On Rates of Convergence in Metric Fixed Point Theory

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Master i matematikk Eyvind Martol Briseid
aus Oslo, Norwegen

Referent:	Prof. Dr. U. Kohlenbach
1. Koreferent:	Prof. Dr. W.A. Kirk
2. Koreferent:	Prof. Dr. T. Streicher
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Abstract

This thesis investigates some effective and quantitative aspects of metric fixed point theory in the light of methods from proof theory. The thesis consists of contributions to the program of proof mining, as developed by Kohlenbach and various collaborators since the early 1990s (but with roots back to Kreisel's program "unwinding of proofs" from the 1950s). The contributions involve both case studies – studying given prima facie ineffective proofs of certain fixed point theorems to extract "hidden" effective information like explicit bounds and rates of convergence for iteration sequences, and also developing further the use of the logical machinery involved. The main theoretical tools involve Gödel's functional ("Dialectica") interpretation combined with negative translation and a variant of Howard's majorizability relation – and specifically the logical metatheorems of Kohlenbach and Gerhardy, where the reach of these techniques is extended to formal systems for analysis with various abstract spaces added as new ground types.

The main contributions of the thesis are twofold:

(1) We construct explicit and effective full rates of convergence for the Picard iteration sequences for two classes of selfmaps on metric spaces. One of these are Kirk's asymptotic contractions, and as a byproduct of the logical analysis we obtain a string of results concerning this class of mappings, including a characterization on nonempty, bounded, complete metric spaces as exactly the mappings for which there exists a point to which all Picard iteration sequences converge with a rate of convergence which is uniform in the starting point. This shows that in the setting of bounded metric spaces the asymptotic contractions in the sense of Kirk in some sense are the most general mappings which still exhibit convergence of the Picard iteration sequences of "Banach type" – to the same point and with strong uniformity with respect to the starting point.

The other class of mappings for which we construct explicit rates of convergence are the so-called uniformly continuous uniformly generalized p-contractive mappings. Logical analysis of the concepts involved – using monotone functional interpretation – allows us to develop an extension of a related fixed point theorem from the case where the space is compact to arbitrary metric spaces. This is possible because monotone functional interpretation automatically leads us to consider the "right" uniform ver-

sion of the corresponding contractive type condition – whereas in the proof of the original theorem the compactness of the space "secretly" upgrades the generalized contractive condition in question to this uniform version. Also in this case we were able to give an effective and highly uniform rate of convergence for the Picard iteration sequences, and by the uniformity features of the resulting rate of convergence it follows that the mappings under consideration are asymptotic contractions in the sense of Kirk.

(2) We develop a method for finding, under general conditions, explicit and highly uniform rates of convergence for the Picard iteration sequences for selfmaps on bounded metric spaces from ineffective proofs of convergence to a unique fixed point. We are able to extract full rates of convergence by extending the use of a logical metatheorem due to Kohlenbach. Our novel method provides an explanation in logical terms for the fact that we in the case studies mentioned above could find such explicit rates of convergence. This amounts, loosely speaking, to general conditions under which we in this specific setting can transform a $\forall\exists\forall$-sentence into a $\forall\exists$-sentence via an argument involving product spaces. This reduction in logical complexity allows us to use the existing machinery to extract quantitative bounds of the sort we need.

Deutsche Zusammenfassung

Diese Dissertation untersucht effektive und quantitative Aspekte metrischer Fixpunkttheorie mit Hilfe von Methoden der Beweistheorie. Sie besteht aus Beiträgen zum "proof mining"-Programm, entwickelt von Kohlenbach und anderen seit Anfang der 1990er Jahre, welches seinerseits seine Ursprüngen in Kreisels "unwinding of proofs"-Programm aus den 1950er Jahren hat. Wir untersuchen prima facie ineffektive Beweise bestimmter Fixpunkttheoreme, um ihnen "versteckte" effektive Informationen, wie zum Beispiel explizite Schranken und Konvergenzraten für Iterationsfolgen, zu entnehmen. Darüber hinaus entwickeln wir die Anwendung der logischen Methoden weiter. Die wichtigsten theoretischen Methoden umfassen Gödels Funktionalinterpretation ("Dialectica") kombiniert mit Negativübersetzung und einer Variante von Howards Majorisierbarkeit, sowie logische Metatheoreme von Kohlenbach und Gerhardy. Diese erweitern die Anwendung der zuerst genannten Techniken auf formale Systeme der Analysis, die verschiedene abstrakte Räume als neu hinzugefügte Grundtypen besitzen.

Die zwei wichtigsten Beiträge sind die folgenden:

(1) Wir konstruieren explizite und effektive Konvergenzraten für die Picard-Iterationsfolgen von zwei Klassen von Selbstabbildungen auf metrischen Räumen. Die eine Klasse sind Kirks asymptotische Kontraktionen. Als Konsequenz der logischen Analyse erhalten wir außerdem eine Reihe qualitative Ergebnisse bezüglich dieser Klasse von Abbildungen. Insbesondere beweisen wir eine Charakterisierung der Klasse der asymptotischen Kontraktionen im Sinne von Kirk für den Fall nichtleerer beschränkter, vollständiger metrischer Räume als genau denjenigen Abbildungen, für welche es einen Punkt gibt, gegen den alle Picard-Iterationsfolgen mit einer Konvergenzrate konvergieren, die gleichmäßig bezüglich des Startpunkts ist. Dies zeigt, dass im Falle von beschränkten metrischen Räumen die asymptotischen Kontraktionen im Sinne von Kirk in gewissem Sinne die allgemeinsten Abbildungen sind, die noch eine Konvergenz der Picard-Iterationsfolgen vom "Banach-Typ" aufweisen, das heißt Konvergenz gegen einen einzelnen Punkt und mit starker Gleichmäßigkeit bezüglich des Startpunktes.

    Die andere Klasse von Abbildungen, für die wir explizite Konvergenzraten konstruieren, sind die sogenannten gleichmäßig stetigen gleichmäßig verallgemeinert p-kontraktiven Abbildungen. Es gelingt uns, ein verwandtes Fixpunkttheorem zu erweitern, bei dem wir nicht länger die Kompaktheit des Raumes (X, d) fordern. Aus den Gleichmäßigkeitseigenschaften der Konvergenzrate folgt, dass diese Abbildungen asymptotische Kontraktionen im Sinne von Kirk sind.

(2) Wir entwickeln Methoden, um unter allgemeinen Bedingungen explizite und stark gleichmäßige Konvergenzraten für die Picard-Iterationsfolgen von Selbstabbildungen auf beschränkten metrischen Räumen aus ineffektiven Beweisen von Konvergenz gegen einen eindeutigen Fixpunkt zu entnehmen. Wir können volle Konvergenzraten extrahieren, indem wir die Anwendung eines logischen Metatheorems von Kohlenbach erweitern. Unsere neuartige Methode liefert eine metamathematische Erklärung für die Tatsache, dass wir in den oben erwähnten Fallstudien solche expliziten Konvergenzraten finden konnten. Dies kommt allgemeinen Bedingungen gleich, unter denen wir in bestimmten Zusammenhängen $\forall\exists\forall$-Sätze mit Hilfe eines Arguments über Produkträume zu $\forall\exists$-Sätzen umformen können. Diese Vereinfachung der logischen Komplexität erlaubt es uns, die vorhandenen Methoden zu nutzen, um quantitative Schranken, wie wir sie brauchen, zu bestimmen.

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Chapter 1

Introduction

This thesis develops further the uses of proof mining in metric fixed point theory, and investigates some effective and quantitative aspects of metric fixed point theory with the help of proof mining. “Proof mining” is a label assigned to a general project of applying methods from that part of mathematical logic known as proof theory to core (or “ordinary”) mathematics, and we will give a general description of this program below.

The main contributions of this thesis can be divided into two parts: In Chapter 3 and Chapter 4 one will find a study of asymptotic contractions in the sense of Kirk and related classes of mappings, where we construct explicit and highly uniform rates of convergence for the Picard iteration sequences; and in Chapter 2 we investigate how these results can be explained in logical terms via a new method for (under general conditions) finding computable and highly uniform rates of convergence for Picard iteration sequences for selfmaps on bounded metric spaces from ineffective proofs of convergence to a unique fixed point. The latter amounts, loosely speaking, to general conditions under which we in this specific setting can transform a $\forall\exists\forall$-sentence into a $\forall\exists$-sentence via an argument involving product spaces. This reduction in logical complexity allows us to use the existing machinery to extract the quantitative bounds we need.

In this chapter we will discuss the context of the work, including both the program of proof mining in general and the relevant aspects of metric fixed point theory.

1.1 Proof mining

“Proof mining” refers to the logical analysis of given mathematical proofs with the help of tools and insights from that part of mathematical logic known as proof theory, with the aim of obtaining relevant information “hidden” in the proofs. This new information can be quantitative or numerical – in the sense that one obtains e.g. explicit bounds or rates of convergence, but it can also yield qualitative improvements of the original theorem through showing that the bounds are uniform with respect to certain parameters, or through weakening of the premises of the theorem. Kohlenbach's recent book [101] provides a wealth of information on the various aspects of proof mining, and among other things the relevant techniques used are described in detail.

Loosely speaking the general structure of proof mining is as follows: Suppose one has a proof $\mathcal{P}$ of a theorem A (of a certain restricted logical form). One then applies an algorithm provided by a logical metatheorem from proof theory to get a new proof $\mathcal{P}'$ of a *stronger* theorem A'. However, strictly speaking this is only possible in the rare situation where the proof $\mathcal{P}$ is completely formalized in a suitable formal system to which the metatheorem applies. In practice one does not deal with completely formalized proofs – rather one identifies only the key steps in the proof, and relies on the original algorithm only as a general guideline in developing the new proof of the new theorem. The proof $\mathcal{P}'$ will again be an ordinary mathematical proof in the sense that it does not rely on the logical metatheorems which provide the algorithm and assure that we can carry out the analysis. We use the prefix "meta-" when refering to these theorems simply to signify that they are theorems which say something about formal systems – in which one can prove theorems. So in comparison to the theorems which one proves *in* the relevant formal systems the theorems which are *about* the formal systems are in some sense one step "higher".

Here there are several things which we should say something more about straight away:

(i) The new theorem A' could be a strengthening in several ways. If $A :\equiv \exists x B(x)$, then it would certainly be an improvement if one could produce a concrete c such that $A' :\equiv B(c)$, or if one could produce a finite number of possible witnesses such that

$$A' :\equiv B(c_1) \vee \ldots \vee B(c_n).$$

If $A :\equiv \forall x \exists y B(x, y)$, then one could try to produce a program p giving a realizer, i.e., such that $A' :\equiv \forall x B(x, p(x))$. And in the case where

$$A :\equiv \forall x \exists n \in \mathbb{N} B(x, n)$$

one could try to produce a function p giving a *bound*, i.e., such that

$$A' :\equiv \forall x \exists n \leq p(x) B(x, n).$$

To illustrate a possible qualitative improvement of the original theorem we can consider the case where

$$A :\equiv \forall x \forall y \exists n \in \mathbb{N} B(x, y, n).$$

Then a new theorem A' of the form

$$A' :\equiv \forall x \forall y \exists n \leq p(x) B(x, y, n),$$

where p is a function which does not take y as an argument, would show that there exists a bound which is uniform in y. This is an improvement which could be of interest even if one has no interest in the numerical or quantitative details of particular bounds or realizers. And as an example where the new theorem has weakened premises we can consider the case where

$$A :\equiv (\forall n \in \mathbb{N} B(n) \to C)$$

and

$$A' :\equiv (\forall n \leq N B(n) \to C),$$

for some given $N \in \mathbb{N}$. Theorems of the forms considered here are common in many areas of mathematics, and we will see examples of this later. In [109] one can find a survey which includes a discussion of different kinds of mathematical statements which could be strengthened via proof mining.

(ii) Our ability to extract information such as computable bounds from a proof of a theorem will be heavily dependent on the logical form of the theorem and on what kind of proof principles has been used in the proof, and there are severe limitations on what we in general can do. It is well-known that given a theorem $\forall x \in \mathbb{N} \exists y \in \mathbb{N} A(x, y)$, it will not in all cases be possible to find a computable bound, i.e., a computable $p : \mathbb{N} \to \mathbb{N}$ such that

$$\forall x \in \mathbb{N} \exists y \leq p(x) A(x, y).$$

And this is the case already in the comparatively simple case where

$$A(x, y) :\equiv \forall z \in \mathbb{N} B_0(x, y, z),$$

with $B_0(x, y, z)$ a quantifier-free formula in the language of elementary arithmetic. This is essentially due to the unsolvability of the halting problem. Namely, letting $T(e, x, y)$ be Kleene's primitive recursive T-predicate, which expresses that the Turing machine e with input x terminates with computation y, we can take

$$B_0(x, y, z) :\equiv (T(x, x, y) \vee \neg T(x, x, z)).$$

Then

$$\forall x \in \mathbb{N} \exists y \in \mathbb{N} \forall z \in \mathbb{N} (T(x, x, y) \vee \neg T(x, x, z))$$

is provable already in first order predicate logic, but a computable bound $p : \mathbb{N} \to \mathbb{N}$ such that

$$\forall x \in \mathbb{N} \exists y \leq p(x) \forall z \in \mathbb{N} (T(x, x, y) \vee \neg T(x, x, z))$$

would allow us to solve the special halting problem, since to decide whether $\exists y \in \mathbb{N} T(x, x, y)$ for given $x \in \mathbb{N}$ we would then only have to check whether $\exists y \leq p(x) T(x, x, y)$, and the latter would be decidable. Thus such a $p : \mathbb{N} \to \mathbb{N}$ cannot exist.

In contrast to this we can consider the case where we are given a theorem $\forall x \in \mathbb{N} \exists y \in \mathbb{N} A_0(x,y)$, where $A_0(x,y)$ is itself a quantifier-free formula in the language of elementary arithmetic, and therefore decidable. Then there always exists a computable bound $p : \mathbb{N} \to \mathbb{N}$ such that

$$\forall x \in \mathbb{N} \exists y \leq p(x) A_0(x,y).$$

Namely, we can take $p(x) := \min\{y \in \mathbb{N} : A_0(x,y)\}$, since such a least y always exists and since $A_0(x,y)$ is decidable. But using this argument we have no control over how fast p grows. In this case the challenge is to extract information from a given proof of the theorem so as to get a subrecursive bound, i.e., a bound which does not use unbounded search.

We will be interested in the borderline between the unproblematic case

$$\forall x \in \mathbb{N} \exists y \in \mathbb{N} A_0(x,y)$$

and the highly problematic

$$\forall x \in \mathbb{N} \exists y \in \mathbb{N} \forall z \in \mathbb{N} A_0(x,y,z),$$

especially in their manifestations as statements about the convergence of iteration sequences in metric fixed point theory. A central question will be in which cases we can predict that a $\forall\exists\forall$-statement will behave like a $\forall\exists$-statement.

(iii) We have already mentioned that, strictly speaking, in order to apply the methods of proof mining the proof of the theorem under consideration must be formalized in one of a number of suitable formal systems, which in most cases is an unrealistic requirement. However, it is often much simpler to establish that a proof *can* be formalized in a certain formal system. This can then give important a priori information about what kind of effective bounds or realizers can be obtained, before any actual proof analysis has taken place. This is often an important step on the way to obtain concrete bounds, which can be produced by more rule-of-thumb or ad hoc methods. Applying proof mining often involves mainly putting the statement of the theorem and the key concepts involved into a suitable logical form and identifying the steps in the proof which need extra consideration. From this one can often infer the existence of uniform bounds based on general metatheorems, and if one wishes one can go on to try to actually extract these.

The tools one uses in proof mining were first developed with a different goal in mind: One wanted to investigate relative consistency between different formal systems for mathematics. The idea of rather applying these methods from proof theory in a different way – to analyze given proofs of theorems in core mathematics – goes back to ideas of Georg Kreisel from the 1950s, and to his program *unwinding of proofs* (see [115, 44, 40] and the references cited therein). Kreisel observed that mathematical proofs of given theorems in many cases carry more information than just the truth of the theorem in question. Furthermore, even though this information might be prima facie hidden, it can often be uncovered in a systematic way through an appropriate logical analysis. His basic question was:

> "What more do we know if we have proved a theorem by restricted means than if we merely know it is true?"

Kreisel suggested that proof theory should shift its focus away from the relative consistency proofs which had been the original motivation for developing the techniques, and that one should use these methods to try to establish in concrete cases what extra information lies hidden in a proof which only uses "restricted means".

Proof theory had developed as a reaction to the perceived foundational crisis in mathematics in the early 20th century, which was brought on both by the inconsistencies which had been discovered in early attempts to develop formal systems for mathematics, and by the criticism of classical logic and set-theoretic mathematics which Brouwer and his school stood for. In an attempt to give mathematics secure foundations, Hilbert together with his followers sought to prove the consistency of the various formal systems in which parts of mathematics could be developed. One originally hoped to be able to carry out such a consistency proof using only "finitistic means", and in this way settle the matter once and for all. However, as a consequence of Gödel's incompleteness theorems, which were published in the early 1930s, it became apparent that the goal of Hilbert's program in its original form had to be modified. Gödel showed that to prove the consistency of even first order arithmetic with full induction, i.e., Peano arithmetic PA, one had to go beyond what was considered strictly finitary. Consequently one thereafter focused on finding the "minimal" abstract notions which sufficed to prove the consistency of e.g. PA. (For historical information concerning the foundational crisis and Hilbert's program one can consult e.g. [170] and [131].) The consistency of arithmetic was soon proved by Gentzen via transfinite induction up to the ordinal ε_0 (see [50]), but an alternative approach developed by Gödel will be of much greater concern to us. In [59] and [60] Gödel introduced two *proof interpretations*: the negative translation (a similar translation was discovered by Gentzen, and there is some preceeding work by Kolmogorov [111] and Glivenko [58]) and the functional (or "Dialectica") interpretation. Together with the negative translation the functional interpretation serves to give a consistency proof of classical arithmetic, and this is not achieved via some kind of transfinite induction, but rather by the extension of primitive recursive arithmetic to all finite types. Negative translation combined with Gödel's functional interpretation form the backbone of the logical metatheorems which will be the basis for our applications of proof mining.

As already mentioned, negative translation and functional interpretation are examples of so-called proof interpretations. In general this means that they are transformations I mapping formulas A and proofs P of one formal system Σ_1 to formulas A^I and proofs P^I in another formal system Σ_2, such that certain properties considered desirable are preserved. Notably, if P is a proof of A, then P^I should be a proof of A^I. Further, there should be some connection between the formula A and the interpretation A^I, at least for certain classes of formulas – typically atomic formulas are left unchanged, for example. In particular, if $(0 = 1)^I$ is just $0 = 1$, then if one could derive $0 = 1$ in Σ_1 one would be able to derive $0 = 1$ already in the target system Σ_2, and if the proof that the interpretation works is itself considered unproblematic (because the transformation I is computable), then one has a relative consistency proof – if Σ_2 is consistent in the sense that one cannot prove $0 = 1$ in the system, then so is Σ_1. Thus in concrete cases one would try to develop proof interpretations between (strong) formal systems Σ_1, which for some reason are considered problematic, and systems Σ_2 for which it is considered easier to justify belief in their consistency.

The first attempt to study proof interpretations as such appears in Kreisel's papers [113, 114] (where he also introduced another proof interpretation: the no-counterexample interpretation). It was Kreisel's idea to apply proof interpretations not to hypothetical proofs of a contradiction such as $0 = 1$, but rather to concrete proofs from mathematics. For more information on Kreisel's unwinding program, where one uses tools from proof theory such as proof interpretations to analyze proofs in mathematics, see [117, 129, 130]. This general project has in later years been dubbed "proof mining". Unwinding of proofs has had applications in algebra ([40]), number theory ([116, 128]), combinatorics ([15, 57]) and computer science ([16, 17]). And from the early 1990s Kohlenbach and various collaborators have systematically applied proof mining to (nonlinear) functional analysis and numerical analysis. For applications to approximation theory, see [88, 89, 90, 110, 145], for applications to ergodic theory and topological dynamics, see [10, 52, 51, 106], and for applications to metric fixed point theory, see [23, 54, 95, 94, 97, 98, 104, 107, 108, 105, 122, 123, 120] (and also [22, 25, 26, 24, 27, 28], which contain material included in this thesis).

The applications in functional analysis and approximation theory have been based on Kohlenbach's monotone functional interpretation (see [91] or Chapter 9 in [101]), which combines Gödel's functional interpretation with Howard's majorizability relation ([72]). Very roughly we might say that monotone functional interpretation is a proof interpretation which systematically transforms the statements appearing in a proof into versions where explicit bounds or moduli (like moduli of uniform continuity) are given or required – in a proof of an implication we must make explicit the bounds or moduli required by (the monotone functional interpretation) of the premise, and monotone functional interpretation then transforms these into bounds or moduli for (the monotone functional interpretation of) the conclusion. In [109] it is argued that monotone functional interpretation in many cases provide the right notion of numerical implication in analysis.

Relatively recently – and in connection with the applications in functional analysis – general logical metatheorems which rather dramatically extend the reach of monotone functional interpretation have been developed by Kohlenbach [99] and Gerhardy–Kohlenbach [56]. These are based on extensions of monotone functional interpretation to formal systems for analysis with various abstract spaces (e.g. metric, normed, uniformly convex normed, Hilbert, CAT(0) or W-hyperbolic spaces) added as new ground types. (Adaptations of these metatheorems to formal theories for $\mathbb{R}$-trees, Gromov hyperbolic spaces and uniformly convex W-hyperbolic spaces are given in [121].) The formal system involves a formal system $\mathcal{A}^\omega$ for analysis, basically Peano arithmetic in all finite types with quantifier-free axiom of choice, dependent choice and countable choice, but with only a certain quantifier-free rule of extensionality instead of the full axiom of extensionality. On top of this one then "adds" e.g. an abstract bounded metric space, obtaining a theory $\mathcal{A}^\omega[X, d]$. In general the metatheorems are of the following form: Suppose a $\forall\exists$-sentence of a certain kind can be proved in one of the formal systems under consideration, then from a sufficiently formal proof one can extract an effective bound which holds in all spaces of the appropriate kind, and moreover this bound is *uniform* in all parameters which satisfy some weak local boundedness criteria. These metatheorems will be crucial both for our concrete results in metric fixed point theory, where we among other things construct explicit and highly uniform rates of convergence for the Picard iteration sequences for Kirk's asymptotic contractions, and also for our results on rates of convergence for Picard iteration sequences in bounded metric spaces in general. Details on this are provided in Chapter 2.

For additional information on applications of proof mining and proof mining in general see also the surveys [102, 103], the PhD theses of Oliva [146] and Gerhardy [53], and the survey [124] by L. Leuştean. For more information on the functional interpretation, including Spector's [166] extension of the interpretation to full classical analysis via bar recursive functionals, which is used in the proofs of the metatheorems, see also [9, 46, 47, 71, 127, 147, 169].

1.2 Some aspects of metric fixed point theory

Metric fixed point theory has its roots in methods from the late 19th century, when successive approximations were used to establish the existence and uniqueness of solutions to equations, and especially differential equations. This approach is particularly associated with the work of Picard, although it was Stefan Banach who in 1922 (in [11]) developed the ideas involved in an abstract setting. Banach's contraction mapping principle is remarkable both for its width of applications in analysis, and for its simplicity.

1.2.1 Contractions and rates of convergence

Notation 1.1. We will throughout this thesis let $\mathbb{N}$ denote the set of nonnegative integers, including 0.

Definition 1.2. A selfmap $f : X \to X$ of a metric space (X, d) is called a *contraction* if there exists $k < 1$ such that

$$d(f(x), f(y)) \leq k \cdot d(x, y)$$

for all $x, y \in X$. Such a $k < 1$ is called a *contraction constant* for f, and the smallest such k is called *the* contraction constant.

Definition 1.3. Let (X, d) be a complete metric space, let $x_0 \in X$, and let $f : X \to X$ be a mapping. The sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} := f(x_n)$ is called the *Picard iteration sequence* with respect to f and x_0.

Theorem 1.4 (Banach)**.** *Let (X, d) be a nonempty complete metric space, and let $f : X \to X$ be a contraction. Then f has a unique fixed point $z \in X$, and for each $x_0 \in X$ the Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to z. Moreover, we have the following error estimate: For all $x_0 \in X$ and all $n \geq 1$ we have*

$$d(f^n(x_0), z) \leq \frac{k^n}{1-k} d(x_0, f(x_0)),$$

where k is a contraction constant for f.

For a proof of this theorem (and a readable survey of different kinds of extensions) see Chapter 1 in [85]. We note that this theorem immediately gives us a rate of convergence for any Picard iteration sequence to the unique fixed point, where by a rate of convergence we mean the following:

Definition 1.5. Let (X, d) be a metric space, let $z \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $\Phi : \mathbb{N} \to \mathbb{N}$ is a *rate of convergence* for $(x_n)_{n\in\mathbb{N}}$ to z if

$$\forall n \in \mathbb{N} \forall m \geq \Phi(n) \left(d(x_m, z) < 2^{-n} \right).$$

Thus a computable rate of convergence gives us complete control over the convergence of a sequence. Corresponding to a rate of convergence we also have the following concept:

Definition 1.6. Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $\Phi : \mathbb{N} \to \mathbb{N}$ is a *Cauchy rate* for $(x_n)_{n\in\mathbb{N}}$ if

$$\forall n \in \mathbb{N} \forall k, m \geq \Phi(n) \left(d(x_k, x_m) < 2^{-n} \right).$$

We next include a related notion which we will call a *rate of proximity*:

Definition 1.7. Let (X, d) be a metric space, let $z \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that $\Phi : \mathbb{N} \to \mathbb{N}$ is a *rate of proximity* for $(x_n)_{n\in\mathbb{N}}$ to z if

$$\forall n \in \mathbb{N} \exists m \leq \Phi(n) \left(d(x_m, z) < 2^{-n} \right).$$

This notion might seem somewhat artificial – and in fact, rates of proximity are of relevance to us mainly as a step on the way to a full rate of convergence. Rates of proximity will turn up in a natural way in the course of our proof theoretic analysis of (ineffective) proofs that for certain kinds of selfmappings on metric spaces all Picard iteration sequences converge to a unique fixed point. In Chapter 2 we will discuss how we can extract rates of proximity from given such proofs of convergence to a unique fixed point for various classes of selfmaps of metric spaces, and we will investigate conditions which allow us to obtain a rate of convergence instead.

Remark 1.8. We will sometimes also say that a function $\Phi : (0, \infty) \to \mathbb{N}$ such that

$$\forall \varepsilon > 0 \forall m \geq \Phi(\varepsilon) \, (d(x_m, z) < \varepsilon)$$

is a rate of convergence for $(x_n)_{n\in\mathbb{N}}$ to z, and similarly we will when this is convenient for notational reasons consider mappings $\Phi : (0, \infty) \to \mathbb{N}$ as Cauchy rates or rates of proximity.

Also the following notion will be relevant later:

Definition 1.9. Given a metric space (X, d) and a mapping $f : X \to X$ we say that a sequence $(x_n)_{n\in\mathbb{N}}$ is an *approximate fixed point sequence* for f if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $m \geq n$ we have $d(x_m, f(x_m)) < \varepsilon$.

Another important aspect of the Banach contraction mapping principle which is worth noting is that the rate of convergence is *uniform* in the starting point $x_0 \in X$ except through an upper bound on the initial displacement, i.e., except through a $b > 0$ such that $d(x_0, f(x_0)) \leq b$. Consequently, if the space is bounded, then the rate of convergence is fully uniform in the starting point. In fact, the rate of convergence does not depend on the space (X, d), the mapping f, or the starting point $x_0 \in X$ except through a contraction constant k and an upper bound b on $d(x_0, f(x_0))$. In contrast to this, it is not in general the case that given a continuous (even nonexpansive, see Definition 1.14) selfmapping $f : X \to X$ on a bounded, complete metric space (X, d) such that all Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ converge to a unique fixed point $z \in X$ of f, then the rate of convergence is uniform in the starting point. Consider e.g., the following example.

Example 1.10. Let

$$X = \{(n, k) \in \mathbb{R}^2 : n, k \in \mathbb{N}, k \leq n\},$$

and consider the discrete metric d on X, i.e., such that

$$d\big((n, k), (n', k')\big) = 1$$

for $(n, k) \neq (n', k')$. Define now $f : X \to X$ by

$$f\big((n, k)\big) = \begin{cases} (0, 0) & \text{if } k = 0, \\ (n, k-1) & \text{if } k \neq 0. \end{cases}$$

Then (X, d) is complete and bounded, and f is uniformly continuous (and in fact nonexpansive). Moreover, all Picard iteration sequences converge to the unique fixed point $(0, 0)$, but there exists no common rate of convergence for all sequences $\big(f^n\big((k,k)\big)\big)_{n\in\mathbb{N}}$, for $k \in \mathbb{N}$.

The definition of a rate of convergence in Definition 1.5 is very different from the "convergence of at least order q" as commonly used in numerical analysis when considering iterative methods:

Definition 1.11. Let (X, d) be a metric space, let $z \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to $z \in X$. Let $q \geq 1$. We say that the convergence of $(x_n)_{n\in\mathbb{N}}$ to z is of at least order q if there exists a null sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals and a $\mu > 0$, with $\mu < 1$ in case $q = 1$, such that

$$\forall n \in \mathbb{N}\, (d(z, x_n) \leq \varepsilon_n)$$

and

$$\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n^q} = \mu.$$

If $q = 1$ then $(x_n)_{n\in\mathbb{N}}$ is said to converge (at least) linearly.

In this definition the μ is often called the (asymptotic) rate of convergence. An order of convergence and a rate of convergence in the sense of Definition 1.11 give only asymptotic information on the convergence, one gets no information on how far one has to go in the sequence to get close to the limit. Consider for example the family of real sequences $(x_n^{(k)})_{n\in\mathbb{N}}$, where for $k \in \mathbb{N}$ we have

$$x_n^{(k)} = \begin{cases} 1 & \text{if } n = k, \\ 2^{-n} & \text{if } n \neq k. \end{cases}$$

All the sequences $(x_n^{(k)})_{n\in\mathbb{N}}$ converge to 0 with at least order 1 and with rate 1/2, but there exists no common rate of convergence in the sense of Definition 1.5. And if we do not know which of the sequences $(x_n^{(k)})_{n\in\mathbb{N}}$ we are given, then simply knowing that the convergence is of at least order 1 does not tell us how far in the sequence we have to go to make sure that e.g. $x_n^{(k)} < 1/2$. Evidently a rate of convergence as given in Definition 1.5 provides important information if we are to approximate the limit in practice, and similarly, uniformity properties of the rate of convergence are important in a setting where our measurements are inaccurate, as well as for various theoretical purposes. We will be concerned with rates of convergence in this strong sense, and unless explicitly otherwise stated "a rate of convergence" will in this thesis refer to the concept in Definition 1.5 rather than the one associated with Definition 1.11. (To reduce ambiguity we could also have used the terminology "modulus of convergence" for the notion in Definition 1.5. However, we will for the most part continue to use "rate of convergence".)

Another concept used in numerical analysis, particularly when considering discretization methods, involves saying that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to z with order $q > 0$ if there exists a constant C such that

$$d(x_n, z) < Cn^{-q} \tag{1.1}$$

for all $n \in \mathbb{N}$, $n \geq 1$. To the extent that one is also interested in determining the constant C this is closer to our Definition 1.5 than Definition 1.11 is, since one does not only consider the limiting behavior as $n \to \infty$, and such a C and q give us a rate of convergence as in Definition 1.5. However, we will not require that a rate of convergence is brought on this form, partly because we will not exclude convergence which is slower than what one gets from (1.1) for any $C, q > 0$, and partly because our rates of convergence Φ will depend on other quantitative information given as parameters in ways which will make the notion in Definition 1.5 more suitable. Treating a rate of convergence as a function from the natural numbers to the natural numbers also gives us a good way of handling questions related to computability. As a general reference on computability theory one might consult [142], and for general information on computability in analysis, see [171]. In relation to this it is worth noting that the rate of convergence for contractions is computable in the sense that we get a *computable* $\Psi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $b, k, n \in \mathbb{N}$, all nonempty and complete (X, d) and all $f : X \to X$ with

$$\forall x, y \in X(d(f(x), f(y)) \leq (1 - 2^{-k})d(x, y))$$

and $x_0 \in X$ with $d(x_0, f(x_0)) \leq b$ we have

$$d(z, f^m(x_0)) < 2^{-n}$$

for all $m \geq \Psi(b, k, n)$, where z is the unique fixed point.

Remark 1.12. Given $k \in \mathbb{N}$ and $b \in \mathbb{N}$ we thus get one fixed rate of convergence $\lambda n.\Psi(b, k, n)$ in the sense of Definition 1.5 which holds for all Picard iteration sequences $(f^n(x_0))_{n \in \mathbb{N}}$ such that (X, d) is a nonempty complete metric space, $f : X \to X$ is a contraction with a contraction constant $c = 1 - 2^{-k}$, and $x_0 \in X$ is a point such that $d(x_0, f(x_0)) \leq b$. We will somewhat loosely say that Ψ itself is a rate of convergence for the Picard iteration sequences of a contraction, whereas the proper thing according to our earlier definition would be to say that Ψ *gives* a rate of convergence for each Picard iteration sequence. For other classes of mappings the quantitative information on which the rate of convergence for each Picard iteration sequence depends might be different – it might be e.g., certain number theoretic functions $\eta, \beta : \mathbb{N} \to \mathbb{N}$ and a number $b \in \mathbb{N}$ rather than the numbers b, k – but we will also in these cases in a similar way speak of rates of convergence Ψ for all Picard iteration sequences, which then take these number theoretic functions (moduli) $\eta, \beta : \mathbb{N} \to \mathbb{N}$ as arguments in addition to b and the desired accuracy n (i.e. n gives the accuracy 2^{-n}). When we say that we obtain effective rates of convergence for a certain class of selfmaps on metric spaces, or for the Picard iteration sequences such mappings give rise to, we refer to the fact that we obtain such a functional Ψ which is computable in some precise sense, and which take the relevant moduli as arguments in addition to the desired accuracy. For the precise statement of this we refer to Chapter 2.

Similarly to the case of rates of convergence we will also call more general functionals Ψ which take suitable moduli for the mapping etc. as arguments and return rates of proximity (respectively Cauchy rates) for a Picard iteration sequence rates of proximity (respectively Cauchy rates).

The contraction mapping principle has been extended or modified in a great many ways, by considering other kinds of conditions on the mapping or the space. But relatively few of these fixed point theorems offer a constructive way to find or approximate the fixed point, and of these even fewer give information on error estimates or effective rates of convergence.

From the point of view of computability there is here a great difference between obtaining a rate of proximity and a rate of convergence. Assume that (X, d) is a metric space, $x_0 \in X$, and $f : X \to X$ a mapping for which we know that $f^n(x_0) \to z$ as $n \to \infty$, where $z \in X$. If we are allowed to treat the predicate $A \subseteq \mathbb{N} \times \mathbb{N}$ given by

$$A(k, n) :\equiv d(f^n(x_0), z) < 2^{-k}$$

as c.e., either because of the way we are able to represent the space (X, d), the mapping f and the real number $d(f^n(x_0), z)$, or because of some oracle, then we get a computable (respectively computable in the oracle) rate of proximity: Namely, since A is c.e. (in an oracle) there is a predicate $C \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which is decidable (in the oracle) such that $A(k, n)$ holds for $k, n \in \mathbb{N}$ if and only if $\exists u \in \mathbb{N} C(u, k, n)$ holds. And since $f^n(x_0) \to z$ we have in particular

$$\forall k \in \mathbb{N} \exists n \in \mathbb{N}(d(f^n(x_0), z) < 2^{-k}),$$

so given $k \in \mathbb{N}$ we can search for the least $m \in \mathbb{N}$ which via the primitive recursive Cantor pairing function $j : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (for a definition see e.g. Definition 3.30 in [101]) codes a pair (u, n) such that $C(u, k, n)$ holds, which gives that

$$d(f^n(x_0), z) < 2^{-k}$$

holds. And from this $m = j(u, n)$ we can get n via the second of the primitive recursive projections associated with the Cantor pairing function. On the other hand, it follows easily from the undecidability of the halting problem that there exist a metric space (X, d), an $x_0 \in X$, and a mapping $f : X \to X$ such that $(f^n(x_0))_{n \in \mathbb{N}}$ converges to the unique fixed point $z \in X$ of f, such that the predicate $A \subseteq \mathbb{N} \times \mathbb{N}$ given by

$$A(k, n) :\equiv d(f^n(x_0), z) < 2^{-k}$$

is decidable, and such that there exists no computable rate of convergence for $(f^n(x_0))_{n \in \mathbb{N}}$ to z. The following is a modification of an example in [10].

Example 1.13. Let $(M_n)_{n \in \mathbb{N}}$ be a computable enumeration of Turing machines, and let $(j_n)_{n \in \mathbb{N}}$ be a computable enumeration of the natural numbers with the property that every natural number appears infinitely often in the enumeration. Let now $(x_n)_{n \in \mathbb{N}}$ be a sequence of distinct points, and let $z \neq x_n$ for all $n \in \mathbb{N}$. Let $X = \{z\} \cup \{x_n : n \in \mathbb{N}\}$, and define a metric on X such that

$$d(x_n, z) = 2^{-j_n}$$

if the following condition holds:

(i) Turing machine M_{j_n}, when started with input 0, halts in less than or equal to n steps, but not in less than or equal to n' steps for any $n' < n$ such that $j_{n'} = j_n$,

and

$$d(x_n, z) = 2^{-n}$$

if (i) does not hold, and such that

$$d(x_n, x_m) = d(x_n, z) + d(x_m, z)$$

for $n \neq m$. Let finally $f : X \to X$ be given by letting $f(x_n) = x_{n+1}$ and $f(z) = z$. Then $(f^n(x_0))_{n\in\mathbb{N}}$ converges to the unique fixed point z. For if $k \in \mathbb{N}$, then we can let $N > k$ be so large that all the machines among $M_1, \ldots, M_k$ that eventually halt have done so in less than N steps, and then for $n > N$ we get $d(x_n, z) \leq 2^{-k}$. And given $k, n \in \mathbb{N}$ we can decide whether $d(f^n(x_0), z) < 2^{-k}$ by first deciding whether (i) holds for n, and if yes, checking whether $j_n > k$, and if no, checking whether $n > k$. But any computable rate of convergence Φ would give us a number $\Phi(n+1)$ such that if M_n halts, then it halts in less than $\Phi(n+1)$ steps, and this would allow us to solve the halting problem.

Notice that in this example the convergence to the fixed point is not monotone, in the sense that it could be that $d(f^m(x_0), z) > d(f^n(x_0), z)$ for $m > n$. This can evidently not happen if the mapping is nonexpansive and the limit is a fixed point:

Definition 1.14. Let (X, d) be a metric space and let $f : X \to X$. We say that f is *nonexpansive* if

$$\forall x, y \in X \, (d(f(x), f(y)) \leq d(x, y)) \, .$$

Since for a nonexpansive mapping a rate of proximity to a fixed point for a Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is already a rate of convergence, it follows that if $f : X \to X$ is nonexpansive and $(f^n(x_0))_{n\in\mathbb{N}}$ converges to a fixed point z, then there always exists a rate of convergence which is computable in an oracle relative to which $A(k, n)$ with $A(k, n) \equiv d(f^n(x_0), z) < 2^{-k}$ is c.e.. This is in marked contrast to the negative result for the general case which we saw in Example 1.13[1]. In the case where $(f^n(x_0))_{n\in\mathbb{N}}$ converges to

[1]Notice that instead of requiring that f is nonexpansive, it is enough that $\forall x \in X(d(f(x), z) \leq d(x, z))$. Then if it holds that $f^n(x_0) \to z$ there would exist a rate of convergence which is computable in an oracle relative to which $d(f^n(x_0), z) < 2^{-k}$ (as a predicate dependent on $k, n \in \mathbb{N}$) is c.e.. Mappings which satisfy $\exists z \forall x \in X(d(f(x), z) \leq d(x, z))$ are called weakly quasi-nonexpansive. Weakly quasi-nonexpansive mappings were introduced (implicitly) by Kohlenbach and Lambov in [104], and a related notion was introduced by Dotson in [41]. The notion of weakly quasi-nonexpansive mappings was considered (independently) under the name J-type mappings by García-Falset et al. in [48], where numerous fixed point results which hold for this class of mappings are given, thus establishing the importance of the notion.

a fixed point $z \in X$ and where the mapping is nonexpansive the sequence of real numbers $(d(f^n(x_0), z))_{n\in\mathbb{N}}$ is monotone decreasing and converges to 0. If on the other hand $(d(f^n(x_0), z))_{n\in\mathbb{N}}$ converges, but not to 0, then it is possible that there exists no computable rate of convergence for $(d(f^n(x_0), z))_{n\in\mathbb{N}}$ to $c := \lim_{n\to\infty} d(f^n(x_0), z)$ even if $(d(f^n(x_0), z))_{n\in\mathbb{N}}$ is a monotone decreasing and computable sequence in $\mathbb{Q}\cap[0, 1]$. Monotone and bounded sequences $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ which are computable but which have no computable rate of convergence are called Specker sequences, and their existence was proved by E. Specker in [165]. As we saw above this cannot happen if the limit of the sequence is 0, and indeed, the limit of a Specker sequence has to be a noncomputable real number. Thus since we are here primarily concerned with selfmaps $f : X \to X$ of metric spaces for which we can prove that $(f^n(x_0))_{n\in\mathbb{N}}$ converges to some $z \in X$, so that $(d(f^n(x_0), z))_{n\in\mathbb{N}}$ converges to 0, the existence of Specker sequences is not a concern. This is in contrast to other cases in metric fixed point theory where one e.g. can prove for some sequence $(x_n)_{n\in\mathbb{N}}$ that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c$$

for some unknown $c \geq 0$. Then because of Specker's result it might be that there is no computable rate of convergence for $(d(x_n, x_{n+1}))_{n\in\mathbb{N}}$ to c even if $(d(x_n, x_{n+1}))_{n\in\mathbb{N}}$ is monotone decreasing and computable.

The study of classes of mappings for which we are able to construct effective and highly uniform rates of convergence to the unique fixed point is the main focus of this thesis, with emphasis both on concrete examples, in particular Kirk's asymptotic contractions, and also on developing a general method – based on methods from proof mining – to find such rates of convergence in various cases from ineffective proofs of convergence to a unique fixed point. Whether this is possible will depend among other things on what formal system we can formalize the proof in, and on certain uniformity features of the moduli and bounds introduced when developing this formal system for the class of selfmappings considered. These moduli will typically be number theoretic functions $\phi : \mathbb{N} \to \mathbb{N}$ (but will sometimes be functionals of higher type, like $\phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$). When it is possible to extract rates of convergence we will typically end up with computable functionals of types of degree 2 which in addition to the desired accuracy n (i.e., 2^{-n}) take only majorants of the moduli and bounds introduced when formalizing the class of selfmaps in question as arguments. The precise meaning of this will be explained in Chapter 2. Here we will only point out that this is what makes the rates of convergence uniform; they do not depend on the mapping, the space or any point in the space except through dependence on majorants of the mentioned moduli and bounds. This uniformity means that we can talk in a meaningful way about the rates of convergence being effective for arbitrary metric spaces; since there is no direct dependence on the points of the space we do not need to first fix a representation for a particular (separable) space and investigate the induced computability concept. Representing various spaces and mappings on these using essentially $\mathbb{N}^{\mathbb{N}}$ or $\{0, 1\}^{\mathbb{N}}$ and mappings $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a central element in the theory of computability on structures other than the natural numbers (see [14, 89, 101, 171]).

The uniformity of the rates of convergence can also lead to new mathematical results: As a byproduct of our treatment of asymptotic contractions in the sense of Kirk we show that in the setting of bounded metric spaces these mappings are in some sense the most general which still exhibit convergence of the Picard iteration sequences of "Banach type" – to the same point and with strong uniformity with respect to the starting point.

For general information on metric fixed point theory one may consult the books [61, 63, 80, 85], and for a survey and comprehensive bibliography of iterative approximations of fixed points, see [18].

1.2.2 Nonexpansive mappings

In the previous section we saw that the fixed point theory for contractions is extremely nice, even from a computational point of view. There exist a large number of results which in some sense extend the contraction mapping principle, and in this section as well as the next ones we will consider some relevant topics.

One of the most natural ways to try to extend the contraction mapping principle is to consider the limiting case when the Lipschitz constant is allowed to be 1, in which case we end up with the nonexpansive mappings from Definition 1.14.

The fixed point theory of nonexpansive mappings is very different from that of contractions, and the study of these mappings has been one of the main research areas of nonlinear functional analysis since the 1950s. Nonexpansive selfmappings of nonempty complete metric spaces do not in general have fixed points – consider e.g. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x+1$, and one consequently considers various geometric conditions on the space in order to ensure the existence of a fixed point. And when fixed points exist, they are in general not unique, since e.g. the identity mapping is nonexpansive. We will not here study the fixed point theory of nonexpansive mappings as such, basically because of the lack of uniqueness of the fixed point. We will here nonetheless include some remarks about this theory – and we will cite negative results concerning the possibility of finding computable rates of convergence in this setting. Instead we will study very general kinds of contractive type mappings – where the requirements on the mappings do guarantee the uniqueness of any fixed points, and where we can find computable and highly uniform rates of convergence via proof mining. It is worth noting that these classes of functions will include mappings which are not nonexpansive.

The most famous result in the theory of nonexpansive mappings is probably the following theorem, which was proved independently by Browder [30], Göhde [65] and Kirk[2] [82]:

Theorem 1.15 (Browder,Göhde,Kirk)**.** *If C is a nonempty, bounded, closed and convex subset of a uniformly convex Banach space $(X, \|\cdot\|)$, and if $f : C \to C$ is nonexpansive, then f has a fixed point.*

[2]Kirk actually proved a more general result, which involved the concept of normal structure.

Even in the cases where fixed points of nonexpansive mappings exist – as for example given by the previous theorem – the Picard iteration scheme can not in general be used to approximate a fixed point. And this is the case even when the fixed point is unique, as can be seen by considering e.g. $X := \mathbb{R}$, $C := [0,1]$, $f(x) = 1-x$ and $x_0 = 0$. Then the Picard iteration sequence alternates between 0 and 1, while the unique fixed point is $1/2$. In the setting of Banach spaces (or hyperbolic spaces) one can then approximate a fixed point via other iteration schemes, such as the Krasnoselski–Mann iteration ([132]), which for a given sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,1]$ and starting point x_0 is defined as follows:

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n f(x_n).$$

(The special case $\lambda_n = 1/2$ was introduced by Krasnoselski in [112].) A central result in this direction is the following theorem by Ishikawa [74], which generalizes a theorem of Krasnoselski:

Theorem 1.16 (Ishikawa)**.** *Let C be a compact convex subset of a Banach space $(X, \|\cdot\|)$, and let $f : C \to C$ be nonexpansive. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,b]$ for some $b < 1$ such that $\sum_{n=0}^{\infty} \lambda_n = \infty$. Then for any starting point $x_0 \in C$ the Krasnoselski–Mann iteration sequence $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of f.*

Thus in this setting there is an effective iteration converging towards a fixed point, but Kohlenbach [98] has shown that (essentially due to lack of uniqueness of the fixed point) there exists no uniform effective rate of convergence:

Theorem 1.17 (Kohlenbach)**.** *There exists a (primitive recursively) computable sequence $(f_l)_{l\in\mathbb{N}}$ of nonexpansive functions $f_l : [0,1] \to [0,1]$ such that for $\lambda_n := 1/2$ and $x_0^l := 0$ and the corresponding Krasnoselski–Mann iterations $(x_n^l)_{n\in\mathbb{N}}$ there is no computable function $\phi : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall m \geq \phi(l) \left(|x_m^l - x_{\phi(l)}^l| \leq 1/2 \right).$$

Here $(f_l)_{l\in\mathbb{N}}$ is a computable sequence in the sense of computability theory, see e.g. [151, 171]. For the iteration sequence in Theorem 1.16 one can still find an effective rate of convergence for $\|x_n - f(x_n)\| \to 0$, and also effective bounds for the Herbrand normal form of the Cauchy property of $(x_n)_{n\in\mathbb{N}}$, i.e., an effective bound on $\exists n \in \mathbb{N}$ in

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \in \mathbb{N} \forall i, j \in [n; n + g(n)](\|x_i - x_j\| \leq 2^{-k}),$$

where $[n;m]$ denotes the subset $\{n, n+1, \ldots, m-1, m\}$ of $\mathbb{N}$ for $m \geq n$. (For details, see [95, 98].)

Notice a crucial difference between the relevance of Theorem 1.17 and Example 1.13: Since we can conclude by Ishikawa's theorem that for all the mappings f_l appearing in Theorem 1.17 the corresponding Krasnoselski–Mann iterations converge we cannot hope to "unwind" a proof of Theorem 1.16 to get a computable functional which given e.g. a bound on the diameter of the space, a modulus governing how quickly $\sum_{n=0}^{\infty} \lambda_n$ diverges, or even a representation[3] of a nonexpansive mapping f on the compact set C as a functional $\Psi_f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, gives a rate of convergence for f. In Example 1.13, on the other hand, we do not conclude that the iteration sequence converges by referring to a general theorem on some class of selfmaps of metric spaces whose proof we would like to "unwind". Thus despite of Example 1.13 it could very well be that the theorems we consider as candidates for proof mining involve extra conditions on the mappings which allow us to obtain effective rates of convergence.

1.2.3 Contractive mappings

In contrast to the case of nonexpansive functions there are other ways of extending the contraction mapping principle which do retain the uniqueness of the fixed point: This is a salient property of various kinds of "mappings of contractive type". We will first mention some results concerning mappings which are contractive, i.e., which satisfy

$$\forall x, y \in X \, (x \neq y \to d(f(x), f(y)) < d(x, y))\,.$$

When we later consider asymptotic contractions and mappings of contractive type we will not require that they are contractive, or even nonexpansive. One of the first extensions of Banach's contraction mapping principle to become widely known is the following theorem due to Rakotch [152]:

Theorem 1.18 (Rakotch)**.** *Let (X, d) be a nonempty, complete metric space, and suppose $f : X \to X$ satisfies*

$$\forall x, y \in X \, (d\,(f(x), f(y)) \leq \alpha\,(d(x, y))\, d(x, y))\,,$$

where $\alpha : [0, \infty) \to [0, 1)$ is monotonically decreasing. Then f has a unique fixed point z, and for all $x_0 \in X$ we have $f^n(x_0) \to z$ as $n \to \infty$.

Rakotch's theorem is related to the following theorem by Edelstein [42]:

Theorem 1.19 (Edelstein)**.** *Let (X, d) be a nonempty, compact metric space, and suppose $f : X \to X$ is contractive, i.e., satisfies*

$$\forall x, y \in X \, (d\,(f(x), f(y)) < d(x, y))\,.$$

Then f has a unique fixed point z, and for all $x_0 \in X$ we have $f^n(x_0) \to z$ as $n \to \infty$.

[3] For information on representation of complete separable metric spaces, in particular compact metric spaces, and mappings on such spaces, using essentially $\mathbb{N}^{\mathbb{N}}$ and mappings $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, see [14, 89, 101, 171].

For a simple proof of this theorem, see e.g. [64]. (Edelstein actually proved a version where X is only assumed to be complete, and where the conclusion states that there exists a unique fixed point z in case there exists an iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ with a convergent subsequence, and that in this case $f^n(x_0) \to z$ as $n \to \infty$.) To illustrate the use of the proof mining techniques in question Kohlenbach and Oliva [109] extracted a full rate of convergence for the Picard iteration sequences from a proof of Edelstein's theorem, and in [55] Gerhardy and Kohlenbach extracted a full rate of convergence in the case of Rakotch's theorem.

A subsequent generalization of Rakotch's result was obtained by Boyd and Wong [21]:

Theorem 1.20 (Boyd,Wong)**.** *Let (X,d) be a nonempty, complete metric space, and suppose $f : X \to X$ satisfies*

$$\forall x, y \in X \left(d\left(f(x), f(y)\right) \leq \phi\left(d(x,y)\right)\right),$$

where $\phi : [0,\infty) \to [0,\infty)$ is upper semicontinuous from the right and satisfies $0 \leq \phi(t) < t$ for $t > 0$. Then f has a unique fixed point z, and for all $x_0 \in X$ we have $f^n(x_0) \to z$ as $n \to \infty$.

