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Foldable Triangulations

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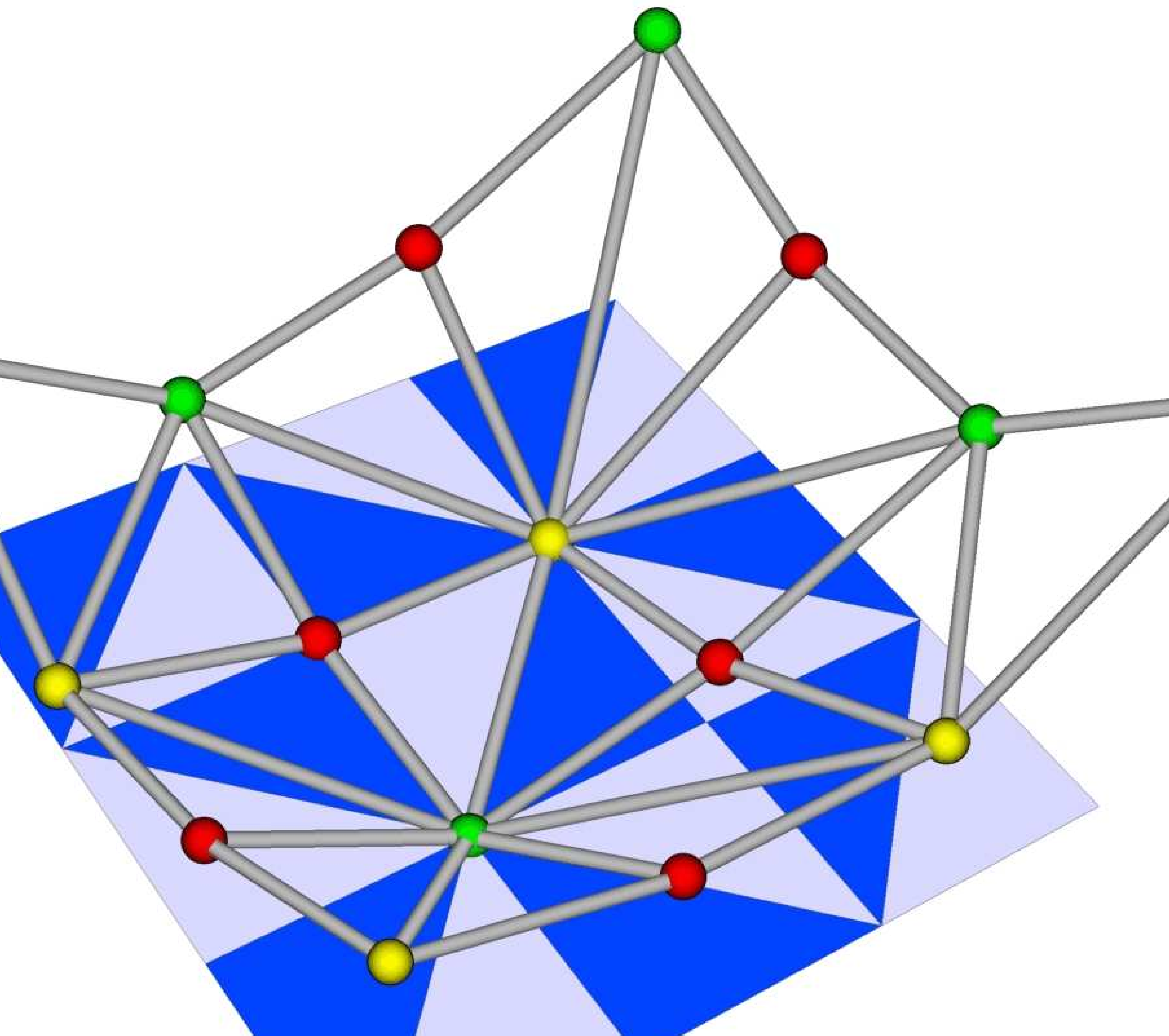

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Contents

Introduction

A simplicial complex of dimension d is *foldable* if it admits a non-degenerate simplicial map to the d-simplex. This is equivalent to the property that its 1-skeleton is colorable in the graph-theoretic sense with the minimally possible number of $d+1$ colors. We apply foldable triangulations to construct simplicial complexes with a specific *odd subcomplex*, the subcomplex defined by all co-dimension 2-faces with a non-bipartite link (and their proper faces). The odd subcomplex controls the behavior of the *unfoldings* introduced by Izmestiev & Joswig [36]. The unfoldings mirror the topological concept of a branched cover with the odd subcomplex as branching set. Hence we are interested in the topology of the odd subcomplex, but certain group theoretic aspects do matter as well. In particular the *partial unfolding* $\widehat{K}$ of a simplicial complex K proves to be the apt gadget for the construction of combinatorial manifolds.

If we consider only simplicial complexes which meet certain connectivity conditions, then "foldability" and "empty odd subcomplex" are equivalent. We make use of this equivalence in the construction of a simplicial complex K with prescribed odd subcomplex by composing K from foldable building blocks. Finally we are able to construct triangulations of closed oriented PL 4-manifolds via unfolding simplicial 4-spheres with prescribed odd subcomplex. One may read this result as a combinatorial analog of the topological construction of closed oriented PL 4-manifolds as branched covers by Piergallini [54]. The construction of closed oriented combinatorial 4-manifolds is the first main result presented here; see Theorem 3.12.

Theorem. For every closed oriented PL 4-manifold M there is a combinatorial manifold S homeomorphic to $\mathbb{S}^4$ such that one of the connected components of the partial unfolding $\widehat{S}$ of S is a combinatorial 4-manifold PL-homeomorphic to M. The canonical projection $\widehat{S} \to S$ is a simple 4-fold branched cover branched over a PL surface with a finite number of cusp and node singularities.

In a different context we construct foldable triangulations with certain additional properties of products of lattice polytopes. A triangulation of a d-dimensional lattice polytope P is *regular* if it can be lifted to $m+1$ dimensions as a lower convex hull. The barycentric subdivision of any regular triangulation is an example of a triangulation which is both regular and foldable. A lattice triangulation of P is *dense* if its vertices are all the lattice points inside P, and, for the sake of brevity, we refer to a regular, dense, and foldable triangulation as an *rdf-triangulation*.

It is known that a triangulation of a polytope (or, more generally, of any simply connected manifold) is foldable if and only if its dual graph is bipartite; see [37]. From an rdf-triangulation K of a lattice polytope P Soprunova & Sottile [62] construct sparse polynomial systems with non-trivial lower bounds for the number of real roots. The polytope P is the common Newton polytope of the polynomials in the system, and the weighted size difference of the bipartition of the dual graph of K is a lower bound for the number of real roots. The size difference is called the *signature* of K, and the polynomial systems constructed by Soprunova & Sottile are called *Wronski systems*.

Given rdf-triangulations of lattice polytopes P and Q, we construct the *simplicial product*, an rdf-triangulation of the product $P \times Q$, and compute its signature. Here the natural ingredient is the *staircase triangulation* of a product of two simplices, studied by Billera, Cushman & Sanders [6], Gel′fand, Kapranov & Zelevinsky [24], and others. The simplicial product already occurs in the work of Eilenberg & Steenrod [18, Section II.8]; see also Santos [58]. The simplicial product of rdf-triangulations of two lattice polytopes yields our second main result; see Theorem 4.17.

Theorem. Let P^λ and Q^μ be rdf-triangulations of an m-dimensional lattice polytope $P \subset \mathbb{R}^m$ and an n-dimensional lattice polytope $Q \subset \mathbb{R}^n$, respectively. For specific vertex orderings of the factors (to be explained later) the simplicial product $P^\lambda \times_{\text{stc}} Q^\mu$ is an rdf-triangulation of the polytope $P \times Q$ with signature

$$\sigma(P^\lambda \times_{\text{stc}} Q^\mu) = \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(Q^\mu) \ ,$$

where $\sigma_{m,n}$ is the signature of the staircase triangulation of the product of simplices $\Delta_m \times \Delta_n$.

For the algebraic applications it is essential that Theorem 4.17 can further be improved. In Theorem 4.29 we show that (with a mild additional assumption) the simplicial product $P^\lambda \times_{\text{stc}} Q^\mu$ meets the geometric requirements of Soprunova & Sottile, provided that both factors do.

This exposition is organized as follows. Chapter 1 reviews the theory of (branched) covers and presents the unfoldings. In particular we examine how the concepts of branching set and monodromy translate to the partial unfolding. This provides us with the key tool for the construction of closed oriented combinatorial 4-manifolds in Chapter 3.

We proceed by introducing the notion of *color equivalence* of simplicial complexes, an equivalence with respect to the partial unfolding: Let K and L be color equivalent simplicial complexes, then the canonical projections $\widehat{K} \to K$ and $\widehat{L} \to L$ are equivalent branched covers. We then use color equivalence to recover (and slightly improve) a result by Izmestiev & Joswig [36], stating that the partial unfolding of the anti-prismatic subdivision of a simplicial complex K is equivalent to the partial unfolding of K. In the last section of Chapter 1 we examine what kind of singularities in the odd subcomplex are allowed if we want to obtain a combinatorial manifold as the unfolding of a combinatorial manifold. This improves a result by Fox [19] for the unfoldings.

The construction of foldable simplicial complexes is studied in Chapter 2. Here the key question is the following: Given a $(d+1)$-colorable combinatorial $(d-1)$-sphere, how to extend the triangulation and coloring to a foldable combinatorial d-ball? This question is answered in Theorem 2.3. Additionally we are interested in preserving further properties like regularity and prove an upper bound for the expected size of the extended triangulation. Theorem 2.3 is then generalized to extending partial triangulations and colorings of CW-complexes and relative handlebody decompositions of dimension at most 4.

On the other hand Chapter 2 examines the odd subcomplex, or rather, we develop techniques to construct some classes of odd subcomplexes. These two techniques, extending partial triangulations and constructing odd subcomplexes, are put to use in the construction of combinatorial 4-manifolds.

Chapter 3 is mostly devoted to the construction of closed oriented combinatorial 4-manifolds. In Section 3.1 we review the topological construction of a closed oriented PL-manifold M as a branched cover $p : M \to \mathbb{S}^4$ branched over an embedded PL-surface with a finite number of cusp and node singularities. This construction is due to Piergallini [54] and earlier results by Montesinos [47, 48], and provides the "blue print" of our construction of combinatorial 4-manifolds. We then construct a triangulation S of the 4-sphere such that the projection $\widehat{S} \to S$ is a branched cover equivalent to p. In particular, $\widehat{S} \cong M$ holds. The construction of S involves a lot of technical details, and it takes up the entire Section 3.2.

We conclude Chapter 3 by applying the techniques developed in the construction of combinatorial 4-manifolds to the construction of combinatorial 3-manifolds. The question how to construct a combinatorial 3-sphere S such that $\widehat{S} \cong M$ holds for a given closed oriented 3-manifold M is answered by Izmestiev & Joswig [36]. In Section 3.3 we give an alternative construction of S, which, starting with an arbitrary triangulation of $\mathbb{S}^3$, relies only on the stellar subdivision of faces and the operation of *twisting*.

The final Chapter 4 is motivated by an application in algebraic geometry, the search for lower bounds for the number of real roots of a sparse polynomial system. Hence it somehow differs from the rather topologically flavored first three chapters. The exact number of complex solutions of a sparse system of polynomials with generic coefficients is known from Kushnirenko's Theorem [40]. To bound the number of real roots from below is significantly more difficult. The 1-dimensional case, that is, a system of one polynomial in one variable, already illustrates the difficulty to bound the number of real roots compared to finding the exact number of complex roots. However, Soprunova & Sottile [62] construct sparse polynomial systems with non-trivial lower bounds from an rdf-triangulation of a lattice polytope P, where P and its rdf-triangulation have to meet additional requirements imposed by the algebraic geometry involved.

We discuss the staircase triangulation of two simplices and the simplicial product, an rdf-triangulation of the product $P \times Q$ of the lattice polytopes P and Q obtained from rdf-triangulations of the factors. Further we prove that the rdf-triangulation of $P \times Q$ meets the algebraic geometry requirements provided both factors do, and compute its signature, which is a lower bound for the number of real roots of the associated sparse polynomial systems.

In the last Section 4.4 we apply our results to obtain triangulations of the d-cube with large signature. For $d \not\equiv 2 \bmod 4$ we give explicit triangulations of the d-cube with signature at least $\lfloor d/2 \rfloor!$ which meet the algebraic geometry requirements. The Wronski systems associated to these cube triangulations are sparse polynomial systems in d unknowns, which have at least $\lfloor d/2 \rfloor!$ real roots compared to exactly $d!$ complex roots by Kushnirenko's Theorem.

The lower bound for the signature of the d-cube partially relies on computational results obtained with `TOPCOM` [55], `polymake` [21, 22, 23], `MAGMA` [13], and `QEPCAD` [30].

Chapter 4 is a joint work with Michael Joswig to appear in *Advances in Mathematics*.

Chapter 1

Covering and Unfolding

Branched covers form a major tool for the study, construction and classification of d-manifolds. First results are by Alexander [2] in 1920, who observed that any closed oriented PL d-manifold M is a branched cover of the d-sphere. Let T be a triangulation of M, and let $b(T)$ be the barycentric subdivision. Then the dual graph $\Gamma^*(b(T))$ is bipartite, and we obtain a bipartition of the facets of $b(T)$ into "black" and "white" facets such that no facets of the same color intersect in a ridge. Now a branched cover $p : M \to \mathbb{S}^d$ is defined by mapping the black facets to the d-simplex $\Delta_d \subset \mathbb{R}^d \subset \mathbb{R}^d \cup \infty \cong \mathbb{S}^d$, and mapping the white facets to the topological closure of the complement of Δ_d, that is, the white facets are mapped to $\mathrm{cl}(\mathbb{S}^d \setminus \Delta_d)$.

Unfortunately Alexander's proof does not allow for any (reasonable) control over the number of sheets of the branched cover, nor over the topology of the branching set: The branching set of p is the co-dimension 2-skeleton of Δ_d, and the number of sheets of p depends on the size of the triangulation T. Further, p is not a simple branched cover.

At least to our knowledge, there are no non-trivial upper bounds for the number of sheets of such a branched cover for $d > 4$. On the contrary, Bernstein & Edmonds [4] showed that at least d sheets are necessary in general (for example the d-torus $(\mathbb{S}^1)^d$ exhibits such a behavior), and that the branching set can not be required to be non-singular for $d \geq 8$.

However, in dimension $d \leq 4$, the situation is fairly well understood; see also Chapter 3. The 2-dimensional case is simple since any closed oriented surface F_g of genus g is a 2-fold (simple) branched cover of $\mathbb{S}^2$ branched over $2g + 2$ isolated points.

By results of Hilden [29] and Montesinos [45] any closed oriented 3-manifold M arises as 3-fold simple branched cover of $\mathbb{S}^3$ branched over a link L. Labeling each bridge b of a diagram of L with the corresponding monodromy action of a meridian around b, we can represent M as a labeled (colored) link diagram.

In dimension 4 the situation becomes increasingly difficult. First Piergallini [54] proved that each closed oriented PL 4-manifold is a 4-fold simple branched cover of $\mathbb{S}^4$ branched over a transversally immersed surface. Then Iori & Piergallini [33] eliminated the singularities, but had to add a fifth sheet to the covering. The question whether each closed oriented PL 4-manifold is a 4-fold branched cover of $\mathbb{S}^4$ branched over a locally flat PL surface is still open.

The main effort of this chapter is dedicated to the topological concept of a branched cover and its combinatorial analog, the unfoldings of a simplicial complex. In particular, we define combinatorial models of the key features of a branched cover, the branching set and the monodromy homomorphism. The idea is to construct a simplicial complex K, such that the canonical projection of the unfolding to K is equivalent to a given branched cover of $|K|$. In particular, the equivalence of the branched covers implies homeomorphy of the covering spaces. The latter is the key tool in the construction of combinatorial 4-manifolds in Chapter 3.

Finally we try to give some insight into what kind of singularities are allowed in the branching set for the covering space to be a manifold, provided the base space is a manifold. Here we improve a result by Fox [19] for the unfoldings.

Combinatorial Manifolds. We clarify some basic definitions and notations. Given some topological manifold M, we call a simplicial complex K homeomorphic to M a *triangulation* of M, or a *simplicial manifold*. A simplicial complex K is a *combinatorial d-sphere* or *combinatorial d-ball* if it is piecewise linear homeomorphic to the boundary of the $(d+1)$-simplex, respectively to the d-simplex. Equivalently, K is a combinatorial d-sphere or d-ball if there is a common refinement of K and the boundary of the $(d+1)$-simplex, respectively the d-simplex. A simplicial complex K is a *combinatorial manifold* if the vertex link of each vertex of K is a combinatorial sphere or a combinatorial ball. Note that combinatorial spheres and balls are combinatorial manifolds.

A manifold M where all charts are piecewise linear is called a *PL-manifold*. Up to dimension 3 there is no difference between topological, PL-, and differential manifolds, that is, every topological manifold allows for a PL- or differential atlas (or structure). The existence of a triangulation of M as a combinatorial manifold is equivalent to the existence of a PL-atlas for M. For an introduction to PL-topology see Björner [7, Part II], Hudson [31], and Rourke & Sanderson [56].

Similarly to the topological situation, there is no difference between the notion of a simplicial and a combinatorial manifold in dimension $d \leq 3$, that is, every simplicial manifold (or sphere, or ball) is a combinatorial manifold (or sphere, or ball). But in dimension 4 the situation becomes more complicated. Freedman & Quinn [20] construct a 4-manifold which does not have a triangulation as a combinatorial manifold. In fact, there are 4-manifolds which can not be triangulated at all [42, p. 9]. The following unanswered question illustrates the subtleties of the 4-dimensional case like no other: Is a combinatorial manifold homeomorphic to the 4-sphere necessarily a combinatorial 4-sphere? Surprisingly, the answer to this question is affirmative in all dimensions $d \neq 4$; see Moise [43] and Kirby & Siebenmann [39].

Neither barycentric subdivision nor anti-prismatic subdivision (of a face) change the PL-type of a simplicial manifold, that is, the subdivision of a simplicial complex K is a combinatorial manifold if and only if K is a combinatorial manifold. The cone of a combinatorial sphere is a combinatorial ball and the suspension of a combinatorial sphere is again a combinatorial sphere.

Connectivity Properties of Simplicial Complexes. The simplicial complexes considered in the following (and throughout this exposition) are always *pure*, that is, all the inclusion maximal faces, called the *facets*, have the same dimension. We call a co-dimension 1-face of a pure simplicial complex a *ridge*, and the *dual graph* $\Gamma^*(K)$ of a pure simplicial complex K has the facets as its node set, and two nodes are adjacent if the corresponding facets share a ridge. Further it is often necessary to restrict ourselves to simplicial complexes with certain connectivity properties.

The connectivity properties in question are as follows: A pure simplicial complex K is *strongly connected* if its dual graph $\Gamma^*(K)$ is connected, and *locally strongly connected* if $\mathrm{st}_K(f)$ is strongly connected for each face $f \in K$. If K is locally strongly connected, then connected and strongly connected coincide. Further we call K *locally strongly simply connected* if for each face $f \in K$ with co-dimension ≥ 2 the link of f is simply connected, and finally, K is *t-nice* if it is strongly connected, locally strongly connected, and locally strongly simply connected. Here we differ slightly from [36], where a nice simplicial complex is not required to be strongly connected. However, strongly connected is only a mild additional condition, since one may treat the strongly connected components individually. Further, to avoid confusion with the concept of a nice triangulation of a lattice polytope introduced in Chapter 4, we resort to the name t-nice to stress the topological flavor of its definition. Observe that connected combinatorial manifolds are always t-nice.

Basic Constructions. The *star*

$$\mathrm{st}_K(f) = \{g \subset \sigma \mid f \subset \sigma \in K\}$$

of a face f of a simplicial complex K is defined by all facets (and their proper faces) containing f. The *link*

$$\mathrm{lk}_K(f) = \{g \setminus f \mid g \in \mathrm{st}_K(f)\}$$

of f is the set of all faces in $\mathrm{st}_K(f)$ not intersecting f. The star and the link of f are simplicial complexes, and we sometimes omit the complex K in the notation. Further we make use repeatedly of the following constructions. The *join* $K * L$ of two simplicial complexes K and L is given by

$$K * L = \{f \cup g \mid f \in K \text{ and } g \in L\} \ .$$

The *cone* $\mathrm{cone}(K) = \{a\} * K$ over K with apex a is the join of K with a single vertex $\{a\}$, and the *suspension* $\mathrm{susp}(K) = \{\{a_1\}, \{a_2\}\} * K$ is the join of K with the 0-dimensional sphere $\{\{a_1\}, \{a_2\}\}$.

1.1 Coverings

A continuous surjective map $p : X \to Y$ is a *covering* if there is an open neighborhood $U_y \subset Y$ for each $y \in Y$ such that the preimage $p^{-1}(U_y)$ is a pairwise disjoint union of open subsets in X, and p maps each of these subsets homeomorphically to U_y. We call X the covering space, and Y the base space. Two coverings $p : X \to Y$ and $p' : X' \to Y'$ are *equivalent* if there are homeomorphisms $\varphi : X \to X'$ and $\psi : Y \to Y'$ such that $\psi \circ p = p' \circ \varphi$ holds.

It is essential for a satisfactory theory of coverings to make certain connectivity assumption for X and Y. The spaces mostly considered are Hausdorff, path connected, and locally path connected; see Bredon [12, III.3.1]. Throughout we will restrict our attention to coverings of manifolds, and we assume X and Y to be connected, hence they meet the connectivity assumptions in [12]. If the base space Y is a d-manifold, then also the covering space X is a d-manifold and the preimage of any point $y \in Y$ has the same cardinality. In the case $k = |p^{-1}(y)| < \infty$ we call p a k-fold covering. From now on we assume X and Y to be connected manifolds and $p : X \to Y$ to be a k-fold covering.

Let $y_0 \in Y$ be an arbitrary but fixed point with preimage $p^{-1}(y_0) = \{x_i\}_{0 \leq i \leq k-1}$, and let $\alpha : [0, 1] \to Y$ be a closed path based at y_0 representing a given element $[\alpha] \in \pi_1(Y, y_0)$. For each $x_i \in p^{-1}(y_0)$ there is a unique lifted path $\alpha_i : [0,1] \to X$ with $\alpha_i(0) = x_i$ and $p \circ \alpha_i = \alpha$; see Munkres [49, Lemma 79.1]. Its end point $\alpha_i(1)$ is again a point in $p^{-1}(y_0)$, and it depends solely on the homotopy class $[\alpha]$ of α. Thus $\pi_1(Y, y_0)$ acts on the set $p^{-1}(y_0)$ by $[\alpha] \cdot x_i = \alpha_i(1)$, which yields the *monodromy homomorphism*

$$\mathfrak{m}_p : \pi_1(Y, y_0) \to \mathrm{Sym}(p^{-1}(y_0)) \ ,$$

where $\mathrm{Sym}(p^{-1}(y_0))$ is the symmetric group on the point set $p^{-1}(y_0)$. The image of $\mathfrak{m}_p$ is denoted by $\mathfrak{M}_p$, the *monodromy group* of p; see Seifert & Threlfall [60, § 58]. If Y is connected the isomorphism type of $\mathfrak{M}_p$ does not depend on the choice of y_0.

Example 1.1. The basic example of a covering is the k-fold covering of the 1-sphere $\mathbb{S}^1$ by itself. For $k \geq 1$ let p_k be the complex map

$$p_k : \mathbb{C} \to \mathbb{C} : z \mapsto z^k \ .$$

If we view $\mathbb{S}^1$ as a subset of the complex numbers then the restriction $p_k|_{\mathbb{S}^1}$ is a k-fold covering of $\mathbb{S}^1$ by itself with the cyclic group of order k as its monodromy group; see Figure 1.1.

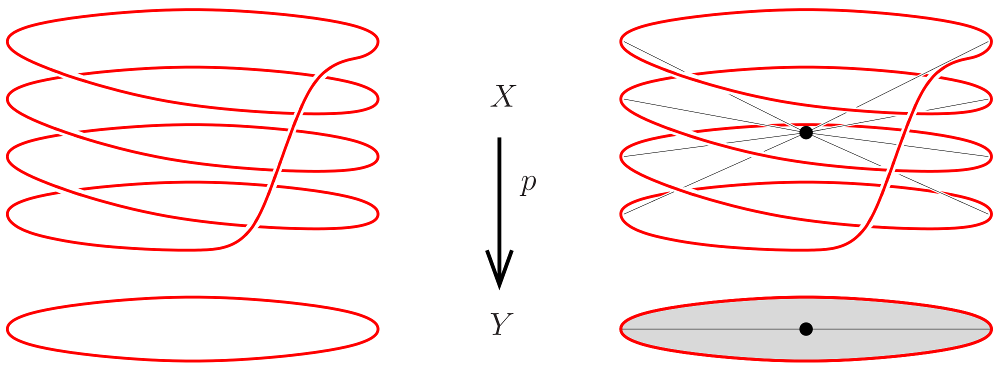


Figure 1.1. *The* 4*-fold cover* $p_4|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ *on the left. By coning one obtains the* 4*-fold branched cover* $p_4|_{\mathbb{D}^2} : \mathbb{D}^2 \to \mathbb{D}^2$ *with branching set* $\{0\}$ *on the right.*

The coverings $p : X \to Y$ of a connected manifold Y are classified (up to equivalence) by the conjugation classes of the subgroups of $\pi_1(Y)$: The induced group homomorphism $p_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ for any $x_0 \in p^{-1}(y_0)$ is injective, and hence $\pi_1(X, x_0)$ is isomorphic to a subgroup of $\pi_1(Y, y_0)$. In general the subgroup $p_*(\pi_1(X, x_0))$ depends on the choice of $x_0 \in p^{-1}(y_0)$, yet we have $p_*(\pi_1(X, x_0)) = g\, p_*(\pi_1(X, x_0'))\, g^{-1}$ for any $x_0, x_0' \in p^{-1}(y_0)$, where

$g \in \pi_1(Y, y_0)$ corresponds to the closed path obtained as the image of a path from x_0 to x'_0. (A covering where $p_*(\pi_1(X, x_0))$ is independent of the choice of $x_0 \in p^{-1}(y_0)$, or equivalently, where $p_*(\pi_1(X, x_0))$ is a normal subgroup, is called *regular*.) Conversely, for every subgroup H of $\pi_1(Y)$ there exists a covering $p : X \to Y$ such that $p_*(\pi_1(X))$ lies in the conjugation class of H.

Alternatively, for a fixed connected manifold Y as the base space, the monodromy homomorphism $\mathfrak{m}_p$ classifies the coverings $p : X \to Y$. For $k \geq 1$ and any group homomorphism $\mathfrak{m} : \pi_1(Y) \to \Sigma_k$ there is a k-fold covering $p : X \to Y$ with $\mathfrak{m}_p = \mathfrak{m}$. Here Σ_k denotes the symmetric group of order k. The covering p is unique up to equivalence, but the covering space X need not be connected. Moreover, conjugation in Σ_k does not change the homeomorphy type of X.

The classifications of coverings of Y via the subgroups of $\pi_1(Y)$, and via the group homomorphisms $\mathfrak{m} : \pi_1(Y) \to \Sigma_k$, $k \geq 1$, hold for a wider class of topological spaces than assumed here, but we will not elaborate; see Munkres [49, § 79, § 82], and Seifert & Threlfall [60, § 58] for a substantial discussion.

Throughout this exposition we will recur to the monodromy group for the classification, since we have the group of projectivities to be introduced in Section 1.2.1 as a combinatorial equivalent of the monodromy group.

1.1.1 Branched Covers

The concept of a covering of a space Y by another space X is generalized by Fox [19] to the notion of the branched cover. Here a certain subset of Y may violate the conditions of a covering map. This allows for a wider application in the construction of topological spaces.

Example 1.2. For $k \geq 1$ consider the map $p_k : \mathbb{C} \to \mathbb{C}$, and the k-fold covering $p_k|_{\mathbb{S}^1}$ of $\mathbb{S}^1$ by itself from Example 1.1. The 2-ball $\mathbb{D}^2$ does not admit such non-trivial coverings, since $\mathbb{D}^2$ has a trivial fundamental group, hence any covering has a trivial monodromy group. However, if we consider the restriction of p_k to the unit disk then $p_k|_{\mathbb{D}^2}$ is a k-fold covering except for the origin: $\mathbb{D}^2 \setminus \{0\}$ is homotopy equivalent to $\mathbb{S}^1$. The map $p_k|_{\mathbb{D}^2}$ is a k-fold branched cover with the single branch point $\{0\}$; see Figure 1.1.

Branched covers can be described in terms of "local models" as in Example 1.2 above: In the case of branched covers of closed surfaces, a map $f : \tilde{F} \to F$ between closed surfaces is a branched cover if it is finite-to-one, and if for every $x \in \tilde{F}$ there exists a neighborhood $U_x \subset \tilde{F}$ such that the restriction $f|_{U_x}$ is homeomorphic to $p_{k(x)}$ for some $k(x) \geq 1$. In the case of branched covers of d-manifolds for $d \geq 3$ more complex local models are needed. We recommend Piergallini [53] for a substantial and easy to read introduction to branched covers.

A different approach is pursued by Fox [19]. Consider a continuous map $h : Z \to Y$, and assume the restriction $h : Z \to h(Z)$ to be a covering. If $h(Z)$ is dense in Y (and meets certain additional connectivity conditions) then there is a surjective map $p : X \to Y$ with $Z \subset X$ and $p|_Z = h$. The map p is called a *completion* of h, and any two completions $p : X \to Y$ and $p' : X' \to Y$ are equivalent in the sense that there exists a homeomorphism $\varphi : X \to X'$ satisfying $p' \circ \varphi = p$ and $\varphi|_Z = \mathrm{Id}$. The map $p : X \to Y$ obtained this way is a *branched cover*, and we call the unique minimal subset $Y_{\mathrm{sing}} \subset Y$ such that the restriction of p to the preimage of $Y \setminus Y_{\mathrm{sing}}$ is a cover, the *branching set* of p. The restriction of p to $p^{-1}(Y \setminus Y_{\mathrm{sing}})$ is called the *associated* cover of p. If $h : Z \to Y$ is a cover, then $X = Z$, and $p = h$ is a branched cover with empty branching set. In this sense the branched cover generalizes the notion of a cover.

We define the *monodromy homomorphism*

$$\mathfrak{m}_p : \pi_1(Y \setminus Y_{\mathrm{sing}}, y_0) \to \mathrm{Sym}(p^{-1}(y_0))$$

of a branched cover for a point $y_0 \in Y \setminus Y_{\mathrm{sing}}$ as the monodromy homomorphism of the associated cover. Similarly, the *monodromy group* $\mathfrak{M}_p$ is defined as the image of $\mathfrak{m}_p$.

Two branched covers $p : X \to Y$ and $p' : X' \to Y'$ are *equivalent* if there are homeomorphisms $\varphi : X \to X'$ and $\psi : Y \to Y'$ with $\psi(Y_{\mathrm{sing}}) = Y'_{\mathrm{sing}}$, such that $p' \circ \varphi = \psi \circ p$ holds. The well known Theorem 1.3 is due to the uniqueness of Y_{sing}, and hence the uniqueness of the associated cover; see [53, p. 2].

Theorem 1.3. Let $p : X \to Y$ be a branched cover of a connected manifold Y. Then p is uniquely determined up to equivalence by the branching set Y_{sing}, and the monodromy homeomorphism $\mathfrak{m}_p$. In particular, the covering space X is determined up to homeomorphy.

Let Y be a connected manifold and Y_{sing} a co-dimension 2 submanifold, possibly with a finite number of singularities. A *meridian* around a point $y \in Y_{\mathrm{sing}}$ is a non-contractable, closed path $m : [0,1] \to Y \setminus Y_{\mathrm{sing}}$ which is contractable in $(Y \setminus Y_{\mathrm{sing}}) \cup \{y\}$. If y is not a singular point, one may picture m as the boundary of a 2-ball embedded transversally to Y_{sing}, and intersecting Y_{sing} in y only. A *meridial loop* around a point $y \in Y_{\mathrm{sing}}$ is a path $\gamma\, m\, \gamma^-$ representing an element in $\pi_1(Y \setminus Y_{\mathrm{sing}}, y_0)$, where m is a meridian around y, and γ is a path in $Y \setminus Y_{\mathrm{sing}}$ from y_0 to $m(0) = m(1)$. Here γ^- is the *inverse path* of γ given by $\gamma^-(t) = \gamma(1-t)$, and $\gamma\, \gamma'$ denotes the *concatenation* of two paths $\gamma : [0,1] \to Y$ and $\gamma' : [0,1] \to Y$ with $\gamma'(0) = \gamma(1)$, that is,

$$\gamma\,\gamma'(t) = \begin{cases} \gamma(2t) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1) & \text{if} \quad \frac{1}{2} < t \le 1 \; . \end{cases}$$

We call a branched cover p *simple* if the image $\mathfrak{m}_p(m)$ of any meridial loop m around a non-singular point of the branching set is a transposition in $\mathfrak{M}_p$. Note that the k-fold (branched) covers $p_k|_{\mathbb{S}^1}$ and $p_k|_{\mathbb{D}^2}$ of $\mathbb{S}^1$, respectively $\mathbb{D}^2$, over themselves presented in Examples 1.1 and 1.2 are not simple for $k \ge 3$.

1.2 Unfoldings

In this section we introduce the notions of the complete $\widetilde{K}$ and partial unfolding $\widehat{K}$ of a simplicial complex K. Unfoldings first appeared in a paper by Izmestiev & Joswig [36], with some of the basic notions already developed in Joswig [37]. Unfoldings are geometric objects defined entirely by the combinatorial structure of K, and come along with canonical projections $r : \widetilde{K} \to K$ and $p : \widehat{K} \to K$.

However, $\widetilde{K}$ and $\widehat{K}$ may not be simplicial complexes. In general the unfoldings are pseudo-simplicial complexes: Let Σ be a collection of pairwise disjoint geometric simplices, and simplicial attaching maps for some pairs $(\sigma, \tau) \in \Sigma \times \Sigma$, mapping a subcomplex of σ isomorphically to a subcomplex of τ. Identifying the subcomplexes accordingly yields the quotient space $\Sigma/\!\sim$, which is called a *pseudo-simplicial complex* if the quotient map $\Sigma \to \Sigma/\!\sim$ restricted to any $\sigma \in \Sigma$ is bijective. The last condition ensures that there are no self-identifications within each simplex $\sigma \in \Sigma$.

1.2.1 The Group of Projectivities

Let σ and τ be neighboring facets of a finite, pure simplicial complex K, that is, $\sigma \cap \tau$ is a ridge. Then there is exactly one vertex in σ which is not a vertex of τ and vice versa, hence a natural bijection $\langle \sigma, \tau \rangle$ between the vertex sets of σ and τ is given by

$$\begin{aligned} \langle \sigma, \tau \rangle : V(\sigma) &\to V(\tau) \\ v &\mapsto \begin{cases} v & \text{if} \quad v \in \sigma \cap \tau \\ \tau \setminus \sigma & \text{if} \quad v = \sigma \setminus \tau \; . \end{cases} \end{aligned}$$

The bijection $\langle \sigma, \tau \rangle$ is called a *perspectivity* from σ to τ.

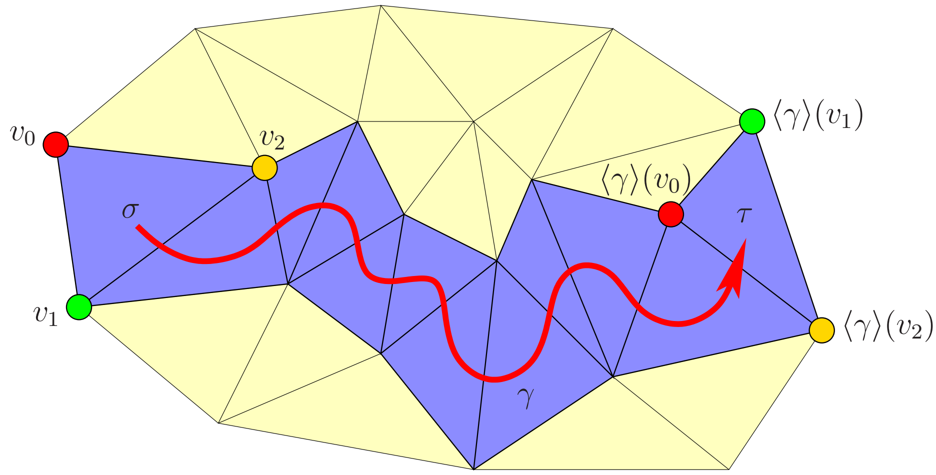


Figure 1.2. *A projectivity from σ to τ along the facet path γ.*

A *facet path* in K is a sequence $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ of facets, such that the corresponding nodes in $\Gamma^*(K)$ form a path, that is, $\sigma_i \cap \sigma_{i+1}$ is a ridge for all $0 \leq i < k$; see Figure 1.2. Now a *projectivity* $\langle\gamma\rangle$ along γ is defined as the composition of perspectivities $\langle\sigma_i, \sigma_{i+1}\rangle$, thus $\langle\gamma\rangle$ maps $V(\sigma_0)$ to $V(\sigma_k)$ bijectively via

$$\langle\gamma\rangle = \langle\sigma_{k-1}, \sigma_k\rangle \circ \cdots \circ \langle\sigma_1, \sigma_2\rangle \circ \langle\sigma_0, \sigma_1\rangle \ .$$

Again, we write $\gamma\,\gamma' = (\sigma_0, \sigma_1, \ldots, \sigma_k, \ldots, \sigma_{k+l})$ for the *concatenation* of two facet paths $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ and $\gamma' = (\sigma_k, \sigma_{k+1}, \ldots, \sigma_{k+l})$, denote by $\gamma^- = (\sigma_k, \sigma_{k-1}, \ldots, \sigma_0)$ the *inverse path* of γ, and we call γ a *closed* facet path based at σ_0 if $\sigma_0 = \sigma_k$. The set of closed facet paths based at σ_0 together with the concatenation form a group, and a closed facet path γ based at σ_0 acts on the set $V(\sigma_0)$ via $\gamma \cdot v = \langle\gamma\rangle(v)$ for $v \in V(\sigma_0)$. Via this action we obtain the *group of projectivities* $\Pi(K, \sigma_0)$ given by all permutations $\langle\gamma\rangle$ of $V(\sigma_0)$. The group of projectivities is a subgroup of the symmetric group $\mathrm{Sym}(V(\sigma_0))$ on the vertices of σ_0.

The projectivities along null-homotopic closed facet paths based at σ_0 generate the subgroup $\Pi_0(K, \sigma_0) < \Pi(K, \sigma_0)$, which is called the *reduced group of projectivities*. Finally, if K is strongly connected then $\Pi(K, \sigma_0)$ and $\Pi(K, \sigma_0')$, respectively $\Pi_0(K, \sigma_0)$ and $\Pi_0(K, \sigma_0')$, are isomorphic for any two facets $\sigma_0, \sigma_0' \in K$. In this case we usually omit the base facet in the notation of the (reduced) group of projectivities, and write $\Pi(K) = \Pi(K, \sigma_0)$, respectively $\Pi_0(K) = \Pi_0(K, \sigma_0)$.

The odd subcomplex. Let K be locally strongly connected; in particular, K is pure. The link of a co-dimension 2-face f is a graph which is connected since K is locally strongly connected, and f is called *even* if $\mathrm{lk}_K(f)$ is bipartite, and *odd* otherwise. We define the *odd subcomplex* of K as all odd co-dimension 2-faces (together with their proper faces), and denote it by K_{odd} or $\mathrm{odd}(K)$.

Assume that K is pure and admits a $(d+1)$-*coloring* of its 1-skeleton $\Gamma(K)$ in the graph theoretic sense, that is, we assign one color of a set of $d+1$ colors to each vertex of $\Gamma(K)$ such that the two vertices of any edge carry different colors. Observe that the $(d+1)$-coloring of K is minimal with respect to the number of colors, and is unique up to renaming the colors if K is strongly connected. Simplicial complexes that are $(d+1)$-colorable are called *foldable*, since a $(d+1)$-coloring defines a non-degenerated simplicial map of K to the $(d+1)$-simplex.

Lemma 1.4. The odd subcomplex of a foldable simplicial complex K is empty, and the group of projectivities $\Pi(K,\sigma_0)$ is trivial. In particular we have $\langle\alpha\rangle = \langle\beta\rangle$ for any two facet paths α and β from σ to τ.

Proof. Assume f is an odd co-dimension 2-face. Then $\mathrm{lk}(f)$ is not bipartite and requires at least three colors for coloring. The $d-1$ vertices of f induce a d-clique in $\Gamma(\mathrm{st}(f))$, and each of the vertices of f is adjacent to each vertex in $\mathrm{lk}(f)$. Hence $\mathrm{st}(f)$ is not $(d+1)$-colorable.

As for $\Pi(K,\sigma_0)$, consider the strongly connected component of σ_0. Observe that the $(d+1)$-coloring of the strongly connected component is unique up to permuting the colors. Hence the equivalence classes of vertices colored the same correspond one-to-one to the orbits of the action of $\Pi(K,\sigma_0)$ on $V(\sigma_0)$. Since there are $d+1$ equivalence classes of vertices, each of the orbits of $\Pi(K,\sigma_0)$ is trivial. Finally, assume $\langle\alpha\rangle \neq \langle\beta\rangle$ for some facet paths α and β from σ to τ, and let γ be any facet path from σ_0 to σ. Now $\langle\alpha\rangle \neq \langle\beta\rangle$ implies that $\langle\gamma\,\alpha\,\beta^-\,\gamma^-\rangle$ is non-trivial, contradicting that $\Pi(K,\sigma_0)$ is trivial. □

Here the odd subcomplex is of interest in particular for its relation to $\Pi_0(K,\sigma_0)$ for a t-nice simplicial complex K. A projectivity *around* an odd face f is a projectivity along a facet path $\gamma\,l\,\gamma^-$, where l is a closed facet path in $\mathrm{st}_K(f)$ based at some facet $\sigma \in \mathrm{st}_K(f)$, and γ is a facet path from σ_0 to σ. The path $\gamma\,l\,\gamma^-$ is null-homotopic since K is locally strongly simply connected.

Theorem 1.5. (Izmestiev & Joswig [36, Theorem 3.2.2]). The reduced group of projectivities $\Pi_0(K,\sigma_0)$ of a t-nice simplicial complex K is generated by projectivities around the odd co-dimension 2-faces. In particular, $\Pi_0(K, \sigma_0)$ is generated by transpositions.

Consider a geometric realization $|K|$ of K. Recall that such a geometric realization always exists if K is finite by assigning one unit vector to each vertex of K. To a given facet path $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ in K we associate a (piecewise linear) path $\overline{\gamma}$ in $|K|$ by connecting the barycenter of σ_i to the barycenters of $\sigma_i \cap \sigma_{i-1}$ and $\sigma_i \cap \sigma_{i+1}$ by a straight line for $1 \leq i < k$, and connecting the barycenters of σ_0 and $\sigma_0 \cap \sigma_1$, respectively σ_k and $\sigma_k \cap \sigma_{k-1}$. The fundamental group $\pi_1(|K| \setminus |K_{\mathrm{odd}}|, y_0)$ of a t-nice simplicial complex K is generated by paths $\overline{\gamma}$, where γ is a closed facet path based at σ_0, and y_0 is the barycenter of σ_0; see [36, Proposition A.2.1]. Furthermore, due to Theorem 1.5 we have the group homomorphism

$$\mathfrak{h}_K : \pi_1(|K| \setminus |K_{\mathrm{odd}}|, y_0) \to \Pi(K, \sigma_0) : [\overline{\gamma}] \mapsto \langle \gamma \rangle \ , \tag{1.1}$$

where $[\overline{\gamma}]$ is the homotopy class of the path $\overline{\gamma}$ corresponding to a facet path γ.

