefer the term *fractional Gaussian white noise*, because we use in this work the term *fractional Gaussian noise* for the increments of the fractional Brownian motion $\beta_{n+1}^H - \beta_n^H$, $n \in \mathbb{N}_0$. Therefore, we also expect that

$$v(x,t) \approx \widetilde{v}(x,t) = \nu^{H-1} \sum_{k \in K} \frac{\sqrt{\lambda_k}}{\alpha_k} \frac{d\beta_k^H(t)}{dt} \nabla^{\perp} e_k(x)$$

for $\nu \gg 1$ and $\tau \ddot{x}(t) \approx -\dot{x}(t)$ as $\nu \to \infty$. As anticipated, if we increase the Hurst parameter, the decorrelation of the velocity field becomes slower.
In the numerical experiment we fix $\tau = 10^{-1}$, H, $\triangle t = 10^{-1}$, $R = 5$, and match the Kolmogorov spectrum, but now vary the values of ν. Figure 15 with $H = 1/3$ and Figure 16 with $H = 3/4$ show the final positions of 10^4 particles in the position space at time $T = 200$ associated to $\nu = 10^k$, $k = 1, -1, -2$ and $\nu = 0$. As seen in Figure 15d.) and Figure 16d.) for the special case $\nu = 0$, the particles converge to a finite number of isolated periodic orbits. By increasing ν we recognize that the clustering becomes less intensive for such initial data. Repeating the experiment for very large and very low values of τ, i.e. for $\tau = 10^2$ and $\tau = 10^{-4}$, we observe almost no clustering.
If we want that the particle notices the velocity field $\widetilde{v}(x,t)$ for $\nu \gg 1$, we have to look at large times and rescale ν, τ in an appropriate way, as it was (also heuristically) done by Sigurgeirsson and Stuart in [88] for $H = 1/2$. We postpone this as well as the analytical investigation of the spacial case $\nu = 0$ for future work.

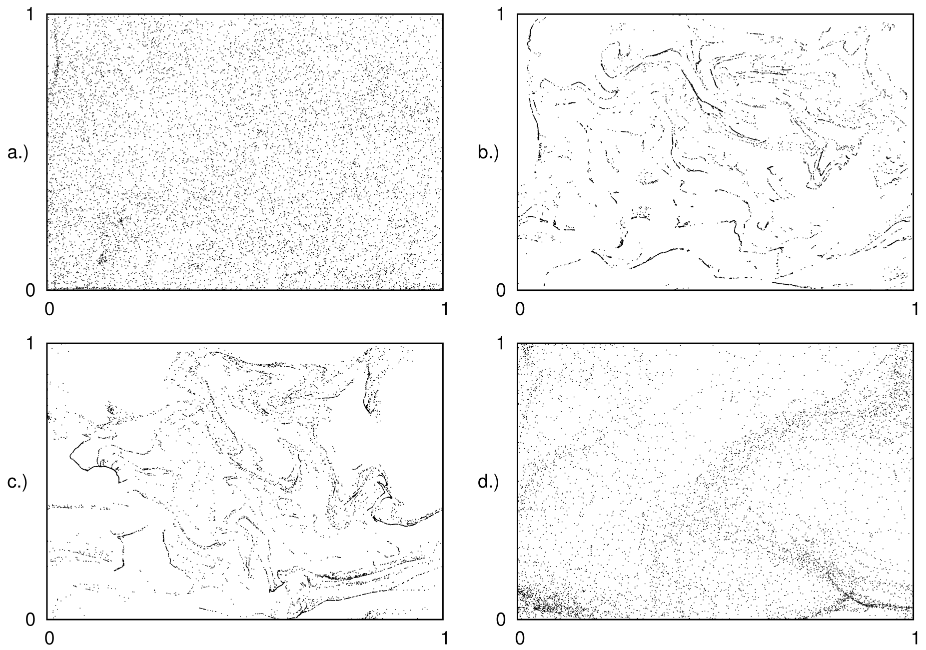

Figure 11: Snapshots of the final positions of 10^4 particles in the phase space with $\nu = 10^{-2}$ and $H = 1/3$ associated to different Stokes' numbers τ. a.) $\tau = 10^{-4}$, b.) $\tau = 10^{-1}$, c.) $\tau = 1$, d.) $\tau = 10^2$.

