Abstract

In this work we address the problems of state stabilization and convergence dynamics for finite-dimensional Quantum Dynamical Semigroups generators. We do so by building on the powerful notions of invariant and attractive quantum subsystems. We first recall some recent results due to Francesco Ticozzi and Lorenza Viola providing explicit algebraic characterizations of invariant and attractive subsystems. Then we present a new result on convergence dynamics.

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Chapter 1

Quantum mechanics

In this chapter we outline the postulates of quantum mechanics using the density operator formalism. We will restrict our discussion to the case of finite-dimensional state-space quantum systems. Indeed this is often the case in many applications; notable examples are tasks related to quantum information and quantum computing ([1], [2]).

In the following we recall the relevant results of linear algebra which are necessary to understand the theory we present. We collect some of the proofs too; this is a useful exercise which helps us introducing the notation we will use in the remainder of this work.

1.1 Hilbert space

Let V_K denote a vector space over the field K. We say that V_K is a *Banach space* if it is complete with respect to a norm $\|\cdot\|_{V_K}$. We say that V_K is an *Hilbert space* if it is a Banach space equipped with an inner product $(\cdot,\cdot)_{V_K}$. As we shall see the field of interest in this context is the set of complex numbers. In the following we let $\mathcal{H}$ denote an Hilbert space on the complex field.

It is customary in the literature on quantum mechanics to define inner products requiring linearity with respect to the *second* argument, that is maps $(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ satisfying the following properties

1. $(v, \sum_i \lambda_i w_i)_{\mathcal{H}} = \sum_i \lambda_i (v, w_i)_{\mathcal{H}}$, $\forall\, \lambda \in \mathbb{C}$ and $v, w_i \in \mathcal{H}$,
2. $(v, w)_{\mathcal{H}} = (w, v)^*_{\mathcal{H}}$, $\forall\, v, w \in \mathcal{H}$,
3. $(v, v)_{\mathcal{H}} \geq 0$, $\forall\, v \in \mathcal{H}$ and with equality iff $v = \mathbf{0}_{\mathcal{H}}$.

where we used the symbol * to denote conjugation and $\mathbf{0}_{\mathcal{H}}$ for the null vector of $\mathcal{H}$.

Notice that any finite-dimensional complex vector space is naturally equipped with the canonical inner-product, is complete and thus is an Hilbert space. For the sake of generality we will prefer the term Hilbert space to that of finite-dimensional complex vector space, though from now on we will implicitly take any vector space to be finite-dimensional. We will often drop suffixes when they are clear from the context.

We call a vector space $\mathcal{H}^\dagger$ of linear functionals acting on $\mathcal{H}$ the *dual vector space* of $\mathcal{H}$. As we shall see it is convenient to consider the dual space $\mathcal{H}^\dagger$ built associating to any element $\psi \in \mathcal{H}$ the unique linear functional $\psi^\dagger(\cdot) : \mathcal{H} \to \mathbb{C}$ mapping the generic element $\phi \in \mathcal{H}$ to $(\psi, \phi)_{\mathcal{H}}$. The functionals $\psi^\dagger$ are easily proved to be well defined and in bijective correspondence with the elements of $\mathcal{H}$.

Following the standard quantum mechanical notation we let vectors be represented by a *ket*: $|\psi\rangle \in \mathcal{H}$, and linear functionals by a *bra*: $\langle\psi| \in \mathcal{H}^\dagger$. In this way the action of a bra on a ket from the left is written $\langle\psi|\phi\rangle$.

1.2 Operators

The by far most relevant class of maps in the context of quantum mechanis are linear endomorphisms, or simply *operators*. In the following we will always require that operators be *bounded*:

Definition 1. Let $\mathcal{A}$ be an operator acting on the complex Hilbert space $\mathcal{H}$. $\mathcal{A}$ is bounded if there exists a real number k such that

$$\|\mathcal{A}|\psi\rangle\| \leq k\||\psi\rangle\| \quad \forall|\psi\rangle \in \mathcal{H}.$$

Indeed any linear operator acting on a finite-dimensional vector space is bounded; again, for the sake of generality, we will prefer the term bounded operator and denote the set of bounded operators acting on the Hilbert space $\mathcal{H}$ by the symbol $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{H}_V$ and $\mathcal{H}_W$ be Hilbert spaces and let $|v\rangle$, $|w\rangle$ be arbitrary elements in $\mathcal{H}_V$ and $\mathcal{H}_W$ respectively. It is possible to consider linear maps of the form $|w\rangle\langle v|(\cdot) : \mathcal{H}_V \to \mathcal{H}_W$ mapping the generic element $\nu \in \mathcal{H}_V$ to $|w\rangle\langle v|\nu\rangle = \langle v|\nu\rangle|w\rangle \in \mathcal{H}_W$. This is known as *outer product representation.* This outer product turns useful for computations in various situations, e.g. when orthogonal projections are involved. One explicit example is given by the following lemma, known as *completeness relation for orthonormal vectors.*

Lemma 1. Let $\{|i\rangle\}$ be an orthonormal basis for the complex Hilbert space $\mathcal{H}$. The following completeness relation holds:

$$\sum_i |i\rangle\langle i| = \mathbf{1}_{\mathcal{H}} \tag{1.1}$$

where $\mathbf{1}_{\mathcal{H}}$ denotes the identity map acting on $\mathcal{H}$.

Proof. Exploiting the orthonormality of $\{|i\rangle\}$ and the linearity of operators we find

$$\left(\sum_i |i\rangle\langle i|\right)|v\rangle = \sum_i |i\rangle\langle i|v\rangle = \sum_i v_i|i\rangle$$

where $v_i \in \mathbb{C}$ are the the coordinates of $|v\rangle$ with respect to the basis $\{|i\rangle\}$. Equality holds for any $|v\rangle \in \mathcal{H}_V$, hence we conclude. □

Now consider a linear map $\mathcal{A} : \mathcal{H}_V \to \mathcal{H}_W$ and let $\{|i\rangle\}$ and $\{|j\rangle\}$ be orthonormal bases for $\mathcal{H}_V$ and $\mathcal{H}_W$ respectively. Then the matrix representation of $\mathcal{A}$ with respect to these basis can be obtained by exploiting the completeness relation twice:

$$\begin{aligned} \mathcal{A} &= I_W \mathcal{A} I_V \\ &= \Big(\sum_j |j\rangle\langle j|\Big)\mathcal{A}\Big(\sum_i |i\rangle\langle i|\Big) \\ &= \sum_{i,j} \langle j|\mathcal{A}|i\rangle |j\rangle\langle i| \end{aligned} \tag{1.2}$$

From which we establish that, in the given basis, $\mathcal{A}$ has matrix representation $A = [\langle j|\mathcal{A}|i\rangle]$.

Most of the time we will use the convention of denoting operators with a calligraphic font, i.e. $\mathcal{A}$ and their matrix representation in a given basis with a regular font, i.e. A. Recall, however, that there is a bijective correspondence between an operator and its matrix representation once the input and output basis are fixed.

From now on we consider the same basis as input and output basis when dealing with the matrix representation of an operator. In particular we say that A is the matrix representation of $\mathcal{A}$ in the basis $\{i\}$ meaning that A is the matrix representation of $\mathcal{A}$ with respect to the same input and output basis $\{i\}$.

Definition 2. Let $\mathcal{A}$ be an operator in $\mathcal{B}(\mathcal{H})$. We call adjoint or Hermitean conjugate operator of $\mathcal{A}$ an operator $\mathcal{A}^\dagger$ acting on $\mathcal{H}$ such that for all vectors $|v\rangle, |w\rangle \in \mathcal{H}$ holds

$$(|v\rangle, \mathcal{A}|w\rangle)_{\mathcal{H}} = (\mathcal{A}^\dagger|v\rangle, |w\rangle)_{\mathcal{H}}.$$

Lemma 2. For any operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$, its adjoint $\mathcal{A}^\dagger$ is uniquely identified.

Furthermore there exists a straightforward relation between the matrix representation of an operator and that of its adjoint.

Lemma 3. Let A be the matrix representation of the operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ with respect to the orthonormal basis $\{|i\rangle\}$. The matrix representation of $\mathcal{A}^\dagger$ with respect to the same basis is given by the conjugate transpose of A.

Proof. By the properties of the inner-product we have:

$$\begin{aligned} A_{i,j} &= \langle j|\mathcal{A}|i\rangle \\ &= (|j\rangle, \mathcal{A}|i\rangle) \\ &= (\mathcal{A}^\dagger|j\rangle, |i\rangle) \\ &= (|i\rangle, \mathcal{A}^\dagger|j\rangle)^* \\ &= \langle i|\mathcal{A}^\dagger|j\rangle^* \\ &= (A^\dagger_{j,i})^* \end{aligned} \tag{1.3}$$

□

It's easily verified that the adjoint operation mapping elements of $\mathcal{H}$ to elements of $\mathcal{H}^\dagger$ satisfies the following properties:

1. $(\lambda\mathcal{A})^\dagger = \lambda^*\mathcal{A}^\dagger$ for $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$;
2. $(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger$ for any $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$.
3. $|\psi\rangle^\dagger = \langle\psi|$.

From now on we denote the real subspace of Hermitian operators in $\mathcal{B}(\mathcal{H})$ with the simbol $\mathfrak{h}(\mathcal{H})$.

Lemma 4. The eigenvalues of a self-adjoint operator are real.

Proof. Let $(\lambda, |v\rangle)$ be an eigenpair of $\mathcal{A} \in \mathfrak{h}(\mathcal{H})$. Then

$$\lambda\|v\|^2 = \lambda(v, v) = (v, \mathcal{A}v) = (\mathcal{A}v, v) = (\lambda v, v) = \bar{\lambda}\|v\|^2 \tag{1.4}$$

Thus $\lambda = \bar{\lambda} \in \mathbb{R}$. □

A particular case of Hermitian operators is that of projectors:

Definition 3. We say that $\Pi \in \mathcal{B}(\mathcal{H})$ is a *projection* or a *projector* if $\Pi \in \mathfrak{h}(\mathcal{H})$ and $\Pi = \Pi^2$.

For any pair of operators $\mathcal{X}, \mathcal{Y} \in \mathcal{B}(\mathcal{H})$ denote with $[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$, that is the *commutator* of $\mathcal{X}$ and $\mathcal{Y}$. It turns out that the commutator relation can be used to characterize many interesting properties of operators.

Definition 4. An operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is normal if it commutes with its adjoint: $[\mathcal{A}, \mathcal{A}^\dagger] = \mathbf{0}_\mathcal{H}$.

Notice that by the definition of normal operator any Hermitian operator is normal too and the following important Theorem holds:

Theorem 1. Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of $\mathcal{A}$ if and only if $\mathcal{A}$ is normal.

Proof. First suppose that $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of $\mathcal{A}$. With respect to this basis, $\mathcal{A}$ and $\mathcal{A}^\dagger$ have the same diagonal matrix representation, thus they commute.

Now suppose that $\mathcal{A}$ is normal. There is an orthonormal basis $\{|e_i\rangle\}$ of $\mathcal{H}$ with respect to which $\mathcal{A}$ has an upper-triangular matrix ([3]). Thus we can write

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_{n,n} \end{bmatrix} \tag{1.5}$$

We see from the matrix above that

$$\|\mathcal{A}e_1\|^2 = |a_{1,1}|^2 \tag{1.6}$$

and

$$\|\mathcal{A}^\dagger e_1\|^2 = |a_{1,1}|^2 + |a_{1,2}|^2 + \cdots + |a_{1,n}|^2 \tag{1.7}$$

Because $\mathcal{A}$ is normal, $\|\mathcal{A}e_1\| = \|\mathcal{A}^\dagger e_1\|$. Thus the two equations above imply that all entries in the first row of the matrix except possibly the first entry $a_{1,1}$ equal 0.

By iterating our reasoning we conclude that all non-diagonal entries must vanish, hence we conclude. □

Before proceeding we recall the following interesting property of Hermitian operators.

Lemma 5. Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$, such that

$$(\mathcal{A}v, v) = 0 \tag{1.8}$$

for all $v \in \mathcal{H}$, then $\mathcal{A} = \mathbf{0}_\mathcal{H}$.

Definition 5. We say that $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is *positive semi-definite* or simply *positive*, denoted by $\mathcal{A} \geq 0$, if $\langle v|\mathcal{A}v\rangle$ is a non negative real number for any $|v\rangle \in \mathcal{H}$. We say that $\mathcal{A}$ is *positive definite*, $\mathcal{A} > 0$, if $\mathcal{A}$ is positive and $\langle v|\mathcal{A}v\rangle$ is zero only for the null vector of $\mathcal{H}$.

It can be proved that positive operators are self-adjoint.

Theorem 2. An operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is positive if and only if $\mathcal{A}$ is Hermitian and all its eigenvalues are non-negative.

Proof. Clearly any positive operator is Hermitean by Lemma 5. Now let $(\lambda, |v\rangle)$ be an eigenpair of $\mathcal{A}$. Then

$$0 \leq \langle v|\mathcal{A}|v\rangle = \lambda\langle v|v\rangle \tag{1.9}$$

and thus λ is bound to be real non-negative. □

The last definition we introduce in this section is that of unitary operator:

Definition 6. An operator $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ is *unitary* if $\mathcal{A}^\dagger\mathcal{A} = \mathcal{A}\mathcal{A}^\dagger = \mathbf{1}_\mathcal{H}$.

In the remainder of this work we will make of the two following properties of unitary operators:

i) unitary operators preserve inner products, that is

$$(\mathcal{U}|\psi\rangle, \mathcal{U}|\psi\rangle) = \langle\psi|\mathcal{U}^\dagger\mathcal{U}|\psi\rangle = \langle\psi|\psi\rangle, \tag{1.10}$$

for any unitary $\mathcal{U} \in \mathcal{B}(\mathcal{H})$ and vector $|\psi\rangle \in \mathcal{H}$;

ii) the columns and the rows of any unitary operator in $\mathcal{B}(\mathcal{H})$ constitute an orthonormal basis of $\mathcal{H}$.