1.2.2 The Complete Unfolding

Let K be a pure simplicial d-complex with fixed base facet $\sigma_0 \in K$, let $\Sigma(K)$ denote the collection of facets of K, and set $\overline{K} = \Sigma(K) \times \Pi(K, \sigma_0)$. Each pair $(\sigma, g) \in \overline{K}$ is a copy of the geometric simplex $|\sigma|$ labeled by the element $g \in \Pi(K, \sigma_0)$. For each facet σ choose a fixed facet path γ_σ from σ_0 to σ, and let σ and τ be neighboring facets of K with $f = \sigma \cap \tau$. Now define the equivalence relation $\sim$ by identifying $(f, g) \subset (\sigma, g) \in \overline{K}$ and $(f, h) \subset (\tau, h) \in \overline{K}$ if the equation

$$gh^{-1} = \langle \gamma_\sigma \rangle \langle \sigma, \tau \rangle \langle \gamma_\tau^- \rangle$$

holds. The resulting pseudo-simplicial complex

$$\widetilde{K} = \overline{K}/{\sim}$$

is called the *complete unfolding* of K. The canonical map $r : \widetilde{K} \to K$ is given by the factorization of the map $\overline{K} \to K : (\sigma, g) \mapsto \sigma$. For an easy example see Figure 1.3, and for simplicial complexes with non-simplicial complete (and partial) unfoldings see Figure 1.4.

Alternatively fix a $(d+1)$-coloring of the vertices of σ_0, and call a $(d+1)$-coloring of an arbitrary facet σ *admissible* if there is a facet path γ from σ_0 to σ such that each vertex $v \in \sigma$ is colored with the same color as its preimage $\langle \gamma \rangle^{-1}(v)$ in the fixed $(d+1)$-coloring of σ_0. Set $\overline{K}$ as the set of all pairs (σ, c), where c is an admissible coloring of σ. Let σ and τ be neighboring facets of K. Now we define the equivalence relation $\sim$ by attaching $(\sigma, c) \in \overline{K}$ and $(\tau, c') \in \overline{K}$ along their common ridge $\sigma \cap \tau$ if c and c' coincide on $\sigma \cap \tau$.

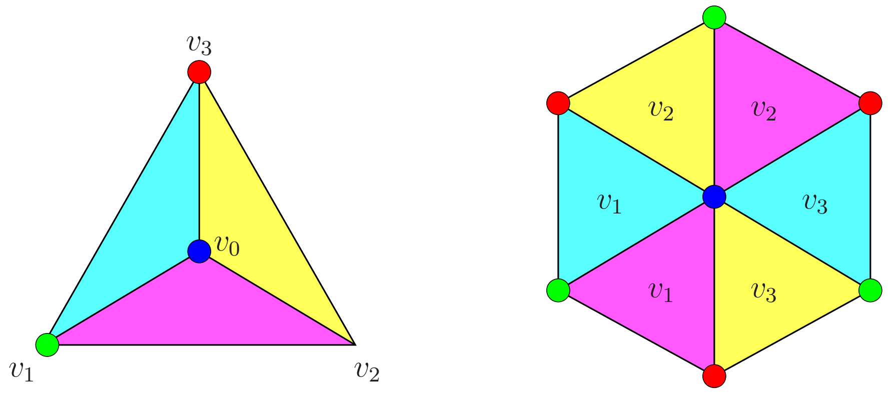


Figure 1.3. *The starred triangle and its unfoldings: The complex on the right is the complete unfolding (indicated by the admissible vertex colorings of the facets), as well as the non-trivial connected component of the partial unfolding (indicated by the labeling of the facets by the vertices v_1, v_2, and v_3). The second connected component of the partial unfolding is a copy of the starred triangle with all facets labeled v_0.*

1.2.3 The Partial Unfolding

The construction of the partial unfolding is similar to the second definition of the complete unfolding described in the previous paragraph. Here we set $\overline{K}$ as the set of all pairs (σ, v), where $v \in \sigma$ is a vertex. Let σ and τ be neighboring facets of K. We define the equivalence relation $\sim$ by attaching $(\sigma, v) \in \overline{K}$ and $(\tau, w) \in \overline{K}$ along their common ridge $\sigma \cap \tau$ if $\langle \sigma, \tau \rangle (v) = w$ holds. Now the *partial unfolding* $\widehat{K}$ is defined as the quotient space $\overline{K}/\sim$. The canonical map $p : \widehat{K} \to K$ is given by the factorization of the map $\overline{K} \to K : (\sigma, v) \mapsto \sigma$; see Figures 1.3 and 1.4.

In contrast to the complete unfolding, the partial unfolding of a connected simplicial complex is not connected in general. We denote by $\widehat{K}_{(\sigma,v)}$ the connected component containing the labeled facet (σ, v). Clearly, $\widehat{K}_{(\sigma,v)} = \widehat{K}_{(\tau,w)}$ holds if and only if there is a facet path γ from σ to τ in K with $\langle \gamma \rangle (v) = w$. It follows that the connected components of $\widehat{K}$ correspond to the orbits of the action of $\Pi(K, \sigma_0)$ on $V(\sigma_0)$. Note that the complete unfolding, as well as each connected component of the partial unfolding is strongly connected and locally strongly connected [65, Satz 3.2.2]. Therefore we do not distinguish between connected and strongly connected components of an unfolding.

The problem that $\widetilde{K}$ and $\widehat{K}$ may not be simplicial complexes can be addressed in several ways. Izmestiev & Joswig [36] suggest barycentrically sub-

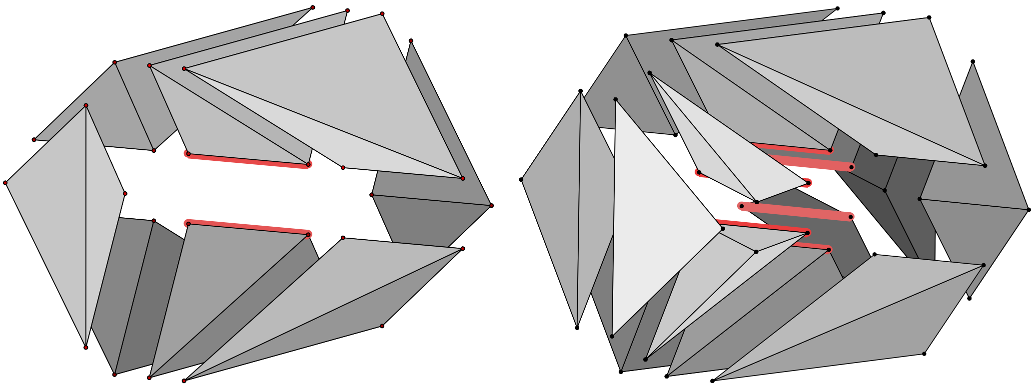


Figure 1.4. *Two simplicial complexes with non-simplicial complete and partial unfoldings. The complex on the left has a trivial fundamental group, and the complete unfolding is obtained by duplicating the marked center edge. The partial unfolding consists of four pairwise disjoint copies of the complete unfolding. On the right a combinatorial 3-ball which exhibits a similar pathological unfolding behavior.*

dividing the unfoldings, or anti-prismatically subdividing K: The Barycentric subdivision of any pseudo-simplicial complex is a simplicial complex, and the unfoldings of the anti-prismatic subdivision of K are PL-homeomorphic to the unfoldings of K (see [36, Corollary A.1.6, Proposition A.1.7], or Corollary 1.13), while the unfoldings of any anti-prismatically subdivided complex are simplicial [36, Proposition A.1.4, Proposition A.1.7]. The anti-prismatic subdivision is defined in Section 1.3.1.

A more efficient solution (with respect to the size of the resulting triangulations) is given in [65]: Rather than subdividing all faces of the unfoldings, only the faces necessary to ensure simpliciality are (stellarly or anti-prismatically) subdivided. This technique relies on the fact that the unfoldings are locally strongly connected as, mentioned before.

1.2.4 Branched Covers and the Unfoldings

The main effort of this section is to relate the topological concept of the branched cover to the canonical maps $r : \widetilde{K} \to K$ and $p : \widehat{K} \to K$ of the unfoldings. As preliminaries to this section we state two Theorems by Fox [19] and Izmestiev & Joswig [36]. Together they imply that under the "usual connectivity assumptions" unfoldings of simplicial complexes are indeed branched covers as suggested in the heading of this section. For simplicial complexes the analog of these topological connectivity requirements (see Section 1.1.1) are t-nice simplicial complexes.

Theorem 1.6. (Izmestiev & Joswig [36, Theorem 3.3.2]). The restriction of the complete unfolding of a simplicial complex to the preimage of the complement of the odd subcomplex is a covering. The same is true for each component of the partial unfolding.

Theorem 1.7. (Fox [19, p. 251]; Izmestiev & Joswig [36, Proposition 4.1.2]). Let $f : J \to K$ be a simplicial map, and let J and K be strongly connected and locally strongly connected. The map f is a branched cover if and only if

$$\operatorname{codim} K_{\text{sing}} \geq 2 \ .$$

Since the complete unfolding and each connected component of the partial unfolding is strongly connected and locally strongly connected Corollary 1.8 follows.

Corollary 1.8. The complete unfolding of a strongly connected and locally strongly connected simplicial complex is a branched cover with the odd subcomplex as its branching set. The same is true for each component of the partial unfolding.

For the rest of this section let K be a t-nice simplicial complex. Also, we restrict ourselves from now on to the partial unfolding $\widehat{K}$. Assume that the action of $\Pi(K, \sigma_0)$ on $V(\sigma_0)$ has only one non-trivial orbit. In this case we refer to the non-trivial connected component of $\widehat{K}$, that is, the unique non-trivial connected component corresponding to the non-trivial orbit, as *the* partial unfolding. Further let y_0 be the barycenter of $|\sigma_0|$. Now $p : \widehat{K} \to K$ is a branched cover by Corollary 1.8, and Izmestiev & Joswig [36] proved that $K_{\text{sing}} = K_{\text{odd}}$ holds, and that there is a bijection $\imath : p^{-1}(y_0) \to V(\sigma_0)$ that induces a group isomorphism $\imath_* \big|_{\mathfrak{M}_p} : \mathfrak{M}_p \to \Pi(S, \sigma_0)$ such that the following Diagram (1.2) commutes.

$$\begin{tikzcd}
\pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \arrow[d, "\mathfrak{h}_K"] \arrow[dr, "\mathfrak{m}_p"] & \\
\Pi(S, \sigma_0) \arrow[d, hook] & \mathfrak{M}_p \arrow[l, "\imath_*|_{\mathfrak{M}_p}"'] \arrow[d, hook] \\
\operatorname{Sym}(V(\sigma_0)) & \operatorname{Sym}(p^{-1}(y_0)) \arrow[l, "\imath_*"']
\end{tikzcd} \tag{1.2}$$

In the case that the $\Pi(K, \sigma_0)$-action has more than one non-trivial orbit, fix a set of generators of $\pi_1(|K| \setminus |K_{\text{odd}}|, y_0)$ corresponding to closed (facet) paths around odd co-dimension 2-faces, and possibly further generators of $\pi_1(|K|, y_0)$. Now each odd co-dimension 2-faces corresponds to exactly one non-trivial orbit of the $\Pi(K, \sigma_0)$-action, and K_{odd} decomposes correspondingly. In this spirit we can think of the empty set as the odd subcomplex corresponding to a trivial orbit. As mentioned above, the connected components of $\widehat{K}$ correspond one-to-one to the orbits of the $\Pi(K, \sigma_0)$-action. Therefore, in the case where the $\Pi(K, \sigma_0)$-action has more than one (non-trivial) orbit, Diagram (1.2) has to be restricted to each orbit of the $\Pi(K, \sigma_0)$-action, its corresponding component of the odd subcomplex, and its corresponding connected component of $\widehat{K}$.

Consider a t-nice simplicial complex K, and a branched cover $r : X \to Z$. Assume that there is a *homomorphism of pairs* $\varphi : (Z, Z_{\mathrm{sing}}) \to (|K|, |K_{\mathrm{odd}}|)$, that is, $\varphi : Z \to |K|$ is a homomorphism with $\varphi(Z_{\mathrm{sing}}) = |K_{\mathrm{odd}}|$. Then Theorem 1.9 gives sufficient conditions for $p : \widehat{K} \to K$ and $r : X \to Z$ to be equivalent branched covers. It is the key tool in the construction of closed oriented combinatorial 4-manifolds in Chapter 3.

Theorem 1.9. Let K be a t-nice simplicial complex, and let $r : X \to Z$ be a branched cover. Further assume that there is a homomorphism of pairs $\varphi : (Z, Z_{\mathrm{sing}}) \to (|K|, |K_{\mathrm{odd}}|)$, and let $z_0 \in Z$ be a point such that $y_0 = \varphi(z_0)$ is the barycenter of $|\sigma_0|$ for some facet $\sigma_0 \in K$. The branched covers $p : \widehat{K} \to K$ and $r : X \to Z$ are equivalent if there is a bijection $\iota : r^{-1}(z_0) \to V(\sigma_0)$ that induces a group isomorphism $\iota_* : \mathfrak{M}_r \to \Pi(K, \sigma_0)$ such that the diagram

$$\begin{array}{ccc} \pi_1(Z \setminus Z_{\mathrm{sing}}, z_0) & \xrightarrow{\varphi_*} & \pi_1(|K| \setminus |K_{\mathrm{odd}}|, y_0) \\ \downarrow{\scriptstyle \mathfrak{m}_r} & & \downarrow{\scriptstyle \mathfrak{h}_K} \\ \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(K, \sigma_0) \end{array} \tag{1.3}$$

commutes. Here φ_* is the group isomorphisms induced by φ. In particular, we have $\widehat{K} \cong X$.

Proof. Corollary 1.8 ensures that $p : \widehat{K} \to K$ is indeed a branched cover, and commutativity of Diagram (1.2) and Diagram (1.3) proves commutativity of their composition:

$$\begin{array}{ccc} \pi_1(Z \setminus Z_{\mathrm{sing}}, z_0) & \xrightarrow{\varphi_*} & \pi_1(|K| \setminus |K_{\mathrm{odd}}|, y_0) \\ \downarrow{\scriptstyle \mathfrak{m}_r} & & \downarrow{\scriptstyle \mathfrak{m}_p} \\ \mathfrak{M}_r & \xrightarrow{\iota_*^{-1} \circ \iota_*} & \mathfrak{M}_p \end{array}$$

Theorem 1.3 completes the proof. $\square$

1.3 Color Equivalence of Simplicial Complexes

Consider two t-nice simplicial complexes K and K'. The (partial) unfoldings of two homeomorphic simplicial complexes need not to be homeomorphic in general. Here we present sufficient criteria for $\widehat{K} \cong \widehat{K'}$ to hold. We remark that what follows are by no means necessary conditions. Assume $K \cong K'$ and that the odd subcomplexes K_{odd} and K'_{odd} are *equivalent*, that is, there is a homeomorphism of pairs $\varphi : (|K|, |K_{\mathrm{odd}}|) \to (|K'|, |K'_{\mathrm{odd}}|)$. In particular, we have $\varphi(|K_{\mathrm{odd}}|) = |K'_{\mathrm{odd}}|$. Let $\sigma_0 \in K$ be a facet, and y_0 the barycenter of σ_0, and assume that the image $y'_0 = \varphi(y_0)$ is the barycenter of $|\sigma'_0|$ for some facet $\sigma'_0 \in K'$. Now K and K' are *color equivalent* if there is a bijection $\psi : V(\sigma_0) \to V(\sigma'_0)$, such that

$$\psi_* \circ \mathfrak{h}_K = \mathfrak{h}_{K'} \circ \varphi_* \tag{1.4}$$

holds, where the maps $\varphi_* : \pi_1(|K| \setminus |K_{\mathrm{odd}}|, y_0) \to \pi_1(|K'| \setminus |K'_{\mathrm{odd}}|, y'_0)$ and $\psi_* : \mathrm{Sym}(V(\sigma_0)) \to \mathrm{Sym}(V(\sigma'_0))$ are the group isomorphisms induced by φ and ψ, respectively. Observe that this is indeed an equivalence relation. The name "color equivalent" suggests that the pairs (K, K_{odd}) and (K', K'_{odd}) are equivalent, and that the "colorings" of K_{odd} and K'_{odd} by the $\Pi(K, \sigma_0)$-action of projectivities around odd faces are equivalent. Proposition 1.10 justifies this name.

Proposition 1.10. Let K and K' be color equivalent simplicial complexes. Then the branched covers $p : \widehat{K} \to K$ and $p' : \widehat{K'} \to K'$ are equivalent.

Proof. With the notation of Equation (1.4) we have that

$$\begin{tikzcd}
& \pi_1(|K| \setminus |K_{\mathrm{odd}}|, y_0) \arrow[r, "\varphi_*"] \arrow[dl, "\mathfrak{m}_p"'] \arrow[d, "\mathfrak{h}_K"] & \pi_1(|K'| \setminus |K'_{\mathrm{odd}}|, y'_0) \arrow[d, "\mathfrak{h}_{K'}"'] \arrow[dr, "\mathfrak{m}_{p'}"] & \\
\mathfrak{M}_p \arrow[r, "\imath_*"] & \Pi(K, \sigma_0) \arrow[r, "\psi_*"] & \Pi(K', \sigma'_0) & \mathfrak{M}_{p'} \arrow[l, "\imath'_*"']
\end{tikzcd}$$

commutes, since the Diagram (1.2) commutes and Equation (1.4) holds. Theorem 1.3 completes the proof. □

1.3.1 The Anti-prismatic Subdivision

Let c_k be the simplicial complex obtained from the boundary complex of the $(k+1)$-dimensional cross polytope by removing one facet. Alternatively, define c_k as the simplicial complex arising from the Schlegel diagram of the $(k+1)$-dimensional cross polytope; see Ziegler [66]. To be more explicit, let $\sigma = \{+v_i\}_{0\le i\le k}$ be the vertices of the k-simplex. Then the facets of c_k are defined as all subsets $\sigma' \neq \sigma$ of $\{\pm v_i\}_{0\le i\le k}$ such that either $+v_i \in \sigma'$ or $-v_i \in \sigma'$ holds. The complex c_k and the k-simplex are PL-homeomorphic with isomorphic boundaries, and c_k is $(k+1)$-colorable by assigning the same color to $+v_i$ and $-v_i$, as $\{+v_i, -v_i\}$ is not an edge. The *anti-prismatic subdivision* $a_f(K)$ of a k-face f of a simplicial d-complex K is obtained from K by replacing $\mathrm{st}_K(f)$ by the join of c_k with $\mathrm{lk}_K(f)$, that is

$$a_f(K) = (K \setminus \mathrm{st}_K(f)) \cup (c_k * \mathrm{lk}_K(f)) .$$

Observe that for a subcomplex $L \subset K$ the anti-prismatic subdivision $a_f(L)$ equals the union of all faces of $a_f(K)$ arising by subdividing faces of L. For an example of a 2-complex with a subdivided edge and triangle see Figure 1.5.

Lemma 1.11. The anti-prismatic subdivision $a_f(K)$ of a face f of any foldable simplicial complex K is again foldable.

Proof. Fix a $(d+1)$-coloring of the vertices of K. Let $\{+v_0, +v_1, \ldots, +v_k\}$ denote the vertex set of f, and let $\sigma = \{+v_0, \ldots, +v_k, w_{k+1}, \ldots, w_d\}$ be a facet in $\mathrm{st}_K(f)$ with $+v_i$, respectively w_i, colored by its index. Now assigning color i to the vertices $\pm v_i$, respectively w_i, in $a_f(\sigma)$ yields a $(d+1)$-coloring of $a_f(\sigma)$. If $\sigma \notin \mathrm{st}_K(f)$ color the vertices of the copy of σ in $a_f(K)$ in the same way as the vertices of σ. (If one thinks of $a_f(K)$ as an refinement of K, then a facet $\sigma \in K$ which is not subdivided appears as a facet in $a_f(K)$. However, here we consider K and $a_f(K)$ as distinct objects and refer to the *copy* of σ in $a_f(K)$.) The colorings of any two facets $\sigma, \tau \in K$ coincide in $\sigma \cap \tau$, since K is foldable. Thus the colorings of $a_f(\sigma)$ and $a_f(\tau)$ coincide on $a_f(\sigma \cap \tau)$. ☐

Let $\sigma \in \mathrm{st}_K(f)$ be a facet, and (with the notation from the proof above) $+v_i \in f$ a vertex of σ. Then we define $-v_i \in a_f(K)$ as the *corresponding vertex* of $+v_i$. The corresponding vertex of $w_i \in \sigma \setminus f$ is its copy in $a_f(K)$. For a vertex $w \in \sigma$ of a facet $\sigma \notin \mathrm{st}_K(f)$ define its corresponding vertex as its copy in $a_f(K)$. The corresponding vertex of a vertex v is denoted by v_*. Note that if $a_f(\sigma)$ is colored as in the proof of Lemma 1.11, v and v_* are colored the same. Induced by the definition of the corresponding vertices we obtain the *corresponding facet* $\sigma_* \in a_f(K)$ for each facet $\sigma \in K$; see Figure 1.5.

The *anti-prismatic subdivision* $a(K)$ of a simplicial complex K is defined by recursively anti-prismatically subdividing all faces of K from the facets down to the edges. Observe that $a_f(K)$, and hence $a(K)$, are PL-homeomorphic to K, and that $a_f(K)$ and $a(K)$ inherit t-niceness from K.

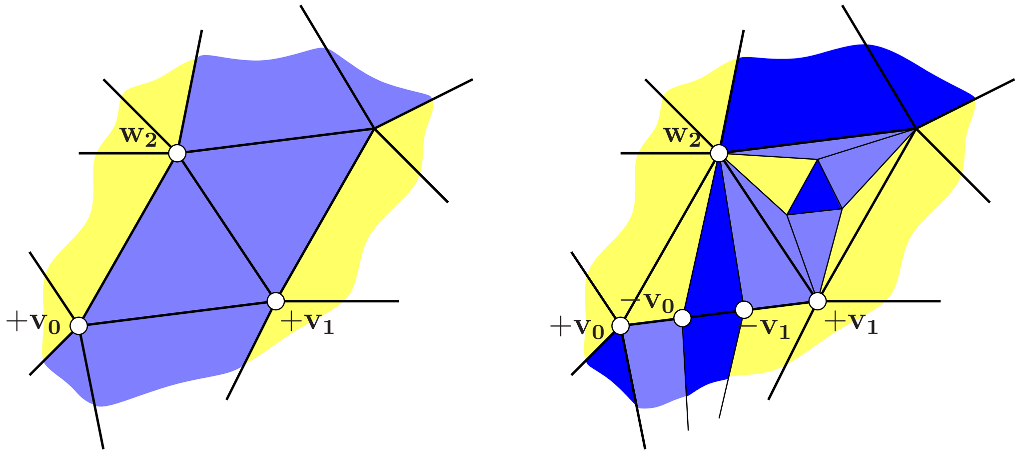


Figure 1.5. *Choosing a corresponding facet path after anti-prismatic subdivision of the edge* $\{+v_0, +v_1\}$ *and a triangle. The corresponding facets are marked.*

Proposition 1.12. Let K be a t-nice simplicial complex. The simplicial complexes $a_f(K)$, $a(K)$ and K are color equivalent.

Proof. It suffices to prove that $a_f(K)$ and K are color equivalent. First observe that $\mathrm{odd}(a_f(K)) = a_f(K_{\mathrm{odd}})$ holds: There are no odd co-dimension 2-faces in $a_f(\sigma \cup \tau)$ for any neighboring facets $\sigma, \tau \in K$ by Lemma 1.4, since $a_f(\sigma \cup \tau)$ is $(d+1)$-colorable by Lemma 1.11. As for a co-dimension 2-face $g' \in a_f(K)$ arising by subdivision of a co-dimension 2-face $g \in K$, we have

$$\mathrm{lk}_{a_f(\sigma)}(g') = c(f \setminus g) * ((\sigma \setminus f) \setminus g) ,$$

for each facet $\sigma \in \mathrm{st}_K(g)$. Here $c(f \setminus g)$ denotes $c_{\dim(f \setminus g)}$ on the vertex set $f \setminus g$. Inspection of the three cases $f \setminus g$ is empty, a vertex, or an edge (g is a co-dimension 2-face) yields that $\mathrm{lk}_{a_f(\sigma)}(g')$ is either an edge (in the first two cases) or c_1. Thus the parity of $\mathrm{lk}_{a_f(\sigma)}(g')$ equals the parity of $\mathrm{lk}_\sigma(g)$, and g' is odd if and only if g is odd.

This establishes the homeomorphism of pairs

$$\varphi : (|K|, |K_{\mathrm{odd}}|) \longrightarrow (|a_f(K)|, |\,\mathrm{odd}(a_f(K))|) ,$$

since $a_f(K)$ and K are PL-homeomorphic.

It remains to prove commutativity of Equation (1.4). Choose $\sigma_0 \in K$ and

$$\psi : V(\sigma_0) \longrightarrow V((\sigma_0)_*) : v \mapsto v_* .$$

For two neighboring facets $\sigma, \tau \in K$, and any facet path $\gamma_{(\sigma_*,\tau_*)}$ in $a_f(\sigma \cup \tau)$ from σ_* to τ_*, we have

$$(\langle \sigma, \tau \rangle (v))_* = \langle \gamma_{(\sigma_*,\tau_*)} \rangle (v_*)$$

by Lemma 1.11 and 1.4. Recall that the map $v \mapsto v_*$ is defined for each facet individually: On the right-hand side of the equation above the star denotes the image of the map $V(\tau) \to V(\tau_*)$, and on the left-hand side the image of the map $V(\sigma) \to V(\sigma_*)$.

For a given facet path $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_l)$ choose a *corresponding path*

$$\gamma_* = \gamma_{((\sigma_0)_*,(\sigma_1)_*)}\, \gamma_{((\sigma_1)_*,(\sigma_2)_*)} \cdots \gamma_{((\sigma_{l-1})_*,(\sigma_l)_*)} \ ,$$

and we have commutativity of the following diagram:

$$\begin{tikzcd}
V(\sigma_0) \arrow[r, "\langle \sigma_0,\sigma_1 \rangle"] \arrow[d, "*"] & V(\sigma_1) \arrow[r, "\langle \sigma_1,\sigma_2 \rangle"] \arrow[d, "*"] & \cdots \arrow[r, "\langle \sigma_{l-1},\sigma_l \rangle"] & V(\sigma_l) \arrow[d, "*"] \\
V((\sigma_0)_*) \arrow[r, "\langle \gamma_{((\sigma_0)_*,(\sigma_1)_*)} \rangle"] & V((\sigma_1)_*) \arrow[r, "\langle \gamma_{((\sigma_1)_*,(\sigma_2)_*)} \rangle"] & \cdots \arrow[r, "\langle \gamma_{((\sigma_{l-1})_*,(\sigma_l)_*)} \rangle"] & V((\sigma_l)_*)
\end{tikzcd}$$

Here the down arrows indicate the maps $v \mapsto v_*$. Composition of the first row yields the projectivity $\langle \gamma \rangle$, and composition of the second row yields the projectivity $\langle \gamma_* \rangle$. For an example of a corresponding path see Figure 1.5. For any closed facet path based at σ_0 commutativity of the diagram above completes the proof. $\square$

Corollary 1.13 is an immediate consequence of Proposition 1.12 and Proposition 1.10. It slightly generalizes an earlier result by Izmestiev & Joswig [36, Corollary A.1.6].

Corollary 1.13. We have $\widehat{a_f(K)} \cong \widehat{K}$ for any t-nice simplicial complex K.

1.4 Manifolds as Unfoldings

Fox [19] shows that the covering space X of a branched cover $p : X \to Y$ is a PL-manifold, provided that Y is a PL-manifold, Y_{sing} is a locally flat submanifold of co-dimension 2, and the index of branching is finite everywhere. However, in the course of this exposition we will encounter branching sets with singularities. In fact, for an arbitrary dimension d there are PL d-manifolds which can not be obtained as branched covers of the d-sphere over a locally flat submanifold of co-dimension 2; see Bernstein & Edmunds [4].

Lemma 1.14 and Proposition 1.15 shed some light on the question, what kind of singularities are allowed such that the covering space is a PL-manifold.

Lemma 1.14. Coning and unfolding commute, that is, for the complete unfolding of a simplicial complex K, and for each connected component $\widehat{K}_{(\sigma,w)}$ of the partial unfolding we have

$$\widetilde{\mathrm{cone}(K)} = \mathrm{cone}(\widetilde{K}) \quad \text{and} \quad \widehat{\mathrm{cone}(K)}_{(\sigma,w)} = \mathrm{cone}(\widehat{K}_{(\sigma,w)}) \ .$$

Proof. A facet path $\gamma = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ in K lifts uniquely to the facet path $\gamma^* = (a * \sigma_0, a * \sigma_1, \ldots, a * \sigma_k)$ in $\mathrm{cone}(K) = a * K$ with apex a. Furthermore, for any closed facet paths γ^* in $\mathrm{cone}(K)$ based at $a * \sigma_0$ we have $\langle\gamma^*\rangle(v) = \langle\gamma\rangle(v) \in \sigma_0$ for any vertex $v \in \sigma_0$, and $\langle\gamma^*\rangle(a) = a$. If one recalls the relation $\sim$ in the construction of $\widetilde{K}$, respectively $\widehat{K}$, it is immediate that two (labeled) simplices $\overline{\sigma}, \overline{\tau} \in \overline{K}$ are identified if and only if $\overline{a * \sigma}$ and $\overline{a * \tau}$ are identified. ☐

Proposition 1.15. Let K be a combinatorial d-manifold. Then $\widetilde{K}$ is a combinatorial d-manifold if and only if $\widetilde{\mathrm{lk}(v)}$ is a combinatorial $(d-1)$-sphere for each vertex $v \in K$. Similarly, a connected component $\widehat{K}_{(\sigma,w)}$ of the partial unfolding is a combinatorial d-manifold if and only if $\widehat{\mathrm{lk}(v)}_{(\sigma,w)}$ is a combinatorial $(d-1)$-sphere for each vertex $v \in K$.

Proof. As a direct consequence of Lemma 1.14 we have

$$\mathrm{lk}_{\widetilde{K}}(w) = \widetilde{\mathrm{lk}_K(v)} \quad \text{and} \quad \mathrm{lk}_{\widehat{K}_{(\sigma,w)}}(w) = \widehat{\mathrm{lk}_K(v)}_{(\sigma,w)}$$

for each vertex w in the preimage of v under the projection $\widetilde{K} \to K$, respectively $\widetilde{K} \to K$. ☐

Remark 1.16. The complete and partial unfolding of the vertex link of $v \notin K_{\mathrm{odd}}$ is always a disjoint union of copies of $\mathrm{lk}_K(v)$. Hence it suffices to verify the conditions of Proposition 1.15 for vertices $v \in K_{\mathrm{odd}}$.

In the case of the partial unfolding, recall that the odd subcomplex decomposes according to the orbit structure of the $\Pi(K)$-action, and that the connected components of $\widehat{K}$ correspond one-to-one to the orbits of the $\Pi(K)$-action; see Section 1.2.4. Hence it suffices to verify the conditions of Proposition 1.15 for vertices in the component of K_{odd} corresponding to $\widehat{K}_{(\sigma,w)}$.

If the odd subcomplex K_{odd} of a combinatorial d-manifold K is locally flat, then $\mathrm{lk}(v)$ is a combinatorial $(d-1)$-sphere with a combinatorial $(d-3)$-sphere as odd subcomplex. From $\pi_1(|K| \setminus |K_{\mathrm{odd}}|) \cong \Sigma_2$ and the classification of the covering spaces via the subgroups of the fundamental group of the base space (see Section 1.1), one deduces that $\widetilde{\mathrm{lk}(v)} \cong \mathbb{S}^{d-1}$ is the connected sum of two combinatorial $(d-1)$-spheres (thus again a combinatorial $(d-1)$-sphere), and $\widehat{\mathrm{lk}(v)}$ is the union of $\widetilde{\mathrm{lk}(v)}$ and $d-1$ copies of $\mathrm{lk}(v)$. Hence Proposition 1.15 implies Fox's statement [19] for the unfoldings.

To conclude this section we examine the case $d=3$. For $d=4$ there are two specific singularities which are essential for the construction of combinatorial 4-manifolds in Chapter 3, and they will be discussed therein. We start the analysis of the case $d=3$ by remarking that following an easy double counting argument any combinatorial 2-sphere has an even number of odd vertices. Thus any singular vertex $v \in K_{\mathrm{odd}}$ is incident to an even number ≥ 4 of odd edges.

We first classify the singular vertices with four incident odd edges, that is, singularities of the form $\mathrm{cone}(C)$, where C is a combinatorial 2-sphere with four vertices as its odd subcomplex. Up to color equivalence, there are only two types C_a and C_b of combinatorial 2-spheres with four odd vertices, which are classified by their group of projectivities: We have $\Pi(C_a) \cong \Sigma_2$ and $\Pi(C_b) \cong \Sigma_3$; see Figure 1.6 and [36, Section 5.3].

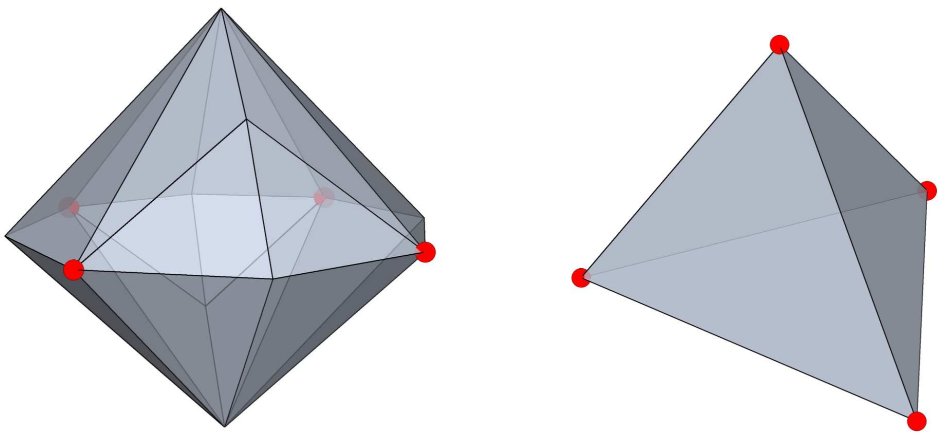


Figure 1.6. *Triangulations of $\mathbb{S}^2$ of type C_a and C_b. The odd subcomplex is marked.*

Proposition 1.17. Let K be a combinatorial 3-manifold. Then $\widetilde{K}$ is a combinatorial 3-manifold if and only if K_{odd} is locally flat. Each connected component of $\widehat{K}$ is a combinatorial 3-manifold if and only if the corresponding component of K_{odd} is locally flat except for singularities of the type $\mathrm{cone}(C_b)$.

Proposition 1.17 in not essential for the understanding of the rest of this exposition, and will only motivate its result. Via computation by hand or aided by `polymake` [21], one establishes $\widetilde{C_a} \cong \widehat{C_a} \cong \widetilde{C_b}$ are homeomorphic to the torus, and $\widehat{C_b}$ is again a 2-sphere. A triangulation of type C_a can be obtained from the boundary complex of the bipyramid over the 8-gon by stellar subdividing two opposite edges, each adjacent to one of the apices. The boundary complex of the 3-simplex is of type C_b; see Figure 1.6. The unfoldings of the boundary of the 3-simplex are discussed in [36, Section 3.3.3]. Now Proposition 1.15 proves Proposition 1.17 for singularities of the types $\operatorname{cone}(C_a)$ and $\operatorname{cone}(C_b)$.

As for a singular vertex v incident to six or more odd edges, there is always a subsingularity of the type $\operatorname{cone}(C_a)$ contained in $\operatorname{st}(v)$. Thus the unfoldings of $\operatorname{lk}(v)$ are not homeomorphic to $\mathbb{S}^2$.

Chapter 2

Foldability and Obstructions

Foldable simplicial complexes are relevant in various fields of mathematics. Here they are of interest since foldable triangulations of lattice polytopes (with some additional properties) yield lower bounds for the number of real roots of certain polynomial systems by Soprunova & Sottile [62]. Their approach and how to construct foldable triangulations of products of lattice polytopes from foldable triangulations of the factors are discussed in Chapter 4.

In Chapter 3 foldable simplicial complexes form the building blocks in the construction of triangulations with a prescribed odd subcomplex. The resulting complexes are not foldable, and the main obstruction to foldability is their odd subcomplex. In general, "foldable" and "empty odd subcomplex" are equivalent for a simply connected, t-nice simplicial complex, hence the title of this chapter.

After recalling some facts about foldability and introducing basic definitions and notations, Section 2.1 provides techniques for extending foldable (partial) triangulations of various representations of topological spaces. First a triangulation S and coloring of the $(d-1)$-sphere is extended to a triangulation and coloring of the d-ball in Theorem 2.3. Here special attention is payed to regular extensions of regular triangulations, that is, S is the boundary complex of a (simplicial) polytope. Further, we examine extensions of lattice triangulations. We also give upper bounds for the size of the extended triangulation. We proceed by extending Theorem 2.3 to partial triangulations of CW-complexes and relative handlebody decompositions of dimension at most 4. These techniques are crucial in the construction of closed oriented combinatorial 4-manifolds in Chapter 3.

Next we present a tool for the construction of simplicial complexes with a prescribed odd subcomplex. The construction begins with a foldable simplicial complex and uses stellar subdivision of edges. The simplicial complexes (and their odd subcomplexes) obtained this way are not the ones used in the construction of combinatorial 4-manifolds via the partial unfolding, but they suffice for the construction of oriented PL 4-manifolds with a handlebody representation of the form $H^0 \cup \lambda_1 H^1 \cup \lambda_2 H^2$.

Finally we examine the connection between k-colorability and the group of projectivities of a t-nice simplicial complexes further. Here the question arises whether $(d+2)$-colorability of a simply connected, t-nice simplicial d-complex K yields non-trivial upper bounds for the size of $\Pi(K)$? Conversely, does $\Pi(K) \cong \Sigma_2$ bound the chromatic number of $\Gamma(K)$ from above? We present counterexamples to both questions.

Foldable simplicial complexes. In Chapter 1 we defined a pure simplicial d-complex K to be *foldable* if K admits a non-degenerate simplicial map to the d-simplex. Equivalently, the 1-skeleton $\Gamma(K)$ of K is $(d+1)$-colorable in the graph-theoretic sense: that is, there is a map c from the vertex set $V(K)$ to the set $[d+1]$ such that for each edge $\{u, v\} \in K$ we have $c(u) \neq c(v)$. Here $[k] = \{0, \ldots, k-1\}$ denotes the set of the first k integers. Notice that there is no coloring of the vertices of K with less than $d+1$ colors, since the $d+1$ vertices of any facet form a clique. If K is strongly connected and foldable then the $(d+1)$-coloring of K is unique up to renaming the colors.

Goodman & Onishi [27] observed that the 4-Color-Theorem [57] is equivalent to the property that each simplicial 3-polytope admits a foldable triangulation (with or without additional vertices in the interior).

Remark 2.1. Other sources, including Billera & Björner [5], Stanley [63], Soprunova & Sottile [62], Izmestiev & Joswig [36], and Joswig [37], call foldable simplicial complexes "balanced." However, this seems to create conflicts with other concepts: A triangulation of a polygon whose dual graph is a balanced tree is sometimes called "balanced", and a minimal set of affinely dependent vertices of a polytope with an equal number of positive and negative coefficients is called a "balanced" circuit in Bayer [3]. Goodman & Onishi call foldable triangulations (of balls and spheres) "even." However, this does not describe the situation in the non-simply connected case. For these reasons we suggest the name "foldable" instead.

Recall that Lemma 1.4 proves that the odd subcomplex K_{odd} of a locally strongly connected and foldable simplicial complex K is empty, and that the group of projectivities $\Pi(K, \sigma_0)$ is trivial. Conversely, for t-nice simplicial complexes, we have that K is foldable if $\Pi(K) = \Pi(K, \sigma_0)$ is trivial. If, additionally, K is simply connected, then $K_{\mathrm{odd}} = \emptyset$ implies foldability; see Theorem 1.5.

Let K be a strongly connected and foldable simplicial complex of dimension d, and fix a $(d+1)$-coloring using the colors $[d+1] = \{0, 1, \ldots, d\}$. Then the $\{i_0, i_1, \ldots, i_k\}$-*skeleton* is the subcomplex of K induced by the vertices colored $\{i_0, i_1, \ldots, i_k\}$. Observe that the $\{i_0, i_1, \ldots, i_k\}$-skeleton is a pure simplicial complex of dimension k. This definition of the $\{i_0, i_1, \ldots, i_k\}$-skeleton is not to be confused with the k-*skeleton*, the collection of all faces of dimension $\leq k$.

A triangulation K of the d-sphere is *regular*, if there is a (simplicial) $(d+1)$-polytope with K as its boundary complex. In the case that K is a triangulation of a d-polytope P, we call K *regular* if K can be lifted to $d+1$ dimensions as a lower convex hull. That is, if there is a convex function $\lambda : \mathbb{R}^d \to \mathbb{R}$ such that K coincides with the polyhedral subdivision of P induced by the lower convex hull of the set

$$\left\{(v, \lambda(v)) \in \mathbb{R}^{d+1} \;\middle|\; v \in V(K)\right\} .$$

In this case λ is called a *lifting function* for K. Regular triangulations are used widely in Chapter 4, and will be discussed therein.

Consider two simplicial complexes K and L, and a continuous map $h : |K| \to |L|$ from a geometric realization $|K|$ of K to a geometric realization $|L|$ of L. A *simplicial approximation* of h is a simplicial map $h' : K \to L$, which for each vertex $v \in K$ maps the open vertex star of v into the open vertex star of $h'(v)$. Although such a simplicial approximation does not exist in general, there is always a simplicial approximation $h' : K' \to L$, where K' is a refinement of K. If K is finite, then the (finitely) iterated barycentric subdivision of K suffices; see Munkres [50, § 16]. Alternatively, the antiprismatic subdivision can be used to refine K if K is finite [36, A.1.1].

Representations of Topological Spaces. We define a *CW-complex* following Hatcher [28, p. 5]. The definition is inductively: The *0-skeleton* X^0 is a discrete set of points, called the *0-cells*. The k-*skeleton* X^k is constructed from the $(k-1)$-skeleton X^{k-1} by attaching k-balls C^k_α via maps $\varphi_\alpha : \partial C^k_\alpha \to X^{k-1}$. Here ∂C^k_α denotes the boundary of C^k_α. That is, X^k is the quotient space obtained from the disjoint union of X^{k-1} and the collection of k-balls $\{C^k_\alpha\}_\alpha$ by the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial C^k_\alpha$. Thus

$$X^k = X^{k-1} \,\dot\cup \dot\bigcup_\alpha e^k_\alpha$$

is the disjoint union of X^{k-1} and an open k-ball e^k_α for each k-ball in $\{C^k_\alpha\}_\alpha$. The open k-balls $\{e^k_\alpha\}_\alpha$ are called the k-*cells*, and the k-balls $\{C^k_\alpha = \mathrm{cl}(e^k_\alpha)\}_\alpha$ are the *closed* k-*cells*. The dimension of the highest non-empty skeleton determines the dimension of X, that is, if all cells are of dimension d or less, then $X = X^d$ is of dimension d. From now on we only consider CW-complexes X with a finite number of cells. In particular the dimension of X is defined.

A CW-complex is *regular* if each attaching map $\varphi_\alpha : \partial C_\alpha^k \to X^{k-1}$ is a homeomorphism, and the image $\varphi_\alpha(\partial C_\alpha^k) \cong \mathbb{S}^{k-1}$ is contained in a finite number of cells of a lower dimension. A (regular) CW-complex X homeomorphic to a given space $|X|$ is called a (regular) *cell decomposition*. As an example consider the cell decomposition of $\mathbb{S}^k$ obtained by attaching two k-cells to a cell decomposition of $\mathbb{S}^{k-1}$.

