UNIVERSITÁ DEGLI STUDI DI PADOVA

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

CORSO DI LAUREA IN INGEGNERIA DELLE TELECOMUNICAZIONI

Adaptive Equalizers and the DFE

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31 marzo 2011

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1 Introduction

This thesis wants to face the subject matter of adaptive equalizers, especially DFEs, that differ from other equalizers because they are non-linear as they use previous detector decision to eliminate the intersymbol interference on pulses that are currently being demodulated.

Starting from the basis of telecommunication systems we introduce unexpected effects that we could find during a transmission, like noise and intersymbol interference [1]. It is because of this inconveniences that we introduce equalization to remove interference or at least to attenuate it. Infact the common way of dealing with ISI is using equalization. The equalizer raises the communication quality without increasing the transmitted signal power and widening the bandwith, because it compensates the signal amplitude and the delay characteristics of received signal.

The adaptive decision feedback equalizer has become a common and useful tool for high-speed digital communications over time, varying frequency-selective channels. The DFE is a good sub-optimal solution and it can be implemented as a combination of simple FIR filters [2]. The decision feedback equalizer is widely used to remove ISI in bandlimited channels because of its advantages, such as a small mean square error (MSE), and low computational cost [3]. This work will also explore one of the most efficients methods to get the best coefficients of the DFE filters (feedback and feedforward) [7]. This is about the Cholesky factorization, that is a decomposition of the symmetric, positive-definite autorrelation matrix $\mathbf{R}$ into the product of a lower triangular matrix and its conjugate transpose [8] [9].

This method is preferable to the direct method because the analitic inversion of the autocorrelation matrix $\mathbf{R}$ has a compuational complexity $O(N^3)$, that makes the direct method inefficient and complex to implement.

2 Telecommunication Systems

A basic telecommunication system (Figure 1) consists of three primary units that are always present in some form:

1. a transmitter, that takes information and converts it to a signal.

2. a transmission medium, also called physical channel, that carries the signal.

3. a receiver, that takes the signal from the channel and converts it back into usable information.

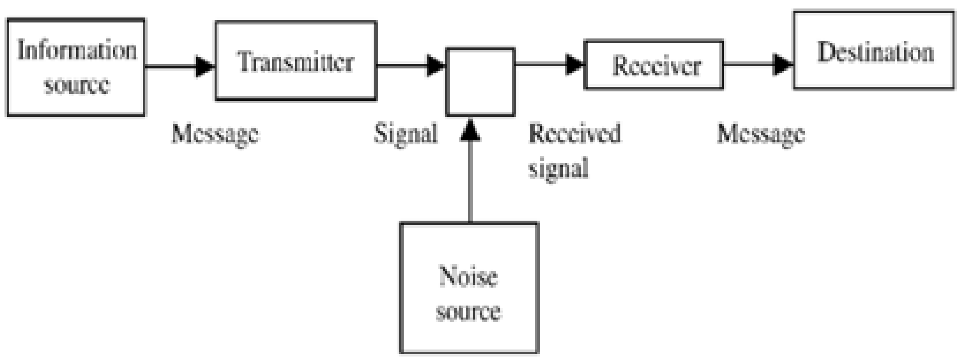

Figure 1: The Block diagram of a telecommunication system.

Sometimes telecommunication systems are "duplex" (two-way) with a single box of electronics working as both a transmitter and a receiver, or a transceiver. For example, a cellular telephone is a transceiver.

Telecommunication over telephone lines is called point-to-point communication because it is between one transmitter and one receiver. Telecommunication through radio broadcasts is called broadcast communication because it is between one powerful transmitter and numerous low-power but sensitive receivers.

A communications network is a collection of transmitters, receivers, and communications channels that send messages to one another.

2.1 Inconveniences in Telecommunication Systems

For almost all types of networks, repeaters may be necessary to amplify or recreate the signal when it is being transmitted over long distances.
This is to combat attenuation that can render the signal indistinguishable from the noise. In communications and electronics, especially in telecommunications, interference is anything which alters, modifies, or disrupts a signal as it travels along a channel between a source and a receiver.
Interference is typically but not always distinguished from noise, for example white thermal noise.

Common examples are:

- Electromagnetic Interference (EMI)
- Co-channel interference (CCI), also known as crosstalk
- Adjacent-channel interference (ACI)
- Intersymbol interference (ISI)
- Inter-carrier interference (ICI), caused by Doppler shift in OFDM modulation

3 ISI Intersymbol Interference

In telecommunication there is another inconveniente, intersymbol interference (ISI), that is a form of distortion of a signal in which one symbol interferes with subsequent symbols. This is an unwanted phenomenon as the previous symbols have similar effect as noise, thus making the communication less reliable.

ISI is usually caused by multipath propagation or the inherent frequency selective response of a channel causing successive symbols to "blur" together.

The presence of ISI in the system introduces errors in the decision device at the receiver output.

Therefore, in the design of the transmitting and receiving filters, the objective is to minimize the effects of ISI, and thereby deliver the digital data to its destination with the smallest error rate possible. Ways to fight intersymbol interference include adaptive equalization and error correcting codes.

3.1 Causes of the Intersymbol Interference

One of the causes of intersymbol interference is what is known as multipath propagation in which a wireless signal from a transmitter reaches the receiver via many different paths. The causes of this include reflection (for instance, the signal may bounce off buildings), refraction (such as through the foliage of a tree) and atmospheric effects such as atmospheric duct and ionospheric reflection. Since all of these paths are different lengths - plus some of these effects will also slow the signal down - this results in the different versions of the signal arriving at different times. This delay means that part or all of a given symbol will be spread into the subsequent symbols, thereby interfering with the correct detection of those symbols. Additionally, the various paths often distort the amplitude and/or phase of the signal thereby causing further interference with the received signal.

Another cause of intersymbol interference is the transmission of a signal through a bandlimited channel, i.e., one where the frequency response is zero above a certain frequency (the cutoff frequency). Passing a signal through such a channel results in the removal of frequency components above this cutoff frequency; in addition, the amplitude of the frequency components below the cutoff frequency may also be attenuated by the channel. This filtering of the transmitted signal affects the shape of the pulse that arrives at the receiver. The effects of filtering a rectangular pulse, not only change the shape of the pulse within the first symbol period, but it is also spread out over the subsequent symbol periods. When a message is transmitted through such a channel, the spread pulse of each individual symbol will interfere with following symbols.