1.3 Tensor products

The *tensor product* is the most general bilinear function mapping elements from the Cartesian product of two vector spaces to a third vector space. In particular, letting $\mathcal{H}_V$ and $\mathcal{H}_W$ be two complex Hilbert spaces, the tensor product of vectors satisfies by definition the following properties:

1. $c\big(|v\rangle \otimes |w\rangle\big) = c|v\rangle \otimes |w\rangle = |v\rangle \otimes c|w\rangle$, $\forall\, c \in \mathbb{C}$, $|v\rangle \in \mathcal{H}_V$, $|w\rangle \in \mathcal{H}_W$;

2. $\big(|v_1\rangle + |v_2\rangle\big) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$, $\forall |v_1\rangle, |v_2\rangle \in \mathcal{H}_V, |w\rangle \in \mathcal{H}_W$;

3. $|v\rangle \otimes \big(|w_1\rangle + |w_2\rangle\big) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$, $\forall |v\rangle \in \mathcal{H}_V, |w_1\rangle, |w_2\rangle \in \mathcal{H}_W$.

We denote with $\mathcal{H}_V \otimes \mathcal{H}_W$ the vector space whose elements are linear combination of tensor products $|v\rangle \otimes |w\rangle$ with $|v\rangle \in \mathcal{H}_V$ and $|w\rangle \in \mathcal{H}_W$. It can be shown that if $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$ are orthonormal bases for $\mathcal{H}_V$ and $\mathcal{H}_W$ respectively then $\{|v_i\rangle \otimes |w_j\rangle\}$ is an orthonormal basis for $\mathcal{H}_V \otimes \mathcal{H}_W$; futhermore by letting $\dim(\mathcal{H}_V) = n$ and $\dim(\mathcal{H}_W) = m$ the dimension of the tensor space is given by $n \cdot m$.

An inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is obtained naturally from the inner products defined on the tensoring spaces in the following way

$$\Big(\sum_i \alpha_i |v_i\rangle \otimes |w_i\rangle, \sum_j \beta_j |v'_j\rangle \otimes |w'_j\rangle\Big)_{\mathcal{H}_V \otimes \mathcal{H}_W} = \sum_{i,j} \alpha_i^* \beta_j \langle v_i|v'_j\rangle \langle w_i|w'_j\rangle. \quad (1.11)$$

Now let $\mathcal{A} \in \mathcal{B}(\mathcal{H}_V)$ and $\mathcal{B} \in \mathcal{B}(\mathcal{H}_W)$. We define the operator $\mathcal{A} \otimes \mathcal{B} \in \mathcal{B}(\mathcal{H}_{V \otimes W})$ as

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) \equiv (\mathcal{A}|v\rangle \otimes \mathcal{B}|w\rangle) \quad (1.12)$$

for all $|v\rangle \in \mathcal{H}_V$ and $|w\rangle \in \mathcal{H}_W$. It can be shown that $\mathcal{A} \otimes \mathcal{B}$ as defined is uniquely identified ([4]).

Interestingly, once the basis are fixed, the matrix representation of the operator $\mathcal{A} \otimes \mathcal{B}$ is given by the Kronecker product between the respective matrix representations:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{bmatrix}. \quad (1.13)$$

In the same way, once the basis are fixed, the tuple of the coordinates of the vector $|v\rangle \otimes |w\rangle \in \mathcal{H}_V \otimes \mathcal{H}_W$ is given by the Kronecker product between the tuple of coordinates of $|v\rangle$ and that of $|w\rangle$.

We conclude our brief introduction to tensor products listing some of the properties of the tensor product we will use in the following:

1. $(A \otimes B)(C \otimes D) = AC \otimes BD$,
2. $(A \otimes B)^\dagger = (A^\dagger \otimes B^\dagger)$,
3. $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$.

1.4 Trace

Let A be the matrix representation of $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ with respect to the orthonormal basis $\{|i\rangle\}$. The *trace of* $\mathcal{A}$ is defined as the sum of the diagonal elements of A:

$$\mathrm{tr}(\mathcal{A}) = \sum_i \langle i|\mathcal{A}|i\rangle. \quad (1.14)$$

It is easily seen that the trace function is linear and cyclic:

1. $\mathrm{tr}(\lambda\mathcal{A} + \mu\mathcal{B}) = \lambda\mathrm{tr}(\mathcal{A}) + \mu\mathrm{tr}(\mathcal{B})$ for any λ, $\mu \in \mathbb{C}$;
2. $\mathrm{tr}(\mathcal{A}\mathcal{B}) = \mathrm{tr}(\mathcal{B}\mathcal{A})$.

By exploiting cyclicity it is easily seen that the trace is well defined in the sense that it is invariant with respect to basis transformations.

1.5 Postulates of Quantum mechanics

Having developed most of the tools we need in this work we may now list the actual postulates of quantum mechanics which, while expressing a surprising reality, are quite straightforward.

Postulate 1. Associated to any isolated physical system is a separable Hilbert space over the field $\mathbb{C}$, known as the *state space* of the system. The state of the system is completely described by a unit vector in such space, also known as state-vector.

Postulate 2. The state of a closed physical system with associated complex Hilbert space $\mathcal{H}$ is described by a probability distribution over $\mathcal{H}$.

Information on the state of a quantum system can be "encoded" in a mathematical object by introducing the density operator formalism.

Definition 7. We say that $\rho \in \mathcal{B}(\mathcal{H})$ is a density operator if the following properties hold:

1. ρ is positive (and hence Hermitian),
2. $\mathrm{tr}(\rho) = 1$.

By the spectral theorem every density operator ρ has representation

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{1.15}$$

where $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$, $\lambda_i \geq 0 \; \forall\, i$ and $\sum_i \lambda_i = 1$.

The unit vectors $|\psi_i\rangle$ have an interpretation as the state-vectors describing the quantum system's state in the state-vector formulation of quantum mechanics. Consider an ensemble which is in the state $|\psi_i\rangle$ with probability p_i with $i \in I$. The density operator ρ associated to such ensemble is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

Thus ρ gives a statistical description of the quantum system; the probabilities p_i may express uncertainty on the state of the system or could be seen as the fractional population of systems in the ensemble which are currently in the *i-th* state.

It is useful to introduce some nomenclature; we call *pure state* a one-dimensional projector, that is density operators in the form $|\psi\rangle\langle\psi|$ for some state vector $|\psi\rangle$, otherwise we call it a *mixed state* or a *mixture* of the different states in the ensemble.

It can be shown that the set of density operators in $\mathcal{B}(\mathcal{H})$ is a convex compact subset of the real subspace of Hermitian acting on such space whose "extreme" points are pure states ([5]).

Interestingly the same density operator might be generated by different statistical ensembles.

Theorem 3 (Unitary freedom in the ensemble for density matrices)**.** Consider the sets $\{|\tilde{\psi}\rangle_i\}$ and $\{|\tilde{\phi}\rangle_i\}$. We may always pad the shorter list by appending null vectors such that their cardinality is the same. Then they generate the same density operator if and only if

$$\tilde{\psi}_i = \sum_j u_{i,j}\tilde{\phi}_j,$$

where $(u_{i,j})$ is a unitary matrix.

Proof. Suppose $|\tilde{\psi}_i\rangle = \sum_j u_{i,j}|\tilde{\phi}_i\rangle$, then

$$\begin{aligned}\sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| &= \sum_{i,j,k} u_{i,j}u^*_{i,k}|\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| \\ &= \sum_{j,k}\big(\sum_i u^*_{k,i}u_{i,j}\big)|\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| \\ &= \sum_{j,k}\delta_{j,k}|\tilde{\phi}_j\rangle\langle\tilde{\phi}_k| \\ &= \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j|\end{aligned} \tag{1.16}$$

and thus the two sets generate the same operator.

Conversely, suppose

$$A = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\phi}_i| = \sum_i |\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|. \tag{1.17}$$

Let $A = \sum_k \lambda|k\rangle\langle k|$ be a decomposition for A such that the set $\{|k\rangle\}$ is orthonormal and λ_k are strictly positive and define $|\tilde{k}\rangle = \sqrt{\lambda_k}|k\rangle$. Now let $|\psi\rangle$ be any any vector orthonormal to the space spanned the $|\tilde{k}\rangle$. Thus $\langle\psi|\tilde{k}\rangle\langle\tilde{k}|\psi\rangle = 0$ and

$$0 = \langle\psi|A|\psi\rangle = \sum_i \langle\psi|\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|\psi\rangle = \sum_i \|\langle\tilde{\psi}_i|\psi\rangle\|^2. \tag{1.18}$$

Thus $\langle\tilde{\psi}_i|\psi\rangle = 0\ \forall i$ and all $|\psi\rangle$ orthonormal to the space spanned by the $|\tilde{k}\rangle$. It follows that each $|\tilde{\psi}_i\rangle$ can be espressed as a linear combination of the $|\tilde{k}\rangle$, $|\tilde{\psi}_i\rangle = \sum_k c_{i,k}|\tilde{k}\rangle$. Since $A = \sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_k |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$ it follows:

$$\sum_k |\tilde{k}\rangle\langle\tilde{k}| = \sum_{k,l}\big(\sum_i c^*_{i,k}c_{i,l}\big)|\tilde{k}\rangle\langle\tilde{l}|. \tag{1.19}$$

The operators $|\tilde{k}\rangle\langle\tilde{l}|$ are easily seen to be linear independent and thus it must be $\sum_i c_{i,k}c_{i,l} = \delta_{k,l}$. This ensures that we may append extra columns to $(c_{i,j})$ to obtain an unitary matrix v such that $|\tilde{\psi_i}\rangle = \sum_k v_{i,k}|\tilde{k}\rangle$, where we have appended zero vectors to the list of $|\tilde{k}\rangle$. Similarly we can find an unitary matrix w such that $|\tilde{\phi_j}\rangle = \sum_k w_{j,k}|\tilde{k}\rangle$. Thus $|\tilde{\psi_i}\rangle = \sum_j u_{i,j}|\tilde{\phi_j}\rangle$, where $u = vw^\dagger$ is unitary. □

Postulate 3. The time evolution of a closed physical system is described by an unitary transformation. That is, given $\mathcal{T}$ the set of time dependent state trajectories generated by the physical system, than for any state $\rho(t) \in \mathcal{T}$ holds the relation: $\rho(t_1) = U\rho(t_2)U^\dagger$ with U unitary operator acting on the state space of the quantum system.

Beware that when working with a description of the quantum system in terms of density operator, applying an operation $\mathcal{M}$ on a given quantum state ρ means that we perform the same transformation on each quantum system of the corresponding statistical ensemble.

Postulate 4. Quantum measurements are described by a set of measurement operators M_m acting on the state space of the system being measured. These operators satisfying the completeness relation $\sum_m M^\dagger M = I$. The index m refers to the measurement outcomes that may occur in the experiment. If the state state of the system is ρ immediately before the measurement the the probability that result m occurs is given by

$$p(m) = \mathrm{tr}(M_m^\dagger M_m \rho)$$

and the state of the systems after the measurement is still a density operator in $\mathcal{D}(\mathcal{H})$ and equals

$$\frac{M_m \rho M_m^\dagger}{p(m)}.$$

The completeness relation expresses the fact that the outcomes probabilities sum to one.

A relevant class of measurement operators are *projective measurements*, that is a measurement described by an Hermitian operator $\mathcal{O}$, also known as *observable*. Since $\mathcal{O}$ is Hermitian it has spectral decomposition of the form

$$\mathcal{O} = \sum_i \lambda_i \Pi_i \tag{1.20}$$

where the operators Π_i are projectors onto the eigenspace U_{λ_i} of $\mathcal{O}$. The possible outcomes of the measurement correspond to the (real) eigenvalues λ_i, as we might aspect on physical grounds. Since $\Pi_i \Pi_j = \delta_{ij}\Pi_i$ and $\Pi_i^2 = \Pi_i$, upon measuring the state the probability of getting result m is given by $\langle\psi|P_m|\psi\rangle$. Given that outcome m occurred, the state of the quantum system immediately after the measurement is $P_m|\psi\rangle/\langle\psi|P_m|\psi\rangle$.

Historically projective measurements have been introduced first: in this sense, the measurements described in the third postulate of quantum mechanics are often referred to as generalized measurement.

We will later be interested in computing the expectation for measurements. It turns out this is particularly easy to do in the case of projective measurements. Indeed, by definition, the expectation value of the measurement with respect to a given state $|\phi\rangle$ is

$$\begin{aligned}\langle \mathcal{O}\rangle := E(\mathcal{O}) &= \sum_i \lambda_i \langle\psi|\Pi_i|\psi\rangle \\ &= \langle\psi|(\sum_i \lambda_i \Pi_i)|\psi\rangle \\ &= \langle\psi|\Pi|\psi\rangle\end{aligned} \tag{1.21}$$

The next postulate addresses the case of composite physical systems. That is systems made up of two or more distinct systems.

Postulate 5. The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and the *i-th* system is prepared in the state ρ_i then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$.

1.6 The reduced density operator

By the last postulate of quantum mechanics the state of a composite quantum system has a non trivial form. In particular to "recover" the state of a quantum subsystem a new tool must be introduced. Before doing so we need to introduce the following theorem.

Theorem 4 (Schmidt decomposition)**.** Suppose $|\psi\rangle\langle\psi|$ is a pure state of a composite system AB. Then there exists orthonormal states $\{|a_i\rangle\} \in \mathcal{H}_A$ and $\{|b_i\rangle\} \in \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_i \lambda_i |a_i\rangle|b_i\rangle \tag{1.22}$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt co-efficients.

