AN ANALYTICAL SOLUTION OF THE REYNOLDS EQUATION FOR THE FINITE JOURNAL BEARING AND EVALUATION OF THE LUBRICANT PRESSURE

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ABSTRACT
The Reynolds equation for the pressure distribution of the lubricant in a journal bearing with finite length is solved analytically. Using the method of the separation of variables in an additive and in a multiplicative form a set of particular solutions of the Reynolds equation is added in the general solution of the homogenous Reynolds equation and a closed form expression for the definition of the lubricant pressure is presented. The Reynolds equation is split in four linear ordinary differential equations of second order with non constant coefficients and together with the boundary conditions they form four Sturm-Liouville problems with the three of them to have direct forms of solution and one of them to be confronted using the method of power series. In this part of the work, the mathematical procedure is presented up to the point that the application of the boundaries for the pressure distribution yield the final definition of the solution with the calculation of the constants. The distributions of the pressure given from the particular solution and the solution of the homogeneous Reynolds equation are presented together with the resulting pressure. Also, the results of an approximate analytical solution using Bessels functions and linearization of the fluid film thickness function are also presented together with the results of the numerical solution using the finite differences method. Diagrams for the pressure profiles under the current study are compared with those from the approximate analytical and the numerical solution. The locations in which the maximum, the zero, and the minimum pressure are presented are given as a function of eccentricity rate with closed form expressions.

KEYWORDS
Journal bearing, Reynolds equation, pressure distribution, analytical solution, separation of variables, Sturm-Liouville, power series.

1 INTRODUCTION
The exact analytical solution of Reynolds equation for lubrication of journal bearings [1] is up to nowadays a problem under investigation. Most investigators achieved to define functions of pressure distribution $P(\theta, x)$ under approximate solutions of Reynolds Eq. (1.1).

$$\frac{\partial}{\partial x}\left(\frac{h(\theta,t)^3}{6\mu}\frac{\partial P(x,\theta)}{\partial x}\right)+\frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\frac{h(\theta,t)^3}{6\mu}\frac{\partial P(x,\theta)}{\partial\theta}\right)=\Omega\frac{\partial h(\theta)}{\partial\theta}+2\frac{\partial h(\theta,t)}{\partial t} \tag{1.1}$$

The approximate analytical solutions of the Reynolds equation are based on the assumptions that one of the two terms in the left side of Eq. (1.1) can be neglected. The first term can be neglected when the journal bearing is considered as a bearing with high length to diameter ratio (long bearing, L/D>>1 [2]) and the second term when the journal bearing is considered as a bearing with low length to diameter ratio (short bearing L/D<1 [3-8]). Such solutions for pressure distribution give considerably simplified mathematical expressions. They are exact solutions in the sense that they satisfy Eq. (1.1) for the case of infinite axial length or of infinite short bearing. However they are regarded as approximate solutions when they are used to determine the pressure distribution in bearings of finite length.

The research contributions in the analytical calculation of the pressure distribution in a journal bearing with finite length are not many and they have not been made recently. Some of them are referenced in brief in the current work. Kingsbury [9] determined the pressure distribution by an experimental electrical analogy. Christopherson [10] determined the pressure distribution by utilizing the mathematical model of “relaxation”. Vogelpohl [11-13] achieved to give closed forms of the functions for the pressure distribution along both directions of the journal bearing, axial and circumferential. Vogelpohl [11-13] assumed a partial solution of the Reynolds equation that corresponds to a “long bearing approximation” and then he used the technique of the separation of variables for the solution of the homogenous Reynolds equation. By adding the two solutions, the pressure distribution was defined in closed form. Cameron and Wood [14] had extended the work of Christopherson [10] to show the effect of length to diameter ratio on eccentricity ratio, attitude angle and friction coefficient. In all cases, these solutions are given to express natural phenomena in the oil film on the basis of Reynolds assumptions regarding lubrication, the most important assumption being that certain terms in the generalized Navier-Stokes equations for flow in a viscous fluid may be neglected.

In the current work the Reynolds equation of the form of Eq. (1.1) is treated similarly to the path that Vogelpohl [11-13] followed, but there are crucial differences that have to do with the way that the particular solution is obtained and with the solutions of the ordinary differential equations that are yielded during the procedure. In the beginning of the current work the Reynolds equation is classified. As expected it turns out that it is an elliptic equation, which is reasonable since it defines a static problem (the time is not considered as an independent variable but as a parameter that gives the eccentricity and its rate of change)

The strategy for obtaining the analytical solution is based in the application of the powerful method of separation of variables. The crucial step is a splitting of the solution in to two parts. The one satisfies the homogeneous Reynolds equation; namely, Reynolds equation without the second part that the pressure does not interfere. For solving this part we assume a multiplicative separation of the independent variables so we obtain two Sturm-Liouville problems. In this point, the contribution of the current work is that the use of the power series method is used in order to obtain the eigenfunctions of the one Sturm-Liuville problem while in the literature the corresponding confrontment of this problem is made with the linear approximation of the fluid film thickness function and the expression of the eigenfunctions using Bessel functions or normal sinusoidal functions. In the current work, the “easy” sturm-Liuville problem is solved and the eigenvalues of it are incorporated in the functions defined with power series.

The second part is a particular solution of the Reynolds equation itself. For finding a particular solution we assume an additive splitting of the independent variables and two Sturm-Liuville problems are obtained. The first has to do with the pressure distribution along the circumferential coordinate and it has a direct solution with a closed form expression taken from the literature for ODE treatment. The other problem has to do with the pressure distribution along the axial coordinate and the boundary conditions which are chosen yield a trivial solution, without this to be problematic for the further progress of the solution. The current particular solution is also a contribution of the current work since it is actually a set of particular solutions that can be different from the solution of the infinitely long bearing as used in [11]. The current particular solution yields the long bearing pressure distribution as a sub case.

The paper is organised as follows. The section 2 contains the basic ingredients of Reynolds equation together with the classification of it and the crucial step where the unknown pressure P is split into two functions g and u. The Section 3 is related with the evaluation of the particular solution of the Reynolds equation, u. In Section 4 we present an analysis for evaluating the function g using again the method of separation of variables, but now in a multiplicative form. The treatment with the Bessel functions and the approximating analytical solution is also given. The last section, section 5 deals with the boundary value problems. The resulting pressure is evaluated for a specific set of values of the physical and geometrical journal bearing characteristics and is compared with the analytical approximating solution and the numerical solution using finite differences method. The article concludes in Section 6 where also the forthcoming results are described.

2 REYNOLDS EQUATION: CLASSIFICATION AND SPLITTING OF THE SOLUTION

The problem of the lubrication of journal bearings with finite length is defined in this work as the calculation of the pressure distribution of the Newtonian lubricant that is assumed to flow under laminar, isoviscous, and isothermal conditions in between the rotating journal and the static bearing. The journal of radius R and length L_b is assumed to be rotating with a constant rotational speed Ω (counter clockwise) and to be constantly located in a point of eccentricity e with respect to the geometric centre of the bearing of radius $R+c_r$ and length L_b after an application of a constant vertical load W as shown in Fig. 1. The journal eccentricity is assumed to have rate of change described by the velocity $\dot{e}$. The journal and the bearing are supposed to be in parallel (aligned bearing) and the fluid film thickness h becomes a function of the unique parameter θ for a time moment of constant e and $\dot{e}$ which means that the function for the fluid film thickness is $h=c_r+e\cos(\theta)$ and its time derivative as $\partial h/\partial t=\dot{e}\cos(\theta)$. The dynamic viscosity of the lubricant is assumed to be constant and equal to μ through the entire control volume (notified with shadow in Fig. 1) that is defined from the bearing and the journal surfaces. The attitude angle of the journal is defined as φ_0 with respect to the vertical coordinate axis (see Fig. 1). The boundary conditions for the pressure distribution are left to be defined in the last section where the numerical application is made.

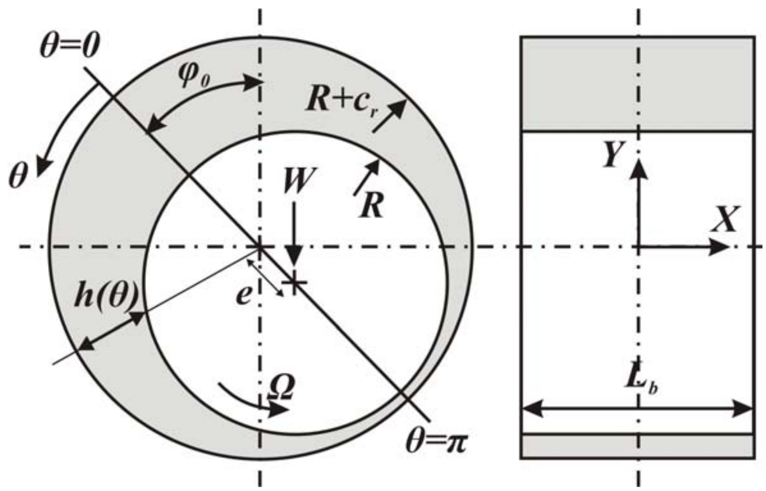

Fig. 1. Definition of the coordinate system and of the parameters of operation and design in a plain cylindrical journal bearing.