A quantitative variant of the Boyd–Wong theorem was proved by Browder [31]:

Theorem 1.21 (Browder)**.** *Let (X,d) be a nonempty, bounded, complete metric space, and suppose $f : X \to X$ satisfies*

$$\forall x, y \in X \left(d\left(f(x), f(y)\right) \leq \phi\left(d(x,y)\right)\right),$$

where $\phi : [0,\infty) \to [0,\infty)$ is monotone nondecreasing and continuous from the right, such that $\phi(t) < t$ for $t > 0$. Then there exists a unique $z \in X$ such that for all $x_0 \in X$ we have $f^n(x_0) \to z$ as $n \to \infty$. Moreover, if d_0 is the diameter of X, then

$$d(f^n(x_0), z) \leq \phi^n(d_0),$$

and $\phi^n(d_0) \to 0$ as $n \to \infty$.

In [133] Meir and Keeler generalize the Boyd–Wong theorem:

Theorem 1.22 (Meir,Keeler)**.** *Let (X,d) be a nonempty, complete metric space, and suppose $f : X \to X$ satisfies*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X \left(\varepsilon \leq d(x,y) \leq \varepsilon + \delta \to d\left(f(x), f(y)\right) < \varepsilon\right). \tag{1.2}$$

Then f has a unique fixed point z, and for all $x_0 \in X$ we have $f^n(x_0) \to z$ as $n \to \infty$.

A mapping $f : X \to X$ on a metric space (X,d) which satisfies the condition (1.2) in the theorem of Meir–Keeler is called a Meir–Keeler contraction. In order to better compare the Boyd–Wong condition with the Meir–Keeler condition, the latter has been characterized by T.C. Lim [125] as follows:

Theorem 1.23 (Lim)**.** *Let (X,d) be a metric space, and let $f : X \to X$ be a mapping. Then f is a Meir–Keeler contraction if and only if there exists a (nondecreasing and right continuous) function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and $\phi(s) > 0$ for $s > 0$, such that*

$$\forall x, y \in X\Big(x \neq y \to d\big(f(x), f(y)\big) < \phi\big(d(x,y)\big)\Big),$$

and such that for every $s > 0$ there exists $\delta > 0$ such that $\phi(t) \leq s$ for all $t \in [s, s+\delta]$.

The mappings in the theorems directly above are all contractive. But there is also a very large amount of literature on various kinds of generalized contractions – where the mappings are no longer contractive. The hope when considering such generalizations is then to obtain corresponding generalizations of the fixed point theorems one has for contractive mappings. We will first consider asymptotic contractions, which were introduced by Kirk in 2003, and afterwards we will discuss how this approach in some sense subsumes much earlier work on contractive type mappings.

1.2.4 Asymptotic contractions

Asymptotic contractions were introduced by Kirk in [83], but asymptotic fixed point theory, where one considers conditions which involve iterates of the mapping, has a long history in nonlinear functional analysis, see for example [32]. Indeed, one of the first variants of Banach's contraction mapping principle considered was the following theorem by Caccioppoli [34], which includes a kind of "asymptotic contraction":

Theorem 1.24 (Caccioppoli)**.** *Let (X,d) be a nonempty, complete metric space, and let $f : X \to X$ be such that for each $n \geq 1$ there exists a constant c_n such that*

$$\forall x, y \in X \left(d\left(f^n(x), f^n(y)\right) \leq c_n d(x,y)\right),$$

with $\sum_{n=1}^{\infty} c_n < \infty$. Then f has a unique fixed point z, and for all $x_0 \in X$ we have $f^n(x_0) \to z$ as $n \to \infty$.

In [83] Kirk introduces a wider class of mappings in order to obtain an asymptotic version of the Boyd–Wong theorem.

Definition 1.25 (Kirk)**.** Let (X,d) be a metric space. A mapping $f : X \to X$ is said to be an *asymptotic contraction* if there exists a sequence of functions $\phi_n : [0,\infty) \to [0,\infty)$ such that

$$\forall n \in \mathbb{N} \forall x, y \in X\big(d\left(f^n(x), f^n(y)\right) \leq \phi_n\left(d(x,y)\right)\big),$$

and such that $\phi_n \to \phi$ uniformly on the range of d, where $\phi : [0,\infty) \to [0,\infty)$ is continuous and satisfies $\phi(s) < s$ for all $s > 0$.

However, in the main theorem of [83] the mappings ϕ_n in the above definition are also assumed to be continuous, and it has been convenient to single out the resulting concept (this was done by e.g. Gerhardy [54]):

Definition 1.26 (Kirk)**.** A function $f : X \to X$ on a metric space (X, d) is called an *asymptotic contraction in the sense of Kirk* with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ if ϕ, ϕ_n are continuous, $\phi(s) < s$ for all $s > 0$ and for all $n \in \mathbb{N}$ and $x, y \in X$,

$$d\left(f^n(x), f^n(y)\right) \leq \phi_n\left(d(x, y)\right),$$

and moreover $\phi_n \to \phi$ uniformly on the range of d.

Note that in the previous two definitions it is irrelevant whether we include 0 in $\mathbb{N}$ or not, since ϕ_0 in any case could be taken to be the identity. Here we use the opportunity to remark on a notational infelicity: In Chapter 3 we will among other things prove results concerning so-called *generalized asymptotic contractions*, which are meant to generalize the concept in Definition 1.26, not the one in Definition 1.25. Asymptotic contractions and various modifications have been widely studied in recent years, see [2, 3, 4, 5, 6, 36, 54, 73, 75, 76, 86, 154, 156, 167, 168, 172, 173, 174, 175, 176, 177], and also [24, 25, 26, 28], which contain material included in this thesis.

We include for reference Kirk's original theorem, as well as its proof, which is a nice application of Banach space ultrapowers. (Note that, as remarked in e.g. [2, 76], in the statement of the theorem in [83] the assumption that the mapping must be continuous was inadvertently left out.)

Theorem 1.27 (Kirk)**.** *Let (X, d) be a complete metric space, and let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk. If for some $x \in X$ the Picard iteration sequence $(f^n(x))_{n\in\mathbb{N}}$ is bounded, then f has a unique fixed point $z \in X$ and for every starting point $x \in X$ the iteration sequence $(f^n(x)))_{n\in\mathbb{N}}$ converges to z.*

Proof. The proof proceeds by first establishing three preliminary steps. For general information on the use of nonstandard methods in fixed point theory one might consult [1, 68] and the chapter on ultra-methods in metric fixed point theory by Khamsi and Sims in [85].

Step 1: We start by isometrically embedding X as a closed subset of a Banach space Y and identifying X with its image in Y. (For example by taking Y to be the space of all real-valued bounded continuous functions on X, for a proof see e.g. [141].)

Step 2: Let now $\tilde{Y}$ be a Banach space ultrapower of Y over some nontrivial ultrafilter $\mathcal{U}$, and let $\tilde{X}$ denote the image of X in $\tilde{Y}$, i.e., let

$$\tilde{X} = \left\{ \tilde{x} = [(x_n)] \in \tilde{Y} : x_n \in X \text{ for each } n \right\}.$$

Let $\tilde{d}$ be the metric on $\tilde{X}$ inherited from the ultrapower norm $\|\cdot\|_{\mathcal{U}}$ on $\tilde{Y}$. Then $(\tilde{X}, \tilde{d})$ is a complete metric space, since it is a closed subset of the Banach space

$\tilde{Y}$. In particular, for $\tilde{x} = [(x_n)], \tilde{y} = [(y_n)] \in \tilde{X}$ it follows that (x_n) and (y_n) are bounded sequences, so that

$$\lim_{\mathcal{U}} d(x_n, y_n) = \tilde{d}(\tilde{x}, \tilde{y})$$

always exists.

Step 3: Define $\tilde{f}, \hat{f} : \tilde{X} \to \tilde{X}$ by for $\tilde{x} = [(x_n)] \in \tilde{X}$ letting

$$\tilde{f}(\tilde{x}) = [(f(x_n))]$$

and

$$\hat{f}(\tilde{x}) = [(f^n(x_n))] .$$

Then $\tilde{f}$ is well-defined since ϕ_1 is continuous, and $\hat{f}$ is well-defined since the orbits of f are bounded.

We can now use that $\hat{f}$ and $\tilde{f} \circ \hat{f}$ are commuting contractive mappings on $\tilde{X}$. Since $\phi_n \to \phi$ uniformly it follows that

$$\begin{aligned} \tilde{d}\left(\hat{f}(\tilde{x}), \hat{f}(\tilde{y})\right) &= \|\hat{f}(\tilde{x}) - \hat{f}(\tilde{y})\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|f^n(x_n) - f^n(y_n)\| \\ &= \lim_{\mathcal{U}} d\left(f^n(x_n), f^n(y_n)\right) \le \lim_{\mathcal{U}} \phi_n\left(d(x_n, y_n)\right) \\ &= \phi\left(\lim_{\mathcal{U}} d(x_n, y_n)\right) = \phi\left(\tilde{d}(\tilde{x}, \tilde{y})\right). \end{aligned}$$

Since ϕ is continuous and satisfies $\phi(s) < s$ for all $s > 0$ it follows by the Boyd–Wong theorem that $\hat{f}$ has a unique fixed point $\tilde{z} \in \tilde{X}$. On the other hand,

$$\begin{aligned} \tilde{d}\left(\tilde{f} \circ \hat{f}(\tilde{x}), \tilde{f} \circ \hat{f}(\tilde{y})\right) &= \|\tilde{f} \circ \hat{f}(\tilde{x}) - \tilde{f} \circ \hat{f}(\tilde{y})\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|f^{n+1}(x_n) - f^{n+1}(y_n)\| \\ &= \lim_{\mathcal{U}} d\left(f^{n+1}(x_n), f^{n+1}(y_n)\right) \le \lim_{\mathcal{U}} \phi_{n+1}\left(d(x_n, y_n)\right) \\ &= \phi\left(\lim_{\mathcal{U}} d(x_n, y_n)\right) = \phi\left(\tilde{d}(\tilde{x}, \tilde{y})\right). \end{aligned}$$

So also $\tilde{f} \circ \hat{f}$ has a unique fixed point, and since $\hat{f}$ and $\tilde{f} \circ \hat{f}$ commute it follows that

$$\hat{f} \circ \tilde{f}(\tilde{z}) = \hat{f} \circ \tilde{f}(\hat{f}(\tilde{z})) = \hat{f} \circ (\tilde{f} \circ \hat{f})(\tilde{z}) = (\tilde{f} \circ \hat{f}) \circ \hat{f}(\tilde{z}) = (\tilde{f} \circ \hat{f})(\tilde{z}) = \tilde{f}(\hat{f}(\tilde{z})) = \tilde{f}(\tilde{z}),$$

so since the fixed point of $\hat{f}$ is unique it follows that $\tilde{f}(\tilde{z}) = \tilde{z}$. From this we conclude that

$$\lim_{\mathcal{U}} d(z_n, f(z_n)) = 0.$$

One can now extract from the sequence (z_n) a sequence (x_n) such that

$$\lim_{n \to \infty} d(x_n, f(x_n)) = 0.$$

Suppose now that (y_n) is a sequence in X for which $\lim_{n\to\infty} d(y_n, f(y_n)) = 0$. Then $\tilde{y} = [(y_n)]$ is also a fixed point of $\tilde{f}$, and so for $k \in \mathbb{N}$ we have

$$\begin{aligned}\tilde{d}(\tilde{z}, \tilde{y}) &= \tilde{d}\left(\tilde{f}^k(\tilde{z}), \tilde{f}^k(\tilde{y})\right) = \lim_{\mathcal{U}} d\left(f^k(z_n), f^k(y_n)\right) \\ &\leq \lim_{\mathcal{U}} \phi_k\left(d\left(z_n, y_n\right)\right) = \phi_k\left(\tilde{d}(\tilde{z}, \tilde{y})\right),\end{aligned}$$

since the moduli ϕ_k are continuous. By letting $k \to \infty$ we obtain

$$\tilde{d}(\tilde{z}, \tilde{y}) \leq \phi\left(\tilde{d}(\tilde{z}, \tilde{y})\right),$$

and since $\phi(s) < s$ for $s > 0$ we conclude that $\tilde{d}(\tilde{z}, \tilde{y}) = 0$. Thus

$$\lim_{\mathcal{U}} d(z_n, y_n) = 0$$

for any approximate fixed point sequence (y_n) of f. Now suppose that

$$\lim_{n\to\infty} d(x_n, f(x_n)) = 0$$

and

$$\lim_{n\to\infty} d(y_n, f(y_n)) = 0,$$

but $\lim_{n\to\infty} d(x_n, y_n) \neq 0$. By if necessary considering subsequences we can assume that $\lim_{n\to\infty} d(x_n, y_n) =: \varepsilon > 0$. This implies

$$\varepsilon = \lim_{\mathcal{U}} d(x_n, y_n) \leq \lim_{\mathcal{U}} d(x_n, z_n) + \lim_{\mathcal{U}} d(y_n, z_n) = 0,$$

which is a contradiction. Thus $\lim_{n\to\infty} d(x_n, y_n) = 0$ for any pair of approximate fixed point sequences for f.

Now for $n \geq 1$ let

$$F_n := \{x \in X : d(x, f(x)) \leq 1/n\}.$$

Since there exists a sequence (x_n) such that $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$ we have that $F_n \neq \emptyset$ for all $n \geq 1$, and since f is continuous each set F_n is closed. Furthermore, $F_{n+1} \subseteq F_n$. Suppose that we do not have $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$. Then there exists a $\rho > 0$ such that for any $n \geq 1$ one can find $x_n, y_n \in F_n$ with $d(x_n, y_n) \geq \rho/2$. Since (x_n) and (y_n) are fixed point sequences for f this contradicts $\lim_{n\to\infty} d(x_n, y_n) = 0$. Thus

$$\lim_{n\to\infty} \operatorname{diam}(F_n) = 0,$$

and since X is complete it follows by Cantor's intersection theorem that $\bigcap_{n=1}^{\infty} F_n$ is a singleton $\{z\}$, and z is necessarily the unique fixed point of f.

Finally we show that the Picard iteration sequences converge to z. Let $x \in X$, and let $i \in \mathbb{N}$. Then

$$\begin{aligned}\limsup_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) &= \limsup_{n\to\infty} d\left(f^{n+i}(x), f^{n+i+1}(x)\right) \\ &\leq \lim_{n\to\infty} \phi_n\left(d\left(f^i(x), f^{i+1}(x)\right)\right) \\ &= \phi\left(d\left(f^i(x), f^{i+1}(x)\right)\right),\end{aligned}$$

and letting $i \to \infty$ we get

$$\limsup_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) \leq \phi\left(\limsup_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right)\right),$$

from which $\lim_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) = \limsup_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) = 0$. Thus given any $k \geq 1$ the sequence $(f^n(x))_{n=1}^{\infty}$ is eventually in F_k, and since the diameters of the sets F_k tend to 0 as $k \to \infty$, we get $\lim_{n\to\infty} f^n(x) = z$.

□

In [54] P. Gerhardy develops a quantitative version of Kirk's theorem by making use of techniques and insights from proof mining in order to analyze the concepts involved. This involves modifying the definition of an asymptotic contraction, subsuming the old definition under the new one, and giving a bound, expressed in the relevant (new) moduli and a bound on the Picard iteration sequence, on how far one must go in the Picard iteration sequence to at least once get close to the fixed point. That is, he constructs a uniform and effective rate of proximity for the Picard iteration sequences to the unique fixed point, and in the process gives a completely elementary proof of Kirk's theorem[4]. This theorem does not, however, give a rate of convergence to the fixed point in the general case. The convergence needs not be monotone, and so for $m > n$ it is not the case that $f^m(x)$ needs to be close to the fixed point if $f^n(x)$ is. For an example of such a function, see Example 2 in [76]. In contrast to this, the results in [54] do give a rate of convergence when the convergence to the fixed point is monotone, and this is the case for a very large class of functions, including the nonexpansive ones. (For further discussion of the logical analysis, see also Chapter 4 in [53].)

In Chapter 3 we give an effective rate of convergence for the Picard iteration sequences, expressed in the relevant moduli and a bound on the sequence, alternatively in the relevant moduli and strictly positive upper and lower bounds on the initial displacement $d(x_0, f(x_0))$, i.e., $b, c > 0$ such that $c \leq d(x_0, f(x_0)) \leq b$. Thus the rate of convergence is uniform in the space, the mapping and the starting point except through dependence on the mentioned moduli and such $b, c > 0$. If the mapping f is not continuous we get the same rates of convergence to the common limit z of all Picard iteration sequences (which needs not be a fixed point), and if the space is not complete we likewise get explicit Cauchy rates for the iteration sequences.

Additionally we prove that there exists a rate of convergence (which we do not give explicitly) which depends on nothing but moduli $\phi, \phi_n : [0,\infty) \to [0,\infty)$ as given in Definition 1.26 such that $\phi_n \to \phi$ uniformly on $[0,\infty)$, and an upper

[4] Previously I.D. Aranđelović had published an elementary proof of a slight generalization of Kirk's theorem in [2]. However, that proof turned out to contain an error, and the theorem as stated is false – see J. Jachymski's note [75], where he also gives conditions which serve to repair the proof in such a way that the resulting theorem still covers Kirk's theorem. Around the same time as Gerhardy's result H.-K. Xu [177] and T. Suzuki [167] developed versions of the theorem with proofs which do not rely on ultrapower techniques. J. Jachymski and I. Jóźwik had earlier given an elementary proof under the additional assumption that the mapping is uniformly continuous, see [76].

bound $b \geq d(x_0, f(x_0))$ for the initial displacement. Thus the convergence does not depend on a lower bound $c > 0$ on the initial displacement.

As a byproduct of the analysis we furthermore give a characterization of asymptotic contractions in the sense of Kirk on bounded, complete metric spaces, showing that they are exactly the mappings for which all Picard iteration sequence converge to the same point with a rate of convergence which is uniform in the starting point. As already mentioned this characterization gives an indication why asymptotic contractions in the sense of Kirk are of interest: In the setting where the space is bounded they are in a sense the most general mappings for which the Picard iteration sequences have convergence of "Banach type", i.e., to the same point and uniformly with respect to the starting point. We also prove that the assumption in Theorem 1.27 that one iteration sequence is bounded is superfluous – since any sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded in any case[5].

The fact that we for asymptotic contractions in the sense of Kirk could obtain a full rate of convergence for the Picard iteration sequences instead of only a rate of proximity can be explained in logical terms (thus far only when we restrict to the setting where the space is bounded) via the work which appears in Chapter 2, where we establish general conditions, under which we can extract such explicit and highly uniform full rates of convergence for the Picard iteration sequences for selfmaps on bounded metric spaces from ineffective proofs of convergence to a unique fixed point. This is done by extending the use of one of Kohlenbach's metatheorems, which concerns the theory $\mathcal{A}^\omega[X,d]$, a formal theory for analysis with an abstract metric space added as new ground type. This metatheorem allows us to extract (via negative translation and monotone functional interpretation) uniform bounds for certain $\forall\exists$-sentences provable in the theory. We will give conditions for when we can transform a $\forall\exists\forall$-sentence expressing that a Picard iteration sequence is Cauchy into a certain $\forall\exists$-sentence via a product space argument. This will allow us to extract full rates of convergence for the iteration sequences in these cases, and by considering Gerhardy's proof of Kirk's theorem on asymptotic contractions we will see that the conditions are satisfied in that particular case.

Similarly we will be able to explain that we in another case study were able to obtain a full rate of convergence for the Picard iteration sequences – namely for the so-called uniformly continuous uniformly generalized p-contractive mappings. This is a particularly general kind of *mapping of contractive type*. We will in the following briefly discuss the background for this.

1.2.5 Mappings of contractive type

One of the earliest definitions of a condition of "contractive type" where the mappings satisfying the condition need not be contractive is due to Kannan (see [78, 79]), who showed that if (X, d) is a nonempty complete metric space

[5] After having published [26], where among other things this is proved, the author became aware that T. Suzuki had already proved that this assumption is superfluous, see [167] and [24].

and $f : X \to X$ a selfmap such that there exists an $a \in (0, 1/2)$ for which

$$\forall x, y \in X\, (d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))])\,,$$

then there exists a unique fixed point to which all Picard iteration sequences converge. It is noteworthy that here the mapping f does not need to be continuous. In [157], B.E. Rhoades compared 25 contraction conditions, most of them previously considered in the literature, and also considered generalizations of the 25 basic conditions to the cases where the condition holds for various iterates of the function. The basic conditions are numbered (1)–(25), and of these Kannan's is number (4). The comparison of the 25 conditions was completed by P. Collaço and J. Carvalho e Silva in [38]. That is, the implications that hold between the different conditions are completely determined. In particular, it is known that condition (25),

$$\forall x, y \in X(x \neq y \to d(f(x), f(y)) < \operatorname{diam}\{x, y, f(x), f(y)\}),$$

is the most general. So if f satisfies one of the conditions (1)–(24), then it also satisfies condition (25), and a fixed point theorem for functions satisfying (25) would entail as corollaries corresponding fixed point theorems for conditions (1)–(24). However, a function on a nonempty complete metric space satisfying (25) need not have a fixed point. If on the other hand f is continuous and X compact and nonempty, then f has a unique fixed point, and for any $x_0 \in X$ the Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ converges to this fixed point, and moreover this also extends to the case where (25) holds for an iterate of the function, i.e., if there exists $p \in \mathbb{N}$ such that

$$\forall x, y \in X(x \neq y \to d(f^p(x), f^p(y)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}).$$

This was proved by Rhoades[6] in [158], and also by Hicks and Sharma[7] in [69] and Kincses and Totik in [81]. The conditions on f obtained by requiring that for some $p \in \mathbb{N}$ the function f^p should satisfy respectively (1)–(25) are numbered respectively (26)–(50). Given $p \in \mathbb{N}$ we will call a function *generalized p-contractive* if it satisfies (25) for f^p, and we will single this out as a definition for ease of reference:

Definition 1.28. Let (X, d) be a metric space, let $f : X \to X$ and let $p \in \mathbb{N}$. We say that f is *generalized p-contractive* if

$$\forall x, y \in X(x \neq y \to d(f^p(x), f^p(y)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}). \tag{1.3}$$

[6] In [158] Rhoades proved a more general theorem: Instead of compactness of the space it is enough if the mapping $f : X \to X$ is a compact map. We will say more about this later. Rhoades also claimed to have proved the theorem for a more general contractive definition, but in his review of his own paper in Zentralblatt MATH this was modified. The results in [158] are proved by noting that the proofs of some theorems by Janos [77] for another contractive definition go through for the new contractive definition.

[7] Without considering the case of iterates f^p.

Notation 1.29. We will also say that f is generalized p-contractive if there exists some $k \in \mathbb{N}$ such that

$$\forall x, y \in X(x \neq y \to d(f^k(x), f^k(y)) < \operatorname{diam}\{x, y, f^k(x), f^k(y)\}), \qquad (1.4)$$

that is, if f satisfies (50). When we say that a mapping f is generalized p-contractive it will be clear from the context whether "p" refers to some given number or whether we mean that there exists k satisfying (1.4).

Theorem 1.30 (Rhoades,Hicks,Sharma,Kincses,Totik)**.** *Let (X, d) be a non-empty compact metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be continuous and generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

One of our case studies in proof mining concerns this theorem: In Chapter 4 we construct an effective and highly uniform Cauchy rate for the Picard iteration sequences. And by using the uniformities of this Cauchy rate we give an improved version of the theorem – where we by isolating the requirements on the mapping, specifically on the contractivity condition, extend the theorem from the compact case to the setting of arbitrary metric spaces, without requiring the map to be compact. The extension from compact metric spaces to arbitrary metric spaces is accomplished by considering a uniform variant of the contractive condition (50), which we are naturally lead to by applying monotone functional interpretation to the condition. In the case of Theorem 1.30 the compactness of the space means that condition (50) is upgraded to this uniform version, much as continuity is upgraded to uniform continuity. And it turns out that we can prove the theorem assuming only that we have such uniform versions of the contractive condition and continuity, along with a bound on the iteration sequence. Here it is essential that the proof does not use completeness or separability of the space in an essential way[8]. For a fuller discussion of the general issues involved – how monotone functional interpretation in a sense systematically transforms certain statements into their "right" uniform versions – and in the process makes it explicit what quantitative information one has to take as input, and how this can be used to remove compactness assumptions, see [55, 101]; and for the use of a certain nonstandard principle of uniform boundedness in this connection, see [93, 100].

In order to tie our results together we then note that by the uniformity of the Cauchy rate given it follows as a special case that all continuous selfmappings on a compact metric space satisfying one of the conditions (1)–(50) are in fact asymptotic contractions in the sense of Kirk. But note that the uniformity of the convergence with respect to the starting point in the cases where one of the conditions (1)–(50) are satisfied and where the space is compact and the mappings continuous was already present in [158].

[8] Except that completeness is used to ensure the actual existence of the common limit of all Picard iteration sequences.

Analogously to the case of the asymptotic contractions in the sense of Kirk the fact that we could obtain a full rate of convergence instead of a rate of proximity can now be explained in logical terms by the results in Chapter 2.

However, Theorem 1.30 is by no means the most general of its kind. We will discuss some other general contractive conditions here, and refer to some relevant literature. The relationships between several general theorems for contractive type mappings which exist in the literature and the version of Theorem 1.30 extended to general metric spaces which we obtained in the course of our case study remain unclear, but to the extent that one is interested in explicit and effective rates of convergence this is not too relevant. For a mapping $f : X \to X$ on a metric space (X, d), and an $x \in X$, we denote by $O(x)$ the orbit of x, i.e.,

$$O(x) = \{f^n(x) : n \in \mathbb{N}\}.$$

Given $x, y \in X$ we let $O(x, y) = O(x) \cup O(y)$. We say that $x \in X$ is *regular* if

$$\operatorname{diam}\big(O(x)\big) < \infty,$$

i.e., if the Picard iteration sequence with starting point x is bounded. One of the comparatively few results which do provide quantitative information is the following theorem by Hegedüs [66].

Theorem 1.31. *Let $c \in [0, 1)$, and let $f : X \to X$ be a selfmap of a nonempty complete metric space (X, d) such that all $x \in X$ are regular, and such that*

$$d\big(f(x), f(y)\big) < c \cdot \operatorname{diam}\big(O(x, y)\big)$$

for all $x, y \in X$. Then f has a unique fixed point $z \in X$, and all Picard iteration sequences converge to z. Furthermore, we have the following error estimates. For all $n \in \mathbb{N}$ and all $x \in X$ we have

$$d(z, f^n(x)) \le c^n \cdot \frac{d(x, f(x))}{1 - c},$$

and if $n \neq 0$ we also have

$$d(z, f^n(x)) \le c \cdot \frac{d(f^{n-1}(x), f^n(x))}{1 - c}.$$

Notice that the existence of such a $c \in [0, 1)$ means that this theorem is in some sense more closely related to Banach's contraction mapping principle than to Edelstein's theorem or Theorem 1.30 above, and indeed, the theorem shows that such f have a very nice and simple rate of convergence. In [149] Park proves the following theorem, which does not give quantitative information:

Theorem 1.32. *Let $f : X \to X$ be a continuous compact selfmap of a nonempty metric space (X, d) satisfying*

$$\forall x, y \in X\Big(x \neq y \to d\big(f(x), f(y)\big) < \operatorname{diam}\big(O(x, y)\big)\Big).$$

Then f has a unique fixed point $z \in X$, and all Picard iteration sequences converge to z, uniformly in the starting point.

An extension of this to the case where the contractive condition holds for an iterate f^p gives a generalization of Theorem 1.30. To prove this theorem also Park uses the approach of Janos [77]. That the convergence in Theorem 1.32 is uniform in the starting point follows from the proof – the statement of the theorem is the weaker claim that for any $c \in (0,1)$ there is a metric ρ on X which is topologically equivalent to d, such that f is a contraction with contraction constant c relative to ρ. This is related to a result by Meyers [135], which is used by both Janos, Rhoades, and Park, and which provides a converse to Banach's contraction mapping principle. (See also [136].) We state this in the form given by Leader in [119], where he essentially rediscovered Meyers' theorem:

Theorem 1.33. *Let (X,d) be a metric space, let $z \in X$, and let $f : X \to X$ be continuous. Then there exists a metric ρ on X which is topologically equivalent to d and relative to which f is a contraction mapping with fixed point z if and only if*

1. $\lim_{n\to\infty} f^n(x_0) = z$ *for each* $x_0 \in X$.
2. *There exists a neighborhood U of z such that $f^n(x_0) \to z$ uniformly for all $x_0 \in U$.*

Notice that a consequence of this theorem is that if we are interested in the rate of convergence to $z \in X$ for a Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ for a selfmap $f : X \to X$ on a metric space (X,d), then knowing that there exists some metric ρ on X which is topologically equivalent to d and relative to which f is a contraction is no big help. In order to draw any conclusions we would at least have to know that $x_0 \in U$, where U is the neighborhood appearing in Theorem 1.33. This is not too surprising, given that e.g., even if (X,d) is unbounded there always exists a topologically equivalent metric on X relative to which X is bounded by 1.

For further information on contractive type mappings see e.g. [18, 66, 67, 126, 134, 148, 149, 150, 159] and the references found there.

Chapter 2

Logical aspects of rates of convergence in metric spaces

This chapter contains material which appears in [22], but the material has been revised, some things have been left out, and additional comments and corollaries have been added. Likewise certain definitions etc. taken from other sources which were only referred to in [22] have now been included.

2.1 Introduction

We will in this chapter develop further the uses of proof mining in metric fixed point theory. Much of the work in proof mining has been centered around applications in (nonlinear) functional analysis, and strong logical metatheorems for functional analysis based on Gödel's functional interpretation and certain notions of majorizability are provided in [99] and [56]. A special case of one of these theorems can be used to get information on the convergence of the Picard iteration sequences $(f^n(x))_{n\in\mathbb{N}}$ to a unique fixed point $z \in X$ of a self-mapping $f : X \to X$ on a bounded metric space (X, d). Before explaining this in more detail we will include the following definitions, which in addition to Definitions 1.5, 1.6, 1.7, and 1.14 will be relevant for our discussion:

Definition 2.1. Let (X, d) be a metric space and let $f : X \to X$. We say that f is *asymptotically regular* if

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall k \geq m \left(d\left(f^k(x_0), f^{k+1}(x_0)\right) < 2^{-n}\right).$$

Definition 2.2. Let (X, d) be a metric space and let $f : X \to X$. We say that $\Phi : \mathbb{N} \to \mathbb{N}$ is a *modulus of uniform asymptotic regularity* for f if

$$\forall x_0 \in X \forall n \in \mathbb{N} \forall m \geq \Phi(n) \left(d(f^m(x_0), f^{m+1}(x_0)) < 2^{-n}\right).$$

Definition 2.3. Let (X,d) be a metric space and let $f : X \to X$. We say that $\Phi : \mathbb{N} \to \mathbb{N}$ is a *modulus of uniform almost asymptotic regularity* for f if

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \leq \Phi(n) \left(d(f^m(x_0), f^{m+1}(x_0)) < 2^{-n} \right).$$

The word "uniform" in the previous two definitions refers to the fact that Φ does not depend on x_0.

Definition 2.4. Let (X,d) be a metric space and let $f : X \to X$. We say that $\Phi : \mathbb{N} \to \mathbb{N}$ is a *modulus of uniqueness* for f if

$$\forall x_1, x_2 \in X \forall n \in \mathbb{N} \left(\bigwedge_{i=1}^{2} d\left(x_i, f(x_i)\right) < 2^{-\Phi(n)} \to d(x_1, x_2) < 2^{-n} \right).$$

The notion of a modulus of uniqueness was defined in full generality by Kohlenbach in [89]. Moduli of uniqueness show up in e.g. approximation theory under the name of strong unicity or rate of strong uniqueness, see [138] for the first investigation of this in the case of Chebysheff approximation, and see [12] for a general discussion of the relevance of the concept.

Now, if one can prove in a suitable[1] formal system for classical analysis with a new ground type for elements of an abstract bounded metric space (X,d) that all $f : X \to X$ from a suitable class of functions are asymptotically regular and that any fixed point of such an f must be unique, then the metatheorem assures that there exists[2] a (not necessarily fixed) point $z \in X$ to which all Picard iteration sequences converge, and we can extract a rate of proximity (cf. Definition 1.7) for all Picard iteration sequences to this point z which is uniform in the starting point (see [99] and [109][3]). Namely, in this case the metatheorem provides an algorithm for extracting such a rate of proximity from given formal (ineffective) proofs of uniqueness and asymptotic regularity. Note that we do *not* require the space to be compact. In practice one does not deal with completely formalized proofs, but the algorithm can then be used as a guideline for actually extracting a uniform and explicit rate of proximity.

Here we develop a general method for finding uniform and explicit full rates of convergence for Picard iteration sequences of selfmaps on (complete) bounded metric spaces (cf. Definition 1.5), as opposed to rates of proximity. Loosely

[1] What is meant by "suitable" will be made clear later.

[2] For convenience we assume here that the space is complete.

[3] [109] is older work, and in that paper the setting involves (i) a formal system which does not include a ground type for an abstract bounded metric space, and (ii) a concrete Polish space which can be represented in the formal system. In that setting one requires compactness to ensure uniformity of the rate of proximity. However, much of the general information in [109] on how logical metatheorems can provide quantitative information which can give us e.g. a rate of proximity is relevant also in the new setting of [99].

speaking our approach will be based on requiring certain uniformity features of the majorants of the moduli introduced when axiomatizing the class of mappings to which f belongs. This will in a sense reduce the ∀∃∀-sentence expressing that the iteration sequence is Cauchy to a ∀∃-sentence. The metatheorem will then guarantee the existence of a uniform full rate of convergence. Earlier one could only get a full rate of convergence from a rate of proximity in the special case where f is required to be nonexpansive. In two case studies we have found such explicit and uniform rates of convergence for Picard iteration sequences for certain classes of (not necessarily nonexpansive) selfmappings of metric spaces, namely for asymptotic contractions in the sense of Kirk and also for so-called uniformly continuous uniformly generalized p-contractive mappings. The results of these case studies are included in Chapters 3 and 4. The results of this chapter provide an explanation for these findings (when restricted to bounded spaces) in logical terms[4]. But it is by no means necessary to acquaint oneself with the material in this chapter in order to appreciate or understand the material in Chapter 3 and Chapter 4. The concrete theorems and the proofs there do not in any way depend on the results in this chapter. Rather, the results here allow us to explain (to the extent noted above) that we could prove the results in Chapters 3 and 4, and it gives us a recipe for proving similar results in other concrete cases.

The general organization of the chapter is as follows: in the next section we will present the formal setting for the metatheorems, in Section 2.3 we will discuss how these theorems relate to questions concerning the convergence of iteration sequences for selfmaps of metric spaces, while in Section 2.4 we will present the main results. The mentioned applications are given in Section 2.5.

2.2 Formal framework and Kohlenbach's metatheorem for bounded metric spaces

We will here present Kohlenbach's metatheorem for bounded metric spaces from [99]. (The metatheorems have been extended in [56], replacing the condition that the space be bounded with some weak local boundedness criteria. We will remark further on this below.) The starting point for the metatheorems in [99] is the formal system $\mathcal{A}^\omega := \mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC} + \mathsf{DC}$, basically Peano arithmetic in all finite types with quantifier free axiom of choice, dependent choice and countable choice, but with only a certain quantifier-free rule of extensionality instead of the full axiom of extensionality. We will for reference present this system below. For more information, see [99, 101, 127, 169].

2.2.1 The system $\mathcal{A}^\omega$

We will begin with a series of definitions.

[4]However, we cannot yet properly explain that we in the concrete cases were able to find rates of convergence also in the setting of *unbounded* metric spaces.

Definition 2.5. The set $\mathbf{T}$ of all finite types is defined inductively by

$$0 \in \mathbf{T}, \text{ and if } \rho, \tau \in \mathbf{T}, \text{ then } (\rho \to \tau) \in \mathbf{T}.$$

The set $\mathbf{P}$ of pure types is generated inductively by

$$0 \in \mathbf{P}, \text{ and if } \rho \in \mathbf{P}, \text{ then } (\rho \to 0) \in \mathbf{P}.$$

Pure types are often denoted by natural numbers by letting $n+1$ denote $(n \to 0)$, so that e.g. 1 denotes $(0 \to 0)$.

Remark 2.6. Any type $\rho \neq 0$ can be written in the normal form

$$\rho = (\rho_1 \to (\rho_2 \to \ldots (\rho_k \to 0) \ldots)),$$

which we usually write as

$$\rho_1 \to \rho_2 \to \ldots \to \rho_k \to 0.$$

Notation 2.7. We usually do not write the outermost brackets for types, and we will often drop other brackets which are uniquely determined.

The intended interpretation of the base type 0 is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, and so we will sometimes blur the distinction and use "$\mathbb{N}$" instead of "0". Likewise we will sometimes write "natural numbers" instead of "objects of type 0". We present first the system $\mathsf{WE\text{-}HA}^\omega$, called weakly extensional Heyting arithmetic in all finite types.

Definition 2.8. The language $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ of $\mathsf{WE\text{-}HA}^\omega$ includes the language of a many-sorted version $\mathsf{IL}^\omega_{-=}$ of first order intuitionistic predicate logic $\mathsf{IL}_{-=}$ without equality, with variables x_n^ρ and quantifiers $\forall x^\rho$, $\exists x^\rho$ for every finite type ρ. Furthermore $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ includes constants 0^0, $S^{0\to 0}$ (successor), and for all finite types δ, ρ, τ a projector $\Pi_{\rho,\tau}^{\rho\to\tau\to\rho}$ and combinator $\Sigma_{\delta,\rho,\tau}$ (of type $(\delta \to \rho \to \tau) \to (\delta \to \rho) \to \delta \to \tau$), and also recursor constants $\underline{R}_{\underline{\rho}}$ for simultaneous primitive recursion in all finite types. $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ also contains a binary predicate constant $=_0$ (equality between objects of type 0).

Definition 2.9. The terms of $\mathsf{WE\text{-}HA}^\omega$ are determined by:

1. Constants and variables of type ρ are terms of type ρ.
2. If $t^{\rho\to\tau}$ is a term of type $\rho \to \tau$ and s^ρ is a term of type ρ, then (ts) is a term of type τ.

We will also when specifying terms sometimes omit uniquely determined brackets. In expressions such as tsw, association is assumed to be to the left. If $t^{\delta\to\rho\to\tau}$, s^δ, w^ρ are terms, we will sometimes write $t(s, w)$ for $((ts)w)$.

Definition 2.10. The formulas of $\mathsf{WE\text{-}HA}^\omega$ are determined by:

1. Prime formulas (also called "atomic formulas") $s^0 =_0 t^0$ are formulas. Also $\bot$ is a prime formula.
2. If A, B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \to B)$ are also formulas.
3. If $A(x^\rho)$ is a formula, then also $(\forall x^\rho A(x))$ and $(\exists x^\rho A(x))$ are formulas.

We will write $\neg A$ for $A \to \bot$, and $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$. We will furthermore write $x \neq_0 y$ for $\neg(x =_0 y)$. We will let $A(\underline{x})$, $B(\underline{x})$, $C(\underline{x})$ and so on denote formulas in $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$ with $x_1, \ldots, x_n$ free, where $\underline{x}$ is the tuple $x_1, \ldots, x_n$. We will let $A_0(\underline{x})$, $B_0(\underline{x})$, $C_0(\underline{x})$ and so on denote quantifier-free formulas in $\mathcal{L}(\mathsf{WE\text{-}HA}^\omega)$.

We note that the only primitive predicate in the language is $=_0$, so that in particular equality between higher type objects is not primitive. In fact, higher type equality is defined extensionally:

Definition 2.11. Higher type equality $=_\rho$ is defined by

$$s^\rho =_\rho t^\rho :\equiv \forall y_1^{\rho_1}, \ldots, y_k^{\rho_k} (s(y_1, \ldots, y_k) =_0 t(y_1, \ldots, y_k)),$$

where $\rho = \rho_1 \to \rho_2 \to \ldots \to \rho_k \to 0$.

Definition 2.12. The axioms and rules of $\mathsf{WE\text{-}HA}^\omega$ are as follows.

1. The axioms and rules of $\mathsf{IL}^\omega_{-=}$. We axiomatize first order intuitionistic predicate logic $\mathsf{IL}_{-=}$ without equality using Gödel's system (introduced in [60], see also [101, 169]).
2. Equality axioms for $=_0$:
   (a) $x =_0 x$,
   (b) $x =_0 y \to y =_0 x$,
   (c) $x =_0 y \wedge y =_0 z \to x =_0 z$.
3. Successor axioms:
   (a) $Sx \neq_0 0$,
   (b) $Sx =_0 Sy \to x =_0 y$.
4. Induction schema:

$$(\mathsf{IA}) : A(0) \wedge \forall x^0 (A(x) \to A(Sx)) \to \forall x^0 A(x),$$

   where $A(x^0)$ is an arbitrary formula of $\mathsf{WE\text{-}HA}^\omega$.
5. Axioms for $\Pi_{\rho,\tau}$, $\Sigma_{\delta,\rho,\tau}$ and $\underline{R}_{\underline{\rho}}$:

   (Π) : $\Pi_{\rho,\tau} x^\rho y^\tau =_\rho x^\rho$,
   (Σ) : $\Sigma_{\delta,\rho,\tau} x^{\delta\to\rho\to\tau} y^{\delta\to\rho} z^\delta =_\tau xz(yz)$,
   (R) : $\underline{R}_{\underline{\rho}} 0 \underline{y}\underline{z} =_{\underline{\rho}} \underline{y}$ and $\underline{R}_{\underline{\rho}}(Sx^0)\underline{y}\underline{z} =_{\underline{\rho}} \underline{z}(\underline{R}_{\underline{\rho}} x \underline{y}\underline{z})x$,

where $\underline{\rho} = \rho_1, \ldots, \rho_k$, y_i is of type ρ_i and z_i of type

$$\rho_1 \to \ldots \to \rho_k \to 0 \to \rho_i.$$

6. The following quantifier-free rule of extensionality:

$$\mathsf{QF\text{-}ER} : \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]}$$

where ρ, τ are arbitrary types and $s^\rho, t^\rho, r[x^\rho]^\tau$ are terms of $\mathsf{WE\text{-}HA}^\omega$.

No effort has been made to eliminate redundancies in this system.

Definition 2.13. We obtain the system $\mathsf{E\text{-}HA}^\omega$, called extensional Heyting arithmetic in all finite types (with the same language, terms and formulas as $\mathsf{WE\text{-}HA}^\omega$) by replacing the rule QF-ER with the following axioms for higher type extensionality:

$$\mathsf{E}_\rho :\equiv \forall z^\rho, x_1^{\rho_1}, y_1^{\rho_1}, \ldots, x_k^{\rho_k}, y_k^{\rho_k} (\bigwedge_{i=1}^{k} (x_i =_{\rho_i} y_i) \to z\underline{x} =_0 z\underline{y}),$$

where $\rho = \rho_1 \to \ldots \to \rho_k \to 0$.

The systems we will use will only have the restricted form of extensionality QF-ER, because one uses Gödel's functional interpretation to extract computational witnesses from proofs as a step in the metatheorems, and this is not possible if we have full extensionality (see [72]). However, the need to restrict to weak extensionality also has a natural mathematical interpretation, see the discussion on extensionality in [99].

Definition 2.14. By adding the law of excluded middle, that is, the schema

$$\mathsf{LEM} : A \vee \neg A,$$

we obtain $\mathsf{WE\text{-}PA}^\omega$ (respectively $\mathsf{E\text{-}PA}^\omega$) from $\mathsf{WE\text{-}HA}^\omega$ (respectively $\mathsf{E\text{-}HA}^\omega$). $\mathsf{E\text{-}PA}^\omega$ and $\mathsf{WE\text{-}PA}^\omega$ are called respectively extensional and weakly extensional Peano arithmetic in all finite types.

Remark 2.15. The fragments $\widehat{\mathsf{(W)E\text{-}PA}}^\omega\!\restriction$, $\widehat{\mathsf{(W)E\text{-}HA}}^\omega\!\restriction$ of respectively $\mathsf{(W)E\text{-}PA}^\omega$, $\mathsf{(W)E\text{-}HA}^\omega$ are obtained by excluding all the recursors $R_{\underline{\rho}}$ except the recursor R_0 for type-0-recursion, and by restricting the induction schema to the schema of quantifier-free induction

$$\mathsf{QF\text{-}IA}: \quad A_0(0) \wedge \forall x^0 \big(A_0(x) \to A_0(S(x))\big) \to \forall x^0 A_0(x),$$

where A_0 is quantifier-free and may contain parameters of arbitrary types. The set-theoretic functionals which are denoted by closed terms of $\widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction$ are called the primitive recursive functionals of finite type in the sense of Kleene, and were first introduced (for pure types) in [87], where they are called S1–S8 computable functionals. It turns out that the Kleene primitive recursive functionals of type 1 are exactly the ordinary primitive recursive functions. This is in contrast to the primitive recursive functionals in the sense of Gödel, i.e., the set-theoretic functionals denoted by closed terms of $\underline{\mathsf{E\text{-}PA}}^\omega$, where the functionals of type 1 form a wider class. The systems $\widehat{\mathsf{(W)E\text{-}HA}}^\omega\!\restriction$ were introduced by Feferman in [43].

Definition 2.16. The schema QF-AC of quantifier-free choice in all finite types is given by

$$\mathsf{QF\text{-}AC} : \forall \underline{x} \exists \underline{y} A_0(\underline{x}, \underline{y}) \to \exists \underline{Y} \forall \underline{x} A_0(\underline{x}, \underline{Y}\, \underline{x}),$$

where A_0 is quantifier-free and $\underline{x}, \underline{y}$ are tuples of variables of arbitrary type. The quantifier-free axiom of choice in types ρ, τ is the schema

$$\mathsf{QF\text{-}AC}^{\rho,\tau} : \forall x^\rho \exists y^\tau A_0(x, y) \to \exists Y^{\rho\to\tau} \forall x^\rho A_0(x, Yx),$$

where x^ρ and y^τ are single variables of the indicated types.

Definition 2.17. The schema of dependent choice[5] DC is defined by DC:= $\bigcup_{\rho\in\mathbf{T}}\{\mathsf{DC}^\rho\}$, where DC^ρ is

$$\forall x^0, y^\rho \exists z^\rho A(x, y, z) \to \exists f^{0\to\rho} \forall x^0 A(x, f(x), f(S(x))),$$

for A an arbitrary formula.

Definition 2.18. The system $\mathcal{A}^\omega$ is defined by

$$\mathcal{A}^\omega := \mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC} + \mathsf{DC}.$$

In $\mathcal{A}^\omega$ one can handle rational numbers and real numbers via an appropriate representation. Rational numbers are represented as pairs (n, m) of natural numbers coded into a single natural number $j(n, m)$ via the Cantor pairing function j. This is done in a way so that each natural number codes a unique rational number. Namely, $j(n, m)$ denotes the rational number $\frac{\frac{n}{2}}{m+1}$ if n is even, and the negative rational number $-\frac{\frac{n+1}{2}}{m+1}$ otherwise. For a rational number of the form 2^{-n} we write $\langle 2^{-n}\rangle$ for the (canonical) representative $j(2, 2^n - 1)$, and for a natural number n we write $\langle n\rangle$ or $n_\mathbb{Q}$ for the (canonical) representative $j(2n, 0)$. An equality relation $=_\mathbb{Q}$ on the representatives of the rational numbers, together with operators $+_\mathbb{Q}$, $-_\mathbb{Q}$, $\cdot_\mathbb{Q}$, etc. and also predicates $<_\mathbb{Q}$ and $\leq_\mathbb{Q}$ are defined primitive recursively in the natural way. Real numbers are represented by type 1 objects $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \left(|f(n) -_\mathbb{Q} f(n+1)|_\mathbb{Q} <_\mathbb{Q} \langle 2^{-n-1}\rangle\right).$$

One ensures that each functional f of type 1 represents a unique real number via the following construction, which can be carried out in $\mathcal{A}^\omega$:

$$\widehat{f}(n) := \begin{cases} f(n) & \text{if } \forall k < n \left(|f(k) -_\mathbb{Q} f(k+1)|_\mathbb{Q} <_\mathbb{Q} \langle 2^{-k-1}\rangle\right) \\ f(k) & \text{for } \min k < n \text{ with } |f(k) -_\mathbb{Q} f(k+1)|_\mathbb{Q} \geq_\mathbb{Q} \langle 2^{-k-1}\rangle \text{ else.} \end{cases}$$

[5]This formulation combines the usual formulation of dependent choice and countable choice, see [101].

Then f represents the real number represented by $\widehat{f}$. Real numbers are thus represented by functionals representing Cauchy sequences of rational numbers with a fixed Cauchy modulus $n \mapsto 2^{-n}$. For natural numbers $b \in \mathbb{N}$ we write $b_{\mathbb{R}}$ for the functional $\lambda n.b_{\mathbb{Q}}$ representing the real number b. For a rational number of the form 2^{-n} for $n \in \mathbb{N}$ we write $(2^{-n})_{\mathbb{R}}$ for the functional $\lambda k.j(2, 2^n - 1)$ representing the real number 2^{-n}. One defines relations $=_{\mathbb{R}}$, $<_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ on representatives of real numbers as follows:

$$\begin{aligned} f_1 =_{\mathbb{R}} f_2 &:\equiv \forall n \left(|\widehat{f}_1(n+1) -_{\mathbb{Q}} \widehat{f}_2(n+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-n} \rangle \right) \\ f_1 <_{\mathbb{R}} f_2 &:\equiv \exists n \left(\widehat{f}_2(n+1) -_{\mathbb{Q}} \widehat{f}_1(n+1) \geq_{\mathbb{Q}} \langle 2^{-n} \rangle \right) \\ f_1 \leq_{\mathbb{R}} f_2 &:\equiv \neg (f_2 <_{\mathbb{R}} f_1). \end{aligned}$$

Thus $=_{\mathbb{R}}$ and $\leq_{\mathbb{R}}$ are Π^0_1-predicates, while $<_{\mathbb{R}}$ is a Σ^0_1-predicate. One can now define operators $+_{\mathbb{R}}$, $-_{\mathbb{R}}$, $\cdot_{\mathbb{R}}$ etc. on representatives of real numbers by primitive recursive functionals (see [99] for details). We include also the following lemma, which appears in [99]:

Lemma 2.19. $\mathcal{A}^\omega \vdash \forall k^0 \left(|f -_{\mathbb{R}} \lambda n^0.\widehat{f}(k)|_{\mathbb{R}} <_{\mathbb{R}} (2^{-k})_{\mathbb{R}} \right)$.

2.2.2 The formal system $\mathcal{A}^\omega[X, d]$ for abstract bounded metric spaces

The theory $\mathcal{A}^\omega[X, d]$ for which the relevant metatheorem is proved is now obtained from $\mathcal{A}^\omega$ by "adding" an abstract metric space (X, d). $\mathcal{A}^\omega[X, d]$ results by (see [99]):

(i) Extending $\mathcal{A}^\omega$ to the set $\mathbf{T}^X$ of all finite types over the two ground types 0 and X, i.e.

$$0, X \in \mathbf{T}^X, \text{ and if } \rho, \tau \in \mathbf{T}^X, \text{ then } (\rho \to \tau) \in \mathbf{T}^X$$

(in particular, the constants $\Pi_{\rho,\tau}$, $\Sigma_{\delta,\rho,\tau}$, $\underline{R}_\rho$ for λ-abstraction and simultaneous primitive recursion (in the extended sense of Gödel [60]) and their defining axioms, and the schemes IA, QF-AC, DC and the weak extensionality rule QF-ER are now taken over the extended set of types (and the extended language)).

(ii) Adding a constant 0_X of type X and a constant b_X of type 0.

(iii) Adding a constant d_X of type $X \to X \to 1$ together with the axioms

(1) $\forall x^X \, (d_X(x, x) =_{\mathbb{R}} 0_{\mathbb{R}})$,
(2) $\forall x^X, y^X \, (d_X(x, y) =_{\mathbb{R}} d_X(y, x))$,
(3) $\forall x^X, y^X, z^X \, (d_X(x, z) \leq_{\mathbb{R}} d_X(x, y) +_{\mathbb{R}} d_X(y, z))$,
(4) $\forall x^X, y^X \, (d_X(x, y) \leq_{\mathbb{R}} (b_X)_{\mathbb{R}})$ $\big($with $(b_X)_{\mathbb{R}} := \lambda k^0.j(2b_X, 0)\big)$.

Still only equality at type 0 is a primitive predicate. One defines $x^X =_X y^X$ as $d_X(x,y) =_{\mathbb{R}} 0_{\mathbb{R}}$ and equality for complex types as extensional equality using $=_0$ and $=_X$ for the ground types.