A straightforward proof of Theorem 4 can be found in [1]. A by-product of this result is that any pure state of a composite system can be written in the form

$$|\psi\rangle\langle\psi| = \sum_{i,j} \lambda_i \lambda_j |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|. \tag{1.23}$$

Consider the composite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces associated to two distinct physical systems A and B.

Let the state of such composite system be described by the density operator $\rho_{AB} = \sum_i p_i \rho_i \otimes \sigma_i$ with $\rho_i \in \mathfrak{D}(\mathcal{H}_A)$ and $\sigma_i \in \mathfrak{D}(\mathcal{H}_B)$. The *partial trace over system B* is the map $\mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A$ defined as

$$\rho_A = tr_B(\rho_{AB}) = \sum_i p_i \rho_i \mathrm{tr}(\sigma_i), \tag{1.24}$$

and the state $\rho_A \in \mathfrak{D}(\mathcal{H}_A)$ is known as *reduced density operator*.

As we shall see the partial trace is the unique operation giving the correct statistical description of observables quantities for subsystems of a composite system with respect to measurement expectation. Let $\mathcal{O}$ be and observable on subsystem A, then $\mathcal{O}$ has spectral decomposition of the form $\sum_i \lambda_i \Pi_i$; moreover be $\tilde{\mathcal{O}}$ the observable for the same measurement performed on the composite system AB. If the composite system is prepared in a product state $|\psi_i\rangle|\phi\rangle$ where $|\psi_i\rangle$ is the eigenstate of $\mathcal{O}$ relative to the eigenvalue λ_i, and $|\phi\rangle$ is any state in $\mathcal{H}_B$, then, the measuring device must yield the outcome λ_i with certainty. Thus for any projector Π_i in the spectral decomposition of $\mathcal{O}$ the corresponding projector in the spectral decomposition of $\tilde{\mathcal{O}}$ is $\Pi_i \otimes I_B$. Therefore

$$\tilde{\mathcal{O}} = \sum_i \lambda_i \Pi_i \otimes I_B = \mathcal{O} \otimes I_B \tag{1.25}$$

For physical consistency we require measurement expectations to be the same whether computed on ρ_A or ρ_{AB}, that is

$$\mathrm{tr}(\mathcal{O}\rho_A) = \mathrm{tr}(\tilde{\mathcal{O}}\rho_{AB}) \tag{1.26}$$

This last equation is clearly satisfied if we choose $\rho_A = \mathrm{tr}_B(\rho_{AB})$.

To prove uniqueness consider a map $f : \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{D}(\mathcal{H}_A)$ such that $\mathrm{tr}(\mathcal{O} f(\rho_{AB})) = \mathrm{tr}(\tilde{\mathcal{O}}\rho_{AB})$ holds for all observables $\mathcal{O}$. Let $\{\mathcal{O}_i\}$ be basis for the real subspace of Hermitian matrices of $\mathcal{H}_{AB}$, then

$$\begin{aligned} f(\rho_{AB}) &= \sum_i M_i \mathrm{tr}(M_i f(\rho_{AB})) \\ &= \sum_i M_i \mathrm{tr}((M_i \otimes I_B)\rho_{AB}) \end{aligned} \tag{1.27}$$

Thus f is uniquely defined by (1.24). Moreover since the partial trace satisfies (1.24), it is the unique function with this property.

Chapter 2

Markovian open quantum systems

In practice no quantum system can be regarded as a perfectly closed physical system. More generally we may think of any "subsystem" of a physical system as an *open quantum system.*

Indeed any open quantum systems can be modeled as the coupling between a "principal system" and its "environment". While such augmented system evolves with unitary dynamics according to Postulate 3, this is not true in general for the reduced state of the principal system. Interestingly, however, it can be shown that quantum information processing is still possible in open quantum systems. Indeed from this standpoint, unwanted correlations with the environment can be characterized as quantum noise acting on the state of the principal system.

In this chapter we will address a relevant class of open quantum systems in which the quantum noise depending on the coupling with the environment is such that the resulting evolution is Markovian.

2.1 Quantum operation

Quantum operations, which we define next, are central to the practice of quantum information processing in that they can be used to describe the stochastic evolution of a quantum system, i.e. in the presence of noise. Let $\hat{\mathfrak{D}}(\mathcal{H})$ denote the set of positive operators acting on $\mathcal{H}$ such that their trace is less or equal to one.

Definition 8 (Quantum operation)**.** Consider a quantum system $\mathcal{S}$ defined on the complex Hilbert space $\mathcal{H}_S$. A quantum operation is defined as any map $\Lambda : \mathfrak{D}(\mathcal{H}_S) \to \hat{\mathfrak{D}}(\mathcal{H}_S)$ with the following properties:

i) The quantity $\mathrm{tr}(\Lambda\rho)$ is the probability that the process expressed by Λ occurs when the initial state is $\rho \in \mathfrak{D}(\mathcal{H}_S)$.

ii) Λ is linear.

iii) Λ is completely-positive. That is, for any quantum system R defined on a complex Hilbert space $\mathcal{H}_R$, with $\dim(\mathcal{H}_R)$ arbitrary, it must hold $(\mathbf{1}_R \otimes \Lambda)\mathcal{X} \geq 0$ for any positive operator $\mathcal{X} \in \mathcal{B}(\mathcal{H}_{RS})$.

Trace-preserving quantum operations, $\mathrm{tr}(\Lambda\rho) = 1$, are thus naturally connected with deterministic processes such unitary evolution. Non-trace-preserving quantum operations describe stochastic processes, i.e. processes involving quantum measurements; in this case the output state is still a density operator up to a renormalization factor.

Conditions (ii) and (iii) appearing in the definition are required for consistency. Linearity make so that Λ does not have "preferred" input states, while complete-positivity is necessary for quantum operations to be well defined even when considering a bigger joint system.

Interestingly quantum operations admit an explicit algebraic characterization. Sufficient and necessary condition are given by the following Theorem due to Kraus.

Theorem 5 (Operator sum representation)**.** A map $\Lambda : \mathfrak{D}(\mathcal{H}_S) \to \hat{\mathfrak{D}}(\mathcal{H}_S)$ is a quantum operation if and only if it can be cast in the form

$$\Lambda\rho = \sum_i E_i \rho E_i^\dagger, \tag{2.1}$$

for some set of operators $\{E_i\}$ such that $\sum_i E_i^\dagger E_i \leq \mathbf{1}_S$.

Proof. Λ is clearly linear with respect to input density operators. To prove that it is completely positive consider an extended system RS and let A be any positive operator acting on the associated augmented Hilbert space. For any vector $|\psi\rangle \in \mathcal{H}_R \otimes \mathcal{H}_S$ we have:

$$\langle\psi|(\mathbf{1}_R \otimes \Lambda)A|\psi\rangle = \sum_i \langle\psi|(\mathbf{1}_R \otimes E_i)A(\mathbf{1}_R \otimes E_i^\dagger)|\psi\rangle \geq 0 \tag{2.2}$$

where the last inequality holds since A is positive, for any choice of $\{E_i\}$ and, in particular, when $\sum_i E_i^\dagger E_i \leq \mathbf{1}_S$. Thus $(\mathbf{1}_R \otimes \Lambda) \geq 0$ and by exploiting the freedom on $\dim(\mathcal{H}_R)$ we conclude that Λ is completely positive.

Suppose next that Λ satisfies the three axioms in Definition 8. Augment again the state space of $\mathcal{S}$ by introducing a quantum system $\mathcal{R}$ such that $\dim(\mathcal{H}_R) = \dim(\mathcal{H}_S)$. Let the sets $\{|i_S\rangle\}$ and $\{|i_R\rangle\}$ be orthonormal basis of $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively, and define an operator σ acting on $\mathcal{H}_{RS}$ by the equation:

$$\sigma = (\mathbf{1}_r \otimes \Lambda) \sum_{i,j} |i_R \otimes i_S\rangle\langle j_R \otimes j_S|. \tag{2.3}$$

Indeed, σ is fully described by Λ alone. To see this, let $|\psi\rangle = \sum_i \psi_i |i_S\rangle$ be an arbitrary unit vector of $\mathcal{H}_S$ and $|\tilde{\psi}\rangle = \sum_i \psi_i^* |i_R\rangle$ the "corresponding" vector in $\mathcal{H}_R$. Then, the following identity holds:

$$\begin{aligned}(\langle\tilde{\psi}| \otimes \mathbf{1}_S)\sigma(|\tilde{\psi}\rangle \otimes \mathbf{1}_S) &= \sum_{i,j} \psi_i \psi_j^* \Lambda |i_S\rangle\langle j_S| \\ &= \Lambda|\psi\rangle\langle\psi|.\end{aligned} \tag{2.4}$$

Now rewrite σ according to its spectral decomposition $\sigma = \sum_i |s_i\rangle\langle s_i|$ and define the maps $E_i : \mathcal{H}_S \to \mathcal{H}_S$ by the equation:

$$E_i|\psi\rangle = \langle\tilde{\psi}|s_i\rangle, \tag{2.5}$$

for all $|\psi\rangle \in \mathcal{H}_S$ and $|\tilde{\psi}\rangle \in \mathcal{H}_R$ in the same one-to-one correspondence we used before. It is not difficult to see that the maps E_i are linear operators acting on $\mathcal{H}_S$. Furthermore, we have

$$\begin{aligned}\sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger &= \sum_i \langle\tilde{\psi}|s_i\rangle\langle s_i|\tilde{\psi}\rangle \\ &= \langle\tilde{\psi}|\sigma|\tilde{\psi}\rangle \\ &= \Lambda|\psi\rangle\langle\psi|.\end{aligned} \tag{2.6}$$

Thus for all unit vectors $|\psi\rangle \in \mathcal{H}_S$ we have

$$\Lambda|\psi\rangle\langle\psi| = \sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger, \tag{2.7}$$

and by linearity it follows that

$$\Lambda\rho = \sum_i E_i \rho E_i^\dagger \tag{2.8}$$

for any state $\rho \in \mathfrak{D}(\mathcal{H}_I)$. The condition $\sum_i E_i^\dagger E_i \leq \mathbf{1}$ follows immediately from the first axiom identifying the trace of $\Lambda\rho$ with a probability. □

The operators $\{E_i\} \in \mathcal{B}(\mathcal{H}_S)$ are often referred to as *operation elements* and (2.1) as *operator-sum representation* or simply *OSR*. As we might expect the *OSR* associated to a quantum operation is not unique.

Theorem 6 (OSR unitary freedom)**.** Suppose $\{E_1, \ldots, E_n\}$ and $\{F_1, \ldots, F_m\}$ are operation elements describing the quantum processes Λ and Γ, respectively. By appending null operators to the shorter list we may ensure $n = m$. Then Λ and Γ are the same quantum process if and only if $E_i = \sum_j u_{i,j} F_j$ and $(u_{i,j})$ is an n by n unitary matrix.

A complete proof might be found in [1]. Interestingly, Theorem 6 enables us to show that it is always possible to express quantum operations with a finite number of operation elements.

Theorem 7. Any quantum operation Λ on a system $\mathcal{S}$ with s-dimensional Hilbert space can be generated by an operator sum representation containing at most s^2 elements.

Proof. Let $\{E_j\}$ be the operation elements of Λ and define the matrix W by the equation:

$$(W_{j,k}) = \langle E_j | E_k \rangle_{\mathcal{B}(\mathcal{H}_S)} \tag{2.9}$$

Notice that the k-th column of W, denote it by $\tilde{E}_k$, is the tuple of coordinates of E_k with respect to the possibly incomplete set of generators $\{E_i\} \in \mathcal{B}(\mathcal{H}_S)$. By the properties of the inner-product W is Hermitian:

$$W_{j,k} = \langle E_j | E_k \rangle_{\mathcal{B}(\mathcal{H}_S)} = \langle E_k | E_s \rangle^*_{\mathcal{B}(\mathcal{H}_S)} = W^*_{k,j}. \tag{2.10}$$

and there exist an unitary matrix U such that $UWU^\dagger$ is diagonal. Furthermore $\mathcal{B}(\mathcal{H}_S)$ is isomorph to $\mathbb{C}^{s \times s}$ and thus any of its basis over the complex field contains s^2 elements. By our previous considerations we must conclude that the rank of W is at most s^2 and thus $UWU^\dagger$ has at most s^2 non zero entries. Now denote by n the cardinality of the set $\{E_j\}$. Since the rows of U form an orthonormal basis of C^n the k-th column of UW, call it $\hat{E}_k$, is the projection of $\tilde{E}_k$ into this basis.

Now define a new set of operation elements $\{F_i\}$ as the columns of $UWU^\dagger$. It easy to verify that $F_i = \sum_{i,j} u_{i,j} \tilde{E}_j$, where $(u_{i,j}) = U$ and $\tilde{E}_j$ is just the tuple of coordinates of E_j for a particular choice of basis of $\mathcal{B}(\mathcal{H}_S)$. Thus $\{E_i\}$ and $\{F_i\}$ describe the same quantum operation by Theorem 6. We conclude by observing that the operation elements $\{F_i\}$ corresponds to the eigenvectors of $UWU^\dagger$ which are at most s^2. □

In open quantum systems Theorem 7 imposes a finite limit on the environment's degrees of freedom which might be necessary to consider. This is a nice feature of quantum operations.

2.2 Quantum Dynamical Semigroups

One possible approach to determining the state evolution of an open-quantum system is that of trying to obtain a statistical description of the quantum noise due to the environment and use to give a statistical description of the principal system's evolution. The simplest case, that we consider here, is that of Markovian open quantum system. The assumption we make is that our best prediction on the system's state at a time $t > t_0$ depends on the state at time t_0 alone. Interestingly a large class of physical phenomena can be described by approximative evolutions which fulfill the Markov property ([6, 4]).