The starting point is the equation of Reynolds which is expressed as in Eq. (2.1).

$$\frac{\partial}{\partial x}\left(\frac{h^3}{6\mu}\frac{\partial P(x,\theta)}{\partial x}\right)+\frac{1}{R^2}\frac{\partial}{\partial \theta}\left(\frac{h^3}{6\mu}\frac{\partial P(x,\theta)}{\partial \theta}\right)=\Omega\frac{\partial h}{\partial \theta}+2\frac{\partial h}{\partial t} \tag{2.1}$$

The constant magnitudes $\Omega, R, \mu, c_r, e, \dot{e}, W$ represent the rotational speed of the journal, the radius of the journal, the lubricant dynamic viscosity, the bearing radial clearance, the journal eccentricity, the journal eccentricity rate of change and the external loading force correspondingly, see Fig. 1.

After substituting the fluid film thickness function of Eq. (2.2) into Eq. (2.1) and performing the derivations one will arrive at Eq. (2.3).

$$h=c_r+e\cos(\theta) \tag{2.2}$$

$$\frac{\left(c_r+e\cos(\theta)\right)^3}{6\mu}\frac{\partial^2 P(\theta,x)}{\partial x^2}-\frac{3\left(c_r+e\cos(\theta)\right)^2 e\sin(\theta)}{6\mu R^2}\frac{\partial P(\theta,x)}{\partial \theta}+\frac{\left(c_r+e\cos(\theta)\right)^3}{6\mu R^2}\frac{\partial^2 P(\theta,x)}{\partial \theta^2}=-e\Omega\sin(\theta)+2\dot{e}\cos(\theta) \tag{2.3}$$

Eq. (2.3) is the one that we are going to work with. This is a non homogenous linear partial differential equation of the second order for the unknown function $P(x,\theta)$ with trigonometric coefficients. Before embarking on the core of our analysis we classify Eq. (2.3). For doing so we need to evaluate the discriminant as in Eq. (2.4).

$$\Delta=B^2-A\Gamma=-\left(\frac{h^3}{6\mu}\right)\left(\frac{1}{R^2}\frac{h^3}{6\mu}\right)=-\frac{h^6}{36R^2\mu^2}<0 \tag{2.4}$$

Whatever the expressions of h, R, μ are, we speak about an elliptic partial differential equation. The crucial step for obtaining the solution is based on the following splitting of the solution. We assume that the unknown function can be written in the form of Eq. (2.5).

$$P(x,\theta)=\underbrace{u(x,\theta)}_{particular}+\underbrace{g(x,\theta)}_{\text{hom}ogenic} \tag{2.5}$$

The function $u(x,\theta)$ is a particular solution of Eq. (2.3) while the function $g(x,\theta)$ describes the set of solutions for the homogeneous Reynolds equation, namely, Eq. (2.3) without the right hand side term. In order to see more clearly that instead of seeking $P(x,\theta)$ we can seek for the functions $g(x,\theta)$ and $u(x,\theta)$ one may write the differential operator of Reynolds equation as in Eq. (2.6).

$$\frac{\left(c_r+e\cos(\theta)\right)^3}{6\mu}\frac{\partial^2}{\partial x^2}-\frac{3\left(c_r+e\cos(\theta)\right)^2 e\sin(\theta)}{6\mu R^2}\frac{\partial}{\partial\theta}+\frac{\left(c_r+e\cos(\theta)\right)^3}{6\mu R^2}\frac{\partial^2}{\partial\theta^2} \tag{2.6}$$

If one applies this operator to both sides of Eq. (2.5) he will immediately verify that equivalently one may seek for the functions $g(x,\theta)$ and $u(x,\theta)$.

3 EVALUATION OF THE PARTICULAR SOLUTION

In order to evaluate the function $u(x,\theta)$ a particular solution of Eq. (2.3) is needed. For doing so we assume that $u(x,\theta)$ can be split in the following additive form of Eq. (3.1).

$$u(x,\theta)=\varphi(x,\theta)+\psi(x,\theta) \tag{3.1}$$

We are looking for a solution where the independent variables can be split in the above form. If we substitute Eq. (3.1) in Eq. (2.3) after some calculations Eq. (3.2) is obtained.

$$-\frac{d^2\psi(x)}{dx^2}=\frac{1}{R^2}\frac{d^2\varphi(\theta)}{d\theta^2}-\frac{3e\sin(\theta)}{R^2\left(c_r+e\cos(\theta)\right)}\frac{d\varphi(\theta)}{d\theta}+\frac{6\mu e\Omega\sin(\theta)}{\left(c_r+e\cos(\theta)\right)^3}-\frac{12\mu\dot{e}\cos(\theta)}{\left(c_r+e\cos(\theta)\right)^3} \tag{3.2}$$

By inspecting the above equation one observes that the right hand side is a function of θ while the left hand side is a function of x only. So, the equality will be feasible only when both sides are equal to the same constant C. From the latter equation we obtain two equations for the functions $\varphi(\theta)$ and $\psi(x)$. These are ordinary differential equations written as in Eqs. (3.3) and (3.4).

$$-\frac{d^2\psi(x)}{dx^2}=C \tag{3.3}$$

$$\frac{1}{R^2}\frac{d^2\varphi(\theta)}{d\theta^2}-\frac{3e\sin(\theta)}{R^2\left(c_r+e\cos(\theta)\right)}\frac{d\varphi(\theta)}{d\theta}+\frac{6\mu e\Omega\sin(\theta)}{\left(c_r+e\cos(\theta)\right)^3}-\frac{12\mu\dot{e}\cos(\theta)}{\left(c_r+e\cos(\theta)\right)^3}=C \tag{3.4}$$

Eq. (3.3) can be solved directly to give Eq. 3.5.

$$\psi(x)=-\frac{Cx^2}{2}+c_1x+c_2 \tag{3.5}$$

The constants c_1, c_2 are arbitrary constants of integration. For solving Eq. (3.4) one observes that we talk about a linear ordinary differential equation with non-constant coefficients where the unknown function $\varphi(\theta)$ is not present explicitly. So, by setting $z(\theta)$ as in Eq. (3.6), instead of Eq. (3.4) we may solve the following linear ordinary differential equation defined in Eq. (3.7) of the first order for the function $z(\theta)$.

$$\frac{d\varphi(\theta)}{d\theta}=z(\theta) \tag{3.6}$$

$$\frac{1}{R^2}\frac{dz(\theta)}{d\theta}-\frac{3e\sin(\theta)}{R^2\left(c_r+e\cos(\theta)\right)}z(\theta)+\frac{6\mu e\Omega\sin(\theta)}{\left(c_r+e\cos(\theta)\right)^3}-\frac{12\mu\dot{e}\cos(\theta)}{\left(c_r+e\cos(\theta)\right)^3}=C \tag{3.7}$$

The generic set of solutions of the last equation is given in Eq. (3.8) [18].

$$z(\theta)=e^{\int\left(\frac{3e\sin(\theta)}{c_r+e\cos(\theta)}\right)d\theta}\left(c_3+\int\left(e^{-\int\left(\frac{3e\sin(\theta)}{c_r+e\cos(\theta)}\right)d\theta}\left(-\frac{6\Omega\mu R^2 e\sin(\theta)}{\left(c_r+e\cos(\theta)\right)^3}+\frac{12\Omega\mu R^2\dot{e}\cos(\theta)}{\left(c_r+e\cos(\theta)\right)^3}+C\right)\right)d\theta\right) \tag{3.8}$$

All the integrals in Eq. (3.8) can be evaluated using direct closed form expressions. By evaluating the integrals that appear in the expression for $z(\theta)$ one can integrate the outcome one time and obtain the function $\varphi(\theta)$, Eq. (3.9).

$$\varphi(\theta)=\int z(\theta)d\theta+c_4 \tag{3.9}$$

The sum of $\varphi(\theta)$ and $\psi(x)$ renders classes of particular solutions by choosing values for the constants C,c_1,c_2,c_3,c_4. By choosing $C=0$ we obtain Eqs. (3.10) and (3.11).

$$\psi(x)=c_1x+c_2 \tag{3.10}$$

$$\varphi(\theta)=c_4+\frac{1}{2}\left(-\frac{2B\left(-18c_re^2R^2\mu\Omega+\left(2c_r^2+e^2\right)c_3\right)}{\left(-c_r^2+e^2\right)^{5/2}}+\frac{3e\left(2c_r^2R^2\mu\Omega-c_rc_3\right)\sin(\theta)}{\left(c_r^2-e^2\right)^2\left(c_r+e\cos(\theta)\right)}+\frac{12\left(-c_r^2+e^2\right)\dot{e}R^2\mu-e^2\left(6c_rR^2\mu\Omega-c_3\right)\sin(\theta)}{\left(-c_r^2e+e^3\right)\left(c_r+e\cos(\theta)\right)^2}\right) \tag{3.11}$$