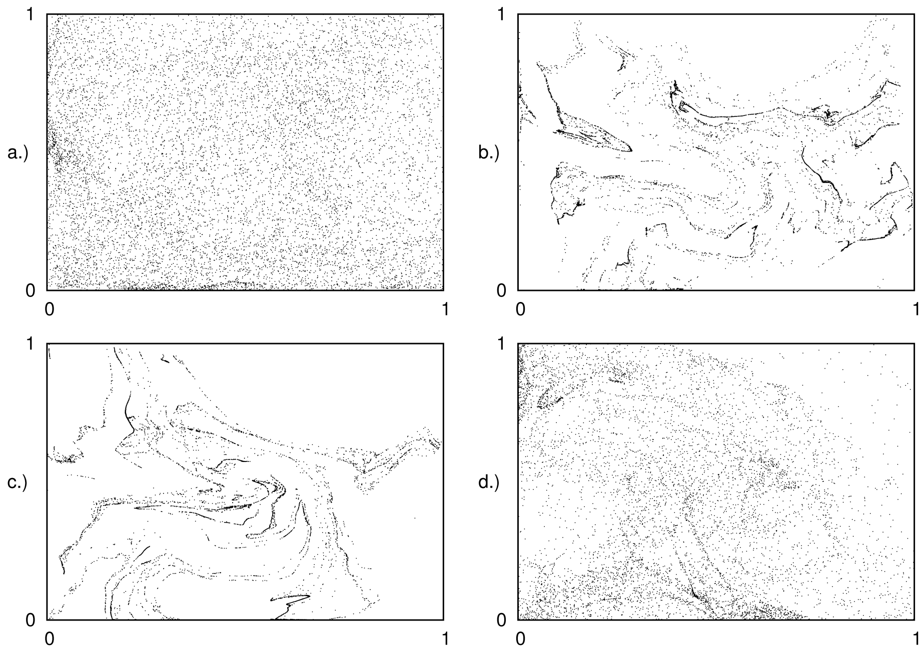

Figure 12: Snapshots of the final positions of 10^4 particles in the phase space with $\nu = 10^{-2}$ and $H = 1/2$ associated to different Stokes' numbers τ. a.) $\tau = 10^{-4}$, b.) $\tau = 10^{-1}$, c.) $\tau = 1$, d.) $\tau = 10^2$.

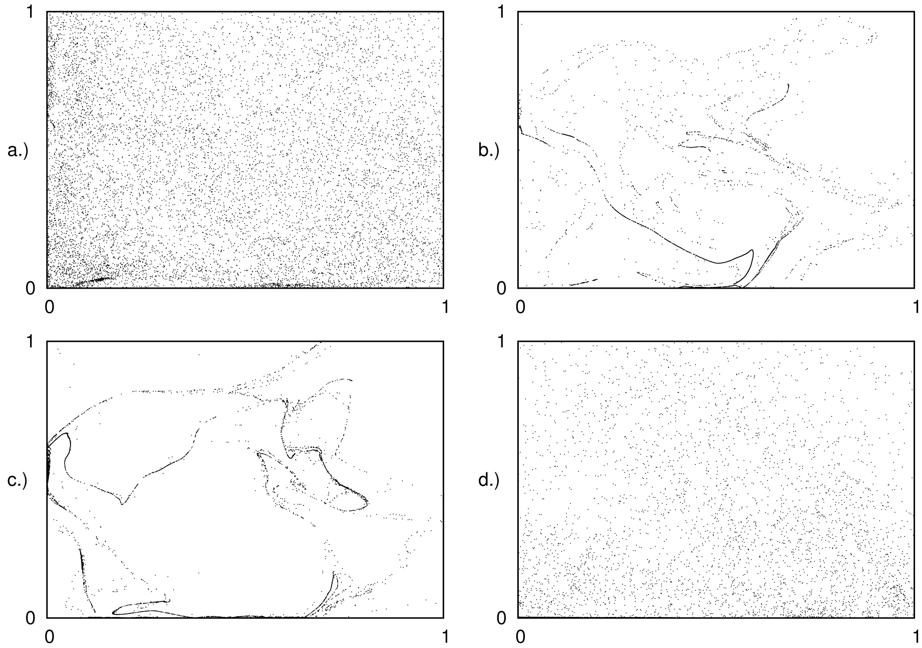

Figure 13: Snapshots of the final positions of 10^4 particles in the phase space with $\nu = 10^{-2}$ and $H = 3/4$ associated to different Stokes' numbers τ. a.) $\tau = 10^{-4}$, b.) $\tau = 10^{-1}$, c.) $\tau = 1$, d.) $\tau = 10^2$.