As opposed to multipath propagation, bandlimited channels are present in both wired and wireless communications. The limitation is often imposed by the desire to operate multiple independent signals through the same area/cable; due to this, each system is typically allocated a piece of the total bandwidth available.
The bandlimiting can also be due to the physical properties of the medium - for instance, the cable being used in a wired system may have a cutoff frequency above which practically none of the transmitted signal will propagate.
Communication systems that transmit data over bandlimited channels usually implement pulse shaping to avoid interference caused by the bandwidth limitation. If the channel frequency response is flat and the shaping filter has a finite bandwidh, it is possible to communicate with no ISI at all. Often the channel response is not known beforehand, and an adaptive equalizer is used to compensate the frequency response.
In communications, in association with the concept of interference, we talk about the Nyquist ISI criterion, that describes the conditions which, when satisfied by a communication channel, result in no intersymbol interference.
It provides a method for constructing band-limited functions to overcome the effects of intersymbol interference. When consecutive symbols are transmitted over a channel by a linear modulation (such as ASK, QAM, etc.), the impulse response (or equivalently the frequency response) of the channel causes a transmitted symbol to be spread in the time domain. This causes intersymbol interference because the previously transmitted symbols affect the currently received symbol, thus reducing

tolerance for noise. The Nyquist theorem relates this time-domain condition to an equivalent frequency-domain condition.

3.2 Methods to overcome Intersymbol Interference

There are several techniques in Telecommunication and data storage that try to work around the problem of intersymbol interference.

- Design systems such that the impulse response is short enough that very little energy from one symbol smears into the next symbol.
- Separate symbols in time with guard periods.
- Apply a sequence detector at the receiver, that attempts to estimate the sequence of transmitted symbols using the Viterbi algorithm.
- Apply an equalizer at the receiver, that, broadly speaking, attempts to undo the effect of the channel by applying an inverse filter.

  There are three types of equalizers that are commonly used:

  - Maximum likelihood (ML) sequence detection which is the optimal detecter but in some cases impractical for implementation due its complexity.
  - Linear equalizers such as the taped-delay-line equalizer (also known as linear transversal equalizer (LTE) which is widely used and simple to implemente.
  - Non-linear equalizers such as decision feedback equalizer.

4 Adaptive Equalizers and the DFE Equalizer

4.1 Equalizers

The equalizer in its basic form is a filter or generally, a set of filters, that aims to remove the undesirable effects of the transmission system, including the channel. In digital communications system, the frequently faced problem is the intersymbol interference.

The equalizer generally models the effect of inverse operation of the transmission system. But, while doing this, an undesirable result may occur. This result happens at the points where the equalizer amplifies the signal to remove ISI. This amplification causes the amplification of the noise as well. So, equalizer design and structure gain importance in order to remove ISI while minimizing the noise. The equalizer can be modeled as a system which has a transfer function. This transfer function will invert the bad effect of transmission system which introduces ISI and noise. Also, some equalizers correct the timing and phase errors to some extent. The simplest equalizer is the linear equalizer which is, generally, implemented with a finite impulse response (FIR) filter. The reason for this filter is its low-complexity and cheap production. But, since its performance is not enough for higher expectations, generally, more sophisticated equalizer schemes were searched. These searches resulted in a wide variety of equalizer types.

In the design of equalizers there exist different types of design criteria.

The two most frequently encountered criteria with their efficiency are illustrated in the sequel. Some equalizers are designed to minimize mean square error (MSE) at

the slicer input with the constraint of zero ISI. These are called zero-forcing (ZF) equalizers. Some equalizers are designed to minimize the MSE at the input of the slicer by reducing the signal slightly at the slicer input. This reduction of signal results in reduction in MSE, so overall MSE is smaller than that of the ZF equalizer. These equalizers are called MSE equalizers. The MSE equalizer is generally preferred against ZF equalizer because of less noise enhancement.
The linear equalizer is cheap in implementation but its noise performance is not very good. So, in the literature, some non linear equalizers types are searched.

The most popular of these nonlinear equalizers is the decision feedback equalizer (DFE), based on the knowledge of the input data. Known the impulsive response of the channel and given a sufficient number of symbols correctly pointed out, the DFE produced the intersymbol interference of the symbol to detect and remove the sample associated. The DFE, however, is sensitive to symbols that are not properly disclosed as the erroneous detection of a symbol produces a wrong calculation of the ISI on next symbols with the possibility of incurring a potential catastrophic error propagation effect on the system performance. In practice, however, this fact has never happend and the incorrect estimate of a symbol affects only few next symbols reducing the global performance, in terms of signal to noise ratio Γ, of 2, 3 dB.
The most popular algorithm from the aspect of performance and complexity is the Least Mean Squares (LMS) algorithm. It has a good performance and low complexity. It is globally convergent if the desired values are given correctly. The handicap of LMS algorithm for equalizer if the desired symbols are not correct, it does not converge. So, the equalizer using LMS algorithm requires a priori known symbols in

case the decisions of the equalizer are wrong.

A better algorithm is the Recursive Least Squares (RLS) algorithm which has better convergence characteristics than the LMS algorithm. But, it has higher computational complexity than the LMS algorithm. The general RLS algorithms complexity grows with N^2 where N is the number of equalizer coefficients. There are also RLS algorithms that have computational complexities that grow linearly with the number of equalizer coefficients. These algorithms are called fast RLS algorithms.

The usual adaptive equalizers need the knowledge of the data sequence which was transmitted. When this knowledge is not present, the equalizer may not converge. Otherwise the solution is the use of blind equalizers. Blind equalizers use different adaptive algorithms that exploit higher order statistical characteristics or cyclostationary statistics of the received signal. For the blind equalizers, the most popular algorithm for its good performance is the constant modulus algorithm (CMA).