To state the metatheorem in the next section we will need some notions and some more terminology. We first recall the definition of the full set-theoretic type structure over $\mathbb{N}$. All of the rest is taken from [99].

Definition 2.20. The full set-theoretic type structure

$$\mathcal{S}^\omega := \langle S_\rho \rangle_{\rho \in \mathbf{T}}$$

over $\mathbb{N}$ is defined by $S_0 := \mathbb{N}$ and $S_{\tau \to \rho} := S_\rho^{S_\tau}$. Here $S_\rho^{S_\tau}$ is the set of all set-theoretic functions $S_\tau \to S_\rho$.

Definition 2.21. Let X be a nonempty set. The full set-theoretic type structure

$$\mathcal{S}^{\omega,X} := \langle S_\rho \rangle_{\rho \in \mathbf{T}^X}$$

over $\mathbb{N}$ and X is defined by $S_0 := \mathbb{N}$, $S_X := X$ and $S_{\tau \to \rho} := S_\rho^{S_\tau}$. Here $S_\rho^{S_\tau}$ is the set of all set-theoretic functions $S_\tau \to S_\rho$.

We note that if $\rho \in \mathbf{T}$, then $\rho \in \mathbf{T}^X$ for any nonempty X, and S_ρ is the same whether thought of as belonging to $\mathcal{S}^\omega$ or $\mathcal{S}^{\omega,X}$.

Definition 2.22. For $x \in [0,\infty)$ define $(x)_\circ \in \mathbb{N}^\mathbb{N}$ by

$$(x)_\circ(n) := j\left(2k_0, 2^{n+1} - 1\right),$$

where

$$k_0 := \max\left\{\frac{k}{2^{n+1}} \leq x : k \in \mathbb{N}\right\}.$$

The following lemma, which lists some of the important properties of the function $(\cdot)_\circ : [0,\infty) \to \mathbb{N}^\mathbb{N}$, is a part of Lemma 2.10 in [99].

Lemma 2.23.

(1) *If $x \in [0,\infty)$, then $(x)_\circ$ is a representation of x in the sense of our representation of real numbers indicated above.*

(2) *If $x, y \in [0,\infty)$ and $x \leq y$ (in the sense of the usual order on $\mathbb{R}$), then $(x)_\circ \leq_{\mathbb{R}} (y)_\circ$ and $(x)_\circ \leq_1 (y)_\circ$.*

Definition 2.24. We will say that a sentence of the language $\mathcal{L}(\mathcal{A}^\omega[X,d])$ of $\mathcal{A}^\omega[X,d]$ holds in a bounded metric space (X,d) if it holds in the models of $\mathcal{A}^\omega[X,d]$ obtained by letting the variables range over the appropriate universes of the full set-theoretic type structure $\mathcal{S}^{\omega,X}$ with the set X as the universe for the base type X, letting 0_X be interpreted by an arbitrary element of X, letting b_X be interpreted as some integer upper bound (also denoted "b") for

d, and by letting d_X be interpreted by $\lambda x, y.(d(x, y))_\circ$, where $(\cdot)_\circ$ refers to the construction in Definition 2.22. We will sometimes denote a model of (an extension of) $\mathcal{A}^\omega[X, d]$ with domain $\mathcal{S}^{\omega,X}$ by $\mathcal{S}^{\omega,X}$, and similarly we will denote a model with domain $\mathcal{M}^{\omega,X}$ (to be defined below) by $\mathcal{M}^{\omega,X}$.

Definition 2.25. A type $\rho \in \mathbf{T}^X$ has degree 1 if

$$\rho = \tau_1 \to \ldots \to \tau_k \to 0$$

(including $\rho = 0$), with $\tau_i = 0$ for $1 \le i \le k$. The type ρ has degree 2 if

$$\rho = \tau_1 \to \ldots \to \tau_k \to 0$$

(including $\rho = 0$), with τ_i of degree 1 for $1 \le i \le k$. A type $\rho \in \mathbf{T}^X$ has degree $(0, X)$ if

$$\rho = \tau_1 \to \ldots \to \tau_k \to X$$

(including $\rho = X$), with $\tau_i = 0$ for $1 \le i \le k$. The type ρ has degree $(1, X)$ if

$$\rho = \tau_1 \to \ldots \to \tau_k \to X$$

(including $\rho = X$), where τ_i has degree 1 or $(0, X)$ for $1 \le i \le k$.

Definition 2.26. A formula F is called a $\forall$-formula (resp. $\exists$-formula) if it has the form $F \equiv \forall \underline{a}^{\underline{\sigma}} F_0(\underline{a})$ (resp. $F \equiv \exists \underline{a}^{\underline{\sigma}} F_0(\underline{a})$), where F_0 does not contain any quantifier and the types in $\underline{\sigma}$ are of degree 1 or $(1, X)$.

Note that when we elsewhere somewhat informally refer to "$\forall\exists\forall$-sentences" or "$\exists\forall$-sentences" then this indicates only the logical complexity of the prefix[6], with no restriction on the degrees of the types.

Definition 2.27. Between functionals x^ρ, y^ρ of type $\rho \in \mathbf{T}$ we define a relation $\le_\rho$ by induction on ρ as follows:

$$\begin{aligned} x \le_0 y \quad &:\equiv \quad x \le y \text{ for the usual (prim. rec.) order on } \mathbb{N}, \\ x \le_{\sigma\to\tau} y \quad &:\equiv \quad \forall z^\sigma \, (x(z) \le_\tau y(z)) \,. \end{aligned}$$

Definition 2.28. The extensional type structure $\mathcal{M}^{\omega,X} := \langle M_\rho \rangle_{\rho \in \mathbf{T}^X}$ of all hereditarily strongly majorizable set-theoretical functionals of type $\rho \in \mathbf{T}^X$ over $\mathbb{N}$ and a set X, together with the relation $x^* \text{ s-maj}_\rho\, x$ ("strong majorizability") between functionals x^*, x of type $\rho \in \mathbf{T}^X$, is defined as follows:

$$\begin{cases} M_0 := \mathbb{N}, \qquad x^* \text{ s-maj}_0\, x :\equiv x^* \ge x \wedge x^*, x \in \mathbb{N}, \\ M_X := X, \qquad x^* \text{ s-maj}_X\, x :\equiv x^*, x \in M_X, \\ x^* \text{ s-maj}_{\sigma\to\tau} x \\ \quad :\equiv x^*, x \in M_\tau^{M_\sigma} \wedge \forall y^*, y \in M_\sigma \, (y^* \text{ s-maj}_\sigma\, y \to x^*(y^*) \text{ s-maj}_\tau\, x^*(y), x(y)) \,, \\ M_{\sigma\to\tau} \quad := \left\{ x \in M_\tau^{M_\sigma} : \exists x^* \in M_\tau^{M_\sigma} (x^* \text{ s-maj}_{\sigma\to\tau}\, x) \right\} \; (\sigma, \tau \in \mathbf{T}^X). \end{cases}$$

Here $M_\tau^{M_\sigma}$ denotes the set of all total set-theoretical mappings from M_σ to M_τ.

[6] With "$\exists\exists\forall$" counted as "$\exists\forall$", etc.

There is a syntactic counterpart of s-maj formulated in $\mathcal{L}(\mathcal{A}^\omega[X,d])$, which we also denote by "s-maj". For x^ρ, y^ρ we define s-maj$_\rho$ as follows:

x^* s-maj$_0$ $x :\equiv x \leq_0 x^*$,

x^* s-maj$_X$ $x :\equiv 0 =_0 0$,

x^* s-maj$_{\sigma\to\tau}$ $x :\equiv \forall y^*, y(y^*$ s-maj$_\sigma$ $y \to x^*y^*$ s-maj$_\tau$ $x^*y, xy)$.

The notion of majorizability was originally introduced by Howard ([72]), and subsequently modified by Bezem ([19]). The version in Definition 2.28, where majorizability is extended to the new types in $\mathbf{T}^X$, is due to Kohlenbach. (We will often write "majorizable" or "majorant" instead of "strongly majorizable" and "strong majorant".)

Remark 2.29. The reason why we can define the majorization relation for objects of type X in this trivial way is that the space (X,d) is bounded, so that we have a common upper bound b for the distance between any two points. Already for normed spaces (which are also treated in [99]) the relation becomes

$$x^*\text{s-maj}_X x :\equiv x^*, x \in M_X \ \wedge \ \|x^*\| \geq \|x\|.$$

In [56] the approach which we are following here is extended to unbounded metric spaces, with associated generalizations of the theorems, and then the majorizability notion used is more involved; roughly speaking it becomes $x^* \geq d(a,x)$, where now x^* is a natural number and $a \in X$ is a reference point. For reasons given later we will not here consider these formal systems where the space is allowed to be unbounded (see Remark 2.44).

The following definition is a special case of a more general construction used in the proofs of the theorems in [99].

Definition 2.30. Define $\phi^{1\to 1}$ by recursion (using R_0) such that

$$\phi(x^1,0) =_0 x(0), \quad \text{and} \quad \phi(x^1,z+1) =_0 \max_0 \left(\phi(x,z), x(z+1)\right),$$

where $\max_0$ is the usual (primitive recursively definable) maximum between natural numbers. We write $x^M := \lambda z^0.\phi(x,z)$.

2.2.3 A metatheorem for bounded metric spaces

The theorem we state below is a part of Theorem 3.7 in [99].

Theorem 2.31 (Kohlenbach)**.** *Let σ, ρ be types of degree 1 and let τ be a type of degree $(1,X)$. Let $s^{\sigma\to\rho}$ be a closed term of $\mathcal{A}^\omega[X,d]$ and let $B_\forall(x^\sigma, y^\rho, z^\tau, u^0)$ be a $\forall$-formula containing only x,y,z,u free. Let also $C_\exists(x^\sigma, y^\rho, z^\tau, v^0)$ be an $\exists$-formula containing only x,y,z,v free. If*

$$\forall x^\sigma \forall y \leq_\rho s(x) \forall z^\tau \left(\forall u^0 B_\forall(x,y,z,u) \to \exists v^0 C_\exists(x,y,z,v)\right) \tag{2.1}$$

is provable in $\mathcal{A}^\omega[X,d]$*, then one can extract a computable functional* $\Phi : S_\sigma \times \mathbb{N} \to \mathbb{N}$ *such that for all* $x \in S_\sigma$ *and all* $b \in \mathbb{N}$

$$\forall y \leq_\rho s(x) \forall z^\tau [\forall u \leq \Phi(x,b) B_\forall(x,y,z,u) \to \exists v \leq \Phi(x,b) C_\exists(x,y,z,v)] \quad (2.2)$$

holds in any (nonempty) metric space (X,d) *whose metric is bounded by* $b \in \mathbb{N}$ *(with* b_X *interpreted by* b*).*

Some additional comments on Theorem 2.31:

(i) The computational complexity of the functional Φ can be estimated in terms of the strength of the $\mathcal{A}^\omega$-principle instances actually used in the proof. Φ can always be defined in the calculus T+(BR) of so-called bar recursive functionals, i.e. as a closed term of WE-PA$^\omega$+(BR). (For the definition of the schema (BR) of bar recursion we refer to e.g. Chapter 11 of [101]. Bar recursion was introduced by Spector in [166].) In particular, if DC is not used in the proof then Φ can be given as a closed term of WE-PA$^\omega$, and so it is primitive recursive in the sense of Gödel.

   The proof of Theorem 2.31 provides an extraction algorithm for Φ.

(ii) Instead of single variables x, y, z, u, v we may also have finite tuples of variables $\underline{x}$, $\underline{y}$, $\underline{z}$, $\underline{u}$, $\underline{v}$ as long as the elements of the respective tuples satisfy the same type restrictions as x, y, z, u, v. Moreover, instead of a single premise of the form $\forall u^0 B_\forall(x,y,z,u)$ we may have a finite conjunction of such premises.

The proof of this theorem is based on an extension of Spector's [166] interpretation of classical analysis by bar recursive functionals to the system $\mathcal{A}^\omega[X,d]$, and we will say more about the proof below. But first we will comment on possible extensions of the theorem which will be of relevance to us. Such extensions were implicit in [99], and explicitly commented on in [56].

An extension of Kohlenbach's metatheorem

Let $\mathcal{A}^\omega[X,d]+\Delta$ be the theory $\mathcal{A}^\omega[X,d]$ extended with new constants $c_1,\ldots,c_m$ of types of degree 2 and new constants $c_{m+1},\ldots,c_n$ of types of degree $(1,X)$ and with purely universal closed axioms with the types of all quantifiers of degree 2 or $(1,X)$. Assume that there exist closed terms $c_1^*,\ldots,c_m^*$ of $\mathcal{A}^\omega[X,d]+\Delta$ with the constants interpreted such that

$$\mathcal{S}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i \qquad \text{for } 1 \leq i \leq m, \quad (2.3)$$

where σ_i is the type of c_i. Then the theorem still holds in the sense that if $\mathcal{A}^\omega[X,d]+\Delta$ proves (2.1) (where $B_\forall$ and $C_\exists$ are now formulas of $\mathcal{A}^\omega[X,d]+\Delta$ and where s is a closed term of $\mathcal{A}^\omega[X,d]+\Delta$) then from a proof of (2.1) we can extract a partial functional

$$\Phi : S_\sigma \times S_{\sigma_1} \times \cdots \times S_{\sigma_m} \times \mathbb{N} \rightharpoonup \mathbb{N},$$

which can be defined in T+BR, and whose restriction to

$$M_\sigma \times M_{\sigma_1} \times \cdots \times M_{\sigma_m} \times \mathbb{N}$$

is a total (bar recursively) computable functional[7], such that

$$\forall y \leq_\rho s(x) \forall z^\tau [\forall u \leq \Phi(x, [c_1^*]_{\mathcal{S}^{\omega,X}}, \ldots, [c_m^*]_{\mathcal{S}^{\omega,X}}, b) B_\forall(x,y,z,u) \to \exists v \leq \Phi(x, [c_1^*]_{\mathcal{S}^{\omega,X}}, \ldots, [c_m^*]_{\mathcal{S}^{\omega,X}}, b) C_\exists(x,y,z,v)]$$

holds in any (nonempty) metric space (X,d) bounded by $b \in \mathbb{N}$ which satisfies the new purely universal axioms (with b_X interpreted by b and with the new constants interpreted by functionals from the appropriate universes of $\mathcal{S}^{\omega,X}$ such that (2.3) holds). Here $[c_i^*]_{\mathcal{S}^{\omega,X}}$ denotes the interpretation of c_i^* in $\mathcal{S}^{\omega,X}$. All of this follows from the proof of Theorem 3.7 in [99], and in the setting of [56] such extensions were explicitly commented on in Remark 4.13 in [56]. In relation to the proof we note that the restriction on the types of the new constants ensures that these can be interpreted by the same functionals in $\mathcal{S}^{\omega,X}$ and in $\mathcal{M}^{\omega,X}$, and that

$$\mathcal{S}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i$$

implies

$$\mathcal{M}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i$$

when all constants are interpreted in the same way in $\mathcal{S}^{\omega,X}$ and in $\mathcal{M}^{\omega,X}$. The restriction on the types of the quantifiers in the new axioms ensures that truth of the axioms in $\mathcal{S}^{\omega,X}$ implies truth of the axioms in $\mathcal{M}^{\omega,X}$. Φ does not depend on interpretations of majorants of $c_{m+1}, \ldots, c_n$, since we can take these majorants to be $\lambda x^\rho.0_X$ for suitable types ρ. And dependence on the interpretations of these terms can be eliminated by an easy extension of the method on page 121 of [99] used to eliminate the dependence on the interpretation of 0_X.

We could be more liberal in our type restrictions for $c_{m+1}, \ldots, c_n$, but types of degree $(1, X)$ are more than enough for our applications. We note also that if $c_1, \ldots, c_m$ are of types of degree 1 then $c_1^*, \ldots, c_m^*$ as required here always exist, by a construction analogous to the one in Definition 2.30. And in this case Φ is total, since then $S_{\sigma_i} = M_{\sigma_i}$ for $1 \leq i \leq m$ (and since $S_\sigma = M_\sigma$).

Remarks on the proof of the metatheorem

We will not present the proof of Theorem 2.31; for full details we refer to [99] and the references cited therein, and also to [101]. The proof is based on an extension of Spector's [166] and Howard's [71] interpretation of the system $\mathcal{A}^\omega$ by Spector's bar recursive functionals (T+BR) to the new formal system. One then interprets these functionals in $\mathcal{M}^{\omega,X}$, makes use of the restricted logical form of the sentences and the low degrees of the types involved to prove that the conclusion holds in $\mathcal{M}^{\omega,X}$, and uses the logical form of the sentences and the low degrees of the types involved to conclude that it holds also in $\mathcal{S}^{\omega,X}$.

[7] In the sense of [87] relativized to the typestructure $\mathcal{M}^\omega$.

Spector's interpretation of classical analysis is an extension of Gödel's interpretation of classical arithmetic by primitive recursive functionals of all finite types, which Gödel accomplished by combining his negative translation, which interprets classical arithmetic in intuitionistic arithmetic, with the functional ("Dialectica") interpretation which he developed in [60]. We will give the definition of a variant of the negative translation due to Kuroda [118], and also the definition of the functional interpretation, both times in the specific setting where the system under consideration is $\mathcal{A}^\omega[X,d]$.

Definition 2.32. Let A be a formula in the language of $\mathcal{A}^\omega[X,d]$. The negative translation A' of A is defined as follows. We let $A' :\equiv \neg\neg A^*$, where A^* is defined by induction on the logical structure of A:

(i) $A^* :\equiv A$, if A is a prime formula,

(ii) $(A \Box B)^* :\equiv (A^* \Box B^*)$, where $\Box \in \{\wedge, \vee, \rightarrow\}$,

(iii) $(\exists x^\rho A(x))^* :\equiv \exists x^\rho (A(x))^*$,

(iv) $(\forall x^\rho A(x))^* :\equiv \forall x^\rho \neg\neg (A(x))^*$.

Definition 2.33 (Functional interpretation of $\mathcal{A}^\omega[X,d]$)**.** To every formula A in the language of $\mathcal{A}^\omega[X,d]$ we assign a translation

$$A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$$

in the same language. The free variables of A^D are the same as those of A. The types and length of $\underline{x}, \underline{y}$ depend only on the logical structure of A, and A_D is a quantifier-free formula. For prime formulas A we let $A^D :\equiv A_D :\equiv A$. Assuming that $A^D \equiv \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$ and $B^D \equiv \exists \underline{u} \forall \underline{v} B_D(\underline{u}, \underline{v})$, we define:

(i) $(A \wedge B)^D :\equiv \exists \underline{x}, \underline{u} \forall \underline{y}, \underline{v} [A_D(\underline{x}, \underline{y}) \wedge B_D(\underline{u}, \underline{v})]$,

(ii) $(A \vee B)^D :\equiv \exists z^0, \underline{x}, \underline{u} \forall \underline{y}, \underline{v} [(z = 0 \rightarrow A_D(\underline{x}, \underline{y})) \wedge (z \neq 0 \rightarrow B_D(\underline{u}, \underline{v}))]$,

(iii) $(\exists z^\rho A(z))^D :\equiv \exists z, \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y}, z)$,

(iv) $(\forall z^\rho A(z))^D :\equiv \exists \underline{X} \forall z, \underline{y} A_D(\underline{X} z, \underline{y}, z)$,

(v) $(A \rightarrow B)^D :\equiv \exists \underline{U}, \underline{Y} \forall \underline{x}, \underline{v} (A_D(\underline{x}, \underline{Y}\, \underline{x}\, \underline{v}) \rightarrow B_D(\underline{U}\, \underline{x}, \underline{v}))$.

One can then combine negative translation and functional interpretation, so that given a formula A the functional interpretation of the negative translation A' of A is $(A')^D \equiv \exists \underline{x} \forall \underline{y} (A')_D(\underline{x}, \underline{y})$. In order to state the next lemma we ought to introduce the extension $\mathcal{A}^\omega[X,d]+(\mathsf{BR})$ of $\mathcal{A}^\omega[X,d]$, which results by for all tuples $\underline{\rho} = \rho_1, \ldots, \rho_m$ and $\underline{\tau} = \tau_1, \ldots, \tau_k$ of types in $\mathbf{T}^X$ adding new constants $\underline{B}^{\underline{\rho},\underline{\tau}}$ to the language – called bar recursors – and new axioms $(\mathsf{BR}^{\underline{\rho},\underline{\tau}})$ for these constants. In the setting of $\mathcal{A}^\omega$ these were introduced by Spector in [166]. However, we will not give the definition of bar recursion, but rather refer to Chapter 11 (and Chapter 17) in [101] for full details. The following lemma, which is crucial in [99], is a simple extension of Spector's deep result for $\mathcal{A}^\omega$ to $\mathcal{A}^\omega[X,d]$.

Lemma 2.34. *Let A be a sentence in the language of $\mathcal{A}^\omega[X,d]$. If*

$$\mathcal{A}^\omega[X,d] \vdash A,$$

then one can construct a tuple of closed terms $\underline{t}$ of $\mathcal{A}^\omega[X,d]$+(BR) such that

$$\mathcal{A}^\omega[X,d] - \mathsf{QF\text{-}AC} + (\mathsf{BR}) \vdash \forall \underline{y} (A')_D(\underline{t}, \underline{y}).$$

Here $\mathcal{A}^\omega[X,d]$−QF-AC+(BR) is the system $\mathcal{A}^\omega[X,d]$+(BR) without QF-AC. The construction of these $\underline{t}$ by recursion on a proof of A is a fundamental part of the algorithm for the extraction of bounds mentioned in Theorem 2.31, together with subsequent majorization. For a full proof of Lemma 2.34 we refer to Kohlenbach's book [101].

2.3 Some proof mining in metric fixed point theory

In this section we will explain how one in certain cases can extract rates of proximity for the Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ for selfmaps $f : X \to X$ on bounded metric spaces (X,d) from ineffective proofs of convergence to a unique fixed point.

2.3.1 Extracting rates of proximity

Let $\mathcal{A}^\omega[X,d] + \Delta$ be an extension of $\mathcal{A}^\omega[X,d]$ as in the discussion after Theorem 2.31 above, with a distinguished constant c_f of type $X \to X$. Then $\mathcal{A}^\omega[X,d] + \Delta$ determines a certain class of selfmappings on a class of nonempty bounded metric spaces in the sense that a selfmapping $f : X \to X$ on a nonempty bounded metric space (X,d) is a member of this class if one can obtain a model of $\mathcal{A}^\omega[X,d] + \Delta$ by letting the variables range over the appropriate universes of the full set-theoretic type structure $\mathcal{S}^{\omega,X}$ with the set X as the universe for the base type X, letting 0_X, b_X and d_X be interpreted as in Definition 2.24, letting c_f be interpreted by f and by letting the other new constants be interpreted such that (2.3) holds.

Let (X,d) be a nonempty bounded metric space, and let $f : X \to X$ be a selfmapping in the class of selfmappings determined by $\mathcal{A}^\omega[X,d] + \Delta$. Suppose we can prove that any fixed point of f is unique if it exists, i.e., that

$$\forall x, y \in X \, (f(x) = x \wedge f(y) = y \to x = y),$$

and furthermore that f is asymptotically regular, i.e., that

$$\forall x \in X \forall k \in \mathbb{N} \exists m \in \mathbb{N} \forall n \geq m \left(d\left(f^n(x), f^{n+1}(x)\right) < 2^{-k} \right). \tag{2.4}$$

Suppose further that this can be proved in $\mathcal{A}^\omega[X,d] + \Delta$, i.e., that

$$\mathcal{A}^\omega[X,d] + \Delta \vdash \forall x^X \forall y^X \, (c_f(x) =_X x \wedge c_f(y) =_X y \to x =_X y) \tag{2.5}$$

and

$$\mathcal{A}^\omega[X,d]+\Delta \vdash \forall x_0^X \forall k^0 \exists m^0 \forall n^0 \left(n \geq_0 m \to d_X(x_n, x_{n+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}}\right), \quad (2.6)$$

where x_n is the n-th member of the Picard iteration sequence starting with x_0, which is definable in the theory[8]. We will hold these assumptions fixed for the remainder of the section.

We can write (2.5) as

$$\mathcal{A}^\omega[X,d]+\Delta \vdash \forall x^X, y^X \forall k^0 \big(\forall m^0 A_0(x,y,m) \to |\widehat{d_X(x,y)}(k+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-k}\rangle\big),$$

where

$$\begin{aligned} A_0(x,y,m) \quad :\equiv \quad & |\widehat{d_X(x,c_f x)}(m+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-m}\rangle \\ & \wedge |\widehat{d_X(y,c_f y)}(m+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-m}\rangle \end{aligned} \quad (2.7)$$

is quantifier-free. So Kohlenbach's metatheorem implies that we can extract (from the corresponding formal proof) a partial functional

$$\Phi : \mathbb{N}\times\mathbb{N}\times S_{\sigma_1}\times\cdots\times S_{\sigma_m} \rightharpoonup \mathbb{N}$$

such that

$$\forall x_1, x_2 \in X \forall k \in \mathbb{N} \left(\bigwedge_{i=1}^{2} d(x_i, f(x_i)) < 2^{-\Phi(k,b,\vec{F})} \to d(x_1,x_2) < 2^{-k}\right) \quad (2.8)$$

holds, where F_i denotes the interpretation of c_i^* and where we use the notation $\Phi(k,b,\vec{F})$ for $\Phi(k,b,F_1,\ldots,F_m)$. (In (2.8) x_1 and x_2 do not denote members of a Picard iteration sequence but rather arbitrary elements of X). Thus f has a modulus of uniqueness $\lambda k.\Phi(k,b,\vec{F})$. And (2.8) holds in fact for all b-bounded metric spaces (X,d) and mappings f satisfying $\mathcal{A}^\omega[X,d]+\Delta$ with the constants suitably interpreted. Note that from (2.4) and (2.8) it follows that all Picard iteration sequences are Cauchy and that

$$\forall x_0, y_0 \in X \forall k \in \mathbb{N} \exists m \in \mathbb{N} \forall n \geq m \left(d(x_n, y_n) < 2^{-k}\right).$$

Thus if the space is complete then there exists $z \in X$ such that all Picard iteration sequences $(x_n)_{n\in\mathbb{N}}$ converge to z, and if in addition f is continuous then we can conclude that z is a fixed point.

To treat (2.6) we first notice that (since we have (2.6)) we in particular have

$$\mathcal{A}^\omega[X,d]+\Delta \vdash \forall x_0^X \forall k^0 \exists m^0 \left(d_X(x_m, x_{m+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}}\right). \quad (2.9)$$

Similarly to the above we can now extract a partial functional

$$\Psi : \mathbb{N}\times\mathbb{N}\times S_{\sigma_1}\times\cdots\times S_{\sigma_m} \rightharpoonup \mathbb{N}$$

[8] Formally we can for example let $x_n := P(x_0,n)$, where $P := \lambda x^X, n^0.\, R_X n^0 x^X z$, with $z := \lambda x^X, m^0.\, c_f x$. Using λ-abstraction is allowed since we in $\mathcal{A}^\omega[X,d]+\Delta$ have closure under functional abstraction by a trivial extension of Lemma 2.4 in [99].

such that

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \leq \Psi(n, b, \vec{F}) \left(d(x_m, x_{m+1}) < 2^{-n} \right), \tag{2.10}$$

with $(x_n)_{n \in \mathbb{N}}$ the Picard iteration sequence with starting point $x_0 \in X$. Thus $\lambda n.\Psi(n, b, \vec{F})$ is a modulus of uniform almost asymptotic regularity for f. We can combine the functional Ψ in (2.10) with a modulus of uniqueness for f as follows. Suppose that f has a fixed point $z \in X$. Then for each $n \in \mathbb{N}$ we have $d(z, f(z)) < 2^{-n}$, and so from (2.8) and (2.10) it follows that

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \leq \Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big) \left(d(x_m, z) < 2^{-n} \right). \tag{2.11}$$

That is, the function $\lambda n.\Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big)$ is a rate of proximity for all Picard iteration sequences $(x_n)_{n \in \mathbb{N}}$ to z. Assume now that X is complete, and let $z \in X$ be the point to which all Picard iteration sequences converge. If z is not a fixed point, then we get

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \leq \Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big) \left(d(x_m, z) \leq 2^{-n} \right).$$

Namely, let $x_0 \in X$ and $n \in \mathbb{N}$. Then given $\varepsilon > 0$ we can let $k \in \mathbb{N}$ be such that $d(x_k, z) < \varepsilon$ and

$$d(x_k, x_{k+1}) < 2^{-\Phi(n,b,\vec{F})}.$$

There exists

$$m \leq \Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big)$$

such that

$$d(x_m, x_{m+1}) < 2^{-\Phi(n,b,\vec{F})},$$

and so by (2.8) we have

$$d(x_m, z) \leq d(x_m, x_k) + d(x_k, z) < 2^{-n} + \varepsilon.$$

Since $\Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big)$ does not depend on ε we get $d(x_m, z) \leq 2^{-n}$ for some

$$m \leq \Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big).$$

If f is nonexpansive and if we assume that f has a fixed point then (2.11) yields a uniform rate of convergence to the fixed point:

$$\forall x_0 \in X \forall n \in \mathbb{N} \forall m \geq \Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big) \left(d(x_m, z) < 2^{-n} \right). \tag{2.12}$$

So in this case a rate of proximity gives a rate of convergence. Note that if f is nonexpansive but if we do not assume that f has a fixed point then we get from (2.10) that

$$\forall x_0 \in X \forall n \in \mathbb{N} \forall m \geq \Psi(n, b, \vec{F}) \left(d(x_m, x_{m+1}) < 2^{-n} \right). \tag{2.13}$$

From this and from (2.8) it follows that

$$\forall x_0, y_0 \in X \forall n \in \mathbb{N} \forall m \geq \Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big) \left(d(x_m, y_m) < 2^{-n} \right). \tag{2.14}$$

Hence if the space is complete then in this case there exists a unique fixed point z such that all Picard iteration sequences converge to z with a rate of convergence which is uniform in the starting point.

The account thus far is based on the discussion in [109]. For the sake of illustrating the concepts in question this approach was used there to obtain a constructive version of Edelstein's theorem for contractive mappings. In [54] this general approach was used to obtain a quantitative version of Kirk's theorem on asymptotic contractions. This involved finding a rate of proximity to the unique fixed point for all Picard iteration sequences. This rate of proximity is then a rate of convergence if we restrict ourselves to nonexpansive mappings. However, asymptotic contractions in the sense of Kirk need not be nonexpansive. We have been able to build on Gerhardy's work to obtain a full rate of convergence for asymptotic contractions in the sense of Kirk without assuming the mappings to be nonexpansive: this work appears in Chapter 3. We have also used the approach outlined here to find a rate of proximity for uniformly continuous uniformly generalized p-contractive mappings, and also in this case we found that we could extend our results so as to get a full rate of convergence without assuming the mappings to be nonexpansive: this work appears in Chapter 4.

These results prompted the investigations in this chapter into the role of uniqueness of the fixed point and the existence of uniform rates of convergence in the general case.

However, before proving our theorem we will comment on a variation of the method above which often works in practice and which amounts to a numerical improvement.

2.3.2 Eliminating the modulus of uniqueness

The rate of proximity above was obtained as a combination of a modulus of uniqueness Φ with the functional Ψ extracted from a proof of (2.9). In fact, in many cases the proof that for any $k \in \mathbb{N}$ and for any $x_0 \in X$ and $x_1 := f(x_0)$ there exists an $m \in \mathbb{N}$ such that $d(f^m(x_0), f^m(x_1)) < 2^{-k}$ does not use that $x_1 = f(x_0)$, i.e., it is exactly the same as a proof that for any $x_0 \in X$ and $y_0 \in X$ there exists an $m \in \mathbb{N}$ such that $d(f^m(x_0), f^m(y_0)) < 2^{-k}$, just with the variables x_1 and y_0 interchanged. Then from the proof of the formalized statement we get that

$$\forall x_0, y_0 \in X \forall n \in \mathbb{N} \exists m \leq \Psi(n, b, \vec{F}) \left(d(x_m, y_m) < 2^{-n} \right). \tag{2.15}$$

Now, if the space is complete then we know that there is a point $z \in X$ to which all Picard iteration sequences converge (by the discussion after (2.8) above, since we still assume that we have (2.5) and (2.6)), and it is easy to see that

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \leq \Psi(n, b, \vec{F}) \left(d(x_m, z) \leq 2^{-n} \right). \tag{2.16}$$

Namely, let $x_0 \in X$ and $n \in \mathbb{N}$. Then given $\varepsilon > 0$ we can let y_0 in (2.15) be such that $d(y_k, z) < \varepsilon$ for all $k \in \mathbb{N}$. This we can do by e.g. letting $y_0 = x_l$ for large enough $l \in \mathbb{N}$. Then by the triangle inequality and (2.15) we have that there exists $m \leq \Psi(n, b, \vec{F})$ with

$$d(x_m, z) < 2^{-n} + \varepsilon.$$

Since $\Psi(n, b, \vec{F})$ does not depend on ε we get $d(x_m, z) \leq 2^{-n}$ for some $m \leq \Psi(n, b, \vec{F})$. In contrast to this, if z is a fixed point then we get directly from (2.15) that

$$\forall x_0 \in X \forall n \in \mathbb{N} \exists m \leq \Psi(n, b, \vec{F}) \left(d(x_m, z) < 2^{-n} \right). \tag{2.17}$$

That is, $\lambda n.\Psi(n, b, \vec{F})$ is a rate of proximity to z for all Picard iteration sequences. Compared to the previously obtained rate of proximity

$$\lambda n.\Psi\big(\Phi(n, b, \vec{F}), b, \vec{F}\big)$$

this is in most cases a significant quantitative improvement, since moduli of uniqueness $\lambda n.\Phi(n, b, \vec{F})$ in practice tend to satisfy $\Phi(n, b, \vec{F}) > n$. We say that we have "eliminated the modulus of uniqueness". For the quantitative version of Edelstein's theorem on contractive mappings mentioned above, and also for the quantitative result on generalized p-contractive mappings, one could numerically improve the convergence rates obtained in exactly this way (see [23] and Chapter 4). Also in the case of the rate of proximity for asymptotic contractions in the sense of Kirk obtained in [54] we could numerically improve the results by eliminating what in that context functions more or less as a modulus of uniqueness (see Chapter 3).

2.4 Main results

We will exploit the fact that the bounds which the metatheorem guarantees are uniform in the space (X, d) and the mapping $f : X \to X$ except through a bound $b \in \mathbb{N}$ on the space and majorants for the new constants one introduces when developing a formal theory for the class of mappings under consideration by adding purely universal axioms to $\mathcal{A}^\omega[X, d]$.

2.4.1 A combinatorial lemma concerning finite product spaces

We will need a lemma which in itself has nothing to do with logic or proof mining, and which might deserve some independent interest.

Definition 2.35. Let (X, d) be a metric space, let $f : X \to X$ and let $m \geq 1$ be a natural number. We define a metric space (X^m, d_m) by supplying the Cartesian product X^m with a metric d_m defined by

$$d_m(\vec{x}, \vec{y}) = \max\left\{d(x^1, y^1), \ldots, d(x^m, y^m)\right\},$$

where $\vec{x} = (x^1, \ldots, x^m)$ and $\vec{y} = (y^1, \ldots, y^m)$. (We here use superscripts instead of subscripts to indicate the i-th coordinate, in order to avoid confusion with members of iteration sequences.) We define the mapping $f_m : X^m \to X^m$ by

$$f_m(\vec{x}) = \left(f(x^1), \ldots, f(x^m)\right).$$

We note that if (X, d) is bounded by $b > 0$ then for all $m \geq 1$ we have that also (X^m, d_m) is bounded by b.

Lemma 2.36 (Main combinatorial lemma)**.** *Let (X, d) be a metric space and let $f : X \to X$ be a mapping. Assume that there exists a function $\Phi : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall k \in \mathbb{N} \forall \vec{x}, \vec{y} \in X^m \exists n \leq \Phi(k) \left(d_m\left(f_m^n(\vec{x}), f_m^n(\vec{y})\right) < 2^{-k-3}\right) \tag{2.18}$$

holds for infinitely many natural numbers $m \geq 1$, where (X^m, d_m) and f_m are respectively the product space and the product mapping introduced in Definition 2.35. Then

$$\forall k \in \mathbb{N} \forall x, y \in X \forall l, n \geq \Phi(k) \left(d\left(f^l(x), f^n(y)\right) < 2^{-k}\right).$$

And so if the space is complete then there exists $z \in X$ such that all Picard iteration sequences $\left(f^n(x)\right)_{n\in\mathbb{N}}$ converge to z with a rate of convergence which is uniform in the starting point, and if f is continuous then z is the unique fixed point of f.

Proof. Let $x, y \in X$ and let $k \in \mathbb{N}$. Let M be the set of natural numbers $m \geq 1$ such that (2.18) holds. Assume for the moment that $1 \in M$. Then taking $m = 1$ and $x, f(x) \in X$ in (2.18) (where we identify (X^1, d_1) with (X, d)) we get that for some $n \leq \Phi(k)$ we have

$$d\left(f^n(x), f^{n+1}(x)\right) < 2^{-k-3}.$$

Likewise, if $2 \in M$, then taking $m = 2$ and $(x, x), \left(f(x), f^2(x)\right) \in X^2$ in (2.18) we get that for some $n \leq \Phi(k)$ we have

$$d\left(f^n(x), f^{n+1}(x)\right) < 2^{-k-3} \wedge d\left(f^n(x), f^{n+2}(x)\right) < 2^{-k-3}.$$

In general, for any $m \in M$ we take $(x, \ldots, x), (f(x), \ldots, f^m(x)) \in X^m$ in (2.18). Then for some $n \leq \Phi(k)$ we have

$$\bigwedge_{i=1}^{m} \left(d\left(f^n(x), f^{n+i}(x)\right) < 2^{-k-3}\right).$$

Since there are only finitely many $n \leq \Phi(k)$ and since $\Phi(k)$ is independent from m it follows that there is some $n_1 \leq \Phi(k)$ such that

$$\bigwedge_{i=1}^{m} \left(d\left(f^{n_1}(x), f^{n_1+i}(x)\right) < 2^{-k-3}\right)$$

holds for infinitely many $m \geq 1$, i.e., such that

$$d\left(f^{n_1}(x), f^{n_1+i}(x)\right) < 2^{-k-3}$$

holds for all $i \in \mathbb{N}$. Then in particular

$$d\big(f^{n_1}(x), f^{\Phi(k)}(x)\big) < 2^{-k-3},$$

and so by the triangle inequality we have that

$$d\big(f^{\Phi(k)}(x), f^{n}(x)\big) < 2^{-k-3} + 2^{-k-3} = 2^{-k-2}$$

holds for all $n \geq \Phi(k)$. Analogously we get that

$$d\big(f^{\Phi(k)}(y), f^{n}(y)\big) < 2^{-k-2}$$

holds for all $n \geq \Phi(k)$. Let $m_1 \in M$. By (2.18) we know that for any $\varepsilon > 0$ there exists $i \in \mathbb{N}$ such that for $\vec{x'} = \big(f^{\Phi(k)}(x), \ldots, f^{\Phi(k)}(x)\big)$ and $\vec{y'} = \big(f^{\Phi(k)}(y), \ldots, f^{\Phi(k)}(y)\big)$ we have

$$d_{m_1}\big(f^{i}_{m_1}(\vec{x'}), f^{i}_{m_1}(\vec{y'})\big) < \varepsilon,$$

and thus

$$d\big(f^{\Phi(k)+i}(x), f^{\Phi(k)+i}(y)\big) < \varepsilon.$$

Hence

$$\begin{aligned} d\big(f^{\Phi(k)}(x), f^{\Phi(k)}(y)\big) &\leq d\big(f^{\Phi(k)}(x), f^{\Phi(k)+i}(x)\big) + \\ &\quad\; d\big(f^{\Phi(k)+i}(x), f^{\Phi(k)+i}(y)\big) + d\big(f^{\Phi(k)+i}(y), f^{\Phi(k)}(y)\big) \\ &< 2^{-k-2} + \varepsilon + 2^{-k-2}, \end{aligned}$$

and since $\varepsilon > 0$ was arbitrary we have

$$d\big(f^{\Phi(k)}(x), f^{\Phi(k)}(y)\big) \leq 2^{-k-1}.$$

So for $l, n \geq \Phi(k)$ we have

$$\begin{aligned} d\big(f^{l}(x), f^{n}(y)\big) &\leq d\big(f^{l}(x), f^{\Phi(k)}(x)\big) + \\ &\quad\; d\big(f^{\Phi(k)}(x), f^{\Phi(k)}(y)\big) + d\big(f^{\Phi(k)}(y), f^{n}(y)\big) \\ &< 2^{-k-2} + 2^{-k-1} + 2^{-k-2} \\ &= 2^{-k}. \end{aligned}$$

□

We note that it is easy to see that if (2.18) in the lemma above holds for some m then it holds for all positive integers $m' < m$. Thus under the assumptions of the lemma we have that (2.18) holds for all $m \geq 1$.

2.4.2 Uniform product space models

The following definition is meant to capture a certain relationship between a formal theory $\mathcal{A}^\omega[X,d]+\Delta$ for some class of selfmappings on bounded metric spaces and a bounded metric space (X,d) together with a mapping $f : X \to X$. Loosely speaking this involves that not only should (X,d) and $f : X \to X$ give rise to a model of $\mathcal{A}^\omega[X,d]+\Delta$, but so should every finite product space (X^m, d_m) and product mapping f_m, and this should happen in a certain uniform way for all $m > 0$. We will in the last section of this chapter see two concrete examples from metric fixed point theory of classes of mappings on bounded metric spaces such that one can find theories $\mathcal{A}^\omega[X,d]+\Delta$ for which every mapping of the kind considered together with the bounded metric space on which it is defined provide a uniform product space model.

Definition 2.37. Let $\mathcal{A}^\omega[X,d]+\Delta$ be the theory $\mathcal{A}^\omega[X,d]$ extended with a new constant c_f of type $X \to X$ and with new constants $c_1,\ldots,c_{n_1}$ of types of degree 2 and new constants $c_{n_1+1},\ldots,c_{n_2}$ of types of degree $(1,X)$, and also with purely universal closed axioms with the types of all quantifiers of degree 2 or $(1,X)$. We say that a nonempty bounded metric space (X,d) and a selfmap $f : X \to X$ together *provide a uniform product space model* for $\mathcal{A}^\omega[X,d]+\Delta$ if:

(∗) There exist closed terms $c_1^*,\ldots,c_{n_1}^*$ of $\mathcal{A}^\omega[X,d]+\Delta$ such that for all m in an infinite set $M \subseteq \mathbb{N}\setminus\{0\}$ one can obtain a model of $\mathcal{A}^\omega[X,d]+\Delta$ by:

(i) letting the variables range over the appropriate universes of the full set-theoretic type structure $\mathcal{S}^{\omega,X^m}$ with the set X^m as the universe for the base type X,

(ii) letting 0_X be interpreted by an arbitrary element of X^m and letting $c_{n_1+1},\ldots,c_{n_2}$ be interpreted by functionals from the appropriate universes of $\mathcal{S}^{\omega,X^m}$,

(iii) letting b_X be interpreted by an integer upper bound b for d_m and letting d_X be interpreted by $\lambda x,y.\,(d_m(x,y))_\circ$,

(iv) letting c_f be interpreted by f_m,

(v) and finally by letting $c_1,\ldots,c_{n_1}$ be interpreted such that

$$\mathcal{S}^{\omega,X^m} \models c_i^* \text{ s-maj}_{\sigma_i}\, c_i \qquad \text{for } 1\le i\le n_1,$$

where σ_i is the type of c_i.

And furthermore for all $m\in M$ the terms $c_1^*,\ldots,c_{n_1}^*$ are interpreted by the same functionals $F_1,\ldots,F_{n_1}$ in the models above.

The concept in this definition will be crucial in our theorems below.

2.4.3 A theorem guaranteeing the extractability of uniform and effective rates of convergence

The following theorem is our main result in this chapter. It uses the notion of a metric space (X, d) and a mapping $f : X \to X$ providing a uniform product space model for a theory (which is meant to capture the class of mappings to which f belongs) to give conditions guaranteeing the extractability of computable and uniform rates of convergence. However, we do not need the full strength of Theorem 2.38 to explain the results of our case studies, and so we include some corollaries which do suffice in many cases.

Theorem 2.38. *Let $\mathcal{A}^\omega[X, d] + \Delta$ be as in Definition 2.37. Suppose that $\mathcal{A}^\omega[X, d] + \Delta$ proves*

$$\forall x^X \forall y^X \left(c_f(x) =_X x \wedge c_f(y) =_X y \to x =_X y\right) \tag{2.19}$$

and

$$\forall x_0^X, y_0^X \forall k^0 \exists n^0 \left(d_X(x_n, x_{n+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}} \wedge d_X(y_n, y_{n+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}}\right), \tag{2.20}$$

where x_n and y_n are the n-th members of the defined Picard iteration sequences[9] *starting with respectively x_0 and y_0. Then from the proofs in $\mathcal{A}^\omega[X, d] + \Delta$ of (2.19) and (2.20) one can extract a partial functional*

$$\Phi : \mathbb{N} \times \mathbb{N} \times S_{\sigma_1} \times \cdots \times S_{\sigma_{n_1}} \rightharpoonup \mathbb{N},$$

which can be given as a closed term of $\mathsf{WE\text{-}PA}^\omega + (\mathsf{BR})$, and whose restriction to $\mathbb{N} \times \mathbb{N} \times M_{\sigma_1} \times \cdots \times M_{\sigma_{n_1}}$ is a total (bar recursively) computable functional, such that whenever we have a nonempty metric space (X, d) bounded by $b \in \mathbb{N}$ and a mapping $f : X \to X$, which together provide a uniform product space model for $\mathcal{A}^\omega[X, d] + \Delta$, then

$$\forall k \in \mathbb{N} \forall x, y \in X \forall l, n \geq \Phi(k, b, \vec{F}) \left(d\left(f^l(x), f^n(y)\right) < 2^{-k}\right)$$

holds in (X, d), where $\vec{F}$ is as in condition $(*)$ in Definition 2.37.

Proof. As commented on after Theorem 2.31 above it follows from the proof of Theorem 3.7 in [99] that Theorem 2.31 can be extended to cover $\mathcal{A}^\omega[X, d] + \Delta$, in the sense that if a sentence of the form (2.1) (where $B_\forall$ and $C_\exists$ are now formulas of $\mathcal{A}^\omega[X, d] + \Delta$ and where s is a closed term of $\mathcal{A}^\omega[X, d] + \Delta$) from Theorem 2.31 is provable in $\mathcal{A}^\omega[X, d] + \Delta$, then we can extract a partial functional (which is total when restricted to the majorizable elements)

$$\Psi : S_\sigma \times S_{\sigma_1} \times \cdots \times S_{\sigma_{n_1}} \times \mathbb{N} \rightharpoonup \mathbb{N},$$

which can be defined by a closed term of $\mathsf{WE\text{-}PA}^\omega + (\mathsf{BR})$, such that

$$\forall y \leq_\rho s(x) \forall z^\tau [\forall u \leq \Psi(x, \vec{F}, b) B_\forall(x, y, z, u) \to \\ \exists v \leq \Psi(x, \vec{F}, b) C_\exists(x, y, z, v)]$$

[9] That is, we write x_n for $P(x_0, n)$, where $P := \lambda x^X, n^0.\, R_X n^0 x^X z$, with $z := \lambda x^X, m^0.\, c_f x$.

holds in any nonempty metric space (X, d) bounded by $b \in \mathbb{N}$ which satisfies the new purely universal axioms with b_X interpreted by b and with the new constants interpreted by functionals from the appropriate universes of $\mathcal{S}^{\omega,X}$ such that there are closed terms $c_1^*, \ldots, c_{n_1}^*$ interpreted by $F_1, \ldots, F_{n_1}$ such that

$$\mathcal{S}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i \qquad \text{for } 1 \leq i \leq n_1. \tag{2.21}$$

Since (2.19) and (2.20) are of the appropriate logical form (when we treat (2.19) as in the discussion at the beginning of Section 2.3) we can extract functionals Ψ_1 and Ψ_2 such that for any nonempty bounded metric space (X, d) and mapping $f : X \to X$ which provide a uniform product space model for $\mathcal{A}^\omega[X, d] + \Delta$ we have for all m in an infinite set $M \subseteq \mathbb{N} \setminus \{0\}$ that

$$\forall \vec{x}_1, \vec{x}_2 \in X^m \forall k \in \mathbb{N} \left(\bigwedge_{i=1}^{2} d_m\left(\vec{x}_i, f_m(\vec{x}_i)\right) < 2^{-\Psi_1(k,b,\vec{F})} \to d_m(\vec{x}_1, \vec{x}_2) < 2^{-k} \right)$$

and

$$\forall \vec{x}_1, \vec{x}_2 \in X^m \forall k \in \mathbb{N} \exists n \leq \Psi_2(k, b, \vec{F}) \left(\bigwedge_{i=1}^{2} d_m\left(f_m^n(\vec{x}_i), f_m^{n+1}(\vec{x}_i)\right) < 2^{-k} \right),$$

where $b \in \mathbb{N}$ is a bound on d and where $F_1, \ldots, F_{n_1}$ are as in $(*)$ in Definition 2.37. It is here essential that for each $m \in M$ the product space (X^m, d_m) gives rise to a model of $\mathcal{A}^\omega[X, d] + \Delta$ as specified in $(*)$ in Definition 2.37, and that the majorants $\vec{F}$ are uniform in m. Hence with Φ defined by

$$\Phi(k, b, \vec{F}) := \Psi_2\left(\Psi_1(k + 3, b, \vec{F}), b, \vec{F}\right)$$

we have

$$\forall k \in \mathbb{N} \forall \vec{x}, \vec{y} \in X^m \exists n \leq \Phi(k, b, \vec{F}) \left(d_m\left(f_m^n(\vec{x}), f_m^n(\vec{y})\right) < 2^{-k-3}\right) \tag{2.22}$$

for all $m \in M$. Thus by Lemma 2.36 we have

$$\forall k \in \mathbb{N} \forall x, y \in X \forall l, n \geq \Phi(k, b, \vec{F}) \left(d\left(f^l(x), f^n(y)\right) < 2^{-k}\right).$$

□

We will include some corollaries. The first one is directly a special case of Theorem 2.38.

Corollary 2.39. *Let $\mathcal{A}^\omega[X, d] + \Delta$ be as in Definition* 2.37*, but such that the new constants $c_1, \ldots, c_{n_1}$ are of types of degree* 1*. Suppose that $\mathcal{A}^\omega[X, d] + \Delta$ proves* (2.19) *and* (2.20) *from Theorem* 2.38*. Then from the proofs in $\mathcal{A}^\omega[X, d] + \Delta$ of* (2.19) *and* (2.20) *one can extract a computable functional*

$$\Phi : \mathbb{N} \times \mathbb{N} \times S_{\sigma_1} \times \cdots \times S_{\sigma_{n_1}} \to \mathbb{N},$$

which can be defined by a closed term of WE-PA$^\omega$+(BR)*, such that whenever we have a nonempty metric space* (X, d) *bounded by* $b \in \mathbb{N}$ *and a mapping* $f : X \to X$*, which together provide a uniform product space model for* $\mathcal{A}^\omega[X, d]+\Delta$*, then*

$$\forall k \in \mathbb{N} \forall x, y \in X \forall l, n \geq \Phi(k, b, \vec{F}) \left(d\left(f^l(x), f^n(y)\right) < 2^{-k}\right)$$

holds in (X, d)*, where* $\vec{F}$ *is as in condition* $(*)$ *in Definition* 2.37*.*

Proof. That Φ is total follows from Theorem 2.38, since $S_{\sigma_i} = M_{\sigma_i}$ for types σ_i of degree 1.

□

Notice that in the case of Corollary 2.39 the new constants can always be majorized – the question is whether these majorants are the same for all product spaces and product mappings.