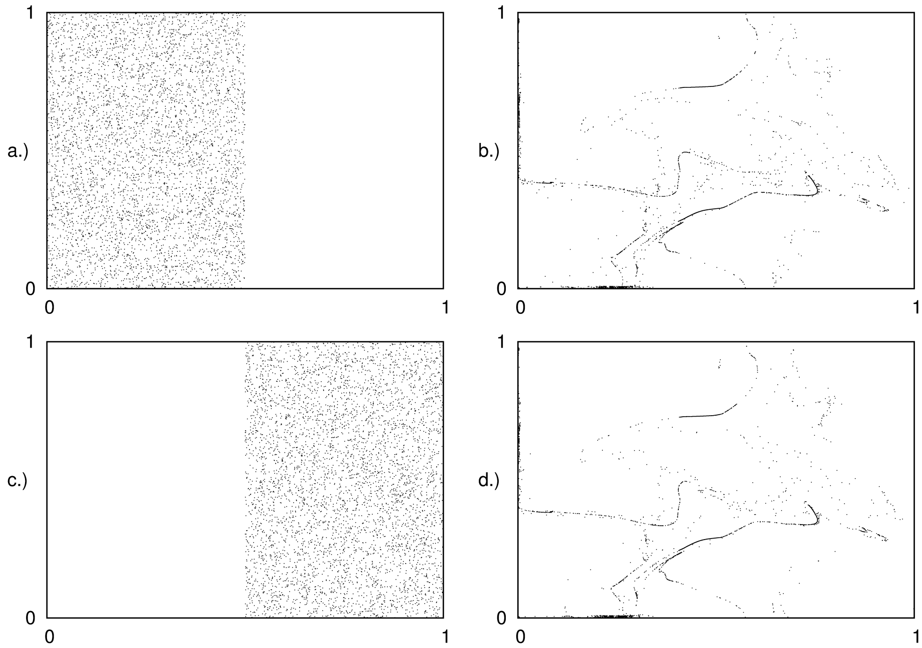

Figure 14: Snapshots of the final positions (b.), d.)) of 5000 particles in the phase space associated to $\tau = 10^{-1}$ and initial positions a.) and c.).

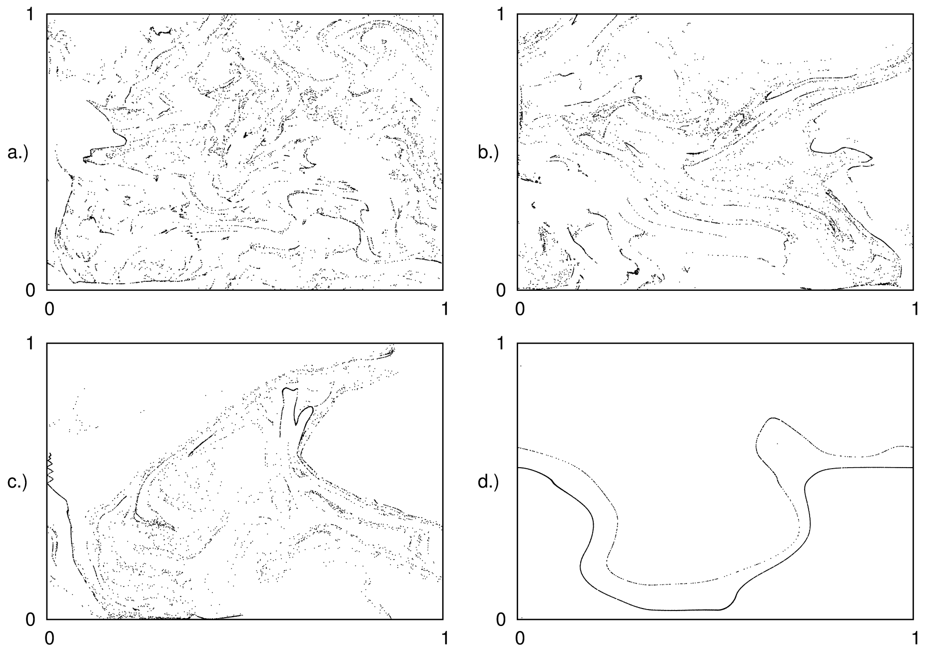

Figure 15: Snapshots of the final positions of 10^4 particles in the phase space with $\tau = 10^{-1}$ and $H = 1/3$ associated to different values of ν. a.) $\nu = 10$, b.) $\nu = 10^{-1}$, c.) $\nu = 10^{-2}$, d.) $\nu = 0$.