For the pressure distribution yielded from the particular solution, the boundary conditions are setting the pressure equal to zero at the both ends of the bearing in axial direction $x=\pm\frac{L_b}{2}$ and at the circumferential location of $\theta=0$. With the following boundary conditions in Eqs. (3.12-3.15) the constants c_1,c_2,c_3,c_4 can be defined as in Eqs. (3.16 – 3.19). The formula in Eq. (3.15) does not correspond to a boundary condition but it expresses the symmetry of pressure distribution to the horizontal axis of Figure 2 and it also coincides with the boundary condition at $\theta=2\pi$. The most known assumptions for pressure distribution through the circumferential direction (along θ) that are given in the literature are: a) the Sommerfeld condition in which $\varphi(0)=0$ and $\varphi(\pi)=0$, b) the Reynolds condition in which $\varphi(0)=0$ and $\varphi(\theta^*)=0$, and c) the Gumbel condition in which $\varphi(0)=0$, $\varphi(\pi)=0$ and $\varphi(\theta)=0$ for $\pi<\theta<2\pi$ (π film bearing). When $\dot{e}=0$ then the Eq. (3.15) yields the Reynolds condition with $\theta^*=\pi$. When $\dot{e}\neq 0$ it is consequently $\theta^*\neq 0$ and for this reason the formula of Eq. (3.15) is used, and instead of predefining the angle θ^* and use it as a boundary of zero pressure it is let to be defined by the global pressure distribution through θ giving to the current analysis a further generality. The values of θ^* as a function of $\dot{e}$ are presented in the section 5 because θ^* is lightly effected also from the values of the pressure yielded by the homogeneous solution $g(x,\theta)$ that is added to $\varphi(\theta)$.

$$\psi\left(\frac{L_b}{2}\right)=0 \tag{3.12}$$

$$\psi\left(-\frac{L_b}{2}\right)=0 \tag{3.13}$$

$$\varphi(0)=0 \tag{3.14}$$

$$\begin{cases} \int_0^{2\pi} \varphi(\theta)\,d\theta = 0, & \dot{e} = 0 \\ \lim\limits_{\dot{e}\to 0} \int_0^{2\pi} \varphi(\theta)\,d\theta = 0, & \dot{e} \neq 0 \end{cases} \tag{3.15}$$

$$c_1 = 0 \tag{3.16}$$

$$c_2 = 0 \tag{3.17}$$

$$c_3 = \frac{6R^2\mu\left(\pi\left(2(c_r-e)^2\sqrt{-c_r^2+e^2}\,\dot{e}+3c_r e^3\Omega\left(\mathrm{Log}\left(-\frac{c_r+e}{\sqrt{-c_r^2+e^2}}\right)-\mathrm{Log}\left(\frac{c_r+e}{\sqrt{-c_r^2+e^2}}\right)\right)\right)+3c_r e^3\Omega\arccos\left(\frac{-c_r}{e}\right)\left(\mathrm{Log}\left(-\frac{c_r+e}{\sqrt{-c_r^2+e^2}}\right)-\mathrm{Log}\left(\frac{c_r+e}{\sqrt{-c_r^2+e^2}}\right)\right)\right)}{e\left(2c_r^2+e^2\right)\left(\pi\left(\mathrm{Log}\left(-\frac{c_r+e}{\sqrt{-c_r^2+e^2}}\right)-\mathrm{Log}\left(\frac{c_r+e}{\sqrt{-c_r^2+e^2}}\right)\right)+\arccos\left(\frac{-c_r}{e}\right)\left(-\mathrm{Log}\left(-\frac{\sqrt{-c_r^2+e^2}}{\sqrt{e}\sqrt{c_r+e}}\right)+\mathrm{Log}\left(\frac{\sqrt{-c_r^2+e^2}}{\sqrt{e}\sqrt{c_r+e}}\right)\right)\right)} \tag{3.18}$$

$$c_4 = -\frac{6\dot{e}R^2\mu}{e(c_r+e)^2} \tag{3.19}$$

The current particular solution corresponds to the pressure distribution developed in an infinitely long bearing and for a numerical application with the parameter values of Table 1 the pressure distribution along the circumferential direction is shown in Figure 2 for three cases of the eccentricity velocity $\dot{e}$. The current particular solution gives a further contribution in Reynolds equation treatment because the additive separation of variables can yield the solutions of the long or of the short bearing without the need of "erasing" the one of two left hand term in Eq. (2.1).

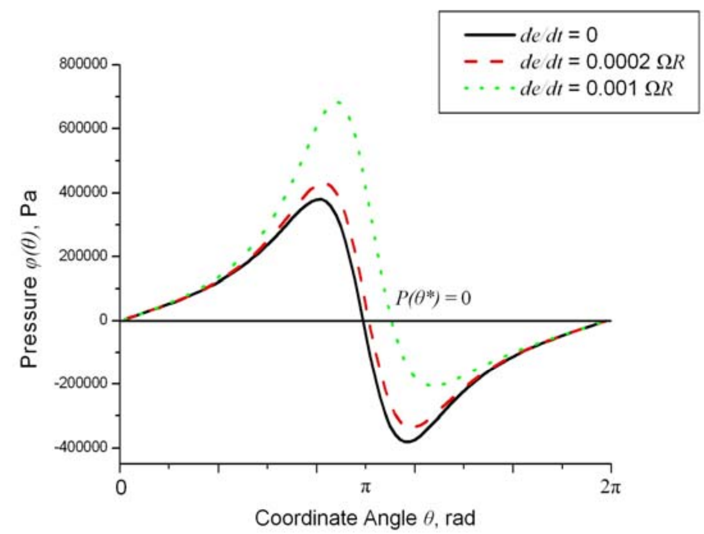

Fig. 2. The pressure distribution along the angular coordinate θ given from the particular solution $\varphi(\theta)$.

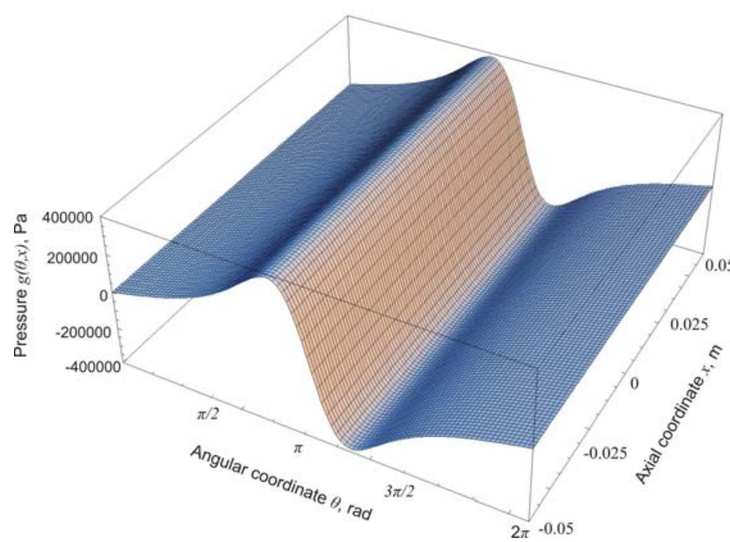

Fig. 3. The pressure distribution along the angular coordinate θ given from the particular solution $\varphi(\theta)$.

$R = 0.05\mathrm{m}$	$e = 0.7c_r$	$\Omega = 100\mathrm{rad/s}$
$c_r = 500\mu\mathrm{m}$	$L_b = 0.10\mathrm{m}$	$\mu = 0.05\mathrm{Pa\,s}$

Table 1. Definition of the values of the geometrical and the physical parameters of the current journal bearing.

As shown in Fig. 2 the pressure distribution of the particular solution becomes zero at the angle θ^* that is a function of the eccentricity velocity. The pressure distribution given by the particular solution is symmetric to the horizontal axis at $y = 0$ for the case of $\dot{e} = 0$. The pressure distribution is intensively affected from $\dot{e}$ and obtains much higher maximum values even for small values of $\dot{e}$ such as $\dot{e} = 0.001\Omega R$ while the domain with negative pressure has an increment in its minimum pressure.

4 EVALUATION OF THE HOMOGENEOUS SOLUTION

The evaluation of $g(x,\theta)$ of Eq. (2.5) is performed in this section. As it was claimed this should be the generic set of solutions for the homogeneous Reynolds equation, namely for Eq. (4.1).

$$\frac{(c_r + e\cos(\theta))^3}{6\mu}\frac{\partial^2 g(x,\theta)}{\partial x^2} - \frac{3(c_r + e\cos(\theta))^2 e\sin(\theta)}{6\mu R^2}\frac{\partial g(x,\theta)}{\partial \theta} + \frac{(c_r + e\cos(\theta))^3}{6\mu R^2}\frac{\partial^2 g(x,\theta)}{\partial \theta^2} = 0 \tag{4.1}$$

We assume that the independent variables of the function $g(x,\theta)$ can be separated in the multiplicative form of Eq. (4.2).

$$g(x,\theta) = f(\theta)m(x) \tag{4.2}$$

By using this assumption in Eq. (4.2) one will arrive at the following outcome of Eq. (4.3).

$$-\frac{m''(x)}{m(x)} = \frac{1}{R^2}\frac{f''(\theta)}{f(\theta)} - \frac{1}{R^2}\frac{3e\sin(\theta)}{(c_r + e\cos(\theta))}\frac{f'(\theta)}{f(\theta)} \tag{4.3}$$

By inspecting the latter equation one realizes that the left hand is a function of x while the right a function of θ only. So, in order this equality to be feasible both sides should be equal to the same constant, set λ. We thus obtain Eqs. (4.4) and (4.5).