We present a different decomposition of topological spaces. Let N be a topological space and $R \subset N$ a subspace. A *relative handlebody decomposition* of the pair (N, R) is a sequence

$$R = N^{-1} \subset N^0 \subset N^1 \subset \cdots \subset N^d = N$$

of subspaces, where N^k is obtained from N^{k-1} by attaching a finite number of k-handles $\{H_\alpha^k\}_\alpha$; see Glaser [25, Vol. II, p. 49]. Each *k-handle* $H_\alpha^k = \mathbb{D}^k \times \mathbb{D}^{d-k}$ is attached to the boundary of N^{k-1} via a PL-embedding $f_\alpha : \partial\mathbb{D}^k \times \mathbb{D}^{d-k} \to \partial N^{k-1}$, and we require the images of the maps $\{f_\alpha\}_\alpha$ to be pairwise disjoint:

$$N^k = N^{k-1} \cup \dot{\bigcup_{f_\alpha}} H_\alpha^k$$

We call a relative handlebody decomposition of the pair $(N, \emptyset)$ a *handle representation* of N.

Example 2.2. Each finite PL d-manifold M admits a handle representation

$$M = H^0 \cup \lambda_1 H^1 \cup \lambda_2 H^2 \cup \cdots \cup \lambda_{d-1} H^{d-1} \cup H^d\ .$$

Here we suppressed the attaching maps of the handles in the notation. The space $N^k = H^0 \cup \lambda_1 H^1 \cup \cdots \cup \lambda_k H^k$ is obtained from N^{k-1} by attaching λ_k copies of a k-handle via their attaching maps. A handle representation may be obtained from any triangulation T of M by choosing an i-handle for each i-faces of T. In order to get a handle representation with a single 0-handle, choose a spanning tree of the 1-skeleton of T. The edges of T contained in the spanning tree together with the vertices of T correspond to a single 0-handle. We then attach the 1-, 2-, ..., d-handles as before. Similarly, all the ridges of T corresponding to edges in a spanning tree of the dual graph of T together with the facets of T correspond to a single d-handle.

2.1 Extending Triangulations

In this section we address the following problem: Given a partial triangulation of some space T, how can we find a triangulation of the entire space T while preserving certain properties. We are interested in particular in preserving k-colorability, which proves crucial in the construction of combinatorial 4-manifolds in Chapter 3, and regularity. The triangulated part might be low dimensional, e.g. parts of the boundary of a PL-manifold. The key example is the extension of a triangulation of $\mathbb{S}^{d-1}$ to a triangulation of $\mathbb{D}^d$; see Theorem 2.3. Here we are interested in preserving (or bounding) the chromatic number of the graph of the extended triangulation, that is, we want to extend the partial triangulation and a given coloring to a triangulation and coloring of T. Additionally we are interested in extending boundary complexes of simplicial polytopes to regular triangulations.

A first assault on this question is by Goodman & Onishi [27], who proved that a 4-colorable triangulation of $\mathbb{S}^2$ may be extended to a 4-colorable triangulation of $\mathbb{D}^3$. Their result was improved independently by Izmestiev [35] and [65] to arbitrary dimensions. The proofs in [35] and [65] are similar, and we only give a sketch of the construction here since Theorem 2.3 is a stronger result. Let S be a k-colored combinatorial $(d-1)$-sphere, and we want to extend S to a $\max\{k, d+1\}$-colored triangulation B of $\mathbb{D}^d$. We have $k \geq d$ and in the case $k = d$ set $B = \operatorname{cone}(S)$. Otherwise observe that the link $\operatorname{lk}_S(v)$ of a vertex $v \in S$ is a $(k-1)$-colored combinatorial $(d-2)$-sphere, and we may extend the triangulation of $\operatorname{lk}_S(v)$ to a $\max\{k-1, d\}$-colored triangulation B_v of $\mathbb{D}^{d-1}$ by induction. Choose one color c_0, and let C be the set of all c_0-colored vertices. Note that the interiors of $\operatorname{st}_S(v)$ and $\operatorname{st}_S(w)$ are disjoint for any two distinct vertices $v, w \in C$, since v and w are colored the same. Now we "cover" each vertex $v \in C$ by adding $v * B_v$ to S, and B can be completed by coning, that is,

$$B = \operatorname{cone}\Big(\big(S \setminus \bigcup_{v\in C} \operatorname{st}_S(v)\big) \cup \bigcup_{v\in C} B_v\Big) \cup \bigcup_{v\in C} v * B_v \ .$$

The apex is colored c_0; see Figure 2.1 (first row).

2.1.1 Extending Regular Triangulations of the Sphere

In Theorem 2.3 we assume S to be the boundary of a simplicial polytope and require B to be a regular triangulation. Surprisingly, the construction in the proof of Theorem 2.3 yields the same triangulation as the proofs in [35] and [65].

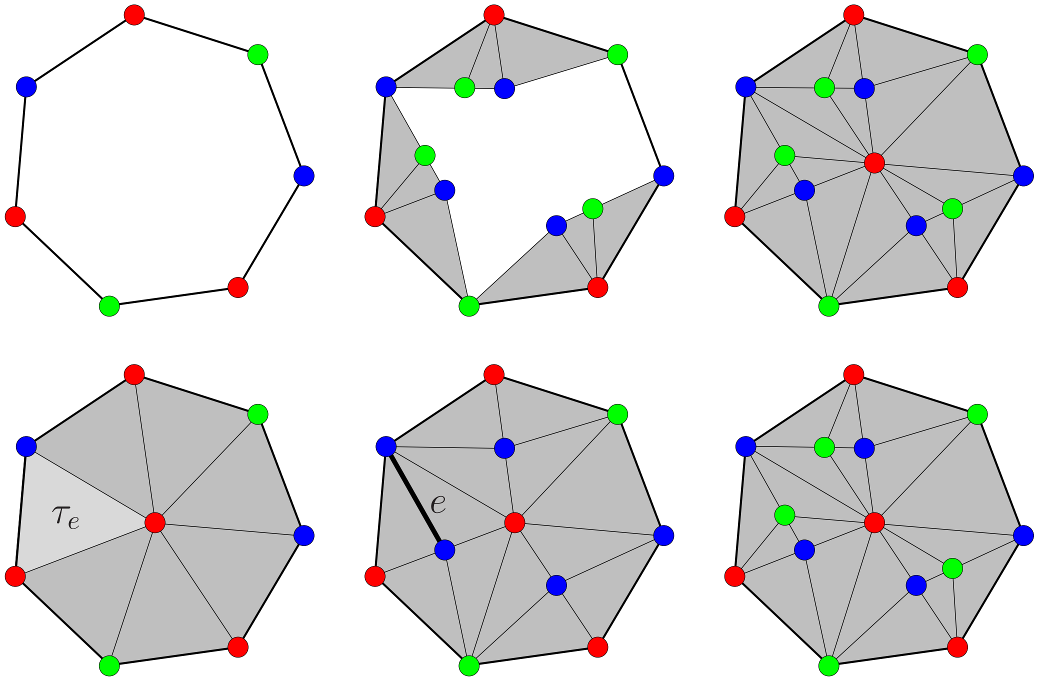


Figure 2.1. *Two ways to extend a triangulation S of $\mathbb{S}^1$ and its coloring to $\mathbb{D}^2$. In the first row vertices with one fixed color are "covered". The triangulation in the second row is obtained by stellar subdivision of conflicting edges in* $\operatorname{cone}(S)$. *A conflicting edge e and its corresponding face τ_e are marked.*

Theorem 2.3. Let S be a k-colored combinatorial $(d-1)$-sphere. Then there exists a combinatorial d-ball B with boundary ∂B equal to S such that the coloring of S may be extended to a $\max\{k, d+1\}$-coloring of B.

The d-ball B can be derived from the cone over S by a finite sequence of stellar subdivisions of edges. In particular, if S is regular then B is regular.

Proof. The d-ball B is constructed from the cone $v_0 * S$ over S with apex v_0 by a finite series of stellar subdivisions of edges. If S is regular then B is regular since $v_0 * S$ is regular, and stellar subdivision of an edge is a polytopal operation; see Ziegler [66]. Similarly, B is a combinatorial d-ball, since $v_0 * S$ is a combinatorial d-ball and stellar subdivision of edges does not change the PL-type.

It remains to show how to successively subdivide edges of $v_0 * S$ such that the resulting triangulation B is colorable using $\max\{k, d+1\}$ colors, such that $\partial B = S$, and the coloring of ∂B and S coincide. Note that the case $k = d$ is trivial as we may assign a new color to v_0. Hence we assume $k \geq d+1$, and we choose any $d+1$ colors from the k colors used to color S. Without loss of generality we assume these $d+1$ colors to be $0, 1, \ldots, d$.

We assign the color 0 to v_0, and subdivide *conflicting* edges, that is, edges with their both vertices colored the same, in a sequence of d steps $v_0 * S = B_0, B_1, \ldots, B_d = B$. To obtain B_i from B_{i-1} we stellarly subdivide all conflicting edges with vertices colored $i-1$ of B_{i-1} and color the new vertices i. Let the set of vertices introduced in the construction of B_i be V_i, and let $V_0 = \{v_0\}$. For the rest of this proof fix a geometric realization $|B_0|$ of B_0. In the case S, and hence B_0, is regular, choose $|B_0|$ to be convex. A geometric realization of B_i is obtained by assigning the barycenter of e to a new vertex introduced when subdividing an edge e, and we choose the induced geometric realization for any subcomplex of B_i. Further let $\|K\|$ be the set union of all geometric simplices of a geometric realization $|K|$ of a simplicial complex K. We prove by induction on $i = 0, 1, \ldots, d$ that the following holds:

(1) $S = \partial B_i$ and the coloring of ∂B_i and S coincide.

(2) The vertices of a conflicting edge $\{v, w\} \in B_i$ are colored i, and $v \in V_i$ and $w \in S$. In particular there are no conflicting edges in S.

(3) For each conflicting edge $e \in B_i$ there is a $(i+1)$-face $\tau_e \in B_0$, such that $\|\operatorname{st}_{B_0}(\tau_e)\| = \|\operatorname{st}_{B_i}(e)\|$ holds, and τ_e is colored $0, 0, 1, \ldots, i$ in the coloring of B_0. We call τ_e the *corresponding* face of e.

(4) The interiors of $|\operatorname{st}_{B_i}(e)|$ and $|\operatorname{st}_{B_i}(e')|$ of two conflicting edges e and e' are disjoint.

We first remark, that (3) implies that $B = B_d$ has no conflicting edges, since the corresponding face $\tau_e \in B_0$ of a conflicting edge e would have dimension $d+1$, yet B_0 is only d-dimensional. In order to make this conclusion more transparent, consider the second to last step B_{d-1}. A conflicting edge $e \in B_{d-1}$ has a facet of B_0 as its corresponding face τ_e, and τ_e is colored $0, 0, 1, \ldots, d-1$. By (3) we have

$$\|\operatorname{st}_{B_0}(\tau_e)\| = \|\tau_e\| = \|\operatorname{st}_{B_i}(e)\| \ ,$$

thus all edges $\{v, w\}$ arising when stellarly subdividing e (in the last step) have a d-colored vertex $v \in V_d$, and w is either in $V_0 \cup V_1 \cup \ldots V_{d-1}$ or a vertex of τ_e. Yet all vertices in $V_0 \cup V_1 \cup \ldots V_d$ and in τ_e are colored with a color less than d, and no conflicts arise in the last step. Thus B is $\max\{k, d+1\}$-colorable, and is an extension of S by (1).

Conditions (1) and (2) are immediate for $B_0 = v_0 * S$ by construction, and for a conflicting edge e we set $\tau_e = e$, hence (3) holds. The link of a conflicting edge $\{v, w\} \in B_0$ with $w \in S$ equals $\operatorname{lk}_S(w)$, thus there are no further 0-colored vertices in $\mathrm{lk}_S(w)$, and no other conflicting edge are contained in $\mathrm{st}_{B_0}(\{v,w\}) = \{v,w\} * \mathrm{lk}_S(w)$. This establishes (4) for B_0.

Now let $i \geq 1$. By (4) we can perform the stellar subdivisions of conflicting edges in B_{i-1} independently. Since there are no conflicting edges in $S \subset B_{i-1}$ by induction, (1) is valid for B_i. In particular $V_i \cap S = \emptyset$ holds. Each conflicting edge e of B_{i-1} is subdivided introducing a new vertex $v_e \in V_i$ colored i. Introducing v_e only causes conflicts with vertices in S, since the vertices in $V_0 \cup V_1 \cup \ldots V_{i-1}$ are colored with colors less than i by induction, thus (2) holds for B_i.

Let $\{v,w\}$ be a conflicting edge with $v \in V_i$ and $w \in S$. The vertex v was introduced by stellar subdivision of an edge $e_v \in B_{i-1}$ in the construction of B_i, and we set $\tau_{\{v,w\}} = w * \tau_{e_v}$. The face $\tau_{\{v,w\}}$ is colored $0,0,1,\ldots,i-1$ by induction, and w is colored i. Further we have

$$\begin{aligned} \|\,\mathrm{st}_{B_i}(\{v,w\})\| &= \|\,\mathrm{st}_{B_i}(v)\| \cap \|\,\mathrm{st}_{B_i}(w)\| = \|\,\mathrm{st}_{B_0}(\tau_{e_v})\| \cap \|\,\mathrm{st}_{B_i}(w)\| \\ &= \|\,\mathrm{st}_{B_0}(w * \tau_{e_v})\| = \|\,\mathrm{st}_{B_0}(\tau_{\{v,w\}})\| \end{aligned} \tag{2.1}$$

by induction, which settles (3). Finally, let $\{v,w\}$ and $\{v',w'\}$ be two conflicting edges in B_i. Then by Equation 2.1 $|\,\mathrm{st}_{B_i}(\{v,w\})| \subset |\,\mathrm{st}_{B_0}(\tau_{e_v})| = |\,\mathrm{st}_{B_{i-1}}(e_v)|$ and $|\,\mathrm{st}_{B_i}(\{v',w'\})| \subset |\,\mathrm{st}_{B_0}(\tau_{e_{v'}})| = |\,\mathrm{st}_{B_{i-1}}(e_{v'})|$. It follows that the interiors of $|\,\mathrm{st}_{B_i}(\{v,w\})|$ and $|\,\mathrm{st}_{B_i}(\{v',w'\})|$ are disjoint, since $|\,\mathrm{st}_{B_{i-1}}(e_v)|$ and $|\,\mathrm{st}_{B_{i-1}}(e_{v'})|$ are disjoint by induction. Thus (4) holds for B_i, and the proof is complete. For an illustration of the case $d=2$ see Figure 2.1 (second row) and Figure 2.2. $\square$

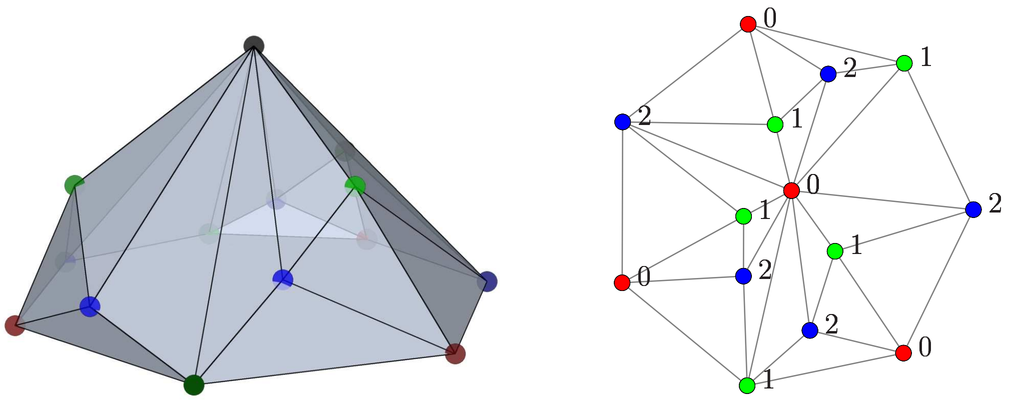


Figure 2.2. *Convex hull of the extended triangulation of a 7-gon and its Schlegel diagram.*

Remark 2.4. In the case that S is the boundary complex of some simplicial polytope P, then the triangulation B of P is actually *schlegel*, that is, there exists a convex lifting of B such that the vertices of ∂B lie in a hyperplane.

Remark 2.5. Algorithmically the construction comes down to consecutive stellar subdivision of all conflicting edges of $\mathrm{cone}(S)$ while taking care which color is assigned to the new vertex. Theorem 2.3 proves termination of the algorithm.

2.1.2 Regular Triangulations of Lattice Polytopes

Regular triangulations of lattice polytopes are instrumental for various applications. Here we would like to point out the algebraic application discussed in Chapter 4. Additionally conditions like certain coloring restrictions may be requested as well. We will apply Theorem 2.3 to vertex colored simplicial lattice polytopes such that triangulation and coloring may be extended to the interior of the polytope using only lattice points as new vertices.

Theorem 2.6. Let P be a simplicial lattice d-polytope with a k-colored graph, that is, we assume that its vertex coordinates are integral. If all vertices of the barycentric subdivision of $\mathrm{cone}(\partial P)$ are lattice points, then there exists a regular lattice triangulation of P which is $\max\{k, d+1\}$-colorable, and the triangulation and the coloring of P extend the triangulation and the coloring of ∂P.

The maximal denominator of the vertex coordinates of the barycentric subdivision of a rational d-polytope P is at most $2d$ times as large as the maximal denominator of the vertex coordinates of P. Hence the following corollary.

Corollary 2.7. If P is obtained from a simplicial lattice d-polytope by multiplication with $2d$, then there exists a regular lattice triangulation of P which is $\max\{k, d+1\}$-colorable, and the triangulation and the coloring of P extend the triangulation and the k-coloring of ∂P.

Proof of Theorem 2.6. We prove that under the conditions of Theorem 2.6 the vertices introduced in the proof of Theorem 2.3 can be realized as lattice points, and we refer to the notation of the proof of Theorem 2.3. Let e be a conflicting edge, and recall the definition of its corresponding face $\tau_e \in \mathrm{cone}(\partial P)$. Conversely, any face $\tau \in \mathrm{cone}(\partial P)$ corresponds to at most one critical edge: Let e and e' be critical edges in some B_i and B_j. First note that if $i \neq j$ the faces τ_e and $\tau_{e'}$ have different dimensions by (3). In the case $i = j$ the interiors of $\|\mathrm{st}_{B_i}(e)\| = \|\mathrm{st}_{B_0}(\tau_e)\|$ and $\|\mathrm{st}_{B_i}(e')\| = \|\mathrm{st}_{B_0}(\tau_{e'})\|$ are disjoint by (4) and (5).

Choosing the barycenter of τ_e as coordinates for the vertex introduced to subdivide a conflicting edge e completes the proof. □

2.1.3 Size of the Extended Triangulation

The triangulation constructed in the proof of Theorem 2.3 is small in the sense that the expected number $f_0(B)$ of vertices of the triangulated d-ball B is bounded by the number $f_1(S)$ of edges of the original $(d-1)$-sphere S for $d \geq 4$. Again we refer to the notation in the proof of Theorem 2.3. In the case $d < 4$ the expected value of $f_0(B)$ is bounded asymptotically by $f_1(S)$.

The triangulation of B depends on the order of the $d+1$ colors assigned to the vertices in $V_0, V_1, \ldots, V_d$. Coloring the vertices in V_i with $\sigma(i)$ for some permutation σ of $[d+1]$ is an *ordering* of the colors.

Theorem 2.8. The expected number $\mathbb{E}(f_0(B))$ (with respect to different orderings of the colors) of vertices of the extended triangulation B of S is bounded by

$$\mathbb{E}(f_0(B)) \leq 1 + f_0(S)\frac{d+2}{d+1} + f_1(S)\frac{2(d-1)}{d(d+1)} \; .$$

For $d \geq 4$ the expected number $\mathbb{E}(f_0(B))$ of vertices of B is bounded by the number of edges of S

$$\mathbb{E}(f_0(B)) \leq f_1(S) \; .$$

Rather than proving Theorem 2.8 directly, we will reformulate the problem as a counting problem for weighted edges of a graph. We want to estimate the number of conflicting edges arising in the construction of B_i from B_{i-1} for $1 \leq i \leq d-1$. Recall that no conflicting edges arise in the construction of B_d, and the conflicting edges in B_0 will be taken care of later.

Let σ be an ordering of the colors $[d+1]$, let $e \in B_{i-1}$ be a conflicting edge, and let u be the end point of e contained in S. Further let $v \in V_i$ be the new vertex introduced when subdividing e, and let $\{v, w\} \in B_i$ be a conflicting edge resulting from the subdivision of e. Then w is colored $\sigma(i)$ and $w \in \mathrm{lk}_S(u) \subset S$, hence there is an edge in S colored $\{\sigma(i-1), \sigma(i)\}$ for each conflicting edge in B_i. Conversely, for each edge $e \in S$ colored $\{\sigma(i-1), \sigma(i)\}$ there is at most one new vertex introduced in the construction of B_i: Let w be the $\sigma(i)$-colored vertex of e. Since $w \in S = \partial B_i$ and since the interiors of the stars of two conflicting edges in B_i are disjoint, w cannot be incident to two conflicting edges.

This observation motivates the following assignment of weights to the edges $\mathrm{E}(\Gamma(S))$ of the graph $\Gamma(S)$ of S. Let $c(v) \in [d+1]$ denote the color of a vertex $v \in S$ and let σ be a fixed permutation of $[d+1]$.

$$w_\sigma : \mathrm{E}(\Gamma(S)) \to \{0,1\}$$
$$\{v,w\} \mapsto \begin{cases} 1 & \text{if } |\sigma^{-1}(c(v)) - \sigma^{-1}(c(w))| = 1 \\ & \text{and } (c(v)) \neq \sigma(d) \neq (c(w)) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.9. The expected value $\mathbb{E}$ of the sum of all edge weights w_σ of $\Gamma(S)$ for a permutation σ of $[d+1]$ is

$$\mathbb{E}\Bigg(\sum_{e \in \mathrm{E}(G_S)} w_\sigma(e) \Bigg) = |\,\mathrm{E}(G_S)|\,\frac{2(d-1)}{d(d+1)}\,.$$

Proof. The expected value of $\sum_{e \in \mathrm{E}(G_S)} w_\sigma(e)$ for some ordering σ equals the average over all orderings

$$\frac{1}{(d+1)!} \sum_{\sigma} \sum_{e \in \mathrm{E}(G_S)} w_\sigma(e) = \frac{1}{(d+1)!} \sum_{e \in \mathrm{E}(G_S)} \sum_{\sigma} w_\sigma(e)\,. \tag{2.2}$$

Note that the two sums are both finite, hence the equality.

For an edge $\{v,w\} \in \mathrm{E}(G_S)$ we count all permutations σ such that $w_\sigma(\{v,w\}) = 1$ holds. There are d choices for $\sigma^{-1}(c(v))$ and another two for $\sigma^{-1}(c(w))$, unless $\sigma^{-1}(c(v)) = 0$ or $\sigma^{-1}(c(v)) = d$, summing up to $2d-2 = 2(d-1)$ choices for $\sigma^{-1}(c(v))$ and $\sigma^{-1}(c(w))$. This fixes two of the $d+1$ choices of σ, hence there is a total of $2(d-1)(d-1)!$ permutations with $w_\sigma(\{v,w\}) = 1$, and Equation 2.2 yields the desired result

$$\mathbb{E}\Bigg(\sum_{e \in \mathrm{E}(G_S)} w_\sigma(e) \Bigg) = \frac{1}{(d+1)!} \sum_{e \in \mathrm{E}(G_S)} 2(d-1)(d-1)! = |\,\mathrm{E}(G_S)|\,\frac{2(d-1)}{d(d+1)}\,.$$

□

Proof of Theorem 2.8. We assume that the $(d-1)$-sphere S is colored with exactly $d+1$ colors since the expected number of vertices of B decreases for colorings using less or more than $d+1$ colors. There is one new vertex in B_0 for the apex and some expected $f_0(S)/(d+1)$ conflicting edges arise in the construction of B_0 regardless of the ordering σ of the colors. In the construction of the triangulations B we add another expected $2f_1(S)\,(d-1)/(d(d+1))$ conflicting edges by Lemma 2.9. This proves the first statement of Theorem 2.8:

$$\begin{aligned} \mathbb{E}(f_0(B)) &\leq 1 + \frac{f_0(S)}{d+1} + f_1(S)\frac{2(d-1)}{d(d+1)} + f_0(S) \\ &= 1 + f_0(S)\frac{d+2}{d+1} + f_1(S)\frac{2(d-1)}{d(d+1)}\,. \end{aligned}$$

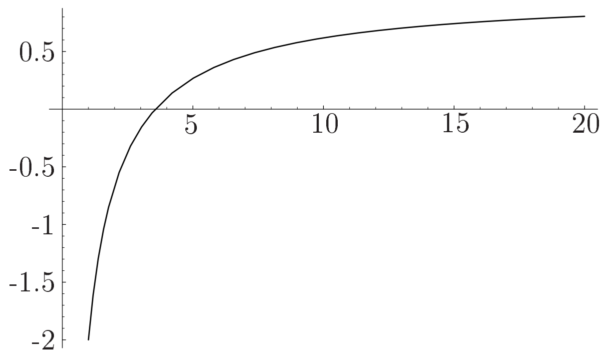
Since S is a $(d-1)$-sphere the vertex degree of each vertex in $\Gamma(S)$ is at least d, hence $d\, f_0(S) \leq 2 f_1(S)$, and

$$\begin{aligned}\mathbb{E}(f_0(B)) &\leq 1 + f_1(S)\frac{2(d+2)}{d(d+1)} + f_1(S)\frac{2(d-1)}{d(d+1)} \\ &= 1 + f_1(S)\frac{4d+2}{d(d+1)}\ .\end{aligned}$$

We want to bound the right hand side of the equation above by $f_1(S)$, which is equivalent to

$$1 \leq f_1(S)\left(1 - \frac{4d+2}{d(d+1)}\right)\ . \tag{2.3}$$

The expression $1-(4d+2)/(d(d+1))$ is positive for all integers $d \geq 4$ (and negative for all positive integers $d \leq 3$).



Recall that $f_1(S) \geq \binom{d+1}{2}$ holds (with equality in the case that S is the boundary of the d-simplex), thus for $d \geq 4$ Equation 2.3 yields

$$\begin{aligned}1 \leq f_1(S)\left(1 - \frac{4d+2}{d(d+1)}\right) &\leq \frac{(d+1)(d)}{2}\left(1 - \frac{4d+2}{d(d+1)}\right) \\ = \frac{d(d+1)(d^2-3d-2)}{2d(d+1)} &= \frac{1}{2}(d^2-3d-2)\ .\end{aligned}$$

The equation above holds for $d \geq 4$, and hence proves the second statement of Theorem 2.8. □

Remark 2.10. The second statement of Theorem 2.8 does not hold for $d \leq 3$. Let S be the boundary of d-simplex, then B is the anti-prismatic subdivision of the d-simplex, where only the d-simplex itself and no low-dimensional faces are subdivided, thus $f_0(B) = 2(d+1)$. The number of edges of S is $\binom{d+1}{2}$ and the bound of the first statement is tight for any ordering σ of the colors:

$$2(d+1) = f_0(B) \leq 1 + (d+1)\frac{d+2}{d+1} + \frac{(d+1)d}{2} \cdot \frac{2(d-1)}{d(d+1)} = 2(d+1) \; .$$

Thus B has six vertices for $d = 2$ and eight vertices for $d = 3$, yet S has only three, respectively six edges. Nevertheless, because of $f_0(S) \leq f_1(S)$ we always have the asymptotic behavior $\mathbb{E}(f_0(B)) = O(f_1(S))$.

2.1.4 Partial Triangulations of Regular CW-Complexes

In this section Theorem 2.3 is exploited to extend partial triangulations and colorings of more general spaces. The main effort is dedicated to extending partial triangulations of regular CW-complexes of dimension at most 4, since this technique is essential in the construction of combinatorial 4-manifolds in Chapter 3. Further we investigate how to extend a partial triangulation along a relative handlebody decomposition of a pair (N, R), that is, given a triangulation and coloring of R, extend the triangulation and coloring to each of the handles of the handlebody decomposition, and subsequently to a triangulation of N.

Let X be a CW-complex of dimension d with l-cells $\{e^l_\alpha\}_\alpha$ and closed cells $\{C^l_\alpha\}_\alpha = \{\mathrm{cl}(e^l_\alpha)\}_\alpha$. We call a simplicial complex $K \cong X$ a *triangulation* of X, if K refines the cell structure of X, that is, the $(d-1)$-skeleton of K is a triangulation of the CW-complex X^{d-1}. A triangulation of a 0-dimensional CW-complex is the set of 0-cells.

A subset $Y \subset \{e^l_\alpha\}_\alpha$ is called a *subcomplex* if for each closed cell $C^l_\alpha \in Y$ all cells in the image of $f_\alpha : C^l_\alpha \to X^{l-1}$ are also in Y. Hence Y is also a CW-complex, and Y is regular if X is regular. For example, any l-skeleton X^l is a subcomplex of X. We call a triangulation of a subcomplex $Y \subset X$ a *partial triangulation* of X.

Proposition 2.11. Let X be a regular CW-complex of dimension at most 4, and let $Y \subset X$ be a subcomplex. Then any triangulation and k-coloring of Y^l can be extended to a triangulation and $\max\{k, l+1\}$-coloring of X^l.

Proof. We prove by induction on $1 \leq i \leq l$ that there exists a triangulation of the i-skeleton X^i which can be colored with $\max\{k, i+1\}$ colors such that the triangulation and coloring of X^i extend the triangulation and coloring of Y^i. This clearly holds for $i = 0$, and for $i = l$ we get Proposition 2.11.

Let $i \geq 1$ and let e^i_α be an i-cell of X^i not contained in Y^i. By induction X^{i-1} is triangulated and colored using $\max\{k, i+1\}$ colors, and the triangulation of X^{i-1} extends triangulation and coloring of Y^{i-1}. Since X is regular, the image of the attaching map $f_\alpha : C^i_\alpha \to X^{i-1}$ is a $(i-1)$-sphere induced by the triangulation of X^{i-1}. Since $i \le d$ is at most 4, every simplicial $(i-1)$-sphere is a combinatorial $(i-1)$-sphere. Now Theorem 2.3 extends triangulation and coloring to the entire i-ball C^i_α. Since the i-balls $\{C^i_\alpha\}_\alpha$ intersect pairwise only in X^{i-1}, extending the triangulation of the boundary of C^i_α to its interior for each i-cell e^i_α yields the desired triangulation of X^i. □

Partial triangulation of a relative handlebody decomposition. The technique of extending a triangulation and coloring along a relative handlebody representation will not be applied in the course of this exposition. However, it is a useful tool in constructing triangulations with certain coloring properties and is applied in the construction of combinatorial 3-manifolds by Izmestiev & Joswig [36]. In the following we generalize this technique to dimension 4.

Proposition 2.12. For $d \le 4$ consider a relative handle decomposition

$$|R| = N_{-1} \subset N_0 \subset \cdots \subset N_d = N$$

of the pair $(N, |R|)$ and let R be a triangulation of $|R|$. Then the triangulation R and a k-coloring of R may be extended to a triangulation and k-coloring of N.

Proof. Observe that we may have to refine the triangulation of ∂N^{i-1} when attaching an i-handle H^i_α via its attaching map $f_\alpha : \partial\mathbb{D}^i \times \mathbb{D}^{d-i} \to \partial N_{i-1}$ in order to find a simplicial approximation of f_α. In general this can be attained by anti-prismatic subdivision of faces in ∂N_{i-1} by Lemma 1.11 and [36, A.1.1], but problems arise if the face to be subdivided is in ∂R, since we do not want to change the triangulation R. To remedy this inconvenience we extend the triangulation ∂R to a triangulation of the CW-complex $\partial R \times [0,1]$ with prisms over $(d-1)$-simplices as d-cells. Now the handles are attached to the (possible refined) triangulation $\partial R \times \{1\}$.

To this end let σ be a non-trivial, cyclic permutation of the colors of the coloring of ∂R. The vertices in $\partial R \times \{0\}$ are colored as in ∂R and the vertex colors in $\partial R \times \{1\}$ are permuted according to σ, defining a k-coloring of the 1-skeleton $\Gamma(\partial R \times [0,1])$. Proposition 2.11 extends the triangulation and k-coloring of $\Gamma(\partial R \times [0,1])$ to $\partial R \times [0,1]$, only we do not subdivide $g \times \{0\}$ for any facet $g \in \partial R$ (it is a simplex after all). Hence we assume the embeddings $\{f_\alpha\}_\alpha$ to be compatible with the triangulation of ∂N_{i-1} from now on.

The rest of the proof is by induction on the dimension d of N. Note that "attaching" a 0-handle is always trivial since we may use any foldable triangulation of $\mathbb{D}^d$, e.g. the d-simplex. Proposition 2.12 holds for $d = 0$ since there are only 0-handles to attach: N (and R) is a collection of discrete points, and all points in $N \setminus R$ may be colored using the same color.

Let $d \geq 1$, and let H^i_α be an i-handle attached to N_{i-1} via some embedding f_α. Since the images of the maps $\{f_\alpha\}_\alpha$ are pairwise disjoint, we may consider each i-handle in $\{H^i_\alpha\}_\alpha$ individually. Now f^{-1}_α induces a triangulation of $\partial\mathbb{D}^i \times \mathbb{D}^{d-i} \subset \partial H^i_\alpha$, and triangulation and coloring of $\partial\mathbb{D}^i \times \mathbb{D}^{d-i}$ may be extended to a k-colored triangulation of ∂H^i_α along a relative handle decomposition of the pair $(\partial H^i_\alpha, \partial\mathbb{D}^i \times \mathbb{D}^{d-i})$ by induction. Theorem 2.3 extends the triangulation and coloring of ∂H^i_α to H^i_α and completes the proof. □

Remark 2.13. Propositions 2.11 and 2.12 are not applicable in higher dimensions. For example, let H be a triangulation of the Poincaré homology sphere; see Björner & Lutz [9, 8] and [65]. The double suspension $\operatorname{susp}^2(H)$ is homeomorphic to $\mathbb{S}^5$, yet not a combinatorial sphere: There are two vertices with $\operatorname{susp}(H) \not\cong \mathbb{S}^4$ as vertex links. Consider the cell decomposition, respectively handlebody decomposition of the 6-ball given by the triangulation $\operatorname{susp}^2(H)$ of $\mathbb{S}^5$ plus an additional 6-cell, respectively 6-handle. Now, if Theorem 2.3 is used when attaching the final 6-cell, respectively 6-handle, one can not apply the inductive argument for the two vertices with $\operatorname{susp}(H)$ as vertex links.

2.2 The Odd Subcomplex

The odd subcomplex plays a crucial role in the study and classification of the unfoldings; see Section 1.2. In a simply connected, t-nice simplicial complex the odd subcomplex is the key obstruction to foldability by Theorem 1.5 and the observations about the group of projectivities and foldability at the beginning of this chapter.

Recall the definition from Chapter 1 of an odd face: A co-dimension 2-face $f \in K$ of a locally strongly connected simplicial complex K is *odd*, if $\Gamma(\operatorname{lk}(f))$ is not bipartite. The *odd subcomplex* is the collection of all odd faces together with their proper faces.

2.2.1 Prescribing the Odd Subcomplex

Theorem 1.9 made it clear, that it is essential to control the odd subcomplex if one tries to determine the unfoldings, e.g. in the construction of combinatorial manifolds as partial unfolding of a triangulation of the sphere. In the following we present the two techniques used in Chapter 3. Additionally we present one negative result concerning the odd subcomplex of the boundary.

Proposition 2.14. Let K be a foldable combinatorial manifold of dimension d and let F be a co-dimension 1-manifold (possibly with more than one connected component) embedded in the $\{i_0, i_1, \ldots, i_{d-1}\}$-skeleton of K. Further assume that all facets (and their proper faces) of ∂F not contained entirely in ∂K, for short the *closure* $\mathrm{cl}(\partial F \setminus \partial K)$, are embedded in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton. Then $\mathrm{cl}(\partial F \setminus \partial K)$ can be realized as the odd subcomplex of some simplicial complex K', that arises from K by stellar subdivision of edges in the $\{i_{d-1}, i_d\}$-skeleton. The complex K' is $(d+2)$-colorable by extending the coloring of K, and the odd subcomplex lies in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton.

Proof. Every $(d-1)$-simplex in F has exactly one i_{d-1}-colored vertex since F is foldable. Hence the vertex stars of all i_{d-1}-colored vertices cover F, that is,

$$F = \bigcup_{v \text{ is } i_{d-1}\text{-colored}} \mathrm{st}_F(v), \tag{2.4}$$

and the vertex stars intersect in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton. Further, a $(d-2)$-face $g \in F$ (a ridge in F) is contained in an odd number of vertex stars of i_{d-1}-colored vertices of F if and only if $g \in \partial F$ since F is an embedded combinatorial manifold.

Observe that stellar subdivision of an edge e changes the parity of $\mathrm{lk}_K(g)$ of each co-dimension 2-face $g \in \mathrm{lk}_K(e) \setminus \partial K$: First, a co-dimension 2-face in the boundary of a combinatorial manifold is never odd. Since K is a combinatorial manifold, the link of any co-dimension 2-face $g \notin \partial K$ is a triangulation of $\mathbb{S}^1$. Hence g is odd if and only if $\mathrm{lk}_K(g)$ has an odd number of edges. Stellar subdivision of e increases the number of edges in $\mathrm{lk}_K(g)$ for any co-dimension 2-face $g \in \mathrm{lk}_K(e) \setminus \partial K$ by one, and an odd face will become even and vise-versa. The odd subcomplex resulting from a series of stellar subdivisions of edges is the symmetric difference of the edge links.

Since K is a combinatorial manifold the vertex star $\mathrm{st}_K(v)$ of an i_{d-1}-colored vertex $v \in F$ is a d-ball, which is the join of v with an $(i_0, i_1, \ldots, i_{d-2}, i_d)$-colored $(d-1)$-ball if $v \in \partial K$, and which is the join with an $(i_0, i_1, \ldots, i_{d-2}, i_d)$-colored $(d-1)$-sphere otherwise. The vertex star $\mathrm{st}_F(v)$ divides $\mathrm{st}_K(v)$ into two connected components, and we will call these two connected components of $|\,\mathrm{st}_K(v)| \setminus |\,\mathrm{st}_F(v)|$ the two *sides* of $\mathrm{st}_F(v)$, mimicking the topological concept of a two-sided manifold (embedded in an orientable space); see Figure 2.3 for a 2-dimensional example. The link $\mathrm{lk}_K(\{v,w\})$ of an $\{i_{d-1}, i_d\}$-colored edge $\{v,w\} \in \mathrm{st}_K(v)$ is a $(d-2)$-sphere in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton of $\partial\,\mathrm{st}_K(v)$. Moreover, the vertex stars of all $\{i_{d-1}, i_d\}$-colored edges $\{v,w\} \in \mathrm{st}_K(v)$ cover $\mathrm{lk}_K(v)$. Thus if we stellar subdivide all $\{i_{d-1}, i_d\}$-edges in one side of $\mathrm{st}_F(v)$ we obtain $\mathrm{lk}_F(v)$ as the odd subcomplex.

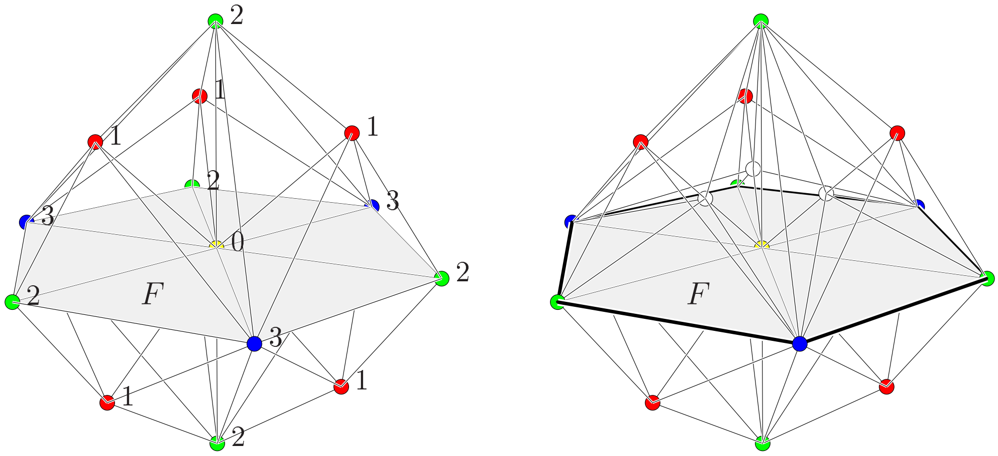


Figure 2.3. *Vertex star of an 0-colored vertex $v \in F$, on the right after stellar subdivisions of all $\{0,1\}$-edges in one side of $\mathrm{st}_F(v)$. The parity of the edges in $\mathrm{lk}_F(v)$ changes.*

Finally we construct the desired odd subcomplex $\mathrm{cl}(\partial F \setminus \partial K)$ as the symmetric difference of vertex links of all i_{d-1}-colored vertices in F.

The resulting complex K' is $(d+2)$-colorable by assigning a new color to the vertices introduced by stellar subdivision of edges. If an edge e gets subdivided twice, use the original colors of e to color the two new vertices; see also Lemma 1.11. $\square$

Observe that a projectivity based at σ_0 around an odd face exchanges the two vertices of σ_0 colored i_{d-1} and i_d.

We conclude this section with a characterization of some co-dimension 2-manifolds which by Proposition 2.14 can be realized as an odd subcomplex in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton.

Lemma 2.15. Let $|K|$ be some geometric realization of a foldable combinatorial d-manifold K. An orientable PL $(d-1)$-manifold F may be embedded in the $\{i_0, i_1, \ldots, i_{d-2}, d-1\}$-skeleton of (a refinement of) K with ∂F embedded in the $\{i_0, i_1, \ldots, i_{d-2}\}$-skeleton if there is an embedding $F \to |K|$. Note that we require the last color in the coloring of the embedding of F to be $d-1$.

Proof. Simplicial approximation of the embedding $F \to |K|$ yields an embedding of F in the co-dimension 1-skeleton of some refinement K' of K. Let $b(K')$ be the barycentric subdivision of K' with each vertex colored by the dimension of its originating face. The embedding $F \to K'$ yields an embedding $\imath F \to b(K')$ of F in the $\{0, 1, \ldots, d-1\}$-skeleton of $b(K')$, with ∂F embedded in the $\{0, 1, \ldots, d-2\}$-skeleton. Further we have that the vertex stars of all $(d-1)$-colored vertices cover F; see Equation (2.4).

It remains to show, how to "push" F into the desired skeleton. The $\{0, 1, \ldots, d-2, d-1\}$-skeleton of $b(K')$ differs from the $\{i_0, i_1, \ldots, i_{d-2}, d-1\}$-skeleton by one color $c = \{0, 1, \ldots, d-2\} \setminus \{i_0, i_1, \ldots, i_{d-2}\}$, that is, replacing c by d in $\{0, 1, \ldots, d-2, d-1\}$ yields $\{i_0, i_1, \ldots, i_{d-2}, d-1\}$. For each $(d-1)$-colored vertex $v \in F$ choose one of the two sides of $\mathrm{st}_F(v)$ consistent with the orientation of F. This may be done since F is orientable. Let $v \in F$ be $(d-1)$-colored, let D_v be the chosen side of $\mathrm{st}_F(v)$, and let V_c denote the set of all c-colored vertices in $\mathrm{lk}_F(v)$; see Figure 2.3. Now we obtain the desired embedding $\imath' : F \to b(K')$ by replacing $\mathrm{st}_F(v)$ with

$$\bigcup_{w \in V_c} v * (\mathrm{lk}_{b(K')}(\{v, w\}) \cap D_v) \cong \mathbb{D}^{d-1} .$$

Here it is important that the triangulation of $b(K')$ may have to be refined further. The map $\imath' : F \to b(K')$ is an embedding of F since we replace $(d-1)$-balls by $(d-1)$-balls, and two $(d-1)$-balls in $\imath'(F)$ intersect as in $\imath(F)$ due to the consistent choice of the sides of $\mathrm{st}_F(v)$. $\square$

The odd subcomplex of the boundary. Let M be a combinatorial manifold. In general, neither $\mathrm{odd}(\partial M) \subset M_{\mathrm{odd}} \cap \partial M$ nor $M_{\mathrm{odd}} \cap \partial M \subset \mathrm{odd}(\partial M)$ holds: Consider a triangulation S of $\mathbb{S}^d$ obtained from a foldable simplicial complex by Proposition 2.14. Then S_{odd} is non-empty, and S is $(d+2)$-colorable. Applying Theorem 2.3 yields a $(d+2)$-colorable (foldable) triangulation B of $\mathbb{D}^{d+1}$ (thus with an empty odd subcomplex), yet $\mathrm{odd}(\partial B) = S_{\mathrm{odd}} \neq \emptyset$. Conversely, let B be a foldable triangulation of $\mathbb{D}^d$, and stellar subdivide an edge $e \notin \partial B$ with $\mathrm{lk}(e) \cap \partial B \neq \emptyset$. Let B' be the resulting simplicial complex. Then $B'_{\mathrm{odd}} \cap \partial B'$ is non empty, yet $\mathrm{odd}(\partial B') = \emptyset$.