Corollary 2.40. *Let* $\mathcal{A}^\omega[X, d]+\Delta-$DC *be as* $\mathcal{A}^\omega[X, d]+\Delta$ *in Definition* 2.37*, but without* DC*, and such that the new constants* $c_1, \ldots, c_{n_1}$ *are allowed to be of arbitrary types* $\sigma_i \in \mathbf{T}$*, and the new purely universal axioms are allowed to have quantifiers of arbitrary types* $\sigma \in \mathbf{T}$ *instead of only types* $\sigma \in \mathbf{T}$ *which are of degree* 2*. Suppose that* $\mathcal{A}^\omega[X, d]+\Delta-$DC *proves* (2.19) *and* (2.20) *from Theorem* 2.38*. Then from the proofs in* $\mathcal{A}^\omega[X, d]+\Delta-$DC *of* (2.19) *and* (2.20) *one can extract a functional*

$$\Phi : \mathbb{N} \times \mathbb{N} \times S_{\sigma_1} \times \cdots \times S_{\sigma_{n_1}} \to \mathbb{N},$$

(which can be defined in T*, i.e. as a closed term of* WE-PA$^\omega$*) such that whenever we have a nonempty metric space* (X, d) *bounded by* $b \in \mathbb{N}$ *and a mapping* $f : X \to X$*, which together provide a uniform product space model for* $\mathcal{A}^\omega[X, d]+\Delta-$DC *in the sense of fulfilling the variant of Definition* 2.37 *one gets by replacing* $\mathcal{A}^\omega[X, d]+\Delta$ *by* $\mathcal{A}^\omega[X, d]+\Delta-$DC*, then*

$$\forall k \in \mathbb{N} \forall x, y \in X \forall l, n \geq \Phi(k, b, \vec{F}) \left(d\left(f^l(x), f^n(y)\right) < 2^{-k}\right)$$

holds in (X, d)*, where* $\vec{F}$ *is as in condition* $(*)$ *in Definition* 2.37*.*

Proof. This follows from the proof of Theorem 3.7 in [99] (which we included as Theorem 2.31), since if the theory does not include DC then we do not need the bar recursive functionals to interpret it, and so we do not need to take the detour via $\mathcal{M}^{\omega,X}$. And the reason for the restriction on the types of the new constants $c_1, \ldots, c_{n_1}$ was that we wanted to ensure that these could be interpreted by the same functionals in $\mathcal{S}^{\omega,X}$ and in $\mathcal{M}^{\omega,X}$, and that

$$\mathcal{S}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i$$

would imply

$$\mathcal{M}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i$$

if all constants were interpreted in the same way in $\mathcal{S}^{\omega,X}$ and in $\mathcal{M}^{\omega,X}$. The restriction on the types $\sigma \in \mathbf{T}$ of the quantifiers in the new axioms was there to ensure that truth of the axioms in $\mathcal{S}^{\omega,X}$ would imply truth of the axioms in $\mathcal{M}^{\omega,X}$.

□

In the following corollaries we will give alternatives to the formal statements which we in Theorem 2.38 required that the theory should prove.

Corollary 2.41. *If we in Theorem 2.38, Corollary 2.39, or Corollary 2.40 remove condition (2.20) but instead require that $\mathcal{A}^\omega[X,d]+\Delta$ proves*

$$\forall x_0^X \forall k^0 \exists m^0 \forall n^0 \left(n \geq_0 m \to d_X(x_n, x_{n+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}}\right),$$

where x_n is the n-th member of the defined Picard iteration sequence starting with x_0, then the conclusion of the theorem (respectively the relevant corollary) still holds.

Proof. The condition in the corollary amounts to requiring asymptotic regularity of any interpretation $f : X \to X$ of c_f. This obviously implies

$$\forall x_1, x_2 \in X \forall k \in \mathbb{N} \exists n \in \mathbb{N} \left(\bigwedge_{i=1}^{2} d\left(f^n(x_i), f^{n+1}(x_i)\right) < 2^{-k} \right), \qquad (2.23)$$

and it is easy to see that the required argument can be formalized in $\mathcal{A}^\omega[X,d]+\Delta$. Thus we have that $\mathcal{A}^\omega[X,d]+\Delta$ proves (2.20). (In (2.23) x_1 and x_2 do not denote respectively the first and second member of a Picard iteration sequence, but rather arbitrary elements of X.)

□

Corollary 2.42. *If we in Theorem 2.38, Corollary 2.39, or Corollary 2.40 replace the condition that $\mathcal{A}^\omega[X,d]+\Delta$ should prove (2.19) and (2.20) by the condition that $\mathcal{A}^\omega[X,d]+\Delta$ should prove*

$$\forall x_0^X, y_0^X \forall k^0 \exists n^0 \left(d_X(x_n, y_n) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}}\right), \qquad (2.24)$$

then the conclusion of the theorem (respectively the relevant corollary) still holds.

Proof. We notice that (2.24) has the appropriate logical form, so that we can extract a functional Φ such that

$$\forall k \in \mathbb{N} \forall x, y \in X \exists n \leq \Phi(k, b, \vec{F})\left(d\left(f^n(x), f^n(y)\right) < 2^{-k}\right)$$

holds in any nonempty metric space (X,d) bounded by $b \in \mathbb{N}$ which satisfies the new purely universal axioms with b_X interpreted by b and with the new constants interpreted by functionals from the appropriate universes of $\mathcal{S}^{\omega,X}$ such that there are closed terms $c_1^*, \ldots, c_{n_1}^*$ interpreted by $F_1, \ldots, F_{n_1}$ such that

$$\mathcal{S}^{\omega,X} \models c_i^* \text{ s-maj}_{\sigma_i} c_i \qquad \text{for } 1 \leq i \leq n_1. \qquad (2.25)$$

Since this Φ is independent of the space and the mapping f it follows that for any nonempty bounded metric space (X,d) and mapping $f : X \to X$ which provide a uniform product space model for $\mathcal{A}^\omega[X,d]+\Delta$ we have for all m in an infinite set $M \subseteq \mathbb{N}\setminus\{0\}$ that

$$\forall k \in \mathbb{N} \forall \vec{x}, \vec{y} \in X^m \exists n \leq \Phi(k,b,\vec{F}) \left(d_m \left(f_m^n(\vec{x}), f_m^n(\vec{y}) \right) < 2^{-k} \right), \tag{2.26}$$

where $b \in \mathbb{N}$ is a bound on d and where $F_1,\ldots,F_{n_1}$ are as in $(*)$ in Definition 2.37. Now we can use Lemma 2.36 as in the proof of Theorem 2.38.

□

Corollary 2.43. *If we in Theorem* 2.38*, Corollary* 2.39*, or Corollary* 2.40 *replace the condition that* $\mathcal{A}^\omega[X,d]+\Delta$ *should prove* (2.19) *and* (2.20) *by the condition that there exists an integer* $N>1$ *such that* $\mathcal{A}^\omega[X,d]+\Delta$ *proves*

$$\forall x_0^X \forall y_0^X \left(x_N =_X x_0 \wedge y_N =_X y_0 \rightarrow x_0 =_X y_0 \right)$$

and

$$\forall x_0^X, y_0^X \forall k^0 \exists n^0 \left(d_X(x_n, x_{n+N}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}} \wedge d_X(y_n, y_{n+N}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}} \right),$$

then the conclusion of the theorem (respectively the relevant corollary) still holds.

Proof. Analogous to the proof of Theorem 2.38.

□

The importance of Theorem 2.38 comes from the fact that it has been possible to find such theories $\mathcal{A}^\omega[X,d]+\Delta$ and uniform majorants $\vec{F}$ of the moduli introduced (i.e. of the interpretations of the new constants of relevant type) such that conditions (2.19) and (2.20) (or the similar conditions in the corollaries above) are provable and such that all members of certain classes of selfmappings of metric spaces considered in the literature satisfy condition $(*)$ in Definition 2.37. By recasting the definitions of the relevant classes of selfmaps by introducing certain moduli and purely universal axioms (to get a suitable formalization $\mathcal{A}^\omega[X,d]+\Delta$) such that if (X,d) and f fulfill the definition then for infinitely many m also (X^m,d_m) and f_m fulfill the definition, and such that there exist majorants for the moduli introduced which are uniform in m, we reduce the question whether all Picard iteration sequences are convergent to the same point[10] (that they are Cauchy is expressed by a $\forall\exists\forall$-sentence) to the question whether certain $\forall\exists$-sentences are provable in a suitable formal theory $\mathcal{A}^\omega[X,d]+\Delta$. This reduction in logical complexity allows us to extract uniform bounds as described in Theorem 2.38.

We will in Section 2.5 below present two such classes of selfmaps on bounded metric spaces. In both of these cases we had already constructed explicit and uniform rates of convergence independently of Theorem 2.38 – these results are contained in Chapter 3 and Chapter 4. However, Theorem 2.38 provides an explanation for why we were able to construct such rates of convergence. And that the conditions in Theorem 2.38 are fulfilled in these cases provides

[10] The existence of the limit of the iteration sequences is not guaranteed unless the space is complete.

a justification for the conditions concerning product spaces and existence of uniform majorants for moduli which we require in the theorem. However, before considering the applications of Theorem 2.38 we include two remarks.

Remark 2.44. The logical metatheorems in [99] (Theorem 2.38 above depends on Theorem 3.7 in [99]) have been extended in [56] so as to no longer require the relevant spaces to be bounded. Instead, some local boundedness criteria are required, and a more involved majorizability relation is used. Our results here hinge on a combinatorial argument (Lemma 2.36) involving a condition which for selfmappings on unbounded metric spaces seems restrictive and somewhat peculiar, but we would not be surprised if the results here could, suitably adapted and restricted, be transfered to the setting in [56].

Remark 2.45. Let $\mathcal{A}^\omega[X,d]+\Delta$ be as in Theorem 2.38. Let ρ_1, ρ_2 be types of degree 1 and let τ_1, τ_2 be types of degree $(1,X)$. Let $s_1^{0\to\rho_1}$ and $s_2^{0\to\rho_2}$ be closed terms of $\mathcal{A}^\omega[X,d]+\Delta$. Let

$$B_\forall(x^X, y^X, k^0, u^0, z_1^{\rho_1}, w_1^{\tau_1})$$

and

$$C_\forall(x_0^X, y_0^X, k^0, u^0, z_2^{\rho_2}, w_2^{\tau_2})$$

be $\forall$-formulas in $\mathcal{L}(\mathcal{A}^\omega[X,d]+\Delta)$ with free variables among those indicated. Suppose we in Theorem 2.38 replace the condition that $\mathcal{A}^\omega[X,d]+\Delta$ should prove (2.19) and (2.20) by the condition that $\mathcal{A}^\omega[X,d]+\Delta$ should prove

$$\begin{aligned}\forall k^0 \forall z_1 \leq_{\rho_1} s_1(k) \forall w_1^{\tau_1}, x^X, y^X \big((\forall u^0 B_\forall(x,y,k,u,z_1,w_1) \\ \wedge \forall m^0 A_0(x,y,m)) \to |\widehat{d_X(x,y)}(k+1)|_{\mathbb{Q}} <_{\mathbb{Q}} \langle 2^{-k}\rangle \big),\end{aligned}$$

where $A_0(x,y,m)$ is as in (2.7), and also

$$\begin{aligned}\forall k^0 \forall z_2 \leq_{\rho_2} s_2(k) \forall w_2^{\tau_2}, \forall x_0^X, y_0^X \big(\forall u^0 C_\forall(x_0,y_0,k,u,z_2,w_2) \\ \to \exists n^0 \big(d_X(x_n,x_{n+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}} \wedge d_X(y_n,y_{n+1}) <_{\mathbb{R}} (2^{-k})_{\mathbb{R}} \big)\big).\end{aligned}$$

These formulas are of the logical form required by (an extension to $\mathcal{A}^\omega[X,d]+\Delta$ of) Theorem 2.31, and similarly to in the proof of Theorem 2.38 we can construct functionals Ψ_1 and Ψ_2 such that for any nonempty bounded metric space (X,d) and mapping $f: X \to X$ which provide a uniform product space model for $\mathcal{A}^\omega[X,d]+\Delta$ we have for all $m \in M$ (where the infinite set $M \subseteq \mathbb{N}\setminus\{0\}$ is as in the definition of a uniform product space model) that

$$\begin{aligned}&\forall k \in \mathbb{N} \forall z_1 \leq_{\rho_1} [s_1]_{\mathcal{S}^{\omega,X^m}}(k) \forall w_1 \in S_{\tau_1} \forall \vec{x}_1, \vec{x}_2 \in X^m \\ &\big(\big(\forall u \leq \Psi_1(k,b,\vec{F}) B_\forall(\vec{x}_1,\vec{x}_2,k,u,z_1,w_1) \wedge \textstyle\bigwedge_{i=1}^2 d_m\left(\vec{x}_i, f_m(\vec{x}_i)\right) < 2^{-\Psi_1(k,b,\vec{F})}\big) \\ &\qquad\qquad \to d_m(\vec{x}_1,\vec{x}_2) < 2^{-k}\big)\end{aligned}$$

and

$$\begin{aligned}&\forall k \in \mathbb{N} \forall z_2 \leq_{\rho_2} [s_2]_{\mathcal{S}^{\omega,X^m}}(k) \forall w_2 \in S_{\tau_2} \forall \vec{x}_1, \vec{x}_2 \in X^m \\ &\quad \big(\forall u \leq \Psi_2(k,b,\vec{F}) C_\forall(\vec{x}_1,\vec{x}_2,k,u,z_2,w_2) \\ &\qquad \to \exists n \leq \Psi_2(k,b,\vec{F}) \big(\textstyle\bigwedge_{i=1}^2 d_m\left(f_m^n(\vec{x}_i), f_m^{n+1}(\vec{x}_i)\right) < 2^{-k}\big)\big)\end{aligned}$$

hold in (X^m, d_m), where $b \in \mathbb{N}$ is a bound on d and where $F_1, \ldots, F_{n_1}$ are as in $(*)$ in Definition 2.37. Thus if for all $m \in M$

$$\forall k \in \mathbb{N} \forall z_1 \leq_{\rho_1} [s_1]_{\mathcal{S}^{\omega}, X^m}(k) \forall w_1 \in S_{\tau_1} \forall \vec{x}_1, \vec{x}_2 \in X^m \\ \forall u \leq \Psi_1(k, b, \vec{F}) B_{\forall}(\vec{x}_1, \vec{x}_2, k, u, z_1, w_1)$$

and

$$\forall k \in \mathbb{N} \forall z_2 \leq_{\rho_2} [s_2]_{\mathcal{S}^{\omega}, X^m}(k) \forall w_2 \in S_{\tau_2} \forall \vec{x}_1, \vec{x}_2 \in X^m \\ \forall u \leq \Psi_2(k, b, \vec{F}) C_{\forall}(\vec{x}_1, \vec{x}_2, k, u, z_2, w_2)$$

hold in (X^m, d_m), then as in the proof of Theorem 2.38 we get a Φ such that

$$\forall k \in \mathbb{N} \forall \vec{x}, \vec{y} \in X^m \exists n \leq \Phi(k, b, \vec{F}) \left(d_m \left(f_m^n(\vec{x}), f_m^n(\vec{y}) \right) < 2^{-k-3} \right)$$

for all $m \in M$, and thus such that

$$\forall k \in \mathbb{N} \forall x, y \in X \forall l, n \geq \Phi(k, b, \vec{F}) \left(d \left(f^l(x), f^n(y) \right) < 2^{-k} \right).$$

This amounts to an improvement of Theorem 2.38, and the significance of this treatment is that we do not for all $m \in M$ require

$$\forall k \in \mathbb{N} \forall z_1 \leq_{\rho_1} [s_1]_{\mathcal{S}^{\omega}, X^m}(k) \forall w_1 \in S_{\tau_1} \forall \vec{x}_1, \vec{x}_2 \in X^m \\ \forall u \in \mathbb{N} B_{\forall}(\vec{x}_1, \vec{x}_2, k, u, z_1, w_1)$$

and

$$\forall k \in \mathbb{N} \forall z_2 \leq_{\rho_2} [s_2]_{\mathcal{S}^{\omega}, X^m}(k) \forall w_2 \in S_{\tau_2} \forall \vec{x}_1, \vec{x}_2 \in X^m \\ \forall u \in \mathbb{N} C_{\forall}(\vec{x}_1, \vec{x}_2, k, u, z_2, w_2).$$

2.5 Applications

We will see that in neither of the two concrete cases considered is it entirely straightforward to give an alternative definition of the class of mappings under consideration which is stable under finite product spaces – in the sense that the definition involves suitable purely universal axioms and moduli such that if (X, d) and f fulfill the definition then for $m \geq 1$ also (X^m, d_m) and f_m fulfill the definition – and moreover such that the moduli are majorizable, uniformly in m. However, this approach allows us to split our goal of obtaining an explicit and uniform rate of convergence into subgoals which might be easier to achieve also in other concrete cases.

It is also worth noting that neither of the two case studies use Theorem 2.38 in its full generality; in neither case is it necessary to include DC in the system, and in the first concrete case all the new constants of types $\rho \in \mathbf{T}$ have types of degree 1. In fact, also in the second case we could have formulated things such that the new constants of types $\rho \in \mathbf{T}$ would have had types of degree 1. We will comment more on this below.

2.5.1 A theorem on contractive type mappings

In Chapter 1 we included a brief discussion of some types of contractive mappings. Much work on extensions of the contraction mapping principle has dealt with a kind of contraction condition which involves a more or less complex relationship between the six distances

$$d(x,y), \quad d(x,f(x)), \quad d(y,f(y)), \quad d(x,f(y)), \quad d(y,f(x)), \quad d(f(x),f(y)),$$

and in the survey [157] B.E. Rhoades compares 25 such contraction conditions for selfmappings of metric spaces, and also considers generalizations of the 25 basic conditions to cases where the condition holds for various iterates of the function. The basic conditions are numbered (1)–(25), and of these condition (25),

$$\forall x,y \in X\,(x \neq y \to d\,(f(x),f(y)) < \mathrm{diam}\,\{x,y,f(x),f(y)\})\,,$$

is the most general, so that if f satisfies one of the conditions (1)–(24) from [157], then it also satisfies condition (25) from [157]. In Definition 1.28 we called mappings which satisfy condition (25) for some given iterate $p \in \mathbb{N}$ generalized p-contractive. If (X,d) is compact and $f : X \to X$ continuous and generalized p-contractive, then f has a unique fixed point z, and for every $x_0 \in X$ we have

$$\lim_{n\to\infty} f^n(x_0) = z,$$

(cf. Theorem 1.30). In Chapter 4 we develop a quantitative version of this theorem, which involves the following notion[11]: Given $p \in \mathbb{N}$ we say that a function $f : X \to X$ on a metric space (X,d) is *uniformly generalized p-contractive* if there exists $\eta : \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall x,y \in X \big(d(x,y) > 2^{-k} \to d\,(f^p(x),f^p(y)) + 2^{-\eta(k)} \leq \mathrm{diam}\,\{x,y,f^p(x),f^p(y)\}\big). \tag{2.27}$$

We call η a *modulus of uniform generalized p-contractivity* for f. When (X,d) is a compact metric space then f is continuous and generalized p-contractive if and only if it is uniformly continuous and uniformly generalized p-contractive. Uniform continuity involves having a modulus $\omega : \mathbb{N} \to \mathbb{N}$ of uniform continuity:

$$\forall x,y \in X \forall k \in \mathbb{N} \left(d(x,y) < 2^{-\omega(k)} \to d\,(f(x),f(y)) \leq 2^{-k}\right). \tag{2.28}$$

Restricted to bounded spaces the quantitative version of Theorem 1.30 then states that for a uniformly continuous uniformly generalized p-contractive mapping $f : X \to X$ on a bounded metric space (X,d) all Picard iteration sequences are Cauchy with a Cauchy rate Φ which is uniform in the starting point, and further that for all $x_0, y_0 \in X$ we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Φ is explicitly constructed and depends only on p, ω, η and a bound b on the space.

[11] In Chapter 4 we use an η mapping positive reals to positive reals, and we have $<$ instead of $\leq$ in the conclusion, but this is unimportant. Similar for ω below.

We now let $\mathcal{A}^\omega[X,d]+\Delta$ be $\mathcal{A}^\omega[X,d]$ extended with constants $c_f^{X\to X}$, c_ω^1 and c_η^1, meant to be interpreted by respectively f, ω and η, together with the obvious formalizations of (2.27) and (2.28) (for some fixed p) as purely universal axioms, where the only point requiring some care is the use of diam $\{\ldots\}$: We notice that we can use

$$\max\{d(x,y), d(x,f^p(x)), d(x,f^p(y)), d(y,f^p(y)), d(y,f^p(x)), d(f^p(x),f^p(y))\}$$

instead of diam $\{x,y,f^p(x),f^p(y)\}$. There is a primitive recursive operator $\max_{\mathbb{Q}}(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ defined on the representatives of rational numbers which among six natural numbers picks out the one representing the largest rational number, so there is a closed term of $\mathcal{A}^\omega[X,d]$ which takes six representatives of real numbers as arguments and gives a representative of the largest of the corresponding real numbers. We use this when formalizing (2.27). Then conditions (2.19) and (2.20) from Theorem 2.38 are provable in $\mathcal{A}^\omega[X,d]+\Delta$, since the proof of Theorem 4.6 in Chapter 4 can clearly be formalized in $\mathcal{A}^\omega[X,d]+\Delta$, along with the trivial argument that the fixed point is unique if it exists. (In fact, as is clear from the proofs, only a fragment of $\mathcal{A}^\omega[X,d]$ is used – in particular, DC is not needed.) Furthermore, let (X,d) be a bounded metric space, and let f, η and ω satisfy (2.27) and (2.28) above (for some fixed p). Let $m\geq 1$, and consider (X^m,d_m) and f_m. Since ω is a modulus of uniform continuity for f we have

$$\forall \vec{x},\vec{y}\in X^m \forall k\in\mathbb{N}\left(\bigwedge_{i=1}^{m} d(x^i,y^i)<2^{-\omega(k)}\to\bigwedge_{i=1}^{m} d\left(f(x^i),f(y^i)\right)\leq 2^{-k}\right),$$

and therefore

$$\forall \vec{x},\vec{y}\in X^m \forall k\in\mathbb{N}\left(d_m(\vec{x},\vec{y})<2^{-\omega(k)}\to d_m\left(f_m(\vec{x}),f_m(\vec{y})\right)\leq 2^{-k}\right).$$

And so ω is also a modulus of uniform continuity for f_m. Let $\tilde{\omega}:\mathbb{N}\to\mathbb{N}$ be given by $\tilde{\omega}(k)=\omega(k+1)$. We will show that the function $\eta':\mathbb{N}\to\mathbb{N}$ defined by

$$\eta'(k):=\max\left\{k+1,\eta\left(\tilde{\omega}^p(k+1)+1\right)\right\}$$

is a modulus of uniform generalized p-contractivity for f_m. Notice that η' does not depend on m. Let now $k\in\mathbb{N}$ and let $\vec{x},\vec{y}\in X^m$ be such that

$$d_m(\vec{x},\vec{y})>2^{-k}.$$

Now let $1\leq j\leq m$. We have two cases:

(i) If $d(x^j,y^j)<2^{-\tilde{\omega}^p(k+1)}$, then

$$\begin{array}{rcl} d\left(f^p(x^j),f^p(y^j)\right) & \leq & 2^{-k-1} \\ & < & d_m(\vec{x},\vec{y})-2^{-k-1} \\ & \leq & \operatorname{diam}\{\vec{x},\vec{y},f_m^p(\vec{x}),f_m^p(\vec{y})\}-2^{-k-1}, \end{array}$$

so that

$$d\left(f^p(x^j),f^p(y^j)\right)+2^{-k-1}\leq \operatorname{diam}\{\vec{x},\vec{y},f_m^p(\vec{x}),f_m^p(\vec{y})\}.$$

(ii) If $d(x^j, y^j) > 2^{-\tilde{\omega}^p(k+1)-1}$, then

$$\begin{aligned} d\left(f^p(x^j), f^p(y^j)\right) + 2^{-\eta(\tilde{\omega}^p(k+1)+1)} &\leq \operatorname{diam}\{x^j, y^j, f^p(x^j), f^p(y^j)\} \\ &\leq \operatorname{diam}\{\vec{x}, \vec{y}, f^p_m(\vec{x}), f^p_m(\vec{y})\}. \end{aligned}$$

Thus with

$$\eta'(k) = \max\{k+1, \eta(\tilde{\omega}^p(k+1)+1)\}$$

we have

$$d\left(f^p(x^j), f^p(y^j)\right) + 2^{-\eta'(k)} \leq \operatorname{diam}\{\vec{x}, \vec{y}, f^p_m(\vec{x}), f^p_m(\vec{y})\}$$

for all $1 \leq j \leq m$, and so

$$d_m\left(f^p_m(\vec{x}), f^p_m(\vec{y})\right) + 2^{-\eta'(k)} \leq \operatorname{diam}\{\vec{x}, \vec{y}, f^p_m(\vec{x}), f^p_m(\vec{y})\}.$$

So for each $m \geq 1$ one can obtain a model of $\mathcal{A}^\omega[X,d] + \Delta$ with the set X^m as the universe for the base type X and with c_f interpreted by f_m and c_ω and c_η interpreted by ω and η'. Since c_ω and c_η are of type 1 they are majorized by the closed terms[12] c^M_ω and c^M_η of $\mathcal{A}^\omega[X,d] + \Delta$. And the interpretations ω^* and η'^* of these terms given by

$$\omega^*(n) := \max_{i \leq n}(\omega(i))$$

and

$$\eta'^*(n) := \max_{i \leq n}(\eta'(i))$$

are the same for all m, so all the conditions of Theorem 2.38 are fulfilled.

In this case we actually have more than we require; also the moduli ω and η' are the same for all $m > 0$, whereas Theorem 2.38 only requires that there should be majorants of the moduli which are the same for all m, in the sense made precise in the definition of a uniform product space model.

We note that in this specific case the mathematical proof we inspect to determine that (2.19) and (2.20) from Theorem 2.38 hold for $\mathcal{A}^\omega[X,d] + \Delta$ in fact already directly supplies the conclusion of Theorem 2.38: We had already constructed an explicit and uniform Cauchy rate. We are thus here using Theorem 2.38 as an a posteriori explanation for the fact that this was possible[13].

2.5.2 Asymptotic contractions

Our second application concerns asymptotic contractions in the sense of Kirk, as given in Definition 1.26, and Kirk's theorem on asymptotic contractions 1.27. For more information on various extensions and modifications of this theorem,

[12] see Definition 2.30.

[13] More information on the analysis of different proofs of Theorem 1.30 appears in [23]. See [100] for an adaption to the metatheorems dealing with abstract metric spaces of a non-standard uniform boundedness principle (due to Kohlenbach) which is used in the analysis.

as well as the rates of convergence reported on in this section, and related results, see Chapter 3. To give an equivalent definition of asymptotic contractions in the sense of Kirk (on bounded spaces) involving moduli and universal axioms as required by Theorem 2.38 we will need the following lemma.

Lemma 2.46. *A function $f : X \to X$ on a metric space (X, d) bounded by $b > 0$ is an asymptotic contraction in the sense of Kirk if and only if there exist continuous and increasing moduli $\phi', \phi'_n : [0, b] \to [0, b]$ such that $\phi'(s) < s$ for all $s > 0$, such that for all $x, y \in X$*

$$d\left(f^n(x), f^n(y)\right) \leq \phi'_n(d(x, y)),$$

and moreover such that $\phi'_n \to \phi'$ uniformly.

Proof. Let (X, d) be a metric space and let $f : X \to X$. First we notice that if the conditions in Lemma 2.46 are satisfied then f is an asymptotic contraction in the sense of Kirk since we can get the moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ required by Definition 1.26 by letting

$$\phi(s) := \phi'\big(\min\{s, b\}\big),$$

and similarly for ϕ_n. Assume now that $f : X \to X$ is an asymptotic contraction in the sense of Kirk with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$. It is not hard to see that in Definition 1.26 we can equivalently assume that $\phi_n \to \phi$ uniformly on $[0, \infty)$. (A proof is included after Proposition 3.2 in Chapter 3.) Define now $\widetilde{\phi}_n : [0, b] \to [0, \infty)$ by

$$\widetilde{\phi}_n(s) := \min\{b, \phi_n(s)\}.$$

Then the $\widetilde{\phi}_n$ are continuous, $\widetilde{\phi}_n(s) \leq b$ for $s \leq b$, $\widetilde{\phi}_n \to \phi$ uniformly on $[0, b]$, and we have

$$d\left(f^n(x), f^n(y)\right) \leq \widetilde{\phi}_n(d(x, y))$$

for all $x, y \in X$, since the space is bounded by b. Define $\phi', \phi'_n : [0, b] \to [0, b]$ by

$$\phi'(s) := \sup\{\phi(\delta) : \delta \leq s\}$$

and

$$\phi'_n(s) := \sup\{\widetilde{\phi}_n(\delta) : \delta \leq s\}.$$

Then ϕ', ϕ'_n are continuous and increasing, $\phi'_n \to \phi'$ uniformly on $[0, b]$, and we have

$$d\left(f^n(x), f^n(y)\right) \leq \phi'_n(d(x, y))$$

for all $x, y \in X$. And since $\phi(s) < s$ for all $s > 0$ and ϕ is continuous we can conclude that $\phi'(s) < s$ for all $s > 0$.

□

From Lemma 2.46 we get the following:

Lemma 2.47. *Let b be a positive integer. A function $f : X \to X$ on a b-bounded metric space (X, d) is an asymptotic contraction in the sense of Kirk if and only if there exist moduli $\alpha : \mathbb{N} \to \mathbb{N}$, $\rho : \mathbb{N} \to \mathbb{N}$, $\phi : [0, b] \to [0, b]$, $\Phi : \mathbb{N} \times [0, b] \to [0, b]$, $\omega : \mathbb{N} \to \mathbb{N}$ and $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that:*

(i) $\forall n \in \mathbb{N} \forall x, y \in X \, (d\,(f^n(x), f^n(y)) \leq \Phi(n, d(x,y)))$,

(ii) $\forall n \in \mathbb{N} \forall s, s' \in [0,b] \, (s' > s \to \Phi(n,s') \geq \Phi(n,s))$,

(iii) $\forall s, s' \in [0,b] \, (s' > s \to \phi(s') \geq \phi(s))$,

(iv) $\forall k \in \mathbb{N} \forall m \geq \rho(k) \forall s \in [0,b] \left(|\Phi(m,s) - \phi(s)| \leq 2^{-k} \right)$,

(v) $\forall n \in \mathbb{N} \forall s \in [0,b] \left(s > 2^{-n} \to \phi(s) + 2^{-\alpha(n)} \leq s \right)$,

(vi) $\forall k \in \mathbb{N} \forall s, s' \in [0,b] \left(|s - s'| < 2^{-\omega(k)} \to |\phi(s) - \phi(s')| \leq 2^{-k} \right)$,

(vii) $\forall k, n \in \mathbb{N} \forall s, s' \in [0,b] \left(|s - s'| < 2^{-\Omega(n,k)} \to |\Phi(n,s) - \Phi(n,s')| \leq 2^{-k} \right)$.

Proof. If $f : X \to X$ is a mapping on a metric space bounded by $b \in \mathbb{N}$, $b > 0$, with moduli α, ρ, ϕ, Φ, ω and Ω as in this lemma, then obviously f has moduli ϕ', ϕ'_n as required in Lemma 2.46. And thus f is an asymptotic contraction in the sense of Kirk.

Assume now that $f : X \to X$ is an asymptotic contraction in the sense of Kirk on a metric space bounded by $b \in \mathbb{N}$, $b > 0$. Then f has continuous and increasing moduli $\phi', \phi'_n : [0,b] \to [0,b]$ as in Lemma 2.46. It is then easy to see that f has moduli ρ, ϕ, Φ, ω and Ω fulfilling (i)–(iv) and (vi)–(vii) in Lemma 2.47, since a continuous function on a compact interval is uniformly continuous. We let $\phi = \phi'$, $\Phi = \lambda n.\phi'_n$, and let ω and $\Omega(n, \cdot)$ be moduli of uniform continuity for respectively ϕ', ϕ'_n. The existence of a suitable ρ follows since $\phi'_n \to \phi'$ uniformly. We must show that there also exists $\alpha : \mathbb{N} \to \mathbb{N}$ as required in Lemma 2.47 such that (v) is satisfied. Since $\phi'(s) < s$ for all $s > 0$ we get

$$\inf\{s - \phi'(s) : s \in [\varepsilon, b]\} > 0$$

for each $\varepsilon > 0$ with $\varepsilon < b$. Thus we can define $\alpha : \mathbb{N} \to \mathbb{N}$ as required by letting

$$\alpha(n) := \min_{k \in \mathbb{N}} \left[2^{-k} \leq \inf \left\{ s - \phi'(s) : s \in [2^{-n}, b] \right\} \right].$$

□

Proposition 2.48. *There exists an extension $\mathcal{A}^\omega[X,d] + \Delta$ of $\mathcal{A}^\omega[X,d]$ as required in Theorem 2.38 which proves (2.24) from Corollary 2.42 and furthermore has the property that whenever (X,d) is a nonempty metric space bounded by $b \in \mathbb{N}$ and $f : X \to X$ is an asymptotic contraction in the sense of Kirk, then (X,d) and f provide a uniform product space model for $\mathcal{A}^\omega[X,d] + \Delta$.*

Proof. We first notice that if (X,d) is a metric space bounded by $b \in \mathbb{N}$ and $f : X \to X$ is an asymptotic contraction with moduli α, ρ, ϕ, Φ, ω and Ω as required in Lemma 2.47, then for each $m \geq 1$ we have that $f_m : X^m \to X^m$ is an asymptotic contraction on (X^m, d_m) with identical moduli: Let $n \in \mathbb{N}$ and $x, y \in X$. Then there exists $i \leq m$ such that

$$d_m\big(f^n_m(\vec{x}), f^n_m(\vec{y})\big) = d\left(f^n(x^i), f^n(y^i)\right) \leq \Phi\left(n, d(x^i, y^i)\right),$$

and since $\lambda s.\Phi(n,s)$ is increasing we have

$$\Phi\left(n, d(x^i, y^i)\right) \le \Phi\left(n, d_m(\vec{x}, \vec{y})\right),$$

and thus

$$d_m\left(f_m^n(\vec{x}), f_m^n(\vec{y})\right) \le \Phi\left(n, d_m(\vec{x}, \vec{y})\right).$$

Regarding the theory $\mathcal{A}^\omega[X,d] + \Delta$: One can via an effective operation represent the elements in the interval $[0,b]$ by type 1 objects in such a way as to reduce quantification over $[0,b]$ to quantification over type 1 objects (without introducing further quantifiers)[14]: We let

$$\tilde{x}_{b_X}(n) := j(2k_0, 2^{n+2} - 1),$$

where

$$k_0 = \max_{k \le_0 b_X \cdot 2^{n+2}} \left[\frac{k}{2^{n+2}} \le_{\mathbb{Q}} \widehat{x}(n+2) \right]$$

(and $k_0 = 0$ if there is no such k), and note that $\lambda x.\tilde{x}_{b_X}$ can easily be given by a closed term in $\mathcal{A}^\omega[X,d]$. Recall that we write $(b_X)_{\mathbb{R}} := \lambda k^0.j(2b_X, 0^0)$. Then, provably in $\mathcal{A}^\omega[X,d]$, for all x of type 1:

$$\begin{array}{l} 0_{\mathbb{R}} \le_{\mathbb{R}} x \le_{\mathbb{R}} (b_X)_{\mathbb{R}} \to \tilde{x}_{b_X} =_{\mathbb{R}} x, \\ 0_{\mathbb{R}} \le_{\mathbb{R}} \tilde{x}_{b_X} \le_{\mathbb{R}} (b_X)_{\mathbb{R}}, \quad \tilde{x}_{b_X} =_{\mathbb{R}} \widetilde{(\tilde{x}_{b_X})}_{b_X}, \\ \tilde{x}_{b_X} \le_1 N_{b_X} := \lambda n.j(b_X \cdot 2^{n+3}, 2^{n+2} - 1). \end{array}$$

We can thus extend $\mathcal{A}^\omega[X,d]$ with constants $c_f^{X\to X}$, c_α^1, c_ρ^1, $c_\phi^{1\to 1}$, $c_\Phi^{0\to 1\to 1}$, c_ω^1, $c_\Omega^{0\to 0\to 0}$, together with purely universal axioms which are the obvious formalizations of (i)–(vii) in Lemma 2.47, where we treat quantification over $[0,b]$ as indicated above. This involves adding also the axioms:

(a) $\forall x^1\, (c_\phi(x) =_{\mathbb{R}} c_\phi(\tilde{x}_{b_X}))$,

(b) $\forall n^0 \forall x^1\, (c_\Phi(n,x) =_{\mathbb{R}} c_\Phi(n, \tilde{x}_{b_X}))$,

(c) $\forall x^1\, (0_{\mathbb{R}} \le_{\mathbb{R}} c_\phi(x) \wedge c_\phi(x) \le_{\mathbb{R}} (b_X)_{\mathbb{R}})$,

(d) $\forall n^0 \forall x^1\, (0_{\mathbb{R}} \le_{\mathbb{R}} c_\Phi(n,x) \wedge c_\Phi(n,x) \le_{\mathbb{R}} (b_X)_{\mathbb{R}})$.

The constants c_f, c_α, c_ρ, c_ω and c_Ω are majorizable since their types are of appropriate degree. To make sure that the constants c_ϕ and c_Φ are majorizable as well we additionally add as axioms

(e) $\forall x^1 (c_\phi(x) \le_1 N_{b_X})$

(f) $\forall n^0 \forall x^1 (c_\Phi(n,x) \le_1 N_{b_X})$.

[14] See [99] for the case $[0,1]$, and [89, 101] for general information on representation of (compact) Polish spaces.

Thus $\lambda x^1 . N_{b_X}^M$ will majorize c_ϕ and $\lambda m^0, x^1 . N_{b_X}^M$ will majorize c_Φ. We will call the resulting theory $\mathcal{A}^\omega[X,d] + \Delta$.

That we have included new constants whose types are of degree 2 is not strictly necessary. For the moduli ϕ and $\Phi(n,\cdot)$ from Lemma 2.47 come equipped with moduli of uniform continuity ω and $\Omega(n,\cdot)$, and so they are uniquely determined by their restrictions to $\mathbb{Q} \cap [0,b]$. Thus we could have made do with new constants $c_{\tilde{\phi}}^{0\to 1}$ and $c_{\tilde{\Phi}}^{0\to 0\to 1}$ instead. We would then in the new axiom corresponding to (i) in Lemma 2.47 not use simply $c_{\tilde{\Phi}}^{0\to 0\to 1}$, but a more involved construction involving also the modulus of uniform continuity, i.e., c_Ω. (For details on how to represent continuous real valued functions on compact intervals as number theoretic functions using a modulus of uniform continuity, see e.g. [92].)

We must now indicate how the new constants should be interpreted in order to ensure that given a nonempty metric space (X,d) bounded by $b \in \mathbb{N}$ and a mapping $f : X \to X$ which is an asymptotic contraction in the sense of Kirk with moduli α, ρ, ϕ, Φ, ω and Ω as in Lemma 2.47, then (X,d) and f provide a uniform product space model for $\mathcal{A}^\omega[X,d] + \Delta$. We will treat the cases c_ϕ and c_Φ. For $x \in \mathbb{N}^\mathbb{N}$ we let $r(x)$ be the unique real number represented by x. Then c_ϕ and c_Φ will be interpreted by for $x \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ letting

$$[c_\phi]_{\mathcal{S}^{\omega,X}}(x) = \Big(\phi\big(r(\tilde{x}_{b_X})\big)\Big)_\circ$$

and

$$[c_\Phi]_{\mathcal{S}^{\omega,X}}(n,x) = \Big(\Phi\big(n, r(\tilde{x}_{b_X})\big)\Big)_\circ,$$

where $\big(\,\cdot\,\big)_\circ$ is as in Definition 2.22 (and where we have written b_X also for the interpretation of the term b_X). When verifying that the new axioms are satisfied we then use the properties of $(\cdot)_\circ$ given in Lemma 2.23. Crucially also the axioms (e) and (f) are satisfied, since indeed

$$(x)_\circ \leq_1 [N_{b_X}]_{\mathcal{S}^{\omega,X}}$$

for any $x \in [0,b]$.

Finally, we treat the question whether (2.24) from Corollary 2.42 is provable in $\mathcal{A}^\omega[X,d] + \Delta$: We note first that we do not have an axiom stating (the formalized version of the statement) that f is a function:

$$\forall x, y \in X\, (x = y \to f(x) = f(y))\,.$$

Stating this would amount to requiring that we should have a modulus of uniform continuity for f; and we do not want to require that f is uniformly continuous or even continuous. For a detailed discussion, see [99]. Thus we need to be aware of this restriction when considering whether certain arguments are formalizable in $\mathcal{A}^\omega[X,d] + \Delta$.

Let now (X,d) be a b-bounded metric space. We assume $b > 0$, since otherwise things are trivial. Let $f : X \to X$ be an asymptotic contraction with α,

ρ, ϕ, Φ, ω and Ω as required in Lemma 2.47. Similarly to the treatment in [54] we define

$$\begin{array}{rcll} \phi^*(s) & := & \dfrac{\phi(s)}{s} & \text{for } s \in (0,b], \\ \phi^*_n(s) & := & \dfrac{\Phi(n,s)}{s} & \text{for } s \in (0,b], \\ \phi^b(2^{-m}) & := & \sup\limits_{t\in[2^{-m},b]} \phi^*(t) & \text{for } m \in \mathbb{N} \text{ such that } 2^{-m} \le b, \\ \phi^b_n(2^{-m}) & := & \sup\limits_{t\in[2^{-m},b]} \phi^*_n(t) & \text{for } m \in \mathbb{N} \text{ such that } 2^{-m} \le b, \end{array}$$

and also $\eta^b : \mathbb{N} \to \mathbb{N}$ and $\beta^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$\eta^b(n) := \min_{k\in\mathbb{N}} \left[2^{-k} \le \frac{2^{-\alpha(n+1)}}{b} \right] \tag{2.29}$$

and

$$\beta^b(m,n) := \rho(n+1+m). \tag{2.30}$$

Then:

(i) For all $x, y \in X$, for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ we have that

$$d(x,y) \ge 2^{-n} \quad \text{gives} \quad d\left(f^k(x), f^k(y)\right) \le \phi^b_k(2^{-n}) \cdot d(x,y).$$

(ii) For each $m \in \mathbb{N}$ the function $\lambda n.\beta^b(m,n)$ is a modulus of uniform convergence for $(\phi^b_n)_{n\in\mathbb{N}}$ on the $2^{-m'}$ such that $2^{-m} \le 2^{-m'} \le b$, i.e., for all $n \in \mathbb{N}$ and for all m' such that

$$2^{-m} \le 2^{-m'} \le b$$

we have

$$\forall k, l \ge \beta^b(m,n) \left(|\phi^b_k(2^{-m'}) - \phi^b_l(2^{-m'})| \le 2^{-n} \right).$$

(iii) For each $m \in \mathbb{N}$ we have

$$\phi^b(2^{-m'}) + 2^{-\eta^b(m)} \le 1$$

for each $m' \in \mathbb{N}$ such that $2^{-m} \le 2^{-m'} \le b$.

It is easy to see that the constructions of η^b and β^b can be carried out in $\mathcal{A}^\omega[X,d]+\Delta$ and that (corresponding formal versions of) (i)–(iii) can be proved in $\mathcal{A}^\omega[X,d]+\Delta$. And in fact, we do not need full comprehension for numbers, as we have in $\mathcal{A}^\omega[X,d]+\Delta$, since weak König's lemma WKL would suffice (see [96]). This will hold also for the rest of the proof, and thus one can predict in advance that the uniform bounds which one can extract will be definable in T rather than T+(BR), i.e., without the use of bar recursion. And so the functional

Φ of type of degree 2 which one can extract, and which will give the rate of convergence, will be a total functional which is primitive recursive in the sense of Gödel. For elimination results concerning WKL from proofs in the fragment WE-PA$^\omega$+QF-AC of $\mathcal{A}^\omega$ of sentences of the form $\forall x^1 \forall y \leq_\rho sx \exists z^\tau A_0(x,y,z)$, with τ of degree 2 and s a closed term, see [88]; and for a discussion of this related to $\mathcal{A}^\omega[X,d]$, see Chapter 17 of [101]. (We make here no attempt to determine exactly what fragment of $\mathcal{A}^\omega[X,d]+\Delta$ is needed in the proof, which would give more a priori information on the complexity of the rate of convergence.)

Before proceeding with the proof we note that (i)–(iii) above mimic closely Definition 2 in [54], which also appears as Definition 3.1 in Chapter 3. We will now prove analogues of Proposition 4 in [54] (which we include as Proposition 3.3) and Lemma 3.25 in Chapter 3. For $n \in \mathbb{N}$ we let

$$K(n) := \beta^b(n, \eta^b(n)+1).$$

Then for all $k \geq K(n)$ we have for all $x, y \in X$ that $d(x,y) \geq 2^{-n}$ implies

$$d\left(f^k(x), f^k(y)\right) \leq \left(1 - 2^{-\eta^b(n)-1}\right) \cdot d(x,y), \tag{2.31}$$

since we have

$$d\left(f^k(x), f^k(y)\right) \leq \phi_k^b(2^{-n}) \cdot d(x,y)$$

and

$$\phi^b(2^{-n}) + 2^{-\eta^b(n)} \leq 1,$$

and since

$$|\phi_k^b(2^{-n}) - \phi^b(2^{-n})| \leq 2^{-\eta^b(n)-1}$$

holds by the definition of K. Let now $x, y \in X$, and let $k \in \mathbb{N}$. Then there exists an $n \in \mathbb{N}$ such that $d\left(f^n(x), f^n(y)\right) \leq 2^{-k}$: Assume that M is such that for all $m \leq M$ we have

$$d\left(f^{m \cdot K(k)}(x), f^{m \cdot K(k)}(y)\right) \geq 2^{-k}.$$

Then repeatedly using (2.31) we get

$$\begin{aligned} d\left(f^{M \cdot K(k)}(x), f^{M \cdot K(k)}(y)\right) &\leq \left(1 - 2^{-\eta^b(k)-1}\right)^M \cdot d(x,y) \\ &\leq \left(1 - 2^{-\eta^b(k)-1}\right)^M \cdot b. \end{aligned}$$

Solving the inequality

$$\left(1 - 2^{-\eta^b(k)-1}\right)^M \cdot b \leq 2^{-k}$$

with respect to M gives an upper bound on an n such that

$$d\left(f^n(x), f^n(y)\right) \leq 2^{-k}.$$

This argument can clearly be carried out in $\mathcal{A}^\omega[X,d]+\Delta$. This ends the proof. □

Thus there exists a functional Ψ such that whenever (X,d) is a metric space bounded by $b \in \mathbb{N}$ and $f : X \to X$ is an asymptotic contraction with moduli α, ρ, ϕ, Φ, ω and Ω as required in Lemma 2.47, then

$$\forall k \in \mathbb{N} \forall x, y \in X \forall m, n \geq \Psi(k, b, \alpha^*, \rho^*, \omega^*, \Omega^*) \left(d\left(f^m(x), f^n(y)\right) < 2^{-k}\right),$$

where α^*, ρ^*, ω^* and Ω^* majorize the respective moduli. (We do not need to indicate dependence on majorants of ϕ and Φ – since $[\lambda x^1.\, N^M_{b_X}]_{\mathcal{S}^\omega,X}$ majorizes ϕ and $[\lambda m^0, x^1.\, N^M_{b_X}]_{\mathcal{S}^\omega,X}$ majorizes Φ.) In particular we can conclude that if $f : X \to X$ is an asymptotic contraction in the sense of Kirk on a (nonempty) complete, bounded metric space (X,d), then there exists a point $z \in X$ such that all Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ converge to z with a rate of convergence which is uniform in the starting point. In Theorem 3.16 we will give an explicit functional which – specialized to the case where the space is bounded – gives such a uniform rate of convergence for all Picard iteration sequences to the common limit $z \in X$. This functional will take as arguments some moduli for the mapping which are closer to the η^b, β^b appearing in (2.29) and (2.30) in the proof of Proposition 2.48 than to Kirk's original moduli.

Remark 2.49. On bounded metric spaces (X,d) we in fact have the following characterization of asymptotic contractions in the sense of Kirk (See Theorem 3.21 and Remark 3.22 in Chapter 3): A mapping $f : X \to X$ is an asymptotic contraction in the sense of Kirk if and only if there exists a function $\Phi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall x, y \in X \forall k \in \mathbb{N} \forall n \geq \Phi(k) \left(d\left(f^n(x), f^n(y)\right) < 2^{-k}\right). \tag{2.32}$$

Condition (2.32) implies that all Picard iteration sequences $(x_n)_{n\in\mathbb{N}}$ are Cauchy, since all members of $(x_n)_{n\geq\Phi(k)}$ have to lie within a distance of 2^{-k} from $x_{\Phi(k)}$, so if the space is complete then there exists $z \in X$ such that all Picard iteration sequences converge to z with a rate of convergence which is uniform in the starting point. And so we have the following easy observation:

Proposition 2.50. *Let $f : X \to X$ be a selfmapping on a nonempty bounded metric space (X,d). Then the following are equivalent:*

(i) *The mapping f is an asymptotic contraction in the sense of Kirk.*

(ii) *There exists a function $\Phi : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall x, y \in X \forall k \in \mathbb{N} \forall n \geq \Phi(k) \left(d\left(f^n(x), f^n(y)\right) < 2^{-k}\right).$$

(iii) *There exists an extension $\mathcal{A}^\omega[X,d]+\Delta$ of $\mathcal{A}^\omega[X,d]$ as required in Theorem 2.38, which proves (2.24) from Corollary 2.42, and such that (X,d) and f provide a uniform product space model for $\mathcal{A}^\omega[X,d]+\Delta$.*

Proof. That (i) and (ii) are equivalent was remarked above, and that (iii) implies (ii) follows from Corollary 2.42. Thus the only thing we have to prove is that (ii) implies (iii). So let $f : X \to X$ be a selfmapping on a nonempty bounded metric space (X, d), and let Φ be as in (ii). We can then extend $\mathcal{A}^\omega[X, d]$ with constants c_f and c_Φ of type $X \to X$ and 1, and with the purely universal axiom

$$\forall x_0^X, y_0^X \forall k^0, n^0 \left(n \geq_0 c_\Phi(k) \to d_X(x_n, y_n) \leq_\mathbb{R} (2^{-k})_\mathbb{R}\right),$$

where x_n again indicates the nth term of the Picard iteration sequence with starting point x_0. Thus (iii) holds.

□

This also means that if we use Theorem 2.38 to prove that a certain class of selfmappings on bounded metric spaces has the property that all Picard iteration sequences $(x_n)_{n\in\mathbb{N}}$ are Cauchy with a uniform Cauchy rate and that furthermore $d(x_n, y_n) \to 0$ for all $x_0, y_0 \in X$, then we in fact establish that these mappings are asymptotic contractions in the sense of Kirk.

In the following chapters we will present the concrete results in metric fixed point theory which we have discussed in Section 2.5, as well as further results concerning these classes of mappings which grew out of the work on rates of convergence. We will end this chapter with a very open-ended remark: Given that we in this chapter have established general conditions for when it is possible to extract a full rate of convergence, rather than only a rate of proximity, it is an interesting question to what extent these methods can be lifted out of the special case of Picard iteration sequences in metric spaces; whether we can use this approach to gain insight into cases when a Π^0_3-sentence proved in a suitable formal system behaves like a Π^0_2-sentence, e.g., in number theory.

Chapter 3

Asymptotic contractions

In this chapter we will build on Gerhardy's analysis from [54] of Kirk's theorem on asymptotic contractions to give an effective and highly uniform rate of convergence for asymptotic contractions in the sense of Kirk, and as a corollary to the analysis we will show among other things that on bounded, complete metric spaces asymptotic contractions in the sense of Kirk are exactly the mappings for which all Picard iteration sequence converge to the same point with a rate of convergence which is uniform in the starting point. Gerhardy modified Kirk's definition in order to get a form suitable for using proof mining to extract quantitative information, and constructed variants of a modulus of uniqueness (see Definition 2.4) and a modulus of uniform almost asymptotic regularity (see Definition 2.3), which he could combine to get an effective and highly uniform rate of proximity (see Definition 1.7) for any Picard iteration sequence to the unique fixed point in the manner outlined in Section 2.3 of Chapter 2.

In the last section of the previous chapter it was shown that it is possible to obtain an effective and highly uniform full rate of convergence, instead of a rate of proximity, by following the approach outlined there. In practice our approach in constructing such a rate of convergence has been somewhat more ad hoc, since we obtained the general method involving product spaces and uniformly majorizable moduli from Chapter 2 by investigating the underlying combinatorial pattern of the arguments used in our concrete investigations presented in this chapter and the next.

The results in this chapter have for the most part appeared in [25, 26, 28] (see also [24]), but the material has been revised, and some discussion has been added. Theorem 3.13 also appeared in the author's Master thesis [23].