$$-\frac{m''(x)}{m(x)} = \lambda \tag{4.4}$$

$$\frac{1}{R^2}\frac{f''(\theta)}{f(\theta)} - \frac{1}{R^2}\frac{3e\sin(\theta)}{(c_r + e\cos(\theta))}\frac{f'(\theta)}{f(\theta)} = \lambda \tag{4.5}$$

The primes denote ordinary derivative with respect to the functions arguments. For treating Eq. (4.4) we distinguish the following cases:

a) If $\lambda = 0$ Eq. (4.4) becomes as in Eq. (4.6) which can be solved to give Eq. (4.7).

$$m''(x) = 0 \tag{4.6}$$

$$m(x) = c_5 x + c_6 \tag{4.7}$$

b) If $\lambda > 0$ then $\lambda = k^2$ and the solution is as in Eq. (4.8).

$$m(x) = c_5\cos(k\,x) + c_6\sin(k\,x) \tag{4.8}$$

c) If $\lambda < 0$ then $\lambda = -k^2$ and one will finally arrive to the following solution as in Eq. (4.9).

$$m(x) = c_5 e^{kx} + c_6 e^{-kx} \tag{4.9}$$

Eq. (4.5) can very easily be solved if $\lambda = 0$ so as to make Eq. (4.5) a first order differential equation (using the corresponding transformation as in Eq. (3.6)) and then to be treated as Eq. (3.7). Such an assumption would yield linear distribution of the pressure through the axial coordinate, since it would be $m(x) = c_5 x + c_6$, thus the case of $\lambda = 0$ is not accepted.

If $\lambda \neq 0$ then the differential equation of Eq. (4.5) has not direct closed form of solution. Vogelpohl in [11] incorporates in detail the treatment of the homogenous Reynolds problem with the assumptions made by Michell [5] and Duffing [15] for the linearization of the fluid film thickness function so as Eq. (4.5) to be solvable. The subsections 4.1 and 4.2 show how the current problem was treated in the past and how it is treated in the current paper.

Two ways are presented in this paper in order to continue with the solution of Eq. (4.9) rewritten as in Eq. (4.10). The first way is to approximate the trigonometric coefficients, introduced by the term $3h'/h$ in Eq. (4.10), with a linear function of first order as Michell [5] and Duffing [15] did so as to obtain a solution using Bessel's functions. This approximation is presented in subsection 4.1 with the use of a linear function for h. The second way is to use the power series method and to give the exact solution of Eq. (4.10) as a sum of infinite series. This way is presented in detail in the subsection 4.2.

$$f''(\theta) + \frac{3h'}{h} f'(\theta) - \lambda R^2 f(\theta) = 0 \tag{4.10}$$

4.1 Linear approximation of the fluid film thickness and use of the Bessel's functions for the definition of $f(\theta)$

Michell in [5] uses the formula $h = \text{const.}\theta$ in order Eq. (4.10) to become as in Eq. (4.11) and its solution to be given using Bessel functions.

$$f''(\theta) + \frac{3}{\theta} f'(\theta) - \lambda R^2 f(\theta) = 0 \tag{4.11}$$

Duffing in [15] used also linear varying fluid film thickness; he used the transformation $f(\theta) = w(\theta)/H$, with $H^2 = h^3/\mu$; and using these transformations the Eq. (4.10) was written as in Eq. (4.12). Since $H^2 = \text{const.}\theta^2 = a^2\theta^2$, this yield $H'' = 0$ and then the eigenfunctions of Eq. (4.12) can be of the form $w_i = \sin(\lambda_i \theta)$.

$$w''(\theta) + \lambda\, w(\theta) = \frac{H''}{H} w(\theta) \tag{4.12}$$

In the current analysis the trigonometric function h is approximated with the linear function h^* that is defined in Eq. (4.13) and shown in Fig. 4 for the set of the values of the Table 1.

$$h^* = (c_r + e) - \frac{2e}{\pi}\theta \tag{4.13}$$

With the use of h^* the solution of Eq. 4.10 is feasible using Bessel functions and the general solution is presented in Eq. (4.14) for both cases of positive and negative value of λ.

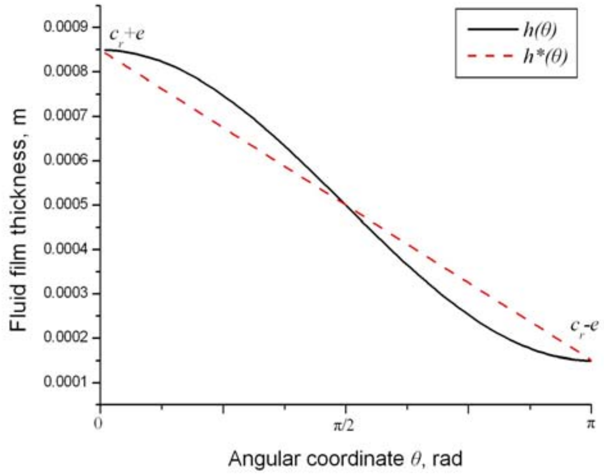

Fig. 4. The trigonometric function of fluid film thickness h and the linear approximation of it h^*, as a function of the angular coordinate θ.

$$f(\theta)=\begin{cases} c_7\dfrac{\mathrm{BesselJ}\left(1,\dfrac{i(c_r+e)k\pi R}{2e}-ikR\theta\right)}{c_r\pi+e\pi-2e\theta}+c_8\dfrac{\mathrm{BesselY}\left(1,\dfrac{i(c_r+e)k\pi R}{2e}-ikR\theta\right)}{c_r\pi+e\pi-2e\theta}, \ \lambda=k^2 \\ -c_7\dfrac{\mathrm{BesselJ}\left(1,\dfrac{(c_r+e)k\pi R}{2e}-kR\theta\right)}{c_r\pi+e\pi-2e\theta}+c_8\dfrac{\mathrm{BesselY}\left(1,\dfrac{-(c_r+e)k\pi R}{2e}+kR\theta\right)}{c_r\pi+e\pi-2e\theta}, \ \lambda=-k^2 \end{cases} \tag{4.14}$$

Since no imaginary solution can be accepted, only the case for $\lambda=-k^2$ is accepted. The solution of Eq. (4.10) given by Eq. (4.15) will form a boundary value problem with results plotted together with those from the exact solution given by the Power Series method, presented in what follows.

$$f(\theta)=-c_7\frac{\mathrm{BesselJ}\left(1,\dfrac{(c_r+e)k\pi R}{2e}-kR\theta\right)}{c_r\pi+e\pi-2e\theta}+c_8\frac{\mathrm{BesselY}\left(1,\dfrac{-(c_r+e)k\pi R}{2e}+kR\theta\right)}{c_r\pi+e\pi-2e\theta} \tag{4.15}$$

4.2 The use of the method of power series for the definition of $f(\theta)$

The method of the Power Series [17] is used in this subsection in order to define a solution for Eq. (4.10). The first step is to convert Eq. (4.10) from a linear ODE with trigonometric coefficients in a linear ODE with polynomial coefficients, thus the transformation $\cos(\theta)=\xi$ is used and we are looking for a function $f(\theta)=\tilde{f}(\xi)$. The derivatives of $\tilde{f}(\xi)$ are defined in Eqs. (4.16) and (4.17).

$$\frac{df}{d\theta}=-\frac{d\tilde{f}}{d\xi}\sin(\theta)=-\tilde{f}'(\xi)\sqrt{1-\xi^2} \tag{4.16}$$

$$\frac{d^2 f}{d\theta^2} = -\frac{d^2 \tilde{f}}{d\xi^2}\sin^2(\theta) - \frac{d\tilde{f}}{d\xi}\cos(\theta) = -\tilde{f}''(\xi)\left(1-\xi^2\right) - \tilde{f}'(\xi)\xi \tag{4.17}$$

By substituting the last two expressions into Eq. (4.10) the following differential equation in Eq. (4.18) is obtained.

$$\left(\alpha\,\xi^3 + \beta\,\xi^2 - \alpha\,\xi - \beta\right)\tilde{f}''(\xi) + \left(4\alpha\,\xi^2 + \beta\,\xi - 3\alpha\right)\tilde{f}'(\xi) + \left(-3\alpha\,\gamma\,\xi - 3\beta\,\gamma\right)\tilde{f}(\xi) = 0 \tag{4.18}$$

The constants α, β, γ are defined as in Eq. (4.19).

$$\alpha = \frac{1}{3}e,\ \beta = \frac{1}{3}c_r,\ \gamma = 2\lambda R^2 \tag{4.19}$$

Study of $\tilde{f}(\xi)$ in the ordinary points

For all the ordinary points of the differential equation in Eq. (4.18) we assume that the solution can be written in the form of Eq. (4.20).

$$\tilde{f}(\xi) = \sum_{n=0}^{\infty} \delta_n \xi^n \tag{4.20}$$

The first and the second derivative of the function $\tilde{f}(\xi)$ are defined in Eq. (4.21) and (4.22) for the correspondingly.

$$\tilde{f}'(\xi) = \sum_{n=1}^{\infty} \delta_n\, n\, \xi^{n-1} \tag{4.21}$$

$$\tilde{f}''(\xi) = \sum_{n=2}^{\infty} \delta_n\, n(n-1)\xi^{n-2} \tag{4.22}$$

By substituting the last three expressions to the differential equation one will find out that the constants that appear in the power series expansion are determined by the following formulas as in Eq. (4.23).