Definition 9 (Quantum dynamical semigroup)**.** Given a complex Hilbert space $\mathcal{H}$, we call quantum dynamical semigroup a one-parameter family of quantum operations $\{\Lambda_t, t \geq 0\} : \mathfrak{D}(\mathcal{H}) \to \mathfrak{D}(\mathcal{H})$ such that:

- $\Lambda_0 = \mathbf{1}_{\mathcal{H}}$;
- Λ_t is trace precerving;
- $\Lambda_t \Lambda_s = \Lambda_{t+s}\ \forall\, s, t \in \mathbb{R}^+$, the semi-group or Markov property;
- $\mathrm{tr}(\Lambda_t \rho \mathcal{A})$ is a continuous function of t for any $\rho \in \mathcal{D}(\mathcal{H}_S)$ and $\mathcal{A} \in \mathcal{B}(\mathcal{H}_S)$.

It can be shown that QDS generators defined on finite-dimensional Hilbert spaces admit an explicit infinitesimal generator such that:

$$\dot{\rho}(t) = \mathcal{L}\rho(t), \tag{2.11}$$

which is called quantum Markovian master equation in the Schrödinger picture. The procedure to find the generator is sketched up in the following; we will follow strictly [6]. Consider a basis $\{F_k\}$, $k = 0, 1, ..., N^{2-1}$ in $\mathcal{B}(\mathcal{H}_S)$ such that F_0 is the identity operator. Then we may write:

$$\Lambda_t \rho = \sum_{k,l=0}^{N^2-1} C_{kl}(t) F_k \rho F_l^\dagger \tag{2.12}$$

where $C_{kl}(t)$ a positive definite matrix. Notice that for consistency we require $C(0) = \mathrm{diag}(1, \ldots, 0)$. Then, by explicitly computing the time derivative of the previous expression for t approaching 0 from the right we find:

$$\begin{aligned}\mathcal{L}\rho = \lim_{\epsilon \to 0} \Big(& \frac{C_{0,0}(\epsilon) - 1}{\epsilon} \rho + \sum_{k=0}^{N^2-1} \frac{C_{0,k}(\epsilon)}{\epsilon} F_k \rho \\ & + \rho \sum_{k=0}^{N^2-1} \frac{C^*_{0,k}(\epsilon)}{\epsilon} F_k^\dagger + \sum_{k,l=1}^{N^2-1} \frac{C_{k,l}(\epsilon)}{\epsilon} F_k \rho F_l^\dagger \Big) \\ = A\rho + \rho A^\dagger + & \sum_{k,l=1}^{N^2-1} \alpha_k l F_k \rho F_l^\dagger \end{aligned} \tag{2.13}$$

where (α_{kl}), is a positive-definite matrix. Since we require $\mathcal{L}$ to be trace preserving we must impose $\mathrm{tr}(\mathcal{L}\rho) = 0\ \forall \rho$ and by the cyclic property of the trace:

$$A + A^\dagger = - \sum_{k,l=1}^{N^2-1} \alpha_{kl} F_l^\dagger F_k, \tag{2.14}$$

leading us to first of two standard forms of the generator $\mathcal{L}$:

$$\mathcal{L}\rho = -i[H, \rho] + \frac{1}{2} \sum_{k,l=0}^{N^2-1} \alpha_{k,l} ([F_k \rho, F_l^\dagger] + [F_k, \rho F_l^\dagger]) \tag{2.15}$$

where $H \in \mathfrak{h}(\mathcal{H}_I)$ denotes the effective Hamiltonian acting of system S and the operators $\{F_i\}$ are referred to as noise-operators and constitute the "dissipative" part of the generator. The second standard form of $\mathcal{L}$, is obtained by replacing the set of operators $\{F_k\}$ by a suitable basis $\{L_k\}$ that diagonalizes $(\alpha_{k,l})$:

$$\mathcal{L}\rho = -i[H, \rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}. \tag{2.16}$$

It can be further shown that (2.16), also know as Lindblad master equation (LME), is the most general generator of quantum dynamical semigroups when finite dimensional Hilbert spaces are considered.

Indeed the LME is the generic form of the infinitesimal generator governing evolution of Markovian open quantum systems that we will consider in the following chapters.

In the following chapter will also consider the Feedback master equation (FME) due to Wiseman and Milburn, as an example QDS generator in the Lindblad form. Details are too involved to be presented here and we limit to give a brief explanation of the ideas behind this feedback scheme.

Wiseman and Milburn consider an atomic-cloud trapped in an optical cavity being stimulated by lasers. A proportional feedback is provided by introducing a measurement apparatus, constituted by an homodyne detector. The measurement output is then used to close the feedback loop by modulating the lasers' waveforms. The infinitesimal generator of the state evolution for such system can be cast in the form of (2.16) in the following way:

$$\dot{\rho}_t = -i\hbar[H + \frac{1}{2}(FM + M^\dagger F),\, \rho_t] + L\rho L^\dagger + \frac{1}{1}\{L^\dagger L, \rho\}, \tag{2.17}$$

where M is the measurement operator describing the quantum measurement realized by the detectors, F is a suitable Hermitian feedback operator and the single noise operator is given by $L = M - iF$.

A mathematically sound derivation of the Markovian feedback master equation can be found in [7].

Chapter 3

Quantum control: Invariance and Attractivity

Quantum state stabilization and partially the problem of preserving quantum information from decoherence can be recast in a natural way as quantum control problems. While different techniques were devised in past decades, here we focus on two recent contributions by Francesco Ticozzi and Lorenza Viola ([8, 9]). Building on the notions of invariance and attractivity we will give sufficient and necessary algebraic conditions for the stabilization of quantum states and looser state control in Markovian open quantum systems.

3.1 Quantum subspaces

A first natural question to ask is what kind of physical "container" is best suited for "storing" quantum information in physical systems. In this section we will concentrate on quantum subspaces, that is, quantum subsystems that have support on a subspace of the full Hilbert space. Our reasoning is further motivated by the existence of decoherence-free subspaces and the fact that universal quantum computation is possible within such protected subspaces , thus giving a a measure of the importance of this approach ([10]).

Throughout this section we consider a physical system $\mathcal{I}$ with associated Hilbert space admitting decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Let $\{|s\rangle\}_{s\in S}$ and $\{|r\rangle\}_{r\in R}$ be orthonormal bases for $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively. A convenient basis of $\mathcal{H}_I$ is given by the set $\{|s\rangle\} \oplus \{|r\rangle\}$; this choice induces the following block-structure on the matrix representation of any operator $\mathcal{X} \in \mathcal{B}(\mathcal{H}_I)$:

$$X = \begin{bmatrix} X_S & X_P \\ X_Q & X_R \end{bmatrix}, \tag{3.1}$$

at which we will continuously refer to in the following.

A requirement of abstract Quantum Information Processing tasks is that of being able to initialize the system in a desired state[1]. The following definition formalizes the concept of initialization in a general way.

Definition 10 (Subspace initialization)**.** Consider a quantum system $\mathcal{I}$ defined on the complex Hilbert space $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ and let $\rho \in \mathfrak{D}(\mathcal{H}_I)$ denote its state. We say that $\mathcal{I}$ is initialized in the subspace $\mathcal{H}_S$ with state $\sigma \in \mathfrak{D}(\mathcal{H}_S)$ if $\rho_S = \sigma$ and $\mathrm{tr}(\Pi_R \rho \Pi_R^\dagger) = 0$.

We denote by $\mathfrak{I}_S(\mathcal{H}_I)$ the set of density operators in $\mathfrak{D}(\mathcal{H}_I)$ satisfying this definition for some $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$.

Notice that with this definition the special case of pure-state initialization is addressed by taking the subspace $\mathcal{H}_S$ to be one-dimensional.

From the control point of view is natural to require that while the initializing-dynamics is enacted evolution of states in the $\mathfrak{I}_S(\mathcal{H}_I)$ stays confined to $\mathfrak{I}_S(\mathcal{H}_I)$ itself.

Definition 11 (Markovian invariant subspace)**.** Let evolution of $\mathcal{I}$ be governed by the QDS generator $\mathcal{L}$. We say that $\mathcal{H}_S$ is an *invariant subspace* of $\mathcal{H}_I$ if, under QDS evolution alone, $\mathfrak{I}_S(\mathcal{H}_I)$ is an invariant subset of $\mathfrak{D}(\mathcal{H}_I)$. That is, evolution of $\rho \in \mathfrak{D}(\mathcal{H}_I)$ obeys:

$$\mathcal{L}\begin{bmatrix}\rho_S & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathcal{L}_S\rho_S & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} \quad \forall t \geq 0, \tag{3.2}$$

where $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$ and $\mathcal{L}_S$ is required to be a QDS generator in its domain.

The just given definition imposes explicit algebraic constraints on the blocks of the operators describing the QDS.

Proposition 1. Consider a QDS with dynamics defined on the Hilbert space with decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Let the generator in the Lindblad form be described by an Hamiltonian H and the set of noise operators $\{L_k\}$. Then $\mathcal{H}_S$ is invariant if and only if:

$$\begin{cases} iH_P - \dfrac{1}{2}\displaystyle\sum_k L_{S,k}^\dagger L_{P,k} = \mathbf{0}, \\ L_k = \begin{bmatrix} L_{S,k} & L_{P,k}\\ \mathbf{0} & L_{R,k}\end{bmatrix}. \end{cases} \tag{3.3}$$

Proof. Consider an initial state $\rho \in \mathfrak{I}_S(\mathcal{H}_I)$. By explicitly computing the generator by blocks we find:

$$\dot{\rho} = \begin{bmatrix}\mathcal{L}_S(\rho) & \mathcal{L}_P(\rho)\\ \mathcal{L}_P(\rho)^\dagger & \mathcal{L}_R(\rho)\end{bmatrix} \tag{3.4}$$

[1] Indeed there are quantum algorithms which do not impose any requirements on the initial state.

where

$$
\begin{aligned}
\mathcal{L}_S(\rho) &= -i[H_S, \rho_S] + \sum_k L_S, k\rho_S L_{S,k}^\dagger \\
&\quad - \frac{1}{2}\sum_k \{L_{S,k}^\dagger L_{S,k} + L_{Q,k}^\dagger L_{Q,k}, \rho_S\}, \\
\mathcal{L}_P(\rho) &= i\rho_S H_P + \sum_k L_{S,k}\rho_S L_{Q,k}^\dagger \\
&\quad - \frac{1}{2}\sum_k \rho_S(L_{S,k}^\dagger L_{P,k} + L_{Q,k}^\dagger L_{R,k}), \\
\mathcal{L}_R(\rho) &= \sum_k L_{Q,k}\rho_S L_{Q,k}^\dagger.
\end{aligned}
\tag{3.5}
$$

And we require (3.2) to hold true. Since $\mathcal{L}_R(\rho)$ is positive at any time t and by exploiting the freedom of choice on ρ_S, i.e. we pick a full-rank density operator in $\mathfrak{D}(\mathcal{H}_S)$, we must conclude that $L_{Q,k} = \mathbf{0}\ \forall k$. Similarly for $\rho_P(t)$ we must impose:

$$iH_P - \frac{1}{2}\sum_k L_{S,k}^\dagger L_{P,k} = \mathbf{0} \tag{3.6}$$

This leaves a S-block of the form:

$$-i[H_{SF}, \rho_{SF}] + \sum_k L_{SF,k}\rho_{SF} L_{SF,k}^\dagger - \frac{1}{2}\sum_k \{L_{SF,k}^\dagger L_{SF,k}, \rho_{SF}\}, \tag{3.7}$$

which is a QDS generator acting on the subspace $\mathcal{H}_S$ alone. □

The following Corollary gives us some insight into the dynamics of invariant subspaces and motivates the definition of *attractive subspaces* which we present next.

Corollary 1. Consider the decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ with $\mathcal{H}_S$ being invariant. Let $\rho(t)$ denote the state of $\mathcal{I}$ at time t. Then, under QDS evolution alone, the trace of $\rho_S(t)$ is non-decreasing.

Proof. By explicit computation of the generator's blocks under the constraint that $\mathcal{H}_S$ be invariant we find:

$$\mathrm{tr}(\Pi_S\dot{\rho}) = \mathrm{tr}\big(\sum_k L_{P,k}\rho_R L_{P,k}^\dagger\big) \geq 0,\ \forall t \geq 0. \tag{3.8}$$

□

Definition 12 (Attractive subspace)**.** Consider a quantum system $\mathcal{I}$ with QDS dynamics defined on the complex Hilbert space $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Let the orthonormal set $\{|\bar{s}\rangle\}$ be a basis of $\mathcal{H}_S$, $\{|s\rangle\}$ its natural extension to $\mathcal{H}_I$, and denote with $\rho(t) \in \mathfrak{D}(\mathcal{H}_i)$ the state of $\mathcal{I}$ at time t. We say that $\mathcal{H}_S$ is *attractive* if the following relation holds for any initial state $\rho(t_0) \in \mathfrak{D}(\mathcal{H}_I)$:

$$\lim_{t\to\infty} \sum_i \langle s|\rho(t)|s\rangle = 1. \tag{3.9}$$

Then attractivity of $\mathcal{H}_S$ ensures that, under the given generator, all trajectories converge to $\mathfrak{I}_S(\mathcal{H}_I)$. Indeed we might think of attractive subspace as being "auto-initialized" in the long time limit.

In the particular case of a one-dimensional subspace $\mathcal{H}_S$, attractivity corresponds to the global stabilization of a pure state. Clearly then, sufficient and necessary algebraic conditions for attractivity become of great interest from the quantum control perspective. Before presenting the main result we introduce the following important Lemma which establishes a correspondence between invariant sets of states and subspaces of the systems's Hilbert space. Let use denote with $\text{supp}(X)$, $X \in \mathcal{B}(\mathcal{H})$, the orthogonal complement of $\ker(X)$.