3.1 Introduction

Asymptotic contractions were introduced by W.A. Kirk in [83], and we gave the definition and stated Kirk's theorem (Theorem 1.27) in Chapter 1. As we saw, Kirk's proof of this theorem uses methods from nonstandard analysis – in particular Banach space ultrapowers. In [54] P. Gerhardy gave an elementary proof of Kirk's theorem on asymptotic contractions, and at the same time developed a quantitative version of the theorem. Here Gerhardy made use of techniques from the program of proof mining, and Gerhardy's paper will be the starting point for our investigations.

Previously J. Jachymski and I. Jóźwik had given an elementary proof of Kirk's theorem on asymptotic contractions – additionally assuming that the mapping is uniformly continuous, see [76]. ("Elementary" is here used in the informal sense that no heavy handed machinery is involved – in particular ultraproducts are not needed.) And in [2] I.D. Aranđelović published an elementary proof of a slight generalization of Kirk's theorem. (However, that proof turned out to contain an error, and the theorem as stated is false – see J. Jachymski's note [75], where he also gives conditions which serve to repair the proof so that the resulting theorem still covers Kirk's theorem.) Around the same time as Gerhardy's result also H.-K. Xu [177] and T. Suzuki [167] developed versions of the theorem, whose proofs do not rely on ultraproduct methods. (Both Gerhardy, Xu and Suzuki accomplished this by subsuming Kirk's definition under a more general definition.) Several other versions of asymptotic contractions have also been studied, e.g., by Y.-Z. Chen [36] and Suzuki [168]. We will comment more thoroughly on this later. However, in contrast to Gerhardy's theorem, none of the above mentioned treatments give explicit numerical information concerning the convergence to the fixed point.

We mentioned that Gerhardy used techniques and insights from proof mining. However, strictly speaking, for the relevant metatheorems from [56, 99] to be applicable and to guarantee that effective bounds can be extracted one would need a proof which does not use ultrapower techniques, since it is problematic to formalize these in the formal systems to which the metatheorems apply. In this particular case it turned out that analyzing the mathematical concepts involved (by proof-theoretic means) provided enough insight to produce the mentioned quantitative version of Kirk's theorem. There are also proof interpretations which are specifically developed for nonstandard theories (see [8]), and in general one may consider these when trying to "unwind" a nonstandard analytical proof. The question to what extent these methods might be combined with or incorporated into the approach of the metatheorems in [56, 99], where one can handle theories for abstract metric spaces etc., and whether this would allow one to systematically unwind proofs based on techniques involving Banach space ultraproducts, is a subject for further research.

An essential part of Gerhardy's analysis consists of giving a modified definition of an asymptotic contraction, so as to make explicit the realizers and bounds which the functional interpretation would ask for – this involves introducing relevant moduli and (essentially) purely universal axioms governing their behavior. Thus the new definition involves other moduli than the ϕ, ϕ_n appearing in Kirk's definition, but the new definition covers the old definition of an asymptotic contraction in the sense of Kirk. The quantitative version of the theorem proved by Gerhardy then involves an explicit rate of proximity to the fixed point which depends on these moduli and a bound on the iteration sequence, i.e., an explicit bound – expressed by the mentioned moduli and a bound on the iteration sequence – on how far one must go in the iteration sequence to at least once get within a specified distance of the fixed point.

However, as already mentioned this theorem does not give a rate of convergence to the fixed point in the general case. The convergence needs not be monotone, and so for $m > n$ it is not the case that $f^m(x_0)$ needs to be close to the fixed point if $f^n(x_0)$ is. For an example of such a function, see Example 2 in [76]. In contrast to this, the results in [54] do give a rate of convergence when the convergence to the fixed point is monotone, and this is the case for a very large class of functions, including the nonexpansive ones.

We will here give for the general case an explicit rate of convergence to the unique fixed point for Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$. The assumptions are in general the same as in [54]. We will, however, consider a slightly more general definition of asymptotic contractions than the one considered by Gerhardy.

In fact, we will provide two explicit rates of convergence: One which depends on the starting point, the space and the function only through a bound on the iteration sequence and the moduli mentioned above, and another which depends on the starting point, the space and the function only through strictly positive *upper and lower* bounds $b, c > 0$ on the initial displacement $d(x_0, f(x_0))$, in addition to the mentioned moduli. That is, the latter rate of convergence depends not only on a $b > 0$ such that

$$b \geq d(x_0, f(x_0)),$$

but also on a $c > 0$ such that

$$c \leq d(x_0, f(x_0)).$$

We note that if the space is not complete or the mapping not continuous, then we have the same rates – the difference is only that the common limit of all Picard iteration sequences might not exist except in the completion, or it might not be a fixed point.

We will also show that for asymptotic contractions in the sense of Kirk there exists a rate of convergence which is uniform in the starting point, the space and the mapping except through an *upper* bound b on the initial displacement $d(x_0, f(x_0))$ and moduli ϕ, ϕ_n as in Kirk's definition (Definition 1.26) such that $\phi_n \to \phi$ uniformly on $[0, \infty)$. In contrast to the rate of convergence referred to above we do not here need a *lower* bound $c > 0$ on the initial displacement. This uniformity will be used to show a result which, specialized to nonempty complete spaces and continuous mappings, says that an asymptotic contraction in the sense of Kirk moves all points which are far from the fixed point by a large distance.

3.2 Preliminaries

In this chapter we will – unless explicitly stated otherwise – by a sequence $(x_n)_{n\in\mathbb{N}}$ mean a Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ in a metric space (X,d), for a selfmap $f : X \to X$ and a starting point $x_0 \in X$.

3.2.1 Gerhardy's rate of proximity

We give for reference the alternative definition of an asymptotic contraction from [54], as well as several results from [54], which we will use repeatedly. We will also repeat the proofs of some of these results. We will call mappings which satisfy Gerhardy's modified definition *asymptotic contractions in the sense of Gerhardy*.

Definition 3.1 (Gerhardy)**.** A function $f : X \to X$ on a metric space (X,d) is called an *asymptotic contraction in the sense of Gerhardy* if for each $b > 0$ there exist moduli $\eta^b : (0,b] \to (0,1)$ and $\beta^b : (0,b] \times (0,\infty) \to \mathbb{N}$ and a sequence of functions $\phi_n^b : (0,\infty) \to (0,\infty)$ such that the following holds:

(1) For each $0 < l \le b$ the function $\beta_l^b := \beta^b(l,\cdot)$ is a modulus of uniform convergence for $(\phi_n^b)_{n\in\mathbb{N}}$ on $[l,b]$, i.e.

$$\forall\varepsilon > 0 \forall s \in [l,b] \forall m,n \ge \beta_l^b(\varepsilon) \left(\left|\phi_m^b(s) - \phi_n^b(s)\right| \le \varepsilon\right).$$

(2) For all $x,y \in X$, for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$ we have that

$$b \ge d(x,y) \ge \varepsilon \text{ gives } d\big(f^n(x), f^n(y)\big) \le \phi_n^b(\varepsilon) d(x,y).$$

(3) Define $\phi^b : (0,\infty) \to (0,\infty)$ by $\phi^b(s) := \lim_{n\to\infty} \phi_n^b(s)$. Then for each $0 < \varepsilon \le b$ we have
$$\phi^b(s) + \eta^b(\varepsilon) \le 1$$
for each $s \in [\varepsilon, b]$.

When there is no risk of ambiguity we will often drop the superscripts from η^b and β^b.

Proposition 3.2 (Gerhardy)**.** *If a function $f : X \to X$ on a metric space (X,d) is an asymptotic contraction in the sense of Kirk with moduli ϕ, ϕ_n, then f is an asymptotic contraction in the sense of Gerhardy with suitable moduli η^b, β^b for all $b > 0$.*

The proof of this proposition in [54] assumes tacitly that Definition 1.26 is equivalent to a version where the sequence of moduli ϕ_n is required to converge to ϕ uniformly on $[0,\infty)$ rather than only on the range of d. It is indeed straightforward to see that given a mapping f on a nonempty metric space (X,d) satisfying Definition 1.26 with moduli ϕ_n, ϕ, one may modify the moduli as follows to get uniform convergence on $[0,\infty)$. Denote by $\overline{\text{ran}(d)}$ the closure of the range of d. For $x \in [0,\infty)$ define $a(x) := \sup\{y \in \overline{\mathrm{ran}(d)} : x > y\}$ and $b(x) := \inf\{y \in \overline{\mathrm{ran}(d)} : x < y\}$ when possible. Define $\phi'_n : [0,\infty) \to [0,\infty)$ by

$$\phi'_n(x) := \begin{cases} \phi_n(x) & \text{if } x \in \overline{\mathrm{ran}(d)}, \\ \phi_n(a(x)) + \left(\frac{x-a(x)}{b(x)-a(x)}\right)(\phi_n(b(x)) - \phi_n(a(x))) & \text{if } x \notin \overline{\mathrm{ran}(d)} \text{ and} \\ & b(x) \text{ exists,} \\ \phi_n(a(x)) & \text{if } x \notin \overline{\mathrm{ran}(d)} \text{ and} \\ & b(x) \text{ does not exist.} \end{cases}$$

Define likewise ϕ' from ϕ. Since ϕ_n, ϕ are continuous, and $\phi_n \to \phi$ uniformly on $\mathrm{ran}(d)$, it follows that ϕ'_n, ϕ' are continuous and that $\phi'_n \to \phi'$ uniformly on $[0,\infty)$.

Now the proof of Proposition 3.2 involves taking moduli $\phi_n, \phi : [0,\infty) \to [0,\infty)$ as in Definition 1.26 such that $\phi_n \to \phi$ uniformly on $[0,\infty)$, letting $b > 0$, and defining $\psi_n, \psi : (0,\infty) \to [0,\infty)$ and $\psi^b_n, \psi^b : (0,b] \to [0,\infty)$ by

$$\psi_n(s) := \frac{\phi_n(s)}{s}, \qquad \psi(s) := \frac{\phi(s)}{s}, \tag{3.1}$$

and

$$\psi^b_n(s) := \sup_{t\in[s,b]} \psi_n(t), \qquad \psi^b(s) := \sup_{t\in[s,b]} \psi(t). \tag{3.2}$$

One then notes that:

1. ψ and ψ_n are continuous on $(0,\infty)$, $\psi(s) < 1$ for $s > 0$, and the sequence $(\psi_n)_{n\in\mathbb{N}}$ converges uniformly to ψ on $[l,\infty)$ for each $l > 0$;
2. ψ^b and ψ^b_n are continuous on $(0,b]$, $\psi^b(s) < 1$ for all $s \in (0,b]$, and the sequence $(\psi^b_n)_{n\in\mathbb{N}}$ converges uniformly to ψ^b on $[l,b]$ for each $l \in (0,b]$.

The following proposition will be used repeatedly throughout our work on asymptotic contractions:

Proposition 3.3 (Gerhardy)**.** *Let (X,d) be a metric space, let f be an asymptotic contraction in the sense of Gerhardy and let $b > 0$ and η^b, β^b be given. Then for every $\varepsilon \in (0,b]$ there is a $K(\eta^b,\beta^b,\varepsilon)$ such that for all $k \geq K$, where $K = \beta^b_\varepsilon(\frac{\eta^b(\varepsilon)}{2})$,*

$$b \geq d(x,y) \geq \varepsilon \quad \to \quad d(f^k(x), f^k(y)) \leq \left(1 - \frac{\eta^b(\varepsilon)}{2}\right) \cdot d(x,y).$$

Proof. Let $K = \beta^b_\varepsilon(\frac{\eta^b(\varepsilon)}{2})$, let a suitable sequence ϕ^b_n be given and let $\phi^b := \lim_{n\to\infty} \phi^b_n$. By the definition of η^b we have that

$$\phi^b(s) + \eta^b(\varepsilon) \leq 1$$

for $s \in [\varepsilon, b]$. By the definition of β^b the function ϕ^b_k is at least $\frac{\eta^b(\varepsilon)}{2}$-close to ϕ^b for all $k \geq K$ and all $s \in [\varepsilon,b]$, and hence we also have $\phi^b_k(s) \leq 1 - \frac{\eta^b(\varepsilon)}{2}$.

□

Lemma 3.4 (Gerhardy). *Let (X,d) be a metric space, let f be an asymptotic contraction in the sense of Gerhardy and let $b>0$ and η^b, β^b be given. Then for every $b \geq \varepsilon > 0$, for all $n \geq N$ and all $x, y \in X$ with $d(x,y) \leq b$*

$$d(x, f^n(x)), d(y, f^n(y)) \leq \delta \quad \rightarrow \quad d(x,y) \leq \varepsilon,$$

where $\delta(\eta^b, \varepsilon) = \frac{\varepsilon \cdot \eta^b(\varepsilon)}{4}$ and $N(\eta^b, \beta^b, \varepsilon) = \beta^b_\varepsilon(\frac{\eta^b(\varepsilon)}{2})$.

Lemma 3.4 provides a kind of "modulus of uniqueness".

Lemma 3.5 (Gerhardy). *Let (X,d) be a metric space, let f be an asymptotic contraction in the sense of Gerhardy and let $b>0$ and η^b, β^b be given. Then for every $\delta \in (0,b]$, for every $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded by b and for every N there exists an $m \leq M$, such that*

$$d(x_m, f^N(x_m)) \leq \delta,$$

where

$$M(\eta^b, \beta^b, \delta, b) = k \left\lceil \frac{\lg(\delta) - \lg(b)}{\lg(1 - \frac{\eta^b(\delta)}{2})} \right\rceil,$$

with $k = \beta^b_\delta(\frac{\eta^b(\delta)}{2})$.

Proof. Let $k = \beta^b_\delta(\frac{\eta^b(\delta)}{2})$. Assume that for some M_0 and all $m < M_0$ we have $d(x_{mk}, f^N(x_{mk})) \geq \delta$, then repeatedly using Proposition 3.3

$$d(x_{M_0 k}, f^N(x_{M_0 k})) \leq \left(1 - \frac{\eta^b(\delta)}{2}\right)^{M_0} d(x_0, f^N(x_0)) \leq b\left(1 - \frac{\eta^b(\delta)}{2}\right)^{M_0}$$

since by assumption $d(x_0, f^N(x_0)) \leq b$. Solving the inequality $b(1 - \frac{\eta^b(\delta)}{2})^{M_0} \leq \delta$ with respect to M_0 yields the described upper bound $M = kM_0$ on an m such that $d(x_m, f^N(x_m)) \leq \delta$.

□

We remark that in [54] the lemma above was stated with $d(x_m, f^N(x_m)) < \delta$ in the conclusion, for which we would have to modify the functional M slightly. This has no further consequences, since Lemma 3.5 is only used in contexts where the modified conclusion $d(x_m, f^N(x_m)) \leq \delta$ works just as well.

Lemma 3.5 will in the elementary proof of Kirk's theorem on asymptotic contractions function somewhat like a modulus of uniform almost asymptotic regularity (see Definition 2.3), but notice that this modulus is the same for all iterations f^N of the mapping f.

Lemma 3.6 (Gerhardy). *Let (X,d) be a metric space, let f be an asymptotic contraction in the sense of Gerhardy and let $b>0$ and η^b, β^b be given. Assume that f has a (unique) fixed point z. Then for every $\varepsilon \in (0,b]$, for every $x_0 \in X$ such that $(x_n)_{n\in\mathbb{N}}$ is bounded by b and $d(x_n, z) \leq b$ for all n there exists an $m \leq M$ such that*

$$d(x_m, z) \leq \varepsilon,$$

where

$$M(\eta^b, \beta^b, \varepsilon, b) = k \left\lceil \frac{\lg(\delta) - \lg(b)}{\lg(1 - \frac{\eta^b(\delta)}{2})} \right\rceil,$$

with $k = \beta^b_\delta(\frac{\eta^b(\delta)}{2})$, $\delta = \frac{\varepsilon \cdot \eta^b(\varepsilon)}{4}$.

Proof. By Lemma 3.4 we have that for every $\varepsilon > 0$ there exist δ and N as described in that lemma such that if $d(x, y) \leq b$ and

$$d(x, f^N(x)) \leq \delta$$

and

$$d(y, f^N(y)) \leq \delta$$

then $d(x, y) \leq \varepsilon$. Now, by Lemma 3.5 for every δ and every N we can find an $m \leq M$ as described above such that $d(x_m, f^N(x_m)) \leq \delta$ and hence x_m is ε-close to the fixed point z.

□

Lemma 3.7 (Gerhardy)**.** *Let (X, d) be a metric space, let f be an asymptotic contraction in the sense of Gerhardy and let $b > 0$ and η^b, β^b be given. Then for every $\delta > 0$, for every $x_0 \in X$ such that $(x_n)_{n \in \mathbb{N}}$ is bounded by b and for every N there exists an M such that for all $m \geq M$*

$$d(x_m, f^N(x_m)) < \delta.$$

Proof. By Lemma 3.5 there exists an m such that $d(x_m, f^N(x_m)) < \delta$. Either $d(x_m, f^N(x_m)) = 0$, then we are done, or $d(x_m, f^N(x_m)) > \varepsilon_0$ for some $\varepsilon_0 > 0$. Let $K = \beta^b_{\varepsilon_0}(\frac{\eta^b(\varepsilon_0)}{2})$, then by Proposition 3.3 it follows that for all $k \geq K$

$$d(x_{m+k}, f^N(x_{m+k})) \leq \left(1 - \frac{\eta^b(\varepsilon_0)}{2}\right) d(x_m, f^N(x_m)) < \delta.$$

Let $M = m + K$ and the result follows.

□

Lemma 3.8 (Gerhardy)**.** *Let (X, d) be a metric space, let f be an asymptotic contraction in the sense of Gerhardy and let $b > 0$ and η^b, β^b be given. If $(x_n)_{n \in \mathbb{N}}$ is bounded by b then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.*

Proof. Let $\varepsilon \in (0, b]$. By Lemma 3.4 there exist $\delta > 0$ and N such that if $d(x, f^N(x)) \leq \delta$ and $d(y, f^N(y)) \leq \delta$, then $d(x, y) \leq \varepsilon$. And by Lemma 3.7 there exists an M such that $d(x_m, f^N(x_m)) < \delta$ for all $m \geq M$. Then $d(x_m, x_n) \leq \varepsilon$ for all $m, n \geq M$.

□

Theorem 3.9 (Gerhardy)**.** *Let* (X,d) *be a complete metric space, let* f *be a continuous asymptotic contraction in the sense of Gerhardy and let* $b>0$ *and* η^b,β^b *be given. If for some* $x_0\in X$ *the sequence* $(x_n)_{n\in\mathbb{N}}$ *is bounded by* b *then* f *has a unique fixed point* z*,* $(x_n)_{n\in\mathbb{N}}$ *converges to* z *and for every* $\varepsilon\in(0,b]$ *there exists an* $m\le M_1$ *such that*

$$d(x_m,z)\le\varepsilon,$$

where

$$M_1(\eta^b,\beta^b,\varepsilon,b)=k\left\lceil\frac{\lg(\delta)-\lg(b)}{\lg(1-\frac{\eta^b(\delta)}{2})}\right\rceil,$$

with $k=\beta^b_\delta(\frac{\eta^b(\delta)}{2})$*,* $\delta=\frac{\varepsilon\cdot\eta^b(\varepsilon)}{4}$*.*

Proof. By Lemma 3.8 every iteration sequence which is bounded is a Cauchy sequence. Since (X,d) is complete the limit z of $(x_n)_{n\in\mathbb{N}}$ exists and using the continuity of f one then easily shows that $f(z)=z$, i.e. that z is a fixed point of f. It is trivial that z is the unique fixed point of f. The bound M_1 follows by Lemma 3.6.

□

3.2.2 Generalized asymptotic contractions

We will in fact work with a slightly generalized definition of asymptotic contractions compared to the one of Gerhardy. This is no wide-ranging generalization, but simply a consequence of the fact that the relevant proofs go through also with a definition where we are allowed to disregard how the mapping behaves for the first few iterates in a Picard iteration sequence. In the definition below we only require

$$b\ge d(x,y)\ge\varepsilon\quad\to\quad d\big(f^n(x),f^n(y)\big)\le\phi^b_n(\varepsilon)d(x,y)$$

to hold for large enough $n\in\mathbb{N}$, namely for $n\ge\beta^b_\varepsilon(1)$. And as a consequence of the technical way in which we require that n is large we also require that $\varepsilon<\varepsilon'$ gives $\beta^b_l(\varepsilon)\ge\beta^b_l(\varepsilon')$. In detail we modify Gerhardy's definition as follows:

Definition 3.10. A function $f:X\to X$ on a metric space (X,d) is called a *generalized asymptotic contraction* if for each $b>0$ there exist moduli $\eta^b:(0,b]\to(0,1)$ and $\beta^b:(0,b]\times(0,\infty)\to\mathbb{N}$ and a sequence of functions $\phi^b_n:(0,\infty)\to(0,\infty)$ such that the following holds:

(1) For each $0<l\le b$ the function $\beta^b_l:=\beta^b(l,\cdot)$ is a modulus of uniform convergence for $(\phi^b_n)_{n\in\mathbb{N}}$ on $[l,b]$, i.e.,

$$\forall\varepsilon>0\forall s\in[l,b]\forall m,n\ge\beta^b_l(\varepsilon)\left(\left|\phi^b_m(s)-\phi^b_n(s)\right|\le\varepsilon\right).$$

Furthermore, if $\varepsilon<\varepsilon'$ then $\beta^b_l(\varepsilon)\ge\beta^b_l(\varepsilon')$.

(2) For all $x,y\in X$, for all $b\ge\varepsilon>0$ and for all $n\in\mathbb{N}$ such that $\beta^b_\varepsilon(1)\le n$, we have:

$$b\ge d(x,y)\ge\varepsilon\quad\to\quad d\big(f^n(x),f^n(y)\big)\le\phi^b_n(\varepsilon)d(x,y).$$

(3) Define $\phi^b : (0,\infty) \to (0,\infty)$ by $\phi^b(s) := \lim_{n\to\infty} \phi^b_n(s)$. Then for each $0 < \varepsilon \le b$ we have

$$\phi^b(s) + \eta^b(\varepsilon) \le 1$$

for each $s \in [\varepsilon, b]$.

If f is an asymptotic contraction in the sense of Gerhardy, then it is also an asymptotic contraction in our sense. However, one might have to modify the moduli β^b_l. The above definition is in fact, as the following simple example shows, strictly more general than Definition 3.1, and therefore also strictly more general than Definition 1.26.

Lemma 3.11. *There exists a complete metric space (X, d) and a continuous mapping $f : X \to X$ which is an asymptotic contraction in the sense of Definition* 3.10*, but not an asymptotic contraction in the sense of Gerhardy.*

Proof. Consider $X \subseteq \mathbb{R}^2$ given by

$$X := \{(0, n) : n \in \mathbb{N}\} \cup \{(1, n) : n \in \mathbb{N}\},$$

and let (X, d) be X equipped with the metric inherited from $\mathbb{R}^2$ equipped with the Euclidean metric. Let $f : X \to X$ be given by letting $f(1, n) = (0, n)$ and $f(0, n) = (0, 0)$ for $n \in \mathbb{N}$. Then f is an asymptotic contraction in our extended sense, with for all $b > 0$ $\phi^b_n(t) := 1/2$, $\phi^b(t) := 1/2$ and $\beta^b_l(t) := 2$ for $t \in (0,\infty)$ and $0 < l \le b$, and with $\eta^b(s) := 1/2$ for $s \in (0, b]$. But f is not an asymptotic contraction in the sense of Gerhardy. For since $1 \ge d((1, n), (0, n)) \ge 1$ for $n \in \mathbb{N}$ we should then have

$$d(f(1, n), f(0, n)) \le \phi^1_1(1) \cdot d((1, n), (0, n)) = \phi^1_1(1)$$

for all $n \in \mathbb{N}$, where ϕ^1_1 is a function as required in Definition 3.1. But if $n > 0$, then $d(f(1, n), f(0, n)) = n$, so

$$\{d(f(1, n), f(0, n)) : n \in \mathbb{N}\}$$

is unbounded.

□

Many of the results for asymptotic contractions in the sense of Gerhardy go through for asymptotic contractions in the sense of Definition 3.10:

Lemma 3.12. *Proposition* 3.3 *as well as Lemmas* 3.4, 3.5, 3.6, 3.7, 3.8 *and Theorem* 3.9 *hold also for generalized asymptotic contractions instead of asymptotic contractions in the sense of Gerhardy.*

Proof. This can be verified easily by inspection of the proofs, and by in so doing noting that $\eta(\varepsilon)/2 < 1$.

□

Because of Lemma 3.12 we will freely refer to Proposition 3.3, Lemmas 3.4, 3.5, 3.6, 3.7, 3.8, and Theorem 3.9 also when the mapping $f : X \to X$ under consideration is an asymptotic contraction in the sense of Definition 3.10.

3.2.3 A discussion of issues concerning computability

As pointed out in Remark 8 in [54], we could have given the moduli η^b, β^b as functions $\tilde{\eta}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, with real numbers approximated from below by rationals 2^{-n}, and $\tilde{\eta}, \tilde{\beta}$ parametrized by $b \in \mathbb{N}$, $b \neq 0$, rather than real numbers $b > 0$. Then a mapping having moduli η^b, β^b would also have corresponding moduli $\tilde{\eta}, \tilde{\beta}$. This is the case for the moduli in Definition 3.10 as well as for the moduli in Definition 3.1. In detail we have that a function $f : X \to X$ on a metric space (X, d) which is a generalized asymptotic contraction in the sense of Definition 3.10 satisfies that for each $b \in \mathbb{N}$, $b \neq 0$, there exist moduli $\tilde{\eta}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and a sequence of functions $\phi_n^b : (0, \infty) \to (0, \infty)$ such that the following hold:

(1) For all $l \in \mathbb{N}$, the function $\tilde{\beta}_l^b := \tilde{\beta}^b(l, \cdot)$ is a modulus of uniform convergence for $(\phi_n^b)_{n \in \mathbb{N}}$ on $[2^{-l}, b]$, i.e.,

$$\forall k \in \mathbb{N} \forall s \in [2^{-l}, b] \forall m, n \geq \tilde{\beta}_l^b(k) \left(\left| \phi_m^b(s) - \phi_n^b(s) \right| \leq 2^{-k} \right).$$

Furthermore, if $k > k'$ then $\tilde{\beta}_l^b(k) \geq \tilde{\beta}_l^b(k')$.

(2) For all $x, y \in X$, for all $l \in \mathbb{N}$ and for all $n \in \mathbb{N}$ such that $\tilde{\beta}_l^b(0) \leq n$, we have:

$$b \geq d(x, y) \geq 2^{-l} \quad \to \quad d\big(f^n(x), f^n(y)\big) \leq \phi_n^b(2^{-l}) d(x, y).$$

(3) Define $\phi^b := \lim_{n \to \infty} \phi_n^b$. Then for each $l \in \mathbb{N}$ we have

$$\phi^b(s) + 2^{-\tilde{\eta}^b(l)} \leq 1$$

for each $s \in [2^{-l}, b]$.

Similarly, an asymptotic contraction $f : X \to X$ in the sense of Gerhardy has for each $b \in \mathbb{N} \setminus \{0\}$ moduli $\tilde{\eta'}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta'}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that there exists a sequence of functions $\phi_n^b : (0, \infty) \to (0, \infty)$ such that conditions (1)–(3) above hold, except that in (1) we do not require that if $k > k'$ then $\tilde{\beta'}_l^b(k) \geq \tilde{\beta'}_l^b(k')$, and in (2) we do not require that $\tilde{\beta'}_l^b(0) \leq n$. Then $\tilde{\beta}^b$ corresponding to β^b in the sense of Definition 3.10 is effectively computable in $\tilde{\beta'}^b$ corresponding to β^b in the sense of Definition 3.1; we simply take

$$\tilde{\beta}_l^b(k) = \max \big\{ \tilde{\beta'}_l^b(k') : k' \in \mathbb{N},\, k' \leq k \big\}.$$

Using the approach with $\tilde{\eta}^b, \tilde{\beta}^b$ as functions $\tilde{\eta}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ one would then get a computable functional $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N} \times \mathbb{N}} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ corresponding to M_1 in Theorem 3.9 such that $\lambda k.\Phi(\tilde{\eta}^b, \tilde{\beta}^b, b, k)$ is a rate of proximity for $(f^n(x_0))_{n \in \mathbb{N}}$ to the fixed point $z \in X$, provided $f : X \to X$ is an asymptotic contraction with moduli $\tilde{\eta}^b, \tilde{\beta}^b$, and $b \in \mathbb{N}$, $b \neq 0$, is a bound on the iteration sequence. This follows easily from inspection of the proofs of the results leading up to Theorem 3.9, whereby one notes that the (from the point of view of computability theory) problematic function $\lceil \cdot \rceil$ in the expression

$$\left\lceil \frac{\lg(\delta) - \lg(b)}{\lg(1 - \frac{\eta^b(\delta)}{2})} \right\rceil$$

in Theorem 3.9 only appears because one needs an upper bound on an $M \in \mathbb{N}$ such that

$$\left(1 - \frac{\eta^b(\delta)}{2}\right)^M \cdot b < \delta. \tag{3.3}$$

And in the setting where we use rationals of the form 2^{-n} and where we have a number theoretic function $\tilde{\eta}^b : \mathbb{N} \to \mathbb{N}$ we would only need to find an upper bound on an $M \in \mathbb{N}$ such that

$$\left(1 - 2^{-\tilde{\eta}^b(k)-1}\right)^M \cdot b < 2^{-k}, \tag{3.4}$$

for some suitable $k \in \mathbb{N}$ such that 2^{-k} takes the place of δ in the proof. Strictly speaking we would have to formulate things in this way to get computable rates of convergence or rates of proximity. However, for the sake of ease of notation we will follow Gerhardy in working with moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ (parametrized by real numbers $b > 0$) rather than number theoretic functions. It will be clear how to reformulate the proofs and the statements of the theorems so as to obtain computable functionals of type 2 which give computable rates of convergence or proximity.

We have yet to comment on Kirk's original moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ from Definition 1.26, and on to what extent one can compute moduli as in Definition 3.1 or Definition 3.10 from these. The discussion here will be very brief, as the issues involved are standard concerns in the theory of computability on the reals or constructively representable complete separable metric spaces. We use a suitable standard representation (see e.g. Chapter 4 in [101]) of the complete separable metric space $[0, \infty)$ as $\mathbb{N}^{\mathbb{N}}$, and for each $b \in \mathbb{N}$, $b > 0$, a representation of the compact (i.e. complete and totally bounded) metric space $[0, b]$ as $C_{N_b} = \{f \in \mathbb{N}^{\mathbb{N}} : f \leq N_b\}$, for some fixed primitive recursive N_b. Functions $g : [0, b] \to [0, \infty)$ are represented by functionals $G : C_{N_b} \to \mathbb{N}^{\mathbb{N}}$ which respect the representation, and – via an operation which primitive recursively in $f \in \mathbb{N}^{\mathbb{N}}$ gives a $g_f \leq N_b$ – by functionals $G : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. If for each $b \in \mathbb{N}$, $b > 0$, we have moduli $\phi^b, \phi^b_n : [0, b] \to [0, \infty)$ such that $\phi^b_n \to \phi^b$ uniformly on $[0, b]$, but which otherwise are as in Definition 1.26, and these are represented as functionals $F_{\phi,b}, F_{\phi_n,b} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, then we have the following relationship with moduli $\tilde{\eta'}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta'}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ corresponding to the moduli in the definition of an asymptotic contraction in the sense of Gerhardy:

(1) We can, effectively in $F_{\phi,b}$ together with a modulus of uniform continuity $\omega_{\phi,b}$ of $F_{\phi,b}$ on C_{N_b} and a modulus $\gamma : \mathbb{N} \to \mathbb{N}$ for the uniform convergence of ϕ^b_n to ϕ^b on $[0, b]$, compute moduli $\tilde{\eta'}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta'}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

We use the modulus of uniform continuity to compute the maximum value taken by ψ on $[s, b]$ – see (3.1) and (3.2) in the discussion concerning the proof of Proposition 3.2 – and to compute the maximum value taken by ψ^b on $[2^{-l}, b]$.

(2) Without a modulus of uniform continuity for $F_{\phi,b}$ the procedure we referred to in (1), which involves computing the maximum value of a continuous function on a compact interval, is not effective. We could then compute moduli $\tilde{\eta'}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta'}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ in the fan functional, by using this to compute a modulus of uniform continuity. For more information on the fan functional and related matters, see [139, 169].

(3) If $F_{\phi,b}$ and a modulus $\gamma : \mathbb{N} \to \mathbb{N}$ for the uniform convergence of ϕ_n^b to ϕ^b on $[0, b]$ are computable, then the moduli $\tilde{\eta'}^b : \mathbb{N} \to \mathbb{N}$ and $\tilde{\beta'}^b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ are computable. This is because we via an encoding of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ as $\mathbb{N}^{\mathbb{N}}$ can regard $F_{\phi,b}$ as a computable type 2 functional, and any computable type 2 functional has a computable associate $\alpha : \mathbb{N} \to \mathbb{N}$. Which means that there also exists a computable modulus of uniform continuity $\omega_{\phi,b}$ of $F_{\phi,b}$ on C_{N_b}.

3.2.4 Eliminating the "modulus of uniqueness"

Our first result here on asymptotic contractions is an improvement of the bound in Theorem 3.9. The following theorem is identical to Theorem 3.9, except that it involves asymptotic contractions in our sense, and that $\eta(\varepsilon) \cdot \varepsilon/4$ is replaced by ε in the definition of M_ε. So the "modulus of uniqueness" from Lemma 3.4 no longer plays any part in the bound. This will in most cases, depending on η, constitute a significant numerical improvement. This is in line with the discussion of elimination of moduli of uniqueness at the end of Section 2.3 in Chapter 2. This result appeared in the paper [25].

Theorem 3.13. *Let (X, d) be a complete metric space, let $f : X \to X$ be a continuous generalized asymptotic contraction and let $b > 0$ and η, β be given. If for some $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded by b then f has a unique fixed point z, $(x_n)_{n\in\mathbb{N}}$ converges to z and for every $\varepsilon > 0$ such that $b \geq \varepsilon$ there exists $m \leq M_\varepsilon$ such that*

$$d(x_m, z) \leq \varepsilon,$$

where

$$M_\varepsilon(\eta, \beta, b) := k \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\left(1 - \frac{\eta(\varepsilon)}{2}\right)} \right\rceil,$$

with $k := \beta_\varepsilon(\frac{\eta(\varepsilon)}{2})$.

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is bounded by b. Let $b \geq \varepsilon > 0$. Let

$$M_\varepsilon := k \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\left(1 - \frac{\eta(\varepsilon)}{2}\right)} \right\rceil,$$

where $k := \beta_\varepsilon(\frac{\eta(\varepsilon)}{2})$. By Theorem 3.9 (and Lemma 3.12) we have that $(x_n)_{n\in\mathbb{N}}$ converges to the unique fixed point z of f. Let $l \in \mathbb{N}$ be arbitrary and let N be such that $d(x_n, z) < 2^{-l}$ for all $n \geq N$. By Lemma 3.5 there exists an $m \leq M_\varepsilon$ such that

$$d\left(x_m, f^N(x_m)\right) \leq \varepsilon.$$

Note that M_ε does not depend on N. Since $f^N(x_m) = x_{m+N}$ and $m + N \geq N$, we have $d(f^N(x_m), z) < 2^{-l}$. Therefore

$$d(z, x_m) \leq d\left(x_m, f^N(x_m)\right) + d\left(f^N(x_m), z\right) < \varepsilon + 2^{-l}.$$

Since there are only finitely many $m \leq M_\varepsilon$ there must exist $m_1 \leq M_\varepsilon$ such that

$$d(z, x_{m_1}) < \varepsilon + 2^{-l}$$

holds for infinitely many l. Hence

$$d(z, x_{m_1}) \leq \varepsilon.$$

□

3.3 Main results

3.3.1 A rate of convergence dependent on a bound on the iteration sequence

Our first main result is an explicit rate of convergence for generalized asymptotic contractions – and therefore also for asymptotic contractions in the senses of Kirk and Gerhardy – which in addition to the required accuracy $\varepsilon > 0$ takes only a bound b on the iteration sequence and the moduli η, β as arguments. We begin with the case where we assume that the mapping is continuous and the space complete. This section corresponds to a part of the paper [25], and the results have appeared there.

Theorem 3.14. *Let (X, d) be a complete metric space, let $b > 0$ be given, and let $f : X \to X$ be a continuous generalized asymptotic contraction with moduli η and β. If for some $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded by b, then $(x_n)_{n\in\mathbb{N}}$ has the following rate of convergence. Let z be the unique fixed point of f. For $b > \varepsilon > 0$ we put $\Phi(b, \eta^b, \beta^b, \varepsilon) := \max\{U, V\}$, where*

$$U := k \cdot \left(2M_\gamma + \beta^b_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right),$$

$$V := (k-1) \cdot \left(2M_\gamma + \beta^b_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\right) + M_\gamma + 1,$$

$$k := \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg\left(1 - \frac{\eta^b(\gamma)}{2}\right)} \right\rceil,$$

$$M_\gamma := K_\gamma \cdot \left\lceil \frac{\lg(\gamma) - \lg(b)}{\lg\left(1 - \frac{\eta^b(\gamma)}{2}\right)} \right\rceil,$$

$$K_\gamma := \beta_\gamma \left(\frac{\eta^b(\gamma)}{2} \right),$$

and $\delta := \min\left\{\varepsilon/2, \eta^b(\varepsilon/2)/2\right\}$, $\gamma := \min\left\{\delta, \delta\varepsilon/4\right\}$. *For* $\varepsilon \geq b$ *we can put* $\Phi(b, \eta^b, \beta^b, \varepsilon) := 0$.

Then for all $n \geq \Phi(b, \eta^b, \beta^b, \varepsilon)$ *we have*

$$d(x_n, z) \leq \varepsilon.$$

Proof. Let $b > \varepsilon > 0$. Let $\delta := \min\{\frac{\varepsilon}{2}, \frac{\eta(\frac{\varepsilon}{2})}{2}\}$ and $\gamma := \min\{\delta, \frac{\delta\varepsilon}{4}\}$. Let $x_0 \in X$ be such that $(x_n)_{n\in\mathbb{N}}$ is bounded by b. For $a > 0$ let

$$B_a := \{x \in X : d(x, z) \leq a\}.$$

By Theorem 3.13 there exists $m' \leq M_\gamma$ such that $x_{m'} \in B_\gamma$. Suppose there exists $m > m'$ such that $x_m \notin B_\varepsilon$. In the first part of the proof we will use this to establish an upper bound on an $n \in \mathbb{N}$ such that

$$d(x_n, z) > \gamma,$$

which is what we need in our further argument. Let now

$$m := \min\{n : n > m' \text{ and } x_n \notin B_\varepsilon\}.$$

Then for $x_n \in B_\gamma$ we get $d(x_n, x_m) > \frac{\varepsilon}{2}$ since

$$d(x_n, x_m) \geq d(x_m, z) - d(x_n, z) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Assume $m - n \geq \beta_{(\frac{\varepsilon}{2})}(\delta)$. Note that $\delta < 1$. Then for all $k \geq m - n$ we have

$$\left|\phi_k^b(\varepsilon/2) - \phi_{m-n}^b(\varepsilon/2)\right| \leq \delta,$$

and hence

$$|\phi^b(\varepsilon/2) - \phi_{m-n}^b(\varepsilon/2)| \leq \delta.$$

The definition of an asymptotic contraction gives

$$\phi^b\left(\frac{\varepsilon}{2}\right) + \eta\left(\frac{\varepsilon}{2}\right) \leq 1,$$

and so

$$\phi^b\left(\frac{\varepsilon}{2}\right) \leq 1 - \eta\left(\frac{\varepsilon}{2}\right),$$

and

$$\phi_{m-n}^b\left(\frac{\varepsilon}{2}\right) \leq 1 - \eta\left(\frac{\varepsilon}{2}\right) + \left|\phi^b\left(\frac{\varepsilon}{2}\right) - \phi_{m-n}^b\left(\frac{\varepsilon}{2}\right)\right|.$$

We therefore have

$$\phi^b_{m-n}\left(\frac{\varepsilon}{2}\right) \leq 1 - 2\delta + \delta = 1 - \delta.$$

Since we by definition have

$$d(x_m, x_{2m-n}) = d(f^{m-n}(x_n), f^{m-n}(x_m)) \leq \phi^b_{m-n}\left(\frac{\varepsilon}{2}\right) \cdot d(x_n, x_m),$$

we get

$$d(x_m, x_{2m-n}) \leq (1-\delta) \cdot d(x_n, x_m).$$

So in this case

$$d(x_{2m-n}, x_n) \geq d(x_n, x_m) - d(x_m, x_{2m-n})$$

gives

$$d(x_{2m-n}, x_n) \geq d(x_n, x_m) - (1-\delta) \cdot d(x_n, x_m) = \delta \cdot d(x_n, x_m) > \frac{\delta\varepsilon}{2}.$$

If $x_{2m-n} \in B_\gamma$ then we would have

$$d(x_{2m-n}, x_n) \leq d(x_{2m-n}, z) + d(z, x_n) \leq 2\gamma \leq \frac{\delta\varepsilon}{2}.$$

So $x_{2m-n} \notin B_\gamma$. Let

$$m'' := \min\{n : n > m' \text{ and } x_n \notin B_\gamma\}.$$

If

$$m'' - m' = M' + \beta_{(\frac{\varepsilon}{2})}(\delta)$$

for some $M' \geq 0$, then since $m \geq m''$ we have

$$m - m', m - (m'+1), \ldots, m - (m' + M') \geq \beta_{(\frac{\varepsilon}{2})}(\delta),$$

and $x_{m'}, x_{m'+1}, \ldots, x_{m'+M'} \in B_\gamma$. By the above argument this gives that respectively $x_{2m-m'}, x_{2m-m'-1} \ldots, x_{2m-m'-M'+1}$ and $x_{2m-m'-M'}$ are not in B_γ. By arranging the indices in increasing order, we have

$$x_{2m-m'-M'}, x_{2m-m'-M'+1}, \ldots, x_{2m-m'} \notin B_\gamma.$$

By taking $x_{2m-m'-M'}$ as the starting point of a b-bounded Picard iteration sequence defined by $x_{n+1} := f(x_n)$, we get by Theorem 3.13 that there exists $m''' \leq M_\gamma$ such that $x_{2m-m'-M'+m'''} \in B_\gamma$. So $M' < M_\gamma$. (And so in this case $0 < M_\gamma$.) So the existence of $m > m'$ such that $x_m \notin B_\varepsilon$ implies that

$$m'' - m' < M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta),$$

and thus that

$$m'' < m' + M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) \leq 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta).$$

Thus in total, if there exists $m > m'$ such that $x_m \notin B_\varepsilon$, then we get that for some

$$n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta)$$

we have

$$\gamma < d(x_n, z).$$

Since $(x_n)_{n\in\mathbb{N}}$ converges to z, we have

$$d(x_n, z) \leq b.$$

So in this case by Proposition 3.3, for $n \in \mathbb{N}$ such that

$$n \geq 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$$

we have

$$d(x_n, z) \leq \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b.$$

Likewise, by then treating $x_{2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1}$ as the starting point y_0 of a Picard iteration sequence $(y_n)_{n\in\mathbb{N}}$ bounded by b with the property that

$$d(y_n, z) \leq \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for all $n \geq 0$, either there exists no $n \in \mathbb{N}$ with

$$n > 3M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$$

such that $x_n \notin B_\varepsilon$, or else for $n \in \mathbb{N}$ such that

$$n \geq 2 \cdot \big(2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1\big)$$

we have

$$d(x_n, z) \leq \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b.$$

We get that we for $n \in \mathbb{N}$ such that

$$n \geq \max\left\{k\cdot\big(2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1\big), (k-1)\cdot\big(2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1\big)+M_\gamma+1\right\},$$

where $k \geq 1$, have

$$x_n \in B_\varepsilon$$

or

$$x_n \in B_{\left(1-\frac{\eta(\gamma)}{2}\right)^k\cdot b}.$$

By letting

$$k := \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg(1 - \frac{\eta(\gamma)}{2})} \right\rceil$$

we get for $n \in \mathbb{N}$ such that

$$n \geq \max\left\{k\cdot\left(2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1\right),(k-1)\cdot\left(2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1\right)+M_\gamma+1\right\},$$

that

$$x_n \in B_\varepsilon.$$

□

Completeness and continuity in the above theorem is only needed to show the existence of a fixed point z. If a fixed point exists, then by Proposition 3.3 every Picard iteration sequence is bounded, irrespectively of completeness and continuity, and hence by Lemma 3.8 it is Cauchy. By Lemma 3.4 it converges to the fixed point z, and by inspection we see that the proof of Theorem 3.14 goes through. Then Theorem 3.14 gives a rate of convergence for a b-bounded Picard iteration sequence. Hence we have the following theorem.

Theorem 3.15. *Let (X, d) be a metric space, let $b > 0$ be given, and let $f : X \to X$ be a generalized asymptotic contraction with moduli η and β. Assume that f has a fixed point z. Then every Picard iteration sequence is bounded, and if for $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded by b then $(x_n)_{n\in\mathbb{N}}$ converges to z with the rate of convergence specified in Theorem* 3.14.

Proof. Follows by the above remarks.

□

If in the metric space (X, d) some iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded, where f is a generalized asymptotic contraction with moduli η^b, β^b for $b > 0$, then by Proposition 3.3 and Lemmas 3.8 and 3.4 all iteration sequences are Cauchy even if none of them converges, and if some $z \in X$ is the limit of one sequence, then z is the limit of all the iteration sequences. Namely, by Lemma 3.8 $(f^n(x_0))_{n\in\mathbb{N}}$ is Cauchy, and if we let $n \in \mathbb{N}$ be such that $m \geq n$ gives $d(f^n(x_0), f^m(x_0)) < 1$, then taking $f^n(x_0)$ as x in Proposition 3.3 gives that any $(f^n(y))_{n\in\mathbb{N}}$ is bounded. Then $(f^n(y))_{n\in\mathbb{N}}$ is Cauchy by Lemma 3.8 and $\lim_{n\to\infty} d(f^n(x_0), f^n(y)) = 0$ by Lemma 3.4. If $(f^n(x_0))_{n\in\mathbb{N}}$ does not converge then we consider the completion $\overline{X}$ of X, in which the limit z exists. We can then extend f to be defined on $X \cup \{z\}$ by letting $f(z) = z$. It is then easy to see that f is a generalized asymptotic contraction with moduli $\eta_1^b : (0, b] \to (0, 1)$ and $\beta_1^b : (0, b] \times (0, \infty) \to \mathbb{N}$ defined by for example $\eta_1^b(\varepsilon) := \eta^{2b}(\varepsilon/2)$, $\beta_1^b(l, \varepsilon) := \beta^{2b}(l/2, \varepsilon)$.

If the b-bounded iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges in X to z, and z is not a fixed point, then we can use the following theorem to conclude that the convergence is uniform. At this point one might refer back to the discussion after the proof of Proposition 2.48, where we noted that we had shown the possibility of obtaining an explicit and uniform rate of convergence for an asymptotic contraction in the sense of Kirk on a bounded metric space.

Theorem 3.16. *Let (X, d) be a metric space, and let $f : X \to X$ be a generalized asymptotic contraction with moduli η^b and β^b for each $b > 0$. Let $x_0 \in X$ be such that the Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded. Then all*

Picard iteration sequences are Cauchy. Assume that $z := \lim_{n\to\infty} f^n(x_0)$ *exists. Then for any* $x_0 \in X$ *the iteration sequence* $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* z*, irrespective of whether* z *is a fixed point or not. If* $(f^n(x_0))_{n\in\mathbb{N}}$ *is bounded by* $b > 0$ *then* $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* z *with the rate of convergence specified in Theorem* 3.14.

Proof. Proposition 3.3 and Lemmas 3.8 and 3.4 still imply that all iteration sequences converge to z. The rate of proximity in Theorem 3.13 only depends on Lemma 3.5 and the fact that the Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$ converges to z, all of which is independent of whether z is a fixed point or not. However, in the proof of Theorem 3.14 we use that z is a fixed point when we use Proposition 3.3 to infer

$$d(x_n, z) \le \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for $n \ge 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$ from the fact that $\gamma < d(x_n, z) \le b$ for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta)$. In this manner we in the proof repeatedly make use of the fact that z is a fixed point. When z is not a fixed point we can proceed as follows. Assuming that there exists $n > M_\gamma$ such that $x_n \notin B_\varepsilon$, we have $\gamma < d(x_n, z) \le b$ for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta)$. Choose such $n \in \mathbb{N}$, and choose a real number $a > 0$. Choose then $K' \in \mathbb{N}$ such that for this n we have

$$d(x_k, z) < \min\big\{a, \big(d(x_n, z) - \gamma\big)\big\}$$

for all $k \ge K'$. We can find such K' since (x_n) converges to z. Then $\gamma < d(x_n, x_{K'}) \le b$, so Proposition 3.3 gives

$$d\big(f^k(x_n), f^k(x_{K'})\big) \le \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for $k \ge K_\gamma$. Now the triangle inequality gives

$$d(x_n, z) \le \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b + a$$

for $n \ge 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$. Since $a > 0$ was arbitrary we get

$$d(x_n, z) \le \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for $n \ge 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$. Then, following the proof of Theorem 3.14 we get that either there does not exist $n \in \mathbb{N}$ with $n > 3M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1$ and $x_n \notin B_\varepsilon$, or else we have

$$\gamma < d(x_n, z) \le \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b$$

for some $n < 2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1 + (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta))$. Choose such $n \in \mathbb{N}$, and choose a real number $a > 0$. Then we can choose $K' \in \mathbb{N}$ as above and get

$$\gamma < d(x_n, x_{K'}) \le \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot b + a.$$

We can assume $(1 - \frac{\eta(\gamma)}{2}) \cdot b + a < b$, so

$$d\big(f^k(x_n), f^k(x_{K'})\big) \leq \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b + \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot a$$

for $k \geq K_\gamma$. And so

$$d(x_n, z) \leq \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b + \left(1 - \frac{\eta(\gamma)}{2}\right) \cdot a + a$$

for $n \geq 2 \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1)$. Since this holds for all sufficiently small $a > 0$ we get for such n that

$$d(x_n, z) \leq \left(1 - \frac{\eta(\gamma)}{2}\right)^2 \cdot b.$$

We can now obviously employ the same strategy each time we have that for a given $k \in \mathbb{N}$ either there does not exist $n \in \mathbb{N}$ with $n > k \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + M_\gamma$ and $x_n \notin B_\varepsilon$, or else we have

$$\gamma < d(x_n, z) \leq \left(1 - \frac{\eta(\gamma)}{2}\right)^k \cdot b$$

for some $n < k \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta))$. We get that for $n \in \mathbb{N}$ such that

$$n \geq \max\left\{k\cdot\big(2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1\big), (k-1)\cdot\big(2M_\gamma+\beta_{(\frac{\varepsilon}{2})}(\delta)+K_\gamma-1\big)+M_\gamma+1\right\},$$

where $k \geq 1$, we have

$$x_n \in B_\varepsilon$$

or

$$x_n \in B_{(1-\frac{\eta(\gamma)}{2})^k \cdot b}.$$

Thus we have the same rate of convergence as in Theorem 3.14.

□

Corollary 3.17. *Under the assumptions in Theorem* 3.16 *we can conclude that the common limit* $z \in X$ *of all Picard iteration sequences* $(f^n(x_0))_{n\in\mathbb{N}}$ *is a fixed point if* f *has a closed graph.*

Proof. Let $x_0 \in X$, and let $(x_n)_{n\in\mathbb{N}}$ be the Picard iteration sequence with respect to f and x_0. Then $x_n \to z$, and also $f(x_n) \to z$, and so if f has a closed graph we get $f(z) = z$.

□

3.3.2 A characterization of asymptotic contractions in the setting of bounded spaces

We will use the rate of convergence given above as well as the following proposition to give a characterization of several variants of asymptotic contractions on nonempty, bounded, complete metric spaces. For this the uniformity features of the rate of convergence will be essential. This section corresponds to a part of [25].

Proposition 3.18. *Let (X,d) be a metric space, and let $f : X \to X$. Let now $\Psi : \mathbb{R}^*_+ \times \mathbb{R}^*_+ \to \mathbb{N}$ satisfy*

$$\forall \varepsilon \in \mathbb{R}^*_+ \forall x, y \in X \forall b \geq d(x,y) \forall n \geq \Psi(\varepsilon, b) \big(d\big(f^n(x), f^n(y)\big) \leq \varepsilon\big).$$

Assume further that $\varepsilon < \varepsilon'$ implies $\Psi(\varepsilon, b) \geq \Psi(\varepsilon', b)$. Then f is a generalized asymptotic contraction.