$$\begin{gathered}\left[\left(\alpha\, n^2 + \alpha\, n - (2+3\gamma)\alpha\right)\delta_{n-1} + \left(\beta\, n^2\right)\delta_n - \left(\alpha\, n^2 + 4\alpha\, n + 3\alpha\right)\delta_{n+1} - \left(\beta\, n^2 + 3n\beta + 2\beta\right)\delta_{n+2}\right]\left(\sum_{n=0}^{\infty}\xi^n\right) + \left(2\alpha + 3\alpha\gamma\right)\delta_{-1} = 0 \\ \delta_{-1} = 0 \\ \left(\alpha\, n^2 + \alpha\, n - (2+3\gamma)\alpha\right)\delta_{n-1} + \left(\beta\, n^2\right)\delta_n - \left(\alpha\, n^2 + 4\alpha\, n + 3\alpha\right)\delta_{n+1} - \left(\beta\, n^2 + 3n\beta + 2\beta\right)\delta_{n+2} = 0\end{gathered} \tag{4.23}$$

Study of $\tilde{f}(\xi)$ in the singular points

The differential equation of Eq. (4.18) is now written in the form of Eq. (4.24).

$$\tilde{f}''(\xi)(\xi-1)(\xi+1)(\alpha\xi+\beta) + \tilde{f}'(\xi)\left(x(\alpha\xi+\beta) + 3\alpha(\xi-1)(\xi+1)\right) + \tilde{f}(\xi)\left(-3d(\alpha\xi+\beta)\right) = 0 \tag{4.24}$$

The term multiplied with the higher order derivative has the following roots as in Eq. (4.25).

$$\xi = 1, \xi = -1, \xi = -\frac{\beta}{\alpha} \qquad (4.25)$$

The first corresponds to the value $\theta = 0$, while the second to the value $\theta = \pi$. The third one is not accepted because it yields $\cos(\theta) < -1$. So, we have two singular points, in the values $\xi = \pm 1$. For them, a separate analysis has to be performed. For the singular point $\xi = 1$ we have that it is a normal singular point since analytic functions $A_0(\xi), A_1(\xi)$ exist such that Eq. (4.26) to be satisfied.

$$\begin{aligned} &(\xi - 1)\left(x(a\xi + \beta) + 3a(\xi - 1)(\xi + 1)\right) = (\xi - 1)(\xi + 1)(a\xi + \beta) A_1(\xi) \\ &(\xi - 1)^2 \left(-3\gamma(a\xi + \beta)\right) = (\xi - 1)(\xi + 1)(a\xi + \beta) A_0(\xi) \end{aligned} \qquad (4.26)$$

The series expansions of the functions $A_0(\xi), A_1(\xi)$ are as in Eq. (4.27).

$$\begin{aligned} &A_1(\xi) = -\sum_{n=0}^{\infty}\left(\frac{-\xi + 1}{2}\right)^{n+1} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-\xi + 1}{2}\right)^{n+1} - 3\sum_{n=0}^{\infty}\left(\frac{a}{a + \beta}(-\xi + 1)\right)^{n+1} \\ &A_0(\xi) = 3\gamma\sum_{n=1}^{\infty}\left(\frac{-\xi + 1}{2}\right)^{n+1} \end{aligned} \qquad (4.27)$$

So, the indicative equation of this normal singular point is as in Eq. (4.28) with the two roots as in Eq. (4.29).

$$p(\lambda) = \lambda^2 - \frac{1}{2}\lambda \qquad (4.28)$$

$$\lambda_1 = 0, \lambda_2 = \frac{1}{2} \qquad (4.29)$$

For this singular point the solutions will be of the form of Eq. (4.30).

$$\begin{aligned} &\tilde{f}_1(\xi) = \sum_{n=0}^{\infty}\delta_n(\xi - 1)^n \\ &\tilde{f}_2(\xi) = (\xi - 1)^{\frac{1}{2}}\sum_{n=0}^{\infty}\varepsilon_n(\xi - 1)^n \end{aligned} \qquad (4.30)$$

After a similar procedure as the one described for the ordinary points, one will be led to the following formulas of Eq. (4.31), for the calculation of the constants of the above power series in Eq. (4.30).

$$\begin{aligned} &\delta_{-1} = 0 \\ &\left(-1 + n + a\left(-2\gamma + (n-1)^2\right)\right)\delta_{n-1} + \left(n(1 + \beta n) - a\left(3\gamma + n - 3n^2\right)\right)\delta_n + (1 + 2n)(a + b + an)\delta_{n+1} = 0 \end{aligned} \qquad (4.31)$$

For the coefficients of the other independent solution one will be led to the following formulas of Eq. (4.32).

$$\begin{aligned} &\varepsilon_{-1} = 0 \\ &a\left(\frac{13}{4} + 3\gamma - 6n + n^2\right)\varepsilon_{n-1} + \left(\frac{1}{4} - 13a - \beta - 12a\gamma - 12\beta\gamma - (32a + 4\beta)n + (12a + 4\beta)n^2\right)\varepsilon_n - (a + \beta)\left(\frac{3}{2} + n\right)\varepsilon_{n+1} \end{aligned} \qquad (4.32)$$

For the normal singular point $\xi = -1$ we have the following independent solutions of Eqs. (4.33) and (4.34).

$$\tilde{f}_1(\xi) = \sum_{n=0}^{\infty} \delta_n (\xi+1)^n \tag{4.33}$$

$$\tilde{f}_2(\xi) = |\xi+1|^{\frac{3}{2}} \sum_{n=0}^{\infty} \varepsilon_n (\xi+1)^n \tag{4.34}$$

For the definition of the constants in Eqs. (4.33) and (4.34), the formulas of Eqs. (4.35) and (4.36) are used correspondingly.

$$\begin{gathered} \delta_{-1} = 0 \\ \left(-1+n+a\left(3-3\gamma-3n+n^2\right)\right)\delta_{n-1} + \left(-3a\gamma^2+\beta n^2+a\left(3\beta\gamma+n-3n^2\right)\right)\delta_n + (1+n)(-2+a-\beta+an+2\beta n)\delta_{n+1} = 0 \end{gathered} \tag{4.35}$$

$$\begin{gathered} \varepsilon_0 = \varepsilon_{-1} = 0 \\ \left(\frac{1}{2}-3a\gamma+n+a\left(\frac{1}{2}+n\right)\right)\varepsilon_{n-1} + \left(a\left(-\frac{11}{2}+3\gamma-8n-2n^2\right)+\beta\left(\frac{9}{4}-3\gamma+3n+n^2\right)\right)\varepsilon_n - \\ \left(\frac{1}{4}(5+2n)(4+a+2an+4\beta(2+n))\right)\varepsilon_{n+1} = 0 \end{gathered} \tag{4.36}$$

5 APPLICATION OF THE BOUNDARY CONDITIONS AND PRESSURE EVALUATION

Two boundary value problems are treated in this section. The first has to do with the distribution of the pressure in the axial direction of the bearing $m(x)$ and the second with the distribution in the circumferential direction $f(\theta)$. The boundary value problem for $m(x)$ is defined is defined from the ODE in Eq. (5.1) and the boundary conditions of Eqs. (5.2) and (5.3).

$$m''(x) - k^2 m(x) = 0 \tag{5.1}$$

$$m\left(\frac{L_b}{2}\right) = 1 \tag{5.2}$$

$$m\left(-\frac{L_b}{2}\right) = 1 \tag{5.3}$$

The explanation for the boundary conditions in Eqs. (5.2) and (5.3) is that the resulting pressure at both ends of the bearing $P(\pm L_b/2,\theta)$ has to be equal to zero (the atmospheric pressure is not incorporated) and thus $u(\pm L_b/2,\theta) = -g(\pm L_b/2,\theta)$, that means $\varphi(\theta) = -m(\pm L_b/2)\, f(\theta)$.

The general solution of Eq. (5.1) is given in Eq. (5.4). The general solution in Eq. (5.1) is substituted in the boundary conditions of Eqs. (5.2) and (5.3) and one can define the constants c_5 and c_6 and obtain the solution as in Eq. (5.4). The eigenfunctions $m_n(x)$ are written as in Eq. (5.5) with k_n to be the eigenvalues of the problem of Eqs. (5.1 – 5.3).

$$m(x) = \frac{e^{\frac{L_b k}{2}}}{1+e^{L_b k}} e^{kx} + \frac{e^{\frac{L_b k}{2}}}{1+e^{L_b k}} e^{-kx} \tag{5.4}$$

$$m_n\left(x\right)=\frac{e^{\frac{L_b k_n}{2}}}{1+e^{L_b k_n}}e^{k_n x}+\frac{e^{\frac{L_b k_n}{2}}}{1+e^{L_b k_n}}e^{-k_n x},$$
$$k_n=\frac{n\pi}{L_b}, n=1,2,... \tag{5.5}$$

The boundary value problem for $f\left(\theta\right)$ is treated in the two following subsections 5.1 and 5.2 for the cases that $f\left(\theta\right)$ is defined using Bessel functions (Section 4.1) and Power Series (Section 4.2) correspondingly.