Lemma 6. Let W be an invariant subset of $\mathfrak{D}(\mathcal{H}_I)$ for the dynamics generated by the QDS generator $\dot{\rho} = \mathcal{L}\rho$, and define:

$$\mathcal{H}_W = \text{supp}(W) = \bigcup_{\rho\in W} \text{supp}(\rho). \tag{3.10}$$

Then $\mathfrak{I}_W(\mathcal{H}_I)$ is an invariant set of such generator.

Proof. Let $\hat{W}$ be the convex hull of W. Any state $\hat{\rho} \in \hat{W}$ can be written in the form $\hat{\rho} = \sum_k p_k\rho_k$, where $0 < p_k \le 1$, $\sum_k p_k = 1$, and $\rho_1, \dots, \rho_k \in W$. By linearity of the dynamics,

$$\mathcal{T}_t\hat{\rho} = \sum_k p_k \mathcal{T}_t \rho_k = \sum_k p_k \rho'_k, \tag{3.11}$$

with $\rho'_1, \dots, \rho'_k \in W$. Hence $\hat{W}$ is invariant. Furthermore, from the definition of $\hat{W}$, there exist a $\bar{\rho} \in \hat{W}$ such that $\text{supp}(\bar{\rho}) = \text{supp}(\hat{W}) = \mathcal{H}_W$. Consider a basis transformation into the decomposition $\mathcal{H}_I = \mathcal{H}_W \oplus \mathcal{H}_W^\perp$. With respect to this partition the block $\bar{\rho}_\mathcal{W}$ of $\bar{\rho}$ is full-rank, while $\bar{\rho}_{P,Q,R}$ are zero-blocks. The trajectory of $\bar{\rho}(t), t \ge 0$ is contained in $\hat{W}$ only if:

$$\frac{d}{dt}\bar{\rho} = \begin{bmatrix} \mathcal{L}_W\bar{\rho}_W & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{3.12}$$

so that, upon computing the generator blocks in the new basis, we must impose:

$$\begin{cases} i\bar{\rho}_W H_P + \sum_k \left(L_{W,k}\bar{\rho}_W L_{Q,k}^\dagger - \frac{1}{2}\bar{\rho}_W L_{W,k}^\dagger L_{P,k} \right) = \mathbf{0}, \\ \sum_k L_{Q,k}\bar{\rho}_W L_{Q,k}^\dagger = \mathbf{0}. \end{cases} \tag{3.13}$$

Since the operators $L_{Q,k}\bar{\rho}_W L_{Q,k}^\dagger$ are positive it must be:

$$\begin{cases} iH_P - \frac{1}{2}\sum_k L_{W,k}^\dagger L_{P,k} = \mathbf{0}, \\ L_{Q,k} = \mathbf{0}, \quad \forall k. \end{cases} \tag{3.14}$$

and by comparison with the conditions given in Proposition 1, we infer that the subspace $\mathcal{H}_W$ is invariant. □

Theorem 8 (Subspace attractivity)**.** Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, and assume that $\mathcal{H}_S$ is an invariant subspace for the QDS dynamics generated by a master equation in the Lindblad form. Define:

$$\mathcal{H}_{R'} = \bigcap_{k=1}^{p} \ker(L_{P,k}), \tag{3.15}$$

Then $\mathcal{H}_S$ is an attractive subspace if and only if $\mathcal{H}_{R'}$ does not support any invariant subspace.

Proof. Clearly, if $\mathcal{H}_{R'}$ supports an invariant subspace then $\mathcal{H}_S$ cannot be attractive.

To prove the other implication recall that $\mathcal{H}_S$ is invariant by hypothesis, and as such evolution of the state's R-block is decoupled and independent. Furthermore, in this case, Definition 12 is satisfied if and only if $\lim_{t\to\infty} \rho_R(t) = \mathbf{0}$. Consider the Lyapunov function $V(\rho_R) = \mathrm{tr}(\rho_R)$; it is positive-definite in any neighborhood of $\mathbf{0}$. By explicit computation we find:

$$\dot{V}(\rho_R) = -\mathrm{tr}(\sum_k L_{P,k}\rho_R L_{P,k}^\dagger). \tag{3.16}$$

$\dot{V}(\rho)$ is negative and vanishes if an only if $\mathrm{supp}(\rho_R) \subseteq \mathcal{H}_{R'}$, with $\mathcal{H}_{R'}$ as defined in (3.15). Hence, by Lyapunov second Theorem on stability, $\rho_R = \mathbf{0}$ is at least Lyapunov stable ([11]). Furthermore by Krasovskii stability Theorem, $\rho_R = \mathbf{0}$ is asymptotically stable if an only if $\mathcal{H}_{R'}$ does not support any invariant set of states ([11]). By Lemma 6 a proper invariant subspace is naturally associated to any non trivial invariant set of states, hence we conclude. □

Theorem 8 fully characterizes subspace attractive generators and it is the most important result we present with respect to the control of quantum states for systems with the mentioned decomposition.

We next provide specific results building mostly on Theorem 8 and focusing on two practical aspects: the ability to infer attractive dynamics by Hamiltonian control alone under some constraints on the noise operators and the stabilization of subspaces in the peculiar case of feedback control.

Theorem 9 (Open-loop attractive subspaces)**.** Consider a quantum system $\mathcal{I}$ defined on the complex Hilbert space $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Let the generator be described by an Hamiltonian H and noise operators $\{L_k\}$. Then it is always possible to render $\mathcal{H}_S$ attractive by Hamiltonian control alone if and only if $L_{Q,k} = \mathbf{0}\ \forall k$ and for at least one of the noise operators holds $L_{P,k} \neq \mathbf{0}$.

Proof. We first assert invariance. By Proposition 1, $\mathcal{H}_S$ supports an invariant subspace if and only if:

$$\begin{cases} iH_P - \dfrac{1}{2}\displaystyle\sum_k L_{S,k}^\dagger L_{P,k} = \mathbf{0}. \\ L_{Q,k} = \mathbf{0}, \quad \forall\, k \end{cases} \tag{3.17}$$

Since an arbitrary control Hamiltonian $\hat{H}$ may be applied, the first condition can always be satisfied by choosing $\hat{H}_P = -iH_P + \frac{1}{2}\sum_k L_{S,k}^\dagger L_{P,k}$, while the second condition always is by hypothesis.

In order to engineer subspace attractivity we observe that if $L_{P,k} = \mathbf{0}\ \forall k$ then $\mathcal{H}_R$ is invariant and thus $\mathcal{H}_S$ cannot be attractive. Suppose instead that at least one of the $L_{P,k} \neq \mathbf{0}$. Let $\mathcal{H}_T$ denote the invariant subspace associated to the set of invariant states in $\mathfrak{I}_R(\mathcal{H}_I)$ and consider the following Hilbert space decomposition:

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_T \oplus \mathcal{H}_Z. \tag{3.18}$$

By requiring that both $\mathcal{H}_S$ and $\mathcal{H}_T$ be invariant, the generator matrices are constrained to the form:

$$\begin{aligned} L_k &= \begin{bmatrix} L_{S,k} & \mathbf{0} & L_{P',k} \\ \mathbf{0} & L_{T,k} & L_{P'',k} \\ \mathbf{0} & \mathbf{0} & L_{Z,k} \end{bmatrix} \\ H &= \begin{bmatrix} H_S & \mathbf{0} & H_{P'} \\ \mathbf{0} & H_T & H_{P''} \\ H_{P'}^\dagger & H_{P''}^\dagger & H_Z \end{bmatrix} \end{aligned} \tag{3.19}$$

and subject to the conditions:

$$\begin{cases} iH_{P'} - \dfrac{1}{2}\displaystyle\sum_k L_{S,k}^\dagger L_{P',k} = \mathbf{0}, \\ iH_{P''} - \dfrac{1}{2}\displaystyle\sum_k L_{T,k}^\dagger L_{P'',k} = \mathbf{0}. \end{cases} \tag{3.20}$$

Thus the most general Hamiltonian control preserving invariance of $\mathcal{H}_S$ has the form:

$$H = \begin{bmatrix} H_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H_2 & H_M \\ \mathbf{0} & H_M^\dagger & H_3 \end{bmatrix}. \tag{3.21}$$

Now consider a control Hamiltonian H_c such that the block H_M has full column-rank, while H_1, H_3 are arbitrary and H_2 is still to be determined. If $\dim(\mathcal{H}_T) \leq \frac{1}{2}\dim(\mathcal{H}_R)$, then $i\rho_T H_M \neq 0$ for every ρ_T, hence $\mathcal{H}_T$ cannot support any invariant subsystem, since conditions in Proposition 1 cannot be satisfied for any subspace of $\mathcal{H}_T$. Conversely, if $\dim(\mathcal{H}_T) > \frac{1}{2}\dim(\mathcal{H}_R)$, by dimension comparison H_M must have a non-trivial left kernel $\mathcal{K}$, $\mathcal{K}H_M = \mathbf{0}$, and thus there exists a $\mathcal{H}_{T'} \subset \mathcal{K}$ that supports an invariant set $\mathfrak{I}_{T'}(\mathcal{H}_I)$, such that $\dim(\mathcal{H}_{T'}) < \dim(\mathcal{H}_T)$. Consider the refined decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_{T'} \oplus \mathcal{H}_{Z'}$, where $\mathcal{H}_{Z'} = \mathcal{H}_Z \oplus \mathcal{H}_T \ominus \mathcal{H}_{T'}$. We may then exploit the freedom on the block H_2 to reduce the dimension of the invariant set. By repeating our reasoning, at each iteration, the procedure either stops rendering $\mathcal{H}_S$ attractive, if $\dim(\mathcal{H}_T) \leq \frac{1}{2}\dim(\mathcal{H}_R)$, or decreases the dimension of the invariant set by at least 1 thus ending in at most $\dim(\mathcal{H}_T)$ steps. □

The potential of open-loop Hamiltonian control is clearly limited by the impossibility to tune the noise parameter. Indeed the proof of Theorem 9 the control Hamiltonian is limited to breaking subspace invariance. Furthermore since Hamiltonian terms do not play a role in (3.8) convergence dynamics are necessarily bounded by the form of the noise operators for any control Hamiltonian.

Such limitations can be overcome by considering measurement-based feedback control techniques. Next we apply the presented results to the case of a generator with the form of a feedback master equation as in (2.17). A general result is found by assuming to have strong control capabilities on both the Hamiltonian term and the feedback operator.

Definition 13 (CHC)**.** We say that a controlled FME of the form (2.17) supports *complete Hamiltonian control* (CHC) if

1. arbitrary feedback Hamiltonians $F \in \mathfrak{h}(\mathcal{H}_I)$ may be enacted;
2. arbitrary constant control perturbations $H_c \in \mathfrak{h}(\mathcal{H}_I)$ may be added to the free Hamiltonian H.

Theorem 10 (Feedback attractive subspaces)**.** Let $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$ and assume CHC capabilities. Then, for any measurement operator M, there exist a feedback Hamiltonian F and a Hamiltonian compensation H_c that make the subspace $\mathcal{H}_S$ attractive for the dynamics generated a feedback master equation if and only if

$$[\Pi_S, (M + M^\dagger)] \neq \mathbf{0}. \tag{3.22}$$

Proof. Since it always is possible to write the measurement operator as $M = M^H + iM^A$ with both M^H and $M^A \in \mathfrak{h}(\mathcal{H}_I)$ we have $L = M^H + i(M^A - F)$. By requiring invariance of $\mathcal{H}_S$ we find:

$$L_Q = M_Q^H + i(M_Q^A - F_Q) = \mathbf{0}, \tag{3.23}$$

and by exploiting the freedom on F we might always choose

$$F_Q = -iM_Q^H + M_Q^A. \tag{3.24}$$

Thus by Hermitian symmetry we find $L_P = 2M_P^H$. Then, if $M_P^H \neq \mathbf{0}$ the subspace $\mathcal{H}_S$ can be made invariant and attractive by Hamiltonian control alone following the procedure outlined in the proof of Theorem 9. Otherwise, $M_P^H = \mathbf{0}$, the subspace $\mathcal{H}_R$ becomes invariant and thus $\mathcal{H}_S$ cannot be attractive.

Since (3.22) holds if and only if M^H is not block diagonal and by observing that the commutator of two matrices vanishes if and only if it vanishes for any choice of basis we conclude. □

3.2 Quantum subsystems

Quantum subsystems are the second type of information preserving structure we consider in this chapter. Indeed quantum subsystems provide a scalable way to perform quantum information processing in physical systems. Systems build by coupled replicas of a given quantum system are studied in the framework of Quantum Error Correcting Codes as the most suited structure for the faithful representation of information in physical systems ([12], [13]). Furthermore relevant situations might be devised where *noiseless quantum subsystem* can be exploited to preserve information against noise in the absence of noiseless subspaces ([14], [15]).

In this context a suitable definition of quantum subsystem is the following:

Definition 14 (Quantum Subsystem)**.** A quantum subsystem of a physical system $\mathcal{I}$ with associated Hilbert space $\mathcal{H}_I$ is an Hilbert space $\mathcal{H}_S$ being a tensor factor of a subspace $\mathcal{H}_{SF}$ of $\mathcal{H}_I$,

$$\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R, \tag{3.25}$$

for some cofactor $\mathcal{H}_F$ and remainder space $\mathcal{H}_R$.

In this section we will consider quantum system with associated Hilbert space admitting a decomposition as in (3.25). As before let $\{|s\rangle\}_{s\in S}$, $\{|f\rangle\}_{f\in F}$ and $\{|r\rangle\}_{r\in R}$ be orthonormal bases for $\mathcal{H}_S$, $\mathcal{H}_F$ and $\mathcal{H}_R$ respectively. A convenient basis of $\mathcal{H}_I$ is given by the set $\{|s\rangle \otimes |f\rangle\}_{(s,f)\in S\times F} \oplus_{r\in R} \{|r\rangle\}$. This choice induces the following block-structure on the matrix representation of any operator $\mathcal{X} \in \mathcal{B}(\mathcal{H}_I)$:

$$X = \begin{bmatrix} X_{SF} & X_P \\ X_Q & X_R \end{bmatrix}. \tag{3.26}$$

Much of the definitions and results we gave in the previous section are easily specialized to the case of quantum subsystems.