However, in the case when M is the cone over a triangulation of the sphere, the following Lemma 2.16 proves that the odd subcomplex of the boundary equals the odd subcomplex intersected with the boundary.

Lemma 2.16. The odd subcomplex of $\mathrm{cone}(K)$ of a simplicial complex K equals $\mathrm{cone}(K_{\mathrm{odd}})$.

Proof. For a face $a * f$ of $\mathrm{cone}(K) = a * K$ we have

$$\mathrm{lk}_{a*K}(a * f) = \mathrm{lk}_K(f) .$$

□

2.2.2 Coloring and the Group of Projectivities

There are several connections between the group of projectivities $\Pi(K)$ of a strongly connected simplicial d-complex K, the chromatic number of its graph $\Gamma(K)$, and its odd subcomplex K_{odd}. If $\Gamma(K)$ is $(d + 1)$-colorable, that is, if K is foldable, then $\Pi(K)$ is trivial. The converse is not true in general, but holds if K is locally strongly connected. If additionally, K is t-nice then the reduced group of projectivities $\Pi_0(K)$ is trivial if and only if the odd subcomplex is empty; see Lemma 1.4 and Theorem 1.5. If K is simply connected then $\Pi(K) = \Pi_0(K)$, and $\Pi(K)$ is trivial if and only if the odd subcomplex is empty.

Thus for a simply connected, t-nice simplicial complex K the properties "$K_{\mathrm{odd}} = \emptyset$", "$\Pi(K)$ is trivial", and "K is foldable" are equivalent. Unfortunately this equivalence does not carry through to a weaker notion of foldability in the following sense: Does $\Pi(K) \cong \Sigma_2$ yield a bound for the chromatic number of the graph $\Gamma(K)$? Conversely, does $(d + 2)$-colorability imply any non trivial restrictions on $\Pi(K)$?

The answer to both of these questions is negative. Let C be a foldable combinatorial d-sphere, and choose for $k \le d$ a set of edges $\{e_i\}_{1 \le i \le k}$ with e_i in the $\{0, i\}$-skeleton. If the stars of the edges $\{e_i\}_{1 \le i \le k}$ are pairwise disjoint, then stellarly subdividing the edges $\{e_i\}_{1 \le i \le k}$ yields a $(d+2)$-colorable complex with $\Pi(K) \cong \Sigma_k$; see Proposition 2.14. This answers the second question. The first question is answered by the following Proposition 2.17 and Example 2.20.

Proposition 2.17. Let K be a t-nice simplicial complex of dimension $d \ge 2$. Then stellar subdivision of all faces of dimension $d, d-1, \ldots, 2$ yields a t-nice simplicial complex K' with $\Pi_0(K') \cong \Sigma_2$, and $\Gamma(K)$ is a subgraph of $\Gamma(K')$ induced by the vertices originating from the vertices of K. The odd subcomplex is the union of all co-dimension 2-faces which correspond to partial flags of K starting with a triangle, that is, the odd subcomplex induced by all "new" vertices.

Stellar subdivision of all faces except the edges is called the *generalized barycentric subdivision* by Munkres [50, p. 90] with 0 assigned to all edges and 1 to all other positive-dimensional simplices.

Corollary 2.18. The odd subcomplex of K' is homotopy equivalent to a graph G_K with a node for each facet of K, and an edge if two facets share a triangle. If K is 3-dimensional then G_K is the dual graph $\Gamma^*(K)$, and K'_{odd} is homeomorphic to $\Gamma^*(K)$. In the case that K is a closed pseudo 3-manifold, the odd subcomplex of K' is the barycentric subdivision of the dual graph of K.

Proof of Proposition 2.17. Let $b(K)$ be the barycentric subdivision of K. The barycentric subdivision $b(K)$ is foldable, and we color each vertex by the dimension of the face it originated from. Stellarly subdividing an edge e twice equals the anti-prismatic subdivision of e, hence K' is color equivalent to the complex obtained from $b(K)$ as follows: For each vertex $v \in b(K)$ colored 2 we stellarly subdivide one of the two incident edges in the $\{0, 1\}$-skeleton. In other words, for each edge in K one of the two corresponding edges in $b(K)$ is stellarly subdivided. Thus the odd subcomplex of K' equals the union of the links of all edges in the $\{0, 1\}$-skeleton of $b(K)$, that is, the odd subcomplex of K' equals the $\{2, 3, \ldots, d\}$-skeleton of $b(K)$.

The reduced group of projectivities $\Pi_0(K')$ is non trivial since the odd subcomplex of K' is non-empty. It remains to show that there are $d-1$ trivial orbits, hence $\Pi_0(K')$ is generated by a single transposition. To this end color each new vertex of K' by the dimension of the face it originated from. These colors mark the $d-1$ trivial orbits, since each facet of K' contains exactly one vertex colored c for each color $c \in \{2, 3, \ldots, d\}$ by construction of K'. $\square$

Example 2.19. The group of projectivities of a 1-dimensional simplicial complex is either trivial or isomorphic to Σ_2. Hence $d = 2$ is the first interesting case where to search for a combinatorial d-sphere with a group of projectivities isomorphic to Σ_2, which is not $(d + 2)$-colorable. An example for $d \geq 3$ is given in Example 2.20. In the case $d = 2$, any 2-sphere has a planar graph, thus it is 4-colorable; see [57]. Yet if we allow other 2-manifolds than spheres, we may choose Möbius torus as our complex K. The graph of Möbius' torus is the complete graph on seven nodes, thus K' is not 4-colorable; see Figure 2.4. (In fact, the chromatic number of $\Gamma(K')$ is 7.)

Example 2.20. Let K be the boundary complex of a 2-neighborly simplicial $(d + 1)$-polytope on $n \geq d + 2$ vertices, hence $\Gamma(K)$ is the complete graph

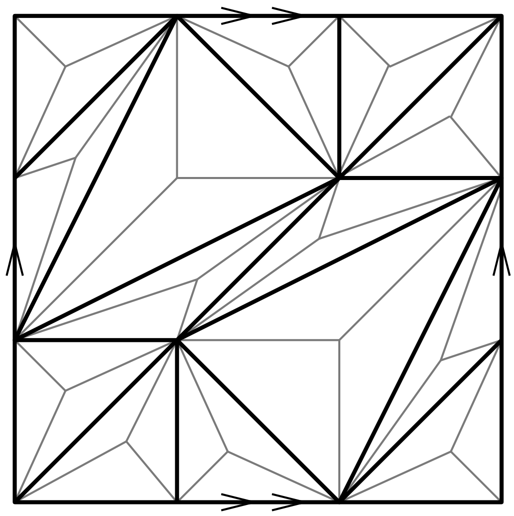


Figure 2.4. *Subdivision of Möbius torus with a group of projectivities isomorphic to* Σ_2*, and a graph with chromatic number 7. The "new" vertices correspond to the trivial orbit of the group of projectivities.*

on n nodes. For example, the cyclic $(d+1)$-polytope on $n \geq d+2$ vertices is in particular 2-neighborly for $d \geq 3$. Then K' is a combinatorial d-sphere (it is even regular) with $\Pi(K') \cong \Sigma_2$, and the chromatic number of $\Gamma(K')$ is at least n since it contains the complete graph on n nodes as a (induced) subgraph.

Chapter 3

Constructing Combinatorial 4-Manifolds

Manifolds of dimension four have been studied widely. Most prominently Simon Donaldson and Michael H. Freedman essentially classified all compact and simply-connected 4-manifolds, winning them the Fields Medal [26]. Piergallini [54] shows how to obtain any closed oriented PL 4-manifold as a 4-fold branched cover of the 4-sphere branched over an immersed PL surface with a finite number of node and cusp singularities. Prior to Piergallini's work Montesinos [47] gave a description of oriented 4-manifolds composed of 0-, 1-, and 2-handles only as a branched cover of the 4-ball. Montesinos' result is essential for Piergallini's construction of closed oriented PL 4-manifolds as branched covers. These two constructions are the "blue print" for the main result of this chapter, the construction of closed oriented combinatorial 4-manifolds obtained as unfoldings of combinatorial 4-spheres, and they are reviewed in Section 3.1. This provides the "topological view" of the situation.

Piergallini [54] and later Iori & Piergallini [33] improved the results on the construction of closed oriented PL 4-manifolds further. First Piergallini [54] eliminated the cusp singularities of the branching set. This yields a branched cover with a transversally immersed PL surface as its branching set. Iori & Piergallini [33] then proved that the branching set may be realized locally flat if one allows for a fifth sheet for the branched cover, thus proving a long-standing conjecture by Montesinos [47]. The question whether any closed oriented PL 4-manifold can be obtained as 4-fold cover of $\mathbb{S}^4$ branched over a locally flat PL-surface is still open. Although these later developments certainly ask for a combinatorial equivalent, we will not investigate these here, nor make use of these observations, since we are primarily interested in the construction of closed oriented combinatorial 4-manifolds.

In the case of 3-dimensional manifolds Hilden [29] and Montesinos [45] show independently that any closed oriented 3-manifold is a 3-fold branched cover of the 3-sphere. On the other hand, any 3-manifold M^3 may be triangulated, that is, there is a (abstract) simplicial complex homeomorphic to M^3; see E. E. Moise [44]. Izmestiev & Joswig [36] show how to obtain any closed oriented 3-manifold as the partial unfolding of a combinatorial 3-sphere S^3 and described how to construct the triangulation S^3. Their result provides a combinatorial version of the work of Hilden and Montesinos since the partial unfolding is a (simplicial) branched cover of S^3, and gives an explicit construction of a triangulation for a given closed oriented 3-manifold.

Returning to the case of 4-dimensional manifolds, it is natural to ask whether any closed oriented PL 4-manifolds may similarly be constructed as partial unfoldings of triangulated 4-spheres. In Section 3.2 we prove that this is indeed possible: For any given closed oriented PL 4-manifold M^4 there is a triangulated 4-sphere S such that one of the connected components of the partial unfolding $\widehat{S}$ is PL-homeomorphic to M. For simplicity, we will refer to the connected component of $\widehat{S}$ PL-homeomorphic to M as the partial unfolding. We proceed by giving an explicit construction of the triangulated 4-sphere S with $\widehat{S} \cong M$.

In contrast to the 3-dimensional case we restrict our attention to closed oriented PL 4-manifolds, since there are 4-manifolds which can not be triangulated; see [42, p. 9]. Again, as previously shown by Izmestiev & Joswig [36] for closed oriented 3-manifolds, Theorem 3.12 is a combinatorial version of the result of Piergallini in the sense that we give a combinatorial description of the branched cover $p : M \to \mathbb{S}^4$ for any closed oriented PL 4-manifold M.

In Section 3.3 we return to the 3-dimensional case. Essentially the problem of how to obtain a given closed oriented 3-manifold M^3 as the partial unfolding of a combinatorial 3-sphere S^3 is answered by Izmestiev & Joswig [36] as mentioned above. We revisit this problem and apply the techniques learned from the construction of closed oriented PL 4-manifolds to the construction of the combinatorial 3-sphere S^3 with $\widehat{S^3} \cong M^3$. This approach differs from the one described in [36], as starting with any combinatorial 3-sphere it uses only stellar subdivision of faces, and the operation of "twisting", which will be explained later. In this sense this new approach is a simplification of the construction by Izmestiev & Joswig [36].

The constructions presented in this chapter are at times technically involved, and we try to clearly separate the ideas from the nitty gritty details of the construction, leaving it to the reader to which extent he exposes himself to the technicalities. Of course, it is a question of personal taste where to draw the line between "ideas" and "technicalities", and we use our own discretion. Nevertheless, we hope the reader finds this structuring of the text helpful in navigating through the sometimes lengthy proofs.

3.1 4-Manifolds as Branched Covers

The main result of this chapter, the construction of a combinatorial 4-sphere S such that the partial unfolding $\widehat{S}$ of S is PL-homeomorphic to a given closed oriented PL 4-manifold M, is developed in Section 3.2. Prior to giving a combinatorial construction of M, we will review the topological situation. The following construction of a closed oriented PL 4-manifold as a branched cover of $\mathbb{S}^4$ is due to Piergallini [54] and earlier results by Montesinos [47, 48].

Let M be a closed oriented PL 4-manifold, and following Example 2.2 let

$$M = H^0 \cup \lambda H^1 \cup \mu H^2 \cup H^3 \cup \gamma H^4$$

be a handle representation of M. With $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$, and $M_B = H^0 \cup \gamma H^1$ by duality, we obtain M as the union $M_A \cup_h M_B$, where h is the attaching map. That is, we paste M_A and M_B together along their common boundary $\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2$, the connected sum of γ copies of $\mathbb{S}^1 \times \mathbb{S}^2$. In fact, Montesinos [46] proved that $H^0 \cup \lambda H^1 \cup \mu H^2$ already topologically determines M. Therefore any attaching map h identifying the boundaries of M_A and M_B yields the same manifold.

Cobordism and a trivial sheet. Let W^3 be a 3-manifold. Following Montesinos [48], we call two given branched coverings $p_1, p_2 : W^3 \to \mathbb{S}^3$ branched over links L_1 and L_2, respectively, *cobordant* if there exists a branched covering $p : W^3 \times [0,1] \to \mathbb{S}^3 \times [0,1]$ which is equal to p_1 in $W^3 \times \{0\}$, and equal to p_1 in $W^3 \times \{1\}$, and is branched over an immersed PL 2-manifold with a boundary equal to the disjoint union $L_1 \cup L_2$. The branched cover p is called a *cobordism.*

A (surprisingly) useful technique is to attach a *trivial sheet.* Given a k-fold branched cover $p_{\mathbb{D}^d} : X \to \mathbb{D}^d$ (with sheets numbered $0, 1, \ldots, k-1$), respectively $p_{\mathbb{S}^d} : X \to \mathbb{S}^d$, of a d-manifold over $\mathbb{D}^d$ or $\mathbb{S}^d$, we want to add another sheet without changing the topology of the covering space X. In the case of $p_{\mathbb{D}^d}$ we add a $(d-2)$-ball D to the branching set of $p_{\mathbb{D}^d}$ such that $D \cap \partial\mathbb{D}^d = \partial D$, and let a meridial loop around D correspond to the transposition $(1, k)$. The covering space X' of the branched cover obtained this way is the union of X and a d-ball attached to ∂X along a $(d-1)$-ball, thus $X' \cong X$.

In the case of $p_{\mathbb{S}^d}$ we add a $(d-2)$-sphere S to the branching set. Again let a meridial loop around S correspond to the transposition $(1, k)$. The covering space X' of the branched cover obtained this way is the direct sum of X and a d-sphere, thus $X' \cong X$.

Construction of the branched cover. The construction of the branched cover $p_M : M \to \mathbb{S}^4$ proceeds in two steps. As described above we have $M = M_A \cup M_B$ with $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$, and $M_B = H^0 \cup \gamma H^1$. In the first step we will construct the 4-manifolds M_A and M_B as 3-fold branched covers p_A and p_B of the 4-ball $\mathbb{D}^4$. Since $M_B = H^0 \cup \gamma H^1$ is of the form $H^0 \cup \lambda H^1 \cup \mu H^2$ it suffices to show how to construct M_A.

Although $\partial M_A = \partial M_B$ holds, the branching sets of p_A and p_B restricted to the common boundary $\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2$ of M_A and M_B may not be equivalent, and $M_A \cup_{\mathrm{Id}} M_B$ is not the covering space of a branched covering $p_M : M \to \mathbb{S}^4$ in general. Hence in the second step we construct a cobordism between $p_A|_{\partial M_A}$ and $p_B|_{\partial M_B}$, that is, a branched cover $p_H : H \to \mathbb{S}^3 \times [0,1]$ with covering space $H \cong (\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2) \times [0,1]$ which satisfies

$$p_H|_{(\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2)\times\{0\}} = p_A|_{\partial M_A} \quad \text{and} \quad p_H|_{(\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2)\times\{1\}} = p_B|_{\partial M_B} \;. \tag{3.1}$$

The cobordism p_H is branched over an immersed PL 2-manifold with a boundary equal to the disjoint union of the branching sets of $p_A|_{\partial M_A}$ and $p_B|_{\partial M_B}$. The boundary of the covering space H is homeomorphic to two disjoint copies of $\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2$, and we have $M \cong M_A \cup_{\mathrm{Id}} H \cup_{\mathrm{Id}} M_B$. Note that p_H is a 4-fold cover in general and we must add a fourth, trivial sheet to p_A and p_B.

The existence of such a cobordism, and hence the representation of M as a branched cover of $\mathbb{S}^4$, was first observed by Piergallini [54]. One can imagine this situation as shrinking the two 4-manifolds M_A and M_B and filling the space obtained by the shrinkage with the 4-manifold H. The following diagram illustrates this approach.

$$\begin{array}{ccccccccc}
M & \cong & M_A & \cup & H & \cup & M_B \\
\big\downarrow{\scriptstyle p_M} & & \big\downarrow{\scriptstyle p_A} & & \big\downarrow{\scriptstyle p_H} & & \big\downarrow{\scriptstyle p_B} \\
\mathbb{S}^4 & \cong & \mathbb{D}^4 & \cup & \mathbb{S}^3 \times [0,1] & \cup & \mathbb{D}^4
\end{array}$$

Finally the closed oriented PL 4-manifold M is obtained by the 4-fold branched cover

$$p_M : M \cong M_A \cup H \cup M_B \to \mathbb{D}^4 \,\cup\, \mathbb{S}^3 \times [0,1] \,\cup\, \mathbb{D}^4 \cong \mathbb{S}^4$$
$$x \mapsto \begin{cases} p_A(x) & \text{if} \quad x \in M_A \\ p_H(x) & \text{if} \quad x \in H \\ p_B(x) & \text{if} \quad x \in M_B. \end{cases}$$

The covering map p_M is well defined since the covering maps p_A and p_H coincide on $M_A \cap M_H$, and the covering maps p_B and p_H coincide on $M_H \cap M_B$ by the construction of the cobordism p_H.

Construction of $\mathbf{M_A}$. In the following we will sketch a construction of $p_A : M_A \to \mathbb{D}^4$ as a 3-fold branched cover branched over a ribbon manifold. This construction is due to Montesinos, and we omit the proofs; refer to [47] for further details.

First, consider the 4-manifold $W = H^0 \cup \lambda H^1 \cong \lambda \sharp\, \mathbb{S}^1 \times \mathbb{D}^3$ which consists of a single 4-ball and 1-handles only. It arises as the *standard branched cover* $p_W : W \to \mathbb{D}^4$ branched along $\lambda + 2$ unlinked and unknotted copies of $\mathbb{D}^2$. We give a sketch of an explicit construction of p_W.

Let $u : \mathbb{R}^4 \to \mathbb{R}^4$ be the reflection on the hyperplane given by $x_1 = 0$, that is, u maps (x_1, x_2, x_3, x_4) to $(-x_1, x_2, x_3, x_4)$. The covering space W is obtained from $[-1,1]^3 \times [-1,2]$ by the following identifications on its boundary. Consider the subset $\mathcal{A}$ of $[-1,1]^3$ consisting of 2λ disjoint rectangles given by

$$\mathcal{A} = \bigcup_{i=1}^{\lambda} \left\{ (x_1, x_2, x_3) \in [-1,1]^3 \;\middle|\; x_3 = 1 \text{ and } x_1 \in \pm \left[\frac{2i-1}{2\lambda+1}, \frac{2i}{2\lambda+1} \right] \right\}.$$

Now identify a point $x \in [-1,1]^3 \times [-1,2]$ with its image $u(x)$ if x lies in the top or bottom facet, that is, if $x_4 = -1$ or $x_4 = 2$, or if $(x_1, x_2, x_3) \in \mathcal{A}$ and $x_4 \in [-\frac{1}{2}, \frac{1}{2}] \cup [\frac{3}{2}, 2]$. If $\mathcal{R}$ denotes the equivalence relation given by these identifications, we have $([-1,1]^3 \times [-1,2])/\mathcal{R} \cong W$.

Similarly we obtain the base space of p_W from $[-1,1]^3 \times [0,1]$ by identifying a point $x \in [-1,1]^3 \times [0,1]$ with its image $u(x)$ if x lies in the top or bottom facet, that is, if $x_4 = 0$ or $x_4 = 1$, or if $(x_1, x_2, x_3) \in \mathcal{A}$ and $x_4 \in [0, \frac{1}{2}]$. Hence we have $([-1,1]^3 \times [0,1])/\mathcal{R}' \cong \mathbb{D}^4$, where $\mathcal{R}'$ is the equivalence relation given by the identifications described above. Figure 3.1 illustrates the 3-dimensional case.

Now we are ready to define the covering map p_W. For simplicity we identify W and $\mathbb{D}^4$ with $([-1,1]^3 \times [-1,2])/\mathcal{R}$, respectively $([-1,1]^3 \times [0,1])/\mathcal{R}'$, and let $[x]$ denote the equivalence class of x in the quotient spaces W and $\mathbb{D}^4$, respectively.

$$p_W : W \to \mathbb{D}^4$$
$$[(x_1, x_2, x_3, x_4)] \mapsto \begin{cases} [(-x_1, x_2, x_3, -x_4)] & \text{if } \quad -1 \le x_4 \le 0 \\ [(-x_1, x_2, x_3, x_4)] & \text{if } \quad 0 < x_4 \le 1 \\ [(-x_1, x_2, x_3, 2 - x_4)] & \text{if } \quad 1 < x_4 \le 2 \,, \end{cases}$$

The covering map p_W is well defined since it is compatible with $\mathcal{R}$ and $\mathcal{R}'$, and p_W is a branched cover. Note that the third sheet $([-1,1]^3 \times [1,2])/\mathcal{R}$ is homeomorphic to $\mathbb{D}^4$ and does not contribute to the construction of the

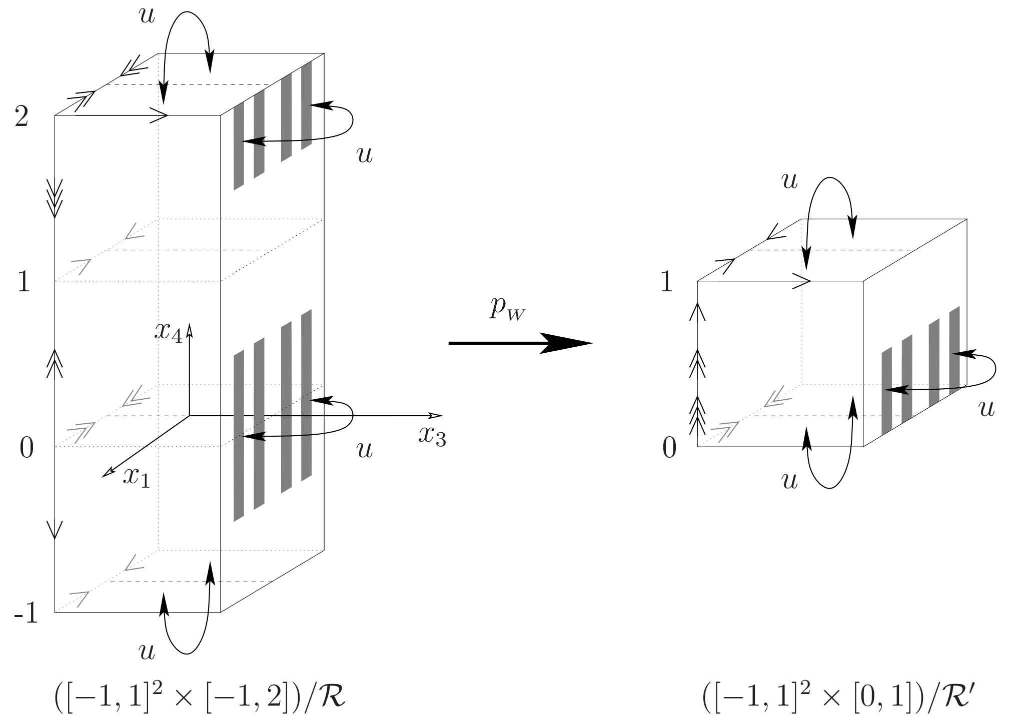


Figure 3.1. *$W = H^0 \cup 2H^1$ as 3-fold branched cover of $\mathbb{D}^4$, illustrated by the 3-dimensional case (with coordinate directions x_1, x_3, and x_4). The areas $\mathcal{A} \times ([-1/2, 1/2] \cup [3/2, 2])$, respectively $\mathcal{A} \times [0, 1/2]$ are shaded, and the arrows on the edges indicate the orientations of the image of an edge under p_W.*

1-handles; it is trivial so far. Yet it will be needed in the process of attaching the 2-handles.

We will distinguish the connected components of the branching set of p_W as follows. The branching set consists of $\lambda + 1$ pairwise disjoint unknotted 2-balls $\{P_i\}_{0 \leq i \leq \lambda}$, and a single unknotted 2-ball Q disjoint to any of the P_i. We denote the $\lambda + 1$ disjoint unknotted 2-balls by $\mathcal{P}$, and they are given by

$$\mathcal{P} = P_0 \cup P_1 \cup \cdots \cup P_\lambda = (\{0\} \times [-1,1]^2 \times \{0\}) \cup ((\mathcal{A}/\mathcal{R}') \times \{0\}) .$$

The single unknotted 2-ball Q is given by

$$Q = \{0\} \times [-1,1]^2 \times \{1\} .$$

The 2- balls $\mathcal{P} \cup Q$ intersect the boundary of $\mathbb{D}^4$ in a system of $\lambda+2$ unknotted and unlinked 1-spheres.

The preimage of a meridian $m \subset \mathbb{D}^4 \setminus (\mathcal{P} \cup Q)$ passing around any $P \in \mathcal{P}$ lies in the first and second sheet of $p_{_W}$, that is, $p_{_W}^{-1}(m)$ is contained in $([-1,1]^3 \times [-1,1])/\mathcal{R}$. On the other hand, if a meridian $m' \subset \mathbb{D}^4 \setminus (\mathcal{P} \cup Q)$ passes around Q we have $p_{_W}^{-1}(m') \subset ([-1,1]^3 \times [0,2])/\mathcal{R}$, and the preimage of m' lies in the second and third sheet of $p_{_W}$. Therefore m and m' correspond to the generators of the monodromy group $\mathfrak{M}_{p_W} \cong \Sigma_3$ of $p_{_W}$, where Σ_k denotes the symmetric group on k elements. In the following we label the the sheets 0, 1, and 2, and assume m and m' to correspond to the transpositions $(0,1) \in \Sigma_3$ and $(0,2) \in \Sigma_3$, respectively.

Attaching 2-handles. A 2-handle $H^2 \cong \mathbb{D}^4$ is attached to $W = H^0 \cup \lambda H^1$ along a solid 3-torus $\mathbb{S}^1 \times \mathbb{D}^2$ in the boundary of H^2. To be more precise, a solid 3-torus $\mathbb{S}^1 \times \mathbb{D}^2 \subset \partial H^2$ is embedded into the boundary ∂W of W via the attaching map $h : \mathbb{S}^1 \times \mathbb{D}^2 \to \partial W$, that is, h is an homeomorphism if restricted to its image. The attaching map h is determined by the image of the meridian $\mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times \mathbb{D}^2$, a knot K in ∂W. Using isotopy the knot K may be placed in ∂W such that its image $A = p_{_W}(K) \subset \partial\mathbb{D}^4$ under $p_{_W}$ is an arc which intersects the branching set $\mathcal{P} \cup Q$ of $p_{_W}$ as follows: The arc A intersects the $\lambda + 1$ connected components $\mathcal{P}$ in its end points only and does not intersect Q at all. Conversely, the preimage $p_{_W}^{-1}(A)$ of A is the knot K and a disjoint arc A'. The restriction $p_{_W}|_K$ is a 2-fold branched cover of A, and $p_{_W}|_{A'}$ is a homeomorphism corresponding to the third, trivial sheet of $p_{_W}$.

In order to represent $W \cup_h H^2$ as a 3-fold branched cover $p_{_{W \cup H^2}}$ of $\mathbb{D}^4$, we attach another 4-ball D to $\mathbb{D}^4$ along the 3-dimensional neighborhood $p_{_W} \circ h(\mathbb{S}^1 \times \mathbb{D}^2)$ of $A \subset \partial\mathbb{D}^4$. This neighborhood of A is homeomorphic to $\mathbb{D}^3$ if the domain $\mathbb{S}^1 \times \mathbb{D}^2 \subset \partial H^2$ of h is chosen sufficiently small, and the resulting base space remains homeomorphic to $\mathbb{D}^4$. The preimage of D under any 3-fold cover p with $p\,|_W = p_W$ is a collection of three "copies" of D, two of which form the 2-handle H^2 attached to W via h. The third "copy" $D' \cong \mathbb{D}^4$ is attached to W via an attaching map h' along a 3-dimensional neighborhood of A', that is, we attach D' to W along a 3-ball, and attaching D' does not alter the homeomorphic type of $W \cup_h H^2$. To be a little more explicit: Rather then attaching the 2-handle H^2 via h we attach the disjoint union $H^2 \dot\cup D'$ via the attaching map

$$h \dot\cup h' : H^2 \dot\cup D' \supset (\mathbb{S}^1 \times \mathbb{D}^2) \dot\cup \mathbb{D}^3 \to W : x \mapsto \begin{cases} h(x) & \text{if} \quad x \in H^2 \\ h'(x) & \text{if} \quad x \in D' \,. \end{cases}$$

Since $p_W|_{K \dot\cup A'}$ is a 3-fold branched cover of A, also the map

$$p_W \circ (h \dot\cup h') : (\mathbb{S}^1 \times \mathbb{D}^2) \dot\cup \mathbb{D}^3 \to \mathbb{D}^4 \cap D \cong \mathbb{D}^3$$

is a 3-fold branched cover, which may be extended to a 3-fold branched cover $p' : H^2 \dot\cup D' \to D$. Finally we are ready to define $p_{W \cup H^2}$ as

$$p_{W \cup H^2} : W \cup_{h \dot\cup h'} (H^2 \dot\cup D') \to \mathbb{D}^4 \cup D : [x] \mapsto \begin{cases} [p_W(x)] & \text{if} \quad x \in W \\ [p'(x)] & \text{if} \quad x \in H^2 \dot\cup D' \end{cases}.$$

The branched cover $p_{W \cup H^2}$ is well defined, since p_W and p' coincide on $W \cap (H^2 \cup D')$ due to the construction of p'.

The resulting branching set is the union of the branching set of p_W, the 2-balls $\mathcal{P} \cup Q$, and a 2-ball $\overline{A} \supset A$ attached to $\mathcal{P}$ along two arcs a and a' in the boundary of $\mathcal{P}$, a *ribbon manifold*; see Figure 3.2. The two arcs a and a' are neighborhoods in $\partial\mathcal{P}$ of the two endpoints $A \cap \mathcal{P}$ of A. Note that $p_{W \cup H^2}$ in general is a "proper" 3-fold branched cover (the third sheet is non-trivial), since although $\overline{A}$ does not intersect Q, it might "weave around" Q (and in fact also around any $P \in \mathcal{P}$).

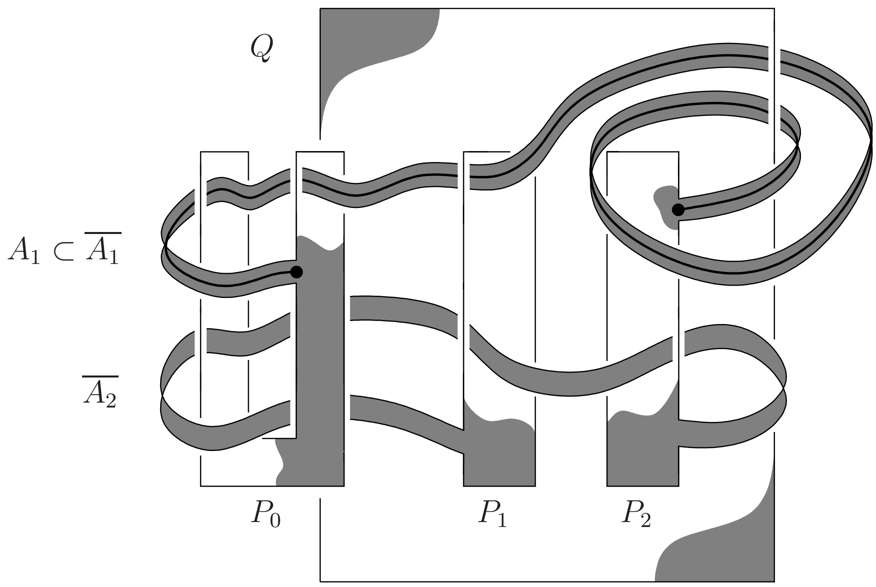


Figure 3.2. *Immersion of a ribbon manifold with two ribbons $\overline{A_1}$ and $\overline{A_2}$. Additionally the arc $A_1 \subset \overline{A_1}$ is pictured.*

Fix a set of meridial loops as generators of $\pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$, that is, choose one meridial loop around each of the 2-balls in $\mathcal{P}$, and one meridial loop around Q. Let $P, P' \in \mathcal{P}$ with $a \subset P$ and $a' \subset P'$, and let $\beta, \beta' \in \pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ be the generators corresponding to the meridial loops around P and P', respectively. Then adding the ribbon $\overline{A}$ to the branching set introduces a new relation to the fundamental group, that is, the group $\pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q \cup \overline{A}), y_0)$ differs from $\pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ by the relation

$$\beta\alpha = \beta' , \tag{3.2}$$

where the element $\alpha \in \pi_1(\mathbb{D}^4 \setminus (\mathcal{P} \cup Q), y_0)$ corresponds to the way $\overline{A}$ weaves around $\mathcal{P} \cup Q$.

We summarize the construction above by the following Theorem 3.1 due to Montesinos [45]. Note the (non-trivial) fact that the 2-handles $\{H_i^2\}_{1 \leq i \leq \mu}$ may be attached independently to $M_A = H^0 \cup \lambda H^1$.

Theorem 3.1. (Montesinos [47, Theorem 6]). Each 4-manifold $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$ is a 3-fold branched cover of $\mathbb{D}^4$, the branching set being a ribbon manifold.

Construction of H. The construction of the cobordism $p_H : H \to \mathbb{S}^3 \times [0, 1]$ is rather straight forward once we have established its existence, which is provided by the Theorems 3.2 and 3.3. Note that the branched cover $p_H : H \to \mathbb{S}^3 \times [0, 1]$ is already defined on the boundary of H by the restrictions given in Equation (3.1). The boundary of H is the disjoint union of two copies of the 3-manifold $\gamma \sharp \mathbb{S}^1 \times \mathbb{S}^2$, and the branching sets of the restrictions $p_A |_{\partial M_A}$ and $p_B |_{\partial M_B}$ are two links L_A and L_B, respectively.

In general, any 3-manifold W^3 arises as a simple 3-fold branched cover of $\mathbb{S}^3$ branched over a link L, and the monodromy group $\mathfrak{M}$ of the branched cover is isomorphic to a subgroup of Σ_3 (generated by transpositions); see [29, 45]. Consider a generic projection of L to the plane with marked over and under crossings. Such a projection is called a *diagram* of L, and we call a strand which is not crossed by other strands of the diagram a *bridge*. Fix a set of meridial loops around the bridges of the diagram as generators of $\pi_1(\mathbb{S}^3 \setminus L)$, and we identify the meridians around the bridges with transpositions in $\mathfrak{M}$ via the monodromy homomorphism $\mathfrak{m} : \pi_1(\mathbb{S}^3 \setminus L) \to \mathfrak{M}$. Hence we can think of L as a *colored link*: A bridge b of the diagram is colored (i, j) if the meridian around b corresponds to the transposition $(i, j) \in \Sigma_3$. Further we define the moves $C^{\pm}$ and $N^{\pm}$ on a colored link as in Figure 3.3.

Theorem 3.2. (Montesinos [48, p. 345]). Let $p_1, p_2 : W^3 \to \mathbb{S}^3$ be 4-fold branched covers (coming from 3-fold covers by the addition of a trivial sheet) such that it is possible to pass from the branching set L_1 of p_1 to the branching set L_2 of p_2 by a sequence of moves $C^{\pm}$ and $N^{\pm}$. Then p_1 and p_2 are cobordant

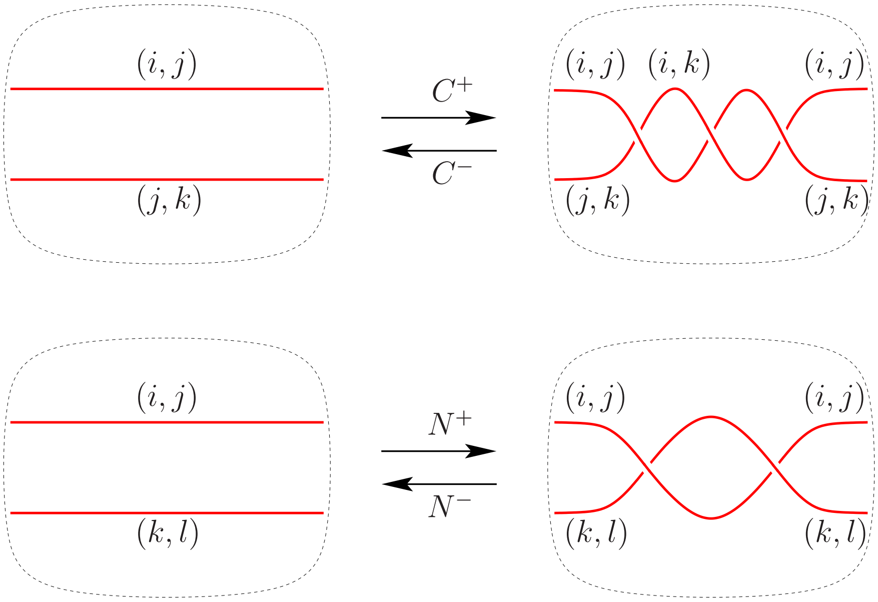


Figure 3.3. *The moves $C^{\pm}$ and $N^{\pm}$.*

and the branching set of the cobordism is an embedded PL 2-manifold with a cusp singularity (a cone over the trefoil) for each $C^{\pm}$-move and a node singularity (a cone over the Hopf link) for each $N^{\pm}$-move; see Figure 3.4.

To understand the main idea of the proof it suffices to look at two branched covers $p_1, p_2 : W^3 \to \mathbb{S}^3$ such that their branching sets L_1 and L_2 differ by exactly one $C^{\pm}$- or $N^{\pm}$-move m. Let $U \subset \mathbb{S}^3$ be a closed *neighborhood* of the move m, that is, $L_1 \setminus U \subset \mathbb{S}^3 \setminus U$ and $L_2 \setminus U \subset \mathbb{S}^3 \setminus U$ are equivalent, and replacing $L_1 \cap U$ by $L_2 \cap U$ is the move m. The branching set in $(\mathbb{S}^3 \setminus U) \times [0, 1]$ is $(L_1 \setminus U) \times [0, 1] \cong (L_2 \setminus U) \times [0, 1]$. If m is a $C^{\pm}$-move then the intersection of the branching set $(L_1 \setminus U) \times [0, 1]$ with the boundary of $U \times [0, 1]$ is the trefoil, otherwise the intersection is the Hopf link. In order to complete the base space of our cobordism, we replace $U \times [0, 1]$ by a 4-ball with a cone over the trefoil or the Hopf link, respectively, as a branching set.

Theorem 3.2 together with the following Theorem 3.3 establish the existence of the cobordism $p_{_H}$, and completes the construction of the branched cover $p_{_M} : M \to \mathbb{S}^4$. As observed by Montesinos [48], Theorem 3.4 then follows immediately.

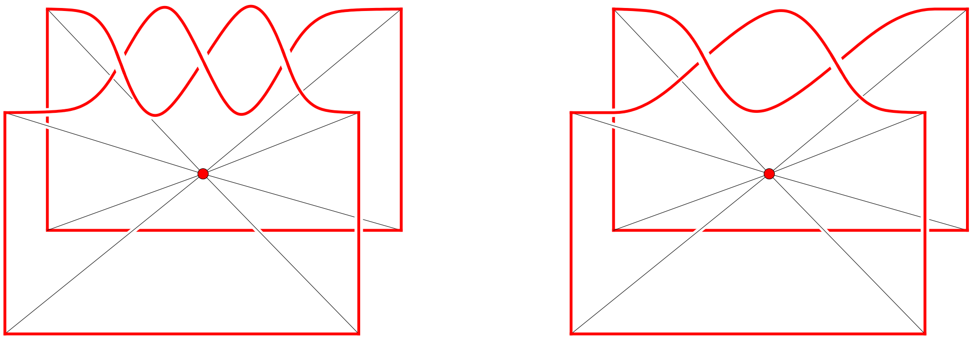


Figure 3.4. *A cusp and a node singularity.*

Theorem 3.3. (Piergallini [54, Theorem A]). Any two branching sets of 4-fold branched covers $p_1, p_2 : W^3 \to \mathbb{S}^3$ obtained from 3-fold branched covers by adding a fourth, trivial sheet, which represent the same 3-manifold W^3, are related by a finite sequence of moves $C^\pm$ and $N^\pm$.

The proof extends over two papers by Piergallini. In [52] the number of different moves needed to relate any two such branching sets via a finite sequence of moves is brought down to four. Then in [54] each of these four moves is realized by a finite sequence of $C^\pm$- and $N^\pm$-moves, and the usage of a fourth, trivial sheet, thus establishing Theorem 3.3.

Theorem 3.4. Every closed oriented PL 4-manifold is a simple 4-fold branched cover of $\mathbb{S}^4$ branched over a immersed PL surface with a finite number of cusp and node singularities.

3.2 4-Manifolds as Partial Unfoldings

Let M be a closed oriented PL 4-manifold, and let $p_{_M} : M \to \mathbb{S}^4$ be the 4-fold branched cover with branching set F described in Section 3.1. Hence F is an immersed PL surface with a finite number of cusp and node singularities by Theorem 3.4. In Theorem 3.12 we construct a triangulation S of $\mathbb{S}^4$ such that the branched cover given by the natural projection of the partial unfolding $p_{_S} : \widehat{S} \to S$ is equivalent to $p_{_M}$. In particular, $\widehat{S}$ is PL-homeomorphic to M. Recall that we refer to the (unique non-trivial) connected component of the partial unfolding PL-homeomorphic to M as *the* partial unfolding.