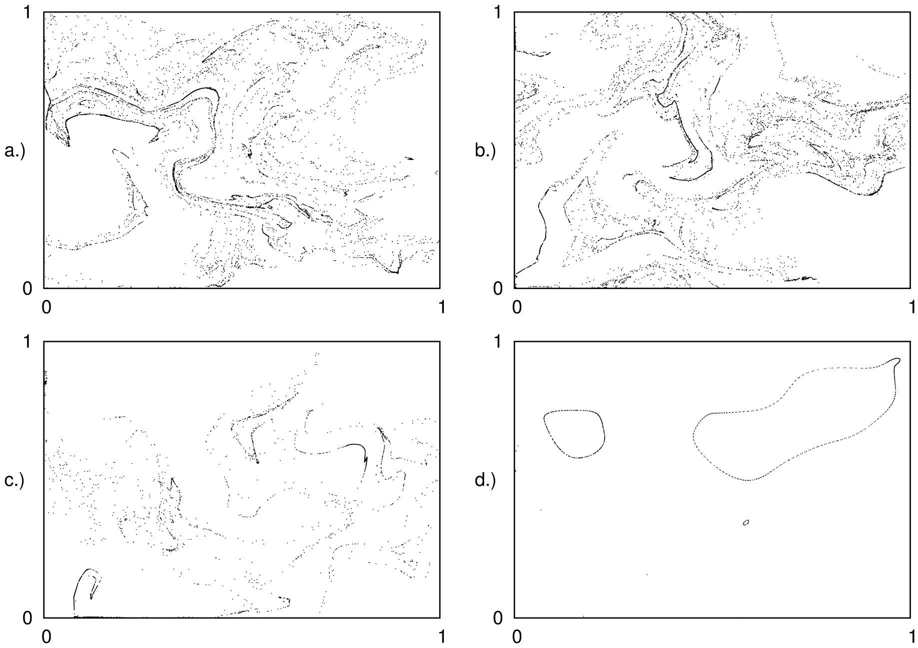

Figure 16: Snapshots of the final positions of 10^4 particles in the phase space with $\tau = 10^{-1}$ and $H = 3/4$ associated to different values of ν. a.) $\nu = 10$, b.) $\nu = 10^{-1}$, c.) $\nu = 10^{-2}$, d.) $\nu = 0$.

A. Fractional Calculus

The fractional Brownian motion is representable as a stochastic integral w.r.t a standard Brownian motion where the integrands are some *fractional integrals* and *derivatives*, respectively. Therefore, we recall some basic definitions and properties of the fractional calculus. For a detailed presentation of these notions we refer to the monograph [79] and [58].

Let $\alpha > 0$. In the following $\Gamma(\cdot)$ denotes the gamma function.
The Riemann–Liouville fractional integrals on $\mathbb{R}$ are defined as

$$(I^{\alpha}_{+} f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\, dt$$

and

$$(I^{\alpha}_{-} f)(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(t)(x-t)^{\alpha-1}\, dt, \tag{A.1}$$

respectively. We say that the function $f \in \mathcal{D}(I^{\alpha}_{\pm})$, where the symbol $\mathcal{D}(\cdot)$ denotes the domain of the corresponding operator, if the corresponding integrals converge for a.a. $x \in \mathbb{R}$.
In particular, we have $L^p(\mathbb{R}) \subset \mathcal{D}(I^{\alpha}_{\pm})$ provided that $0 < \alpha < 1$ and $1 \le p < \frac{1}{\alpha}$. Also note that $I^{\alpha}_{+}(L^p(\mathbb{R})) = I^{\alpha}_{-}(L^p(\mathbb{R}))$ if $1 < p < \frac{1}{\alpha}$ (see Corollary 1 of Theorem 11.4 in [79]). Moreover, the following theorem holds.

Theorem A.1 (Hardy–Littlewood)**.** *Let $1 \le p, q < \infty$, $0 < \alpha < 1$. Then the operators $I^{\alpha}_{\pm}$ are bounded from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ if and only if $q = \frac{p}{1-\alpha p}$ and $1 \le p < \frac{1}{\alpha}$.*

Proof. Theorem 5.3 in [79]. □

The next result is evident.

Lemma A.2. *Let $0 < \alpha < 1$, $f \in L^p(\mathbb{R})$, $1 \le p < \frac{1}{\alpha}$ and $I^{\alpha}_{\pm} f = 0$. Then $f = 0$ for a.a. $x \in \mathbb{R}$.*