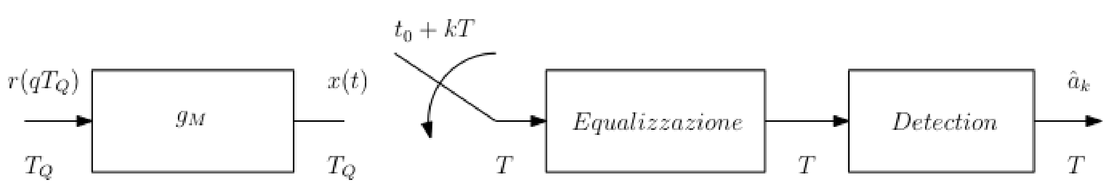

Figure 2: Receiver scheme

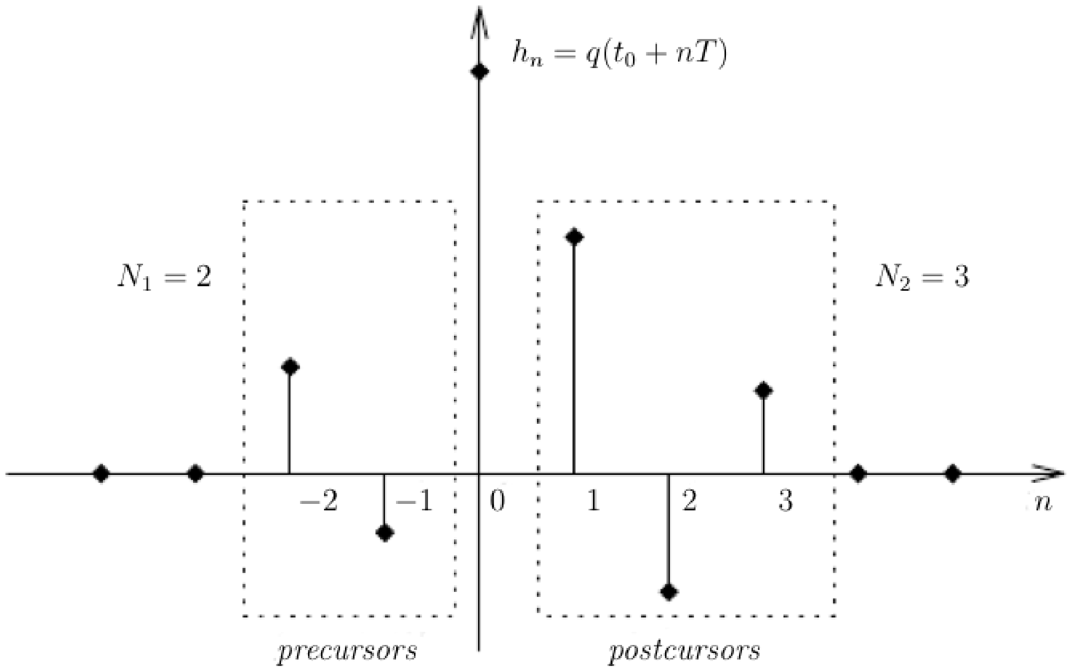

Figure 3: Impulsive response of the system at the input of the feedforward filter FF.

4.2 The DFE structure

The general structure of the DFE in composed by two filters: *feedforward* filter (FF) and the *feedback* filter (FB). Consider the output sampled signal of the analog receive filter given by

$$s_k = s_R\left(t_0 + kT\right) = \sum_{i=-\infty}^{\infty} a_i h_{k-i} \tag{1}$$

where the impulsive sequence $\{h_n\}$ is the equivalent impulse response, that is ob-

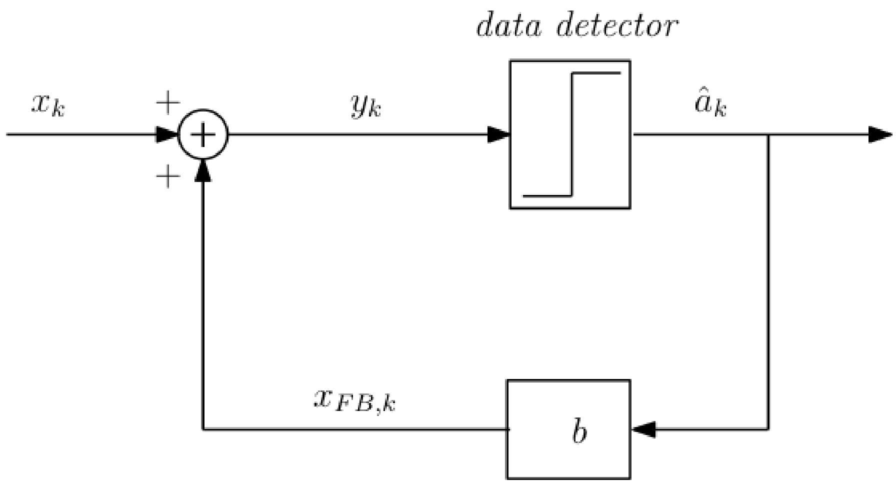

Figure 4: Simplified scheme of the DFE receiver (only feedback filter).

tained at the decision point. The global signal is:

$$x_k = s_k + \tilde{w}_k \tag{2}$$

where $\tilde{w}_k$ is the additive noise, that is supposed to be Gaussain and white.

As illustrated in Figure 3, assume that $\{h_n\}$ has a finite duration with support $\{-N_1, -N_1 + 1, \ldots, N_2 - 1, N_2\}$, we define postcursors the samples on the right of the origin and precursors the ones on the left of the origin.

(2) can be written also as:

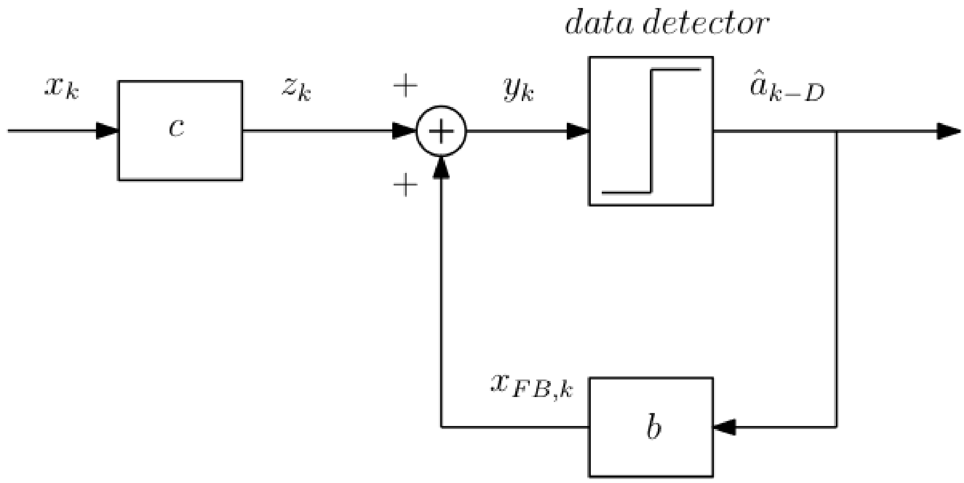

Figure 5: DFE.