We outline the construction of S. The branched cover $p_{_M}$ is characterized by F and the monodromy isomorphism $\mathfrak{m}_{p_M} : \pi_1(\mathbb{S}^4 \setminus F, y_0) \to \mathrm{Sym}(p^{-1}(y_0))$, where y_0 is a point in $\mathbb{S}^4 \setminus F$; see Section 1.1.1 and Theorem 1.3. Therefore we construct S such that there is a homeomorphism of pairs $\varphi : (\mathbb{S}^4, F) \to (|S|, |S_{\mathrm{odd}}|)$, hence in particular $\varphi(F) = |S_{\mathrm{odd}}|$, and φ induces a group isomorphism $\varphi_* : \pi_1(\mathbb{S}^4 \setminus F, y_0) \to \pi_1(|S| \setminus |S_{\mathrm{odd}}|, \varphi(y_0))$. Further, assume that $\varphi(y_0)$ is the barycenter of some facet $\sigma_0 \in S$. We construct S such that the following Diagram (3.3) commutes for some bijection $\iota : p_{_M}^{-1}(y_0) \to V(\sigma_0)$ and the induced group isomorphism $\iota_* : \mathfrak{M}_{p_M} \to \Pi(S, \sigma_0)$.

$$\begin{tikzcd}
\pi_1(\mathbb{S}^4 \setminus F, y_0) \arrow[r, "\varphi_*"] \arrow[d, "\mathfrak{m}_{p_M}"] & \pi_1(|S| \setminus |S_{\mathrm{odd}}|, \varphi(y_0)) \arrow[d, "\mathfrak{h}_S"] \\
\mathfrak{M}_{p_M} \arrow[r, "\iota_*"] & \Pi(S, \sigma_0)
\end{tikzcd} \tag{3.3}$$

This establishes Theorem 3.12, since the (partial) unfolding of a t-nice simplicial complex is a branched cover by Corollary 1.8, and since $\varphi(F) = |S_{\mathrm{odd}}|$ and commutativity of Diagram (3.3) ensures that $p_{_S}$ and $p_{_M}$ are indeed equivalent by Theorem 1.9. The PL-properties follow once we proved S to be a combinatorial manifold. The following Diagram illustrates the proof of Theorem 3.12.

$$\begin{tikzcd}
M \arrow[d, "p_M"] & \widehat{S} \arrow[d, "p_S"] \\
(\mathbb{S}^4, F) \arrow[r, "\varphi"] & (|S|, |S_{\mathrm{odd}}|)
\end{tikzcd}$$

The construction of S follows closely the construction of the branched cover $p_{_M} : M = M_A \cup H \cup M_B \to \mathbb{D}^4 \cup (\mathbb{S}^3 \times [0,1]) \cup \mathbb{D}^4$ reviewed in Section 3.1: First the combinatorial 4-balls D_A and D_B are constructed such that $\widehat{D}_A \cong M_A$ and $\widehat{D}_B \cong M_B$, respectively. The resulting complex T_1 is the disjoint union of D_A and D_B, and $|T_1| \subset \mathbb{S}^4$. For each $C^{\pm}$- and $N^{\pm}$-move m needed to relate the odd subcomplexes of ∂D_A and ∂D_B we then attach a 4-ball D_m to D_A such that the partial unfolding of $D_A \cup \bigcup_m D_m$ is PL-homeomorphic to $M_A \cup H$. We refer to the simplicial complex constructed as T_2, and we have $T_1 \subset T_2$ and $|T_2| \subset \mathbb{S}^4$. In a last step we triangulate the remaining space $\mathbb{S}^4 \setminus |T_2| \cong \mathbb{S}^3 \times [0,1]$, attaching D_B to $D_A \cup \bigcup_m D_m$. This yields $T_3 = S$. In each step T_1, T_2, and T_3 of the construction of S we have to ensure

$$\begin{aligned}
&(1) \text{ that } \varphi(F) \cup |T_i| = |\operatorname{odd}(T_i)|, \text{ and} \\
&(2) \text{ that Diagram (3.3) restricted to } T_i \text{ commutes.}
\end{aligned} \tag{3.4}$$

Note that each of the complexes T_i has to be t-nice for $\mathfrak{h}_{T_i}$ to be well defined. Finally we may assume T_i to be a sufficiently fine triangulation. A fine triangulation may be obtained by anti-prismatic subdivision of faces at any stage of the construction by Proposition 1.12.

Construction of $\mathbf{T_1 = D_A \cup D_B}$. We begin with constructing a triangulation D_W of the base space of the branched cover $p_W : W = H^0 \cup \lambda H^1 \to \mathbb{D}^4$, that is, $\widehat{D_W} \cong W$. Then we modify odd(D_W) by adding the branching set which produces the μ 2-handles in order to construct a triangulation D_A of the base space of $p_A : M_A = H^0 \cup \lambda H^1 \cup \mu H^2 \to \mathbb{D}_4$, that is $\widehat{D_A} \cong M_A$. To this end let C be a sufficiently fine triangulated foldable combinatorial 4-ball obtained via the iterated barycentric subdivision of a 4-simplex. Since C arises as a barycentric subdivision there is a natural 5-coloring of the vertices of C by coloring each vertex $v \in C$ by the dimension of the original face subdivided by v. Therefore ∂C lies in the $\{0,1,2,3\}$-skeleton, and vertices colored 4 appear only in the interior of C. The triangulation D_W of $\mathbb{D}^4$ (and later the triangulation D_A) is obtained from C by a series of stellar subdivisions of edges. To cut down on notation we keep referring to our complex by C throughout all stages of the construction, and C is 6-colorable assigning a new color to all new vertices while preserving the original coloring otherwise; see Proposition 2.14.

In order to specify the isomorphism $\iota_* : \mathfrak{M}_{p_M} \to \Pi(S, \sigma_0)$ in Equation (3.3) fix a facet $\sigma_0 \in C$ and let ι map the element $x_i \in p^{-1}(y_0)$ contained in the i-th sheet of p_M to the vertex of σ_0 colored $j \in \{0, \ldots, 4\}$ via the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 & 0 \end{pmatrix}$$

We will keep σ_0 fixed throughout the construction of S. Although the choice for ι may seem arbitrary, it turns out to be useful when applying Lemma 2.15 in the construction of D_W.

Recall that subdividing an edge e in the $\{i, j\}$-skeleton yields lk(e) as the odd subcomplex in the complementary skeleton, that is, in the $(\{0, \ldots, 4\} \setminus \{i, j\})$-skeleton; see Proposition 2.14. A projectivity around a triangle in lk(e) exchanges the two vertices of σ_0 colored i and j. Via ι^{-1} such a projectivity corresponds to exchanging the elements of $p_M^{-1}(y_0)$ contained in the sheets of p_M labeled $\iota^{-1}(i)$ and $\iota^{-1}(j)$.

We first realize the 2-balls in $\mathcal{P}$ as the odd subcomplex in the $\{0,2,4\}$-skeleton, since they correspond via ι^{-1} to the transposition $(0,1)$ in $\mathfrak{M}_{p_W}$. To this end we embed for each $P \in \mathcal{P}$ a 3-ball F_P in the $\{0,2,3,4\}$-skeleton with ∂F_P in the $\{0,2,4\}$-skeleton, and $P \cong \mathrm{cl}(\partial F_P \setminus \partial C)$. Such an embedding of F_P exists by Lemma 2.15 since we assume C to be sufficiently finely triangulated, and we choose the $\{F_P\}_{P \in \mathcal{P}}$ pairwise disjoint. Now we obtain $\mathcal{P}$ as the odd subcomplex by stellar subdivision of $\{1,3\}$-edges following Proposition 2.14.

The odd subcomplex representing Q is built in a similar fashion in the $\{0,1,4\}$-skeleton, since Q corresponds via ι^{-1} to the transposition $(0,2)$

in $\mathfrak{M}_{p_W}$. The 3-ball F_Q with $Q \cong \mathrm{cl}(\partial F_Q \setminus \partial C)$ is embedded in the $\{0,1,3,4\}$-skeleton with ∂F_Q in the $\{0,1,4\}$-skeleton. Proposition 2.14 is applicable since $\mathcal{P}$ and F_Q are disjoint. Now Q is realized as the odd subcomplex in the $\{0,1,4\}$-skeleton by subdividing $\{2,3\}$-edges. This completes the construction of D_W. The odd subcomplex intersects ∂C in a system of $\lambda+1$ unknotted and unlinked $\mathbb{S}^1$ in the $\{0,2\}$-skeleton representing $\partial\mathcal{P}$, and a single unknotted and unlinked $\mathbb{S}^1$ in the $\{0,1\}$-skeleton representing ∂Q.

Finally we have to add the μ ribbons to the odd subcomplex in order to construct D_A. To this end let y_0 be the barycenter of σ_0, and fix a set of meridial loops as generators of $\pi_1(C \setminus (\mathcal{P} \cup Q), y_0)$, that is, choose one meridial loop around each of the 2-balls in $\mathcal{P}$, and one meridial loop around Q. Further assume that the generators do not intersect the collection of 3-balls $\{F_P\}_{P\in\mathcal{P}} \cup F_Q$. Then a projectivity along the image under $\mathfrak{h}_C$ of a generator around a 2-ball $P \in \mathcal{P}$ exchanges the vertices colored 1 and 3 of σ_0, and a projectivity along the image under $\mathfrak{h}_C$ of the generator around the 2-ball Q exchanges the vertices colored 2 and 3.

Now let $A \in \partial C$ be the arc corresponding to a ribbon $\overline{A}$ and let $a \subset P$ and $a' \subset P'$ be the intersection of $\overline{A}$ with $\mathcal{P}$ as described in Section 3.1. Further let β and β' be the elements of $\pi_1(C \setminus (\mathcal{P} \cup Q), y_0)$ corresponding to the meridial loops around P and P'. In order to apply Proposition 2.14 choose a simplicial 4-dimensional neighborhood U_A of A in C. (A *simplicial neighborhood* of a subcomplex L of a simplicial complex K is a subcomplex $N_L \subset K$ such that $|N_L|$ is a regular neighborhood of $|L|$ in $|K|$. Provided that K is sufficiently fine triangulated one may choose $N_L = \bigcup_{v\in L} \mathrm{st}_K(v)$ as the union of all stars of vertices in L.) The neighborhood U_A is 5-colorable since $\mathrm{odd}(U_A) = \emptyset$, and we may choose the coloring such that it coincides with the coloring of C in neighborhoods of a and a', respectively. The later assumption holds since $\beta\alpha = \beta'$ holds due to Equation (3.2), where $\alpha \in \pi_1(C \setminus (\mathcal{P} \cup Q), y_0)$ corresponds to the way A weaves around $\mathcal{P} \cup Q$. Observe that the 5-coloring U_A does not coincide with the coloring of C in general. It changes corresponding to the way A weaves around the 2-balls $\mathcal{P} \cup Q$.

Now choose a 3-ball $F_{\overline{A}}$ according to Proposition 2.14 in the $\{0,2,3,4\}$-skeleton of U_A with $\partial F_{\overline{A}}$ in the $\{0,2,4\}$-skeleton, and $\overline{A} \cong \mathrm{cl}(\partial F_{\overline{A}} \setminus \partial C)$. If we color the vertices of U_A by the coloring of C, then in general $\partial F_{\overline{A}}$ is partly embedded in the $\{0,2,4\}$-, $\{0,1,4\}$-, and $\{0,3,4\}$-skeleton, reflecting the fact that different parts of the ribbon correspond to different transposition $(0,1)$, $(0,2)$, and $(1,2)$. The intersection of $\overline{A}$ with $\mathcal{P}$ however is always contained in the $\{0,2\}$-skeleton.

The ribbon $\overline{A}$ is added to the odd subcomplex by stellar subdividing edges in the $\{1,3\}$-skeleton of U_A by Proposition 2.14. Adding all ribbons $\bigcup_{i=1}^{\mu} \overline{A}_i$ to the odd subcomplex completes the construction of D_A.

The simplicial 4-balls D_A and D_B are indeed combinatorial 4-balls (and hence t-nice) since they are constructed by subdivision of faces from the 4-simplex, and they meet conditions (1) and (2) from Equation (3.4) by construction. We have $T_1 = D_A \dot{\cup} D_B$, and we summarize the construction of D_A by the following proposition.

Proposition 3.5. For each PL 4-manifold $M_A = H^0 \cup \lambda H^1 \cup \mu H^2$ there is a combinatorial 4-ball D_A such that one of the connected components of the partial unfolding $\widehat{D_A}$ is PL-homeomorphic to M_A. The canonic projection $\widehat{D_A} \to D_A$ is a simple 3-fold branched cover with a ribbon manifold as a branching set.

Construction of $\mathbf{T_2 = D_A \cup D_H \dot{\cup} D_B}$. For the construction of the cobordism $p_H \to \mathbb{S}^3 \times [0,1]$ we need $p_{{}_A}$ and $p_{{}_B}$ to be 4-fold branched covers obtained from a 3-fold branched cover by adding a trivial sheet. The fourth sheet is obtained by adding a 2-ball in the $\{0,1,2\}$-skeleton to the odd subcomplex via stellarly subdividing edges in the $\{3,4\}$-skeleton by Proposition 2.14 and Lemma 2.15. A projectivity along a closed facet path based at σ_0 around a triangle of the newly added odd subcomplex exchanges the vertices of σ_0 colored 3 and 4, and corresponds via ι^{-1} to the transposition $(0,3)$ in $\mathfrak{M}_{p_M}$.

We first construct $D_A \cup D_H$ such that its partial unfolding yields $M_A \cup H$. In particular the odd subcomplex of the boundary of $D_A \cup D_H$ is equivalent to the odd subcomplex of D_B. To this end a combinatorial 4-ball D_m is attached to ∂D_A successively for each of the $C^{\pm}$- and $N^{\pm}$-moves required to relate the odd subcomplex of ∂D_A and ∂D_B. The 4-ball D_m realizes m in the sense that the odd subcomplexes of ∂D_A and $\partial(D_A \cup D_m)$ differ by the move m. This produces the triangulation T_2. We then identify the boundaries of D_B and $D_A \cup (\partial D_A \times [0,1])$, thus completing the triangulation $S = T_3$. Keep in mind that we have to ensure conditions (1) and (2) from Equation (3.4) to be valid throughout the construction.

The combinatorial 4-ball D_m is constructed as the cone over a combinatorial 3-sphere S_m with a trefoil knot or Hopf link as (colored!) odd subcomplex, respectively. In general, the sphere S_m may be obtained following the construction by Izmestiev & Joswig [36].

In both cases the partial unfolding $\widehat{S_m}$ is again a triangulation of $\mathbb{S}^3$, hence a combinatorial 3-sphere. With Proposition 1.15 this explains why the partial unfolding of $D_A \cup D_H$, and subsequently $\widehat{S}$ are combinatorial 4-manifolds, although the odd subcomplex is not locally flat.

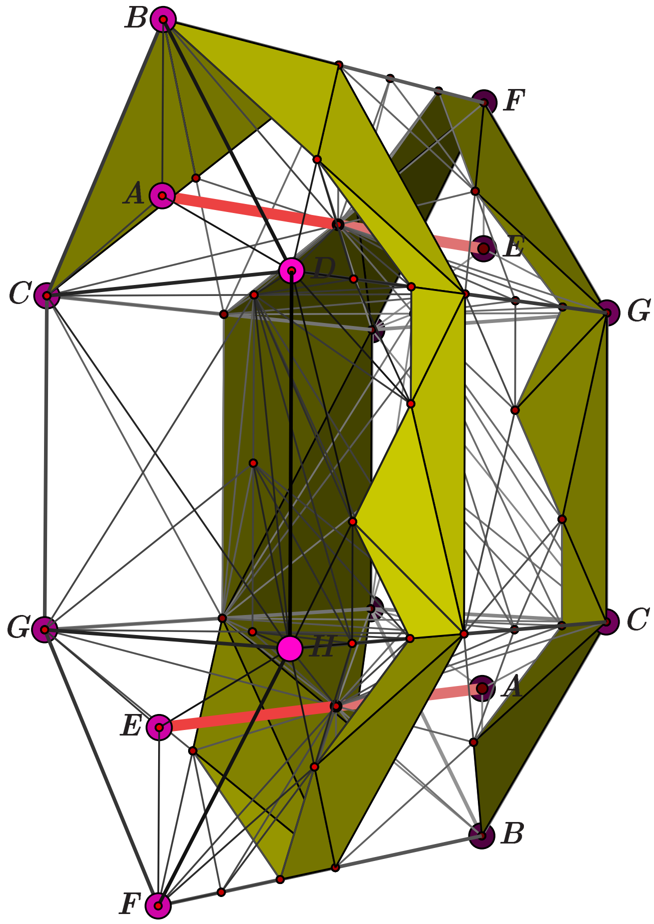


Figure 3.5. *Construction of S_m with the trefoil knot as odd subcomplex and $\Pi(S_m) \cong \Sigma_3 < \Sigma_4$. The odd subcomplex is marked.*

Example 3.6. Alternatively, in the case where m is a $C^{\pm}$-move, that is, $\mathrm{odd}(S_m)$ is the trefoil knot with the appropriate coloring of the edges (see Figure 3.3), and $\Pi(S_m) \cong \Sigma_3 < \Sigma_4$, an explicit triangulation with an f-vector (the vector with the number of i-faces as i-th entry) equal to $(68, 430, 724, 362)$ is available as an electronic model (`polymake` [21] file) by [65]. We give a sketch of the construction.

The 3-complex depicted in Figure 3.5 is homeomorphic to $\mathbb{D}^3$, has two pairs of edges as odd subcomplex, and a group of projectivities isomorphic to Σ_3. The triangulation S_m is obtained by first identifying the vertices $A, B, \ldots, H$ with a twist of 3π, thus constructing a solid torus with an trefoil knot as odd subcomplex. Now a 2-handle is attached to the area shaded in Figure 3.5, creating a 3-ball B with a trefoil knot as odd subcomplex. (Convince yourself that after identification with the 3π twist, a 2-handle may be attached to the shaded area.) Here Proposition 2.12 is used to extend a 4-coloring of a simplicial neighborhood of the shaded area to a 4-colored triangulation of the 2-handle in order to ensure that no new odd edges arise. Adding a final 3-handle to B completes S_m by extending a 4-coloring of a simplicial neighborhood of ∂B to a 4-coloring of the 3-handle by Theorem 2.3.

Example 3.7. In the case where m is an $N^{\pm}$-move, a triangulation of S_m with the Hopf link as odd subcomplex and $\Pi(S_m) \cong \Sigma_2 \times \Sigma_2 < \Sigma_4$ may be obtained from any foldable triangulation of $\mathbb{S}^3$ by first stellar subdividing an arbitrary edge e, and then stellar subdividing an edge in $\mathrm{lk}(e)$. A more efficient triangulation of S_m is obtained by taking the cone over the (unique) triangulation of the bipyramid over the triangle consisting of three tetrahedra grouped around one edge; see Figure 3.6 (left). The resulting triangulation S_m has f-vector $(6, 15, 18, 9)$, and S_m is the same triangulation as the boundary complex of the direct-sum of two triangles; see Figure 3.6 (right).

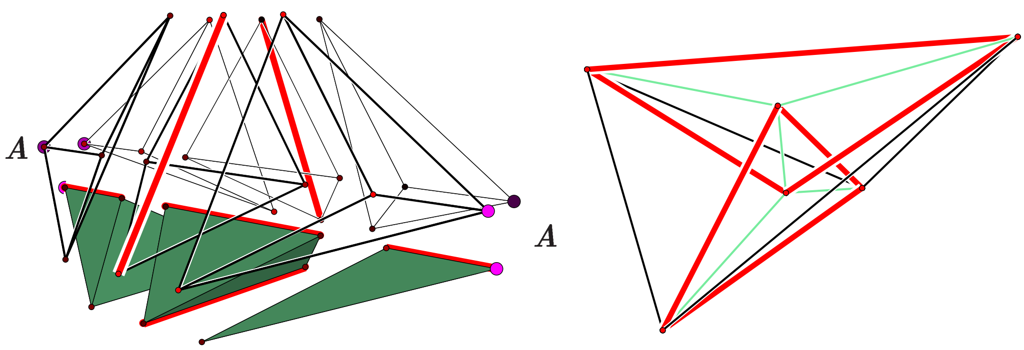


Figure 3.6. *Construction of S_m with the Hopf link as odd subcomplex and $\Pi(S_m) \cong \Sigma_2 \times \Sigma_2 < \Sigma_4$ as the cone (with apex A) over the triangulated bipyramid, and as the boundary complex of the direct-sum of two triangles, pictured as its Schlegel diagram. The odd subcomplex is marked.*

Now D_m is obtained as the cone over S_m. The resulting odd subcomplex $\mathrm{odd}(D_m)$ is a cusp or a node singularity depending on whether m is a $C^{\pm}$- or $N^{\pm}$-move by Lemma 2.16.

It remains to show how to attach D_m to D_A. Choose 3-dimensional neighborhoods $U \subset \partial D_A$ and $U' \subset \partial D_m = S_m$, such that replacing U by $S_m \setminus U'$ realizes the move m. Now the move m is realized by identifying $|U|$ and $|U'|$. Since the triangulations U and U' are non-equal in general, we triangulate the space $|U| \times [0,1]$, such that U triangulates $|U| \times \{0\}$ and U' triangulates $|U| \times \{1\}$, and such that the odd subcomplex is equivalent to the prism over $|U_{\mathrm{odd}}|$.

Attaching D_m to D_A by identifying $|U|$ to $|U'|$ is similar to the last remaining step in the construction of S, where M_B is attached to $M_A \cup H$ via identifying $|\partial(M_A \cup H)|$ and $|\partial M_B|$. We explain how to realize the identification of $|U|$ and $|U'|$, respectively of $|\partial(M_A \cup H)|$ and $|\partial M_B|$, via extending the triangulations U and U', respectively ∂M_B and $\partial(M_A \cup H)$ in a more general setting, thus completing the construction of S.

Attaching along color equivalent subcomplexes. Consider two combinatorial manifolds K and K' of dimension $d \in \{3,4\}$, and combinatorial $(d-1)$-manifolds (possibly with boundary) $U \subset \partial K$ and $U' \subset \partial K'$ with $U \cong U'$. Assume that there are color equivalent simplicial neighborhoods N and N' of U, respectively U', such that $|N_{\mathrm{odd}}|$ (and hence also $|N'_{\mathrm{odd}}|$) is equivalent to $|U \cup N_{\mathrm{odd}}| \times [0,1]$, and N_{odd} is a locally flat combinatorial $(d-2)$-manifold. Recall that $U_{\mathrm{odd}} = U \cap N_{\mathrm{odd}}$ does not hold in general; see Section 2.2.1. Further let $\varphi : |N| \to |N'|$, $\sigma_0 \in N$, $\sigma'_0 \in N'$, and $\psi : V(\sigma_0) \to V(\sigma'_0)$ as in Equation (1.4), defining the color equivalence of N and N'.

Proposition 3.8. There is a triangulation T of $|U| \times [0,1]$ with $|T_{\mathrm{odd}}|$ equivalent to $|U \cap N_{\mathrm{odd}}| \times [0,1]$, such that T equals U on $|U| \times \{0\}$ and U' on $|U| \times \{1\}$, and such that $\mathrm{odd}(K \cup T \cup K') = K_{\mathrm{odd}} \cup T_{\mathrm{odd}} \cup K'_{\mathrm{odd}}$, thus in effect attaching K' to K via identification of U and U'. Here $K \cup T \cup K'$ denotes the union of K, K', and T, attaching T to K and K' along U, respectively U'. The simplicial complex $K \cup T \cup K'$ is a combinatorial d-manifold.

In order to make the proof digestible it is split into the three Lemmas 3.9, 3.10, and 3.11. We denote a face $f \in N$ which intersects U in all except one vertex by $f = \{g, x_g\}$, where g is a face of U and x_g the one remaining vertex. Faces of N' intersecting U' in all except one vertex are denoted similarly. Throughout, $\tau \in U$ will be a facet of U, that is, a $(d-1)$-face of N.

After possible refinements of N and N' via anti-prismatic subdivision there is a simplicial approximation $\varphi' : N \to N'$ of φ which does not degenerate σ_0. Note that any simplicial approximation of φ maps N_{odd} to N'_{odd}, and U to U'; see [50, Lemma 14.4, Theorem 16.1]. Let $\sigma \in N$ be a facet, γ a facets path in n from σ_0 to σ, and let γ' be the facet path in N' defined by the non-degenerated images of facets in γ. Let κ_σ be the last facet of γ', hence $\sigma' = \varphi'(\sigma) \subset \kappa_\sigma$ in general, and $\sigma' = \kappa_\sigma$ if φ' does not degenerate σ.

We define the bijective map $\psi_\sigma : V(\sigma) \to V(\kappa_\sigma)$ by

$$\psi_\sigma = \langle \gamma' \rangle \circ \psi \circ \langle \gamma \rangle^{-1} .$$

Since N and N' are color equivalent, ψ_σ is independent of the choice of γ and hence well defined. Further note that $\psi_\sigma^{-1}|_{\sigma'}$ is injective, and that $\psi_\sigma(\sigma \cap N_{\text{odd}}) \subset N'_{\text{odd}}$ since N and N' are color equivalent.

Consider the following regular cell decomposition of $|U| \times [0,1]$. First the i-faces of U and U' form closed i-cells in the natural way. In particular, the vertices of U and U' are the 0-cells. Now we add a closed $(i+1)$-cell C_f^{i+1} for each i-face $f \in U$. The $(i+1)$-cell C_f^{i+1} is attached to the union of all i-cells along the cell decomposition of $\mathbb{S}^i$ given by the cells f (and its proper faces), $\varphi'(f)$ (and its proper faces), and all cells C_g^{j+1} with $g \subset f$ is a j-face. The top dimensional cells are the d-cells $\{C_\tau^d\}_{\tau \in U}$ corresponding to facets of U. Any two cells C_f^{i+1} and C_g^{j+1} intersect properly, that is, in the common cell corresponding to $f \cap g$, and the union of all cells equals $|U| \times [0,1]$.

We describe how to triangulate C_τ^d for each facet $\tau \in U$. Note that apart from τ and $\tau' = \varphi'(\tau)$ there might be already a triangulation induced on some cells of ∂C_τ^d via the triangulation of neighboring cells of C_τ^d. Fix a $(d+1)$-coloring on the vertices of $\{\tau, x_\tau\} \in N$, and color each vertex of τ' with the color of its preimage under $\psi_{\{\tau, x_\tau\}}$.

Lemma 3.9. The $(d+1)$-coloring of τ and τ' can be extended to a $(d+1)$-coloring of the cells of ∂C_τ^d already triangulated.

Proof. Let us call any strongly connected subcomplex of N with trivial group of projectivities which contains a facet $\sigma \in N$ a *trivial domain* of σ, and consider the trivial domain of $\{\tau, x_\tau\}$

$$O = \bigcup_{v \in \{\tau, x_\tau\} \setminus N_{\text{odd}}} \text{st}_N(v) ,$$

defined by the union of the stars of all vertices of $\{\tau, x_\tau\}$ not contained in N_{odd}. This is indeed a trivial domain if N is triangulated sufficiently fine (there are no identifications in ∂O), since no star of an odd co-dimension 2-face is contained in O, and since any facet path in O is contractable. For each cell C_f^{i+1} of ∂C_τ^d already triangulated there is a facet $\rho \in U$ in with $f = \tau \cap \rho$, and in the case $f \notin N_{\text{odd}}$ we have $\{\rho, x_\rho\} \in O$. Hence the $(d+1)$-coloring of $\{\tau, x_\tau\}$ extends uniquely to the triangulation of C_f^{i+1}. Furthermore, if there are two facets ρ and $\overline{\rho}$ with $f = \tau \cap \rho = \tau \cap \overline{\rho}$, both facets ρ and $\overline{\rho}$ produce the same coloring of the triangulation of C_f^{i+1} since $\text{st}_N(f) \subset O$, and since O is a trivial domain.

In the case where $f \in N_{\text{odd}}$ consider the subcomplex $\overline{O} = \bigcup_{v \in \{\tau, x_\tau\}} \operatorname{st}_N(v)$, a simplicial neighborhood of $\{\tau, x_\tau\}$. Assuming a sufficiently fine triangulation of N and that N_{odd} is locally flat, we have $\overline{O} \cong \mathbb{D}^d$, $\overline{O}_{\text{odd}} \cong \mathbb{D}^{d-2}$ with $\overline{O}_{\text{odd}} \cap \partial\overline{O} \cong \mathbb{S}^{d-3}$, and $\Pi(\overline{O}) \cong \Sigma_2$. Therefore $d-1$ colors of the $(d+1)$-coloring of $\{\tau, x_\tau\}$ corresponding to the $d-1$ trivial orbits of $\Pi(\overline{O})$, and let us call these $d-1$ colors the *stable colors*. Propagating the $(d+1)$-coloring of $\{\tau, x_\tau\}$ along any facets path in $\overline{O}$ from $\{\tau, x_\tau\}$ to any facet $\{\rho, x_\rho\} \in \overline{O}$ with $f = \tau \cap \rho$ yields the same coloring for the triangulation of C_f^{i+1} using only the $d-1$ stable colors, since the vertices of $\{f, x_f\}$ correspond to trivial orbits of $\Pi(\overline{O})$. ☐

Now the partial triangulation and $(d+1)$-coloring of ∂C_τ^d is extended to a triangulation and $(d+1)$-coloring of the entire cell C_τ^d using Proposition 2.11. The triangulation and $(d+1)$-coloring of C_τ^d is extended in two steps. First, let $f = \tau \cap N_{\text{odd}}$, and triangulate C_f^{i+1} applying Proposition 2.11 using only the $d-1$ stable colors, unless, of course, C_f^{i+1} is already triangulated; see Figure 3.7 for an example in $d = 4$. Then using Proposition 2.11 once more, the triangulation and $(d+1)$-coloring is extended to the entire cell C_τ^d.

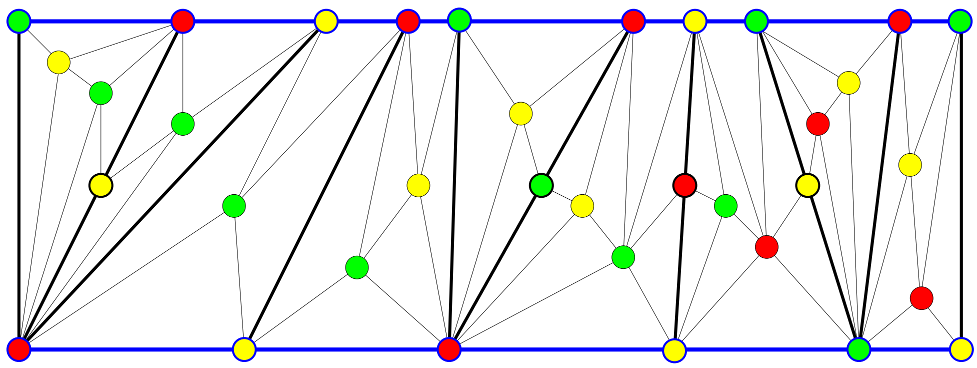


Figure 3.7. *Extending the triangulation to $\bigcup_{f \in U \cap N_{\text{odd}}} C_f^{i+1}$ using only the three stable colors.*

Lemma 3.10. The odd subcomplex of $K \cup T \cup K'$ is $K_{\text{odd}} \cup K'_{\text{odd}}$, and the union of all co-dimension 2-faces in $\bigcup_{f \in U \cap K_{\text{odd}}} C_f^{i+1}$.

Proof. We first prove that a co-dimension 2-face t in the interior of a cell C_f^{i+1} is even if $f \notin K_{\text{odd}}$. To this end let $\tau \in U$ be a facet with $f \in \tau$ and let O be the trivial domain of $\{\tau, x_\tau\}$ as described above. By construction of T there is a $(d+1)$-coloring of the triangulation of $\bigcup_{\tau \in O} C_\tau^d \supset \operatorname{st}_T(t)$, thus t is even. Any co-dimension 2-face t in U, respectively U', is even in $K \cup T \cup K'$, since for any facet $\tau \in U$ the $(d+1)$-coloring of the cell C_τ^d extends the $(d+1)$-coloring of $\{\tau, x_\tau\}$ and $\{\tau', x_{\tau'}\}$ by construction of T, hence $\mathrm{st}_{K\cup T\cup K'}(t)$ is $(d+1)$-colorable and t is even.

It remains to determine the parity of the co-dimension 2-faces in the union $\bigcup_{f\in U\cap N_{\mathrm{odd}}} C_f^{i+1}$, which form a PL $(d-2)$-manifold (with boundary) equivalent to $|U \cap N_{\mathrm{odd}}| \times [0,1]$, and we denote the co-dimension 2-faces in question suggestively by T_O. Let e be an interior co-dimension 3-face of a combinatorial manifold, hence we have $\mathrm{lk}(e) \cong \mathbb{S}^2$. It is immediate by double counting facet-ridge incidences in any simplicial pseudo manifold without boundary, that the number of facets is even, thus $\mathrm{lk}(e)$, and consequently $\mathrm{st}(e)$ has an even number of facets. We double count the number of incidences of co-dimension 2-faces $\{e,x\} \in \mathrm{st}(e)$ incident to e, and facets of $\mathrm{st}(e)$

$$\sum_{\{e,x\}\in\mathrm{st}(e)} \sharp\,\{\sigma \in \mathrm{st}(e) \mid \{e,x\} \subset \sigma\} \;=\; \sum_{\sigma\in\mathrm{st}(e)} 3\,.$$

The left hand side equals the number of odd co-dimension 2-faces incident to e modulo 2, and the right hand side is even since there is an even number of facets $\sigma \in \mathrm{st}(e)$.

Returning to our triangulation $K\cup T\cup K'$, we have that any co-dimension 3-face $e \notin \partial T_O$ is contained in none or two odd co-dimension 2-faces (e is a ridge of the $(d-2)$-manifold T_O). Therefore if there is one odd co-dimension 2-face in a (strongly) connected component of T_O, then all co-dimension 2-faces in the connected component of T_O must be odd faces of T, and each connected component of T_O intersects K_{odd} in at least one co-dimension 3-face. Thus all co-dimension 2-faces in $\bigcup_{f\in U\cap K_{\mathrm{odd}}} C_f^{i+1}$ are odd, and we proved $T_O = T_{\mathrm{odd}}$. □

Lemma 3.11. The simplicial complex $K \cup T \cup K'$ is a combinatorial d-manifold. In particular, $K \cup T \cup K'$ is a t-nice simplicial complex.

Proof. It suffices to prove that the vertex link of each vertex in T is a $(d-1)$-sphere or $(d-1)$-ball (in $K \cup T \cup K'$). The combinatorial properties follow from $d-1 \leq 3$. Let $f \in U$ be an i-face, C_f^{i+1} the corresponding closed cell of the regular cell decomposition of $|U| \times [0,1]$, and let $v \in T$ be a vertex contained in the triangulation of C_f^{i+1}. Further let $g \in U$ be an j-face containing f (thus $i \leq j$). In the case $v \in U \subset T$ (or $v \in U' \subset T$) we have

$$D(v,g) = \bigl|\,\mathrm{lk}_T(v)\bigr| \cap C_g^{j+1} \;\cong\; \mathrm{cone}\,\bigl(\partial\,\mathrm{st}_U(f) \cap g\bigr)\ ,$$

and otherwise

$$D(v,g) = \bigl|\,\mathrm{lk}_T(v)\bigr| \cap C_g^{j+1} \;\cong\; \mathrm{susp}\,\bigl(\partial\,\mathrm{st}_U(f) \cap g\bigr)\ .$$

Observe that if $i = d-1$, that is, f is a $(d-1)$-face (a facet of U), then $\mathrm{lk}_T(v)$ is a $(d-1)$-ball (if $v \in \partial T$) or $(d-1)$-sphere completely contained in C_f^d. Otherwise $D(v,g)$ is a $(d-1)$-ball in the case $v \in U \cup U'$ as well as in the case $v \notin U \cup U'$ for $j \neq 0$. In the remaining case $v \notin U \cup U'$ and $j = 0$ we have $D(v,g) \cong \mathbb{S}^0$. (Recall that $\mathrm{cone}(\emptyset) \cong \mathbb{D}^0$ and $\mathrm{susp}(\emptyset) \cong \mathbb{S}^0$ holds by definition.)

For $i < d-1$ let $\tau, \tau' \in \mathrm{st}_U(f)$ be facets intersecting in $g = \tau \cap \tau' \supset f$. Then the two $(d-1)$-balls $D(v,\tau)$ and $D(v,\tau')$ intersect in $D(v,g)$. Assume that $f \notin \partial U$ holds. Since $\mathrm{st}_U(f)$ is a combinatorial $(d-1)$-ball (and $\partial\, \mathrm{st}_U(f)$ a combinatorial $(d-2)$-sphere) we have

$$\mathrm{lk}_T(v) \cong \bigcup_{\tau \in \mathrm{st}_U(f)} D(v,\tau) \cong \mathrm{cone}(\partial\, \mathrm{st}_U(f))$$

if $v \in U \cup U'$, and

$$\mathrm{lk}_T(v) \cong \bigcup_{\tau \in \mathrm{st}_U(f)} D(v,\tau) \cong \mathrm{susp}(\partial\, \mathrm{st}_U(f))$$

otherwise. The case $f \in \partial U$ is treated similarly, except we consider the $(d-2)$-ball $\mathrm{cl}(\partial\, \mathrm{st}_U(f) \setminus \partial U)$ instead of the entire boundary of $\mathrm{st}_U(f)$. Thus T is a combinatorial d-manifold.

It remains to prove that $\mathrm{lk}_{K \cap T \cap K'}(v)$ is a $(d-1)$-sphere or $(d-1)$-ball for a vertex $v \in U \subset T$ (or $v \in U' \subset T$). This follows since $\mathrm{lk}_{K \cap T \cap K'}(v)$ is the union of the two combinatorial $(d-1)$-balls $\mathrm{lk}_T(v)$ and $\mathrm{lk}_K(v)$, respectively $\mathrm{lk}_{K'}(v)$. $\square$

It remains to verify conditions (1) and (2) from Equation (3.4). As for condition (1), T_{odd} is homotopy equivalent to U_{odd}. Further, any path around an odd triangle in the triangulation of some cell C_f^{i+1}, where f is an edge in $U \cap N_{\mathrm{odd}}$, is homotopy equivalent to a path around the (unique) triangle $\{f, x_f\} \in N_{\mathrm{odd}}$. This settles condition (2).

Attaching $\bigcup_m D_m$ to D_A producing $D_A \cup D_H$, and then attaching D_B to $D_A \cup D_H$ as described above completes the construction of S, a combinatorial manifold homeomorphic to $\mathbb{S}^4$ (note the difference to a combinatorial 4-sphere) with $\widehat{S} \cong M$ for a given closed oriented PL 4-manifold M. The partial unfolding $\widehat{S}$ is a combinatorial manifold by Proposition 1.15, thus $\widehat{S}$ is PL-homeomorphic to M.

We summarize the construction in the following Theorem 3.12 which states the main result of this chapter.

Theorem 3.12. For every closed oriented PL 4-manifold M there is a combinatorial manifold S homeomorphic to $\mathbb{S}^4$ such that one of the connected components of the partial unfolding $\widehat{S}$ of S is a combinatorial 4-manifold PL-homeomorphic to M. The canonical projection $\widehat{S} \to S$ is a simple 4-fold branched cover branched over a PL surface with a finite number of cusp and node singularities.

Remark 3.13. Let M be a simplicial manifold obtained as the partial unfolding of a combinatorial sphere. In general, we can not assume M to be a combinatorial manifold. As an example consider a combinatorial 3-sphere S with $H = \widehat{S}$ is a triangulation of the Poincaré homology sphere, see [65, Section 5.2.2] and [36]. The double suspension $\operatorname{susp}^2(S)$ is again a combinatorial sphere, yet its partial unfolding is the double suspension $\operatorname{susp}^2(H)$, a simplicial 5-sphere which is not a combinatorial manifold.

3.3 Constructing Combinatorial 3-Manifolds

Izmestiev & Joswig [36] show how to construct combinatorial 3-manifolds as partial unfoldings of combinatorial 3-spheres. Apart from defining the partial unfolding, they show how to construct the base space, a combinatorial 3-sphere with a colored link as its odd subcomplex. We want to reconsider the construction of the base space, applying the new techniques used in the construction of 4-manifolds above.

We will prove that the desired odd subcomplex can be constructed by stellar subdivision of faces and *twisting*, which will be introduced in the following.

The branching set. Recall the definition of a diagram of a link from Section 3.1. A diagram is 3-*colored*, if each bridge is colored with one color, and the three bridges meeting at a crossing are either colored the same or no two of them have the same color; see Figure 3.8. For an introduction to knot theory see Adams [1]. The notion of 3-coloring of a link diagram presented here differs from [1], since we allow at most three colors, rather then exactly three colors. Thus every diagram of a link has a (trivial) 3-coloring in our sense by using the same color for all bridges, which is not true if one requires all three colors to be used. In particular no diagram of the trivial knot allows for a 3-coloring using all three colors [1, p. 25].

Consider a closed oriented 3-manifold M^3, and a simple 3-fold branched cover $p_{M^3} : M^3 \to \mathbb{S}^3$ branched over a link L. The branched cover p_{M^3} exists by Hilden [29] and Montesinos [45]. Fixing a diagram of L yields a set

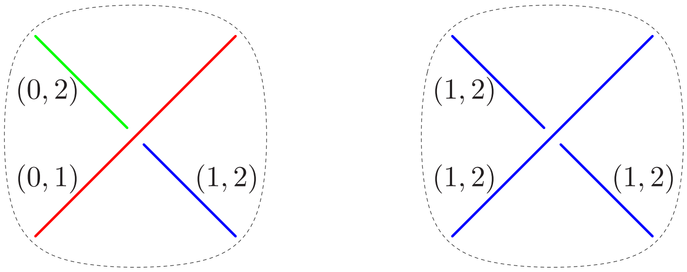


Figure 3.8. *Crossing of strands in a 3-colored link diagram.*

of meridial loops around bridges of the diagram as generators of $\pi(\mathbb{S}^3 \setminus L)$. Thus we obtain a 3-coloring of L, coloring each bridge b by the monodromy action of the meridial loop around b. We will use the color red for a bridge corresponding to the transposition $(0,1)$ in $\mathfrak{M}_{p_{M^3}}$, green for a bridge corresponding to the transposition $(0,2)$, and blue for a bridge corresponding to the transposition $(1,2)$.

In addition to the moves $C^{\pm}$ and $N^{\pm}$ from Section 3.1, we define the move $M^{\pm}$ in Figure 3.9. Further we allow colored Reidemeister moves; see [1]. The following Lemma 3.14 is probably known.

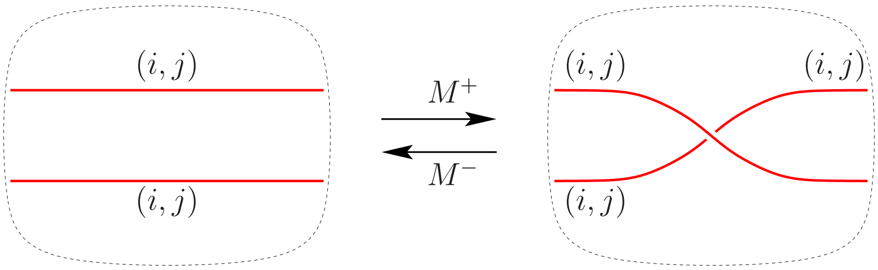


Figure 3.9. *The $M^{\pm}$-move.*

Lemma 3.14. Every 3-colored diagram of a link L can be constructed from a collection of unlinked and unknotted $\mathbb{S}^1$ colored red and green by a finite sequence of moves $C^{\pm}$, $M^{\pm}$, and Reidemeister moves.

Proof. First we eliminate all crossings using only one color by applying an M^--move to each crossing. Then the blue colored bridges are removed, eliminating remaining crossings, and all blue colored strands.

Applying colored Reidemeister moves we isotopy all unlinked and unknotted $\mathbb{S}^1$ such that they neither cross or are crossed by any other strand. If any one of these $\mathbb{S}^1$ is colored blue, we isotopy it underneath a red or green colored strand, hence changing its color. We are left with the task of eliminating the blue bridges. Without loss of generality we may assume that a blue bridge does not cross any other strand of the diagram (Figure 3.10), and eliminate blue bridges which are crossed by bridges of the same color at their ends using isotopy, one M^+-, and one M^--move (Figure 3.11).