Let $p \ge 1$ and $f \in I^{\alpha}_{\pm}(L^p(\mathbb{R}))$ be a function that can be represented as a Riemann–Liouville integral, i.e. $f = I^{\alpha}_{\pm}\phi$ for some $\phi \in (L^q(\mathbb{R}))$, $q \ge 1$. Then Lemma A.2 ensures the uniqueness of such a function ϕ, more exactly, for $0 < \alpha < 1$ it coincides for a.a. $x \in \mathbb{R}$ with the left– (or right–)sided Riemann–Liouville fractional derivative of f of order α. These derivatives are denoted by

$$(I^{-\alpha}_{+} f)(x) = (D^{\alpha}_{+} f)(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} f(t)(x-t)^{-\alpha}\, dt$$

and

$$(I_-^{-\alpha} f)(x) = (D_-^\alpha f)(x) := \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_x^\infty f(t)(t-x)^{-\alpha}\, dt, \tag{A.2}$$

respectively. For $1 \le p < \frac{1}{\alpha}$, $0 < \alpha < 1$, the class of functions $f = I_\pm^\alpha \phi$, $\phi \in (L^p(\mathbb{R}))$ coincides (see Theorem 6.2 in [79]) with the class of those functions $f \in (L^q(\mathbb{R}))$, $q = \frac{p}{1-\alpha p}$, for which the integrals

$$(D_\pm^\alpha f)(x) = \lim_{\epsilon \to 0 \text{ in } (L_p)} (D_{\pm,\epsilon}^\alpha f)(x), \tag{A.3}$$

converge in $L^p(\mathbb{R})$ as $\epsilon \to 0$, where

$$(D_{\pm,\epsilon}^\alpha f)(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_\epsilon^\infty \frac{f(x) - f(x \mp t)}{t^{1+\alpha}}\, dt.$$

The limit in (A.3) also exists almost everywhere and the integrals $(D_\pm^\alpha f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x)-f(x\mp t)}{t^{1+\alpha}}\, dt$ are called *Marchaud fractional derivatives.*

Fractional integration admits the following composition formulas for fractional integrals (see page 96 in [79]):

$$I_\pm^\alpha I_\pm^\beta f = I_\pm^{\alpha+\beta} f$$

for $f \in L^p(\mathbb{R})$, $\alpha > 0$, $\beta > 0$ such that $\alpha + \beta < \frac{1}{p}$.
If $f \in I_\pm^\alpha(L^p(\mathbb{R}))$, $0 < \alpha < 1$, $p \ge 1$, then

$$I_\pm^\alpha D_\pm^\alpha f = f.$$

Moreover, for $f \in L^1(\mathbb{R})$ we have (see §6.2 in [79])

$$D_\pm^\alpha I_\pm^\alpha f = f.$$

B. Some Definite Integrals

Here we give representations of some definite integrals which were used in previous sections. In the following $\Gamma(x) = \int_0^\infty s^{x-1}\exp(-s)ds$, $x \in \mathbb{C}, Re(x) > 0$, denotes the *gamma function* and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y \in \mathbb{C}$, $Re(x), Re(y) > 0$, denotes the *beta function*. Here $Re(z)$ represents the real part of $z \in \mathbb{C}$. Further, the *hyperbolic cosine* is defined by $\cosh(x) := (e^x + e^{-x})/2$, $x \in \mathbb{C}$, and the *generalized hypergeometric function* ${}_1F_2$ is given by

$$ {}_1F_2(a_1; b_1, b_2; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!}, $$

where $(c)_0 := 1$, $(c)_k := c \cdot (c+1) \cdot \ldots (c+k-1)$ for $c \in \mathbb{C}$ and $k \in \mathbb{N}$ denotes the *Pochhammer symbol*, a_1, b_1, b_2 are real or complex parameters of ${}_1F_2$ such that none of b_1, b_2 is a non–positive integer and z is the complex variable of ${}_1F_2$.

Lemma B.1. *We have*

(i) 3.241.2, p. 322 in [40]:

$$ \int_0^\infty \frac{x^{\mu-1}}{1+x^\nu} dx = \frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right) \quad \text{for } Re(\nu) > Re(\mu) > 0. $$

(ii) 3.1.3.2, p. 567 in [75]:

$$ \int_0^\infty \int_0^\infty f(|x-y|) e^{-px-qy} dx dy = \frac{1}{pq(p+q)} \left[p \int_0^\infty e^{-qy} f(y) dy + q \int_0^\infty e^{-px} f(x) dx \right] $$

for $p, q > 0$.

(iii) 2.5.9.2, p. 394 in [75]:

$$ \int_0^\infty \frac{x^{\beta-1}}{x^2+z^2} \cos(bx) dx $$
$$ = \frac{\pi z^{\beta-2}}{2} \frac{\cosh(bz)}{\sin(\pi\beta/2)} - \Gamma(\beta-2)\cos(\pi\beta/2) b^{2-\beta} {}_1F_2\left(1; \frac{3-\beta}{2}, 2-\frac{\beta}{2}; \frac{b^2 z^2}{4}\right) $$

with $b, Re(z) > 0$ and $0 < Re(\beta) < 3$.

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