5.1 Boundary value problem for $f\left(\theta\right)$ using Bessel Functions

The boundary value problem is consisted from the ODE in Eq. (5.6) and of the boundary conditions of Eqs. (5.7) and (5.8).

$$f''\left(\theta\right)+\frac{3h^{*\prime}}{h^*}f'\left(\theta\right)+k^2R^2f\left(\theta\right)=0 \tag{5.6}$$

$$f\left(0\right)=0 \tag{5.7}$$

$$f\left(\pi\right)=0 \tag{5.8}$$

The explanation about the boundary conditions in Eqs. (5.7) and (5.8) is that in the beginning of the oil film ($\theta=0$) the pressure $P\left(x,0\right)$ has to be zero and thus $\varphi\left(0\right)=-m\left(x\right)f\left(0\right)$. Since $\varphi\left(0\right)$ is chosen to be zero (see section of the particular solution) then $f\left(0\right)=0$. Also, since $f\left(\theta\right)$ is the homogenous solution there is no influence from $\dot{e}$ (this means $\theta^*=\pi$) and the pressure yielded from the homogeneous problem becomes also zero at $\theta=\theta^*=\pi$.

The general solution of Eq. (5.6) is given in Eq. (5.9). The substitution of the general solution in the boundary conditions gives the system of equations in Eq. (5.10).

$$f\left(\theta\right)=-c_7\frac{\mathrm{BesselJ}\left(1,\frac{\left(c_r+e\right)k\pi R}{2e}-kR\theta\right)}{c_r\pi+e\pi-2e\theta}+c_8\frac{\mathrm{BesselY}\left(1,\frac{-\left(c_r+e\right)k\pi R}{2e}+kR\theta\right)}{c_r\pi+e\pi-2e\theta} \tag{5.9}$$

$$\begin{cases} -c_7\dfrac{\mathrm{BesselJ}\left(1,\dfrac{\left(c_r+e\right)k\pi R}{2e}\right)}{c_r\pi+e\pi}+c_8\dfrac{\mathrm{BesselY}\left(1,\dfrac{-\left(c_r+e\right)k\pi R}{2e}\right)}{c_r\pi+e\pi}=0 \\ c_7\dfrac{\mathrm{BesselJ}\left(1,\dfrac{-\left(c_r+e\right)k\pi R}{2e}-kR\pi\right)}{c_r\pi-e\pi}+c_8\dfrac{\mathrm{BesselY}\left(1,\dfrac{-\left(c_r+e\right)k\pi R}{2e}+kR\pi\right)}{c_r\pi-e\pi}=0 \end{cases} \tag{5.10}$$

The constants c_7 and c_8 can be determined as the solution of the system of Eq. (5.10) as in Eq. (5.11).

$$c_7 = -\frac{\mathrm{BesselY}\left(1, \frac{-(c_r+e)k\pi R}{2e}\right)}{\mathrm{BesselJ}\left(1, \frac{(c_r+e)k\pi R}{2e}\right)} \tag{5.11}$$

$$c_8 = 1$$

The characteristic equation of the system in Eq. (5.10) is given in Eq. (5.12) and is plotted in Fig. 5 as a function of k for the set of values of Table 1. The eigenvalues κ_n are determined as the roots of it and for the current set of values of Table 1 they are very well approximated by the formula $\kappa_n = 20n$. The eigenfunctions $f_n(\theta)$ are given in Eq. (5.13) together with the eigenvalues κ_n for the current set of values of Table 1.

$$\begin{vmatrix} -\frac{\mathrm{BesselJ}\left(1, \frac{(c_r+e)k\pi R}{2e}\right)}{c_r\pi + e\pi} & \frac{\mathrm{BesselY}\left(1, \frac{-(c_r+e)k\pi R}{2e}\right)}{c_r\pi + e\pi} \\ \frac{\mathrm{BesselJ}\left(1, \frac{-(c_r+e)k\pi R}{2e} - kR\pi\right)}{c_r\pi - e\pi} & \frac{\mathrm{BesselY}\left(1, \frac{-(c_r+e)k\pi R}{2e} + kR\pi\right)}{c_r\pi - e\pi} \end{vmatrix} = 0 \tag{5.12}$$

$$f_n(\theta) = -c_7 \frac{\mathrm{BesselJ}\left(1, \frac{(c_r+e)\kappa_n\pi R}{2e} - \kappa_n R\theta\right)}{c_r\pi + e\pi - 2e\theta} + c_8 \frac{\mathrm{BesselY}\left(1, \frac{-(c_r+e)\kappa_n\pi R}{2e} + \kappa_n R\theta\right)}{c_r\pi + e\pi - 2e\theta} \tag{5.13}$$

$$\kappa_n = 20n, n = 1, 2, \ldots$$

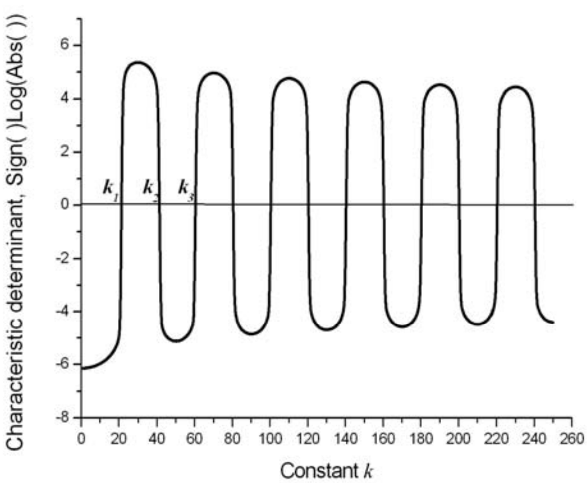

Fig. 5. Values of the characteristic determinant of the boundary value problem for $f(\theta)$ as a function of k, for the case of use of Bessel functions

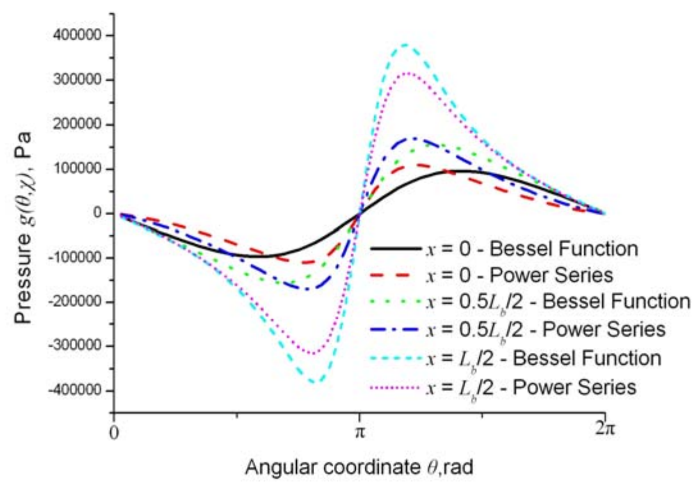

Fig. 6. The pressure distribution along the angular coordinate θ given from the homo0genous solution $g(\theta, x)$ for variable values of axial coordinate x and for the two different solutions using Bessel functions and Power Series method.

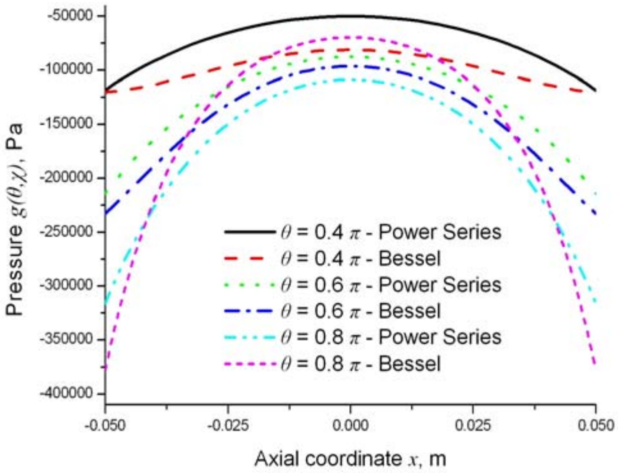

Fig. 7. The pressure distribution along the axial coordinate x given from the homogenous solution $g(\theta, x)$ for variable values of angular coordinate θ.

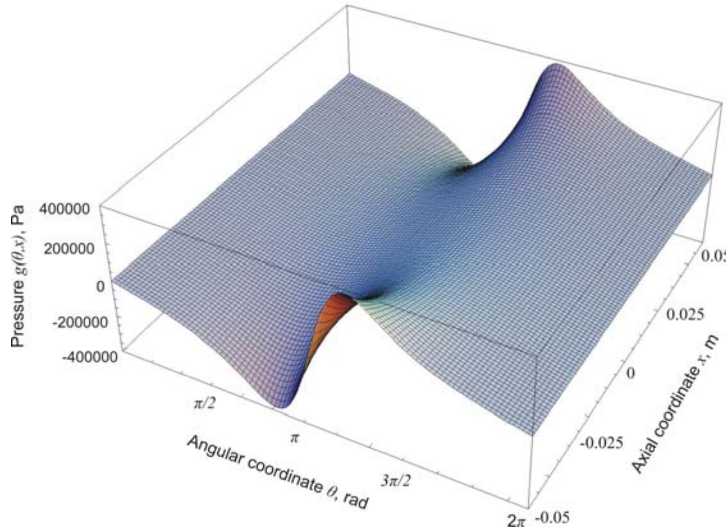

Fig. 8. The pressure distribution $g(\theta, x)$ given from the homogenous solution using the Power Series Method as a function of angular and axial coordinates.