Definition 15 (State subsystem initialization)**.** Consider a quantum system $\mathcal{I}$ defined on $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R$ and let $\rho \in \mathfrak{D}(\mathcal{H}_I)$ denote the its state. We say that $\mathcal{I}$ is initialized in subsystem $\mathcal{S}$ with state $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$ if the blocks of ρ satisfy:

i) $\rho_{SF} = \rho_S \otimes \rho_F$ for some $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$,

ii) $\mathrm{tr}(\rho_R) = 0$.

We denote by $\mathfrak{I}_S(\mathcal{H}_I)$ the set of density operators in $\mathfrak{D}(\mathcal{H}_I)$ satisfying this definition for some $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$.

We keep the notation we already introduced, since the concepts are similar and discrimination between the two cases is already provided by the different Hilbert space decomposition we consider here.

The counterpart of invariant and attractive subspaces arise naturally in the form of invariant and attractive subsystems.

Definition 16 (Markovian invariant subsystems)**.** Let $\mathcal{I}$ evolve under TPCP maps, we say that $\mathcal{S}$ is an invariant subsystem if $\mathfrak{I}_S(\mathcal{H}_I)$ is an invariant subset of $\mathfrak{D}(\mathcal{H}_I)$, that is, evolution of $\rho \in \mathfrak{D}(\mathcal{H}_I)$ obeys:

$$\rho(t) = \begin{bmatrix} \mathcal{T}_t^S \rho_S \otimes \mathcal{T}_t^F \rho_F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \ \forall t \geq 0, \tag{3.27}$$

where $\mathcal{T}_t^S$ and $\mathcal{T}_t^F$ are required to be QDSs on their respective domain.

As we might expect invariant subsystems impose stricter algebraic conditions on the Hamiltonian term and the noise operators than subspace invariance does.

Proposition 2 (Markovian invariant subsystem)**.** Consider a QDS with dynamics defined on the Hilbert space with decomposition $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R$. Then $\mathcal{S}$ is an invariant subsystem for the given generator if and only if for any initial state $\rho_S \otimes \rho_F$, with $\rho_S \in \mathfrak{D}(\mathcal{H}_S)$ and $\rho_F \in \mathfrak{D}(\mathcal{H}_F)$, the following conditions hold:

$$\begin{cases} \dot\rho(t) = \begin{bmatrix} \mathcal{L}_{SF}\rho_{SF} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \ \forall t \geq 0 \\ \mathrm{tr}_F(\mathcal{L}_{SF}\rho_{SF}(t)) = \mathcal{L}_S(\rho_S(t)), \ \ \forall t \geq 0 \end{cases} \tag{3.28}$$

where $\mathcal{L}_{SF}$ and $\mathcal{L}_S$ are QDS generators on their respective domains.

Proof. By definition of invariant subsystem an by the properties of the infinitesimal generator $\mathcal{L}$ it must hold:

$$\dot\rho(0) = \begin{bmatrix} (\mathcal{L}_S \otimes \mathbf{1}_F + \mathbf{1}_S \otimes \mathcal{L}_F)\rho_S \otimes \rho_F & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{3.29}$$

Thus, by taking the trace over $\mathcal{F}$, it is easily seen that both conditions in (3.28) hold.

To prove the converse let the generator be described, in the Lindblad form, by an Hamiltonian H and the set $\{L_k\}$ of noise operators. Then by straightforward specialization of Proposition 1 we have:

$$\begin{cases} iH_P - \frac{1}{2}\sum_k L_{SF,k}^\dagger L_{P,k} = \mathbf{0} \\ L_k = \begin{bmatrix} L_{SF,k} & L_{P,k} \\ \mathbf{0} & L_{R,k} \end{bmatrix} \end{cases} \tag{3.30}$$

Furthermore by definition of invariant subsystem we require that evolution of subsystem $\mathcal{S}$ be decoupled from that of $\mathcal{F}$, that is, ρ_{SF} is factorisable at any time $t \geq 0$ and such that $\rho_S = \mathrm{tr}_F(\rho_S F)$ and $\rho_F = \mathrm{tr}_S(\rho_S F)$. Rewrite the SF-block of H as $H_{SF} = \sum_i N_i \otimes M_i$. By recalling (3.7), for the Hamiltonian part of the generator we have:

$$-i[H_{SF}, \rho_{SF}] = -i\sum_i M_i\rho_S \otimes N_i\rho_F - \rho_S M_i \otimes \rho_F N_i, \tag{3.31}$$

and by taking the partial trace over $\mathcal{F}$ we find

$$\mathrm{tr}_F(-i[H_{SF}, \rho_{SF}]) = -i\sum_i [M_i, \rho_S]\mathrm{tr}(N_i\rho_F). \tag{3.32}$$

Since we require that ρ_S be independent of ρ_F, we find that for each i either $\mathrm{tr}(N_i\rho_F) = 1$ or $[M_i, \rho_S] = \mathbf{0}$. Thus by the freedom of choice on the states ρ_S and ρ_F the following relation must hold:

$$H = H_S \otimes \mathbf{1}_F + \mathbf{1}_S \otimes H_F. \tag{3.33}$$

By re-iterating this procedure on the noise operators we find:

$$L_{SF,k} = L_{S,k} \otimes L_{F,k} \tag{3.34}$$

where for each k either $L_{S,k} = \mathbf{1}_S$ or $L_{F,k} = \mathbf{1}_F$ or both. Thus the generator must have the declared form. □

The proof of Proposition 2 lends us an explicit algebraic characterization of subsystem invariant generators.

Corollary 2. Consider a QDS with dynamics defined on the Hilbert space with decomposition $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R$. Then $\mathcal{S}$ is an invariant subsystem if and only if the generator, in the Lindblad form, has the following algebraic structure:

$$\begin{cases} iH_P - \frac{1}{2}\sum_k L_{SF,k}^\dagger L_{P,k} = \mathbf{0} \\ L_k = \begin{bmatrix} L_{SF,k} & L_{P,k} \\ \mathbf{0} & L_{R,k} \end{bmatrix} \\ H_{SF} = H_S \otimes \mathbf{1}_F + \mathbf{1}_S \otimes H_F \\ L_{SF,k} = L_{S,k} \otimes L_{F,k} \end{cases} \tag{3.35}$$

where for each k either $L_{S,k} = \mathbf{1}_S$ or $L_{F,k} = \mathbf{1}_F$ or both.

Definition 17 (Attractive subsystem)**.** Consider a quantum system $\mathcal{I}$ defined on $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R$. We say that $\mathcal{S}$ is an *attractive subsystem* with respect to a family $\{\mathcal{T}_t\}_{t\geq 0}$ of TPCP maps if every trajectory in $\mathfrak{D}(\mathcal{H}_I)$ converges asymptotically to $\mathfrak{I}_S(\mathcal{H}_I)$:

$$\begin{cases} \lim_{t\to\infty}\left(\mathcal{T}_t\rho - \begin{bmatrix} \bar\rho_S(t)\otimes\bar\rho_F(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right) = \mathbf{0} \\ \bar\rho_S(t) = \mathrm{tr}_F[\bar\Pi_{SF}\mathcal{T}_t\rho\bar\Pi_{SF}^\dagger] \\ \bar\rho_F(t) = \mathrm{tr}_S[\bar\Pi_{SF}\mathcal{T}_t\rho\bar\Pi_{SF}^\dagger]. \end{cases} \tag{3.36}$$

An algebraic characterization of subsystem attractive generators is not obtained as easily as before. Some partial results are in order as follows.

Proposition 3. Assume $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$ ($\mathcal{H}_R = 0$), and let $\mathcal{H}_S$ be invariant under a QDS of the form

$$\mathcal{L} = \mathcal{L}_S \otimes \mathbf{1}_F + \mathbf{1}_S \otimes \mathcal{L}_F. \tag{3.37}$$

If $\mathcal{L}_F$ has a unique attractive state $\hat\rho_F$, then $\mathcal{H}_S$ is attractive.

Proof. Let $\rho \in \mathcal{H}_I$. We may always write $\rho = \sum_i P_i \otimes Q_i$ and decompose the Q_i in their Hermitian and anti-Hermitian parts, $Q_i = Q_i^H + iQ_i^A$. Then for each of $Q_i^{H,A}$ consider their spectral representation, separate the positive and negative eigenvalues such that $Q_i = Q_i^+ + Q_i^-$ and normalize Q_i^+ and Q_i^- to trace 1, -1, respectively; reabsorb the normalization coefficients and the minus sign, in P_i. Thus, we can write $\rho = \sum_i \tilde P_i \otimes \rho_{F,i},$. By applying the given generator we find:

$$\begin{aligned} \lim_{t\to\infty}\rho_t &= \sum_i \lim_{t\to\infty}\left(\mathcal{T}_S^t\tilde P_i \otimes \mathcal{T}_F^t\rho_{F,i}\right) \\ &= \big(\sum_i \lim_{t\to\infty}\mathcal{T}_S^t\tilde P_i\big)\otimes\hat\rho_F \\ &= \lim_{t\to\infty}\mathcal{T}_S^t(\bar\rho_{NS})\otimes\hat\rho_F. \end{aligned} \tag{3.38}$$

□

Interestingly it turns out that uniqueness of the attractive state for the dynamics reduced to $\mathcal{H}_F$ constitutes a necessary condition for subsystem attractivity of $\mathcal{S}$.

Proposition 4. Assume $\mathcal{H}_I = \mathcal{H}_S \otimes \mathcal{H}_F$ ($\mathcal{H}_R = 0$), and let $\mathcal{S}$ be invariant under a QDS generator of the form:

$$\mathcal{L} = \mathcal{L}_S \otimes \mathbf{1}_F + \mathbf{1}_S \otimes \mathcal{L}_F. \tag{3.39}$$

If $\mathcal{L}_F$ admits at least two invariant states, then $\mathcal{S}$ is not attractive.

Proof. A proof is given by counter example. Consider the state in $\mathfrak{D}(\mathcal{H}_I)$:

$$\rho = p\rho_S^{(1)} \otimes \rho_F^{(1)} + (1-p)\rho_S^{(2)} \otimes \rho_F^{(2)}, \;\; 0 < p < 1 \tag{3.40}$$

where $\rho_S^{(1)}, \rho_S^{(2)}$ are orthogonal pure states on $\mathcal{H}_S$, and $\rho_F^{(1)}, \rho_F^{(2)}$ are two invariant states of $\mathcal{L}_F$. By using the linearity of the generator we have:

$$\begin{aligned} \rho(t) &= p\mathcal{T}_S\rho_S^{(1)} \otimes \mathcal{T}_F\rho_F^{(1)} + (1-p)\mathcal{T}_S\rho_S^{(2)} \otimes \mathcal{T}_F\rho_F^{(2)} \\ &= pU_S(t)\rho_S^{(1)}U_S^\dagger(t) \otimes \rho_F^{(1)} + (1-p)U_S(t)\rho_S^{(2)}U_S^\dagger(t) \otimes \rho_F^{(2)}, \end{aligned} \tag{3.41}$$

and thus $\rho(t)$ does not factorize for any $t \geq 0$. □

While we are not able to provide further results on subsystem attractivity in the open-loop case, a complete characterization can be given in the case of a generator in the case of the feedback master equation.

Theorem 11 (Feedback attractive subsystems)**.** Let $\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = \mathcal{H}_S \otimes \mathcal{H}_F \oplus \mathcal{H}_R$ ($\dim(\mathcal{H}_S), \dim(\mathcal{H}_F) \geq 2$), and assume CHC capabilities. Then for any M there exist a feedback Hamiltonian F and an Hamiltonian compensation H_c that make the subsystem $\mathcal{S}$ attractive for the FME (2.17) if and only if the following conditions hold:

i) $[\Pi_{SF}, M + M^\dagger] \neq 0$,

ii) $M_{SF} = M_S \otimes \mathbf{1}_F$ or $M_{SF} = \mathbf{1}_S \otimes M_F$,

iii) The matrix $M_{SF} + M_{SF}^\dagger$ is not diagonal.

Proof. By Theorem 10, condition (i) is necessary and sufficient to render $\mathcal{H}_{SF}$ attractive, which is a necessary condition for attractivity of $\mathcal{S}$. To ensure invariance of $\mathfrak{I}_S(\mathcal{H}_I)$, by Proposition 2, the block L_{SF} of $L = M - iF$ has to satisfy $L_{SF} = L_S \otimes L_F$, with $L_S = \mathbf{1}_S$ or $L_F = \mathbf{1}_F$ (or both). Then, this holds in particular for M_{SF} and thus (ii) follows.

If (iii) is not satisfied then L_{SF} is unitarily similar to a diagonal matrix for any choice of feedback-operator F that ensures invariance of $\mathfrak{I}_S(\mathcal{H}_I)$. Hence, the dynamics restricted to $\mathcal{H}_F$ admits at least two different steady states ($\dim(\mathcal{H}_F) \geq 2$ by hypothesis) and by Lemma 4 subsystem $\mathcal{S}$ cannot be attractive.