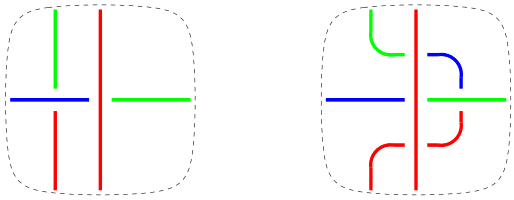


Figure 3.10. *Using isotopy to ensure that no blue bridges cross other strands.*

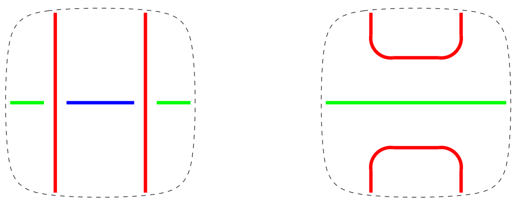


Figure 3.11. *Removing a blue bridge crossed by bridges of the same color at its ends using isotopy, one M^+-, and one M^--move.*

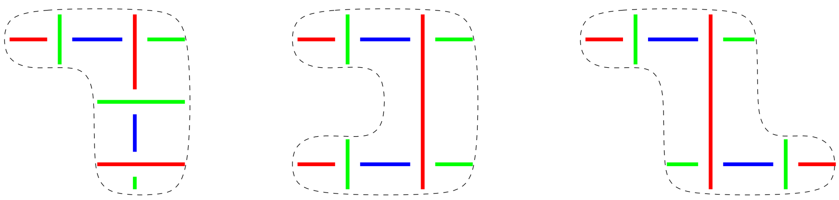


Figure 3.12. *The three cases of a pair of blue bridges.*

The hard part is to eliminate the blue bridges which are crossed by a green bride at the one end, and by a red one at the other end. But the situation has been simplified by the previous steps. The colors of a strand in the diagram of L appear in the cyclic order blue, red, blue, green, and again blue. This is due to the fact that there are no crossings using only one color, that the blue bridges do not cross any strand, and that each blue bridge is crossed by different colored bridges at its ends. Further this implies that the number of blue bridges is even, and we eliminate pairs of blue bridges using $C^{\pm}$ and $M^{\pm}$-moves.

Choose any blue bridge b. There are three different cases to consider. First the red bridge crossing b does not cross any other strands. Otherwise consider a strand next to b crossed by the red bridge. There are two ways how this strand can change its color: either from blue to green or vice versa, which yields the cases two and three; see Figure 3.12.

In the first case the diagram may be transformed using isotopy and $M^{\pm}$-moves to the diagram in Figure 3.13, first row. Then isotopy and a final C^{-}-move eliminate the pair of blue bridges. The second case is easily reduced (using isotopy and $M^{\pm}$-moves) to two blue bridges which are crossed at both ends by bridges of the same color; Figure 3.13, second row. Finally, reduce the third case to one blue bridge with a single red and green colored strand running underneath it, again using isotopy and $M^{\pm}$-moves. The remaining blue bridge is eliminated by a single C^{-}-move; Figure 3.13, third row. □

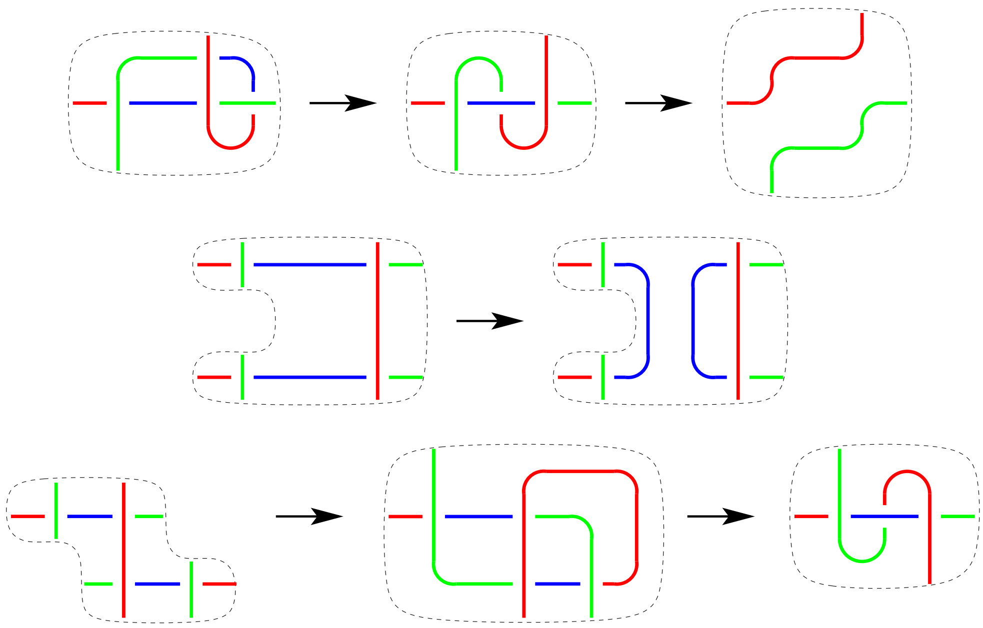


Figure 3.13. *Eliminating pairs of blue bridges.*

The collection of red and green unlinked and unknotted 1-spheres can be simplified further. For example, we may use $M^{\pm}$-moves and isotopy to change it to one unknotted and unlinked 1-sphere for each color red, green and blue.

The twisting operation. Consider a simplicial neighborhood $U_m \subset S$ of a $C^{\pm}$- or $M^{\pm}$-move. The odd subcomplex of U_m consists of two disjoint strands, which in the case of a C^--move wind around each other by a twist of 3π (see Figure 3.3), and in the case of an M^--move wind around each other by a twist of π (see Figure 3.9). Now the move m is realized by removing U_m from S, and gluing U_m back in with a twist of $\pm 3\pi$, respectively $\pm\pi$. That is, in the case of a C^+- or M^+-move the odd subcomplex of U_m is twisted by 3π, respectively π, and in the case of a C^-- or M^--move the twist of the odd subcomplex of U_m is annihilated.

To this end choose simplicial neighborhoods N and N' of the boundaries of $\mathrm{cl}(S \setminus U_m)$ and U_m, respectively. If there is a homeomorphism of pairs $\varphi_\alpha : (|N|, |N_{\mathrm{odd}}|) \to (|N'|, |N'_{\mathrm{odd}}|)$ which twists $\mathrm{odd}(U_m)$ by an angle of α, and N and N' are color equivalent with respect to φ_α, then $\mathrm{cl}(S \setminus U_m)$ and U_m may be attached along their boundaries via φ_α by Proposition 3.8. The operation of removing U_m and replacing it with a twist of α is called an *α-twist*. Observe that N and N' are color equivalent with respect to φ_α if we choose $\alpha = \pm 3\pi$ in case of a $C^{\pm}$-move, and $\alpha = \pm\pi$ in case of an $M^{\pm}$-move; see Figure 3.9.

Theorem 3.15. Let M^3 be a closed oriented 3-manifold. Then a combinatorial 3-sphere S with $\widehat{S} \cong M^3$ can be constructed from any triangulation of $\mathbb{S}^3$ using only finitely many stellar subdivisions of faces, $\pm 3\pi$-, and $\pm\pi$-twists.

Proof. Starting with a sufficiently fine and foldable triangulation S of the 3-sphere, e.g. obtained by barycentric subdivision, we first construct the collection of red and green 1-spheres required by Lemma 3.14 as odd subcomplex. To this end choose any 4-coloring of S, and stellarly subdivide an edge in the $\{0,1\}$-skeleton for each red 1-sphere, and stellarly subdivide an edge in the $\{0,2\}$-skeleton for each green 1-sphere. Then the triangulation S is completed using a $\pm 3\pi$-twist for each $C^{\pm}$-move, and a $\pm\pi$-twist for each $M^{\pm}$-move. □

Problem 3.16. The question if stellar subdivision of faces alone suffices to construct a combinatorial 3-sphere S with $\widehat{S} \cong M^3$ for a given closed oriented 3-manifold M^3 remains open.

Chapter 4

Products of Foldable Triangulations

A lattice triangulation of a lattice m-polytope P is *dense* if its vertices are all the lattice points inside P, and, for the sake of brevity, we refer to a regular, dense, and foldable triangulation as an *rdf-triangulation*. It is known that a triangulation of a polytope (or more generally, any simply connected manifold) is foldable if and only if its dual graph is bipartite; see [37]. From rdf-triangulations of lattice polytopes Soprunova and Sottile [62] construct sparse polynomial systems with non-trivial lower bounds for the number of real roots.

For generic coefficients the exact number of complex solutions of a sparse system of polynomials is known from Kushnirenko's Theorem [40]. To estimate the number of real solutions however, is considerably more delicate. The lower bound in the approach of Soprunova and Sottile is the degree of a map on the oriented double cover of the real part Y_P of the toric variety associated with the lattice polytope P, where P comes in as the common Newton polytope of the polynomials in the system. In combinatorial terms this map degree translates into the size difference of the two color classes of facets of an rdf-triangulation K of P. More precisely, only those facets of K count in the size difference, called the *signature*, which have odd normalized volume. We sketch this approach in Section 4.3.1.

We focus mainly on the combinatorial aspects, but apply our results to sparse polynomial systems in Section 4.3.2. We form rdf-triangulations of products of lattice polytopes from rdf-triangulations of the factors. As an application we construct triangulations of the d-cube $C_d = [0,1]^d$, which is the product of d line segments. Here we find rdf-triangulations of C_d with a super exponentially large signature. Optimizing triangulations of cubes for combinatorial parameters is often difficult, and basic questions are still open: Most prominently, for the minimal number of facets in a d-cube triangulation for $d > 7$ only partial asymptotic results are known; see Anderson & Hughes [32], Smith [61], Orden & Santos [51], Bliss & Su [10], and Zong [67]. The question whether the constructed triangulations of the d-cube have maximal signature is not addressed.

We start out with studying products of simplices because these naturally form the building blocks in our product triangulations. The key player here is the staircase triangulation studied by Billera, Cushman, & Sanders [6], Gel′fand, Kapranov, & Zelevinsky [24], and others. Then we focus on products of arbitrary simplicial complexes. These simplicial products, which depend on linear orderings of the vertices of the factors, already occur in the work of Eilenberg & Steenrod [18, Section II.8]; see also Santos [58]. It turns out that the product of two foldable simplicial complexes again has a foldable triangulation, and we compute the signature of the simplicial product (Theorem 4.17). Here it is important that there are still some choices left, a fact which plays a role in the construction of the cube triangulations. Further more, if the factors of the simplicial product are rdf-triangulations of lattice polytopes P and$Q$, then the simplicial product is a rdf-triangulations of $P \times Q$, provided we choose specific vertex orderings of the factors (to be explained later).

For the algebraic applications it is essential to improve these results further. In Theorem 4.29 we show that (with a mild additional assumption) the simplicial product $K \times_{\mathrm{stc}} L$ meets the geometric requirements of Soprunova and Sottile, provided that both factors do.

As an application of our Product Theorems Section 4.4 continues with an explicit construction of rdf-triangulations of the d-cube with signature in $\Omega(\lfloor d/2 \rfloor !)$. This lower bound partially relies on computational results obtained with `TOPCOM` [55], `polymake` [21, 22, 23], `MAGMA` [13], and `QEPCAD` [30].

This chapter is a joint work with Michael Joswig to appear in *Advances in Mathematics*.

4.1 Products of Simplicial Complexes

Let $\Delta_m = \mathrm{conv}(0, e_1, \ldots, e_m)$ be the standard m-simplex, where e_i denotes the i-th unit vector of $\mathbb{R}^m$. We define its normalized volume $\nu(\Delta_m)$ as $\nu(\Delta_m) = \mathrm{vol}(\Delta_m)\, m! = 1$.

The product $\Delta_m \times \Delta_n$ is an $(m+n)$-dimensional convex polytope with $(m+1)(n+1)$ vertices and $m+n+2$ facets. As one key feature $\Delta_m \times \Delta_n$ has the property that it is *totally unimodular*, that is, each facet of any triangulation which uses no additional vertices has normalized volume 1. As a consequence the size of an arbitrary such triangulation of $\Delta_m \times \Delta_n$ is

$$\nu(\Delta_m \times \Delta_n) = \mathrm{vol}(\Delta_m)\,\mathrm{vol}(\Delta_n)\,(m+n)! = \binom{m+n}{m}.$$

The staircase triangulation. We are interested in one particular triangulation of $\Delta_m \times \Delta_n$, the *staircase triangulation* $\mathrm{stc}_{m,n} = \mathrm{stc}(\Delta_m \times \Delta_n)$, which can be described as follows. Consider a rectangular grid of size $m+1$ by $n+1$. Each node in the grid corresponds to one vertex of $\Delta_m \times \Delta_n$. The facets of $\mathrm{stc}_{m,n}$, described as subsets of these nodes, correspond to the non-descending and not-returning paths from the lower left node to the upper right node. These paths, which go only right or up, but never left nor down, look like staircases, and hence the name; see Figure 4.1 (left).

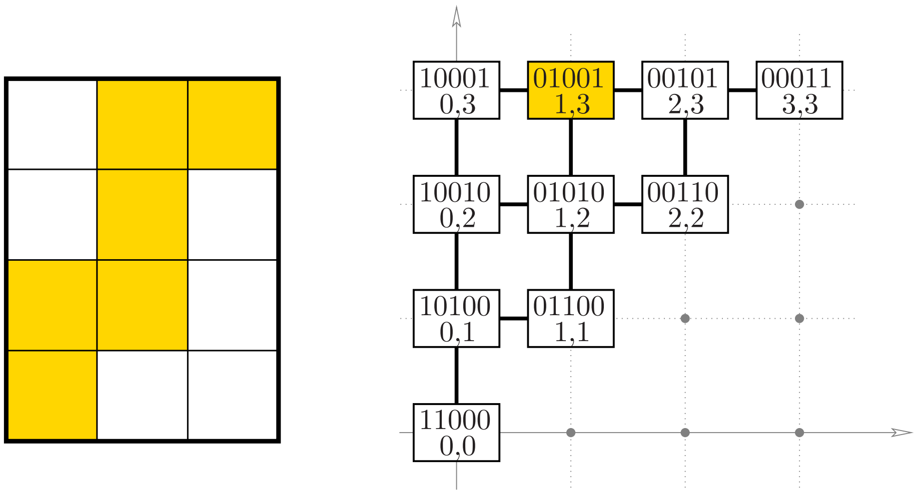


Figure 4.1. *The facet* 01001 *of* $\mathrm{stc}(\Delta_2 \times \Delta_3)$ *and the dual graph of* $\mathrm{stc}(\Delta_2 \times \Delta_3)$ *with the facet* 01001 *marked.*

The choice of "right" and "up" in the definition of $\mathrm{stc}_{m,n}$ implicitly assumes an ordering of the vertices of both factors. Throughout this chapter we will keep this ordering fixed. The staircase triangulation of $\Delta_m \times \Delta_n$ is the same as the placing triangulation induced by the *lexrev ordering*, that is, the lexicographic ordering of the vertices with the reversed ordering of the vertices of the second factor. In particular, $\mathrm{stc}_{m,n}$ is a regular triangulation.

Each such staircase can be encoded as a *shuffle* of "up" and "right" moves. The name "shuffle" reflects the fact that the number of "up" and "right" moves is always the same, but their order is all that matters. We write the shuffle in Figure 4.1 as the bit-string 01001, where 0 means "up" and 1 means "right". The staircase triangulations occurred in Eilenberg & Steenrod [18, Section II.8]; see also Billera, Cushman, & Sanders [6], Gel′fand, Kapranov, & Zelevinsky [24, §7.D], and Santos [58].

Yet another way to encode a facet F of $\mathrm{stc}_{m,n}$ is to assign a vector $s(F) \in \mathbb{N}^m$ as follows. The bit-string $11\ldots100\ldots0$ corresponds to the origin, and for an arbitrary facet F the k-th entry $s(F)_k$ measures the difference between the position of the k-th one in the bit-representation of F and k. This difference may be viewed as the number of "shifts to the right" of the k-th one, starting with the bit-string corresponding to the origin. For example, the bit-string 01001 in Figure 4.1 is mapped to $(1, 3)$.

Via the map s the facets of $\mathrm{stc}_{m,n}$ correspond to the integer points in the polytope

$$\mathcal{S}_{m,n} = \left\{ s \in \mathbb{R}^m \;\middle|\; \begin{array}{c} 0 \le s_k \le n \text{ for } 1 \le k \le m \\ s_k \le s_l \text{ for } k < l \end{array} \right\} .$$

This provides us with a convenient description of the dual graph of $\mathrm{stc}_{m,n}$; see Figures 4.1 (right) and 4.2. Let $\mathcal{L}_m$ be the m-dimensional cubic grid, that is, the infinite graph with node set $\mathbb{Z}^m$, and two nodes are adjacent if they differ in exactly one coordinate by one.

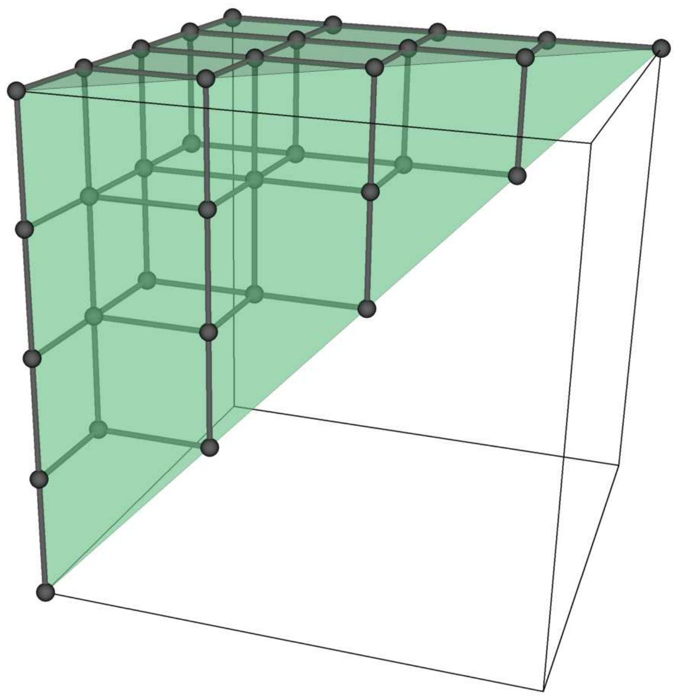


Figure 4.2. *The dual graph of* $\mathrm{stc}_{3,4}$ *as the subgraph of* $\mathcal{L}_3$ *induced by* $\mathcal{S}_{3,4} \cap \mathbb{Z}^3$.

Proposition 4.1. The dual graph $\Gamma^*(\mathrm{stc}_{m,n})$ is the subgraph of $\mathcal{L}_m$ induced by the node set $\mathcal{S}_{m,n} \cap \mathbb{Z}^m$. In particular, this graph is bipartite.

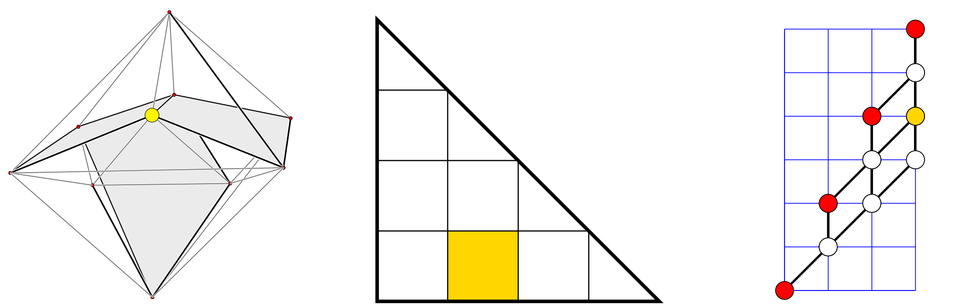
To conclude this section we mention further aspects of the staircase triangulations, which are however, inessential for the understanding of the rest of this chapter.

Bit-strings of length $m+n$ with precisely m ones correspond to the vertices of the hypersimplex

$$H(m+n, m) = \left\{ x \in [0,1]^{m+n} \;\middle|\; \sum x_i = m \right\} .$$

The graph $\Gamma^*(\mathrm{stc}_{m,n})$ is a (not induced) subgraph of the vertex-edge graph of $H(m+n,m)$. See Figure 4.3 (left) for the case $\Gamma^*(\mathrm{stc}_{2,3})$.

The *Cayley trick* establishes a one-to-one correspondence between the regular triangulations of $\Delta_m \times \Delta_n$ and the fine mixed subdivisions of $(n+1)\Delta_m$; see Santos [59]. For an example see Figure 4.3 (center).



As a node of $\Gamma^*(\mathrm{stc}_{2,3})$ in the Schlegel diagram of $H(5,2)$. | As a cell in a fine mixed subdivision of $4\Delta_2$. | As a pseudo-vertex in a tropical cyclic polytope.

Figure 4.3. *The facet* 01001 *of* $\mathrm{stc}(\Delta_2 \times \Delta_3)$ *from Figure 4.1.*

In a different context regular triangulations of $\Delta_m \times \Delta_n$ recently reappeared as the *tropical convex hulls* of $n+1$ points in the *tropical projective space* $\mathbb{TP}^m$; see Develin & Sturmfels [17]. The staircase triangulations arise as the *tropical cyclic polytopes* of Block & Yu [11].

Example 4.2. The staircase triangulation $\mathrm{stc}_{2,3}$ is dual to the tropical convex hull of the four points $(0,0,0)$, $(0,1,2)$, $(0,2,4)$, and $(0,3,6)$ in $\mathbb{TP}^2$. The facet $F = 01001$ with $s(F) = (1,3)$ from Figure 4.1 corresponds to the pseudo-vertex $(-s(F,2), 0, s(F,1)) = (-3,0,1) = (0,3,4)$; see Figure 4.3 (right).

4.1.1 The Simplicial Product

Let K and L be two abstract simplicial complexes. Then the product space $|K| \times |L|$ is equipped with the structure of a cell complex whose cells are the products $f \times g$, where f is a face of K and g is a face of L. This section is about the study of triangulations of $|K| \times |L|$ which refine this natural cell structure.

Assume that $\dim K = m$ and $\dim L = n$, and denote the vertex sets of K and L by V_K and V_L, respectively. We choose a linear ordering O_K of V_K and another linear ordering O_L of V_L. The product $O_K \times O_L$, defined by

$$(v, w) \geq (v', w') \;\Leftrightarrow\; v \geq v' \text{ and } w \geq w' \,,$$

is a partial ordering of the set $V_K \times V_L$. Let $\pi_K : V_K \times V_L \to V_K$ and $\pi_L : V_K \times V_L \to V_L$ be the canonical projections.

We define the *simplicial product* (with respect to the vertex orderings O_K and O_L) of K and L as

$$K \times_{\mathrm{stc}} L = \left\{ F \subset V_K \times V_L \;\middle|\; \begin{array}{l} \pi_K(F) \in K \text{ and } \pi_L(F) \in L\,, \\ \text{and } O\,|_F \text{ is a total ordering} \end{array} \right\} .$$

The simplicial product $K \times_{\mathrm{stc}} L$ appeared earlier in Eilenberg and Steenrod [18, Section II.8] as the “Cartesian product”, and in Santos [58], who calls it the “staircase refinement”. Both sources prove the staircase triangulation to be a triangulation of the space $|K| \times |L|$ on the vertex set $V_K \times V_L$.

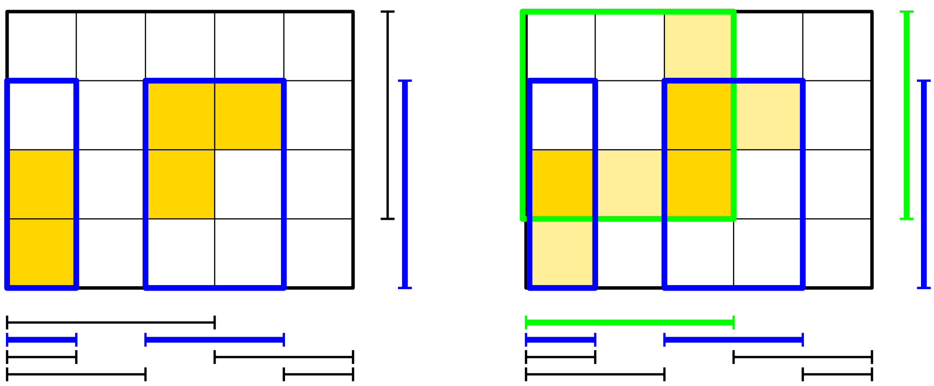


Figure 4.4. *A facet defining path of the simplicial product of two different triangulations of the square. On the right two facets intersecting in a low dimensional face.*

Let $k = |V_K|$ and $l = |V_L|$ denote the number of vertices of K and L, respectively. There is a convenient way to visualize the simplicial product in the $(k \times l)$-grid $\mathcal{R}$: We label the columns of $\mathcal{R}$ with the vertices of K

according to the vertex order O_K, and we label the rows of $\mathcal{R}$ with the vertices of L according to the vertex order O_L. For each $f \in K$ and $g \in L$ let $\mathcal{R}_{f,g}$ be the minor of $\mathcal{R}$ induced by f and g. Then we may think of the facets of the simplicial product as the collection of all ascending paths in $\mathcal{R}_{f,g}$ starting bottom-left and finishing top-right. This is a direct generalization of the staircase triangulation of the product of two simplices; see Figure 4.4. More precisely, we may view the simplicial product $K \times_{\text{stc}} L$ as a subcomplex of the staircase triangulation of the product of a $(k-1)$-simplex and an $(l-1)$-simplex.

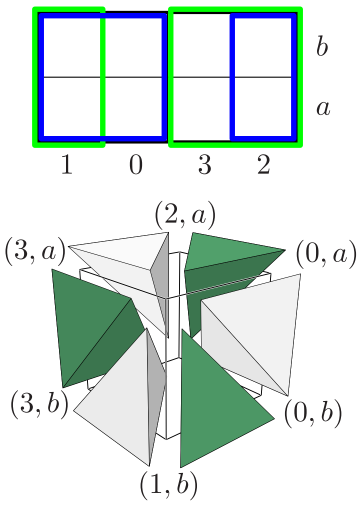


Figure 4.5. *Three different orderings of the vertices of the triangulated square* $\{\{0,1,2\},\{1,2,3\}\}$ *and the resulting regular triangulations of the* 3*-cube. The vertices* 0 *and* 3 *of the square are colored the same, and the top-front vertex of the* 3*-cube is labeled* $(1,a)$*, and the bottom-back vertex is labeled* $(2,b)$*. The second and third* 3*-cube are labeled the same.*

The ordering of the vertices of K and L are crucial to $K \times_{\text{stc}} L$. Figure 4.5 depicts the product of the triangulated unit square with the unit interval. The three distinct orderings of the vertices of the triangulated square yield three pairwise non-isomorphic triangulations of the 3-cube C_3 decomposed as $C_2 \times I$.

The multi staircase triangulation. The staircase triangulation generalizes naturally to a triangulation of the product of finitely many simplices. Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be simplices of dimension $m_1, m_2, \ldots, m_k$, and identify the integer points of the cuboid $C(m_1, \ldots, m_k) = [0, m_1] \times [0, m_2] \times \cdots \times [0, m_k]$ with the vertices of $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$: For $1 \leq i \leq k$ assign the vertices of Δ_i

to the integer points in $[0, m_i]$. Now the faces of the *multi staircase triangulation* $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k)$ are given by all monotone ascending paths in the subgraph of $\mathcal{L}_k$ induced by the node set $C(m_1, \ldots, m_k) \cap \mathbb{Z}^k$. In particular, the facets correspond to monotone ascending paths from the origin to the point $(m_1, m_2, \ldots, m_k)$. (Note that for simplicity we choose the integer points in $C(m_1, \ldots, m_k)$ rather then unit cubes in $C(m_1 + 1, \ldots, m_k + 1)$ to represent the vertices of $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$.) The multi staircase triangulation appears in Orden & Santos [51, p. 516]; see Figure 4.6 for a facet of $\mathrm{mstc}(\Delta_4 \times \Delta_3 \times \Delta_2)$.

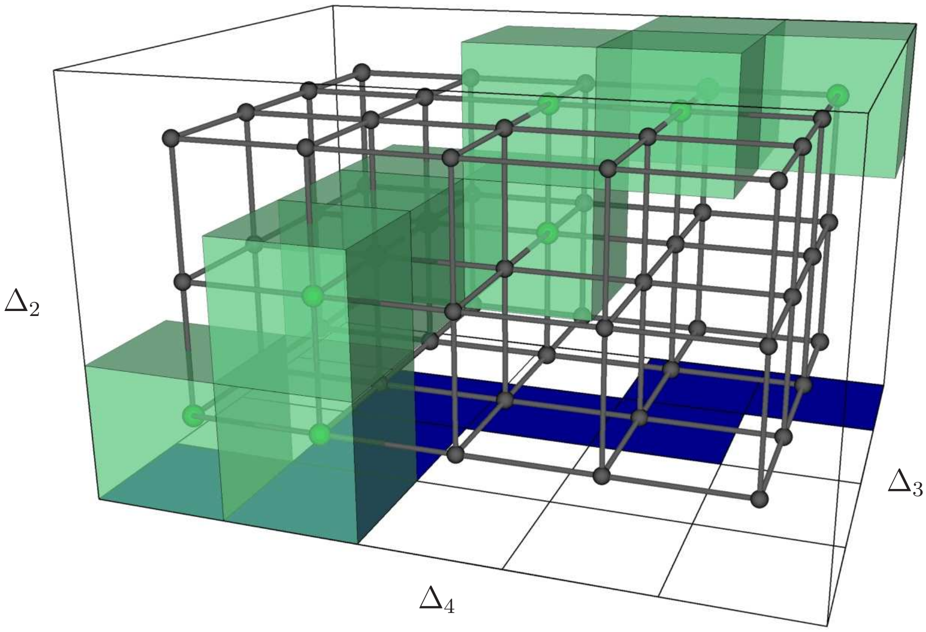


Figure 4.6. *A facet of* $\mathrm{mstc}(\Delta_4 \times \Delta_3 \times \Delta_2) = \mathrm{mstc}(\Delta_4 \times \Delta_3) \times_{\mathrm{stc}} \Delta_2$*, and its projection to* $\mathrm{mstc}(\Delta_4 \times \Delta_3)$.

Proposition 4.3. The multi staircase triangulation may be obtained as an iterated simplicial product, that is,

$$\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k) = \mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{k-1}) \times_{\mathrm{stc}} \Delta_k \,.$$

Proof. Assigning the vertices V_i of Δ_i to the integer points in $[0, m_i]$ fixes an ordering of V_i for $1 \leq i \leq k$, and we choose the lexicographic ordering for the vertices of $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{k-1})$.

The numbers of facets of $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k)$ and $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{k-1}) \times_{\mathrm{stc}} \Delta_k$ are easily calculated to be the same:

$$\frac{\left(\sum_{i=1}^{k} m_i\right)!}{\prod_{i=1}^{k} m_i!} = \frac{\left(\sum_{i=1}^{k-1} m_i\right)!}{\prod_{i=1}^{k-1} m_i!} \cdot \frac{\left(\sum_{i=1}^{k} m_i\right)!}{m_k!\,\left(\sum_{i=1}^{k-1} m_i\right)!} = \frac{\left(\sum_{i=1}^{k-1} m_i\right)!}{\prod_{i=1}^{k-1} m_i!} \cdot \binom{\sum_{i=1}^{k} m_i}{m_k}$$

It remains to show that each facet f of $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k)$ appears as a facet in $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{k-1}) \times_{\mathrm{stc}} \Delta_k$. The vertices of f come in a natural order, and f projects to a facet f' of $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{k-1})$. The ordering of the vertices of f' inherited from f coincides with the vertex ordering of f' induced by lexicographic ordering of the vertices of $\mathrm{mstc}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{k-1})$. Hence f appears as a facet in the staircase triangulation of $f' \times \Delta_k$. $\square$

Remark 4.4. Since we may choose different vertex orderings for the factors, iterating the simplicial product produces many triangulations of $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$ different from the multi staircase triangulation. In particular, not all triangulations obtained by iterating the simplicial product are foldable, but it is easily seen that the multi staircase triangulation is foldable.

4.1.2 Foldable Simplicial Complexes

Let $[k] = \{0, \ldots, k-1\}$ be the vertex set of a foldable simplicial m-complex K. Assume that there is a coloring of K given by a weakly monotone map $c_K : [k] \to [m+1]$. Then we call the natural ordering on $[k]$ *color consecutive*. Any foldable complex admits (many) color consecutive orderings.

Proposition 4.5. If K and L are foldable simplicial complexes with color consecutive orderings of their vertices then the corresponding simplicial product $K \times_{\mathrm{stc}} L$ is foldable.

Proof. Let the vertex sets of K and L be $[k]$ and $[l]$, respectively, with weak monotone coloring maps $c_K : [k] \to [m+1]$ and $c_L : [l] \to [n+1]$. We define

$$c : [k] \times [l] \to [m+n+1] : (v, w) \mapsto c_K(v) + c_L(w) .$$

In order to show that c is a coloring of $K \times_{\mathrm{stc}} L$ it suffices to check that each facet contains each color at most once. Each facet F of $K \times_{\mathrm{stc}} L$ is contained in a unique cell $f \times g$ where f is a facet of K and g is a facet of L. Let $v \times w$ and $v' \times w'$ be distinct vertices of F. We may assume $v < v'$; then $w \leq w'$ since F is a facet of the staircase triangulation of $f \times g$. As the restrictions $c_K|_f$ and $c_L|_g$ are strictly monotone we have $c(v, w) = c_K(v) + c_L(w) < c_K(v') + c_L(w') = c(v', w')$. For an example see Figure 4.7. $\square$

In what follows below it is essential that it is not necessary to have color consecutive orderings for the factors in order to obtain a foldable simplicial product triangulation.

Example 4.6. Let B_n be the triangulation of the bipyramid over the $(n-1)$-simplex Δ_{n-1} formed of two n-simplices sharing a facet. Combinatorially, B_n is the join of Δ_{n-1} with the zero-dimensional sphere $\mathbb{S}^0$ consisting of two isolated points. The triangulation B_n is obviously foldable. The *symmetric vertex ordering* S_n on B_n starts with one of the two apices and ends with the other apex, the vertices of Δ_{n-1} come in between. That is to say, we take $[n+2]$ as the vertex set of B_n, where 0 and $n+1$ are the apices, and a coloring map $s_n : [n+2] \to [n+1] : w \mapsto w \bmod(n+1)$. Because of the symmetry properties of B_n the precise ordering of the vertices $1, 2, \ldots, n$ does not matter. Likewise it is not necessary to distinguish the two apices.

The triangulation B_n with the symmetric vertex ordering will be used in the construction of certain cube triangulations in Section 4.4.

Proposition 4.7. Let K be a foldable simplicial complex with a color consecutive ordering O_K. Then the simplicial product $K \times_{\text{stc}} B_n$ with respect to O_K and S_n is foldable.

Proof. We use almost the same coloring scheme as in Proposition 4.5. Let $[k]$ be the vertex set of K, and let $c_K : [k] \to [m+1]$ be a weak monotone coloring map. We define

$$c : [k] \times [n+2] \to [m+n+1] : (v, w) \mapsto c_K(v) + w \bmod(m+n+1).$$

This, indeed, is a coloring since there is no facet of $K \times_{\text{stc}} B_n$ containing both, a vertex of the type $(v, 0)$ and a vertex of the type $(v, n+1)$. □

We refer to Figure 4.5 for the three different simplicial products of an interval with a square arising from the two color consecutive and the symmetric vertex ordering of the square (which is a bipyramid over a 1-simplex).

4.1.3 Regular Triangulations of Polytopes

Let P be an m-dimensional convex polytope in $\mathbb{R}^m$, and let K be a triangulation of P with vertex set V. Recall that the triangulation K is *regular* if there is a convex function $\lambda : \mathbb{R}^m \to \mathbb{R}$ such that K coincides with the polyhedral subdivision of P induced by the lower convex hull of the set $\{(v, \lambda(v)) \in \mathbb{R}^{m+1} \mid v \in V\}$. In this case λ is called a *lifting function* for K. Since we want to stress that a regular triangulation only depends on P and λ we denote such a triangulation as P^λ.

Choose points $p_1, \dots, p_k$ in P such that $\operatorname{conv}\{p_1, \dots, p_k\} = P$, and assume $p_1, \dots, p_k$ to be pairwise distinct. This implies that the vertices of P occur among the chosen points. Then the *placing triangulation* of P with respect to the chosen points in the given ordering is the regular triangulation of P with vertex set $\{p_1, \dots, p_k\}$ and a lifting function λ such that $(p_l, \lambda(p_l))$ is above all affine hyperplanes spanned by points in $\{(p_1, \lambda(p_1)), \dots, (p_{l-1}, \lambda(p_{l-1}))\}$. A point $(p, \lambda(p))$ lies *above* the affine hyperplane $H \subset \mathbb{R}^{m+1}$ spanned by the points $\{(p_1, \lambda(p_1)), \dots, (p_{m+1}, \lambda(p_{m+1}))\}$ if and only if the unique $\lambda' \in \mathbb{R}$ with

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ p & p_1 & p_2 & \dots & p_{m+1} \\ \lambda' & \lambda(p_1) & \lambda(p_2) & \dots & \lambda(p_{m+1}) \end{pmatrix} = 0 \tag{4.1}$$

satisfies $\lambda' < \lambda(p)$.

Example 4.8. Consider the standard simplices $\Delta_m = \operatorname{conv}\{0, e_1, \dots, e_m\}$ and $\Delta_n = \operatorname{conv}\{0, e_1, \dots, e_n\}$. To simplify the formulae below we set $e_0 = 0$. Then the lexrev ordering on the vertices of the product $\Delta_m \times \Delta_n$ is given as

$$O : \{e_0, \dots, e_m\} \times \{e_0, \dots, e_n\} \to [(m+1)(n+1)] : (e_i, e_j) \mapsto (n+1)i + (n-j) \ .$$

A lifting function ω of $\mathrm{stc}_{m,n}$ corresponding to (any) placing order O is given by

$$\omega : \{e_0, \dots, e_m\} \times \{e_0, \dots, e_n\} \to \mathbb{R} : (v, w) \mapsto 2^{O(v,w)} \ ,$$

that is, $(\Delta_m \times \Delta_n)^\omega = \mathrm{stc}_{m,n}$ holds. Let $x \in \mathbb{R}^{m+n}$ be a vertex of $\Delta_m \times \Delta_n$, and let the vertices $\{x_1, x_2, \dots, x_{m+n+1}\} \subset \Delta_m \times \Delta_n$ appear prior to x in the placing ordering O, that is, $O(x) > O(x_i)$ for $1 \le i \le m+n+1$. We prove that Equation 4.1 holds for some $\lambda' < 2^{O(x)}$. We have

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x & x_1 & x_2 & \dots & x_{m+n+1} \\ \lambda' & 2^{O(x_1)} & 2^{O(x_2)} & \dots & 2^{O(x_{m+n+1})} \end{pmatrix}$$

and we want to compute the value λ' for which $\det A$ vanishes. To this end let A' be any co-dimension 1-minor of A not involving the last row of A. The absolute value of the determinant of A' equals the volume of a simplex in some triangulation of $\Delta_m \times \Delta_n$. Since $\Delta_m \times \Delta_n$ is totally unimodular we have $\det A' = \pm 1$. Laplace expansion with respect to the last row yields

$$\lambda' \pm 2^{O(x_1)} \pm 2^{O(x_2)} \pm \dots \pm 2^{O(x_{m+n+1})} = 0 \ ,$$

and solving for λ' gives the following estimate

$$\lambda' \le 2^{O(x_1)} + 2^{O(x_2)} + \dots + 2^{O(x_{m+n+1})} \le \sum_{k=1}^{O(x)-1} 2^k = 2^{O(x)} - 1 < 2^{O(x)} \ .$$

This shows that the point $(x, \lambda(x))$ lies above the hyperplane spanned by any set of $m+n+1$ previous points.

Proposition 4.9. Let P^λ and Q^μ be regular triangulations of an m-polytope $P \subset \mathbb{R}^m$ and an n-polytope $Q \subset \mathbb{R}^n$, respectively. Then the simplicial product $P^\lambda \times_{\mathrm{stc}} Q^\mu$ is a regular triangulation of the polytope $P \times Q$ for any vertex orderings O_{P^λ} and O_{Q^μ}.

Proof. Let V_{P^λ} be the vertex set of P^λ equipped with a linear ordering O_{P^λ}, and let V_{Q^μ} be the vertex set of Q^μ with a linear ordering O_{Q^μ}. The simplicial product $P^\lambda \times_{\mathrm{stc}} Q^\mu$ (with respect to O_{P^λ} and O_{Q^μ}) is a triangulation of the product $P \times Q$ on the vertex set $V_{P^\lambda} \times V_{Q^\mu}$.