$$x_k = (h_{-N_1} a_{k+N_1} + \ldots + h_{-1} a_{k+1}) + h_0 a_k + (h_1 a_{k-1} + \ldots + h_{N_2} a_{k-N_2}) + \tilde{w}_k \quad (3)$$

Over the present symbol a_k, that we want to detect depending on x_k, in (3) there are two terms in brackets: one depends only on past symbols $a_{k-1}, \ldots, a_{k-N_2}$ and the other one depends only on future symbols $a_{k+1}, \ldots, a_{k+N_1}$.

If past symbols and the impulse responce were perfectly known, a scheme of elimination of the ISI limited only to postcursors will be able to be used.

Substituiting past symbols with their estimate $\hat{a}_{k-1}, \ldots, \hat{a}_{k-N_2}$, a scheme to cancel

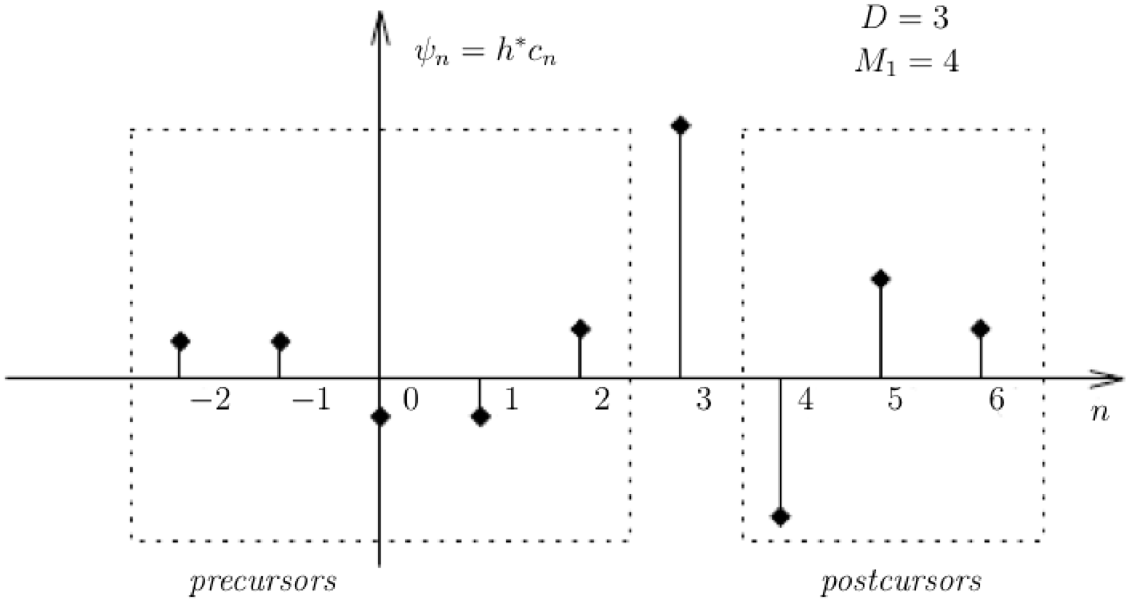

Figure 6: Impulsive response of the system at the output of the feedforward filter FF.

a part of the ISI is reported in Figure 4 where, generically, the feedback filter (FB) has impulsive responce $\{b_n\}, n = 1, \ldots, M_2$, and output:

$$x_{FB,k} = b_1 \hat{a}_{k-1} + \ldots + b_{M_2} \hat{a}_{k-N_2} \tag{4}$$

If $M_2 \geq N_2$, $b_n = -h_n$, if $n = 1, \ldots, N_2$, $b_n = 0$, if $n = N_2 + 1, \ldots, M_2$ and $\hat{a}_{k-i} = a_{k-i}$, if $i = 1, \ldots, N_2$, then the DFE scheme actually eliminates the ISI caused by the postcursors. Note that this is made without alterating the noise $\tilde{w}_k$ present in x_k.

The general structure of a DFE is shown in Figure 5 , where we can individuate two filters and a delay in the decision:

1. *Feedforward* filter (FF), c, made by M_1 coefficients,

$$z_k = x_{FF,k} = \sum_{i=0}^{M_1-1} c_i x_{k-i} \tag{5}$$

2. *Feedback* filter (FB), b, made by M_2 coefficients,

$$x_{FB,k} = \sum_{i=1}^{M_2} b_i x_{k-i-D} \tag{6}$$

Moreover:

$$y_k = x_{FF,k} + x_{FB,k} \tag{7}$$

The aim of the feedforward FF is to the make the transfer function of the global system having a minimum phase. So the global impulsive responce $\{\psi_n = h * \ c_n\}$ has really small precursors, as shown in Figure 6, so almost all ISI is cancelled by the feedback filter FB. Moreover, the feedforward filter may work with fractions of the time T, while the feedback filter works only at T.