Proof. For $b > 0$ and $n \in \mathbb{N}$ define $\phi^b_n : (0,\infty) \to (0,\infty)$ by $\phi^b_n(\varepsilon) := 1/2$. Define further $\eta^b : (0,b] \to (0,1)$ by $\eta^b(\varepsilon) := 1/2$ and $\beta^b_l : (0,\infty) \to \mathbb{N}$ by $\beta^b_l(\varepsilon) := \Psi(l/2, b)$. These moduli satisfy Definition 3.10.

□

Corollary 3.19. *Let (X,d) be a metric space, and let $f : X \to X$. Let $z \in X$ and assume that for each $x_0 \in X$ the Picard iteration sequence converges to the point $z \in X$ with a rate of convergence which is uniform in the starting point x_0. Then f is a generalized asymptotic contraction.*

Proof. By assumption there exists $\Psi : \mathbb{R}^*_+ \to \mathbb{N}$ such that

$$\forall \varepsilon \in \mathbb{R}^*_+ \forall x, y \in X \forall n \geq \Psi(\varepsilon) \big(d\big(f^n(x), f^n(y)\big) \leq \varepsilon\big).$$

We can furthermore assume that $\varepsilon < \varepsilon'$ implies $\Psi(\varepsilon) \geq \Psi(\varepsilon')$. Thus Proposition 3.18 applies.

□

The above corollary also follows from Proposition 3 in [76], which implies that f in this case is an asymptotic contraction in the sense of Kirk.

Corollary 3.20. *Let (X,d) be a nonempty, bounded, complete metric space, and let $f : X \to X$. Then f is a generalized asymptotic contraction if and only if there exists $z \in X$ such that for each $x_0 \in X$ the Picard iteration sequence converges to z with a rate of convergence which is uniform in the starting point.*

Proof. That f is a generalized asymptotic contraction if such a $z \in X$ exists follows from Corollary 3.19. The other implication follows from Theorem 3.16, since we assume that the space is bounded.

□

Theorem 3.21. *Let (X,d) be a nonempty, bounded, complete metric space, and let $f : X \to X$. Then the following are equivalent:*

(1) *The function f is a generalized asymptotic contraction.*

(2) *The function f is an asymptotic contraction in the sense of Gerhardy.*

(3) *The function f is an asymptotic contraction in the sense of Kirk.*

(4) *There exists $z \in X$ such that for each $x_0 \in X$ the Picard iteration sequence converges to z with a rate of convergence which is uniform in the starting point.*

(5) *There exists $\alpha : (0, \infty) \to \mathbb{N}$ such that*

$$\forall x, y \in X \forall \varepsilon > 0 \forall n \geq \alpha(\varepsilon) \left(d(x, y) \geq \varepsilon \to d\big(f^n(x), f^n(y)\big) \leq \frac{1}{2} d(x, y) \right).$$

Proof. Assume first (1). Then by the previous corollary, (4) holds. Furthermore, by Theorem 3.16 and the proofs of Corollary 3.19 and Proposition 3.18, (5) holds. Now assume that (4) holds. Then $\mathrm{diam}\big(f^n(X)\big) \to 0$. Following the proof of Proposition 3 in [76], we define $\phi, \phi_n : [0, \infty) \to [0, \infty)$ by $\phi_n(t) := \mathrm{diam}\big(f^n(X)\big)$ and $\phi(t) := 0$. These moduli satisfy Definition 1.26, so f is an asymptotic contraction in the sense of Kirk, i.e., (3) holds. Thus (1)-(4) are equivalent. Now assume that (5) holds. For $b > 0$ and $n \in \mathbb{N}$ define $\phi^b_n : (0, \infty) \to (0, \infty)$ by $\phi^b_n(\varepsilon) := 1/2$. Define further $\eta^b : (0, b] \to (0, 1)$ by $\eta^b(\varepsilon) := 1/2$ and $\beta^b_l : (0, \infty) \to \mathbb{N}$ by $\beta^b_l(\varepsilon) := \alpha(l)$. These moduli satisfy Definition 3.10, so (5) is equivalent to (1).

□

Theorem 3 in [76] gives a characterization of continuous asymptotic contractions in the sense of Kirk on *compact* metric spaces, showing among other things that they are exactly the continuous functions such that the core $Y := \bigcap_{n \in \mathbb{N}} f^n(X)$ is a singleton (assuming the space is nonempty). If we in Theorem 3.21 require that f be continuous, then we get a generalization of this fact from the compact case to the case where the space is bounded and complete. Namely, we get by Theorem 3.14 that if a continuous f is an asymptotic contraction (in one of the three senses considered), then there exists a fixed point z, and $Y = \{z\}$. If on the other hand the core Y is a singleton $\{z\}$, then Theorem 3.21 implies that f is an asymptotic contraction (in all three senses considered).

Remark 3.22. In Theorem 3.21 completeness is only needed to establish the existence of the common limit of all Picard iteration sequences. If the space (X, d) is bounded and nonempty, and if f is a generalized asymptotic contraction such that no Picard iteration sequence is convergent, then we “add” the limit point z, define $f(z) = z$, and get that f is again a generalized asymptotic contraction as in the discussion preceeding Theorem 3.16. Thus the convergence to z is uniform in the starting point, and there exists a function $\Phi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall x, y \in X \forall k \in \mathbb{N} \forall n \geq \Phi(k) \left(d\left(f^n(x), f^n(y)\right) < 2^{-k}\right). \tag{3.5}$$

Hence diam $\big(f^n(X)\big) \to 0$, and as in the proof of Theorem 3.21 it follows that f is an asymptotic contraction in the sense of Kirk . And so in the setting of bounded metric spaces the three versions of asymptotic contraction we have considered are all equivalent, and all equivalent to the existence of a function Φ satisfying 3.5.

The above theorem also has consequences for the relationship between asymptotic contractions in the sense of Kirk and other kinds of contractive type mappings on bounded, complete metric spaces – in particular in relation to the 25 basic definitions of various contractive type mappings systematized by Rhoades in [157], and to some of the standard generalizations of these. In Chapter 4 we treat so-called uniformly generalized p-contractive mappings, and we now get the following result regarding these mappings.

Corollary 3.23. *Let (X, d) be a bounded, complete metric space, let $p \in \mathbb{N}$, and let $f : X \to X$ be uniformly generalized p-contractive and uniformly continuous. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. Let b be a bound on the space. Then for each $x_0 \in X$ we have that b is a bound on the Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$. We can assume X nonempty, for otherwise the proof is trivial. Thus Theorem 4.6 in Chapter 4 (and the comments directly following it) assures the existence of a fixed point $z \in X$ and a rate of convergence for Picard iteration sequences $(x_n)_{n\in\mathbb{N}}$ to z, and this rate is moreover uniform in the starting point x_0. Then by Theorem 3.21 we have that f is an asymptotic contraction in the sense of Kirk.

□

We state also another corollary, although the uniformity of the convergence of the Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ with respect to the starting point $x_0 \in X$, which is the essence of the following corollary, was already present in Rhoades' paper [158].

Corollary 3.24. *Let (X, d) be a compact metric space. Let $f : X \to X$ be continuous and such that it satisfies one of the conditions* (1)–(50) *from* [157]*. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. Since f satisfies one of the requirements (1)–(50) we know from [157] and [38] that there exists $p \in \mathbb{N}$ such that f^p satisfies (25). Then in the terminology of Chapter 4 f is generalized p-contractive. Since X is compact we know that f is uniformly continuous, and by Proposition 4.3 in Chapter 4 f is uniformly generalized p-contractive. Thus by the previous corollary it follows that f is an asymptotic contraction in the sense of Kirk.

□

3.3.3 A rate of convergence dependent on strictly positive upper and lower bounds on the initial displacement

This section corresponds to the paper [26], and the results have appeared there. Given the complete characterization of asymptotic contractions in the sense of

Kirk in the bounded setting it is natural to investigate what happens when the underlying space is allowed to be unbounded. In this section we will prove that the requirement in Kirk's theorem on asymptotic contractions that one Picard iteration sequence is bounded is redundant, since any Picard iteration sequence is bounded in any case.

That this assumption is superfluous was already proved by T. Suzuki in [167] (and see also the interesting results in [168]), but the present author only became aware of Suzuki's result after having published [26], where the result mentioned above appears. The ways in which this is proved in the two cases are not very similar, and we will comment further on Suzuki's work below. (In [36] and [76] conditions are given which allow one to remove the requirement that some iteration sequence is bounded from the corresponding theorems on variants of asymptotic contractions, but this is done only by introducing further limit requirements on the relevant moduli. We need here no such extra conditions.)

We will also construct an explicit rate of convergence for the Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$, for a generalized asymptotic contraction, which does not depend on a bound on the iteration sequence, but which instead depends on (strictly positive) upper and lower bounds $b, c > 0$ on the initial displacement $d(x_0, f(x_0))$. This is thus in a sense an improvement of Theorem 3.14 (and Theorem 3.16), where the rate of convergence depends on a bound on the iteration sequence. In both cases the rate of convergence also depends on the moduli η, β for the mapping[1] (which appear as parameters), but is again in both cases otherwise fully uniform.

We will later show that for asymptotic contractions in the sense of Kirk there exists a rate of convergence which depends on the starting point only through an upper bound $b \geq 0$ on the initial displacement $d(x_0, f(x_0))$, but we do not construct that rate of convergence explicitly, and it is not clear whether that result extends to generalized asymptotic contractions.

We will need the following lemma, which draws heavily on Lemma 3.5.

Lemma 3.25. *Let (X, d) be a metric space, let $f : X \to X$ be a generalized asymptotic contraction and let $b > 0$ and η, β be given. For $b \geq \delta > 0$ let $K_\delta := \beta_\delta(\eta(\delta)/2)$ and*

$$M_\delta := K_\delta \cdot \left\lceil \frac{\lg(\delta) - \lg(b)}{\lg(1 - \frac{\eta(\delta)}{2})} \right\rceil .$$

Then for all $x_0, y_0 \in X$ such that for all $n \geq 0$ we have $d(x_n, y_n) \leq b$ there exists an $m \leq M_\delta$ such that

$$d(x_m, y_m) \leq \delta.$$

Proof. Let $K_\delta := \beta_\delta(\eta(\delta)/2)$. Assume that for some M and all $m < M$ we have $d(x_{mK_\delta}, y_{mK_\delta}) \geq \delta$. Then repeatedly using Proposition 3.3 (and Lemma 3.12)

[1] More precisely, the rates of convergence are dependent on η^b, β^b for some values of $b > 0$ (but not for all).

we have

$$d(x_{MK_\delta}, y_{MK_\delta}) \leq \left(1 - \frac{\eta(\delta)}{2}\right)^M \cdot d(x_0, y_0) \leq \left(1 - \frac{\eta(\delta)}{2}\right)^M \cdot b.$$

Solving the inequality $(1-\eta(\delta)/2)^M \cdot b \leq \delta$ with respect to M gives the described upper bound $M_\delta = K_\delta \cdot M$ on an m such that $d(x_m, y_m) \leq \delta$.

□

Now we can prove that any Picard iteration sequence is bounded.

Theorem 3.26. *Let (X, d) be a metric space and let $f : X \to X$ be a generalized asymptotic contraction. Let $x_0 \in X$. Then the Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.*

Proof. Assume $d(x_0, f(x_0)) > 0$, for else there is nothing to prove. Let $b := d(x_0, f(x_0))$, and let η, β be the associated moduli of f from Definition 3.10. We first prove that $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$. Since $b \geq d(x_0, f(x_0)) \geq b$ we can conclude by Proposition 3.3 that

$$d(x_n, f(x_n)) \leq \left(1 - \frac{\eta^b(b)}{2}\right) \cdot b$$

for $n \geq K_b$, where $K_b = \beta_b^b(\eta^b(b)/2)$. Now let $b \geq \varepsilon > 0$. Then by considering x_{K_b} and x_{K_b+1} as the starting points x'_0 and y'_0 of two Picard iteration sequences $(x'_n)_{n\in\mathbb{N}}$ and $(y'_n)_{n\in\mathbb{N}}$ with the property that $d(x'_n, y'_n) < b$ for $n \geq 0$, we know by Lemma 3.25 that there exists $m \leq K_b + M_\varepsilon$ such that $d(x_m, f(x_m)) \leq \varepsilon$. Here M_ε is as in Lemma 3.25. Let $c := d(x_m, f(x_m))$ for some particular such m. If $c = 0$, then x_m is a fixed point, and $(x_n)_{n\in\mathbb{N}}$ is bounded. So assume $c > 0$. Then Proposition 3.3 gives that

$$d(x_n, f(x_n)) \leq \left(1 - \frac{\eta^b(c)}{2}\right) \cdot c \leq \left(1 - \frac{\eta^b(c)}{2}\right) \cdot \varepsilon < \varepsilon,$$

for $n \geq K_b + M_\varepsilon + K_c$, with $K_c = \beta_c^b(\eta^b(c)/2)$. So $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$.

Let now $N := \beta_{1/2}^1(\eta^1(1/2)/2)$ and $\delta := \eta^1(1/2) \cdot 1/8$. Since $(x_n)_{n\in\mathbb{N}}$ is the Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ we can now let M be so large that for $n \geq M$ we have

$$d(x_n, f(x_n)) < 1/2$$

and

$$d(x_n, f^N(x_n)) \leq \delta.$$

Then Lemma 3.4 yields that for $m, n \geq M$ we have

$$d(x_m, x_n) > 1$$

or

$$d(x_m, x_n) \leq 1/2.$$

So in particular, for $n \geq M$ we have $d(x_M, x_n) > 1$ or $d(x_M, x_n) \leq 1/2$. If for all $n \geq M$ we have $d(x_M, x_n) \leq 1/2$, then $(x_n)_{n\in\mathbb{N}}$ is bounded. So suppose there exists $n \geq M$ such that $d(x_M, x_n) > 1$. Let $n' > M$ be the first such $n \in \mathbb{N}$. Then

$$d(x_{n'-1}, x_{n'}) + d(x_{n'-1}, x_M) \geq d(x_{n'}, x_M) > 1,$$

so $d(x_{n'-1}, x_M) \leq 1/2$ gives $d(x_{n'-1}, x_{n'}) > 1/2$. But

$$d(x_{n'-1}, x_{n'}) = d(x_{n'-1}, f(x_{n'-1})) < 1/2,$$

which thus contradicts our choice of M and n'. Thus $d(x_M, x_n) \leq 1/2$ for all $n \geq M$, and hence $(x_n)_{n\in\mathbb{N}}$ is bounded.

□

Corollary 3.27. *Let (X, d) be a nonempty, complete metric space, and let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk. Then f has a unique fixed point $z \in X$, and for every starting point $x \in X$ the iteration sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to z.*

Proof. Since $f : X \to X$ is an asymptotic contraction in the sense of Kirk it is also an asymptotic contraction in the sense of Definition 3.10. Hence Theorem 3.26 yields that all Picard iteration sequences are bounded. The rest follows from Theorem 1.27.

□

Similarly we can improve the results we gave earlier in this chapter. As an instance of this we give the following improvement of Theorem 3.16.

Corollary 3.28. *Let (X, d) be a metric space, and let $f : X \to X$ be a generalized asymptotic contraction with moduli η^b and β^b for each $b > 0$. Then all Picard iteration sequences are Cauchy. Assume that for some $x_0 \in X$ the limit $z := \lim_{n\to\infty} x_n$ exists. Then for any $x_0 \in X$ the iteration sequence $(x_n)_{n\in\mathbb{N}}$ converges to z, irrespective of whether z is a fixed point or not. If $(x_n)_{n\in\mathbb{N}}$ is bounded by $b > 0$ then $(x_n)_{n\in\mathbb{N}}$ converges to z with the rate of convergence specified in Theorem* 3.14.

Proof. Immediate from Theorem 3.26 and from Theorem 3.16.

□

The rate of convergence in Theorem 3.14 which is referred to in Corollary 3.28 depends on a bound b on the iteration sequence in question, and also on the moduli η^b, β^b. We recall that we in Theorem 3.14 denote the functional giving this rate of convergence by Φ, so that given a metric space (X, d), an asymptotic contraction $f : X \to X$ with moduli η and β, an $x_0 \in X$ such that $\lim_{n\to\infty} x_n = z$ and such that $(x_n)_{n\in\mathbb{N}}$ is bounded by $b > 0$, and also a real number $b \geq \varepsilon > 0$, then $n \geq \Phi(b, \eta^b, \beta^b, \varepsilon)$ gives $d(x_n, z) \leq \varepsilon$.

We now give the details of the promised rate of convergence which does not depend on a bound on the iteration sequence, but which instead depends on (strictly positive) upper and lower bounds on $d(x_0, f(x_0))$, and also on the moduli $\eta^{b'}$, $\beta^{b'}$ for some specific values b'. Specifically, we have the following[2].

[2] The definition of N in Proposition 3.29 is slightly changed compared to in Proposition 2.6 in [26], since we here let $0 \in \mathbb{N}$.

Proposition 3.29. *Let (X,d) be a metric space, and let $f : X \to X$ be a generalized asymptotic contraction with moduli η^b and β^b for each $b > 0$. Assume that for some $x_0 \in X$ the limit $z := \lim_{n\to\infty} x_n$ exists. Let $b \geq c > 0$ and let $x_0 \in X$ be such that $b \geq d(x_0, f(x_0)) \geq c$. Then $(x_n)_{n\in\mathbb{N}}$ has the following rate of convergence. Let $\varepsilon > 0$. Let Φ be as in Theorem 3.14, and let*

$$K_c := \beta^b_c(\eta^b(c)/2),$$

$$N := \max\{\beta^b_{b/2}(\eta^b(b/2)/2), 1\},$$

$$\alpha := \eta^b(b/2) \cdot b/8,$$

$$K' := \left\lceil \frac{\lg(\alpha/N) - \lg(b)}{\lg(1 - \eta^b(\alpha/N)/2)} \right\rceil,$$

$$K_{\alpha/N} := \beta^b_{\alpha/N}(\eta^b(\alpha/N)/2),$$

$$b' := \max\{\varepsilon, K' \cdot (5b/2 + b \cdot K_{\alpha/N}) + b\}.$$

Let $n \in \mathbb{N}$ satisfy $n \geq \Phi(b', \eta^{b'}, \beta^{b'}, \varepsilon) + K_c$. Then

$$d(x_n, z) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$. Let $x_0 \in X$, and let $b, c > 0$ be such that $b \geq d(x_0, f(x_0)) \geq c$. (If $d(x_0, f(x_0)) = 0$ then $x_0 = z$.) Let $K_c := \beta^b_c(\eta^b(c)/2)$. Then Proposition 3.3 gives that

$$d(x_n, f(x_n)) \leq (1 - \eta^b(c)/2) \cdot d(x_0, f(x_0)) \leq (1 - \eta^b(c)/2) \cdot b < b \qquad (3.6)$$

for $n \geq K_c$. Let now

$$N := \max\{\beta^b_{b/2}(\eta^b(b/2)/2), 1\},$$

$$\alpha := \eta^b(b/2) \cdot b/8.$$

Notice that $\alpha/N < b/(2N)$, since $0 < \eta^b(b/2) < 1$. Assume that for some integer $M \geq 0$ we have $d(x_n, f(x_n)) \leq \alpha/N$ for all $K_c \leq n \leq K_c + M$. Let $M = k \cdot N + m$, with $k \geq 0$ and with $m \geq 0$ an integer strictly smaller than N. If $k < 2$ then it follows by the triangle inequality that $d(x_n, x_{n'}) < b$ for $K_c \leq n, n' \leq K_c + M + 1$, since $\alpha/N < b/(2N)$. If $k = 2$ then likewise

$$d(x_n, x_{n'}) < \frac{3b}{2}$$

for $K_c \leq n, n' \leq K_c + M + 1$. Assume $k > 2$, and assume that for some integer $k' > 0$ such that $k \geq k' + 2$ we have

$$d(x_{K_c}, x_{K_c + k'N}) \leq \frac{b}{2}.$$

(Notice that this holds for $k' = 1$.) Then

$$d(x_{K_c+k'N}, x_{K_c+(k'+1)N}) \leq \frac{b}{2},$$

and so

$$d(x_{K_c}, x_{K_c+(k'+1)N}) \leq b.$$

We also have

$$d(x_{K_c}, x_{K_c+N}) \leq \alpha$$

and

$$d(x_{K_c+(k'+1)N}, x_{K_c+(k'+2)N}) \leq \alpha,$$

and thus Lemma 3.4 gives that

$$d(x_{K_c}, x_{K_c+(k'+1)N}) \leq \frac{b}{2}.$$

Thus we have $d(x_n, x_{n'}) \leq 3b/2$ for $K_c \leq n, n' \leq K_c + (k-1)N$, and hence $d(x_n, x_{n'}) < 5b/2$ for $K_c \leq n, n' \leq K_c + M + 1$. If $d(x_n, f(x_n)) \leq \alpha/N$ for all $n \geq K_c$, then we get that $d(x_n, x_{n'}) \leq b$ for all $n, n' \geq K_c$. Namely, if we assume

$$d(x_{K_c}, x_{K_c+kN}) \leq \frac{b}{2}$$

for some $k \in \mathbb{N}$ with $k \geq 1$ (notice that this holds for $k = 1$), then

$$d(x_{K_c}, x_{K_c+(k+1)N}) \leq b,$$

and also

$$d(x_{K_c}, x_{K_c+N}) \leq \alpha$$

and

$$d(x_{K_c+(k+1)N}, x_{K_c+(k+2)N}) \leq \alpha.$$

So by Lemma 3.4 we have

$$d(x_{K_c}, x_{K_c+(k+1)N}) \leq \frac{b}{2},$$

and hence $d(x_n, x_{n'}) \leq b$ for all $n, n' \geq K_c$. (Then $d(x_n, x_{n+N}) \leq \alpha$ and $d(x_{n'}, x_{n'+N}) \leq \alpha$ in fact imply that $d(x_n, x_{n'}) \leq b/2$ for all $n, n' \geq K_c$.)

Thus by letting $x'_0 := x_{K_c}$ we can conclude that either $(x'_n)_{n\in\mathbb{N}}$ is bounded by b or else we have

$$\alpha/N \leq d(x'_m, f(x'_m)) \leq (1 - \eta^b(c)/2) \cdot b \tag{3.7}$$

for an m such that $(x'_n)_{n\leq m}$ is bounded by $5b/2$. So by Proposition 3.3 we get an $N_1 \in \mathbb{N}$ such that

$$d(x'_n, f(x'_n)) \leq (1 - \eta^b(\alpha/N)/2) \cdot (1 - \eta^b(c)/2) \cdot b < (1 - \eta^b(\alpha/N)/2) \cdot b$$

for $n \geq N_1$ and such that $(x'_n)_{n \leq N_1}$ is bounded by $5b/2 + b \cdot K_{\alpha/N}$. (Where $K_{\alpha/N} = \beta^b_{\alpha/N}(\eta^b(\alpha/N)/2)$. If we are in the case where there exists $m \in \mathbb{N}$ such that (3.7) holds and such that $(x'_n)_{n \leq m}$ is bounded by $5b/2$, then we can take $N_1 := m + K_{\alpha/N}$. Note that for $n \in \mathbb{N}$ we have $d(x'_n, f(x'_n)) < b$, since we took $x'_0 = x_{K_c}$ and because of the property of K_c given in 3.6.) By considering x'_{N_1} as the starting point of a Picard iteration sequence $(x''_n)_{n\in\mathbb{N}}$ with the property that

$$d(x''_m, f(x''_m)) < (1 - \eta^b(\alpha/N)/2) \cdot b$$

for all $m \geq 0$, we get by the above argument that either $(x''_n)_{n\in\mathbb{N}}$ is bounded by b or else we get an $N_2 \in \mathbb{N}$ such that

$$d(x''_m, f(x''_m)) < (1 - \eta^b(\alpha/N)/2)^2 \cdot b$$

for $m \geq N_2$ and such that $(x''_n)_{n \leq N_2}$ is bounded by $5b/2 + b \cdot K_{\alpha/N}$. Thus either $(x'_n)_{n\in\mathbb{N}}$ is bounded by $5b/2 + b \cdot K_{\alpha/N} + b$ or else we have that

$$d(x'_m, f(x'_m)) < (1 - \eta^b(\alpha/N)/2)^2 \cdot b$$

for $m \geq N_1 + N_2$, and furthermore that $(x'_n)_{n \leq N_1 + N_2}$ is bounded by $2 \cdot (5b/2 + b \cdot K_{\alpha/N})$. Solving the inequality

$$(1 - \eta^b(\alpha/N)/2)^k \cdot b \leq \alpha/N$$

with respect to k leads us to consider

$$K' := \left\lceil \frac{\lg(\alpha/N) - \lg(b)}{\lg(1 - \eta^b(\alpha/N)/2)} \right\rceil.$$

We get that either $(x'_n)_{n\in\mathbb{N}}$ is bounded by $(K' - 1) \cdot (5b/2 + b \cdot K_{\alpha/N}) + b$, or else we get an $N' \in \mathbb{N}$ such that

$$d(x'_m, f(x'_m)) < (1 - \eta^b(\alpha/N)/2)^{K'} \cdot b \leq \alpha/N$$

for $m \geq N'$, and furthermore such that $(x'_n)_{n \leq N'}$ is bounded by $K' \cdot (5b/2 + b \cdot K_{\alpha/N})$. Hence $(x'_n)_{n\in\mathbb{N}}$ is bounded by $K' \cdot (5b/2 + b \cdot K_{\alpha/N}) + b$. Let

$$b' := \max\{\varepsilon, K' \cdot (5b/2 + b \cdot K_{\alpha/N}) + b\}.$$

Then Corollary 3.28 gives that $d(x'_n, z) \leq \varepsilon$ for $n \geq \Phi(b', \eta^{b'}, \beta^{b'}, \varepsilon)$, so

$$d(x_n, z) \leq \varepsilon$$

for $n \geq \Phi(b', \eta^{b'}, \beta^{b'}, \varepsilon) + K_c$. (Here Φ is as in Theorem 3.14.)

□

We remark that in the above proposition there is room for some numerical improvement. And we can also easily adapt the proposition to cover the case where $\lim_{n\to\infty} x_n$ does not exist, as explained in the remarks following Theorem 3.15.

3.3.4 A further uniformity for asymptotic contractions in the sense of Kirk

This section corresponds more or less to the paper [28], and most of the results have appeared there.

In this section we will prove a result concerning asymptotic contractions in the sense of Kirk which, specialized to the case where the space (X, d) is complete (and nonempty) and the mapping $f : X \to X$ continuous, says that all Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ converge to the unique fixed point z with a rate of convergence which only depends on an upper bound b on the initial displacement $d(x_0, f(x_0))$ and some moduli for the mapping appearing as parameters. That is, if we fix the moduli for the mapping then there exists $\Psi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that for all (X, d), all asymptotic contractions $f : X \to X$ (in the sense of Kirk) with these moduli, all $b \in \mathbb{N}$, all $x, y \in X$ with $d(x, f(x)) \leq b$ and $d(y, f(y)) \leq b$, all $k \in \mathbb{N}$ and all $m, n \geq \Psi(b, k)$ we have $d(f^m(x), f^n(y)) < 2^{-k}$.

This gives a new uniformity of the convergence as $n \to \infty$ compared to the rates of convergence we saw in previous sections, since these were dependent on respectively a bound on the iteration sequence and strictly positive upper and lower bounds on the initial displacement $d(x_0, f(x_0))$. However, the results here do not make the previous rates of convergence irrelevant, both because the results here concern asymptotic contractions in the sense of Kirk rather than the more general mappings from Definition 3.10, and because we do not construct an explicit and effective rate of convergence - we only show that the convergence is uniform in the mentioned way.

As a corollary we prove a result which (specialized to nonempty complete spaces and continuous mappings) says that far from the fixed point an asymptotic contraction in the sense of Kirk moves all points by a large distance. Specifically, we prove that for an asymptotic contraction in the sense of Kirk all points x such that $d(x, f(x)) \leq k$ lie in a set whose diameter is bounded by an integer $b(k)$ which depends only on $k \in \mathbb{N}$ and some moduli for the mapping.

Our result here that the rate of convergence only depends on (some moduli for the mapping and) an upper bound on $d(x_0, f(x_0))$ also gives a new uniformity compared to a previous result by M. Arav, F.E. Castillo Santos, S. Reich and A.J. Zaslavski. They have shown that for a continuous asymptotic contraction in the sense of Kirk on a complete metric space the convergence to the fixed point z is uniform on every bounded set $B_n(z) = \{x \in X : d(x, z) \leq n\}$ of the space (see [5])[3]. Proposition 3.31 will show that our result here subsumes this result. Our result is an improvement (when it comes to calculating the

[3] This follows from Theorem 1.2 in [5] (combined with the fact that any continuous asymptotic contraction in the sense of Kirk on a complete (nonempty) metric space has a fixed point). The main theorem in [5] is a similar result for a variant of asymptotic contractions considered by Chen, where some assumptions on the mappings are weakened and others strengthened. (We will comment more on Chen's work later.) Some of the authors have since extended this by further weakening assumptions on the mappings, see [6] and [156]. But the convergence to z is in all these cases shown to be uniform in the starting point except through dependence on an upper bound on $d(x_0, z)$, not an upper bound on $d(x_0, f(x_0))$.

fixed point) in the sense that for unbounded spaces it seems easier in general to calculate a bound on $d(x_0, f(x_0))$ rather than a bound on $d(x_0, z)$.

We have earlier remarked that the characterization of asymptotic contractions in the sense of Kirk on complete, bounded metric spaces shows that they "behave similarly" to Banach contractions on such spaces - all Picard iteration sequences converge to the same point, and the rate of convergence is uniform in the starting point. Our results in this section show that also on unbounded, complete metric spaces these maps have much in common with ordinary contractions - the rate of convergence is uniform in the starting point except through dependence on an upper bound on $d(x_0, f(x_0))$, and far away from the common limit of all Picard iteration sequences any point is moved by a large distance by the mapping.

A natural further question which we do not answer here is how to suitably extend the characterization theorem mentioned above for asymptotic contractions in the sense of Kirk from the bounded setting to the case of unbounded spaces - in the sense of giving a decent characterization, in terms of a kind of asymptotic contractions, of the mappings $f : X \to X$ on nonempty, complete (possibly unbounded) metric spaces (X, d) such that all Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ converge to the same point with a rate of convergence which is uniform in the starting point except through dependence on an upper bound b on the initial displacement $d(x_0, f(x_0))$.

A technical lemma

We need a lemma which is reminiscent of Lemma 2.46.

Lemma 3.30. *Let $\phi, \phi_n : [0, \infty) \to [0, \infty)$ be moduli as in Definition 1.26 for an asymptotic contraction $f : X \to X$ in the sense of Kirk on a metric space (X, d), such that $\phi_n \to \phi$ uniformly on $[0, \infty)$. Then for each $b \in \mathbb{N}$ there exist continuous and increasing moduli $\phi'_b, \phi'_{n,b} : [0, b] \to [0, \infty)$ such that*

(i) *$\phi'_b(s) < s$ for all $s > 0$,*

(ii) *the function $h : [0, b] \to [0, \infty)$ defined by $h(s) = s - \phi'_b(s)$ is increasing,*

(iii) *$\phi'_{n,b} \to \phi'_b$ uniformly,*

and such that for all metric spaces (X, d) and all asymptotic contractions $f : X \to X$ (in the sense of Kirk) having $\phi, \phi_n : [0, \infty) \to [0, \infty)$ as moduli we have:

$$\forall n \in \mathbb{N} \forall x, y \in X \left(d(x, y) \le b \to d\left(f^n(x), f^n(y)\right) \le \phi'_{n,b}(d(x, y)) \right). \tag{3.8}$$

Proof. The statement and proof of this lemma are somewhat similar to the statement and proof of Lemma 2.46. Let (X, d) be a metric space and let $f : X \to X$ be an asymptotic contraction in the sense of Kirk with moduli

$\phi, \phi_n : [0,\infty) \to [0,\infty)$ such that $\phi_n \to \phi$ uniformly on $[0,\infty)$. Define now $\phi''_b, \phi''_{n,b} : [0,b] \to [0,\infty)$ by

$$\phi''_b(s) := \sup\{\phi(\delta) : \delta \le s\}$$

and

$$\phi''_{n,b}(s) := \sup\{\phi_n(\delta) : \delta \le s\}.$$

Then $\phi''_b, \phi''_{n,b}$ are continuous and increasing, $\phi''_{n,b} \to \phi''_b$ uniformly on $[0,b]$, and we have

$$d\left(f^n(x), f^n(y)\right) \le \phi''_{n,b}(d(x,y)) \tag{3.9}$$

for all $x, y \in X$ with $d(x,y) \le b$. And since $\phi(s) < s$ for all $s > 0$ and ϕ is continuous we can conclude that $\phi''_b(s) < s$ for all $s > 0$. Furthermore, for each $b \ge \varepsilon > 0$ the continuous function $h'' : [0,b] \to [0,\infty)$ given by

$$h''(s) := s - \phi''_b(s)$$

assumes its infimum on the compact interval $[\varepsilon, b]$. And since we have $\phi''_b(s) < s$ for all $b \ge s > 0$ we get

$$\inf\left\{s - \phi''_b(s) : s \in [\varepsilon, b]\right\} > 0.$$

Define $\phi'_b, \phi'_{n,b} : [0,b] \to [0,\infty)$ by

$$\phi'_b(\varepsilon) := \begin{cases} 0 & \text{if } \varepsilon = 0, \\ \varepsilon - \inf\left\{s - \phi''_b(s) : s \in [\varepsilon, b]\right\} & \text{if } \varepsilon > 0 \end{cases}$$

and

$$\phi'_{n,b}(\varepsilon) := \max\left\{\phi''_{n,b}(\varepsilon), \phi'_b(\varepsilon)\right\}.$$

We will prove that $\phi'_b, \phi'_{n,b}$ fulfill the requirements in Lemma 3.30. It is easy to see that $\phi'_b, \phi'_{n,b}$ are continuous, and the function $h : [0,b] \to [0,\infty)$ defined by $h(s) = s - \phi'_b(s)$ is increasing by the definition of ϕ'_b. And since we saw that $\inf\left\{s - \phi''_b(s) : s \in [\varepsilon, b]\right\} > 0$ for all $b \ge \varepsilon > 0$ it follows that $\phi'_b(s) < s$ for all $b \ge s > 0$. Now for all $x, y \in X$ with $d(x,y) \le b$ we have

$$d\left(f^n(x), f^n(y)\right) \le \phi'_{n,b}(d(x,y)).$$

This follows since we for all $x, y \in X$ with $d(x,y) \le b$ have

$$d\left(f^n(x), f^n(y)\right) \le \phi''_{n,b}(d(x,y))$$

and

$$\phi''_{n,b}(d(x,y)) \le \phi'_{n,b}(d(x,y)).$$

Since $\phi'_b(\varepsilon) \ge \phi''_b(\varepsilon)$ for all $\varepsilon \in [0,b]$ and since $\phi''_{n,b} \to \phi''_b$ uniformly on $[0,b]$ we have that $\phi'_{n,b} \to \phi'_b$ uniformly on $[0,b]$. Finally we prove that $\phi'_b, \phi'_{n,b}$ are increasing. Let $\varepsilon, \varepsilon' \in [0,b]$ with $\varepsilon' > \varepsilon$. Assume that $\phi'_b(\varepsilon) > \phi'_b(\varepsilon')$, i.e., that $\phi'_b(\varepsilon) - \phi'_b(\varepsilon') > 0$. Then

$$(\varepsilon - \varepsilon') - \left(\inf h''\left([\varepsilon, b]\right) - \inf h''\left([\varepsilon', b]\right)\right) > 0,$$

so

$$\inf h''([\varepsilon', b]) - \inf h''([\varepsilon, b]) > \varepsilon' - \varepsilon. \tag{3.10}$$

If the restriction of h'' to $[\varepsilon, b]$ takes its infimum for an $s \in [\varepsilon, b]$ such that $s \in [\varepsilon', b]$, then

$$\inf h''([\varepsilon, b]) = \inf h''([\varepsilon', b]),$$

contradicting (3.10). If on the other hand the restriction of h'' to $[\varepsilon, b]$ takes its infimum for an $s_1 \in [\varepsilon, b]$ such that $s_1 \in [\varepsilon, \varepsilon']$, then since ϕ_b'' is increasing we have

$$(\varepsilon' - \phi_b''(\varepsilon')) - (s_1 - \phi_b''(s_1)) \leq \varepsilon' - s_1 \leq \varepsilon' - \varepsilon.$$

And since

$$\inf h''([\varepsilon, b]) = s_1 - \phi_b''(s_1)$$

and

$$\inf h''([\varepsilon', b]) \leq \varepsilon' - \phi_b''(\varepsilon')$$

we get

$$\inf h''([\varepsilon', b]) - \inf h''([\varepsilon, b]) \leq \varepsilon' - \varepsilon,$$

again contradicting (3.10). Thus ϕ_b' is increasing, and since $\phi_{n,b}''$ is increasing for each n it follows that also $\phi_{n,b}'$ is increasing.

□

As an application of Lemma 3.30 we include an observation to the effect that our main theorem (Theorem 3.32) below covers the already mentioned result of M. Arav, F.E. Castillo Santos, S. Reich and A.J. Zaslavski in [5].

Proposition 3.31. *Let (X, d) be a nonempty, complete metric space, let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk, and let $k \in \mathbb{N}$. Then there exists $b(k) > 0$ such that for all*

$$x \in B_k(z) = \{y \in X : d(y, z) \leq k\},$$

where $z \in X$ is the unique fixed point of f, we have

$$d(x, f(x)) \leq b(k).$$

Moreover, the bound $b(k)$ does not depend on the space (X, d) or the mapping $f : X \to X$ except through moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ for f as in Definition 1.26 such that $\phi_n \to \phi$ uniformly on $[0, \infty)$.

Proof. Let $z \in X$ be the unique fixed point of f, let $k \in \mathbb{N}$ and let $x \in B_k(z)$. Let $\phi_k', \phi_{n,k}' : [0, k] \to [0, \infty)$ be moduli as given in Lemma 3.30. Then

$$d(f(x), z) = d(f(x), f(z)) \leq \phi_{1,k}'(d(x, z)) \leq \phi_{1,k}'(k).$$

Thus $d(x, f(x)) \leq k + \phi_{1,k}'(k)$. We note that the moduli $\phi_k', \phi_{n,k}'$ do not depend on (X, d) and f except through the moduli ϕ, ϕ_n.

□

The convergence is uniform in x_0 except through dependence on an upper bound $b \geq d(x_0, x_1)$

Theorem 3.32 and the following corollaries will show that asymptotic contractions in the sense of Kirk share some important properties with ordinary contractions, even in the setting of unbounded metric spaces.

Theorem 3.32. *Let (X, d) be a metric space and let $f : X \to X$ be an asymptotic contraction in the sense of Kirk. Then all Picard iteration sequences are Cauchy, and there exists $\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $k, b \in \mathbb{N}$ and all $x, y \in X$, if*

$$d(x, f(x)) \leq b$$

and

$$d(y, f(y)) \leq b,$$

then for all $m, n \geq \Psi(b, k)$ we have

$$d(f^m(x), f^n(y)) < 2^{-k}.$$

Moreover, this Ψ does not depend on the space (X, d) or the mapping f except through moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ for f as in Definition 1.26 such that $\phi_n \to \phi$ uniformly on $[0, \infty)$.

Proof. Let (X, d) be a metric space and let $f : X \to X$ be an asymptotic contraction in the sense of Kirk with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ for f (as in Definition 1.26) such that $\phi_n \to \phi$ uniformly on $[0, \infty)$. For the case $b = 0$ we can let $\Psi(0, k) = 0$ for all $k \in \mathbb{N}$, since any fixed point of f is necessarily unique. So we can concentrate on the case $b > 0$.

We will show first that for all $b \in \mathbb{N}$ with $b > 0$ and for all $b > \varepsilon > 0$ there exists a natural number $M(b, \varepsilon)$ such that for all $x \in X$ with $d(x, f(x)) \leq b$ we have

$$d(f^n(x), f^{n+1}(x)) < \varepsilon$$

for $n \geq M(b, \varepsilon)$. And moreover, this $M(b, \varepsilon)$ will depend on (X, d) and $f : X \to X$ only through the moduli ϕ, ϕ_n.

Let $b \in \mathbb{N}$ with $b > 0$, and let $x_0 \in X$ satisfy $d(x_0, f(x_0)) \leq b$. Let $\phi'_b, \phi'_{n,b} : [0, b] \to [0, \infty)$ be moduli as provided by Lemma 3.30. We note that these moduli do not depend on (X, d) and f except through the moduli ϕ, ϕ_n. Given $\varepsilon > 0$ with $b > \varepsilon$ we let

$$\delta = \frac{\varepsilon - \phi'_b(\varepsilon)}{2},$$

and we let $N \in \mathbb{N}$ be so large that $n \geq N$ gives

$$|\phi'_{n,b}(s) - \phi'_b(s)| < \delta$$

for all $s \in [0, b]$. We note that $\delta \leq (s - \phi'_b(s))/2$ for all $\varepsilon \leq s \leq b$, and conclude that for $n \geq N$ we have

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \phi'_{n,b}(d(x_0, f(x_0))) \leq \phi'_{n,b}(b) < \phi'_b(b) + \delta \leq b - \delta.$$

Similarly, if $b-\delta \geq \varepsilon$, then for $n \geq 2N$ we have

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\leq \phi'_{n-N,b}(d(f^N(x_0), f^{N+1}(x_0))) \\ &\leq \phi'_{n-N,b}(b-\delta) \\ &< \phi'_b(b-\delta)+\delta \\ &\leq b-2\delta. \end{aligned}$$

Let $k_1 = \min\{n \in \mathbb{N} : 0 \leq b - n\delta < \varepsilon\}$. Notice that such k_1 exists, since $0 < \delta < \varepsilon/2$. By induction we conclude that for $1 \leq k' \leq k_1$ and for $n \geq k'N$ we have

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\leq \phi'_{n-(k'-1)N,b}\left(d(f^{(k'-1)N}(x_0), f^{(k'-1)N+1}(x_0))\right) \\ &\leq \phi'_{n-(k'-1)N,b}\,(b-(k'-1)\delta) \\ &< \phi'_b\,(b-(k'-1)\delta)+\delta \\ &\leq b-k'\delta. \end{aligned}$$

Thus for $M(b,\varepsilon) = k_1 N$ and $n \geq M(b,\varepsilon)$ we have $d(f^n(x_0), f^{n+1}(x_0)) < \varepsilon$. We note that $M(b,\varepsilon)$ does not depend on x_0 except through the bound b on $d(x_0, f(x_0))$, i.e., for all $x \in X$ with $d(x, f(x)) \leq b$ we have

$$d(f^n(x), f^{n+1}(x)) < \varepsilon \tag{3.11}$$

for $n \geq M(b,\varepsilon)$. And furthermore, this $M(b,\varepsilon)$ depends on (X,d) and f only through the moduli $\phi'_b, \phi'_{n,b}$, and thus only through the moduli ϕ, ϕ_n.

We can now recycle a part of the proof of Theorem 3.26. Let η, β be moduli for f as given in Definition 3.10. We remark that Proposition 7 in [54] shows that we can assume η, β to be independent of (X,d) and f except through moduli $\phi, \phi_n : [0,\infty) \to [0,\infty)$ for f as in Definition 1.26 such that $\phi_n \to \phi$ uniformly on $[0,\infty)$. Let furthermore

$$N' := \beta^1_{1/2}(\eta^1(1/2)/2),$$

$$\delta' := \eta^1(1/2) \cdot 1/8,$$

and

$$M = M(b, \delta'/(N'+1)). \tag{3.12}$$

Note that $\delta'/(N'+1) < 1/2$. Now for all $x \in X$ with $d(x, f(x)) \leq b$ and for all $n \geq M$ we have

$$d(f^n(x), f^{n+1}(x)) < 1/2 \tag{3.13}$$

and

$$d(f^n(x), f^{n+N'}(x)) < \delta'. \tag{3.14}$$

So Lemma 3.4 yields that for all $x \in X$ with $d(x, f(x)) \leq b$ and for all $m, n \geq M$ we have

$$d(f^m(x), f^n(x)) > 1 \tag{3.15}$$

or

$$d(f^m(x), f^n(x)) \leq 1/2. \tag{3.16}$$

Let $x_0 \in X$ with $d(x_0, f(x_0)) \leq b$. Then, in particular, for $n \geq M$ we have $d(f^M(x_0), f^n(x_0)) > 1$ or $d(f^M(x_0), f^n(x_0)) \leq 1/2$. Let $y_0 = f^M(x_0)$. If for all $n \geq M$ we have

$$d(y_0, f^n(x_0)) \leq 1/2,$$

then $(f^n(y_0))_{n\in\mathbb{N}}$ is bounded by 1. Suppose now that there exists $n \geq M$ such that $d(y_0, f^n(x_0)) > 1$. Let $n' > M$ be the first such $n \in \mathbb{N}$. Then

$$d(f^{n'-1}(x_0), f^{n'}(x_0)) + d(f^{n'-1}(x_0), y_0) \geq d(f^{n'}(x_0), y_0) > 1,$$

so

$$d(f^{n'-1}(x_0), y_0) \leq 1/2$$

gives

$$d(f^{n'-1}(x_0), f^{n'}(x_0)) > 1/2.$$

But by (3.13) we have

$$d(f^{n'-1}(x_0), f^{n'}(x_0)) = d(f^{n'-1}(x_0), f(f^{n'-1}(x_0))) < 1/2,$$

which thus contradicts our choice of M and n'. Thus $d(y_0, f^n(x_0)) \leq 1/2$ for all $n \geq M$, and hence $(f^n(y_0))_{n\in\mathbb{N}}$ is bounded by 1. By Corollary 3.28 we know that all Picard iteration sequences are Cauchy, and if for some $x \in X$ we have that $z := \lim_{n\to\infty} f^n(x)$ exists, then all Picard iteration sequences converge to z. Furthermore, all Picard iteration sequences bounded by 1 converge to z with a common rate of convergence, i.e. there exists $\Phi : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ and for all $x \in X$ such that $(f^n(x))_{n\in\mathbb{N}}$ is bounded by 1 we have that $n \geq \Phi(k)$ gives

$$d(f^n(x), z) < 2^{-k}.$$

And Φ depends on (X, d) and f only through the moduli η, β, and thus only through the moduli ϕ, ϕ_n.

If $(f^n(x))_{n\in\mathbb{N}}$ does not converge for any $x \in X$ then we consider the completion $\overline{X}$ of X, in which the limit z exists. We can then extend f to be defined on $X \cup \{z\}$ by letting $f(z) = z$. As previously noted it is then easy to see that f is a generalized asymptotic contraction with moduli $\widetilde{\eta}^b : (0, b] \to (0, 1)$ and $\widetilde{\beta}^b : (0, b] \times (0, \infty) \to \mathbb{N}$ defined by for example $\widetilde{\eta}^b(\varepsilon) := \eta^{2b}(\varepsilon/2)$, $\widetilde{\beta}^b(l, \varepsilon) := \beta^{2b}(l/2, \varepsilon)$ for each $b > 0$. Thus also in this case there exists a common rate of convergence (to the same point) $\Phi : \mathbb{N} \to \mathbb{N}$ for all Picard iteration sequences bounded by 1. (We note that even if the limit $z \in X$ existed to begin with we could have used the modified moduli $\widetilde{\eta}^b$ and $\widetilde{\beta}^b$ instead of η^b and β^b to determine a rate of convergence Φ.)

Hence for all $x \in X$ with $d(x, f(x)) \leq b$, for all $k \in \mathbb{N}$ and for all $n \geq M + \Phi(k)$ (where M is as in (3.12)) we have

$$d(f^n(x), z) < 2^{-k}.$$

And so for all $x, y \in X$ with $d(x, f(x)) \leq b$ and $d(y, f(y)) \leq b$, for all $k \in \mathbb{N}$, and for all $m, n \geq M + \Phi(k+1)$ we have

$$d(f^m(x), f^n(y)) < 2^{-k}.$$

And the number $M + \Phi(k+1)$ depends only on the moduli of the mapping f, on b and on k. Hence we can define $\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as desired by letting

$$\Psi(b, k) = M + \Phi(k+1)$$

if $b > 0$ and by letting $\Psi(0, k) = 0$.

□

Corollary 3.33. *Let (X, d) be a (nonempty) complete metric space and let $f : X \to X$ be an asymptotic contraction in the sense of Kirk. Then all Picard iteration sequences $(f^n(x))_{n \in \mathbb{N}}$ converge to the same point z with a rate of convergence which is uniform in the starting point except through dependence on an upper bound on the initial displacement, i.e., there exists $\Psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $k, b \in \mathbb{N}$ and all $x \in X$, if*

$$d(x, f(x)) \leq b$$

then for all $n \geq \Psi(b, k)$ we have

$$d(f^n(x), z) < 2^{-k}.$$

Proof. Immediate from Theorem 3.32.

□

We include also a result concerning a rate of asymptotic regularity. This is a corollary to the proof of Theorem 3.32.

Corollary 3.34. *Let (X, d) be a metric space, and let $f : X \to X$ be an asymptotic contraction in the sense of Kirk, which for each $b \in \mathbb{N}$ has moduli $\phi'_b, \phi'_{n,b} : [0, b] \to [0, \infty)$ as provided by Lemma 3.30. Let $\gamma : \mathbb{N} \times (0, \infty) \to \mathbb{N}$ give for each $b \in \mathbb{N}$ a rate of convergence for $\phi'_{n,b}$ to ϕ'_b on $[0, b]$, i.e.,*

$$\forall b \in \mathbb{N} \forall \varepsilon > 0 \forall n \geq \gamma(b, \varepsilon) \forall s \in [0, b] \left(|\phi'_{n,b}(s) - \phi'_b(s)| < \varepsilon \right).$$

Let $b \in \mathbb{N}$, $b > 0$, and let $x_0 \in X$ be such that $d(x_0, f(x_0)) \leq b$. Define $(x_n)_{n \in \mathbb{N}}$ by letting $x_{n+1} := f(x_n)$. Then $(x_n)_{n \in \mathbb{N}}$ has a rate of asymptotic regularity as follows: Let $\varepsilon \in (0, b)$, and let

$$\delta := \frac{\varepsilon - \phi'_b(\varepsilon)}{2},$$

$$N := \gamma(b, \delta),$$

$$k_1 := \left\lceil \frac{b - \varepsilon}{\delta} \right\rceil + 1,$$

and $M(b, \varepsilon, \gamma, \phi'_b) = k_1 N$. Then $n \geq M(b, \varepsilon, \gamma, \phi'_b)$ gives

$$d(x_n, f(x_n)) < \varepsilon.$$

Proof. This follows by considering the proof of Theorem 3.32 up to (3.11). □

The following corollary is a counterpart to Proposition 3.31. Loosely speaking it implies that far from the fixed point an asymptotic contraction in the sense of Kirk moves all points by a large distance.

Corollary 3.35. *Let (X, d) be a metric space, let $f : X \to X$ be an asymptotic contraction in the sense of Kirk, and let $k \in \mathbb{N}$. Then there exists $b(k) > 0$ such that for all $x, y \in X$ with*

$$d(x, f(x)) \leq k \quad \text{and} \quad d(y, f(y)) \leq k$$

we have

$$d(x, y) \leq b(k).$$

Moreover, the bound $b(k)$ does not depend on the space (X, d) or the mapping $f : X \to X$ except through moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ for f (as in Definition 1.26) such that $\phi_n \to \phi$ uniformly on $[0, \infty)$.