5.2 Boundary value problem for $\tilde{f}(\theta)$ using Power Series

The current boundary value problem is consisted from the ODE in Eq. (5.14) and of the boundary conditions of Eqs. (5.15) and (5.16).

$$\left(\alpha\,\xi^3+\beta\,\xi^2-\alpha\,\xi-\beta\right)\tilde{f}''(\xi)+\left(4\alpha\,\xi^2+\beta\,\xi-3\alpha\right)\tilde{f}'(\xi)+\left(-3\alpha\,\gamma\,\xi-3\beta\,\gamma\right)\tilde{f}(\xi)=0 \tag{5.14}$$

$$\tilde{f}(-1)=0 \tag{5.15}$$

$$\tilde{f}(1)=0 \tag{5.16}$$

The explanation of the current boundary conditions corresponds to this of Eqs. (5.7) and (5.8). Using the solution of Eq. (4.20) and the eigenvalues $k_n=\dfrac{n\,\pi}{L_b}, n=1,2,...$ the eigenfunctions $\tilde{f}_n\left(\cos(\theta)\right)$ are defined using the iterative formula Eq. (4.23) and the two boundary conditions for every $n=1,2,...$. For the current evaluation a total number of $n=100$ was used. The analysis around the singular points is used also in the evaluation of the eigenfunctions $\tilde{f}_n\left(\cos(\theta)\right)$ when the higher eigenvalues are incorporated.

The normalized eigenfunctions of both problems (for $m(x)$ and $f(\theta)$) are given in Eqs. (5.17) and (5.18) with the constants α_n or $\tilde{\alpha}_n$ and β_n to be defined in Eqs. (5.19) and (5.20) correspondingly [17].

$$\overline{f}_n(\theta)=\alpha_n\,f_n(\theta) \;\text{ or }\; \overline{\tilde{f}}_n(\theta)=\tilde{\alpha}_n\,\tilde{f}_n(\theta) \tag{5.17}$$

$$\bar{m}_n(\theta)=\beta_n m_n(\theta) \tag{5.18}$$

$$\int_0^{\pi}\left(\alpha_n f_n(\theta)\right)^2 d\theta=1, n=1,2,.... \text{ or } \int_0^{\pi}\left(\tilde{\alpha}_n \tilde{f}_n(\theta)\right)^2 d\theta=1, n=1,2,.... \tag{5.19}$$

$$\int_0^{\pi}\left(\beta_n m_n(\theta)\right)^2 d\theta=1, n=1,2,.... \tag{5.20}$$

5.3 Evaluation of the resulting pressure

The solution of $g(x,\theta)$ can now be written as in Eq. (5.21). The constants δ_n are defined in Eq. (5.23) using the boundary condition for zero pressure at the ends of the bearing which yields $\varphi(\theta)=-m(\pm L_b/2)\, f(\theta)$ or since $m(\pm L_b/2)=1$, $f(\theta)=-\varphi(\theta)$ see Eq. (5.22).

$$g(x,\theta)=\sum_{n=1}^{\infty}\left(\delta_n \bar{f}_n \bar{m}_n\right) \text{ or } \tilde{g}(x,\theta)=\sum_{n=1}^{\infty}\left(\tilde{\delta}_n \bar{\bar{f}}_n \bar{m}_n\right) \tag{5.21}$$

$$\delta_1 \bar{f}_1(\theta)\bar{m}_1\left(\frac{L_b}{2}\right)+\delta_2 \bar{f}_2(\theta)\bar{m}_2\left(\frac{L_b}{2}\right)+\delta_3 \bar{f}_3(\theta)\bar{m}_3\left(\frac{L_b}{2}\right)+....=-\varphi(\theta) \text{ for } f \text{ or } \tilde{f} \text{ and } \delta \text{ or } \tilde{\delta}. \tag{5.22}$$

$$\delta_n=\int_0^{2\pi}\left(-\varphi(\theta)\,\bar{f}_n(\theta)\bar{m}_n\left(\frac{L_b}{2}\right)\right)d\theta \text{ or } \tilde{\delta}_n=\int_0^{2\pi}\left(-\varphi(\theta)\,\bar{\bar{f}}_n(\theta)\bar{m}_n\left(\frac{L_b}{2}\right)\right)d\theta \tag{5.23}$$

The pressure distribution defined in Eq. (5.21) is plotted in Figs. 6-8 for both cases of the eigenfunctions $\bar{f}_n(\theta)$ (Bessel functions) and $\bar{\bar{f}}_n(\theta)$ (Power Series method). As it can be seen in Figs. 6 and 7 the Bessel functions yield differences in the pressure distribution along both the axial and circumferential coordinate. The two distributions have similar maximum and minimum pressure value but in different locations of θ. The differences in the pressure distribution are more intense in the domain around the axial centre of the bearing ($x=0$) but both distributions tent to be equal near the axial ends of the bearing ($x=\pm\frac{L_b}{2}$) and this is because in these locations both distributions are forced to tent to $\varphi(\theta)$.

The resulting fluid film pressure distribution is then defined explicitly as in Eq. (5.24) and is plotted in Figs. 9-12 through angular and axial coordinate, together with numerical results obtained during the current work from the solution of Eq. (1.1) using the Finite Differences Method.

$$\begin{aligned} P(x,\theta)&=\varphi(\theta)+\psi(x)+\sum_{n=1}^{\infty}\left(\delta_n \bar{f}_n(\theta)\bar{m}_n(x)\right) \text{ (approximate analytical)}\\ \tilde{P}(x,\theta)&=\varphi(\theta)+\psi(x)+\sum_{n=1}^{\infty}\left(\tilde{\delta}_n \bar{\bar{f}}_n(\theta)\bar{m}_n(x)\right) \text{ (exact analytical)} \end{aligned} \tag{5.24}$$

In Figs. 9-11 it can be seen that the resultant pressure distribution presents differences comparing the numerical solution with the approximate analytical solution (using Bessel functions) and the exact analytical solution (using the Power Series method) especially in the values of maximum and minimum pressure. The divergence in the value of the maximum pressure becomes more intensive when the parameter of eccentricity velocity obtains higher values (see Fig. 11) while the minimum pressure value is not affected as much as this of the maximum.

The resulting pressure distribution through the axial coordinate is plotted in Fig. 12 for the three cases of evaluation. The differences are becoming more intensive for values of θ where the pressure tents to maximize (eg. $\theta = 0.8\pi$) while in the other domains of θ the pressure diverge less. What those differences mean for the resulting impedance forces of the fluid film in the journal is a concept that is left to be discussed in future work.

A three dimensional plot for the resulting pressure distribution is given in Fig. 13. Having also a look in Figs. 3 and 8 one can realize how their distributions contribute in the resulting pressure of Fig. 13. Using Eq. (5.24) the values for the angles of the maximum and the minimum pressure can be calculated by finding the roots of Eq. (5.25), while the roots of Eq. (5.26) are the angles of zero pressure. The roots of Eqs. (5.25) and (5.26) are evaluated numerically and they are plotted in Fig. 14 as function of the parameter $\dot{e}$ for the three cases of study. The three cases of evaluation have differences in where the pressure obtains the maximum and the zero pressure especially as $\dot{e}$ takes higher values.

$$\frac{d}{d\theta}\left(\varphi(\theta)+\psi(0)+\sum_{n=1}^{\infty}\left(\tilde{\delta}_n \bar{\tilde{f}}_n(\theta)\bar{m}_n(0)\right)\right)=0 \tag{5.25}$$

$$\varphi(\theta)+\psi(0)+\sum_{n=1}^{\infty}\left(\tilde{\delta}_n \bar{\tilde{f}}_n(\theta)\bar{m}_n(0)\right)=0 \tag{5.26}$$

6 CONCLUDING REMARKS, CONTRIBUTION AND FUTURE CONCEPTS

The current work gives the path of obtaining an exact analytical solution of the Reynolds equation for the lubrication of journal bearings with finite length. The contribution of the current work can be described with the following comments which highlight the differences to the solutions given in the past.

a) A set of particular solutions of the Reynolds equation is given with a sub case of them to correspond to the pressure distribution of the infinitively long bearing without the demand of excluding any partial derivative in Reynolds equation. This means that a particular solution that describes either the infinitely long or the infinitely short bearing can be defined regarding the boundary conditions that are applied in the function of the particular solution for the finite bearing.
b) The homogeneous Reynolds equation is treated without the use of any approximating function for the fluid film thickness and the use of Power Series yield a closed form solution for the homogeneous Reynolds Equation for the finite bearing.
c) The analytical expression of the resulting pressure as a function of eccentricity rate of change $\dot{e}$ gives the ability to define a formula that together with the initial condition for the pressure distribution in the circumferential beginning of the film lets the angle of zero pressure to be self defined. An additional benefit is that there exist closed form formulas for the definition of the angles of maximum and minimum pressure of the lubricant in the finite bearing.

A study of the pressure distribution under the approximation of the fluid film thickness function with a linear function was motivated from the literature in order to show the differences between the approximate analytical solution using Bessel functions and the exact analytical solution using the method of power series. The current work does not intent to estimate the importance of the differences in the pressure distribution between the methods of evaluation (approximating analytical or numerical) because there are cases of operational conditions that these differences are minimized and cases that can yield higher divergence between the theories.