Now assume (i) and (iii) and let $M_{SF} = \mathbf{1}_S \otimes M_F$ (the other case may be treated specularly). Notice that neither the invariance of $\mathcal{S}$ nor attractivity of $\mathcal{H}_{SF}$ require any tuning of the SF-blocks in H or F. Since (iii) holds there exist a state $|f\rangle \in \mathcal{H}_F$ such that $[|f\rangle\langle f|, M_{SF} + M_{SF}^\dagger] \neq \mathbf{0}$. By Theorem 10 we can find an Hamiltonian term H_F and a feedback operator F_F such that by choosing $H_{SF} = \mathbf{1}_S \otimes H_F$ and $F_{SF} = \mathbf{1}_S \otimes F_F$, the subspace $\mathcal{H}_S \otimes \mathrm{span}(|f\rangle)$ is attractive. We conclude by recalling Proposition 3. □

3.3 Convergence dynamics

In this section we provide further results dealing with the convergence dynamics to attractive subspaces. Before proceeding we recall the definition of vectorization and some of its properties.

Definition 18. Consider the n by m matrix $A = (a_{i,j})_{(i,j)\in\{1\dots n\}\times\{1\dots m\}}$. The vectorization of A, $\mathrm{vec}(A)$, is defined as the (linear) transformation mapping matrices to vectors in the following way:

$$\mathrm{vec}(A) = \begin{bmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} & a_{1,2} & \dots & a_{n,2} & \dots & a_{n,m} \end{bmatrix}^T \tag{3.42}$$

Vectorization is a powerful tool when used to express matrix multiplications as linear transformations acting on vectors.

Lemma 7. For any matrices X, Y and Z such that their composition XYZ is well defined the following relation holds:

$$\mathrm{vec}(XYZ) = (Z^T \otimes X)\mathrm{vec}(Y). \tag{3.43}$$

The following Theorem exploits vectorization to provide sufficient and necessary conditions for subspace attractivity. This is done by recasting the QDS generator into a LTI state-space model and by exploiting basic notions of control theory.

Theorem 12 (Attractive quantum subspaces)**.** Consider a quantum system $\mathcal{I}$ defined on an Hilbert space with decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Denote the state of $\mathcal{I}$ with $\rho \in \mathfrak{D}(\mathcal{H}_I)$ and let $\mathcal{H}_S$ be an invariant subspace for the generated dynamics. Under this constraint, the linear map governing the evolution of $\mathrm{vec}(\rho_R)$ alone is

$$\begin{aligned} Z = &-\frac{i}{\hbar}\left(\mathbf{1}_R \otimes H_R - H_R^T \otimes \mathbf{1}_R\right) + \sum_k L_{R,k}^* \otimes L_{R,k} \\ &- \frac{1}{2}\sum_k \mathbf{1}_R \otimes (L_{P,k}^\dagger L_{P,k} + L_{R,k}^\dagger L_{R,k}) \\ &- \frac{1}{2}\sum_k (L_{P,k}^T L_{P,k}^* + L_{R,k}^T L_{R,k}^*) \otimes \mathbf{1}_R. \end{aligned} \tag{3.44}$$

and $\mathcal{H}_S$ is attractive if and only if Z has an eigenvalue in 0.

Before presenting the proof of Theorem 12 we need to introduce the following trivial lemmas.

Lemma 8. Let $\mathcal{L}$ be a QDS generator defined on the complex Hilbert space $\mathcal{H}$ and be $\hat{\mathcal{L}}$ its "vectorization", that is:

$$\hat{\mathcal{L}}\text{vec}(X) = \text{vec}(\mathcal{L}X) \;\; \forall X \in \mathcal{B}(\mathcal{H}) \;\wedge\; t \geq 0. \tag{3.45}$$

Then (λ, X) with $X \in \mathcal{B}(\mathcal{H})$ is an eigenpair of $\mathcal{L}$ if and only if $(\lambda, \text{vec}(X))$ is an eigenpair of $\hat{\mathcal{L}}$.

The proof by contradiction is trivial; since (3.45) holds at any time $t \geq 0$, both implications arise naturally.

Lemma 9. Let $\mathcal{L}$ be a QDS generator defined on the complex Hilbert space $\mathcal{H}$. Then for any operator $X \in \mathcal{B}(\mathcal{H})$ the following identity holds:

$$(\mathcal{L}X)^\dagger = \mathcal{L}(X^\dagger). \tag{3.46}$$

Proof. By explicit computation of the generator we find:

$$\begin{aligned}(\mathcal{L}X)^\dagger &= +i(X^\dagger H - HX^\dagger) + \sum_k \left(L_k X^\dagger L_k^\dagger - \frac{1}{2}\{X^\dagger, L_k^\dagger L_k\}\right) \\ &= -i[H, X^\dagger] + \sum_k \left(L_k X^\dagger L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, X^\dagger\}\right) \\ &= \mathcal{L}X^\dagger\end{aligned} \tag{3.47}$$

□

Lemma 10. Let $\mathcal{L}$ be a QDS generator defined on the complex Hilbert space $\mathcal{H}$. For any eigenpair (λ, X), $X \in \mathcal{B}(\mathcal{H})$, there exist an Hermitian operator Y such that (λ, Y) is an eigenpair of $\mathcal{L}$ only if $\lambda \in \mathbb{R}$.

Proof. Let (λ, X) be an eigenpair of $\mathcal{L}$. We look for an operator $Y \in \mathfrak{h}(\mathcal{H})$ such that (λ, Y) is an eigenpair of $\mathcal{L}$. If X is already Hermitian or anti-Hermitian then such Y exists and is trivial.

In the general case write $X = X^H + iX^A$ where both X^H and X^A are Hermitian operators. By Lemma 9 and using linearity we find:

$$\begin{aligned}\mathcal{L}(X + X^\dagger) &= \lambda X + \lambda^* X^\dagger \\ &= \lambda(X^H + iX^A) + \lambda^*(X^H - iX^A)\end{aligned} \tag{3.48}$$

Now let $\lambda = a + ib$, with $a, b \in \mathbb{R}$. Then :

$$\mathcal{L}(X + X^\dagger) = \mathcal{L}(2X^H) = 2aX^H - 2bX^A. \tag{3.49}$$

Since at this point it must be $X^{H,A} \neq \mathbf{0}$, we find that if $\lambda \in \mathbb{R}$, the pair (λ, X^H) is an eigenpair of $\mathcal{L}$, hence we conclude. □

We are now in the position to prove Theorem 12.

Proof. By explicitly computing the blocks it's easily found that the dynamics of $\rho_R = \bar{\Pi}_R \rho \bar{\Pi}_R^\dagger$ are governed by:

$$\begin{aligned}\dot{\rho}_R = &- i([H_R, \rho_R] + H_P^\dagger \rho_P - \rho_P^\dagger H_P) \\ &+ \sum_k (L_{Q,k}\rho_S + L_{R,k}\rho_P^\dagger)L_{Q,k}^\dagger + (L_{Q,k}\rho_P + L_{R,k}\rho_R)L_{R,k}^\dagger \\ &- \frac{1}{2}\sum_k (L_{P,k}^\dagger L_{S,k} + L_{R,k}L_{Q,k})\rho_P + \rho_P^\dagger(L_{S,k}^\dagger L_{P,k} + L_{Q,k}^\dagger L_{R,k}) \\ &- \frac{1}{2}\sum_k \left\{L_{P,k}^\dagger L_{P,k} + L_{R,k}^\dagger L_{R,k}, \rho_R\right\}.\end{aligned} \tag{3.50}$$

Since $\mathcal{H}_S$ is invariant by hypothesis, by substituting (3.35) in the previous expression we find:

$$\begin{aligned}\dot{\rho}_R = &-i[H_R, \rho_R] + \sum_k L_{R,k}\rho_R L_{R,k}^\dagger \\ &- \frac{1}{2}\sum_k \left\{L_{P,k}^\dagger L_{P,k} + L_{R,k}^\dagger L_{R,k}, \rho_R\right\}.\end{aligned} \tag{3.51}$$

Now let $\hat{\rho}_R = \mathrm{vec}\left(\bar{\Pi}_R \rho \bar{\Pi}_R^\dagger\right)$. By exploiting (3.43) we find $\dot{\hat{\rho}}_R = Z\hat{\rho}_R$, where Z is the map defined in (3.44).

If $(0, \mathrm{vec}(X))$, $X \in \mathcal{B}(\mathcal{H}_R)$, is an eigenpair of Z then by Lemma 8 we have that $(0, X)$ is an eigenpair of $\mathcal{L}_R$. Notice that $\mathrm{vec}(X) \neq \mathbf{0}$ by definition of eigenvector. Then by Lemma 10, we can always find a non trivial Hermitian operator Y with support on $\mathcal{H}_R$ such that $\mathcal{L}Y = \mathbf{0}$. Since any initial state ρ_R with non vanishing projection on such state cannot converge to $\mathbf{0}$ we conclude that $\mathcal{H}_S$ is not attractive.

To prove the converse, let the state of $\mathcal{I}$ be described by $\rho \in \mathfrak{D}(\mathcal{H}_I)$ such that $\bar{\Pi}_R \rho \bar{\Pi}_R^\dagger$ is non trivial. Since $\mathcal{H}_R$ supports an invariant subspace, $\dot{\rho}_R$ maps a compact convex set of non-trace normalized density-operators into itself. Thus by Brouwer's fixed point Theorem such a map has at least one fixed point. By rephrasing in control theory terminology, $\dot{\rho}_R$ has at least one non trivial steady state. □

Building on Theorem 12 the following result gives an asymptotic worst-case bound on the convergence time.

Corollary 3 (Asymptotic convergence bound)**.** Consider a quantum system $\mathcal{I}$ defined on an Hilbert space with decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Let $\mathcal{H}_S$ be an attractive subspace for the given QDS generator. Then, any state $\rho \in \mathfrak{D}(\mathcal{H}_I)$ is converges to a state with support on $\mathcal{H}_S$ asymptotically bounded by $e^{z_0 t}$, where $z_0 := \max_\lambda\{\lambda \in \delta_Z(\lambda) \wedge \lambda \in \mathbb{R}\}$.

Still the dimension of Z grows exponentially with that of $\mathcal{H}_R$. A partial solution to this problem is to discard the redundant information stored in ρ_R by considering only its upper-diagonal terms.

Definition 19. Consider the n-dimensional square matrix $A = (a_{i,j})$. The half-vectorization of A, $\text{hvec}(A)$, is defined as the (linear) transformation mapping matrices to vectors in the following way:

$$\text{hvec}(A) = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{2,2} & a_{1,3} & \dots & a_{3,3} & \dots & a_{1,n} & \dots & a_{n,n} \end{bmatrix}^T \tag{3.52}$$

We may think of half-vectorization as the "full" vectorization where under-diagonal elements are simply skipped. For symmetric matrices, vectorization and half-vectorization can be mapped into each-oder by making use of the elimination and the duplication matrices. In the following we extend these concepts to Hermitian matrices. Before doing so however we need to introduce a *conjugation operator*.

Definition 20 (Conjugation operator)**.** Define $\mathfrak{i} : \mathbb{C} \to \mathbb{C}$ as the linear function mapping an arbitrary element of its domain λ into its complex conjugate λ^*.

Definition 21. Let A be the matrix representation of an hermitian operator acting on an n-dimensional complex Hilbert space. The $n^2 \times n(n+1)/2$ dimensional duplication matrix $\mathfrak{h}_d$ and the $n(n+1)/2 \times n^2$ dimensional elimination matrix $\mathfrak{h}_e$ are the unique linear functionals such that $\mathfrak{h}_d \text{hvec}(A) = \text{vec}(A)$ and $\mathfrak{h}_e \text{vec}(A) = \text{hvec}(A)$.

As an example consider the 2-dimensional complex Hilbert space $\mathcal{H}$. Then the duplication and elimination matrices, acting on the vectorization and half-vectorization respectively, of operators in $\mathfrak{h}(\mathcal{H})$ are given by:

$$\mathfrak{h}_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathfrak{i} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathfrak{h}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.53}$$

Then, the reduced Z matrix governing evolution of the upper-diagonal elements of ρ_R alone is easily found to be:

$$Z' = \mathfrak{h}_e Z \mathfrak{h}_d. \tag{3.54}$$

Clearly Theorem 12 and Corollary 3 still hold when substituting Z'. Still the reduced complexity might help in both symbolic and numerical resolution of the eigenvalue problem.

Chapter 4

Examples of QDS evolution

In this chapter we study the convergence dynamics of three example QDS. The first two of them are three-level toy examples we will use to achieve some insight into the generated dynamics. We will differentiate between the case of subspace attractivity achieved trough dissipative dynamics alone as opposed to the case when Hamiltonian dynamics are strictly necessary to establish attractivity. The third QDS we shall consider is a parameterized four-level physical system where three degenerate stable states are coupled to an excited state by resonant laser fields.

4.1 Pure dissipative dynamics

Consider a Markovian open quantum system with dynamics defined on the complex Hilbert space $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$. Let the sets $\{|\bar{s}\rangle\}$ and $\{|\bar{r}_1\rangle, |\bar{r}_2\rangle\}$ be orthonormal basis of $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively and associate to $\mathcal{H}_I$ the orthonormal basis $\{|s\rangle, |r_1\rangle, |r_2\rangle\}$, build by the natural extension to $\mathcal{H}_I$ itself of the former sets of vectors[1]. In this basis we consider a master equation described by the generator:

$$\mathcal{L}_1(l_1, l_2)[\rho] = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}, \tag{4.1}$$

where the noise-operator L is parameterized by the complex valued variables l_1 and l_2 as follows

$$L = \begin{bmatrix} 0 & l_1 & 0 \\ 0 & 0 & l_2 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.2}$$

Our intent is to find whether, and under what constraints on the parameters l_1 and l_2, $\mathcal{H}_S$ becomes attractive. Since the subspace $\mathcal{H}_S$ is one

[1] We use the same block-structure notation we introduced in the last chapter when dealing with operators acting on $\mathcal{H}_I$.

dimensional, this corresponds to being able to render GAS the pure state $|s\rangle\langle s|$.

Sufficient and necessary conditions for subspace invariance are given in Proposition 1. Since the relations (3.35) hold true for any choice of parameters $(l_1, l_2) \in C^2$, $\mathcal{H}_S$ is indeed invariant for all the generators parameterized by $\mathcal{L}_1$.