Let $\lambda : V_{P^\lambda} \to \mathbb{R}$ and $\mu : V_{Q^\mu} \to \mathbb{R}$ be lifting functions of P^λ and Q^μ. We construct a lifting function $\omega : V_{P^\lambda} \times V_{Q^\mu} \to \mathbb{R}$ of $P^\lambda \times_{\mathrm{stc}} Q^\mu$ in two steps. First consider the map

$$\omega_0 : V_{P^\lambda} \times V_{Q^\mu} \to \mathbb{R} : (x, y) \mapsto \lambda(x) + \mu(y) ,$$

which is a lifting function for the polytopal complex $P^\lambda \times Q^\mu$. In the second step ω_0 has to be perturbed such that it induces a staircase triangulation on each cell of $P^\lambda \times Q^\mu$. To this end recall that the staircase triangulations are placing, and that the lexrev ordering O on $V_{P^\lambda} \times V_{Q^\mu}$ induces a placing order on each product of simplices $f \times g$ where $f \in P^\lambda$ and $g \in Q^\mu$. Now define ω as an ϵ-perturbation of ω_0 by the lifting function from Example 4.8 corresponding to O:

$$\omega : V_{P^\lambda} \times V_{Q^\mu} \to \mathbb{R} : (v, w) \mapsto \lambda(v) + \mu(w) + \epsilon 2^{O(v,w)} , \qquad (4.2)$$

for a sufficiently small $\epsilon > 0$. Viewing the simplicial product again as subcomplex of the staircase triangulation of two large simplices, shows that, indeed $(P \times Q)^\omega = P^\lambda \times_{\mathrm{stc}} Q^\mu$. For an example see Figure 4.7. $\square$

In general, there may be several perturbations which lead to different lifting functions but which induce the same triangulations. An important special case occurs if the triangulations P^λ and Q^λ additionally are foldable. In this case it is possible to define a perturbation which only depends on the color classes of the vertices of the factors:

Example 4.10. Let $c_{P^\lambda} : V_{P^\lambda} \to [m+1]$ and $c_{Q^\mu} : V_{Q^\mu} \to [n+1]$ be coloring maps. Using color consecutive vertex orderings for V_{P^λ} and V_{Q^μ} and the resulting lexrev ordering O for the vertices of $P \times_{\mathrm{stc}} Q$ we may choose

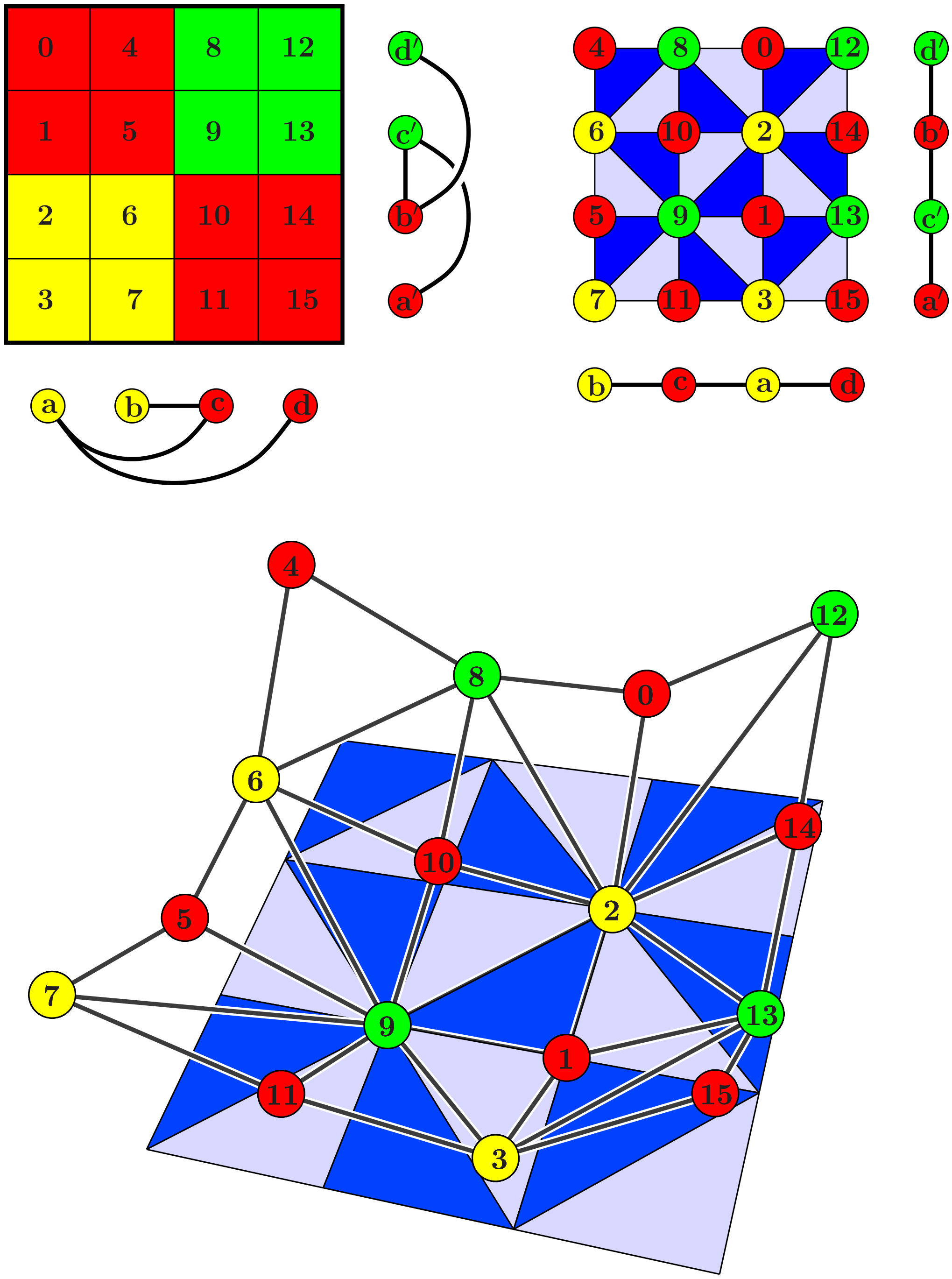


Figure 4.7. *Simplicial product of a path I of length 3 with itself, using color consecutive vertex orderings. The vertices of the product are colored according to the color scheme from the proof of Proposition 4.5 and are labeled in lexrev order.*

a different perturbation than in Equation (4.2). This yields the following lifting function

$$\omega : V_{P^\lambda} \times V_{Q^\mu} \to \mathbb{R} : (v,w) \mapsto \lambda(v) + \nu(w) + \epsilon\, 2^{(n+1)c_{P^\lambda}(v)+(n-c_{Q^\mu}(w))}\ , \quad (4.3)$$

for $\epsilon > 0$ chosen sufficiently small. Note that we use the same perturbation $\epsilon 2^{(n+1)i+(n-j)}$ for all vertices (v,w) with $c_{P^\lambda}(v) = i$ and $c_{Q^\mu}(w) = j$. Let us restrict our attention to a cell $f \times g$ for facets $f \in P^\lambda$ and $g \in Q^\mu$. Since any color $i \in [m+1]$ appears exactly once in the coloring of f and any color $j \in [n+1]$ appears exactly once in the coloring of g, respectively, there is exactly one vertex $(v,w) \in f \times g$ with $c_{P^\lambda}(v) = i$ and $c_{Q^\mu}(w) = j$ for each $(i,j) \in [m+1] \times [n+1]$. Hence ω restricted to $f \times g$ induces the staircase triangulation $f \times_{\mathrm{stc}} g$ from Example 4.8, and ω induces the simplicial product triangulation $(P \times Q)^\omega = P^\lambda \times_{\mathrm{stc}} Q^\mu$ on $P^\lambda \times Q^\mu$.

4.2 Triangulations of Lattice Polytopes

Let P be an m-dimensional *lattice polytope*, that is, we assume that its vertex coordinates are integral. Since the determinant of an integral matrix is an integer it follows that the *normalized volume* $\nu(P) = m!\,\mathrm{vol}(P)$ is an integer, where $\mathrm{vol}(P)$ is the usual m-dimensional volume of P. A lattice simplex is called *even* or *odd* depending on the parity of its normalized volume. A triangulation K of a lattice polytope P is *dense* if it uses all lattice points inside P, that is, its vertex set is $P \cap \mathbb{Z}^m$. In the case that K is additionally regular, say with lifting function λ, we again write P^λ for K since it only depends on P and λ.

Let P^λ be an rdf-triangulation of P, that is, P^λ is regular, dense, and foldable. In particular P^λ is a lattice triangulation. Recall that P^λ is foldable if and only if its dual graph is bipartite. Usually we refer to the two color classes as “black” and “white”. Then the *signature* $\sigma(P^\lambda)$ of P^λ is defined as the absolute value of the difference of the odd black and the odd white facets in P^λ. Note that the even facets are not accounted for in any way. Moreover, in the important special case where P^λ is *unimodular*, that is, where all the facets have a normalized volume equal to 1, all facets are odd. For examples of unimodular triangulations of the 3-cube with signatures equal to 0 and 2 see Figure 4.5; note that all triangulations of the 3-cube without additional vertices are regular.

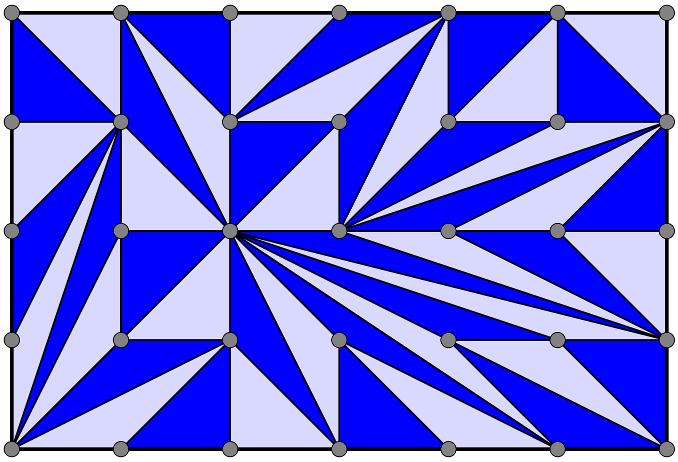


Figure 4.8. *Dense and foldable triangulation of the rectangular grid $G_{4,6}$.*

Example 4.11. Consider the rectangular grid $G_{k,l} = [0,k] \times [0,l]$. Note that each triangulation of the grid is unimodular if and only if it is dense. Triangulations of the rectangular grid $G_{k,l} = [0,k] \times [0,l]$ are an interesting subject on their own; see, for instance, Kaibel & Ziegler [38] and the references therein.

Proposition 4.12. Even without the assumption of regularity there are no dense and foldable triangulation of $G_{k,l}$ with a positive signature.

The following surprisingly simple proof is by Günter M. Ziegler, personal communication. See Figure 4.8 for an example of a dense and foldable triangulation of $G_{4,6}$.

Proof. Let K be a dense triangulation of $G_{k,l}$. Label each edge of K with x if the difference of the x-coordinates of its endpoints is odd. Similarly, label each edge of K with y if the difference of the y-coordinates of its endpoints is odd. All edges are labeled since K is dense, and convince yourself that the edges of each triangle are labeled x, y, and xy. No edges in the boundary of $G_{k,l}$ are labeled xy, hence we match triangles which share an xy-labeled edge. In the case that K is foldable, each black triangle is matched via its xy-labeled edge to a white one, and we have $\sigma(K) = 0$. $\square$

Example 4.13. Dense and foldable triangulations do not exist for all lattice polytopes. For instance, in any dimension $m \geq 2$ there are lattice simplices of arbitrarily large volume which admit exactly one dense triangulation (which is regular), but which is not foldable.

For $k \geq 1$ let $\Delta_2(k) = \operatorname{conv}\{(0,1),(1,0),(2k,2)\}$, a triangle with normalized volume $\nu(\Delta_2(k)) = 2k+1$; see Figure 4.9. For $m \geq 3$ we define $\Delta_m(k)$ as the cone over $\Delta_{m-1}(k)$ with the m-th unit vector as its apex; this is an m-simplex with normalized volume $\nu(\Delta_m(k)) = \nu(\Delta_{m-1}(k)) = \ldots = \nu(\Delta_2(k)) = 2k+1$.

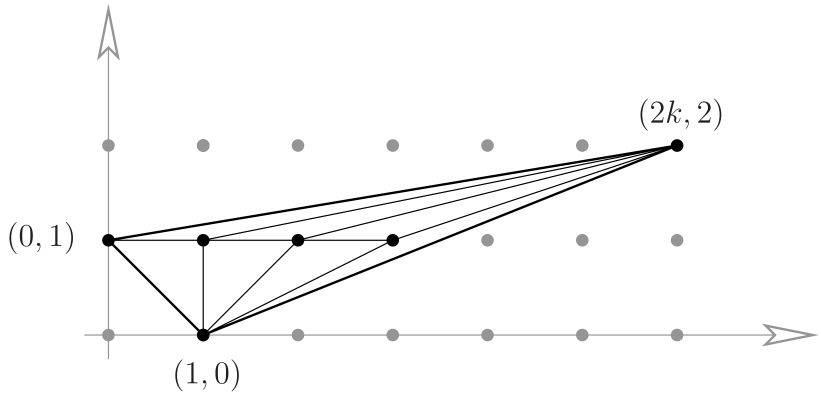


Figure 4.9. *A lattice triangle without a dense and foldable triangulation.*

The interior point $(k,1) \in \Delta_2(k)$ is a degree-3-vertex in the unique (regular and) dense triangulation of $\Delta_2(k)$, hence there is no dense and foldable triangulation of $\Delta_2(k)$. The cone over a triangulation K of $\Delta_{m-1}(k)$ is foldable if and only if K is foldable, and any triangulation of $\Delta_m(k)$ arises as a cone over a triangulation of $\Delta_{m-1}(k)$. Therefore there is no rdf-triangulation of $\Delta_m(k)$ by induction.

Example 4.14 (Signature of the Staircase Triangulation)**.** Let Δ_m and Δ_n be odd simplices of dimension m and n, respectively. From the description of $\Gamma^*(\mathrm{stc}_{m,n})$ as the subgraph of $\mathcal{L}_m$ induced by the node set $\mathcal{S}_{m,n} \cup \mathbb{Z}^m$ (see Proposition 4.1) one can read off that $\Gamma^*(\mathrm{stc}_{m,n})$ is bipartite and extract a recursive formulae for the signature of $\mathrm{stc}_{m,n}$. Remember that $\mathrm{stc}_{m,n}$ is unimodular, hence $\sigma_{m,0} = \sigma_{0,n} = 1$ and

$$\begin{aligned}\sigma_{m,n} &= \left| \sum_{i=0}^{n} (-1)^i \, \sigma_{m-1,i} \right| = \left| \sum_{i=0}^{n-1} (-1)^i \, \sigma_{m-1,i} \; + \; (-1)^n \, \sigma_{m-1,n} \right| \\ &= \left| \, \sigma_{m,n-1} \; + \; (-1)^n \, \sigma_{m-1,n} \, \right| = \sigma_{m,n-1} \; + \; (-1)^n \, \sigma_{m-1,n} \, .\end{aligned}$$

A careful inspection of the four cases arising from the two choices each for the parities of m and n gives the last equation. This recursion then yields the explicit formulae for $\sigma_{m,n}$ given by White [64] and stated in Proposition 4.15. Observe that $\Delta_m \times \Delta_n$ is the order polytope of the poset of the disjoint union of a path of length $m+1$ and a path of length $n+1$. The staircase triangulation $\mathrm{stc}_{m,n}$ coincides with the *canonical triangulation* of the order polytope; see Soprunova and Sottile [62, Section 4].

Proposition 4.15. The signature of the staircase triangulation of the product of two simplices of odd normalized volume is

$$\sigma_{2k,2l} = \binom{k+l}{k}, \quad \sigma_{2k,2l+1} = \binom{k+l}{k} \quad \text{and} \quad \sigma_{2k+1,2l+1} = 0 .$$

If at least one of the simplices is even then this signature vanishes.

Throughout the rest of the section let $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$ be an m- and n-dimensional lattice polytopes, respectively. Further we assume that there are rdf-triangulations P^λ and Q^μ. Suppose now that we have linear orderings O_P and O_Q of the vertex sets $V_P = P \cap \mathbb{Z}^m$ and $V_Q = Q \cap \mathbb{Z}^n$ such that the corresponding simplicial product $P^\lambda \times_{\mathrm{stc}} Q^\mu$ is again foldable. Note that such orderings always exist due to Proposition 4.5. By Proposition 4.9, $P^\lambda \times_{\mathrm{stc}} Q^\mu$ is also regular and dense.

The rest of this section is devoted to computing the signature of $P^\lambda \times_{\mathrm{stc}} Q^\mu$. The dual graph Γ^* of the cell complex $P^\lambda \times Q^\mu$ is the product of the dual graphs of P^λ and Q^μ. Further the dual graph of the simplicial product $P^\lambda \times_{\mathrm{stc}} Q^\mu$ arises from Γ^* by replacing each node by a copy of $\Gamma^*(\mathrm{stc}_{m,n})$ in a suitable way.

Recall that only odd simplices contribute to the signature. Since the staircase triangulation is unimodular for each facet F of $\mathrm{stc}(f \times g)$ we have $\nu(F) = \nu(f)\nu(g)$. Therefore we have

$$\sigma(P^\lambda \times_{\mathrm{stc}} Q^\mu) = \sigma_{m,n} \left| \sum_{f \times g \text{ facet of } P^\lambda \times Q^\mu} \delta(f,g)\, \overline{\nu}(f)\, \overline{\nu}(g) \right| , \tag{4.4}$$

where $\delta(f,g) = \pm 1$ and $\overline{\nu}(h) = \nu(h) \bmod 2$ denotes the parity of the normalized volume of h. So it remains to determine the sign $\delta(f,g)$. This only depends on the vertex orderings O_P and O_Q.

As a *point of reference* inside $\mathrm{stc}(f \times g)$ we choose the facet $F_0(f,g)$ corresponding to the origin in the notation from Section 4.1; this corresponds to the staircase $F_0 = 11 \ldots 100 \ldots 0$ which first goes all the way to the right and then all the way up in Figure 4.1. To determine the sign $\delta(f,g)$ amounts to determining the color of the facet $F_0(f,g)$ in $P^\lambda \times_{\mathrm{stc}} Q^\mu$.

We first consider the case where P^λ is a lattice m-simplex Δ_m (without interior lattice points) and Q^μ consists of two neighboring n-simplices (without interior lattice points), that is, Q^μ is the rdf-triangulation B_n of the bipyramid over the $(n-1)$-simplex from Example 4.6. Note that Δ_m is an rdf-triangulation of itself. Further, the signature of Δ_m vanishes if the normalized volume of Δ_m is even and equals 1 otherwise.

Lemma 4.16. The simplicial product $\Delta_m \times_{\mathrm{stc}} B_n$ is an rdf-triangulation of the product of Δ_m and a lattice bipyramid over the $(n-1)$-simplex with signature

$$\sigma(\Delta_m \times_{\mathrm{stc}} B_n) \;=\; \begin{cases} \sigma_{m,n}\,\sigma(\Delta_m)\,\sigma(B_n) & \text{if the vertex ordering on } B_n \text{ is color consecutive or if } m \text{ is even,} \\ \sigma_{m,n}\,\sigma(\Delta_m)\,\omega & \text{if the vertex ordering on } B_n \text{ is symmetric and } m \text{ is odd.} \end{cases}$$

Here $\omega \in \{0,1,2\}$ counts the number of odd simplices in B_n.

Proof. It is a consequence of Propositions 4.5 and 4.9 that $\Delta_m \times_{\mathrm{stc}} B_n$ is an rdf-triangulation.

Let g and g' be the two facets of B_n. In both cases we get a contribution of $\delta(\Delta_m, g)\,\sigma_{m,n}\,\sigma(\Delta_m)$ to $\sigma(\Delta_m \times_{\mathrm{stc}} B_n)$ if g is odd, and similarly a contribution of $\delta(\Delta_m, g')\,\sigma_{m,n}\,\sigma(\Delta_m)$ to $\sigma(\Delta_m \times_{\mathrm{stc}} B_n)$ if g' is odd; see Equation (4.4).

It remains to compare $\delta(\Delta_m, g)$ and $\delta(\Delta_m, g')$, which depends on the vertex ordering of B_n. We have $\delta(\Delta_m, g) = -\delta(\Delta_m, g')$ if and only if $F_0(\Delta_m, g)$ and $F_0(\Delta_m, g')$ are colored differently which in turn holds if and only if the distance between $F_0(\Delta_m, g)$ and $F_0(\Delta_m, g')$ in $\Gamma^*(\Delta_m \times_{\mathrm{stc}} B_n)$ is odd.

Since $\Gamma^*(\Delta_m \times_{\mathrm{stc}} B_n)$ is bipartite, each path from $F_0(\Delta_m, g)$ to $F_0(\Delta_m, g')$ has the same parity, and we may choose any path to determine the parity of the distance. Let $\tilde{F}_0(\Delta_m, g) \in \mathrm{stc}(\Delta_m \times g)$ and $\tilde{F}_0(\Delta_m, g') \in \mathrm{stc}(\Delta_m \times g')$ be neighboring facets. Then the distance between $F_0(\Delta_m, g)$ and $F_0(\Delta_m, g')$ is odd if and only if the distance between $F_0(\Delta_m, g)$ and $\tilde{F}_0(\Delta_m, g)$ has the same parity as the distance between $F_0(\Delta_m, g')$ and $\tilde{F}_0(\Delta_m, g')$ (keep in mind that the distance between $\tilde{F}_0(\Delta_m, g)$ and $\tilde{F}_0(\Delta_m, g')$ is 1).

We first consider the case where the vertex ordering of B_n is color consecutive. Let c be the color of the unique vertex $v \in g \setminus g'$ (which is the same as the color of the unique vertex $v' \in g' \setminus g$). All columns in the lattice grid defining $\Delta_m \times_{\mathrm{stc}} B_n$ corresponding to vertices colored c are consecutive and hence v and v' follow one after another in the vertex ordering of B_n. We distinguish the two cases where v and v' appear somewhere in the middle or at the beginning of the vertex ordering of B_n and where v and v' appear

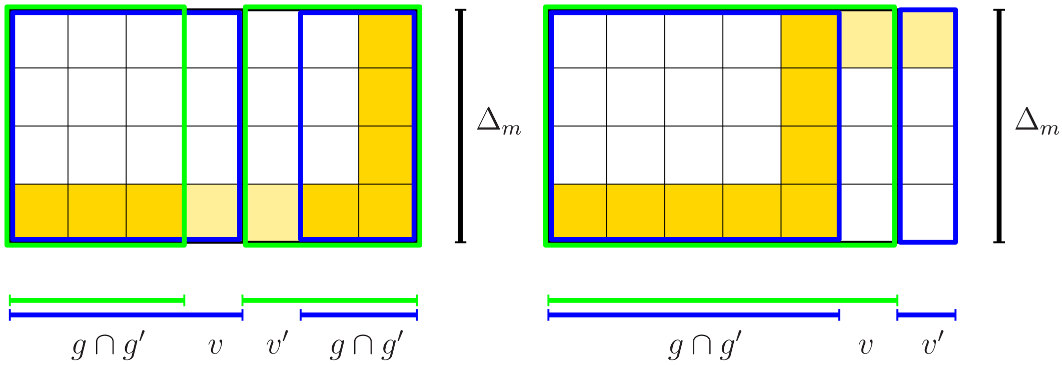


Figure 4.10. *Distance of the facets of reference $F_0(\Delta_m, g)$ and $F_0(\Delta_m, g')$ in $\Gamma^*(\Delta_m \times_{\mathrm{stc}} B_n)$ for color consecutive orderings of B_n. The facets $\tilde{F}_0(\Delta_m, g)$ and $\tilde{F}_0(\Delta_m, g')$ and their intersection is shaded. On the left the two apices v, v' occur somewhere in the middle or at the beginning of the vertex ordering of B_n, on the right at the end.*

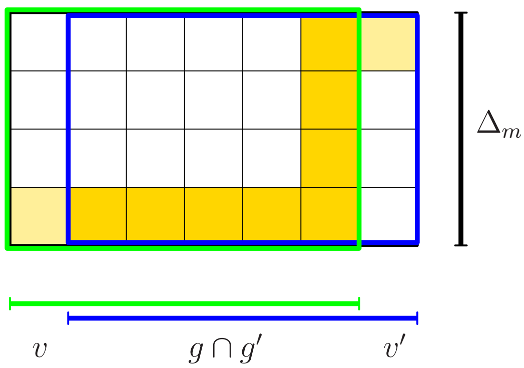


Figure 4.11. *Distance of the facets of reference $F_0(\Delta_m, g)$ and $F_0(\Delta_m, g')$ in $\Gamma^*(\Delta_m \times_{\mathrm{stc}} B_n)$ for the symmetric ordering of the vertices of B_n. The facets $\tilde{F}_0(\Delta_m, g)$ and $\tilde{F}_0(\Delta_m, g')$ and their intersection is shaded.*

at the end of the vertex ordering; see Figure 4.10. In the first case we may choose $F_0(\Delta_m, g) = \tilde{F}_0(\Delta, g)$ and $F_0(\Delta_m, g') = \tilde{F}_0(\Delta_m, g')$ and the distance between $F_0(\Delta_m, g)$ and $F_0(\Delta, g')$ is 1. In the second case the distance between $F_0(\Delta_m, g)$ and $\tilde{F}_0(\Delta_m, g)$ equals the distance between $F_0(\Delta_m, g')$ and $\tilde{F}_0(\Delta_m, g')$. Therefore we obtain $\delta(\Delta_m, g) = -\delta(\Delta_m, g')$ in the color consecutive case.

Let the vertex ordering on B_n be symmetric. We have $F_0(\Delta_m, g) = \tilde{F}_0(\Delta_m, g)$ and the distance of $F_0(\Delta_m, g')$ and $\tilde{F}_0(\Delta_m, g')$ is m, hence $\delta(\Delta_m, g) = -\delta(\Delta_m, g')$ if and only if m is even; see Figure 4.11. $\square$

We refer to Figure 4.5 for an example of three triangulations of $[0, 1] \times B_2$ resulting from different vertex orders of B_2.

Theorem 4.17 (Combinatorial Product Theorem)**.** Let P^λ and Q^μ be rdf-triangulations of an m-dimensional lattice polytope $P \subset \mathbb{R}^m$ and an n-dimensional lattice polytope $Q \subset \mathbb{R}^n$, respectively. For color consecutive vertex orderings O_P and O_Q the simplicial product $P^\lambda \times_{\mathrm{stc}} Q^\mu$ is an rdf-triangulation of the polytope $P \times Q$ with signature

$$\sigma(P^\lambda \times_{\mathrm{stc}} Q^\mu) = \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(Q^\mu)\ .$$

Proof. Again, by Propositions 4.5 and 4.9, $P^\lambda \times_{\mathrm{stc}} Q^\mu$ is an rdf-triangulation.

Let $f, f' \in P^\lambda$ and $g, g' \in Q^\mu$ be facets such that $f \times g$ and $f' \times g'$ are neighboring cells of $P^\lambda \times Q^\mu$. We may assume that $f = f'$ and $g \cap g'$ is a ridge. Hence $g \cup g'$ is a bipyramid over the common ridge $g \cap g'$. Applying Lemma 4.16 to $f \times_{\mathrm{stc}} (g \cup g')$ yields $\delta(f,g) = -\delta(f,g')$, and we may label the cells of $P^\lambda \times Q^\mu$ with $\delta(f,g)$ by assigning $+1$ (black) and -1 (white) according to the bipartition of the dual graph $\Gamma^*(P^\lambda \times Q^\mu)$ of $P^\lambda \times Q^\mu$.

We may think of $\Gamma^*(P^\lambda \times Q^\mu)$ as a copy of $\Gamma^*(P^\lambda)$ for each node of $\Gamma^*(Q^\mu)$. Each copy of $\Gamma^*(P^\lambda)$ may be 2-colored using the bipartition of $\Gamma^*(P^\lambda)$, but we must use the inverse coloring for a copy of $\Gamma^*(P^\lambda)$ if the corresponding node of $\Gamma^*(Q^\mu)$ is colored white. Therefore a node $f \times g$ of $\Gamma^*(P^\lambda \times Q^\mu)$ is labeled $+1$ if and only if the facets $f \in P^\lambda$ and $g \in Q^\mu$ are colored the same, and using Equation (4.4) we have

$$\begin{aligned}
&\sigma(P^\lambda \times_{\mathrm{stc}} Q^\mu) \\
&= \sigma_{m,n} \Bigg| \sum_{f \in P^\lambda \text{ black}} \Big(\overline{\nu}(f) \sum_{g \in Q^\mu \text{ black}} \overline{\nu}(g) \Big) + \sum_{f \in P^\lambda \text{ white}} \Big(\overline{\nu}(f) \sum_{g \in Q^\mu \text{ white}} \overline{\nu}(g) \Big) \\
&\qquad - \sum_{f \in P^\lambda \text{ black}} \Big(\overline{\nu}(f) \sum_{g \in Q^\mu \text{ white}} \overline{\nu}(g) \Big) - \sum_{f \in P^\lambda \text{ white}} \Big(\overline{\nu}(f) \sum_{g \in Q^\mu \text{ black}} \overline{\nu}(g) \Big) \Bigg| \\
&= \sigma_{m,n} \Bigg| \sum_{f \in P^\lambda \text{ black}} \overline{\nu}(f) - \sum_{f \in P^\lambda \text{ white}} \overline{\nu}(f) \Bigg| \Bigg| \sum_{g \in Q^\mu \text{ black}} \overline{\nu}(g) - \sum_{g \in Q^\mu \text{ white}} \overline{\nu}(g) \Bigg| \\
&= \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(Q^\mu)\ .
\end{aligned}$$

□

Finally we consider the case where Q^μ is the rdf-triangulation B_n of the bipyramid over the $(n-1)$-simplex from Example 4.6. While this seems to cover a very special case only, the result is instrumental for the construction of triangulations of the d-cube with non-trivial signature in Section 4.4.

Proposition 4.18. Let P^λ be an rdf-triangulation of an m-dimensional lattice polytope $P \subset \mathbb{R}^m$ with a color consecutive ordering on its vertex set $V_P = P \cap \mathbb{Z}^m$. Then $P^\lambda \times_{\text{stc}} B_n$ is an rdf-triangulation of the product of P with a lattice bipyramid over the $(n-1)$-simplex with signature

$$\sigma(P^\lambda \times_{\text{stc}} B_n) \;=\; \begin{cases} \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(B_n) & \text{if the vertex ordering on } B_n \text{ is color consecutive or if } m \text{ is even,} \\ \sigma_{m,n}\, \sigma(P^\lambda)\, \omega & \text{if the vertex ordering on } B_n \text{ is symmetric and } m \text{ is odd.} \end{cases}$$

Here $\omega \in \{0, 1, 2\}$ counts the number of odd simplices in B_n.

One can show that for other vertex orderings of B_n the simplicial product $P^\lambda \times_{\text{stc}} B_n$ is not foldable. In this sense the two cases listed exhaust all the possibilities.

Proof. Propositions 4.5 and 4.9 ensure that $P^\lambda \times_{\text{stc}} Q^\mu$ is an rdf-triangulation. Let g and g' be the two facets of B_n, and let us think of $P^\lambda \times B_n$ as the union of two copies of $P^\lambda \times \Delta_n$, which we denote as $P^\lambda \times g$ and $P^\lambda \times g'$. Further let $f \in P^\lambda$ be an arbitrary but fixed facet. We get a contribution of $\delta(f, g)\, \sigma(P^\lambda)\, \sigma_{m,n}$ to $\sigma(P^\lambda \times_{\text{stc}} B_n)$ if g is odd by Theorem 4.17. Similarly we get a contribution of $\delta(f, g')\, \sigma(P^\lambda)\, \sigma_{m,n}$ to $\sigma(P^\lambda \times_{\text{stc}} B_n)$ if g' is odd. It remains to compare $\delta(f, g)$ and $\delta(f, g')$. The simplicial product $f \times_{\text{stc}} (g \cup g')$ is a triangulation of the product of an m-simplex and B_n and by Lemma 4.16 we have $\delta(f, g) = -\delta(f, g')$ in the first and $\delta(f, g) = \delta(f, g')$ in the second case. □

A referee suggested the following generalization of Proposition 4.18, which we state without a proof. Let P^λ and Q^μ be rdf-triangulations of the full dimensional lattice polytopes $P \subset \mathbb{R}^m$ and $Q \subset \mathbb{R}^n$, respectively. Further let the vertices of P^λ be ordered color consecutive, and let the vertices of Q^μ be partitioned into subsets $V_0, V_1, \ldots, V_n$ according to their colors. An *almost color consecutive* ordering of the vertices of Q^μ is obtained by splitting V_0 into two subsets V_0' and V_0'' and taking any vertex ordering compatible with $V_0' < V_1 < \cdots < V_n < V_0''$. The vertex sets V_0' and V_0'' induce a bipartition on the facets of Q^μ denoted by L' and L'', and let the facets of L', respectively L'' be colored "black" and "white" according to the coloring of the facets of Q^μ (neither L' nor L'' is strongly connected in general). Finally we set the *signed signature* $\tilde{\sigma}(L)$ of a geometric simplicial complex L with facets colored "black" and "white" as the number of odd "black" facets minus the number of odd "white" facets.

Proposition 4.19. The simplicial product $P^\lambda \times_{\text{stc}} Q^\mu$ (with respect to the color consecutive vertex ordering of P^λ and the almost color consecutive vertex ordering of Q^μ) is a rdf-triangulation of $P \times Q$ with signature

$$\sigma(P^\lambda \times_{\text{stc}} Q^\mu) = \begin{cases} \sigma_{m,n}\ \sigma(P^\lambda)\ \sigma(Q^\mu) & \text{if } m \text{ is even,} \\ \sigma_{m,n}\ \sigma(P^\lambda)\ |\tilde\sigma(L') - \tilde\sigma(L'')| & \text{if } m \text{ is odd.} \end{cases}$$

4.3 Lower Bounds for the Number of Real Roots of Polynomial Systems

Triangulations which are regular, dense, and foldable are interesting since they yield non-trivial lower bounds for the number of real roots of associated polynomial systems, provided that a number of additional geometric conditions are met. To discuss these issues we first review the construction of Soprunova and Sottile [62].

4.3.1 Triangulations and Lower Bounds

Let $P \subset \mathbb{R}^m_{\geq 0}$ be a lattice m-polytope contained in the positive orthant, and let $\lambda : P \cap \mathbb{Z}^m \to \mathbb{R}$ be a lifting function such that the induced triangulation P^λ is an rdf-triangulation. Further let the vertices $P \cap \mathbb{Z}^m$ of P^λ be colored by the map $c : P \cap \mathbb{Z}^m \to [m+1]$. We define the *coefficient polynomial* $F_{P^\lambda,i,s} \in \mathbb{R}[t_1, \ldots, t_m]$ of a color i and an additional parameter $s \in (0, 1]$ as

$$F_{P^\lambda,i,s}(t) \ = \sum_{v \in c^{-1}(i)} s^{\lambda(v)}\, t^v, \tag{4.5}$$

where $t = (t_1, \ldots, t_m)$ and $t^v = t_1^{v_1} \ldots t_m^{v_m}$. Choosing a real number a_i for each color $i \in [m+1]$ defines a *Wronski polynomial*

$$\mathcal{F}_{P^\lambda,s}(t) \ = \ a_0 F_{P^\lambda,0,s}(t) + a_1 F_{P^\lambda,1,s}(t) + \ldots + a_m F_{P^\lambda,m,s}(t) \ \in \mathbb{R}[t_1, \ldots, t_m]\ ,$$

for fixed $s \in (0, 1]$. A *Wronski system* associated with P^λ is a sparse system of m Wronski polynomials which is *generic* in the sense that it attains Kushnirenko's bound [40], that is, it has exactly $\nu(P)$ distinct complex solutions.

Let $M = |P \cap \mathbb{Z}^m|$ denote the number of integer points in P and let $\mathbb{CP}^{M-1}$ be the complex projective space with coordinates $\{x_v \mid v \in P \cap \mathbb{Z}^m\}$. The toric projective variety $X_P \subset \mathbb{CP}^{M-1}$ parameterized by the monomials $\{t^v \mid v \in P \cap \mathbb{Z}^m\}$ is given by the closure of the image of the map

$$\varphi_P : (\mathbb{C}^\times)^m \to \mathbb{CP}^{M-1} : t \mapsto [t^v \mid v \in P \cap \mathbb{Z}^m]\ , \tag{4.6}$$

where $[t^{v_1}, \dots, t^{v_m}]$ is a point in $\mathbb{CP}^{M-1}$ written in homogeneous coordinates. Via φ_P a Wronski system on $(\mathbb{C}^\times)^m$ corresponds to a system of m linear equations on the toric variety $X_P \subset \mathbb{CP}^{M-1}$.

Let $Y_P = X_P \cap \mathbb{RP}^{M-1}$ be the real points of the variety X_P. For $s \in (0,1]$ the *s-deformation* $s.Y_P$ is obtained as the closure of the image of the deformed map

$$s.\varphi_P : (\mathbb{C}^\times)^m \to \mathbb{CP}^{M-1} : t \mapsto [s^{\lambda(v)}\, t^v \mid v \in P \cap \mathbb{Z}^m]$$

intersected with $\mathbb{RP}^{M-1}$. The s-deformation $s.Y_P$ interpolates between $Y_P = 1.Y_P$ and its homotopic image $0.Y_P$, which is defined as the initial variety $\mathrm{in}_\lambda(Y_P)$; the whole family $\{s.Y_P \mid s \in [0,1]\}$ is called the *toric degeneration* of Y_P; for the details see [62, Section 3]. A Wronski polynomial corresponds to the image of $s.Y_P$ under the linear *Wronski projection*

$$\pi_E : \begin{array}{rcl} \mathbb{CP}^{M-1} \setminus E & \to & \mathbb{CP}^m \\ {[x_v \mid v \in P \cap \mathbb{Z}^m]} & \mapsto & \Big[\displaystyle\sum_{v \in c^{-1}(i)} x_v \mid i = 0, 1, \dots, m \Big] \end{array}$$

with *center*

$$E = \left\{ x \in \mathbb{CP}^{M-1} \;\middle|\; \sum_{v \in c^{-1}(i)} x_v = 0 \quad \text{for } i = 0, 1, \dots, m \right\}.$$

The toric degeneration *meets* the center of the projection π_E if there are $s \in (0,1]$ and $t \in \mathbb{R}^m$ such that

$$F_{P^\lambda,0,s}(t) \;=\; F_{P^\lambda,1,s}(t) \;=\; \dots \;=\; F_{P^\lambda,m,s}(t) \;=\; 0\,.$$

The sphere $\mathbb{S}^{M-1}$ is a double cover of $\mathbb{RP}^{M-1}$. Let $Y_P^+ \subset \mathbb{S}^{M-1}$ be the pre-image of Y_P under the covering map. Note that Y_P^+ is not necessarily smooth nor connected. Nonetheless, its orientability is well defined. The following theorem is a slightly simplified version of what is proved in [62].

Theorem 4.20 (Soprunova & Sottile)**.** Let $P \subset \mathbb{R}^m_{\geq 0}$ be a non-negative lattice m-polytope such that Y_P^+ is oriented, and let P^λ be an rdf-triangulation of P induced by the lifting function λ. Suppose that there is a number $s_0 \in (0,1]$ such that the s-deformation $s.Y_P$ does not meet the center of the Wronski projection π_E for all $s \in (0, s_0]$ and all $t \in \mathbb{R}^m$. Then for all $s \in (0, s_0]$ the number of real solutions of any associated Wronski system in $\mathbb{R}[t_1, \dots, t_m]$ is bounded from below by the signature $\sigma(P^\lambda)$.

In general, it seems difficult to decide the orientability of Y_P^+. To this end Soprunova and Sottile suggest to consider the following sufficient condition:

Let (A, b) be an integral facet description of $P = \{x \in \mathbb{R}^m \mid Ax + b \geq 0\}$ such that the i-th row of the matrix A is the unique inward pointing primitive normal vector of the i-th facet of P. This way, up to a re-ordering of the facets, A and b are uniquely determined. Denote by Λ_A the lattice spanned by the columns of A. Suppose that the lattice spanned by $P \cap \mathbb{Z}^m$ has odd index in $\mathbb{Z}^m$ and that Λ_A has odd index in its saturation $\Lambda_A \otimes_{\mathbb{Z}} \mathbb{Q}$, that is, A has a maximal minor $\tilde{A}$ with $\det \tilde{A}$ odd. If these two parity conditions are satisfied and if, additionally, there is a vector v with only odd entries in the integer column span of (A, b) then Soprunova and Sottile call the double cover Y_P^+ *Cox-oriented*.

We call the rdf-triangulation P^λ *geometrically nice* or *g-nice* for the value s_0 if all the conditions of Theorem 4.20 are satisfied. (This definition of a g-nice rdf-triangulation differs from the topologically motivated definition of a t-nice simplicial complex in Section 1.2.) Note that the (Cox-)orientability of Y_P^+ solely depends on the polytope P.

Example 4.21. The unique rdf-triangulation of the line segment $[k, l]$, where $0 \leq k < l$, is g-nice for $s_0 = 1$ (and any lifting function) if and only if $k = 0$. We have $\sigma([0, l]) \in \{0, 1\}$ depending on l being even or odd. This is a sharp lower bound for the number of real roots in the one-dimensional case.

Example 4.22. The staircase triangulation of $\Delta_m \times \Delta_n$ is g-nice for $s_0 = 1$. This is true if at least one of the two vertices whose color occurs only once is located at the origin.

Example 4.23. Let P^λ be an rdf-triangulation of a lattice polytope $P \subset \mathbb{R}^m_{\geq 0}$, and let Y_P^+ be Cox-oriented. The cone $0 * P^\lambda$ of the triangulation P^λ (embedded into $\mathbb{R}^{m+1}$ via the map $(v_1, \ldots, v_m) \mapsto (1, v_1, \ldots, v_m)$) with apex $0 \in \mathbb{R}^{m+1}$ is g-nice for $s_0 = 1$. The signature of $0 * P^\lambda$ equals the signature of P^λ.

4.3.2 Products of Toric Varieties

Let us consider the *Segre embedding*

$$\iota : \begin{array}{rcl} \mathbb{CP}^{M-1} \times \mathbb{CP}^{N-1} & \to & \mathbb{CP}^{MN-1} \\ ([x_1, \ldots, x_M], [y_1, \ldots, y_N]) & \mapsto & [x_1 y_1, \ldots, x_i y_j, \ldots, x_M y_N] \, , \end{array}$$

which is the tensor product. The restriction $\iota : \mathbb{RP}^{M-1} \times \mathbb{RP}^{N-1} \to \mathbb{RP}^{MN-1}$ lifts to the double covers $\iota^+ : \mathbb{S}^{M-1} \times \mathbb{S}^{N-1} \to \mathbb{S}^{MN-1}$.

Proposition 4.24. Let P be an m-dimensional lattice polytope with M lattice points, and let Q be an n-dimensional lattice polytope with N lattice points. Then we have

$$\iota(Y_P \times Y_Q) = Y_{P\times Q} \quad \text{and} \quad \iota^+(Y_P^+ \times Y_Q^+) = Y_{P^+\times Q^+} .$$

Proof. Let $\varphi_P : (\mathbb{C}^\times)^m \to \mathbb{CP}^{M-1}$ denote the map in Equation (4.6) which defines the toric variety X_P. Observe that $\varphi_{P\times Q} = \iota \circ (\varphi_P, \varphi_Q)$. This readily implies $\iota(X_P \times X_Q) = X_{P\times Q}$ and also $\iota(Y_P \times Y_Q) = Y_{P\times Q}$. Now $\iota^+(Y_P^+ \times Y_Q^+) = Y_{P^+\times Q^+}$ follows since the map ι lifts to the coverings. □

Corollary 4.25. Let P and Q be lattice polytopes such that Y_P^+ and Y_Q^+ are oriented. Then $Y_{P\times Q}^+$ is oriented.

Proof. The orientability of $Y_{P\times Q}^+$ depends on the orientability of its smooth part, which is the ι^+-image of the product of the smooth parts of Y_P^+ and Y_Q^+. The product of orientable manifolds is orientable. □

Remark 4.26. As a further consequence, if Y_P^+ and Y_Q^+ are Cox-oriented, then $Y_{P\times Q}^+$ is oriented. However, $Y_{P\times Q}^+$ does not have to be Cox-oriented itself. For an example consider products $\Delta_m \times \Delta_n$ of standard simplices for m even and n odd.

The question under which conditions the toric degeneration of $Y_{P\times Q}$ meets the center of the Wronski projection is a little harder to answer. The lifting function ω determines the triangulation of $P \times Q$ and we write $(P \times Q)^\omega = P^\lambda \times_{\mathrm{stc}} Q^\mu$ if we want to emphasize the particular lifting function ω defined in Equation (4.2). Recall that a vertex (v, w) of $(P \times Q)^\omega$ is colored $k = c_{P^\lambda}(v) + c_{Q^\mu}(w)$ where $c_{P^\lambda} : P \cap \mathbb{Z}^m \to [m + 1]$ and $c_{Q^\mu} : Q \cap \mathbb{Z}^n \to [n + 1]$ denote the coloring maps; see Proposition 4.5. Therefore for $s \in (0, 1]$ the coefficient polynomial (Equation (4.5)) of $(P \times Q)^\omega$ for $k \in [m + n + 1]$ has the form

$$\begin{aligned} F_{(P\times Q)^\omega,k,s}(t) &= \sum_{c_{P^\lambda}(v)+c_{Q^\mu}(w)=k} s^{\lambda(v)+\mu(w)+\epsilon(v,w)}\ t^{(v,w)} \\ &= \sum_{c_{P^\lambda}(v)+c_{Q^\mu}(w)=k} s^{\lambda(v)}(t_1,\dots,t_m)^v\ s^{\mu(w)}(t_{m+1},\dots,t_{m+n})^w\ s^{\epsilon(v,w)} . \end{aligned}$$

As in Example 4.8 we may choose the same perturbation $\epsilon(i, j) = \epsilon 2^{(n+1)i+(n-j)}$ (for sufficiently small $\epsilon > 0$) for all vertices (v, w) with $c_{P^\lambda}(v) = i$ and $c_{Q^\mu}(w) = j$ if we choose color consecutive orderings of the vertices of P^λ and Q^μ; see Equation (4.3). Summing over all colors i of P^λ and all colors j of Q^μ with $i + j = k$ yields

$$F_{(P\times Q)^\omega,k,s} = \sum_{i+j=k} F_{P^\lambda,i,s}\ F_{Q^\mu,j,s}\ s^{\epsilon(i,j)} . \tag{4.7}$$

The s-degeneration $s.Y_P$ meets the center of the Wronski projection in the points

$$V_s(P^\lambda) = \left\{ t \in \mathbb{R}^m \;\middle|\; F_{P^\lambda,i,s}(t) = 0 \text{ for all } i \in [m+1] \right\} ,$$

the real variety generated by the coefficient polynomials of P^λ. Treating the parameter s as an additional indeterminate we arrive at

$$V(P^\lambda) = \left\{ (s,t) \in \mathbb{R}^{1+m} \;\middle|\; F_{P^\lambda,i,s}(t) = 0 \text{ for all } i \in [m+1] \text{ and } s \in (0,1] \right\} .$$

Lemma 4.27. Choose color consecutive orderings of the vertices of P^λ and Q^μ. Then there is a lifting function ω of $P^\lambda \times_{\mathrm{stc}} Q^\mu = (P \times Q)^\omega$, such that the points in the variety $V_s((P\times Q)^\omega)$ are exactly the points $(t,t') = (t_1,\ldots,t_{m+n}) \in \mathbb{R}^{m+n}$ with $t \in V_s(P^\lambda)$ or $t' \in V_s(Q^\mu)$, that is,

$$V_s((P\times Q)^\omega) = (V_s(P^\lambda) \times \mathbb{R}^n) \cup (\mathbb{R}^m \times V_s(Q^\mu)) .$$

Remark 4.28. The variety $V_s(P^\lambda)$ may be infinite, in general.