4.3 DFE with a finite number of coefficients: direct method

Known channel, in terms of the impulsive response $\{h_i\}$ and of the noise autocorrelation $r_{\hat{w}}(n)$, for a functional J, that indicates the MSE,

$$J = E\left[\left|a_{k-D} - y_k\right|^2\right] \tag{8}$$

the Wiener theory [1] lets us determinate the DFE filter's coefficients in case of $\hat{a}_k = a_k$ and in the case of i.i.d. symbols and statisticaly indipendent from the noise. For a generic sequence $\{h_i\}$:

1. mutual correlation between a_k and x_k:

$$r_{ax}(n) = {\sigma_a}^2 h^*_{-n} \tag{9}$$

2. autocorrelation of x_k

$$r_x(n) = {\sigma_a}^2 r_h(n) + r_{\tilde{w}}(n) \tag{10}$$

where

$$r_h(n) = \sum_{j=-N_1}^{N_2} h_j h^*_{j-n} \qquad r_{\tilde{w}}(n) = N_0 r_{gM}(nT) \tag{11}$$

Set

$$\psi_p = h * \ c_p = \sum_{l=0}^{M_1-1} c_l h_{p-l} \tag{12}$$

and remembering that

$$y_k = \sum_{i=0}^{M_1-1} c_i x_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j} \tag{13}$$

using (1) and (2) we obtain

$$y_k = \sum_{i=0}^{M_1-1} \psi_p a_{k-p} + \sum_{i=0}^{M_1-1} c_i \tilde{w}_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j} \tag{14}$$

Under (14), the best choice for the feedback filter's coefficients is given by

$$b_i = -\psi_{i+D}, \qquad i = 1, \ldots, M_2 \tag{15}$$

The substitution of (15) into (13) gives

$$y_k = \sum_{i=0}^{M_1-1} c_i \left(x_{k-i} - \sum_{j=1}^{M_2} h_{j+D-i} a_{k-j-D} \right) \tag{16}$$

The solution of Wiener-Hopf [1] requires the following correlations:

$$[\mathbf{p}]_p = E\left[a_{k-D}\left(x_{k-p} - \sum_{j=1}^{M_2} h_{j+D-i}a_{k-j-D}\right)^*\right] = \sigma_a^2 h_{D-p}^* \tag{17}$$

when $p = 0, 1, \ldots, M_1 - 1$

$$[\mathbf{R}]_{p,q} = E\left[\left(x_{k-q} - \sum_{j_1=1}^{M_2} h_{j_1+D-q}a_{k-D-j_1}\right)\left(x_{k-p} - \sum_{j_2=1}^{M_2} h_{j_2+D-p}a_{k-D-J_2}\right)^*\right]$$

$$= \sigma_a^2\left(\sum_{j=-N_1}^{N_2} h_j h_{j-(p-q)}^* - \sum_{j=1}^{M_2} h_{j+D-q}h_{j+D-p}^*\right) + r_{\tilde{w}}(p-q) \tag{18}$$

when $p, q = 0, 1, \ldots, M_1 - 1$.

For the determination of the feedforward filter $c = [c_0, c_1, \ldots, c_{M-1}]^T$ we have

$$\mathbf{c}_{opt} = \mathbf{R}^{-1}\mathbf{p} \tag{19}$$

while from (15) the feedback filter is given by

$$b_i = -\sum_{l=0}^{M_1-1} c_{opt,l}h_{i+D-l} \qquad i = 1, 2, \ldots, M_2 \tag{20}$$

Finally, using (17) we arrive at

$$J_{min} = \sigma_a^2 - \sum_{l=0}^{M_1-1} c_{opt,l}\,[\mathbf{p}]_l^* = \sigma_a^* \left(1 - \sum_{l=0}^{M_1-1} c_{opt,l}h_{D-l}\right) \tag{21}$$

The design of the DFE with the direct method exposed in this paragraph needs the knowledgement, or at least the estimate, of the mutual correlation vector $\mathbf{p}$ between the desired output and the input of the equalizator and of the autocorrelation matrix $\mathbf{R}$, that needs to be inverted to reach the best solution for the feedforward filter's coefficients' vector $\mathbf{c}_{opt}$.

The matrix $\mathbf{R}$ is only Hermitian and so its analitic inversion has a computational complexity $O\left(N^3\right)$ that makes the direct method inefficient and complex to implement.

5 Equalization Algorithms

We have just said that the best solution by the Winer-Hopf theory [3] for the design of a DFE equalizer with the direct method requires the computation of the inverse matrix of the autocorrelation matrix $\mathbf{R}$. This inversion requires a high computational complexity and it is hardly implementable in practice. So it is indispensable the development of alternative procedures with a lower computational complexity. It's not possible to use iterative algorithm like, for example, the LMS algorithm (Least Mean Square), because of the poor performance, in terms of low convergence to the optimal solution. Here we will consider an approach based on the Cholesky factorization of the autocorrelation matrix $\mathbf{R}$ (19), with a much reduced computational complexity to determine $\mathbf{R}^{-1}$.

5.1 Determination of the best feedback and feedforward filters

Consider the the block diagram of the DFE in Figure 7.

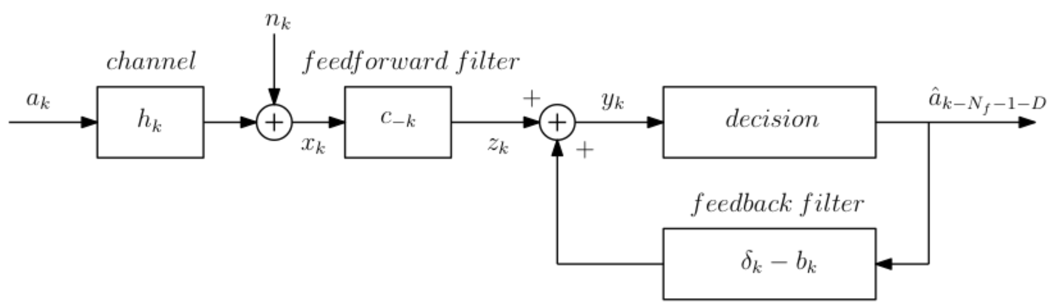

Figure 7: scheme of the receiver MMSE-DFE.