Proof. Assume $k > 0$, for otherwise the claim is trivial. Let Ψ be the common Cauchy rate from Theorem 3.32, and for each $b > 0$ let $\phi'_b, \phi'_{n,b}$ be moduli as given in Lemma 3.30. Then for all $x \in X$ with

$$d(x, f(x)) \leq k$$

we have

$$d(f^m(x), f^{\Psi(k,1)}(x)) < 1/2$$

for all $m \geq \Psi(k, 1)$. Hence we have

$$d(f^m(x), x) \leq k + \sum_{i=1}^{\Psi(k,1)-1} \phi'_{i,k}(k) + 1/2$$

for all $x \in X$ with $d(x, f(x)) \leq k$ and for all $m \geq \Psi(k, 1)$. So let $x, y \in X$ be such that $d(x, f(x)) \leq k$ and $d(y, f(y)) \leq k$, let $\varepsilon > 0$ and let $m \geq \Psi(k, 1)$ be such that

$$d(f^m(x), f^m(y)) < \varepsilon.$$

Then

$$\begin{aligned} d(x, y) &\leq d(x, f^m(x)) + d(f^m(x), f^m(y)) + d(f^m(y), y) \\ &< d(x, f^m(x)) + d(y, f^m(y)) + \varepsilon, \end{aligned}$$

and so

$$d(x, y) < 2k + 2 \cdot \sum_{i=1}^{\Psi(k,1)-1} \phi'_{i,k}(k) + 1 + \varepsilon.$$

And since $\varepsilon > 0$ was arbitrary we get

$$d(x, y) \leq 2k + 2 \cdot \sum_{i=1}^{\Psi(k,1)-1} \phi'_{i,k}(k) + 1.$$

So we let

$$b(k) = 2k + 2 \cdot \sum_{i=1}^{\Psi(k,1)-1} \phi'_{i,k}(k) + 1.$$

And this bound does not depend on the space, the mapping or the points x, y except through moduli $\phi, \phi_n : [0,\infty) \to [0,\infty)$ for the mapping (as in Definition 1.26) such that $\phi_n \to \phi$ uniformly on $[0,\infty)$.

□

Corollary 3.36. *Let (X, d) be a nonempty, complete metric space, let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk, and let $k \in \mathbb{N}$. Then there exists $b(k) > 0$ such that for all $x \in X$ with*

$$d(x, f(x)) \leq k$$

we have

$$x \in B_{b(k)}(z),$$

where z is the unique fixed point of f. Moreover, the bound $b(k)$ does not depend on the space (X, d) or the mapping $f : X \to X$ except through moduli $\phi, \phi_n : [0,\infty) \to [0,\infty)$ for f (as in Definition 1.26) such that $\phi_n \to \phi$ uniformly on $[0,\infty)$.

Proof. Immediate from Corollary 3.35 since (X, d) nonempty and complete and f continuous ensures the existence of a unique fixed point.

□

3.4 Other results

In this section we will briefly survey some of the results which have been proved by others concerning various versions of asymptotic contractions, and we will comment on the relationship between this work and the results we have presented earlier in the chapter.

3.4.1 A condition by Chen giving the existence of a fixed point without assuming continuity

In [36], Y.-Z. Chen proves a theorem on asymptotic contractions which is similar to Kirk's. Several of the conditions are weaker than the ones in [83]. In particular, it is no longer assumed that f is continuous, and it is enough that ϕ and one particular ϕ_{n_*} are upper semicontinuous (here ϕ, ϕ_{n_*} are as in Kirk's definition). It is furthermore enough that $\lim_{n\to\infty} \phi_n = \phi$ uniformly on any bounded interval $[0, b]$. (A condition which allows one to drop the requirement that one iteration sequence is bounded is also specified.) It is, however, assumed that $\phi_{n_*}(0) = 0$. We give for reference the theorem.

Theorem 3.37 (Chen)**.** *Let (X, d) be a complete metric space, and let $f : X \to X$ be such that*

$$\forall n \in \mathbb{N} \forall x, y \in X \Big(d\big(f^n(x), f^n(y)\big) \leq \phi_n\big(d(x, y)\big)\Big), \tag{3.17}$$

where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and that $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and

$$\phi_{n_*}(0) = 0.$$

If there exists $x_0 \in X$ such that the Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded, then f has a unique fixed point z, and $\lim_{n\to\infty} f^n(x) = z$ for all $x \in X$.

We will sometimes call a function $f : X \to X$ on a metric space satisfying the conditions in Theorem 3.37 asymptotic contractions in the sense of Chen, more precisely:

Definition 3.38. A function $f : X \to X$ on a metric space (X, d) is called an *asymptotic contraction in the sense of Chen* with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ if there exists $n_* \geq 1$ such that $\phi_{n_*}(0) = 0$, such that ϕ and ϕ_{n_*} are upper semicontinuous, $\phi(t) < t$ for all $t > 0$ and for all $n \in \mathbb{N}$ and $x, y \in X$,

$$d\left(f^n(x), f^n(y)\right) \leq \phi_n\left(d(x, y)\right),$$

and moreover $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$.

We note in connection with this theorem that the arguments in [54] which allow us to subsume Definition 1.26 under Definition 3.1 would work just as well if we assume that the moduli ϕ, ϕ_n in Definition 1.26 are upper semicontinuous instead of continuous and that the ϕ_n converge uniformly on bounded intervals $[0, b]$ instead of on the range of d, since upper semicontinuous functions $\phi, \phi_n : [0, \infty) \to [0, \infty)$ are bounded on bounded closed intervals $[s, b]$. Definition 6 in [54] would be unchanged, in Proposition 7 one would have to say that the sequence[4] $(\tilde{\phi}_n)_{n\in\mathbb{N}}$ converges uniformly to $\tilde{\phi}$ on $[l, b]$ for all $b > l > 0$ instead of saying that it converges uniformly on $[l, \infty)$ for all $l > 0$, but the second part of Proposition 7 and also Proposition 9 would remain unchanged, except that one would change every instance of "continuous" to "upper semicontinuous". This does not, of course, show that Definition 3.38 is subsumed by Definition 3.1.

However, in [5], M. Arav, F.E. Castillo Santos, S. Reich and A.J. Zaslavski have shown that for an asymptotic contraction $f : X \to X$ in the sense of Chen (with a bounded Picard iteration sequence) on a nonempty complete metric space (X, d) the convergence to the fixed point z is uniform on every bounded set $B_n(z) = \{x \in X : d(x, z) \leq n\}$ of the space, i.e., there is a rate of convergence for the Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ which only depends on the starting point x_0 through an upper bound on $d(x_0, z)$. Thus if the space is required to be bounded and complete we have the following.

[4] We included the construction appearing in Definition 6 in [54] in (3.1) and (3.2) in our discussion after Proposition 3.2, but called the resulting functions ψ_n, ψ rather than $\tilde{\phi}_n, \tilde{\phi}$.

Proposition 3.39. *Let (X, d) be a bounded, complete metric space, and let $f : X \to X$ be an asymptotic contraction in the sense of Chen. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. Since the space is bounded this follows by the mentioned result of Arav et al. together with Theorem 3.21. (If the space is empty this is trivial.)

□

The condition $\phi_{n_*}(0) = 0$ in the definition of an asymptotic contraction in the sense of Chen is a real restriction, as the following example shows.

Example 3.40. This example will show that there exist a bounded, complete metric space (X, d) and a continuous mapping $f : X \to X$ which is an asymptotic contraction in the sense of Kirk, so that Theorem 1.27 applies, but which is not an asymptotic contraction in the sense of Chen. Let

$$A := \{x_0^n : n \in \mathbb{N}\},$$

where the x_0^n are distinct points, i.e., such that $x_0^n \neq x_0^m$ for $m \neq n$, and let

$$B := \left\{x_n^m : n \in \mathbb{N},\, m \in \{1, 2\}\right\},$$

where $x_n^m \neq x_{n'}^{m'}$ if $n \neq n'$ or $m \neq m'$, and where $A \cap B = \emptyset$. Let z be such that $z \notin A \cup B$, and let

$$X := \{z\} \cup A \cup B.$$

Define a metric d on X such that

$$\begin{aligned}
d(z, x_0^n) &= 2^{-n} \quad \text{for } n \in \mathbb{N}, \\
d(x_0^n, x_0^m) &= 2^{-n} - 2^{-m} \quad \text{for } m, n \in \mathbb{N},\ m \neq n, \\
d(x_n^1, x_n^2) &= 2^{-n-1} \quad \text{for } n > 0, \\
d(x_n^2, x_0^m) &= 1 + (1 - 2^{-m}) \quad \text{for } n > 0 \text{ and } m \in \mathbb{N}, \\
d(x_n^2, z) &= 2 \quad \text{for } n > 0, \\
d(x_n^1, x_0^m) &= d(x_n^2, x_0^m) + d(x_n^1, x_n^2) \quad \text{for } n > 0 \text{ and } m \in \mathbb{N}, \\
d(x_n^1, z) &= d(x_n^2, z) + d(x_n^1, x_n^2) \quad \text{for } n > 0, \\
d(x_n^m, x_{n'}^{m'}) &= d(x_n^m, x_0^0) + d(x_0^0, x_{n'}^{m'}) \quad \text{for } n, n' > 0 \text{ and } n \neq n'.
\end{aligned}$$

Then (X, d) is a bounded, complete metric space. Let now $f : X \to X$ be given by

$$f(x) := \begin{cases} z & \text{if } x = z, \\ x_0^{m+1} & \text{if } x = x_0^m,\ m \in \mathbb{N}, \\ x_n^2 & \text{if } x = x_n^1,\ n > 0, \\ x_0^0 & \text{if } x = x_n^2,\ n > 0. \end{cases}$$

Then all Picard iteration sequences converge to the unique fixed point z with a rate of convergence which is uniform in the starting point, so f is an asymptotic contraction in the sense of Kirk. We note also that f is continuous. But f is not an asymptotic contraction in the sense of Chen. To see this we assume for a contradiction that there exists a positive integer n_* such that $\phi_{n_*} : [0,\infty) \to [0,\infty)$ is upper semicontinuous and satisfies $\phi_{n_*}(0) = 0$ and

$$\forall x, y \in X\Big(d\big(f^{n_*}(x), f^{n_*}(y)\big) \leq \phi_{n_*}\big(d(x,y)\big)\Big).$$

Let $\varepsilon := 2^{-n_*+1}$. Since ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$ there exists a $\delta > 0$ such that for all $t > 0$ with $t < \delta$ we have $\phi_{n_*}(t) < \varepsilon$. This implies that if $d(x,y) < \delta$ then

$$d\big(f^{n_*}(x), f^{n_*}(y)\big) \leq \phi_{n_*}\big(d(x,y)\big) < \varepsilon = 2^{-n_*+1}.$$

Let $m > 0$ be so large that $2^{-m-1} < \delta$. Then for $x := x_m^1$, $y := x_m^2$ we have

$$d(x,y) = 2^{-m-1} < \delta$$

and

$$d\big(f^{n_*}(x), f^{n_*}(y)\big) = 2^{-n_*+1} = \varepsilon,$$

which is a contradiction.

Thus requiring that $\phi_{n_*}(0) = 0$ puts a restriction on what mappings are covered, but it also allows one to obtain the existence of a fixed point without requiring that f is continuous. We can adapt a part of Chen's argument in [36] to get a similar result for asymptotic contractions in the sense of Gerhardy. In the following proposition we develop a criterion which allows us to infer the existence of a fixed point without the assumption of continuity. This will in a sense work like the condition $\phi_{n_*}(0) = 0$ in [36].

Proposition 3.41. *Let (X,d) be a complete metric space, and let $f : X \to X$ be an asymptotic contraction in the sense of Gerhardy with moduli η^b and β^b for each $b > 0$. For each $b > 0$ let $(\phi_n^b)_{n\in\mathbb{N}}$ be a sequence of functions which satisfy Definition 3.1. Let $b' > 0$ and let $x_0 \in X$ be such that the Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$ is b'-bounded. Let $z := \lim_{n\to\infty} x_n$. Let $m \in \mathbb{N}$ be such that $\limsup_{t\to 0} \phi_m^{b'}(t) < \infty$. Then $f(z) = z$.*

Proof. We have for each $n \in \mathbb{N}$ that

$$d\big(f^{n+m}(x_0), f^m(z)\big) \leq \phi_m^{b'}\big(d\big(f^n(x_0), z\big)\big) \cdot d\big(f^n(x_0), z\big).$$

Since $\lim_{n\to\infty} d(f^n(x_0), z) = 0$ and $\limsup_{t\to 0} \phi_m^{b'}(t) < \infty$, we get

$$\lim_{n\to\infty} d\big(f^{n+m}(x_0), f^m(z)\big) = 0,$$

i.e., $\lim_{n\to\infty} f^{n+m}(x_0) = f^m(z)$. Thus $f^m(z) = z$. We know by Lemma 3.8 that $(f^n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence, hence $f(z) = z$.

□

We note that in the case covered by Proposition 3.41 each iteration sequence converges to z, and the rate of convergence from Theorem 3.14 applies. This follows from Theorem 3.15 or Theorem 3.16.

Several authors have taken the paper by Chen as a starting point for further studies. I.D. Aranđelović and D.S. Petković have developed versions of Corollary 2.4 in [36], where a condition is given which allows one to remove the assumption that one Picard iteration sequence is bounded from the theorem on asymptotic contractions in the sense of Chen. We give here first Chen's corollary:

Corollary 3.42 (Chen)**.** *Let (X, d) be a (nonempty) complete metric space, and let $f : X \to X$ be an asymptotic contraction in the sense of Chen with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$. Assume that*

$$\limsup_{t\to\infty} \frac{\phi(t)}{t} < 1.$$

Then f has a unique fixed point z, and $\lim_{n\to\infty} f^n(x_0) = z$ for each $x_0 \in X$.

Aranđelović and Petković weaken some assumptions in this setting - for more information on this we refer to [3] and [4]. The main theorem in [4] has very weak requirements on the sequence of moduli $(\phi_n)_{n\in\mathbb{N}}$, but requires of the limit ϕ that $\liminf_{t\to\infty} \phi(t)/t < 1$. In detail the result reads as follows:

Theorem 3.43 (Aranđelović,Petković)**.** *Let (X, d) be a (nonempty) complete metric space, let $f : X \to X$ be continuous, and let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of functions $\phi_n : [0, \infty) \to [0, \infty)$ such that*

$$\forall n \in \mathbb{N}\forall x, y \in X\Big(d\big(f^n(x), f^n(y)\big) \le \phi_n\big(d(x, y)\big)\Big), \tag{3.18}$$

and such that there exists an upper semicontinuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for any $t > 0$, $\phi(0) = 0$, and $\phi_n \to \phi$, uniformly on any bounded interval. If

$$\liminf_{t\to\infty} \frac{\phi(t)}{t} < 1,$$

then f has a unique fixed point $z \in X$, and all Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ converge to z, uniformly on each bounded subset of X.

Remark 3.44. In the statement of Theorem 3.43 the sequence $(\phi_n)_{n\in\mathbb{N}}$ is only required to converge uniformly on each bounded interval $[0, b]$, but in the proof of this theorem given in [4] it is explicitly stated, and seemingly used, that $(\phi_n)_{n\in\mathbb{N}}$ converges uniformly on the range of d. Whether this is essential is unclear.

Corollary 3.45. *Let (X, d) be a bounded, complete metric space, and let $f : X \to X$ satisfy the conditions in Theorem 3.43. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. Immediate from Theorem 3.43 and Theorem 3.21.

□

Here we wish to remark on an oversight: In [3] and [4] it is claimed that continuity of the mapping $f : X \to X$ is necessary in the proof of Theorem 2.1

in [36] (i.e. Theorem 3.37). And in line with this it is claimed that Theorem 1 in [3] and Theorem 1 in [4] (i.e. Theorem 3.43) generalize Corollary 2.4 in [36] (i.e. Corollary 3.42), even though continuity of $f : X \to X$ is assumed in the theorems of Aranđelović and Petković. The argument that continuity of f should be necessary in Theorem 3.37 is based on the following example, which was used by Jachymski and Jóźwik in [76] to show that continuity of $f : X \to X$ is necessary in Kirk's original theorem: Consider $X = [0,1]$ with the natural metric and $f : [0,1 \to [0,1]$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ x/2 & \text{if } x \neq 0, \end{cases}$$

and also the sequence of moduli $\phi_n : [0,\infty) \to [0,\infty)$ defined by $\phi_n(t) = 2^{-n+1}$. In [3] and [4] it is claimed that this mapping with these moduli satisfy all the conditions in Theorem 3.37. However, this is not the case, since there exists no positive integer n_* such that $\phi_{n_*}(0) = 0$. And the existence of such an n_* is one of the conditions in Theorem 3.37.

3.4.2 Some theorems giving uniformity of the convergence on bounded subsets

We have already mentioned the theorem by Arav et al. [5], which says that for an asymptotic contraction $f : X \to X$ in the sense of Chen (with a bounded Picard iteration sequence) on a nonempty complete metric space (X, d) the convergence to the fixed point z is uniform on every bounded set $B_n(z) = \{x \in X : d(x, z) \leq n\}$. Some of the authors of [5] have continued this line of investigation in [6] and [156]. In [6] conditions are given which ensures that all Picard iteration sequences converge to the unique fixed point z with a rate of convergence which only depends on the starting point x_0 through an upper bound on $d(x_0, z)$, in a setting where one already assumes that there exists a fixed point of the mapping $f : X \to X$.

Theorem 3.46 (Arav,Reich,Zaslavski). *Let (X, d) be a complete metric space, and let $f : X \to X$. Let $z \in X$ be a fixed point of f. Assume that*

$$\forall x_0 \in X\Big(d\big(f^n(x_0), z\big) \leq \phi_n(d(x_0, z))\Big)$$

for all natural numbers n, where the functions $\phi_n : [0,\infty) \to [0,\infty)$ satisfy the following conditions:

1. *For each $b > 0$ there is a natural number n_b such that*

$$\sup\{\phi_n(t) : t \in [0, b] \text{ and } n \geq n_b\} < \infty,$$

2. *there exists an upper semicontinuous function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) < t$ for all $t > 0$ and a strictly increasing sequence of natural numbers $(m_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \phi_{m_k} = \phi$, uniformly on any bounded interval $[0, b]$.*

Then $f^n(x_0) \to z$ *for all* $x_0 \in X$*, uniformly on each bounded subset of* X*.*

They also show that this theorem has a converse in the following sense: if $f : X \to X$ and $z \in X$ is such that $f^n(x_0) \to z$ for all $x_0 \in X$, uniformly on each bounded subset of X, and if in addition $f(C)$ is bounded for any bounded subset $C \subseteq X$, then f satisfies all the conditions of Theorem 3.46 for suitable $(\phi_n)_{n\in\mathbb{N}}$.

In [156] the following theorem concerning asymptotic contractions is proved.

Theorem 3.47 (Reich,Zaslavski)**.** *Let* (X, d) *be a complete metric space, and let* $f : X \to X$*. Assume that*

$$\forall x, y \in X\Big(d\big(f^n(x), f^n(y)\big) \le \phi_n(d(x, y))\Big)$$

for all natural numbers n*, where* $\phi_n : [0,\infty) \to [0,\infty)$*,* $n \in \mathbb{N}$*. Suppose that:*

1. *For each* $b > 0$ *there is a natural number* n_b *such that*
$$\sup\{\phi_n(t) : t \in [0, b] \text{ and } n \ge n_b\} < \infty.$$
2. *There exists an upper semicontinuous function* $\phi : [0,\infty) \to [0,\infty)$ *satisfying* $\phi(t) < t$ *for all* $t > 0$ *and a strictly increasing sequence of natural numbers* $(m_k)_{k\in\mathbb{N}}$ *such that* $\lim_{k\to\infty} \phi_{m_k} = \phi$*, uniformly on any bounded interval* $[0, b]$*.*
3. *There exists* $x_0 \in X$ *such that* $(f^n(x_0))_{n\in\mathbb{N}}$ *is bounded.*

Then there exists a unique point $z \in X$ *such that* $f^n(x_0) \to z$ *for all* $x_0 \in X$*, uniformly on each bounded subset of* X*. And if there exists a natural number* $n_* > 0$ *such that* f^{n_*} *is continuous at* z*, then* z *is a fixed point.*

Similarly to the case for Theorem 3.43 we get as a corollary that for bounded spaces the mappings satisfying the conditions in this theorem are asymptotic contractions in the sense of Kirk[5].

Corollary 3.48. *Let* (X, d) *be a bounded, complete metric space, and let* $f : X \to X$ *satisfy the conditions in Theorem* 3.47*. Then* f *is an asymptotic contraction in the sense of Kirk.*

Proof. Immediate from Theorem 3.47 and Theorem 3.21.

□

3.4.3 Suzuki's asymptotic contractions of the final type

When first discussing Kirk's asymptotic contractions in Chapter 1 we mentioned that the purpose was to provide an asymptotic version of the Boyd–Wong theorem (Theorem 1.20). Thus it is a natural question whether one also can prove

[5]Both of the above theorems were originally formulated in a context where $0 \notin \mathbb{N}$, but this is inessential.

an asymptotic version of the Meir–Keeler theorem, which generalizes the theorem of Boyd and Wong. This was in fact mentioned as an open problem in [83]. In [167] T. Suzuki showed that it is indeed the case that one can obtain an asymptotic version of the Meir–Keeler theorem. He introduced asymptotic contractions of Meir–Keeler type, and showed that these mappings include both the asymptotic contractions from Definition 1.25 (which include the asymptotic contractions in the sense of Kirk) and the Meir–Keeler contractions, i.e., the mappings $f : X \to X$ on a metric space (X, d) which satisfy (1.2) from Theorem 1.22. For the latter he made use of Lim's characterization from [125], which we repeated as Theorem 1.23. Suzuki's definition and main theorem from [167] are as follows.

Definition 3.49 (Suzuki)**.** Let (X, d) be a metric space and let $f : X \to X$. The mapping f is called an *asymptotic contraction of Meir–Keeler type* ($ACMK$, for short) if there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of functions $\phi_n : [0,\infty) \to [0,\infty)$ such that

1. $\limsup_{n\to\infty} \phi(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$.
2. $\forall \varepsilon > 0 \exists \delta > 0 \exists \nu \in \mathbb{N} \forall t \in [\varepsilon, \varepsilon + \delta]\big(\phi_\nu(t) \leq \varepsilon\big)$.
3. $\forall n \in \mathbb{N} \forall x, y \in X\Big(x \neq y \to d\big(f^n(x), f^n(y)\big) < \phi_n(d(x, y))\Big)$.

Theorem 3.50 (Suzuki)**.** *Let (X, d) be a complete metric space and let $f : X \to X$ be an ACMK on X. Assume that there exists a natural number $m > 0$ such that f^m is continuous. Then f has a unique fixed point z, and $f^n(x_0) \to z$ for all $x_0 \in X$.*

Since every asymptotic contraction in the sense of Kirk is an $ACMK$ it follows from this theorem that one can drop the assumption in Kirk's original theorem that one iteration sequence is bounded[6]. It is unclear to what extent the convergence in Theorem 3.50 is uniform. However, for a subsequent further generalization of the asymptotic contractions of Meir–Keeler type this is settled. In [168] Suzuki introduced the so-called asymptotic contractions of the final type, and proved that this notion is strictly more general than the notion of an asymptotic contraction of Meir–Keeler type.

Definition 3.51 (Suzuki)**.** Let (X, d) be a metric space and let $f : X \to X$. The mapping f is called an *asymptotic contraction of the final type* (ACF, for short) if

1. $\limsup_{\delta\to+0} \left\{ \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) : d(x, y) < \delta \right\} = 0.$
2. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with
$$\varepsilon < d(x, y) < \varepsilon + \delta$$
there exists $\nu \in \mathbb{N}$ such that $d\big(f^\nu(x), f^\nu(y)\big) \leq \varepsilon$.

[6] This is our Corollary 3.27, but it was proved first in [167].

3. For all $x, y \in X$ with $x \neq y$ there exists $\nu \in \mathbb{N}$ such that
$$d\big(f^\nu(x), f^\nu(y)\big) < d(x, y).$$
4. For all $x \in X$ and $\varepsilon > 0$ there exist $\delta > 0$ and $\nu \in \mathbb{N}$, $\nu > 0$, such that
$$\varepsilon < d\big(f^i(x), f^j(x)\big) < \varepsilon + \delta \quad \text{implies} \quad d\big(f^{i+\nu}(x), f^{j+\nu}(x)\big) \leq \varepsilon$$
for all $i, j \in \mathbb{N}$ with $i > 0$ and $j > 0$.

A main result in [168] is a characterization of the *ACF*s on a metric space (X, d) as exactly the mappings $f : X \to X$ such that all Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ are Cauchy, and such that $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$ for all $x, y \in X$. Thus on a nonempty complete metric space the *ACF*s are the mappings such that all Picard iteration sequences converge to the same point. Hence even in the setting of bounded metric spaces the concept of an *ACF* is strictly more general than the concept of an asymptotic contraction in the sense of Kirk, since e.g., the mapping in Example 1.10 is an *ACF*, but not an asymptotic contraction in the sense of Kirk. And because of this characterization the *ACF*s are indeed in some sense the most general asymptotic contractions possible. But this also means that in general it is not possible to find a rate of convergence for *ACF*s as such which has uniformity properties with respect to the starting point. The convergence of $(f^n(x_0))_{n\in\mathbb{N}}$ will in the general case depend essentially on x_0. This sets Suzuki's theorem in [168] apart from the results presented earlier in this chapter, where we could prove that the convergence is highly uniform, and also calculate explicit and uniform rates of convergence.

3.4.4 Other variants

Other work on variants of asymptotic contractions includes considering *asymptotic pointwise contractions*, where the moduli take a point $x \in X$ as an argument. These were first considered in the setting of Banach spaces, and a fixed point theorem for asymptotic pointwise contractions defined on a bounded, closed, convex subset of a superreflexive Banach space was announced by Kirk in [84]. This was proved via ultrapower techniques, but an elementary proof (in the sense that ultrapower methods are not needed) was given by Kirk and Xu in [86]. Asymptotic pointwise contractions in the setting of metric spaces are studied in [73], and it is proved that a so-called strongly asymptotic pointwise contraction $f : X \to X$ on a bounded metric space (X, d) for which the convexity structure $\mathcal{A}(X)$ of admissible subsets is compact, has a fixed point to which all Picard iteration sequences converge. Since dependence on $x \in X$ is built in in the definition of an asymptotic pointwise contraction it seems unlikely that the convergence should be uniform.

Lastly we mention that K. Wlodarczyk and various coauthors have studied asymptotic contractions in the context of set-valued dynamic systems in uniform spaces, see [172, 173, 174, 175, 176], and Razani et al. have studied asymptotic contractions in the modular space, see [154].

Chapter 4

Generalized contractive mappings

In this chapter we will present the results of a case study in proof mining concerned with a general class of mappings of contractive type. These results have appeared in [27], but the material has here been somewhat revised[1] and some comments have been added. Proposition 4.3, Theorem 4.6, and Lemmas 4.12, 4.13, and 4.16 have also (essentially) appeared in the author's Master thesis [23].

Our starting point is a theorem which was proved by Rhoades [158], by Kincses and Totik [81], and in a less general form by Hicks and Sharma [69], concerning fixed points of a very general class of mappings of contractive type. By isolating the requirements on the mapping, specifically on the contractivity condition in question, we develop an extension of the theorem from the compact case to the setting of arbitrary metric spaces. This is accomplished by using (negative translation and) monotone functional interpretation to study the concepts in question (along with the theorem and one of its proofs), and below we will include some remarks on how the extension from the case where the space is compact to the case of arbitrary metric spaces can be viewed as an instance of a general phenomenon.

We also supply numerical information concerning the convergence of the Picard iteration sequence to the fixed point, in the form of a rate of convergence[2] for the Picard iteration sequences to the fixed point. That we could obtain a full rate of convergence is in line with what we proved in Chapter 2 (when we restrict to bounded metric spaces), and a discussion of how the methods from Chapter 2 are relevant for the mappings we consider here was included in Section 2.5.

The uniformity features of the rate of convergence exhibited means that we can relate the results in this chapter to the material in Chapter 3. This was already included in Chapter 3, but the proofs there made reference to results in

[1] Notably because we here let $0 \in \mathbb{N}$.

[2] If the space is not complete we have a Cauchy rate instead.

this chapter.

4.1 Introduction

If a function $f : X \to X$ on a nonempty compact metric space (X, d) is contractive, i.e., satisfies

$$\forall x, y \in X(x \neq y \to d(f(x), f(y)) < d(x, y)),$$

then it has a unique fixed point, and for every starting point $x_0 \in X$ the iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to this fixed point. This well-known theorem due to Edelstein has led to the study of many generalizations of the notion of contractivity. (For a simple proof of Edelstein's theorem, see e.g. [64].) The hope when considering such generalizations is then to obtain corresponding generalizations of the fixed point theorem. These generalized contraction properties are also considered as conditions on functions $f : X \to X$ on complete metric spaces, or on metric spaces in general. In [157], B.E. Rhoades compares 25 contraction conditions, most of them previously considered in the literature, and also considers generalizations of the 25 basic conditions to the cases where the condition holds for various iterates of the function. The basic conditions are numbered (1)–(25). P. Collaço and J. Carvalho e Silva completes the comparison of the 25 conditions in [38], so that the implications that hold between the different conditions are completely determined. Specifically, it is known that condition (25),

$$\forall x, y \in X(x \neq y \to d(f(x), f(y)) < \operatorname{diam}\{x, y, f(x), f(y)\}),$$

is the most general. So if f satisfies one of the conditions (1)–(24), then it also satisfies condition (25). Hence a fixed point theorem for functions satisfying (25) would give as corollaries corresponding fixed point theorems for conditions (1)–(24). However, a function on a complete metric space satisfying (25) need not have a fixed point, consider for example $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x + 1$. But if one in addition assumes that f is continuous and X compact, then f has a unique fixed point, and for any $x_0 \in X$ the Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to this fixed point. This result also extends to the case where (25) holds for an iterate of the function, i.e., if there exists $p \in \mathbb{N}$ such that

$$\forall x, y \in X(x \neq y \to d(f^p(x), f^p(y)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}).$$

These conditions, where we require that for some $p \in \mathbb{N}$ we should have that f^p satisfies respectively (1)–(25), are numbered respectively (26)–(50). This theorem − which we stated as Theorem 1.30 in Chapter 1 − was proved by Rhoades[3] in [158], and also by Hicks and Sharma (without considering the case of iterates $p \neq 1$) in [69] and Kincses and Totik in [81].

[3]As already mentioned Rhoades claimed in [158] to have proved the theorem for a more general contractive condition, but in his review of his own paper in Zentralblatt MATH this was modified.

Rhoades actually proved a more general theorem; instead of requiring the space (X, d) to be compact it is enough if the mapping $f : X \to X$ is a compact map (and the space nonempty). In [158] it is also proved that if $f : X \to X$ is a generalized p-contractive condensing self-mapping of a complete, bounded (nonempty) metric space (X, d), then f has a unique fixed point $z \in X$, and $f^n(x_0) \to z$ for each $x_0 \in X$. Both of these theorems are proved by noting that the proofs of two theorems proved by Janos in [77] work just as well with the contractive condition under consideration in place of the less general condition considered by Janos. Our extension of Theorem 1.30 from the setting of a compact space to arbitrary metric spaces will not be covered by these theorems. It should be noted that there are theorems for mappings of contractive type more general than Theorem 1.30 – we included a discussion of this at the end of the subsection on mappings of contractive type in Chapter 1, and in particular one of Park's theorems in [149] (which we included as Theorem 1.32) provides a generalization. There are also variants of Theorem 1.32 where neither the space nor the mapping are assumed to be compact, but with various other restrictions on the contractive condition and/or e.g. conditions concerning the existence of regular cluster points – see the references given in the discussion in Chapter 1. The relationships between several of these theorems and the extension of Theorem 1.30 which we obtain in the course of our case study (as a consequence of the uniformity properties of the explicit Cauchy rate constructed) are unclear. However, to the extent that our focus is on explicit rates of convergence this is not too relevant. Several of these theorems might be candidates for proof mining – with the goal of extracting quantitative information.

Theorem 1.30 does not hold if the contractive condition is replaced by the other standard generalizations treated in [157]. For example, a continuous function on a compact space satisfying the condition that there exists $p : X \times X \to \mathbb{N}$ such that for all $x, y \in X$ with $x \neq y$ we have

$$d(f^{p(x,y)}(x), f^{p(x,y)}(y)) < \operatorname{diam}\{x, y, f^{p(x,y)}(x), f^{p(x,y)}(y)\},$$

or satisfying the condition that there exists $p, q \in \mathbb{N}$ such that

$$\forall x, y \in X (x \neq y \to d(f^p(x), f^q(y)) < \operatorname{diam}\{x, y, f^p(x), f^q(y)\}),$$

does not necessarily have a fixed point. The second of these cases involves two iterates, the first involves one iterate which is not uniform in x and y. Given $p \in \mathbb{N}$ we will call a function *generalized p-contractive* if it satisfies (25) for an iterate f^p. With an abuse of notation we will also sometimes say that f is generalized p-contractive to mean that there exists *some* $p \in \mathbb{N}$ for which f^p satisfies (25), without having specified any $p \in \mathbb{N}$ in advance. (See Definition 1.28 and the remark on notation following the definition.)

With the help of techniques and insights from proof mining we develop a quantitative version of the fixed point theorem discussed above. This involves finding a rate of convergence for the Picard iteration sequences to the unique fixed point[4], and compared to the theorem of Rhoades and Kincses–Totik we

[4] Assuming the space is complete.

also obtain new qualitative information, insofar as we show that the convergence of the iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ depends on conditions which are satisfied if the space is compact, but conditions which we can also single out and see satisfied in other cases. Namely, we require uniform continuity and a uniform version of generalized p-contractivity, and also the existence of a bounded iteration sequence for some starting point. Furthermore, we show that the rate of convergence is highly uniform in the sense that it only depends on the starting point x_0, the space (X, d), and the function f through suitable moduli expressing uniform continuity and uniform generalized p-contractivity and a bound on the iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$. If the space is not complete we still get a Cauchy rate for the iteration sequence.

As indicated above, some of the results in this chapter are taken from the author's Master thesis [23], which also contain a fuller account of the logical analysis of Theorem 1.30 together with (variants of) its proof as given by Kincses and Totik. The main tool is (negative translation) combined with monotone functional interpretation, and by applying monotone functional interpretation to the contractive condition in the theorem one is lead to consider a uniform version of this condition – which we will call *uniform generalized p-contractivity* (see Definition 4.4). Similarly, the continuity of the mapping is upgraded by monotone functional interpretation to uniform continuity, and monotone functional interpretation also makes it explicit what quantitative information one requires as input in place of the conditions of continuity and generalized p-contractivity, namely a *modulus of uniform continuity* (see Definition 4.1) and a *modulus of uniform generalized p-contractivity* (see Definition 4.2). In the case of Theorem 1.30 the compactness of the space means that the condition of generalized p-contractivity is upgraded to this uniform version, much as the continuity of the mapping is turned into uniform continuity – see Proposition 4.3. It then turns out that we can prove that all Picard iteration sequences are Cauchy (with a Cauchy rate which only depends on the starting point x_0 through a bound on $(f^n(x_0))_{n\in\mathbb{N}}$), and that $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$ for any x and y in the space, by assuming only that we have such uniform versions of the contractive condition and continuity, along with the existence of one bounded iteration sequence. This corresponds loosely speaking to the fact that the proof of the theorem does not use the separability or the completeness of the space (as supplied by the compactness) in any essential way[5] – but that compactness essentially is used to ensure that continuity and generalized p-contractivity imply uniform continuity and uniform generalized p-contractivity – and more precisely, to the fact that the proof can be formalized in the system $\mathcal{A}^\omega[X, d]$ extended with a uniform boundedness principle $\exists\text{-UB}^X$ for abstract bounded metric spaces. The principle $\exists\text{-UB}^X$ systematically transforms certain kinds of statements into their uniform variants; and even though the principle implies numerous results which are true only for compact metric spaces and continuous functions one can show that for a large class of consequences the conclusion A is true in arbitrary bounded metric spaces even when $\exists\text{-UB}^X$ has been used in

[5]Except to ensure the existence of the common limit of the Picard iteration sequences.

the proof of A. And for a more restricted class of consequences one can also extract effective bounds from proofs involving $\exists$-UB^X via negative translation and monotone functional interpretation. This then gives us a systematic way of removing compactness assumptions from certain kinds of theorems where the compactness is used to obtain uniform versions of the notions involved. The principle $\exists$-UB^X was introduced by Kohlenbach in [100] as an extension to theories like $\mathcal{A}^\omega[X,d]$, which involve abstract bounded metric spaces, of his principle Σ^0_1-UB of Σ^0_1-boundedness, treated e.g. in [93][6]. For further discussion of the role of compactness assumptions in upgrading e.g. contractivity conditions to their uniform variants, and how monotone functional interpretation handles this, see e.g. [55] and Chapter 17 in [101].

So far we have discussed a treatment which allows one to extract rates of proximity for the Picard iteration sequences. For the explanation why we could instead obtain a full rate of convergence we refer to the discussion in Chapter 2.

We have mentioned that the explicit Cauchy rate which we have constructed depends on a bound on the iteration sequence. However, if the space satisfies a certain further structural condition we get a Cauchy rate for $(f^n(x_0))_{n\in\mathbb{N}}$ which does not depend on a bound on the iteration sequence, but rather depends on an upper bound $b > 0$ on $d(x_0, y)$, where $y \in X$ is a point which is moved a sufficiently short distance by the mapping. This class of spaces will include many spaces commonly used in analysis, such as spaces of hyperbolic type in the sense of [62], hyperbolic spaces in the sense of [155] and also hyperbolic spaces in the sense of [99] – and therefore e.g., normed linear spaces, Hadamard manifolds and CAT(0)-spaces.

4.2 Preliminaries

For the definition of a generalized p-contractive mapping we refer to Definition 1.28. Notice that a generalized p-contractive function is not necessarily nonexpansive; take for instance $f : (0,\infty) \to (0,\infty)$ defined by $f(x) := 2x$. Then f is generalized 1-contractive. For a statement of the theorem of Rhoades and Kincses–Totik we refer to Theorem 1.30. To give a quantitative version of this theorem, we express the requirements on f by the following moduli.

Definition 4.1. Let (X,d) be a metric space, and let $f : X \to X$. We say that $\omega : (0,\infty) \to (0,\infty)$ is a *modulus of uniform continuity* for f if for all $\varepsilon \in (0,\infty)$ and for all $x,y \in X$ with $d(x,y) < \omega(\varepsilon)$ we have $d(f(x), f(y)) < \varepsilon$.

Definition 4.2. Let (X,d) be a metric space, let $p \in \mathbb{N}$, and let $f : X \to X$. We say that $\eta : (0,\infty) \to (0,\infty)$ is a *modulus of uniform generalized p-contractivity* for f if for all $\varepsilon \in (0,\infty)$ and for all $x,y \in X$ with $d(x,y) > \varepsilon$ we have

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < \mathrm{diam}\,\{x, y, f^p(x), f^p(y)\}.$$

[6] In the analysis in [23] we did not use the principle $\exists$-UB^X, but rather Σ^0_1-UB, together with an approach which does not involve abstract metric spaces, but rather complete, separable metric spaces representable in the formal system.

When X is a compact metric space, f having such moduli coincides with f being continuous and generalized p-contractive.

Proposition 4.3. *Let (X, d) be a compact metric space, let $p \in \mathbb{N}$, and let $f : X \to X$ be continuous and generalized p-contractive. Then f has moduli ω and η of uniform continuity and uniform generalized p-contractivity.*

Proof. We can without loss of generality assume that $\operatorname{diam}(X) > 0$, since otherwise everything is trivial. Existence of a modulus of uniform continuity follows since f is uniformly continuous. For the other modulus, consider for $\varepsilon > 0$ such that $\operatorname{diam}(X) > \varepsilon$ the set

$$A_\varepsilon := \{(x, y) \in X \times X : d(x, y) \geq \varepsilon\}.$$

Then A_ε is closed and therefore compact, and the continuous function $g : X \times X \to \mathbb{R}$ defined by

$$g(x, y) := \operatorname{diam}\{x, y, f^p(x), f^p(y)\} - d(f^p(x), f^p(y))$$

assumes its infimum on A_ε. That is, there exists $(x, y) \in A_\varepsilon$ such that $g(x, y) = \inf g(A_\varepsilon)$. Therefore $\inf g(A_\varepsilon) \neq 0$, since we otherwise would have

$$\operatorname{diam}\{x, y, f^p(x), f^p(y)\} = d(f^p(x), f^p(y)),$$

contradicting the fact that f is generalized p-contractive and $d(x, y) \geq \varepsilon$. So we can define a modulus of uniform generalized p-contractivity η by for $0 < \varepsilon < \operatorname{diam}(X)$ letting $\eta(\varepsilon)$ be some positive real number smaller than $\inf g(A_\varepsilon)$ and for $\varepsilon \geq \operatorname{diam}(X)$ letting $\eta(\varepsilon)$ be e.g. 1.

□

The following is just a way of rephrasing the statement that f has a modulus of uniform generalized p-contractivity.

Definition 4.4. Let (X, d) be a metric space, let $p \in \mathbb{N}$ and let $f : X \to X$. We say that f is *uniformly generalized p-contractive* if for all real $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) > \varepsilon$ we have

$$\operatorname{diam}\{x, y, f^p(x), f^p(y)\} - d(f^p(x), f^p(y)) > \delta.$$

We note that for a metric space (X, d) examples of uniformly generalized p-contractive mappings $f : X \to X$ are e.g. the mappings f such that f^p fulfills condition (24) from [157], i.e., the condition that there should exist $0 \leq h < 1$ such that

$$d(f(x), f(y)) \leq h \cdot \operatorname{diam}\{x, y, f(x), f(y)\}$$

holds for all $x, y \in X$. This condition was introduced by L.B. Ćirić in [37], and a mapping on a complete metric space satisfying this condition is there called a quasi-contraction. Likewise mappings satisfying one of those conditions (1)–(23) which in [38] are listed as stronger than condition (24) are uniformly generalized p-contractive for $p = 1$. Ćirić proved the following theorem concerning mappings satisfying condition (24).

Theorem 4.5 (Ćirić). *Let (X,d) be a nonempty complete metric space, and let $f : X \to X$ and $h \in [0,1)$ be such that*

$$d(f(x), f(y)) \leq h \cdot \operatorname{diam}\{x, y, f(x), f(y)\}$$

holds for all $x, y \in X$. Then f has a unique fixed point z, and $f^n(x_0) \to z$ for every $x_0 \in X$.

This theorem is different from Theorem 1.30 in much the same way that Banach's contraction mapping principle is different from Edelstein's theorem on contractive mappings.

4.3 Main results

4.3.1 A Cauchy rate for uniformly continuous uniformly generalized p-contractive mappings

Our theorem will concern arbitrary metric spaces instead of compact ones. We will later improve this theorem by showing that if one Picard iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded, then any Picard iteration sequence is bounded (see Theorem 4.18), and we will construct rates of convergence which do not depend on a bound on the iteration sequence, but rather on various more local bounds (see Corollary 4.19, 4.20, 4.21, 4.27, 4.28, and 4.29).

Theorem 4.6. *Let (X,d) be a nonempty metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus ω of uniform continuity, and a modulus η of uniform generalized p-contractivity. Let $x_0 \in X$ be the starting point of a sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_{n+1} := f(x_n)$. Suppose $(x_n)_{n\in\mathbb{N}}$ is bounded by b. Let $\rho : (0,\infty) \to (0,\infty)$ be defined by*

$$\rho(\varepsilon) := \min\{\eta(\varepsilon), \varepsilon/2, \eta(1/2 \cdot \omega^p(\varepsilon/2))\}.$$

Let $\phi : (0,\infty) \to \mathbb{N}$ be defined by

$$\phi(\varepsilon) := \begin{cases} p\lceil (b-\varepsilon)/\rho(\varepsilon)\rceil & \text{if } b > \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

Then ϕ is a Cauchy rate for $(x_n)_{n\in\mathbb{N}}$. Given p, ω, η and b we will denote this Cauchy rate also by $\Phi(p,\omega,\eta,b,\cdot)$, so that given $\varepsilon > 0$ we get that

$$m, n \geq \Phi(p,\omega,\eta,b,\varepsilon)$$

gives $d(x_n, x_m) \leq \varepsilon$.

Thus the appropriate moduli, together with the existence of a bounded iteration sequence, guarantee the existence of a Cauchy sequence which is an approximate fixed point sequence. If the space is complete, then $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point z, and ϕ is a rate of convergence for the sequence. The fixed point is unique if it exists, for if x and y were fixed points with $x \neq y$, we would have

$$d(x, y) = d(f^p(x), f^p(y))$$

and

$$d(f^p(x), f^p(y)) = \operatorname{diam}\{x, y, f^p(x), f^p(y)\},$$

contradicting the fact that f is generalized p-contractive. The rate ϕ only depends on the function f and the starting point $x_0 \in X$ through p and the moduli ω and η, and also through a bound b on $(x_n)_{n\in\mathbb{N}}$. If b is a bound on the whole space then the rate does not depend on x_0, and gives if the fixed point exists a rate of convergence for f, or else a Cauchy rate for f.

We note in passing that the moduli in Definition 4.1 and Definition 4.2 might be equivalently given as functions $\omega : \mathbb{N} \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ with conditions of the form that e.g. $d(x, y) < 2^{-\omega(k)}$ should give $d(f(x), f(y)) < 2^{-k}$. Likewise the Cauchy rate in Theorem 4.6 can be given as a function $\Phi : \mathbb{N} \to \mathbb{N}$. In this case we have that with b an integer and with ω and η computable, then Φ is computable. In fact, it is clear that a Cauchy rate as in Theorem 4.6 could be given as an effectively computable functional $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ taking ω and η as two of its arguments. The functional Φ could be taken to be primitive recursive in the sense of Kleene, and for fixed p it could even be taken to be of a low level in the Grzegorczyk hierarchy (as a functional in the remaining variables). (See also the comments in Section 2.5.)

Before proving this theorem we give some corollaries and a definition, and we also prove some lemmas.

Corollary 4.7. *Let (X, d) be a nonempty, bounded, complete metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be uniformly continuous and uniformly generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

Together with Proposition 4.3, this corollary implies Theorem 1.30 as a special case.

Corollary 4.8 (Theorem of Rhoades and Kincses–Totik)**.** *Let (X, d) be a nonempty compact metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be continuous and generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

Notice that if (X, d) is a nonempty compact metric space and $f : X \to X$ is continuous and satisfies one of the conditions (1)–(24) from [157], then f has moduli of uniform continuity and uniform generalized 1-contractivity, and hence also a rate of convergence as given in Theorem 4.6. As an application of Theorem 4.6 we note also the following relationship with asymptotic contractions in the sense of Kirk. The following two corollaries already appeared as Corollary 3.23 and Corollary 3.24 in Chapter 3, but the proofs there made reference to and were dependent on Theorem 4.6 and Proposition 4.3 in the present chapter. The uniformity of the convergence of the Picard iteration sequences $(f^n(x_0))_{n\in\mathbb{N}}$ with respect to the starting point $x_0 \in X$ in the case where (X,d) is compact and $f : X \to X$ satisfies one of the conditions (1)–(50) from Rhoades' paper [157] was already present in [158].

Corollary 4.9. *Let (X,d) be a bounded, complete metric space, let $p \in \mathbb{N}$, and let $f : X \to X$ be uniformly generalized p-contractive and uniformly continuous. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. See the proof of Corollary 3.23.

□

Corollary 4.10. *Let (X,d) be a compact metric space. Let $f : X \to X$ be continuous and such that it satisfies one of the conditions* (1)–(50) *from* [157]. *Then f is an asymptotic contraction in the sense of Kirk.*

Proof. See the proof of Corollary 3.24.

□

We will in the following let X, b, f, p, ω and η be as in Theorem 4.6.

Definition 4.11. We say that $\rho : (0,\infty) \to (0,\infty)$ is a *modulus of modified uniform generalized p-contractivity* for f if for all $\varepsilon > 0$ and for all $x, y \in X$ with

$$\operatorname{diam}\{x, y, f^p(x), f^p(y)\} > \varepsilon$$

we have

$$d(f^p(x), f^p(y)) + \rho(\varepsilon) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}.$$

Lemma 4.12. *Define $\rho : (0,\infty) \to (0,\infty)$ by*

$$\rho(\varepsilon) := \min\left\{\eta(\varepsilon), \frac{\varepsilon}{2}, \eta\left(1/2 \cdot \omega^p(\varepsilon/2)\right)\right\}.$$

Then ρ is a modulus of modified uniform generalized p-contractivity for f.

Proof. We consider the different cases.

1. If $d(x,y) > \varepsilon$ then

$$d(f^p(x), f^p(y)) + \rho(\varepsilon) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}, \tag{4.1}$$

since $\rho(\varepsilon) \le \eta(\varepsilon)$.

2. If $d(f^p(x), x) > \varepsilon$ we again look at the different cases.

(a) If $d(x,y) < \omega^p(\varepsilon/2)$, then

$$d(f^p(x), f^p(y)) < \varepsilon/2,$$

and (4.1) holds since $\rho(\varepsilon) \le \varepsilon/2$ and

$$\operatorname{diam}\{x, y, f^p(x), f^p(y)\} > \varepsilon.$$

(b) If $d(x,y) \geq \omega^p(\varepsilon/2)$, then by definition of η we have

$$d(f^p(x), f^p(y)) + \eta(1/2 \cdot \omega^p(\varepsilon/2)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}.$$

Then (4.1) holds since $\rho(\varepsilon) \leq \eta(1/2 \cdot \omega^p(\varepsilon/2))$. (This holds in fact whether $d(f^p(x), x) > \varepsilon$ or not.)

The cases where $d(f^p(y), y) > \varepsilon$, $d(f^p(x), y) > \varepsilon$ or $d(f^p(y), x) > \varepsilon$ are treated in exactly the same way as the case $d(f^p(x), x) > \varepsilon$.

□

Lemma 4.13. *Let (X, d) be a nonempty metric space, and let $x_0 \in X$ be such that b is a bound on the Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$. Let $p \in \mathbb{N}$, and let ρ be a modulus of modified uniform generalized p-contractivity for f. Let $\phi : (0, \infty) \to \mathbb{N}$ be defined by*

$$\phi(\varepsilon) := \begin{cases} p\,\lceil (b-\varepsilon)/\rho(\varepsilon) \rceil & \text{if } b > \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

Then ϕ satisfies

$$\forall \varepsilon > 0 \forall m, n \geq \phi(\varepsilon)(d(x_m, x_n) \leq \varepsilon).$$

Proof. The proof of this lemma comes essentially from the proof of the first theorem in [81]. If $\varepsilon \geq b$, then

$$\forall m, n \geq \phi(\varepsilon)(d(x_m, x_n) \leq \varepsilon).$$

So let $\varepsilon < b$. Let $x_0 \in X$, and let $n, k, l \in \mathbb{N}$. Let $n_0 := np + k$, $m_0 := np + l$. For $0 \leq i < n$ we define n_{i+1} and m_{i+1} inductively so that

$$n_{i+1}, m_{i+1} \in \{n_i, n_i - p, m_i, m_i - p\},$$
$$d(x_{n_{i+1}}, x_{m_{i+1}}) = \operatorname{diam}\{x_{n_i}, x_{n_i - p}, x_{m_i}, x_{m_i - p}\}.$$

We write d_i for $\operatorname{diam}\{x_{n_i}, x_{n_i - p}, x_{m_i}, x_{m_i - p}\}$ for $i < n$. If for some i we have $d_i = 0$, then

$$d(x_{np+k}, x_{np+l}) = 0.$$

So suppose not. Since ρ is a modulus of modified uniform generalized p-contractivity we have

$$d(x_{n_0}, x_{m_0}) + \rho(\varepsilon_0) < d_0$$

for all $\varepsilon_0 > 0$ with $\varepsilon_0 < d_0$. Furthermore, we have

$$d_0 + \rho(\varepsilon_1) < d_1$$

for all $\varepsilon_1 > 0$ with $\varepsilon_1 < d_1$. And in general

$$d_i + \rho(\varepsilon_{i+1}) < d_{i+1}$$

for all $\varepsilon_{i+1} > 0$ with $\varepsilon_{i+1} < d_{i+1}$. Therefore, for $0 \leq i < n$,

$$d(x_{n_0}, x_{m_0}) < d_i - \sum_{j=0}^{i} \rho(\varepsilon_j),$$

for $\varepsilon_j > 0$ with $\varepsilon_j < d_j$ for $j \leq i$. If for some $0 \leq i < n$ we have $d_i \leq \varepsilon$, then

$$d(x_{np+k}, x_{np+l}) = d(x_{n_0}, x_{m_0}) < \varepsilon.$$

If on the other hand we have $d_i > \varepsilon$ for all $0 \leq i < n$, then we get

$$d(x_{n_0}, x_{m_0}) < d_i - \sum_{j=0}^{i} \rho(\varepsilon).$$

Thus

$$d(x_{np+k}, x_{np+l}) < b - n\rho(\varepsilon).$$

Now let

$$n := \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil .$$

Then $d(x_{np+k}, x_{np+l}) < \varepsilon$. And this n does not depend on x_0, except through the bound b. By letting

$$m := p \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil ,$$

we get $d(x_{m+k}, x_{m+l}) < \varepsilon$. And since $\varepsilon < b$ we have

$$\phi(\varepsilon) = p \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil .$$

Since k and l were arbitrary, we get

$$\forall \varepsilon > 0 \forall m, n \geq \phi(\varepsilon)(d(x_m, x_n) < \varepsilon).$$

□

Proof of Theorem 4.6. The lemmas give directly that ϕ as defined in the theorem is a Cauchy rate for $(x_n)_{n\in\mathbb{N}}$.