Proof of Lemma 4.27. For a point $t \in V_s(P^\lambda)$ we have $(t,t') \in V_s((P\times Q)^\omega)$ for all $t' \in \mathbb{R}^n$ by Equation (4.7). Similarly we have $(t,t') \in V_s((P\times Q)^\omega)$ for $(s,t') \in V_s(Q^\mu)$ and all $t \in \mathbb{R}^m$.

For the reverse, let us assume there is a point $(t,t') \in V_s((P\times Q)^\omega)$ but $t \notin V_s(P^\lambda)$ and $t' \notin V_s(Q^\mu)$. Choose $i_0 \in [m+1]$ and $j_0 \in [n+1]$ minimal such that $F_{P^\lambda,i_0,s}(t) \neq 0$ and $F_{Q^\mu,j_0,s}(t') \neq 0$. Further let us assume $i_0 \geq j_0$. We prove by induction on i that $i_0 > m$, or alternatively that $F_{P^\lambda,i,s}(t) = 0$ for all $i \in [m+1]$, contradicting our assumption $t \notin V_s(P^\lambda)$.

We have $F_{P^\lambda,i,s}(t) = 0$ for all $i < j_0$. Note that this is also true for $j_0 = 0$. Now let $F_{P^\lambda,i',s}(t) = 0$ for all $i' < i$. Equation (4.7) yields for $k = i + j_0$

$$\begin{aligned} F_{(P\times Q)^\omega,i+j_0,s}(t,t') &= \sum_{i'+j'=i+j_0} F_{P^\lambda,i',s}(t)\; F_{Q^\mu,j',s}(t')\; s^{\epsilon(i',j')} \\ &= \sum_{i'+j'=i+j_0,i'<i} F_{P^\lambda,i',s}(t)\; F_{Q^\mu,j',s}(t')\; s^{\epsilon(i',j')} \\ &\quad +\; F_{P^\lambda,i,s}(t)\; F_{Q^\mu,j_0,s}(t')\; s^{\epsilon(i,j_0)} \\ &\quad + \sum_{i'+j'=i+j_0,i'>i} F_{P^\lambda,i',s}(t)\; F_{Q^\mu,j',s}(t')\; s^{\epsilon(i',j')} \\ &= 0 , \end{aligned}$$

since we assumed $(t,t') \in V_s((P\times Q)^\omega)$.

We have $F_{P^\lambda,i',s}(t) = 0$ for $i' < i$ by induction and $i' > i$ implies $j < j_0$ hence $F_{Q^\mu,j,s}(t') = 0$ for $i' > i$. We are left with $F_{P^\lambda,i,s}(t)\; F_{Q^\mu,j_0,s}(t')\; s^{\epsilon(i,j_0)} = 0$ which in turn yields $F_{P^\lambda,i,s}(t) = 0$ since $s^{\epsilon(i,j_0)} > 0$ and $F_{Q^\mu,j_0,s}(t') \neq 0$; see Figure 4.12. □

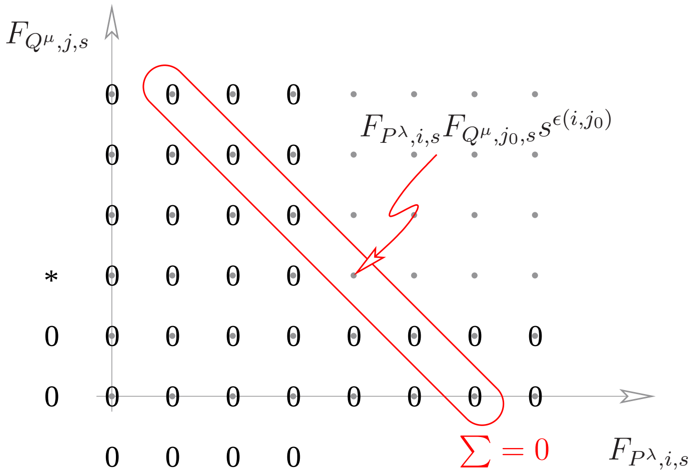


Figure 4.12. *The inductive step in the proof of Lemma 4.27. Here $*$ denotes the non-zero value of $F_{Q^\mu,j_0,s}(t')$.*

Now we are ready to state and prove our main result.

Theorem 4.29 (Algebraic Product Theorem)**.** Let $P \subset \mathbb{R}^m_{\geq 0}$ and $Q \subset \mathbb{R}^n_{\geq 0}$ be non-negative full-dimensional lattice polytopes with rdf-triangulations P^λ and Q^μ which are g-nice for some value $s_0 \in (0,1]$. Further choose any color consecutive vertex orderings for P^λ and Q^μ. Then there is a lifting function $\omega : (P \times Q) \cap \mathbb{Z}^{m+n} \to \mathbb{R}$ such that $(P \times Q)^\omega = P^\lambda \times_{\text{stc}} Q^\mu$ is g-nice for s_0. Moreover, the number of real solutions of any Wronski polynomial system associated with $(P \times Q)^\omega$ is bounded from below by

$$\sigma\left((P \times Q)^\omega\right) = \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(Q^\mu)\ .$$

Proof. The orientability of $Y^+_{P\times Q}$ is a consequence of Corollary 4.25. Now Lemma 4.27 provides a lifting function $\omega : (P\times Q)\cap\mathbb{Z}^{m+n} \to \mathbb{R}$ of $P^\lambda \times_{\text{stc}} Q^\mu$ such that the s-degeneration $s.Y_{(P\times Q)^\omega}$ does not meet the center of the Wronski projection for $s \in (0, s_0]$ and $(t,t') \in \mathbb{R}^{m+n}$: Since $V_s(P^\lambda) = V_s(Q^\mu) = \emptyset$ for all $s \in (0,s_0]$ we have $V_s((P\times Q)^\omega) = (V_s(P^\lambda)\times\mathbb{R}^n)\cup(\mathbb{R}^m\times V_s(Q^\mu)) = \emptyset$ for all $s \in (0, s_0]$. The claim hence follows from Theorem 4.20 and our Combinatorial Product Theorem 4.17. $\square$

Remark 4.30. The decomposition $\sigma(P^\lambda \times_{\text{stc}} Q^\mu) = \sigma_{m,n}\, \sigma(P^\lambda)\, \sigma(Q^\mu)$ from Theorems 4.17 and 4.29 reflects the geometric situation in the following sense: Let $M = |P \cap \mathbb{Z}^m|$ and $N = |Q \cap \mathbb{Z}^n|$ denote the number of lattice points of P and Q, respectively. The Wronski projection $\pi_E : \mathbb{CP}^{M-1} \setminus E \to \mathbb{CP}^m$ (and its center E) depends solely on the lifting function $\lambda : \mathbb{R}^m \to \mathbb{R}$ which induces the rdf-triangulation P^λ on P. Hence we will denote the Wronski projection π_E associated with P^λ by π_{P^λ}, and its lifting to $\mathbb{S}^{M-1}$ by $\pi^+_{P^\lambda}$. To give a lower bound on the number of real roots of the Wronski system associated with $(P \times Q)^\omega = P^\lambda \times_{\mathrm{stc}} Q^\mu$ we have to bound the topological degree of the map $\pi^+_{(P\times Q)^\omega}$ restricted to $Y^+_{P\times Q}$. A decomposition of $\pi^+_{(P\times Q)^\omega}$ by the maps $\pi^+_{P^\lambda}$, $\pi^+_{Q^\mu}$, $\pi^+_{\Delta_m \times_{\mathrm{stc}} \Delta_n}$, and the covers of the Segre embeddings is given by the following diagram which commutes provided that the lifting functions match as in Equation (4.3). Here the vertical arrows indicate the covers of the Segre embeddings of the appropriate dimensions.

$$\begin{tikzcd}
\mathbb{S}^{M-1} \times \mathbb{S}^{N-1} \arrow[d, "\iota^+"'] & Y^+_P \times Y^+_Q \arrow[l, hook'] \arrow[rr, "\pi^+_{P^\lambda} \times \pi^+_{Q^\mu}"] \arrow[d, "\iota^+"'] & & Y^+_{\Delta_m} \times Y^+_{\Delta_n} \arrow[r, equal] \arrow[d, "\iota^+"'] & \mathbb{S}^m \times \mathbb{S}^n \arrow[d, "\iota^+"'] \\
\mathbb{S}^{MN-1} & Y^+_{P\times Q} \arrow[l, hook'] \arrow[r, "\pi^+_{(P\times Q)^\omega}"] & \mathbb{S}^{m+n} & Y^+_{\Delta_m\times\Delta_n} \arrow[l, "\pi^+_{\mathrm{stc}\,m,n}"'] \arrow[r, hook] & \mathbb{S}^{mn+m+n}
\end{tikzcd}$$

This decomposition of $\pi^+_{(P\times Q)^\omega}$ yields the decomposition of $\sigma(P^\lambda \times_{\mathrm{stc}} Q^\mu)$ given in the Theorems 4.17 and 4.29.

4.4 Cubes

We define the *signature* of a lattice polytope P, denoted as $\sigma(P)$, as the maximum of the signatures of all rdf-triangulations of P. The signature is undefined if P does not admit any such triangulation as in Example 4.13. However, here we are concerned with cubes, which do have rdf-triangulations: This is an immediate consequence of the Product Theorem 4.17 since $C_d = [0,1]^d = I \times \cdots \times I$ can be triangulated as the d-fold simplicial product $I \times_{\mathrm{stc}} \ldots \times_{\mathrm{stc}} I$ with zero signature.

Since C_d does not contain any non-vertex lattice points, each lattice triangulation of C_d is dense. Note that C_d does have non-regular triangulations for $d \geq 4$; see De Loera [16].

4.4.1 Triangulations with Large Signature

Since the simplicial product of unimodular triangulations is again unimodular it follows that each d-fold simplicial product $I \times_{\mathrm{stc}} \ldots \times_{\mathrm{stc}} I$ has $d!$ facets, which is the maximum that can be obtained for the d-cube without introducing new vertices. On the other hand the minimal number of facets in a triangulation of C_d is known only for $d \leq 7$; see Anderson and Hughes [32]. The best currently known upper and lower bounds are due to Smith [61], Orden & Santos [51], and Bliss & Su [10]. For a recent survey on cubes, their triangulations, and related issues see Zong [67]. Rambau's program `TOPCOM` allows to enumerate all regular triangulations of C_d for $d \leq 4$ [55]. This then yields the following result.

Proposition 4.31. We have the signatures $\sigma(C_1) = 1$, $\sigma(C_2) = 0$, $\sigma(C_3) = 4$, and $\sigma(C_4) = 2$.

The cases of $C_1 = I$ and C_2 are trivial. The unique (regular and) foldable triangulation of C_3 with the maximal signature 4 is the unique minimal triangulation; it has one (black) facet of normalized volume 2 and four (white) facets of normalized volume 1.

There is one further ingredient which relies on an explicit construction, a triangulation of C_6 with a non-trivial signature. We give more details on our experiments in Section 4.4.3 below.

Proposition 4.32. We have $\sigma(C_6) \geq 4$.

Theorem 4.33. The signature of C_d for $d \geq 3$ is bounded from below by

$$\sigma(C_d) \;\geq\; \begin{cases} 2^{\frac{d+1}{2}} \left(\frac{d-1}{2}\right)! & \text{if } d \equiv 1 \bmod 2 \\ \left(\frac{d}{2}\right)! & \text{if } d \equiv 0 \bmod 4 \\ \frac{2}{3}\left(\frac{d}{2}\right)! & \text{if } d \equiv 2 \bmod 4\,. \end{cases}$$

Proof. Let us start with the case d odd. Here for C_3 we choose the rdf-triangulation with signature 4 from Proposition 4.31. For $d \geq 5$ we factorize C_d as $C_2 \times C_{d-2}$ and choose a color consecutive vertex ordering for C_{d-2}. There is only one triangulation to choose for C_2, but we take the symmetric ordering of the vertices; see Example 4.6. The signature of $stc_{2,d-2}$ equals $(d-1)/2$ by Proposition 4.15 and the second case of Proposition 4.18 inductively gives

$$\sigma(C_d) \;\geq\; 2\,\sigma_{d-2,2}\,\sigma(C_{d-2}) \;\geq\; 2\,\frac{d-1}{2}\,2^{\frac{d-3}{2}}\left(\frac{d-3}{2}\right)! \;=\; 2^{\frac{d+1}{2}}\left(\frac{d-1}{2}\right)!\,.$$

If $d \equiv 0 \bmod 4$ then we inductively prove that $\sigma(C_d) \geq \left(\frac{d}{2}\right)!$. The induction starts with $d = 4$ by Proposition 4.31. For $d \geq 8$ we decompose C_d as $C_4 \times C_{d-4}$. The signature of $\mathrm{stc}_{4,d-4}$ equals $d(d-2)/8$ by Proposition 4.15. Choosing color consecutive orderings for C_4 and C_{d-4} Theorem 4.17 now yields

$$\sigma(C_d) \;\geq\; \sigma_{4,d-4}\,\sigma(C_4)\,\sigma(C_{d-4}) \;\geq\; \frac{d(d-2)}{8}\,2\left(\frac{d-4}{2}\right)! \;=\; \left(\frac{d}{2}\right)!\,.$$

In the remaining case where $d \equiv 2 \bmod 4$ we construct C_d as a simplicial product of C_6 and C_{d-6}. By the explicit construction in Proposition 4.32

the signature of C_6 is at least 4. The signature of C_{d-6} is bounded from below by $(d-6)/2!$ as just proved. Proposition 4.15 yields $\sigma_{6,d-6} = \binom{\frac{d}{2}}{3}$, and Theorem 4.17 completes the proof:

$$\sigma(C_d) \ \geq \ \sigma_{6,d-6}\sigma(C_6)\sigma(C_{d-6}) \ \geq \ \frac{\frac{d}{2}\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-2\right)}{3!} 4\left(\frac{d}{2}-3\right)! \ = \ \frac{2}{3}\left(\frac{d}{2}\right)! \, .$$

□

4.4.2 Nice Triangulations

Our main result, the Algebraic Product Theorem 4.29, asserts that the simplicial product of two g-nice triangulations P^λ and Q^μ is again g-nice, provided that the vertex ordering of P^λ and Q^μ are color consecutive. So what about the triangulations of the d-cube with signature in $\Omega(\lceil d/2 \rceil !)$ constructed in Section 4.4.1 above? Since the construction for d odd was based on the symmetric vertex ordering for the square, which is not color consecutive, Theorem 4.29 does not apply. The goal of this section is thus to construct g-nice cube rdf-triangulations from a decomposition into different factors.

The *geometric signature* $\sigma^+(P)$ of a lattice polytope P is defined as the maximum of the signatures of all rdf-triangulations of P which are g-nice for some parameter $s \in (0, 1]$. Clearly, $\sigma^+(P) \leq \sigma(P)$. Note that $Y^+_{C_d}$ is always oriented by Corollary 4.25 since $C_d = I \times I \times \cdots \times I$, and I is Cox-oriented.

Let us examine two cases of low dimension explicitly: There is a lifting function $C_3 \cap \mathbb{Z}^3 \to \mathbb{N}$ such that the induced triangulation is the unique minimal triangulation of the 3-cube from Proposition 4.31, and the toric degeneration meets the center only for $s = 1$; see [62]. This implies $\sigma^+(C_3) = 4$. In the subsequent Section 4.4.3 a triangulation C_4^λ of the 4-cube with signature equal to 2 is constructed explicitly via a lifting function $\lambda : C_4 \cap \mathbb{Z}^4 \to \mathbb{N}$. The variety $V(C_4^\lambda)$ (see Section 4.3.2), describing the values of s for which the center of the projection is met, consists of two isolated points for some $s_1 > 1$ and some $s_2 < 0$, hence C_4^λ is g-nice for any $s_0 \in (0, 1]$. A complete enumeration of all regular triangulation of C_4 shows that $\sigma^+(C_4) = 2$.

We want to avoid splitting off factors which are squares, since neither of its two vertex orderings can be used for our purposes: The color consecutive vertex ordering has signature zero, and products with respect to the symmetric vertex ordering are not known to be g-nice. Hence we factorize

$$C_d \ = \ \begin{cases} C_1 \times C_{d-1} & \text{if } d \equiv 1 \bmod 4 \\ C_3 \times C_{d-3} & \text{if } d \equiv 3 \bmod 4 \, , \end{cases}$$

which means that we reduced the cases $d \equiv 1 \bmod 4$ and $d \equiv 3 \bmod 4$ to the case $d \equiv 0 \bmod 4$. Proposition 4.15 and Theorem 4.17 yield for $d \equiv 1 \bmod 4$

$$\sigma^+(C_d) \;\geq\; \sigma_{1,d-1}\, \sigma^+(C_1)\, \sigma^+(C_{d-1}) \;=\; \sigma^+(C_{d-1}) \;\geq\; \left(\frac{d-1}{2}\right)! \,.$$

For $d \equiv 3 \bmod 4$ we have

$$\sigma^+(C_d) \;\geq\; \sigma_{3,d-3}\, \sigma^+(C_3)\, \sigma^+(C_{d-3}) \;\geq\; \frac{d-1}{2}\, 4\, \left(\frac{d-3}{2}\right)! \;=\; 4\, \left(\frac{d-1}{2}\right)! \,,$$

and we obtain an overall lower bound in $\Omega(\lfloor d/2 \rfloor!)$ for the geometric signature of the d-cube. Observe that this lower bound for the signature in the case of d odd is significantly weaker than the bound given in Theorem 4.33, which does not take the geometric properties of the Wronski projection into account.

Corollary 4.34. For $d \not\equiv 2 \bmod 4$ there are rdf-triangulations of the d-cube with signature at least $\lfloor d/2 \rfloor!$ which are g-nice for any $s_0 \in (0, 1)$.

Proving that the triangulation of the 6-cube with signature 4 from Proposition 4.32 (together with its generating lifting function) is g-nice for some $s_0 \in (0, 1]$ would also settle the $d \equiv 2 \bmod 4$ case. However, with the techniques of Section 4.4.3 one needs to solve a system of seven polynomials in the seven unknowns $s, x_1, \dots, x_6$ of maximal total degree 386; see Problem 4.37. This is beyond the scope of this investigation.

4.4.3 Constructions and Computer Experiments

We completely enumerated all regular triangulations of the d-cube C_4 up to symmetry using `TOPCOM` [55]. These 235,277 triangulations were then checked whether they are foldable by `polymake` [21, 22, 23]; it turns out that their total number is 454. For all the foldable ones we computed the signature, and we found 36 triangulations with signature 2, all other foldable triangulations of C_4 have a vanishing signature. The regularity of Example 4.35 was independently verified by the explicit construction of a lifting function.

Example 4.35. We now give an explicit description of an rdf-triangulation C_4^λ of the 4-cube with signature two. To this end we encode the vertices of C_4, that is, the 0/1-vectors of length 4 as the hexadecimal digits $0, 1, \dots, 9, a, b, c, d, e, f$. The lifting function λ and the vertex 5-coloring is given in Table 4.1. The facets of C_4^λ are listed in Table 4.2, and the f-vector reads $(16, 64, 107, 81, 23)$.

As mentioned before, the double cover $Y_{C_d}^+$ of the associated real toric variety of the d-cube is indeed oriented for all dimensions d. To prove that C_4^λ

Table 4.1. *The vertex* 5*-coloring* c *and a lifting function* λ *for* C_4^λ *described in Example 4.35. The vertices of the first facet* 01248 *are chosen as the colors.*

v	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f
$\lambda(v)$	0	0	0	4	0	2	8	8	10	11	19	19	10	19	24	31
$c(v)$	0	1	2	4	4	0	0	1	8	2	1	0	2	4	4	8

Table 4.2. *Facets of the triangulation* C_4^λ.

01248	12358	12458	13589	2378b	23578	24578	24678
2678e	278be	28abe	35789	3789b	4578c	4678c	5789d
578cd	678ce	789bd	78bcd	78bce	7$bcef$	7$bcdf$	

is g-nice for any $s_0 \in (0,1]$ we examine the variety $V(C_4^\lambda)$, describing the values of s for which the center of the projection is met; see Section 4.3.2. The variety $V(C_4^\lambda)$ is the solution set of the ideal $I(C_4^\lambda)$ generated by the five coefficient polynomials

$$\begin{aligned}
F_{C_4^\lambda,0,s} &= 1 + s^2x_1x_3 + s^8x_2x_3 + s^{19}x_1x_2x_4 \ , \\
F_{C_4^\lambda,1,s} &= x_1 + s^8x_1x_2x_3 + s^{19}x_2x_4 \ , \\
F_{C_4^\lambda,2,s} &= x_2 + s^{10}x_3x_4 + s^{11}x_1x_4 \ , \\
F_{C_4^\lambda,3,s} &= x_3 + s^4x_1x_2 + s^{19}x_1x_3x_4 + s^{24}x_2x_3x_4 \ , \text{ and} \\
F_{C_4^\lambda,4,s} &= x_4 + s^{31}x_1x_2x_3x_4 \ .
\end{aligned}$$

For the lexicographical ordering $x_4 > x_3 > x_2 > x_1 > s$ a Gröbner basis of $I(C_4^\lambda)$ reads (computed by `MAGMA` [13])

$$\{\, x_4 + g_4(s),\ x_3 + g_3(s),\ x_2 + g_2(s),\ x_1 + g_1(s),\ g(s) \,\} \ ,$$

for certain polynomials $g, g_1, \dots, g_4 \in \mathbb{Q}[s]$. The polynomial $g(s)$ is displayed in Figure 4.13, and the others are by far too large to be listed. The essential feature of this Gröbner basis is that knowing the (real) roots of the polynomial $g(s)$ of degree 444 allows to compute the values for $x_1, \dots, x_4$ directly.

It turns out that $g(s)$ has exactly two real roots s_1 and s_2 with $s_1 > 1$ and $-1 < s_2 < 0$. Given $g(s)$ this can be verified with any standard computer algebra program by computing all 444 distinct (complex) solutions. Additionally, this was counter-checked via Collins' method of cylindrical algebraic decomposition [15], as implemented in `QEPCAD` [30]. Approximate values for the two real zeroes of $g(s)$ are given in Table 4.3. It follows that C_4^λ is g-nice for any $s_0 \in (0,1]$.

$$s^{444}-2s^{418}-4s^{417}-4s^{415}-2s^{412}-6s^{401}+s^{400}-s^{399}-5s^{398}+5s^{397}+3s^{396}-6s^{394}+
3s^{393}+3s^{392}-4s^{391}+5s^{390}+10s^{389}+10s^{388}+12s^{386}+8s^{385}+5s^{383}+13s^{380}+4s^{379}-
15s^{375}+31s^{374}-8s^{373}+14s^{372}+29s^{371}-32s^{370}+19s^{369}+29s^{368}-28s^{367}+4s^{366}+
45s^{365}-18s^{364}-8s^{363}+42s^{362}-12s^{361}-20s^{360}-6s^{359}-13s^{358}-26s^{357}-12s^{356}+
24s^{355}-17s^{354}-87s^{353}+21s^{352}+5s^{351}-59s^{350}+131s^{349}+36s^{348}-125s^{347}+
142s^{346}-36s^{345}-86s^{344}+46s^{343}-113s^{342}-4s^{341}+20s^{340}-131s^{339}+43s^{338}+
43s^{337}-142s^{336}-55s^{335}-7s^{334}-60s^{333}+124s^{332}+56s^{331}-54s^{330}+23s^{329}+
13s^{328}-202s^{327}+84s^{326}+185s^{325}-292s^{324}+32s^{323}+191s^{322}-189s^{321}-20s^{320}-
77s^{319}-147s^{318}+104s^{317}-188s^{316}-93s^{315}+467s^{314}-50s^{313}-269s^{312}+236s^{311}+
29s^{310}-433s^{309}+349s^{308}+203s^{307}-449s^{306}+74s^{305}+178s^{304}+69s^{303}-165s^{302}-
260s^{301}+625s^{300}-455s^{299}-430s^{298}+1018s^{297}-661s^{296}-493s^{295}+1170s^{294}-
790s^{293}-411s^{292}+1222s^{291}-432s^{290}-201s^{289}+605s^{288}-624s^{287}+243s^{286}+
938s^{285}-352s^{284}-553s^{283}+1328s^{282}-560s^{281}-1343s^{280}+1506s^{279}-1263s^{278}-
826s^{277}+1988s^{276}-1423s^{275}+828s^{274}+2093s^{273}-1779s^{272}+1129s^{271}+686s^{270}-
2280s^{269}+1292s^{268}+938s^{267}-1279s^{266}-48s^{265}+1606s^{264}-595s^{263}-1445s^{262}+
1409s^{261}-876s^{260}-1256s^{259}+1340s^{258}+325s^{257}+1433s^{256}+29s^{255}+571s^{254}+
1933s^{253}-3175s^{252}+181s^{251}+1768s^{250}-3124s^{249}+1204s^{248}+432s^{247}-1215s^{246}+
2103s^{245}-683s^{244}-521s^{243}+786s^{242}-1184s^{241}-355s^{240}+1889s^{239}+1888s^{238}-
2616s^{237}+3311s^{236}+2553s^{235}-6876s^{234}+3628s^{233}+886s^{232}-6562s^{231}+4543s^{230}-
1364s^{229}-2218s^{228}+5371s^{227}-2353s^{226}+292s^{225}+2304s^{224}-2830s^{223}+540s^{222}+
1685s^{221}+641s^{220}-2651s^{219}+3260s^{218}+2777s^{217}-6771s^{216}+3916s^{215}+837s^{214}-
6602s^{213}+4239s^{212}-2085s^{211}-611s^{210}+4945s^{209}-3172s^{208}+3461s^{207}+978s^{206}-
4176s^{205}+3841s^{204}-909s^{203}-2110s^{202}+416s^{201}+789s^{200}+1019s^{199}-2635s^{198}+
1849s^{197}+595s^{196}-3099s^{195}+859s^{194}-1946s^{193}+2463s^{192}+870s^{191}-2980s^{190}+
6933s^{189}-1758s^{188}-4228s^{187}+6606s^{186}-2718s^{185}-4392s^{184}+2695s^{183}-875s^{182}-
1806s^{181}+455s^{180}+1139s^{179}-1102s^{178}-156s^{177}+846s^{176}-2773s^{175}+2989s^{174}+
43s^{173}-3244s^{172}+5688s^{171}-1833s^{170}-3051s^{169}+5638s^{168}-2460s^{167}-3614s^{166}+
2791s^{165}-1135s^{164}-2479s^{163}+796s^{162}+1119s^{161}-1792s^{160}-403s^{159}+1850s^{158}-
1662s^{157}+756s^{156}+588s^{155}-1355s^{154}+2376s^{153}-1103s^{152}-1312s^{151}+3206s^{150}-
1518s^{149}-2313s^{148}+1869s^{147}-343s^{146}-1914s^{145}+575s^{144}+1203s^{143}-1568s^{142}-
506s^{141}+1542s^{140}-753s^{139}-540s^{138}+759s^{137}-254s^{136}+119s^{135}+24s^{134}-
68s^{133}+692s^{132}-463s^{131}-306s^{130}+156s^{129}-209s^{128}-127s^{127}+94s^{126}+215s^{125}-
444s^{124}+15s^{123}+274s^{122}-211s^{121}-339s^{120}+240s^{119}-159s^{118}-132s^{117}+133s^{116}+
127s^{115}+49s^{114}-173s^{113}+197s^{112}-114s^{111}-180s^{110}+203s^{109}+78s^{108}-109s^{107}-
53s^{106}+191s^{105}-80s^{104}-20s^{103}-160s^{102}+s^{101}-191s^{100}-75s^{99}+15s^{98}+61s^{97}-
57s^{96}+43s^{94}+2s^{93}-34s^{92}+43s^{91}+10s^{90}-27s^{89}-2s^{88}+44s^{87}-38s^{86}+70s^{85}-
105s^{84}-16s^{83}-83s^{82}-31s^{81}-25s^{80}+44s^{79}-89s^{78}+28s^{77}-15s^{76}+16s^{75}-23s^{74}+
24s^{73}-11s^{72}-9s^{71}+14s^{70}-s^{69}+2s^{68}+20s^{67}-29s^{66}-8s^{65}-16s^{64}-20s^{63}+6s^{62}+
18s^{61}-42s^{60}+10s^{59}-s^{57}-18s^{56}+16s^{55}-19s^{54}-3s^{53}+3s^{52}+5s^{51}+3s^{49}-5s^{48}+
s^{47}-2s^{46}-3s^{45}+2s^{44}+6s^{43}-9s^{42}+2s^{41}-6s^{38}+4s^{37}-15s^{36}+2s^{33}-6s^{18}-1$$

Figure 4.13. *The polynomial $g(s)$ of the Gröbner basis of $I(C_4^\lambda)$.*

Table 4.3. *Approximate coordinates for the two points in the variety $V(C_4^\lambda)$.*

s	-0.9955941875452	1.0003839818262
x_1	1.3469081499925	-1.1340421741317
x_2	0.7663015145691	-1.8447577233888
x_3	1.1109881050869	-0.4723488390037
x_4	3.4823714929884	-1.1436761629897

While, with current computers, it seems to be out of reach to completely enumerate all triangulations of most polytopes in dimension 5 and beyond, `TOPCOM` can still be used to enumerate large numbers of triangulations. We let `TOPCOM` compute altogether 59,083 different triangulations which originate from randomly chosen placing triangulations by successive flipping. Not a single triangulation among these was foldable. Next we took the triangulation of C_5 with signature 16 that comes from Theorem 4.33 and we inspected 102,184 triangulations by random flipping. This way we found only two more foldable triangulations, one with signature 14 and a second one with signature 16.

For C_6 the situation is more complicated. None of our results so far directly yields any foldable triangulation with a positive signature: All the simplicial product triangulations of C_6 arising from decomposing C_6 as a product of two (or more) cubes of smaller dimensions do not yield a non-trivial lower bound since at least one factor vanishes in the corresponding expressions in Proposition 4.18 and Theorem 4.17. And, as can be expected from the 5-dimensional case, `TOPCOM` did not find a foldable triangulation with a positive signature either. Therefore we took a detour in that we used `TOPCOM` to study triangulations of the product of the 4-simplex and the square. This time we were lucky to find a foldable triangulation with signature 2, which also turned out to be regular.

Proposition 4.36. We have $\sigma(\Delta_4 \times C_2) \geq 2$

In the sequel we denote this rdf-triangulation of $\Delta_4 \times C_2$ with signature 2 by S, and let C_4^λ be the rdf-triangulation of C_4 with signature 2 from Proposition 4.31. Then the product $C_6 = C_4 \times C_2$ inherits a polytopal subdivision into facets of type $\Delta_4 \times C_2$ from C_4^λ. Each of these facets can now be triangulated using S in such a way that one obtains an rdf-triangulation of C_6 with signature 4. Its f-vector equals $(64, 656, 2640, 5298, 5676, 3115, 690)$. This establishes Proposition 4.32.

Problem 4.37. In order to decide whether the triangulation of C_6 from Proposition 4.32 (together with its generating lifting function) is g-nice for some $s_0 \in (0,1]$, it suffices to prove that the real variety generated by

$$\begin{aligned}
F_{C_6,0,s} &= 1 + s^2x_5x_6 + s^8x_1x_6 + s^{55}x_1x_3 + s^{57}x_1x_3x_5x_6 + s^{124}x_2x_3 \\
&\quad + s^{151}x_2x_3x_5x_6 + s^{157}x_1x_2x_3x_6 + s^{197}x_1x_2x_4 + s^{218}x_2x_4x_6 \\
&\quad + s^{224}x_1x_2x_4x_5x_6, \\
F_{C_6,1,s} &= x_6 + s^4x_1x_5 + s^{41}x_2x_5x_6 + s^{55}x_1x_3x_6 + s^{122}x_1x_4x_5x_6 \\
&\quad + s^{128}x_1x_2x_3x_5 + s^{149}x_2x_3x_6 + s^{167}x_3x_4x_5x_6 + s^{189}x_2x_4x_5 \\
&\quad + s^{222}x_1x_2x_4x_6, \\
F_{C_6,2,s} &= x_5 + s^8x_1x_5x_6 + s^{55}x_1x_3x_5 + s^{124}x_2x_3x_5 + s^{157}x_1x_2x_3x_5x_6 \\
&\quad + s^{197}x_1x_2x_4x_5 + s^{218}x_2x_4x_5x_6, \\
F_{C_6,3,s} &= x_1 + s^8x_2x_5 + s^{35}x_3x_6 + s^{55}x_4x_5x_6 + s^{89}x_1x_4x_5 + s^{92}x_1x_2x_6 \\
&\quad + s^{124}x_1x_2x_3 + s^{134}x_3x_4x_5 + s^{185}x_2x_4 + s^{218}x_1x_3x_4x_6 \\
&\quad + s^{311}x_2x_3x_4x_6 + s^{380}x_1x_2x_3x_4x_5x_6, \\
F_{C_6,4,s} &= x_2 + s^{10}x_3x_5 + s^{39}x_4x_6 + s^{67}x_1x_2x_5 + s^{81}x_1x_4 + s^{126}x_3x_4 \\
&\quad + s^{193}x_1x_3x_4x_5 + s^{286}x_2x_3x_4x_5 + s^{364}x_1x_2x_3x_4x_6, \\
F_{C_6,5,s} &= x_3 + s^{12}x_4x_5 + s^{37}x_2x_6 + s^{57}x_1x_2 + s^{118}x_1x_4x_6 + s^{163}x_3x_4x_6 \\
&\quad + s^{183}x_1x_3x_4 + s^{276}x_2x_3x_4 + s^{337}x_1x_2x_3x_4x_5, \text{ and} \\
F_{C_6,6,s} &= x_4 + s^{49}x_3x_5x_6 + s^{106}x_1x_2x_5x_6 + s^{325}x_1x_2x_3x_4 \\
&\quad + s^{325}x_2x_3x_4x_5x_6 + s^{232}x_1x_3x_4x_5x_6
\end{aligned}$$

is empty for all $s \in (0, s_0]$. We leave this as an open problem.

Concluding Remarks

In Theorem 3.12 we stated a combinatorial analog of the Piergallini [54] result, in the sense that the partial unfolding $\widehat{S}$ of the simplicial 4-sphere S constructed is PL-homeomorphic to a given closed oriented PL 4-manifold, and the canonical projection $\widehat{S} \to S$ is a simple 4-fold branched cover of $\mathbb{S}^4$ branched over a PL surface with a finite number of cusp and node singularities.

The Piergallini [54] result can be improved such that the branching set is locally flat, if one allows for a fifth sheet [33]. In general the number of sheets of the branched cover $\widehat{K} \to K$ of a combinatorial d-manifold K is at most $d+1$, since the sheets correspond to the vertices of an arbitrary but fixed facet $\sigma_0 \in K$. Thus it is possible to obtain a 5-fold branched cover via the partial unfolding of a triangulation of $\mathbb{S}^4$. The results of Iori & Piergallini [33] suggest that branched covers obtained via the partial unfolding indeed produce all closed oriented PL 4-manifolds as 5-fold branched covers of $\mathbb{S}^4$ branched over a locally flat PL surface.

In the case of 3-manifolds it remains unclear whether stellar subdivision of faces suffices to construct a triangulation S of $\mathbb{S}^3$ with $\widehat{S} \cong M$ for any given closed oriented 3-manifold M. An affirmative answer to the following problem would allow us to construct S by stellar subdivision of faces only, since the moves $C^{\pm}$ and $M^{\pm}$ from Lemma 3.14 may be realized as the symmetric difference with the boundary of an embedded 2-ball; see Proposition 2.14.

Problem. Starting with the boundary of the 4-simplex, is it possible to construct a triangulation of $\mathbb{S}^3$ with the trefoil knot as odd subcomplex and group of projectivities isomorphic to Σ_3 via stellar subdivision of faces only?

Proposition 1.15 may be exploited further to understand what kind of singularities may appear in the odd subcomplex if we require the unfolding to be a combinatorial manifold. Conversely it might be useful in finding a combinatorial proof of the Edwards result [41, 14], stating that suspending a homology 3-sphere twice yields a 5-sphere.

The simplicial product of g-nice rdf-triangulations (with color consecutive vertex orderings) of lattice polytopes P and Q provides means to construct sparse polynomial systems with non-trivial lower bounds for the number of real roots following Soprunova & Sottile [62]. Further we gained inside in the product structure of the topological degrees of the maps involved in the Soprunova & Sottile bound for the simplicial product, see Remark 4.30.

We gave explicit g-nice rdf-triangulations of the d-cube C_d with signature at least $\lfloor d/2 \rfloor !$ for the case $d \not\equiv 2 \bmod 4$. The associated Wronski systems of d polynomials in d unknowns have at least $\lfloor d/2 \rfloor !$ real roots compared to $d!$ complex roots. The case of $3 \leq d \equiv 2 \bmod 4$ remains open since we are unable to verify g-niceness for our explicit rdf-triangulation of C_6 with positive signature. Constructing any g-nice rdf-triangulation of C_6 with positive signature would yield g-nice rdf-triangulation of C_d with signature in $O(\lfloor d/2 \rfloor !)$ for all dimensions $d \geq 3$.

Finally there is the quest for lower bounds for the number of positive real roots of a sparse polynomial system. Here it might be possible to combine results by Itenberg & Roy [34] with the work of Soprunova & Sottile [62] to construct sparse polynomial systems with non-trivial lower bounds for the number of positive real roots.

Zusammenfassung

Ein simplizialer Komplex der Dimension d ist *faltbar*, wenn es eine nicht degenerierte simpliziale Abbildung in den d-Simplex gibt. Wir verwenden faltbare Triangulierungen zur Konstruktion simplizialer Komplexe mit einem vorgegebenen *ungeraden Teilkomplex*. Letzterer ist der durch alle Kodimension 2-Seiten mit nicht bipartitem Link gegebene Teilkomplex. Der ungerade Teilkomplex kontrolliert das Verhalten der von Izmestiev & Joswig [36] eingeführten *Entfaltungen*. Die Entfaltungen definieren simpliziale Abbildungen, die im topologischen Sinne verzweigte Überlagerungen mit dem ungeraden Teilkomplex als Verzweigungsmenge sind. Somit gilt unser Interesse der Topologie des ungeraden Teilkomplexes, aber gewisse gruppentheoretische Überlegungen sind darüber hinaus von Bedeutung.

Im Fall simplizialer Komplexe mit gewissen zusätzlichen Zusammenhangsbedingungen sind die Eigenschaften „faltbar“ und „leerer ungerader Teilkomplex“ äquivalent. Wir verwenden diese Äquivalenz in der Konstruktion eines simplizialen Komplexes K mit vorgeschriebenem ungeraden Teilkomplex, indem K aus faltbaren Triangulierungen als Bausteinen zusammen gesetzt wird. Dies ermöglicht die Konstruktion aller geschlossenen, orientierbaren PL 4-Mannigfaltigkeiten mit Hilfe der (partiellen) Entfaltung. In diesem Sinne ist unser Resultat eine kombinatorische Version der Arbeit von Piergallini [54].

An anderer Stelle konstruieren wir faltbare Triangulierungen von Produkten ganzzahliger Polytope. Insbesondere werden reguläre und dichte Triangulierungen konstruiert. Eine Triangulierung eines ganzzahligen d-Polytops P ist *regulär*, wenn sie sich als untere konvexe Hülle in den $\mathbb{R}^{d+1}$ heben läßt, und *dicht*, wenn die Eckenmenge der Triangulierung mit den ganzzahligen Punkten in P übereinstimmt. Reguläre, dichte und faltbare Triangulierungen benennen wir der Kürze halber als *rdf-Triangulierungen*.

Eine Triangulierung K eines ganzzahligen Polytops P ist genau dann faltbar, wenn der duale Graph von K bipartit ist; siehe [37]. Soprunova & Sottile [62] konstruieren Polynomsysteme ausgehend von rdf-Triangulierungen von P, und geben eine nicht triviale untere Schranke für die Anzahl der reellen Nullstellen. Das Polytop P ist das gemeinsame Newton Polytop der

Polynome des Systems. Die Anzahl der reellen Nullstellen ist größer als die gewichtete Größendifferenz der zwei Klassen der Bipartition des dualen Graphen der rdf-Triangulierung. Die Größendifferenz ist die *Signatur* der Triangulierung.

Gegeben seien rdf-Triangulierungen zweier ganzzahliger Polytope P und Q. Wir konstruieren das *simpliziale Produkt*, eine rdf-Triangulierung des Polytops $P \times Q$, und berechnen dessen Signatur.

Das simpliziale Produkt wird angewandt, um für $d \not\equiv 2 \bmod 4$ Triangulierungen des d-Würfels mit Signatur mindestens $\lfloor d/2 \rfloor!$ zu erhalten. Die korrespondierenden Polynomsysteme sind Systeme in d Unbekannten, welche mindestens $\lfloor d/2 \rfloor!$ reellen Nullstellen im Gegensatz zu $d!$ komplexen Nullstellen haben.

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Notation Index

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^{\times}$	set of complex numbers without zero
$\mathbb{RP}^d$	d-dimensional real projective space
$\mathbb{CP}^d$	d-dimensional complex projective space
$\mathbb{S}^d$	d-dimensional sphere
$\mathbb{D}^d$	d-dimensional ball
Δ_d	d-simplex
∞	infinity
$\emptyset$	empty set
$[a, b]$	interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
$A \setminus B$	set minus, all elements in A and not in B
$A \dot{\cup} B$	disjoint union of A and B
$f = O(g)$	O-notation
$\mathrm{conv}(V)$	convex hull of a set V
$\mathrm{vol}(P)$	d-dimensional volume of a polytope $P \subset \mathbb{R}^d$
$V(K)$	vertex set of a simplicial complex K
$\dim(f)$	dimension of a face f
$\mathrm{codim}(f)$	co-dimension of a face f
∂X	boundary (complex) of X
$\mathrm{cl}(X)$	topological closure of X
$X \times Y$	product space
$X/{\sim}$	quotient space

Index

Curriculum Vitae

General

2007 *Promotion* at "Fachbereich Mathematik der TU Darmstadt", with distinction.
2006 five month research scholarship at the "Mathematical Sciences Research Institute", Berkeley, California.
2005 begin of a three years research position at TU Darmstadt.

2004 *Mathematik Diplom* at TU Berlin, main subject *discrete geometry*.
2001 begin participation in the development of the software package `polymake`.
1999 commence studies of mathematics at TU Berlin.

1993–1999 dance training and engagements; see *Dance Education and Engagements*.
1992–1993 civilian service at hospital "am Urban", Berlin.

1991 *Abitur* (A-level equivalent) at "Gymnasium an der Hamburger Straße", Bremen.
1978 begin of primary school education at "Grundschule an der Lessingstrasse".
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Publications

Products of Foldable Triangulations (with Michael Joswig), to appear in Advances in Mathematics, available at arXiv math.CO/0508180.
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Publications at `www.eg-models.de`:

- *A counterexample to the Perles conjecture*, EG-Models number 2002.04.001.
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Dance Education and Engagements

2001 participation in *Ring* by Felix Rueckert.
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1997–1999 engagement with "Kibbutz Contemporary Dance Company", Israel. Performances in Israel, USA, France, Germany, and Mexico.

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Scholarships and Distinctions

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