$\{a_k\}$ is the sequence of the input data of the channel, $\{h_k\}$ is the equivalent impulsive response of the channel and $\{n_k\}$ is the additive noise, that is supposed to be white and Gaussian. The feedforward and the feedback filter are indicated respectively as $\mathbf{c}_{-k}$ and $\mathbf{b}_k$. If T is the symbol period, the channel output is sampled at $\frac{T}{Q}$, with Q a positive integer. Grouping consecutive Q samples at the output of the channel into $\mathbf{x}_n$, we can write the input of the DFE as

$$\mathbf{x}_k = \sum_{m=0}^{\nu} \mathbf{h}_m a_{k-m} + \mathbf{n}_k \tag{22}$$

where ν is the memory channel and

$$\mathbf{x}_k = \begin{bmatrix} x\left(\left(kQ+Q-1\right)\frac{T}{Q}\right) \\ \vdots \\ x\left(kT\right) \end{bmatrix} \mathbf{h}_m = \begin{bmatrix} h\left(\left(mQ+Q-1\right)\frac{T}{Q}\right) \\ \vdots \\ h\left(mT\right) \end{bmatrix} \mathbf{n}_k = \begin{bmatrix} n\left(\left(kQ+Q-1\right)\frac{T}{Q}\right) \\ \vdots \\ n\left(kT\right) \end{bmatrix} \tag{23}$$

If we consider N_f input vectors, each one having Q samples, (22) can be written as

$$\begin{bmatrix} \mathbf{x}_{k+N_f-1} \\ \mathbf{x}_{k+N_f-2} \\ \vdots \\ \mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{h}_0 & \mathbf{h}_1 & \dots & \mathbf{h}_\nu & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_0 & \mathbf{h}_1 & \dots & \mathbf{h}_\nu & \mathbf{0} & \dots \\ \vdots & & & & & & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{h}_0 & \mathbf{h}_1 & \dots & \mathbf{h}_\nu \end{bmatrix} \begin{bmatrix} a_{k+N_f-1} \\ a_{k+N_f-2} \\ \vdots \\ a_{k-\nu} \end{bmatrix} \begin{bmatrix} \mathbf{n}_{k+N_f-1} \\ \mathbf{n}_{k+N_f-2} \\ \vdots \\ \mathbf{n}_k \end{bmatrix} \tag{24}$$

expressed in compact form as

$$\mathbf{x}_{k+N_f-1:k} = \mathbf{H}\mathbf{a}_{k+N_f-1:k-\nu} + \mathbf{n}_{k+N_f-1:k} \tag{25}$$

For the next analysis it is supposed that:

- the channel is linear and time invariant with finite memory ν,

$$h(D) = h_0 + h_1 D + h_2 D^2 + \ldots + h_\nu D^\nu \tag{26}$$

- $\{a_k\}$ is a sequence of i.i.d. symbols with variance σ_a^2;

- the additive noise is white and Gaussian with power spectral density N_0;

- the feedforward filter $\mathbf{c}$ is linear and anti-causal, with N_f coefficients,

$$\mathbf{c} = \left[c_{-\left(N_f-1\right)}, c_{-\left(N_f-2\right)}, \ldots, c_0\right]^T \tag{27}$$

- the feedback filter $\mathbf{b}$ is linear and causal, with N_b coefficients,

$$\mathbf{b} = \left[-b_1, -b_2, \ldots, -b_{N_b}\right]^T \tag{28}$$

  In order to have better performances and to simplify the analysis we assume $N_b = \nu$;

- decoded symbols preceeding the actual symbol have been correctly revealed, i.e.

  $\hat{a}_{k-D} = a_{k-D}$, where D is the decision delay due to the presence of the feedforward filter.

5.2 Analysis

The error sequence is given by ($*$ denotes transpose conjugate):

$$e_k = a_k - \left(\sum_{i=0}^{N_f-1} \mathbf{c}_{-i}^* \ \mathbf{x}_{k+i} + \sum_{j=1}^{\nu} b_j^* \ a_{k-j} \right) \tag{29}$$

$$= \mathbf{b}^* \ \mathbf{a}_{k:k-\nu} - \mathbf{c}^* \ \mathbf{x}_{k+N_f-1:k} \tag{30}$$

The matrix of the mutual correlations between the input and the output of the channel is:

$$\mathbf{R}_{ax} = E\left[\mathbf{a}_{k:k-\nu}\mathbf{x}_{k+N_f-1:k}^* \ \right] = \sigma_a^2 \left[\mathbf{0}_{(\nu+1)\times\left(N_f-1\right)}\mathbf{I}_{\nu+1}\right]\mathbf{H}^* \tag{31}$$

where $\mathbf{I}_{\nu+1}$ is the identity matrix of order $\nu + 1$. The correlatic matrix of the input x is given by:

$$\mathbf{R}_{xx} = E\left[\mathbf{x}_{k+N_f-1:k}\mathbf{x}_{k+N_f-1:k}^* \ \right] = \sigma_a^2 \mathbf{H}\mathbf{H}^* \ + QN_0\mathbf{I}_{QN_f} \tag{32}$$

In order to minimize MSE we need to apply the principle of orthogonality, which says that the best sequence of error is incorrelated with the observed data,

$$E\left[e_k\mathbf{x}_{k+N_f-1:k}^* \ \right] = \mathbf{0} \tag{33}$$

from which we obtain

$$\mathbf{b}^* \ \mathbf{R}_{ax} = \mathbf{c}^* \ \mathbf{R}_{xx} \tag{34}$$

From (30) and (34), the MSE is given by:

$$J = E\left[|e_k|^2\right] = \mathbf{b}^* \ \mathbf{R}^{\perp}\mathbf{b} \tag{35}$$

where $\mathbf{R}^{\perp}$ indicates the correlation matrix of the estimated error's vector and it is given by

$$\mathbf{R}^{\perp} = \mathbf{R}_{aa} - \mathbf{R}_{ax}\mathbf{R}_{xx}^{-1}\mathbf{R}_{xa} = \sigma_a^2\left(\mathbf{I}_{\nu+1} - \begin{bmatrix} \mathbf{0} & \mathbf{I}_{\nu+1} \end{bmatrix} \mathbf{H}^* \ \left(\mathbf{H}\mathbf{H}^* \ + \tfrac{1}{SNR}\mathbf{I}_{QN_f}\right)^{-1} \mathbf{H} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{\nu+1} \end{bmatrix}\right)$$

$$= \sigma_a^* \ \begin{bmatrix} \mathbf{0} & \mathbf{I}_{\nu+1} \end{bmatrix} \left(\mathbf{I}_{N_f+1} - \mathbf{H}^* \ \left(\mathbf{H}\mathbf{H}^* \ + \frac{1}{SNR}\mathbf{I}_{QN_f}\right)^{-1}\right) \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{\nu+1} \end{bmatrix} \tag{36}$$

and $SNR' = \frac{SNR}{Q} = \frac{\sigma_a^2}{\sigma_n^2}Q$ is the signal to noise ratio. Using the inverse matrix lemma

$$\mathbf{H}^* \ \left(\mathbf{H}\mathbf{H}^* \ + \frac{1}{SNR}\mathbf{I}_{QN_f}\right)^{-1} \mathbf{H} = \left(\mathbf{H}^* \ \mathbf{H} + \frac{1}{SNR}\mathbf{I}_{N_f+\nu}\right)^{-1} \mathbf{H}^* \ \mathbf{H} \tag{37}$$