Two linear ordinary differential equations of 2nd order and with trigonometric coefficients were solved in this work. They were extracted and formed under the proper assumptions of splitting the solutions and the independent variables. The trigonometric coefficients that were introduced by the function for the fluid film thickness introduced the great difficulty in solving the Reynolds equation. Since the one of them, which corresponds to the particular solution, has direct form of solution whatever the thickness function is, and the other, which corresponds to the homogenous solution, can be confronted using the power series method, a new concept comes up and has to do with the introduction of a different function for the fluid film thickness that corresponds to a worn journal bearing.

This part of the work for the analytical solution of the Reynolds equation was planned to give the formulas of the pressure distribution of the lubricant in a finite journal bearing. These formulas happen to have functions that can be integrated in space domain with closed form expressions so as to give the resulting impedance forces of the lubricant to the bearing. This fact gives the ability to define the closed form expressions for the stiffness and damping coefficients of the lubricant film and this is a concept to be made in the future work together with the definition of the formulas for the operational parameters such as the eccentricity and the attitude angle as a function of Sommerfeld number.

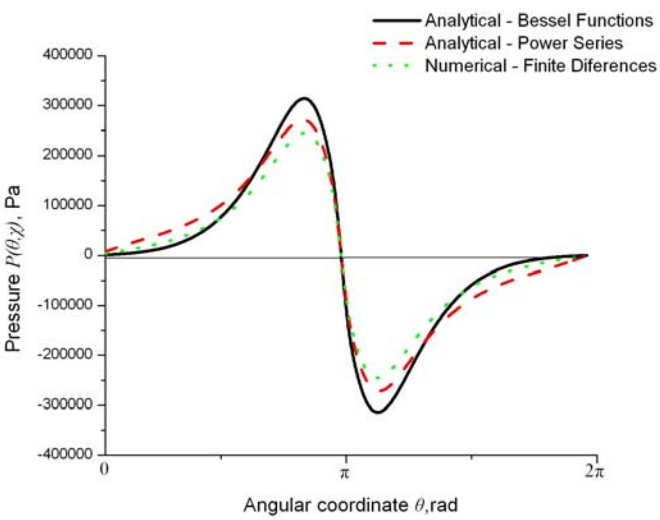

Fig. 9. The pressure distribution $P(\theta, x)$ along the angular coordinate θ in the axial centre of the bearing ($x=0$) for $\dot{e}=0$

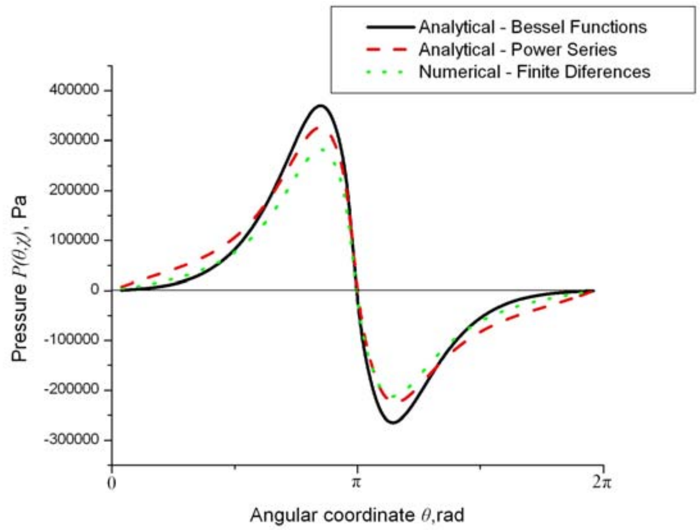

Fig. 10. The pressure distribution $P(\theta, x)$ along the angular coordinate θ in the axial centre of the bearing ($x=0$) for $\dot{e}=0.0002\Omega R$

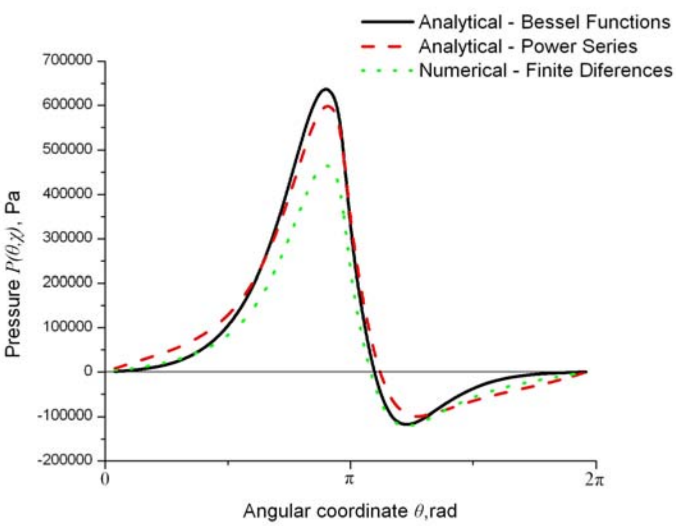

Figure 11. The pressure distribution $P(\theta, x)$ along the angular coordinate θ in the axial centre of the bearing ($x=0$) for $\dot{e}=0.001\Omega R$

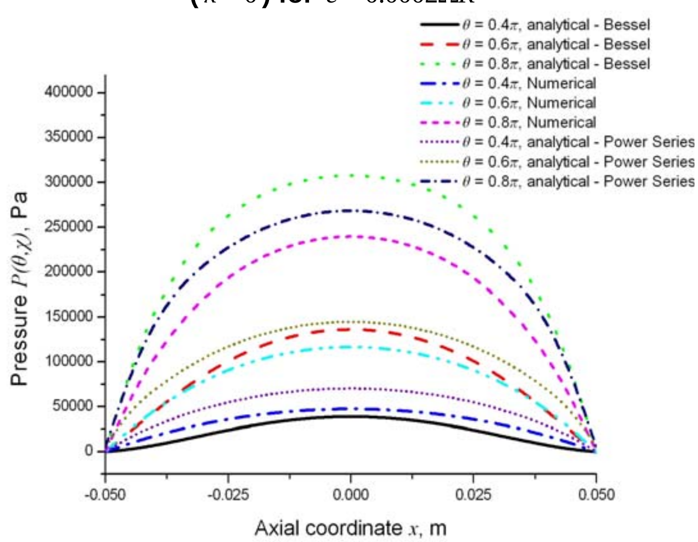

Fig. 12. The pressure distribution $P(\theta, x)$ along the axial coordinate x for variable values of angular coordinate θ.

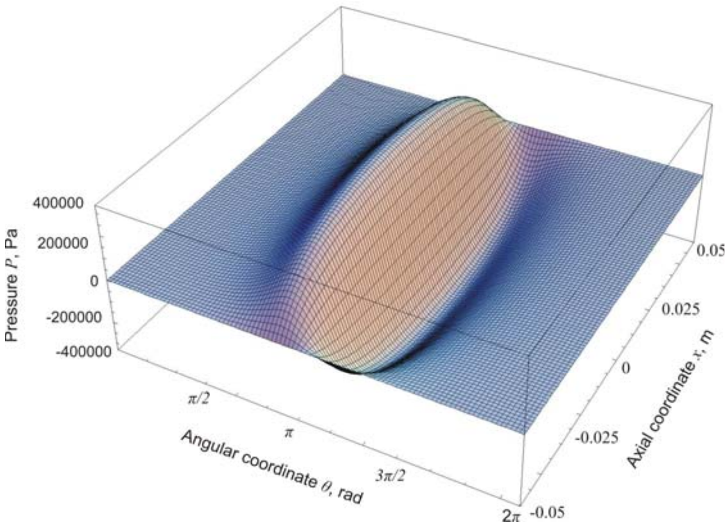

Fig. 13. The pressure distribution $P(\theta, x)$ using the power series method, as a function of the angular and the axial coordinate (in the case of $\dot{e} = 0$)

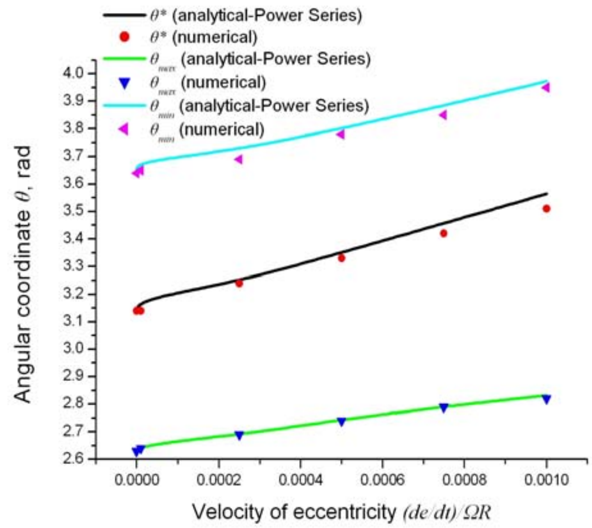

Fig. 14. Values of the angles of the maximum pressure $\theta_{\max}$, of the minimum pressure $\theta_{\min}$, and of the zero pressure θ^{*} as a function of eccentricity velocity yielded from the analytical solution using Power Series method and the numerical solution with finite differences.

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