□

Below we include an example which helps to set Theorem 4.6 apart from other results. This example is rather artificial and messy – since it tries to do many things at the same time. In short: Example 4.14 provides us with an unbounded complete metric space (X, d) and a selfmapping $f : X \to X$ where the conditions in Theorem 4.6 are satisfied, so that there exists a fixed point $z \in X$, where

$$\{x \in X : d(z, x) \leq 1\}$$

is not compact, and where we cannot remove either $d(x, y)$, $d(y, f(x))$, $d(x, f(y))$, $d(x, f(x))$ or $d(y, f(y))$ in the formulation of the condition that for all real $\varepsilon > 0$ there should exist $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) > \varepsilon$ we have

$$\max\{d(x, y), d(x, f(y)), d(y, f(x)), d(x, f(x)), d(y, f(y))\} - d(f^p(x), f^p(y)) > \delta,$$

and where in addition the mapping f does not satisfy the condition (24) from [157], so that theorem 4.5 does not apply.

Example 4.14. Let $a, b, c, d \notin \mathbb{R}$ be pairwise distinct. Let $Y = \{0, a, b, c, d\}$ and

$$Y' = \{3k+1 : k \geq 0\} \cup \{-3k-1 : k \geq 0\} \cup \{2^{-k} : k \geq 0\} \cup \{-2^{-k} : k \geq 0\} \cup \{0\}.$$

Equip Y' with the natural metric, and define a metric d_Y on Y such that $d_Y(0, a) = 3$, $d_Y(0, b) = 3$, $d_Y(0, c) = 1$, $d_Y(0, d) = 1$, $d_Y(c, d) = 2$, $d_Y(a, c) = 2$, $d_Y(b, c) = 2$, $d_Y(a, d) = 2$, $d_Y(b, d) = 2$ and $d_Y(a, b) = 3$. Let X be the set of sequences $(x_n)_{n\in\mathbb{N}}$ with $x_0 \in Y$ and with $x_n \in Y'$ for $n \geq 1$ such that $\{|x_n| : n \geq 1\}$ is bounded. Define a metric on X by for $x, y \in X$ with $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}}$ letting

$$d(x, y) = \max\big\{d_Y(x_0, y_0), \sup\{|x_n - y_n| : n \geq 1\}\big\}.$$

Given $x = (x_n)_{n\in\mathbb{N}} \in X$ and x_n with $n \geq 1$, consider the condition:

$$\text{There is } m \geq 1 \text{ with } x_m > x_n. \tag{4.2}$$

Define $f : X \to X$ by for $x = (x_n)_{n\in\mathbb{N}} \in X$ letting $f(x) = (y_n)_{n\in\mathbb{N}}$ be given by

$$y_n = \begin{cases} 0 & \text{if } x_n = 0,\ x_n = c \text{ or } x_n = d, \\ 3k-2 & \text{if } x_n = 3k+1,\ k > 0 \text{ an integer}, \\ -3k+2 & \text{if } x_n = -3k-1,\ k > 0 \text{ an integer}, \\ -2^{-k-1} & \text{if } x_n = -2^{-k},\ k \geq 0 \text{ an integer}, \\ 2^{-k} & \text{if } x_n = 2^{-k} \text{ for an integer } k \geq 0 \text{ and (4.2) holds}, \\ -2^{-k} & \text{if } x_n = 2^{-2k} \text{ for an integer } k \geq 0 \text{ and (4.2) does not hold}, \\ -2^{-k} & \text{if } x_n = 2^{-2k-1} \text{ for an integer } k \geq 0 \text{ and (4.2) does not hold}, \\ c & \text{if } x_n = a, \\ d & \text{if } x_n = b. \end{cases}$$

Then $\{x \in X : d(x, 0) \leq 1\}$ is not compact, where 0 denotes the sequence which is constant 0, and it is easy to see that f is uniformly continuous. We leave out the verification that f is uniformly generalized 1-contractive. We have the following.

1. For $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ with $x_0 = 0$, $y_0 = 0$ and with $x_n = 1$ and $y_n = 0$ for all $n \geq 1$ we have
$$d(f(x), f(y)) < d(x, f(x)),$$
but
$$d(f(x), f(y)) \geq d(x, y), d(y, f(y)), d(x, f(y)), d(y, f(x)).$$
2. For $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ with $x_0 = 0$, $y_0 = 0$ and with $x_n = 4$ and $y_n = 7$ for all $n \geq 1$ we have
$$d(f(x), f(y)) < d(y, f(x)),$$
but
$$d(f(x), f(y)) \geq d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)).$$

3. For $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ with $x_0 = a$, $y_0 = b$ and with $x_n = 0$ and $y_n = 0$ for all $n \geq 1$ we have

$$d(f(x), f(y)) < d(x, y),$$

but

$$d(f(x), f(y)) \geq d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x)).$$

Furthermore, f does not satisfy the condition (24) from [157], i.e. there does not exist $0 \leq h < 1$ such that

$$d(f(x), f(y)) \leq h \cdot \operatorname{diam}\{x, y, f(x), f(y)\}$$

holds for all $x, y \in X$. For given $0 \leq h < 1$ we can let $m \in \mathbb{N}$ and consider $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ with $x_0 = 0$, $y_0 = 0$ and with $x_n = 2^{-2m}$ and $y_n = 0$ for all $n \geq 1$. Then $d(f(x), f(y)) = 2^{-m}$ and

$$\operatorname{diam}\{x, y, f(x), f(y)\} = 2^{-m} + 2^{-2m}.$$

So for $m \in \mathbb{N}$ large enough we have

$$d(f(x), f(y)) > h \cdot \operatorname{diam}\{x, y, f(x), f(y)\}.$$

And since the closure of $f(X)$ is not compact it follows that f is not a compact map and thus that the strengthened version of Theorem 1.30 proved by Rhoades in [158], where the map is assumed to be compact instead of the space, does not apply. (We note that this means that also Theorem 1.32 does not apply.) Finally, since the space is unbounded also the other variant of Theorem 1.30 proved in [158], where the mapping is assumed to be condensing and the space bounded and complete, does not apply.

Some comments on the Cauchy rate

We note that contrary to the case where f is contractive and we are given a modulus of uniform contractivity (see [55]), we cannot in Theorem 4.6 replace the bound b on $(x_n)_{n\in\mathbb{N}}$ by a bound on $d(x_0, x_1)$. Even if we have a b which for all $x \in X$ bounds $d(x, f(x))$, we are not guaranteed to have a fixed point. Take for instance $X = \mathbb{R}$, $p = 1$ and $f(x) := x + 1$. Then the identity is a modulus of uniform continuity for f, and the function $\eta : (0, \infty) \to (0, \infty)$ defined by $\eta(\varepsilon) := 1/2$ is a modulus of uniform generalized 1-contractivity for f. Now $d(x, f(x))$ is bounded by 1, but the function has no fixed point, and no Picard iteration is a Cauchy sequence. It is also easy to see that given a uniformly continuous and uniformly generalized p-contractive f and bounded iteration sequences, we cannot in general construct a common Cauchy rate involving only p and the moduli of uniform continuity and uniform generalized p-contractivity. Consider e.g. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := \frac{x}{2}$.

Furthermore, as the following example shows, we cannot do without the modulus of uniform generalized p-contractivity. Let $X := \mathbb{N} \setminus \{0\}$ and define a metric on X by

$$d(i,j) = 1 + \frac{1}{i \cdot j}$$

for $i \neq j$. Let $f : X \to X$ be defined by $f(i) := i + 1$. Then (X, d) is bounded (complete, separable) and f is uniformly continuous and generalized 1-contractive, but no Picard iteration sequence is Cauchy. This example is taken from [81], where it is used to show that a function satisfying condition (25) need not have a fixed point. Notice that f in this case is not uniformly generalized 1-contractive. Now consider uniformly continuous and uniformly generalized p-contractive functions with the same modulus of uniform continuity, and bounded Picard iteration sequences $(x_n)_{n\in\mathbb{N}}$ with a common bound. In this case we cannot in general construct a common Cauchy rate for all the $(x_n)_{n\in\mathbb{N}}$ involving only p, the bound b, and the modulus of uniform continuity ω, as shown by the following example. Define for each $k \in \mathbb{N}\setminus\{0\}$ a metric space (X_k, d_k) by letting $X_k := \{n \in \mathbb{N} : 1 \leq n \leq k\}$ and by letting

$$d_k(i,j) := 1 + \frac{1}{i \cdot j}$$

for $i \neq j$. Let $f_k : X_k \to X_k$ be defined by

$$f_k(i) := \begin{cases} i+1 & \text{for } i < k, \\ k & \text{for } i = k. \end{cases}$$

It is easy to see that all the mappings f_k are uniformly generalized 1-contractive. And for all k we have the same bound b on $(f_k^n(1))_{n\in\mathbb{N}}$, and we can moreover find a modulus of uniform continuity which is the same for all f_k. But there exists no common Cauchy rate for all the sequences $(f_k^n(1))_{n\in\mathbb{N}}$.

Also, as we show in the following proposition, the modulus of uniform continuity contributes in an essential way to the Cauchy rate.

Proposition 4.15. *There exists a bounded metric space (X, d), a family of uniformly continuous functions $f_i : X \to X$, $i \in \mathbb{N}$, and an $\eta : (0,\infty) \to (0,\infty)$ which is a modulus of uniform generalized 1-contractivity for all the f_i, such that for some $x_0 \in X$ the Picard iterations with starting point x_0 do not have a common Cauchy rate.*

Proof. Consider $X := \{(\frac{1}{2})^n : n \geq 0\} \bigcup \{-(\frac{1}{2})^n : n \geq 0\}$ with the natural metric, and define $f_i : X \to X$ by

$$f_i(x) := \begin{cases} -(\frac{1}{2})^{n+1} & \text{if } x = -(\frac{1}{2})^n, \\ (\frac{1}{2})^{n+1} & \text{if } x = (\frac{1}{2})^n \text{ and } n \neq i, \\ -1 & \text{if } x = (\frac{1}{2})^n \text{ and } n = i. \end{cases}$$

Then each f_i is uniformly continuous. And $\eta : (0,\infty) \to (0,\infty)$ defined by $\eta(\varepsilon) := \frac{\varepsilon}{2}$ is a modulus of uniform generalized 1-contractivity for each f_i. To see this, we fix i and consider different cases. If $x, y \in X$ are such that $d(x,y) > \varepsilon$, and if $x \neq (\frac{1}{2})^i$ and $y \neq (\frac{1}{2})^i$, then

$$d(f_i(x), f_i(y)) = \frac{d(x,y)}{2}.$$

Therefore

$$\operatorname{diam}\{x, y, f_i(x), f_i(y)\} - d(f_i(x), f_i(y)) > \frac{\varepsilon}{2}.$$

If $x, y \in X$ are such that $d(x,y) > \varepsilon$, and if $x = (\frac{1}{2})^i$, then we have one of the following.

1. If $y = -(\frac{1}{2})^n$, then

$$\begin{aligned} d(f_i(x), x) - d(f_i(x), f_i(y)) &= \left(\frac{1}{2}\right)^i + \left(\frac{1}{2}\right)^{n+1} \\ &> \left(\frac{1}{2}\right)^{i+1} + \left(\frac{1}{2}\right)^{n+1} \\ &> \frac{\varepsilon}{2}. \end{aligned}$$

2. If $y = (\frac{1}{2})^n$ and $n < i$, then

$$d(f_i(x), y) - d(f_i(x), f_i(y)) = \left(\frac{1}{2}\right)^{n+1} > \frac{(\frac{1}{2})^n - (\frac{1}{2})^i}{2} > \frac{\varepsilon}{2}.$$

3. If $y = (\frac{1}{2})^n$ and $n > i$, then

$$\begin{aligned} d(f_i(x), x) - d(f_i(x), f_i(y)) &= \left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^{n+1} \\ &> \left(\frac{1}{2}\right)^{i+1} - \left(\frac{1}{2}\right)^{n+1} \\ &> \frac{\varepsilon}{2}. \end{aligned}$$

So in all cases we have

$$\operatorname{diam}\{x, y, f_i(x), f_i(y)\} - d(f_i(x), f_i(y)) > \frac{\varepsilon}{2},$$

and η is a modulus of uniform generalized 1-contractivity. Let $x_0 := 1$. Then there does not exist a Cauchy rate valid for all the sequences $(f_i^n(x_0))_{n \in \mathbb{N}}$.

□

A modulus of uniqueness

As the following lemma shows, if $f : X \to X$ is a mapping with a modulus of uniform generalized p-contractivity, then f^p has what has been called a modulus of uniqueness. This notion was defined in full generality by U. Kohlenbach in [89].

Lemma 4.16. *Let (X, d) be a metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity. Define $\psi : (0,\infty) \to (0,\infty)$ by $\psi(\varepsilon) := \eta(\varepsilon)/2$. Then for all $\varepsilon \in (0,\infty)$ and for all $x, y \in X$, if*

$$d(x, f^p(x)) \leq \psi(\varepsilon)$$

and

$$d(y, f^p(y)) \leq \psi(\varepsilon),$$

then $d(x, y) \leq \varepsilon$.

Proof. Since η is a modulus of uniform generalized p-contractivity, it follows that if $d(x, y) > \varepsilon$ then we have one of the following:

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(x, y), \tag{4.3}$$
$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(x), y), \tag{4.4}$$
$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(y), x), \tag{4.5}$$
$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(x), x), \tag{4.6}$$
$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(y), y). \tag{4.7}$$

We show that if

$$d(x, f^p(x)) \leq \eta(\varepsilon)/2$$

and

$$d(y, f^p(y)) \leq \eta(\varepsilon)/2,$$

then $d(x, y) \leq \varepsilon$. So let $d(x, f^p(x)) \leq \eta(\varepsilon)/2$ and $d(y, f^p(y)) \leq \eta(\varepsilon)/2$. Then it is obvious that (4.6) and (4.7) do not hold. Furthermore, we have

$$d(x, y) \leq d(f^p(x), f^p(y)) + d(f^p(x), x) + d(f^p(y), y) \leq d(f^p(x), f^p(y)) + \eta(\varepsilon),$$

so (4.3) does not hold. In the same way it follows by the triangle inequality that (4.4) and (4.5) do not hold. It follows that we have $d(x, y) \leq \varepsilon$.

□

Corollary 4.17. *Let (X, d) be a nonempty metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity. If the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ (which are not necessarily Picard iteration sequences) satisfy*

$$\forall \varepsilon > 0 \exists n \forall m \geq n (d(x_m, f^p(x_m)) < \varepsilon) \tag{4.8}$$

and

$$\forall \varepsilon > 0 \exists n \forall m \geq n (d(y_m, f^p(y_m)) < \varepsilon), \tag{4.9}$$

then the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges to 0, and in addition the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are in fact Cauchy sequences.

Proof. Suppose the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ satisfy (4.8) and (4.9). Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that for all $m \geq n$ we have

$$d(x_m, f^p(x_m)) < \eta(\varepsilon)/2.$$

Let $m_1, m_2 \geq n$. Then $d(x_{m_1}, f^p(x_{m_1})) < \eta(\varepsilon)/2$ and $d(x_{m_2}, f^p(x_{m_2})) < \eta(\varepsilon)/2$. And so by Lemma 4.16 it follows that $d(x_{m_1}, x_{m_2}) \leq \varepsilon$. Thus we have that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. In the same way it follows that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $n' \in \mathbb{N}$ be such that for all $m \geq n'$ we have

$$d(y_m, f^p(y_m)) < \eta(\varepsilon)/2.$$

Then for $m \geq \max\{n, n'\}$ we have $d(x_m, f^p(x_m)) < \eta(\varepsilon)/2$ and $d(y_m, f^p(y_m)) < \eta(\varepsilon)/2$. So by Lemma 4.16 it follows that $d(x_m, y_m) \leq \varepsilon$. Hence the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges to 0.

□

4.3.2 One iteration sequence bounded implies all iteration sequences bounded

We now prove that if the iteration sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded for one $x_0 \in X$, then it is bounded for any $x_0 \in X$.

Theorem 4.18. *Let (X, d) be a nonempty metric space, let $p \in \mathbb{N}$, and let $f : X \to X$ be uniformly generalized p-contractive and uniformly continuous. For $x_0 \in X$ define the iteration sequence $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} := f(x_n)$. Suppose for some $x_0 \in X$ the iteration sequence is bounded. Then for every choice of $x_0 \in X$ the Picard iteration sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, and in fact Cauchy. Also, for all $x_0, y_0 \in X$ we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. If (X, d) is complete all Picard iteration sequences converge to the unique fixed point of f.*

Proof. We prove first the special case where $p = 1$ and the space is complete. We know from Lemma 4.12 that f has a modulus of modified uniform generalized 1-contractivity. Call this modulus ρ. Likewise from Theorem 4.6 we know that f has a unique fixed point z. Let now $x_0 \in X$ be arbitrary. We want to prove that $(x_n)_{n\in\mathbb{N}}$ is bounded. Let $n > 0$ and consider $\operatorname{diam}\{x_0, \ldots, x_n\}$. Assume $x_n \neq z$, for else $(x_n)_{n\in\mathbb{N}}$ is bounded. Of course $x_n \neq z$ also implies $z \neq x_i$ for $i < n$. Similarly, we can assume that $x_i \neq x_j$ for any $i \neq j$, for otherwise $x_i = z$. Thus

$$d(x_i, x_j) > 0$$

for any $0 \leq i, j \leq n$ with $i \neq j$. For some $0 \leq i \leq n$ we have

$$\operatorname{diam}\{x_0, \ldots, x_n\} = d(x_0, x_i),$$

for if we for $0 < i, j \leq n$ had $\operatorname{diam}\{x_0, \ldots, x_n\} = d(x_i, x_j)$, then we would have

$$\operatorname{diam}\{x_i, x_j, x_{i-1}, x_{j-1}\} > d(x_i, x_j) = \operatorname{diam}\{x_0, \ldots, x_n\}.$$

In the same way, for $i > 0$ we have

$$\operatorname{diam}\{z, x_0, \ldots, x_n\} \neq d(z, x_i),$$

for we have for such i

$$d(z, x_i) < \max\{d(z, x_{i-1}), d(x_i, x_{i-1})\}.$$

Assume

$$\operatorname{diam}\{z, x_0, \ldots, x_n, x_{n+1}\} > \operatorname{diam}\{z, x_0, \ldots, x_n\}.$$

By the above we have

$$\operatorname{diam}\{z, x_0, \ldots, x_{n+1}\} = d(x_0, x_{n+1}).$$

Assume $d(x_0, x_{n+1}) > 2d(x_0, z)$. Since

$$d(z, x_{n+1}) + d(x_0, z) \geq d(x_0, x_{n+1}),$$

we have

$$d(z, x_{n+1}) > d(x_0, z).$$

Let $\varepsilon > 0$ satisfy $\varepsilon \leq d(x_0, z)$. Since ρ is a modulus of modified uniform generalized 1-contractivity, we have either

$$d(z, x_n) > d(z, x_{n+1}) + \rho(\varepsilon)$$

or

$$d(x_{n+1}, x_n) > d(z, x_{n+1}) + \rho(\varepsilon).$$

Let $m_0 := n + 1$ and $m'_0 := -1$. We will let x_{-1} denote z. For $0 \leq i < n$ we define m_{i+1} and m'_{i+1} inductively such that the following holds.

1. If $m'_i = -1$, then $m'_{i+1} \in \{m_i, m_i - 1, m'_i\}$ and $m_{i+1} \in \{m_i, m_i - 1\}$ such that
$$d(x_{m_{i+1}}, x_{m'_{i+1}}) = \operatorname{diam}\{x_{m_i}, x_{m_i-1}, z\}.$$
2. If $m'_i \neq -1$, then $m'_{i+1}, m_{i+1} \in \{m_i, m_i - 1, m'_i, m'_i - 1\}$ such that
$$d(x_{m_{i+1}}, x_{m'_{i+1}}) = \operatorname{diam}\{x_{m_i}, x_{m_i-1}, x_{m'_i}, x_{m'_i-1}\}.$$

Then since $d(x_{m_0}, x_{m'_0}) > \varepsilon$ we can prove by induction on i that

$$d(x_{m_{i+1}}, x_{m'_{i+1}}) > d(x_{m_i}, x_{m'_i}) + \rho(\varepsilon)$$

and $d(x_{m_i}, x_{m'_i}) > \varepsilon$ for all $0 \leq i < n$. And so

$$d(x_{m_i}, x_{m'_i}) > d(x_{m_0}, x_{m'_0}) + i\rho(\varepsilon)$$

for $0 < i < n$. Hence, if n satisfies $n\rho(\varepsilon) > d(x_0, z)$ then we get

$$d(x_{m_n}, x_{m'_n}) > d(x_{n+1}, z) + d(x_0, z) \geq d(x_{n+1}, x_0) = \operatorname{diam}\{z, x_0, \ldots, x_{n+1}\}.$$

Specifically, we may take

$$n := \left\lceil \frac{b}{\rho(\varepsilon)} \right\rceil,$$

for any $b > d(x_0, z)$. But we have

$$d(x_{m_n}, x_{m'_n}) \leq \operatorname{diam}\{z, x_0, \ldots, x_{n+1}\},$$

and hence for large enough n we have $d(x_0, x_{n+1}) \leq 2d(x_0, z)$ or

$$\operatorname{diam}\{z, x_0, \ldots, x_n, x_{n+1}\} \leq \operatorname{diam}\{z, x_0, \ldots, x_n\}.$$

Thus for the special case where the space is complete and $p = 1$ we have proved that if one Picard iteration sequence is bounded, then any Picard iteration sequence is bounded. And so in this case it follows by Theorem 4.6 that all Picard iteration sequences converge to the unique fixed point z.

Now let $p \neq 1$, but assume still that the space is complete. Then by the above f^p has a unique fixed point z' and moreover for any $x_0 \in X$ we have

$$\lim_{n \to \infty} f^{np}(x_0) = z'.$$

Let $x_0 \in X$ and let $N \in \mathbb{N}$ be such that for $m \geq N$ we have

$$d(z', f^{mp}(x_0)) < \min\{1, \omega(1), \omega^2(1), \ldots, \omega^p(1)\}.$$

Then $d(z', f^k(x_0)) < 1$ for $k \geq Np$, and so the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded. Thus $(f^n(x_0))_{n\in\mathbb{N}}$ converges to the unique fixed point $z \in X$ of f (and hence it follows that $z' = z$).

Next suppose the space X is not complete. We consider the completion of X and the canonical extension of the uniformly continuous function f. Then the extension of f still has moduli of uniform continuity and uniform generalized p-contractivity, and the bounded Picard iteration sequence we presupposed stays the same. So by the above every Picard iteration sequence in the completion of X converges to the unique fixed point z of f. And so for all $x_0, y_0 \in X$ we have that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy and in particular bounded, and furthermore that $\lim_{n\to\infty} d(x_n, y_n) = 0$. This ends the proof.

□

4.3.3 Rates of convergence not dependent on a bound on the iteration sequence

In this section we will use the proof of Theorem 4.18 to construct rates of convergence or Cauchy rates for uniformly continuous uniformly generalized p-contractive mappings which do not depend on a bound on the iteration sequence, but rather on bounds of a more local nature. The three first corollaries will concern (complete) metric spaces in general, but the rates of convergence given will depend essentially on the mapping $f : X \to X$. The last three corollaries, on the other hand, will only concern spaces with more structure. But the rates of convergence constructed in these cases will have uniformity properties with respect to the mapping $f : X \to X$ in the sense that they only depend on certain bounds and certain moduli for the mapping and the space.

Corollary 4.19. *Let (X, d) be a nonempty complete metric space, and let $p \in \mathbb{N}$, $p \geq 1$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let z be the unique fixed point of f. Let $x_0 \in X$ and for $0 \leq i < p$ let $b_i > 0$ and $\delta_i > 0$ satisfy $\delta_i \leq d(x_i, z) < b_i$. Let $\varepsilon > 0$. Let Φ be as in Theorem 4.6 and let*

$$N_i = \left\lceil \frac{b_i}{\rho(\delta_i)} \right\rceil,$$

where $\rho : (0, \infty) \to (0, \infty)$ is defined by

$$\rho(\gamma) = \min \left\{ \eta(\gamma), \frac{\gamma}{2}, \eta(1/2 \cdot \omega^p(\gamma/2)) \right\}.$$

Let

$$M_i = \max \left\{ 2b_i, \operatorname{diam} \{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{N_i p}(x_i)\} \right\},$$

and let

$$M = \max\{M_0, \ldots, M_{p-1}\}.$$

Then for all $m, n \in \mathbb{N}$ we have that

$$m, n \geq \Phi(p, \omega, \eta, 2M, \varepsilon)$$

gives

$$d(x_n, x_m) \leq \varepsilon$$

and so

$$d(x_n, z) \leq \varepsilon.$$

Proof. We first note that since f^p has moduli η and ω^p of uniform generalized 1-contractivity and uniform continuity, we have by Lemma 4.12 that ρ is a modulus of modified uniform generalized 1-contractivity for f^p. Then by the proof of Theorem 4.18 we can infer that for $0 \leq i < p$ the iteration sequence $(f^{pn}(x_i))_{n \in \mathbb{N}}$ is bounded by M_i. Namely, we proved that

$$\operatorname{diam} \{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{(n+1)p}(x_i)\} = \operatorname{diam} \{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{np}(x_i)\}$$

if

$$\operatorname{diam} \{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{(n+1)p}(x_i)\} > 2d(x_i, z)$$

and $n \geq N_i$. Thus $(f^n(x_0))_{n \in \mathbb{N}}$ is bounded by $2M$. Now the claim follows by Theorem 4.6.

□

Corollary 4.20. *Let (X, d) be a nonempty complete metric space, and let $p \in \mathbb{N}$, $p \geq 1$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let z be the unique fixed point of f. Let $x_0 \in X$ and let $\delta, b > 0$ be such that $\delta \leq d(x_0, z) < b$. Let $\varepsilon > 0$. Let Φ be as in Theorem 4.6. Let*

$$N = \left\lceil \frac{b}{\rho(\delta)} \right\rceil,$$

where ρ is as in Corollary 4.19. Let

$$M = \max\{2b, \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\}\},$$

and let

$$K = \Phi(1, \omega^p, \eta, M, 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}).$$

Let

$$M' = \operatorname{diam}\{x_n : 0 \leq n \leq Kp\} + 2.$$

Then for all $m, n \in \mathbb{N}$ we have that

$$m, n \geq \Phi(p, \omega, \eta, M', \varepsilon)$$

gives

$$d(x_n, x_m) \leq \varepsilon$$

and so

$$d(x_n, z) \leq \varepsilon.$$

Proof. As in the proof of Corollary 4.19 we note that f^p has moduli η of uniform generalized 1-contractivity, ω^p of uniform continuity and ρ of modified uniform generalized 1-contractivity. Furthermore, as in the proof of Corollary 4.19 we get that $(f^{pn}(x_0))_{n\in\mathbb{N}}$ is bounded by M. Then for $m, n \geq K$ we have

$$d(x_{mp}, x_{np}) \leq 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}$$

and

$$d(z, x_{np}) \leq 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}.$$

Since ω is a modulus of uniform continuity for f we have in particular that

$$\begin{array}{rcl} d(x_{np}, z) & < & 1, \\ d(x_{np+1}, z) & < & 1, \\ d(x_{np+2}, z) & < & 1, \\ & \vdots & \\ d(x_{np+(p-1)}, z) & < & 1, \end{array}$$

for $n \geq K$. And so for $n \geq Kp$ we have in fact $d(x_n, z) < 1$. Let now m, n be nonnegative integers. We distinguish three cases:

1. If $m, n \leq Kp$, then
$$d(x_n, x_m) \leq \operatorname{diam}\{x_k : 0 \leq k \leq Kp\} < M'.$$
2. If $m, n \geq Kp$, then
$$d(x_n, x_m) \leq d(x_n, z) + d(x_m, z) < 2 \leq M'.$$
3. If $m < Kp$ and $n > Kp$, then
$$d(x_n, x_m) \leq d(x_m, x_{Kp}) + d(x_n, x_{Kp}) < \operatorname{diam}\{x_k : 0 \leq k \leq Kp\} + 2 = M'.$$

It follows that M' is a bound on $(x_n)_{n\in\mathbb{N}}$. Now Theorem 4.6 gives the conclusion. □

In the previous two corollaries we gave rates of convergence which were dependent on strictly positive upper and lower bounds on $d(z, x_i)$ for some i (and on the diameter of a set consisting of some of the initial members of the Picard iteration sequence). We will now improve this as detailed in the following corollary. Essentially we will here instead require a lower bound $\delta > 0$ on the initial displacement $d(x_0, x_1)$ and an upper bound $b > 0$ on the distance to a point which is an approximate fixed point. Additionally we require an upper bound on the diameter of a set consisting of some of the initial members of the Picard iteration sequence.

Corollary 4.21. *Let (X, d) be a nonempty complete metric space, and let $p \in \mathbb{N}$, $p \geq 1$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let $x_0 \in X$ and let $\delta > 0$ be such that $\delta \leq d(x_0, x_1)$. Let $b, c, \varepsilon > 0$. Let Φ be as in Theorem 4.6. Assume that there is $y \in X$ such that $d(x_0, y) < b$ and $d(x_1, y) < b$, and such that either*
$$d(y, f^p(y)) < \frac{\eta(c)}{2}$$
or $(f^n(y))_{n\in\mathbb{N}}$ is bounded by c. Let
$$N = \left\lceil \frac{b + c}{\rho(\delta/2)} \right\rceil,$$
where ρ is as in Corollary 4.19. Let
$$M_0 = \max\Big\{2(b + c), \operatorname{diam}\big\{x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\big\} + b + c\Big\},$$
$$M_1 = \max\Big\{2(b + c), \operatorname{diam}\big\{x_1, f^p(x_1), f^{2p}(x_1), \ldots, f^{Np}(x_1)\big\} + b + c\Big\},$$
and let
$$K = \Phi\Big(1, \omega^p, \eta, \max\big\{M_0, M_1\big\}, 1/2 \cdot \min\big\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\big\}\Big).$$

Let

$$M' = \text{diam}\left\{x_n : 0 \leq n \leq Kp + 1\right\} + 2.$$

Then for all $m, n \in \mathbb{N}$ *we have that*

$$m, n \geq \Phi(p, \omega, \eta, M', \varepsilon)$$

gives

$$d(x_n, x_m) \leq \varepsilon.$$

And so

$$d(x_n, z) \leq \varepsilon,$$

where z *is the unique fixed point.*

Proof. By Lemma 4.16 it follows that $d(y, z) \leq c$, where z is the unique fixed point. Thus by the triangle inequality

$$d(x_0, z) < b + c$$

and

$$d(x_1, z) < b + c.$$

Furthermore, either $\delta/2 \leq d(x_0, z)$ or $\delta/2 \leq d(x_1, z)$. As in the proof of Corollary 4.19 we get that either $(f^{pn}(x_0))_{n\in\mathbb{N}}$ is bounded by M_0 or $(f^{pn}(x_1))_{n\in\mathbb{N}}$ is bounded by M_1. So we have that

$$d(f^n(x_0), z) < 1$$

for all $n \geq Kp$ or

$$d(f^n(x_1), z) < 1$$

for all $n \geq Kp$, and so we have $d(f^n(x_0), z) < 1$ for all $n \geq Kp + 1$. Hence, M' is a bound on $(x_n)_{n\in\mathbb{N}}$, and the conclusion follows by Theorem 4.6.

□

The Cauchy rates appearing in the last three corollaries depend heavily on f. If the space satisfies a further structural condition we may find Cauchy rates with uniformity properties with respect to f. This will include for instance spaces of hyperbolic type in the sense of [62], as well as hyperbolic spaces in the sense of [155] and hyperbolic spaces in the sense of [99], and therefore e.g., normed linear spaces, Hadamard manifolds and CAT(0)-spaces.

Definition 4.22. Let (X, d) be a metric space. Let $\varepsilon > 0$ and $x, y \in X$. We say that x is *ε-step-equivalent to* y if there exist points $x_0 = x, x_1, \ldots, x_n = y$, belonging to X, with

$$d(x_i, x_{i+1}) \leq \varepsilon$$

for $i < n$. This defines for each $\varepsilon > 0$ an equivalence relation on X. We call the equivalence classes *ε-step-territories*.

The notions in Definition 4.22 are taken from [144]. The condition on a metric space which in the terminology of Definition 4.22 amounts to requiring that the space should be an ε-step-territory was already treated by M. Edelstein. We will employ a uniform version of ε-step-territories.

Definition 4.23. Let (X, d) be a metric space, and let $\varepsilon > 0$. A subset T_ε of X is a *uniform ε-step-territory* if there exists $\alpha_\varepsilon : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in T_\varepsilon$ and all $n \in \mathbb{N}$, if $d(x, y) < n\varepsilon$, then there exist $x_0 = x, x_1, \ldots, x_{\alpha_\varepsilon(n)} = y \in T_\varepsilon$ with

$$d(x_i, x_{i+1}) < \varepsilon$$

for $i < \alpha_\varepsilon(n)$.

Definition 4.24. Let (X, d) be a metric space. A subset T of X is called a *territory* if it is an ε-step-territory for each $\varepsilon > 0$. A subset T of X is called a *uniform territory* if it is a uniform ε-step-territory for each $\varepsilon > 0$.

Definition 4.25. Let (X, d) be a metric space, and let T be a subset of X. A function $\alpha : (0, \infty) \times \mathbb{N} \to \mathbb{N}$ is called a *uniform territory modulus* for T if for each $\varepsilon > 0$ and for all $x, y \in T$ and $n \in \mathbb{N}$ such that

$$d(x, y) < n\varepsilon,$$

there exist $x_0 = x, x_1, \ldots, x_{\alpha(\varepsilon,n)} = y \in T$ with $d(x_i, x_{i+1}) < \varepsilon$ for $i < \alpha(\varepsilon, n)$.

We note that if T has a uniform territory modulus then T is a uniform territory.

Remark 4.26. If (X, d) is a geodesic space, then X is a uniform territory with a uniform territory modulus α given by $\alpha(\varepsilon, n) = n$.

Corollary 4.27. *Let (X, d) be a nonempty complete metric space which is a uniform territory with a uniform territory modulus α. Let $p \in \mathbb{N}$, $p \geq 1$, and let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose that for some starting point the Picard iteration sequence is bounded. Let z be the unique fixed point of f. Let $x_0 \in X$ and let $b > 0$ satisfy $d(x_0, z) < b$. Then for all $n \in \mathbb{N}$,*

$$d(z, f^n(x_0)) < K^{p-1}(b + K^{Np}(b)),$$

where $K : (0, \infty) \to (0, \infty)$ is defined by

$$K(\gamma) := \max\left\{\alpha\left(\omega(1), \left\lceil \frac{\gamma}{\omega(1)} \right\rceil\right), \gamma\right\},$$

$$N := \left\lceil \frac{b}{\rho(\delta)} \right\rceil,$$

$\rho : (0, \infty) \to (0, \infty)$ is defined by

$$\rho(\gamma) := \min\left\{\eta(\gamma), \frac{\gamma}{2}, \eta\left(\frac{1}{2}\omega^p\left(\frac{\gamma}{2}\right)\right)\right\},$$

and $\delta := \min\{b, \omega(b)\}$. *Let* $\varepsilon > 0$ *and let* Φ *be as in Theorem* 4.6. *Then*

$$m, n \geq \Phi(p, \omega, \eta, 2K^{p-1}(b + K^{Np}(b)), \varepsilon)$$

gives $d(x_n, x_m) \leq \varepsilon$.

Proof. Since f^p has moduli η and ω^p of uniform generalized 1-contractivity and uniform continuity, we have that ρ is a modulus of modified uniform generalized 1-contractivity for f^p. Now, if $d(x_0, z) < \delta$, then if we do not for all $n \in \mathbb{N}$ have

$$d(f^n(x_0), z) < \delta,$$

it follows from the definition of δ that we for some $m \in \mathbb{N}$ have

$$\delta \leq d(f^m(x_0), z) < b.$$

For if $f^m(x_0)$ is the first member of the sequence which is not an element of the set $\{x \in X : d(x, z) < \delta\}$, then $d(f^{m-1}(x_0), z) < \delta \leq \omega(b)$. So since ω is a modulus of uniform continuity for f we have $d(f^m(x_0), z) < b$. So in total

$$\delta \leq d(f^m(x_0), z) < b.$$

We can take $f^m(x_0)$ as the starting point x'_0 of a new Picard iteration sequence. If we can establish the bound on $d(f^n(x'_0), z)$ for this sequence, then it is also proved for our original sequence, since

$$d(f^i(x_0), z) < \delta < K^{p-1}(b + K^{Np}(b))$$

for $i < m$. The last inequality follows since $K(\gamma) \geq \gamma$ for $\gamma > 0$. Hence, we may assume

$$\delta \leq d(x_0, z) < b.$$

Then as in the proof of Corollary 4.19 we can infer that the iteration sequence $(f^{pn}(x_0))_{n\in\mathbb{N}}$ is bounded by

$$M := \max\{2d(x_0, z), \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\}\}.$$

From the proof of Theorem 4.18 it follows that if $z \neq f^{kp}(x_0)$ for all $0 \leq k \leq N$, then

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} = d(x_0, z)$$

or

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} = d(x_0, f^{ip}(x_0)),$$

for some $i \leq N$. And thus we in fact have

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} = d(x_0, z)$$

or

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} = d(x_0, f^{ip}(x_0)),$$

for some $i \leq N$, for if $z = f^{kp}(x_0)$ for some $0 \leq k \leq N$, then either $x_0 = z$ or else for $k = \min\{n \in \mathbb{N} : z = f^{np}(x_0)\}$ we have

$$\begin{aligned} \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} &= \\ \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{(k-1)p}(x_0)\} &= d(x_0, z) \end{aligned}$$

or

$$\begin{aligned} \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} &= \\ \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{(k-1)p}(x_0)\} &= d(x_0, f^{ip}(x_0)), \end{aligned}$$

for some $i \leq k-1$. Therefore

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} \leq d(x_0, z) + d(z, f^{ip}(x_0)), \quad (4.10)$$

for some $i \leq N$. Since

$$d(z, x_0) < \left\lceil \frac{b}{\omega(1)} \right\rceil \cdot \omega(1),$$

we have by definition of K and by the assumed property of the space that

$$d(z, f(x_0)) < K(b).$$

This follows since with

$$m := \alpha\left(\omega(1), \left\lceil \frac{b}{\omega(1)} \right\rceil\right),$$

we have that there exist $x'_0 = x_0, x'_1, \ldots, x'_m = z \in X$ with

$$d(x'_i, x'_{i+1}) < \omega(1)$$

for $i < m$. And so $d(f(x_0), z) < K(b)$. Furthermore,

$$d(z, f(x_0)), d(z, f^2(x_0)) < K^2(b),$$

since $K(\gamma) \geq \gamma$. And in general,

$$d(z, f(x_0)), d(z, f^2(x_0)), \ldots, d(z, f^k(x_0)) < K^k(b).$$

So by (4.10) we have

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} < b + K^{Np}(b).$$

Thus $M < b + K^{Np}(b)$. Hence for any $n \in \mathbb{N}$ we have

$$d(z, f^{np}(x_0)) < b + K^{Np}(b),$$

and so

$$d(z, f^{np}(x_0)), d(z, f^{np+1}(x_0)) < K(b + K^{Np}(b)).$$

For all $n \in \mathbb{N}$ we have

$$d(z, f^{np}(x_0)), d(z, f^{np+1}(x_0)), \ldots, d(z, f^{np+p-1}(x_0)) < K^{p-1}(b + K^{Np}(b)).$$

That is, for all $n \in \mathbb{N}$ we have

$$d(z, f^n(x_0)) < K^{p-1}(b + K^{Np}(b)).$$

Hence, $2K^{p-1}(b + K^{Np}(b))$ is a bound on $(x_n)_{n\in\mathbb{N}}$, and the conclusion follows by Theorem 4.6.

□

Notice that the Cauchy rate in the preceeding corollary only depends on p, ω, η, α and b. Given these the rate is uniform in the space, the mapping and the starting point.

We can treat the situation where the space is not complete as follows. We consider a metric space (X, d) and a function $f : X \to X$ with moduli ω and η of uniform continuity and uniform generalized p-contractivity. We denote by f also the canonical extension of f to the completion of X. We can then define e.g. $\omega' : (0, \infty) \to (0, \infty)$ by $\omega'(\varepsilon) := \omega(\varepsilon/2)$ and $\eta' : (0, \infty) \to (0, \infty)$ by $\eta'(\varepsilon) := \eta(\varepsilon)/2$. It is easy to see that ω' and η' are moduli of uniform continuity and uniform generalized p-contractivity for f considered as a function on the completion of X. We can thus find Cauchy rates for $(x_n)_{n\in\mathbb{N}}$ with $x_0 \in X$ by considering the completion and the suitably modified moduli.

We will now improve Corollary 4.27 similarly to the way Corollary 4.21 is an improvement of Corollary 4.20, and at the same time spell out the details for what happens in this case when the space is not complete. Notice that in the following corollary the Cauchy rate does not depend on an upper bound on the distance $d(x_0, z)$ between the starting point and the fixed point, but rather on an upper bound on the distance $d(x_0, y)$ between the starting point and a point y which is moved a sufficiently short distance by the mapping.

Corollary 4.28. *Let (X, d) be a nonempty metric space which is a uniform territory with a uniform territory modulus α. Let $p \in \mathbb{N}$, $p \geq 1$, and let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose that for some starting point the Picard iteration sequence is bounded. Let ω' and η' be defined as above, and let $x_0 \in X$. Let $b, c \in (0, \infty)$ be such that there is $y \in X$ with*

$$d(y, f^p(y)) < \frac{\eta'(c)}{2},$$

such that $d(x_0, y) < b$. Then $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded by

$$2K^{p-1}(b + c + K^{Np}(b + c)),$$

where $K : (0, \infty) \to (0, \infty)$ is defined by

$$K(\gamma) := \max\left\{\alpha\left(\frac{1}{2}\cdot\omega'(1), \left\lceil\frac{\gamma}{1/2\cdot\omega'(1)}\right\rceil\right), \gamma\right\},$$

$$N := \left\lceil \frac{b+c}{\rho'(\delta)} \right\rceil,$$

$\rho' : (0,\infty) \to (0,\infty)$ is defined by

$$\rho'(\gamma) := \min\left\{\eta'(\gamma), \frac{\gamma}{2}, \eta'\left(\frac{1}{2}\omega'^p\left(\frac{\gamma}{2}\right)\right)\right\},$$

and $\delta := \min\{b+c, \omega'(b+c)\}$. Let $\varepsilon > 0$ and let Φ be as in Theorem 4.6. Then

$$m, n \geq \Phi(p, \omega, \eta, 2K^{p-1}(b+c+K^{Np}(b+c)), \varepsilon)$$

gives $d(x_n, x_m) \leq \varepsilon$.

Proof. We consider the completion $(\widehat{X}, \widehat{d}\,)$ of (X, d) and the canonical extension of f, which we also denote by f. We have that ω' and η' are moduli of uniform continuity and uniform generalized p-contractivity for f. Now $(\widehat{X}, \widehat{d})$ satisfies the condition that for each $\varepsilon' > 0$ and for all $x, y \in \widehat{X}$ and $n \in \mathbb{N}$, if $\widehat{d}(x,y) < n\varepsilon'$, then there exist $x'_0 = x, x'_1, \ldots, x'_{\alpha(\varepsilon',n)} = y \in \widehat{X}$ with $\widehat{d}(x'_i, x'_{i+1}) < 2\varepsilon'$ for $i < \alpha(\varepsilon', n)$. Let z be the unique fixed point. By assumption we have

$$d(y, f^p(y)) < \eta'(c)/2,$$

and so by Lemma 4.16 we get that $\widehat{d}(y,z) \leq c$. And since we assume that $d(x_0, y) < b$ we get

$$\widehat{d}(x_0, z) < b + c.$$

Our new definition of K serves the same purpose as the version in Corollary 4.27, i.e. for $x \in \widehat{X}$ and $b' > 0$ with $\widehat{d}(x,z) < b'$, we get $\widehat{d}(f(x), z) < K(b')$. This follows since

$$\left\lceil \frac{b'}{1/2 \cdot \omega'(1)} \right\rceil \cdot 1/2 \cdot \omega'(1) > \widehat{d}(x,z),$$

so with

$$m := \alpha\left(1/2 \cdot \omega'(1), \left\lceil \frac{b'}{1/2 \cdot \omega'(1)} \right\rceil\right),$$

we have that there exist $x'_0 = x, x'_1, \ldots, x'_m = z \in \widehat{X}$ with

$$\widehat{d}(x'_i, x'_{i+1}) < 2 \cdot 1/2 \cdot \omega'(1)$$

for $i < m$. And so $\widehat{d}(f(x), z) < K(b')$. Now by identical reasoning as in Corollary 4.27 we get that for all $n \in \mathbb{N}$ we have

$$\widehat{d}(f^n(x_0), z) < K^{p-1}(b+c+K^{Np}(b+c)).$$

Thus

$$2K^{p-1}(b+c+K^{Np}(b+c))$$

is a bound on $(f^n(x_0))_{n\in\mathbb{N}}$ in $\widehat{X}$, and hence also in X. The conclusion follows by Theorem 4.6.

$\square$

Corollary 4.29. *Let (X, d) be a nonempty metric space which is a uniform territory with a uniform territory modulus α. Let $p \in \mathbb{N}$, $p \geq 1$, and let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose that for some starting point the Picard iteration sequence is bounded. Let $x_0, y_0 \in X$, and let $b, c \in (0, \infty)$ be such that $d(x_0, y_0) < b$ and such that c is a bound on $(f^n(y_0))_{n\in\mathbb{N}}$. Then $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded by*

$$2K^{p-1}(b + c + K^{Np}(b + c)),$$

where K, N, ρ' and δ are defined as in Corollary 4.28. Let $\varepsilon > 0$ and let Φ be as in Theorem 4.6. Then

$$m, n \geq \Phi(p, \omega, \eta, 2K^{p-1}(b + c + K^{Np}(b + c)), \varepsilon)$$

gives $d(x_n, x_m) \leq \varepsilon$.

Proof. We have in the completion $(\widehat{X}, \widehat{d})$ of (X, d) that $\widehat{d}(x_0, z) < b + c$, where z is the unique fixed point. Now the result follows as in Corollary 4.28.

□

Finally we include some remarks on applications of fixed point theorems for mappings satisfying contractive type conditions more general than e.g. the one due to Banach or the one due to Edelstein. Such contractive type conditions have been extensively studied as part of an attempt to conceptually understand the fixed point theory of selfmappings of abstract metric spaces, but they are often difficult to apply in other areas of mathematics.

It is a relevant point in this connection that Banach's original contraction mapping principle is so frequently used in analysis precisely because the contractive condition involved is so simple. The results in this chapter show that the classes of mappings studied here are asymptotic contractions in the sense of Kirk (when we restrict the treatment to bounded spaces), and indeed, the characterization of asymptotic contractions in the sense of Kirk on bounded spaces, which we gave in Chapter 3, shows that we in some sense have found conditions which are so general that they cover all classes of mappings on bounded metric spaces which we would consider "nice". A natural line of further research would be to investigate possible applications of the theorems concerning asymptotic contractions, and this might thus involve trying to find *less* general conditions on the mappings – that is, one can try to find conditions which are sufficient for a mapping to be an asymptotic contraction, but conditions which are more easy to check in various cases of interest, and which lend themselves easily to applications.

As an example of the kind of result one easily obtains from a fixed point theorem for mappings satisfying a general contractive condition we will consider how we can formulate a more general version of Picard's theorem for differential equations using Theorem 4.5. This will illustrate the fact that often, when one wishes to apply fixed point theorems for more general contractive conditions to obtain in turn more general versions of theorems outside of metric fixed point theory proper, the results are indeed more general, but also seemingly not always very practical.

Picard's theorem (for a proof, see e.g. [29]) tells us that given a bounded, continuous real-valued function $f : G \to \mathbb{R}$ defined on an open subset G of $\mathbb{R}^2$, if f satisfies a Lipschitz condition with respect to the second variable, i.e., if there exists $M \geq 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

holds for all $(x, y_1), (x, y_2) \in G$, then for any $(x_0, y_0) \in G$ the differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$ has a unique solution ϕ in some interval $I = [x_0 - \delta, x_0 + \delta]$. Here $\delta > 0$ is chosen such that $M\delta < 1$ and such that

$$\{(x, y) : |x - x_0| \leq \delta, |y - y_0| \leq K\delta\} \subseteq G,$$

where $K > 0$ is such that $|f(x, y)| \leq K$ for all $(x, y) \in G$. The proof involves considering the complete metric space (X, d) of all continuous functions

$$g : I \to [y_0 - K\delta, y_0 + K\delta],$$

with the metric d defined by $d(g, h) = \max_{t \in I} |g(t) - h(t)|$, and the mapping $T : X \to X$ defined by

$$(Tg)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt$$

for all $g \in X$ and $x \in I$, and then showing that T is a contraction. As a consequence we also get that $(T^n g)_{n \in \mathbb{N}}$ converges to the unique solution ϕ for any $g \in X$. A crucial step in the proof involves showing that $d(Tg, Th) \leq M\delta \cdot d(g, h)$ by showing that

$$|(Tg)(x) - (Th)(x)| \leq \left| \int_{x_0}^{x} [f(t, g(t)) - f(t, h(t))]\, dt \right| \leq M\delta \cdot d(g, h)$$

for all $g, h \in X$ and all $x \in I$. This follows since f is Lipschitzian with constant M with respect to the second variable. Now from Ćirić's theorem we can deduce that if we remove the condition that f is Lipschitzian with respect to the second variable (but still assume that f is continuous and bounded) then for $(x_0, y_0) \in G$ and initial condition $y(x_0) = y_0$ we can still conclude the existence of a unique solution ϕ in $I = [x_0 - \delta, x_0 + \delta]$, with $\delta > 0$ such that

$$\{(x, y) : |x - x_0| \leq \delta, |y - y_0| \leq K\delta\} \subseteq G,$$

if for some $M \geq 0$ with $M\delta < 1$ we have for all $x \in I$ and all $g, h \in X$ that *one* of the following conditions holds:

$$\left| \int_{x_0}^{x} [f(t, g(t)) - f(t, h(t))]\, dt \right| \leq M\delta \cdot \max_{t \in I} |g(t) - h(t)|,$$

$$\left| \int_{x_0}^{x} [f(t, g(t)) - f(t, h(t))]\, dt \right| \leq M\delta \cdot \max_{t \in I} \left| y_0 + \int_{x_0}^{t} f(u, h(u))\, du - h(t) \right|,$$

$$\left|\int_{x_0}^{x}[f(t,g(t)) - f(t,h(t))]\,dt\right| \le M\delta \cdot \max_{t\in I}\left|y_0 + \int_{x_0}^{t} f(u,g(u))\,du - g(t)\right|,$$

$$\left|\int_{x_0}^{x}[f(t,g(t)) - f(t,h(t))]\,dt\right| \le M\delta \cdot \max_{t\in I}\left|y_0 + \int_{x_0}^{t} f(u,g(u))\,du - h(t)\right|$$

or

$$\left|\int_{x_0}^{x}[f(t,g(t)) - f(t,h(t))]\,dt\right| \le M\delta \cdot \max_{t\in I}\left|y_0 + \int_{x_0}^{t} f(u,h(u))\,du - g(t)\right|.$$

We have also in this situation that $(T^n g)_{n\in\mathbb{N}}$ converges to the unique solution ϕ for any $g \in X$.

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Curriculum vitae

- Born December 23, 1980 in Oslo, Norway.
- 1998: Graduation from Stabekk Videregående Skole (High school) in Bærum, Akershus, Norway.
- 1998–2005: Study of mainly mathematics and philosophy at the University of Oslo, Norway.
- December 16, 2005: Master degree in mathematics ("Master i matematikk") from the University of Oslo. Thesis supervisor: Prof. Dr. Dag Normann.
- 2005–present: "Wissenschaftlicher Mitarbeiter" at Technische Universität Darmstadt, Germany. Working towards the degree Doctor rerum naturalium. Thesis supervisor: Prof. Dr. Ulrich Kohlenbach.