(36) gives

$$\mathbf{R}^{\perp} = QN_0 \begin{bmatrix} \mathbf{0} & \mathbf{I}_{\nu+1} \end{bmatrix} \left(\mathbf{H}^* \ \mathbf{H} + \frac{1}{SNR'} I_{N_f+\nu} \right)^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{\nu+1} \end{bmatrix} \tag{38}$$

We can now introduce the Cholesky factorization

$$\mathbf{R} = \mathbf{H}^* \ \mathbf{H} + \frac{1}{SNR'} \mathbf{I}_{N_f+\nu} \tag{39}$$

$$\mathbf{R} = \mathbf{LDL}^* \tag{40}$$

where $\mathbf{L}$ is the inferior triangular matrix, that has all 1 in its main diagonal, and $\mathbf{D}$ is the diagonal matrix. They are given as

$$\mathbf{L} = \begin{bmatrix} \mathbf{I}_1 & \dots & \mathbf{I}_{N_f} + \nu - 1 \end{bmatrix}; \qquad \mathbf{L}^{-1} = \begin{bmatrix} \mathbf{u}_0^* \\ \vdots \\ \mathbf{u}_{N_f+\nu-1}^* \end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix} d_0 & & \\ & \ddots & \\ & & d_{N_f+\nu-1} \end{bmatrix} \tag{41}$$

Substituting (38) and (39) in (35) we have

$$MSE = \mathbf{b}^* \left(\mathbf{L}^{-1}\right)^* \mathbf{D}^{-1}\mathbf{L}^{-1}\mathbf{b}$$

$$= QN_0 \begin{bmatrix} \mathbf{0} & \mathbf{b}^* \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 & \dots & \mathbf{u}_{N_f+\nu-1} \end{bmatrix} \begin{bmatrix} d_0^{-1} & & \\ & \ddots & \\ & & d_{N_f+\nu-1}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0^* \\ \vdots \\ \mathbf{u}_{N_f+\nu-1}^* \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} \tag{42}$$

from the analysis of the MSE with the new settings we have

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{b}_{opt} \end{bmatrix} = \mathbf{I}_{N_f-1} \tag{43}$$

that is the best solution for the feedback filter's coefficients and it is given by the N_f-th coloumn of the matrix $\mathbf{L}$. Best MSE is given by

$$J_{opt} = \frac{QN_0}{d_{N_f} - 1} \tag{44}$$

and it depends only on the number of samples Q, on the power spectral density of the noise N_0 and on the $(N_f - 1)$-nth coefficients of the diagonal matrix $\mathbf{D}$.

In order to obtain the best vector $\mathbf{c}$ of the feedforward filter's coefficients we use equation (34)

$$\mathbf{c}_{opt}^* = \mathbf{b}_{opt}^* \mathbf{R}_{ax}\mathbf{R}_{xx}^{-1} \tag{45}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{b}^*_{opt} \end{bmatrix} \mathbf{H}^* \left(\mathbf{H}\mathbf{H}^* + \frac{1}{SNR}\mathbf{I}_{QN_f} \right)^{-1} \tag{46}$$

This vector of the feedforward filter's coefficients is obtainable by the BSM algorithm given in Paragraph 5.5.

5.3 Structured matrix

We have just seen that by the Cholesky factorization it is possible to obtain the coefficients of the filters of the DFE starting with the matrix $\mathbf{R} = \mathbf{LDL}^*$. The efficiency of this factorization is given by the peculiarity of $\mathbf{R}$ that is a structured matrix [8] [9].

Definition 5.1 *A semi-infinite Hermitian matrix* $\boldsymbol{R} = [R_{ij}, 0 \leq i, j < \infty]$ *is called structured matrix if its generating function*

$$R(D, w) = \begin{bmatrix} 1 & D & D^2 & \dots \end{bmatrix} \mathbf{R} \begin{bmatrix} 1 & w & w^2 & \dots \end{bmatrix}^* = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} R_{ij} D^i (w^*)^j \quad (47)$$

can be written as

$$R(D, w) = \frac{\mathbf{G}(D) \mathbf{J} \mathbf{G}^*(w)}{d(D, w)} \quad (48)$$

where

- $\mathbf{J}$, diagonal matrix of dimension $p + q$, has a number p of 1 and a number q of -1 on the main diagonal;

- $d(D, w)$ is a polynomial of the form $\alpha(D) \alpha^*(w) - \beta(D) \beta^*(w)$;

- $\mathbf{G}(D)$ is a line vector composed of ρ elements; ρ is a displacement rank of $\mathbf{R}$ and $\mathbf{G}(D)$ is a generator for $\mathbf{R}$.

The determination of the generating matrix of the structured matrix to be factorized $\mathbf{R}$ in (39) gives

$$R(D,w) = \frac{\mathbf{G}(D)\,\mathbf{J}\mathbf{G}^{*}\;(w)}{1 - Dw^{*}} \tag{49}$$

in which the vector $\mathbf{G}(D)$ is given by

$$\mathbf{G}(D) = \left[\frac{1}{\sqrt{SNR}}, \quad h^{*}\;(D^{*}\;), \quad D^{N_f}\tilde{h}^{*}\;(D^{*}\;)\right] \tag{50}$$

which is composed of three elements and it allows to attribute to the matrix $\mathbf{R}$ a displacemement rank of 3. This low value of the displacement rank simpifies the determination of the coefficients of the feedback filter.

5.4 Feedback filter calculation

The iterative algorithm basically uses the generating vector $\mathbf{G}(D)$ of the structurated matrix $\mathbf{R}$ and at the i-th recursion calculates the i-th elements d_i of the diagonal matrix $\mathbf{D}$ and the i-th column l_i of the matrix $\mathbf{L}$, of the Cholesky factorization $\mathbf{R} = \mathbf{LDL}^*$.

Definied the polynomial

$$l_i = (D) = \begin{bmatrix} 1 & D & D^2 & \dots \end{bmatrix} \mathbf{l}_i \qquad i = 0, 1, \dots \tag{51}$$

associated with the columns $\mathbf{l}_i$ of the inferior triangular matrix $\mathbf{L}$, the algorithm requires N_f iterations to calculate the N_f-th coloumn of $\mathbf{L}$, $l_{N_f-1}(D)$, whose elements coincide to the coefficients of the feedback filter that we want to realize.

General condition:

$$\mathbf{G}_0(D) = \mathbf{G}(D) = \left[\frac{1}{\sqrt{SNR}}, \quad h^*\ (D^*\ ) \right] \tag{52}$$

Iterations:

For $i = 0, 1, \dots, N_f - 1$

$$d_i = |G_i(0)|^2 \tag{53}$$

$$l_i(D) = D^i \mathbf{G}_i(D)\, \mathbf{G}_i^*\ (0)\, d_i^{-1} \tag{54}$$

$$\begin{bmatrix} \alpha_i & \beta_i \end{bmatrix} = d_i^{-\frac{1}{2}} \mathbf{G}_i(0) \tag{55}$$

$$\mathbf{G}_{i+1}(D) = \frac{1}{D} \mathbf{G}_i(D) \begin{bmatrix} \alpha_i D & -\beta_i \\ \beta_i^*\ D & \alpha_i \end{bmatrix} \tag{56}$$

Output:

$$b_{opt}(D) = l_{N_f-1}(D) \tag{57}$$

Elements, α_i real and positive for $i = 0, 1, \ldots, N_f - 1$, β_i complex, are matched by the relation $|\alpha_i|^2 + |\beta_i|^2 = 1$ [9].

5.5 Feedforward filter calculation

Once obtained the vector of the feedback filter's coefficients, the feedforward filter can be calculated from (44) using the BSM method (back substitution method). We define the vector

$$\mathbf{v}^*_{N_f-1} = [v^*\ (0) \quad v^*\ (1) \quad \dots \quad 1] \tag{58}$$

ad the submatrix N_f x N_f of the triangolar matrix $\mathbf{L}$, previously calculated with the Cholesky factorization (41)

$$L_{N_f \times N_f} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \mathbf{L}(1,0) & 1 & 0 & \dots \\ \vdots & & \ddots & \vdots \\ \mathbf{L}(N_f-1, \quad 0) & \mathbf{L}(N_f-1, \quad 1) & \dots & 1 \end{bmatrix}$$

The elements of the vector $\mathbf{v}^*_{N_f-1}$ are calculated from:

$$v^*\ (k) = -\sum_{j=k+1}^{N_f-1} \mathbf{L}(j,k)\, v^*\ (j)\,, \qquad k = N_f-2, N_f-3; \dots, 0 \tag{59}$$

If $\mathbf{L}$ has the value 1 in all the elements of the main diagonal, then $\mathbf{v}^*_{N_f-1} = 1$. After the determination of $\mathbf{v}^*_{N_f-1}$, the coefficients c^*_i of the feedforward filter $\mathbf{c}^*_{opt}$ are given by:

$$c_i^* = d_{N_f-1}^{-1} \sum_{k=0}^{min\left(\nu, N_f-1-i\right)} v^* \; (k+i)\, h_k^* \qquad i = 0, 1, \ldots, N_f - 1 \tag{60}$$

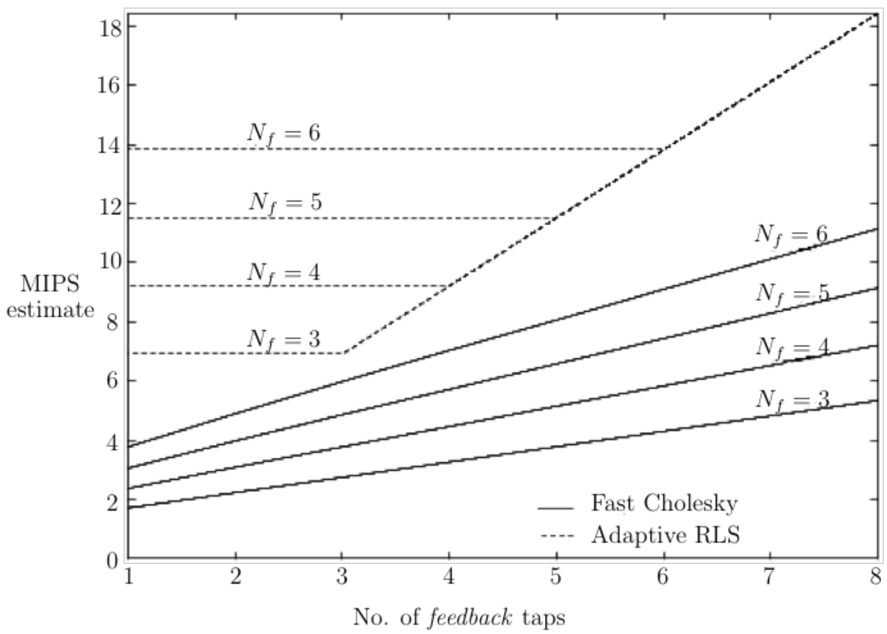

Figure 8: Comparison of the computational complexity to filter design by the Cholesky factorization and adaptive RLS.

5.5.1 Analysis of computational complexities

The algorithm that uses the Cholesky factorization for the determination of the feedback filter requires $(6N_f\nu + 12N_f - 4\nu - 8)$ complex multiplications and $(3N_f\nu + 4N_f - 2\nu - 3)$ complex additions; this yields a $O\left(N^2\right)$ computational complexity. The algorithm that uses the BSM method for the computation of the feedforward filter also presents a $O\left(N^2\right)$ computational complexity. Therefore the overall complexity for the DFE design is $O\left(N^2\right)$. In Figure 8 there is a comparison, in terms of computational complexity expressed in MIPS (million instructions per second) between the algorithm of the analysis and the RLS [1], with different choises of the feedforward and feedback filters. The algorithm that uses the Cholesky factorization results to be superior, in particular for a high number of coefficients of the two filters.

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