Attractivity can be asserted by exploiting Theorem 8. Using the definition of $\mathcal{H}_{R'}$ given in (3.15) we find

$$\mathcal{H}_{R'} = \ker(L_P) = \begin{cases} \mathrm{span}(|\bar{r}_2\rangle) & \text{if } l_1 \in \mathbb{C} \setminus \{0\} \\ \mathcal{H}_R & \text{if } l_1 = 0. \end{cases} \tag{4.3}$$

If l_1 vanishes, $\mathcal{H}_R$ becomes an invariant subspace and thus $\mathcal{H}_S$ cannot be attractive. Suppose instead that $l_1 \neq 0$, and define $\rho_{r_2} = |r_2\rangle\langle r_2| \in \mathfrak{D}(\mathcal{H})$; ρ_{r_2} is the only density operator acting on $\mathcal{H}_I$ with support on $\mathcal{H}_{R'}$ alone. Applying the infinitesimal generator to the state ρ_{r_2} we find

$$\mathcal{L}\rho = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \|l_2\|^2 & 0 \\ 0 & 0 & -\|l_2\|^2 \end{bmatrix} \tag{4.4}$$

By requiring $\mathcal{L}\rho$ to have a non trivial support on $\mathcal{H}_{R'}^{\perp}$ we find the condition $l_2 \neq 0$. This, together with $l_1 \neq 0$, ensures that $\mathcal{H}_S$ is an attractive subspace by Theorem 8. In fact if l_2 vanishes ρ_{r_2} becomes a steady state of the generator and thus $\mathcal{H}_S$ cannot be attractive.

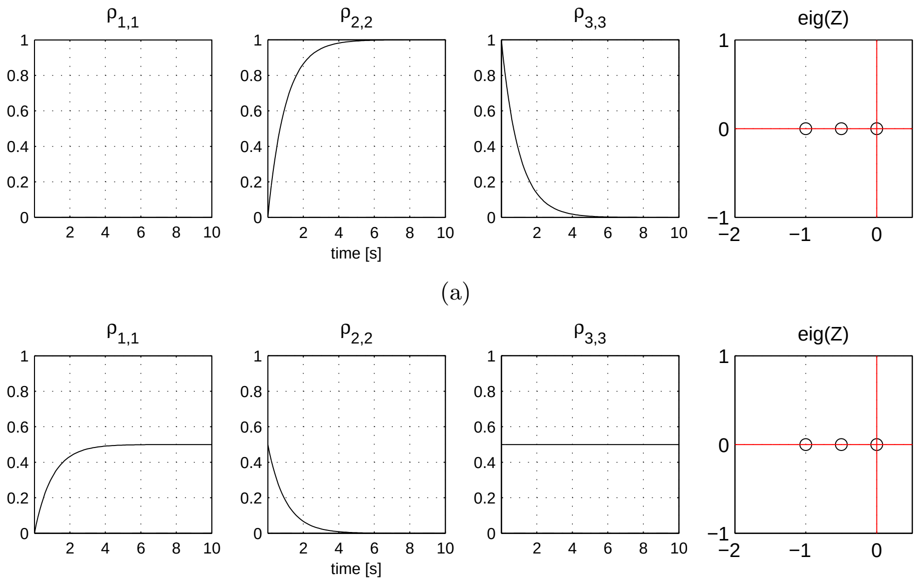
A modal analysis of the convergence dynamics can be carried out in the light of Theorem 12; this is possible since the subspace $\mathcal{H}_S$ is invariant. By recalling the definition of the Z matrix in (3.44) we find

$$Z = \begin{bmatrix} -|l_1|^2 & 0 & 0 & |l_2|^2 \\ 0 & -\frac{|l_1|^2+|l_2|^2}{2} & 0 & 0 \\ 0 & 0 & -\frac{|l_1|^2+|l_2|^2}{2} & 0 \\ 0 & 0 & 0 & -|l_2|^2 \end{bmatrix}, \tag{4.5}$$

Its spectrum is easily found to be

$$\delta_Z(\lambda) = \{-|l_1|^2, -|l_2|^2, -\frac{|l_1|^2 + |l_2|^2}{2}, -\frac{|l_1|^2 + |l_2|^2}{2}\}. \tag{4.6}$$

Thus Z has only real valued eigenvalues, and such are the modes appearing in the R-block evolution of any initial state in $\mathfrak{D}(\mathcal{H})$. Furthermore such eigenvalues are all strictly negative if and only if both the parameters, l_1 and l_2, belong to $\mathbb{C} \setminus \{0\}$. Under this constraint $\mathcal{H}_S$ is an attractive subspace by



(a)

(b)

Figure 4.1: Density matrix evolution and eigenvalues of Z: $\mathcal{L}_1(0,1)$ with initial state ρ_{r_2} in (a), $\mathcal{L}_1(1,0)$ with initial state ρ_m in (b).

Theorem 12, in agreement with what we already found.

It adds to our analysis to plot some selected trajectories generated by $\mathcal{L}_1$ for some choices of the parameters. Before doing so we introduce an analogy with hydraulics which might help with visualizing QDS dynamics. Think of the subspaces of $\mathcal{H}_I$ as basins; then, in this particular example, the trace/probability is pumped with "intensity" l_2 from the subspace $\mathcal{H}_{r_2}$ to the subspace $\mathcal{H}_{r_1}$ and with "intensity" l_1 from $\mathcal{H}_{r_1}$ to $\mathcal{H}_S$. In this simple case the analogy immediately suggests that both l_1 and l_2 are required to be non zero to establish attractivity of $\mathcal{H}_S$ (further elaboration on this lines is provided in [16] and [17]).

Some details on the following simulations are in order as follows. State trajectories are computed by integrating[2] the Lindblad master equation with the fourth-order Runge-Kutta method ([18, 19]). We consider three different initial states: $\rho_{r_1} = |r_1\rangle\langle r_1|$, $\rho_{r_2} = |r_2\rangle\langle r_2|$ and $\rho_m = \frac{1}{2}(\rho_{r_1} + \rho_{r_2})$. Since the state's diagonal terms trajectories are indeed of greatest interest when engineering attractive subspaces, in the following we take the approach of only depicting such trajectories for a given state evolution.

[2]For these simulations a software package was written in the python programming language. The package, named *python-qds* and licensed under GNU GPLv3, might be found at the url: `https://launchpad.net/python-qds`.

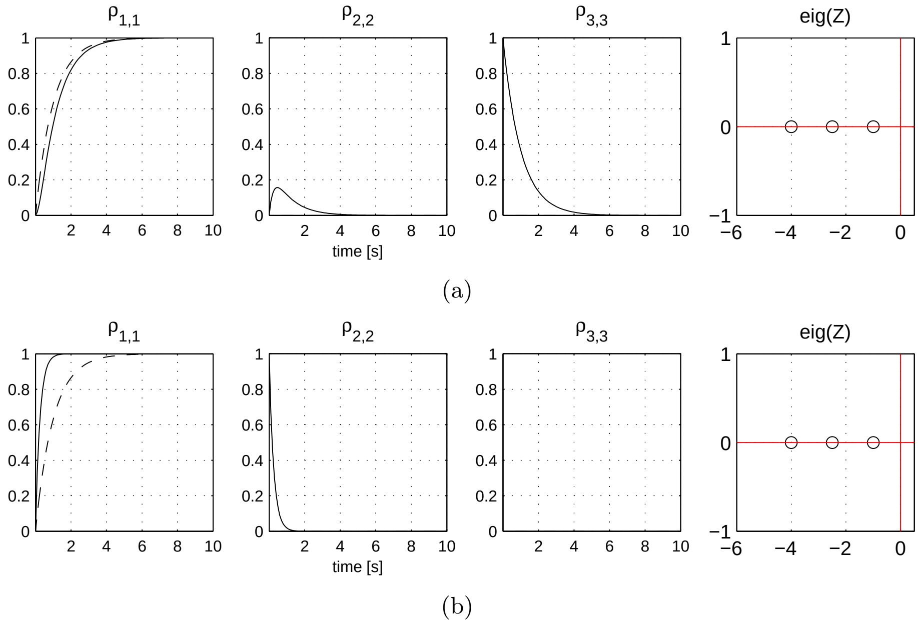


Figure 4.2: Density matrix evolution with asymptotic bound to convergence in dashed lines and eigenvalues of Z. The QDS is $\mathcal{L}_1(2,1)$, exp. bound $z_0 = -1$. Initial state are ρ_{r_2} in (a) and ρ_{r_1} in (b).

In Figure 4.1 we plot two evolutions relative to generators which do not fulfill the conditions ensuring attractivity of $\mathcal{H}_S$. In the first case, $l_1 = 0$ and $l_2 = 1$, evolution is constrained to happen in the subspace $\mathcal{H}_R$ alone (Figure 4.1(a)). Since $\mathcal{H}_{R'} \equiv \mathcal{H}_R$ the state's S and R-blocks are fully decoupled and the quantity $\mathrm{tr}(\Pi_R \rho(t) \Pi_R^\dagger)$ is bound to be constant in time. Clearly then $|s\rangle\langle s|$ cannot be GAS. In the second case we choose $l_1 = 1$ and $l_2 = 0$ (Figure 4.1(b)). Since the one-dimensional subspace $\mathcal{H}_{R'} \subset \mathcal{H}_R$ is invariant we find again that $\mathrm{tr}(\Pi_{R_2} \rho(t) \Pi_{R_2}^\dagger)$ is constant in time and the same conclusion of above is due.

In Figure 4.2 we plot two examples of attractive dynamics with the asymptotic bound given by Corollary 3 plotted in dashed lines. Both evolutions are generated by $\mathcal{L}_1(l_1 = 2, l_2 = 1)$. By recalling (4.6), the asymptotic exponential bound is easily found to be given by $e^{z_0 t}$ where $z_0 = \max_\lambda(\Re\{\delta_Z(\lambda)\}) = -1$. In Figure 4.2(b) the bound might appear broken; recall that the bound expressed in Corollary 3 has a worst-case nature. Indeed there might be states which do not solicit the slower modes of the generator and have asymptotically faster convergence than z_0, though asymptotically slower convergence is not possible.

In Figure 4.3 we graph one last evolution of $\mathcal{L}_1$. In this case, all the

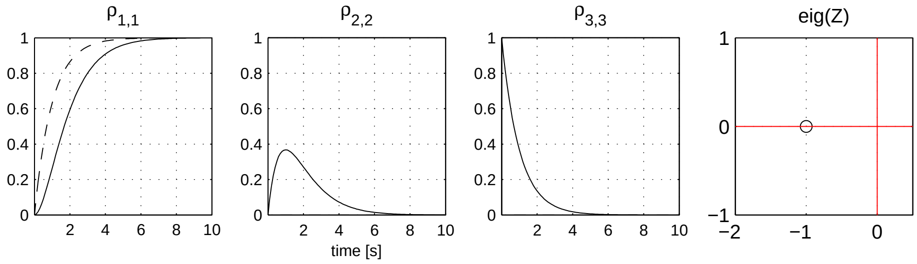


Figure 4.3: Density matrix trajectories (asymptotic bound in dashed lines) and eigenvalues of Z. The QDS is $\mathcal{L}_1(1,1)$, exp. bound $z_0 = -1$ and initial state ρ_{r_2}.

eigenvalues of Z are placed in -1; the higher algebraic multiplicity of z_0 as a root of the characteristic polynomial of Z does effects convergence by initially slowing it. Since the bound is asymptotic care must be taken when approximating the convergence time by $k \cdot z_0$ for some positive k in order to achieve the requested fidelity.

4.2 Hamiltonian and dissipative dynamics

Let $\mathcal{L}_2$ denote our second three-level example quantum system. The associated Hilbert space has decomposition $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$. As we did before let the sets $\{|\bar{s}\rangle\}$ and $\{|\bar{r}_1\rangle, |\bar{r}_2\rangle\}$ be orthonormal basis of $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively, and associate to $\mathcal{H}_I$ the basis $\{|s\rangle, |r_1\rangle, |r_2\rangle\}$ constituted by the natural extensions of the former sets of vectors to $\mathcal{H}_I$. In this basis the QDS generator in the Lindblad form is given by:

$$\mathcal{L}_2(h,l)[\rho] = -i[H,\rho] + L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\} \tag{4.7}$$

where the complex valued variables h and l parameterize the Hamiltonian term H and the single noise-operator L as:

$$\begin{aligned} H &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & h^* & 0 \end{bmatrix} \\ L &= \begin{bmatrix} 0 & l & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{4.8}$$

We want to render the state $|s\rangle\langle s|$ GAS constraining to the degrees of freedom given by the parameters h and l. To do so we proceed by first verifying that $\mathcal{H}_S$ is invariant. Indeed (3.35) holds for any choice of parameters h and $l \in \mathbb{C}$, and thus $\mathcal{H}_S$ is an invariant subspace for the dynamics generated by $\mathcal{L}_2$.

To study attractivity recall the definition of $\mathcal{H}_{R'}$ in (3.15); we have

$$\mathcal{H}_{R'} = \ker(L_P) = \begin{cases} \text{span}(|r_2\rangle) & \text{if } l \in \mathbb{C} \setminus \{0\} \\ \mathcal{H}_R & \text{otherwise.} \end{cases} \tag{4.9}$$

If l vanishes, the subspace $\mathcal{H}_R$ becomes invariant and thus $\mathcal{H}_S$ cannot be attractive. Suppose instead that $l \neq 0$ and let $\rho_{r_2} = |r_2\rangle\langle r_2| \in \mathfrak{D}(\mathcal{H})$; clearly ρ has support on $\mathcal{H}_{R'}$ alone. By explicit computation of the state's time derivative we find

$$\mathcal{L}\rho = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & -h^* & 0 \end{bmatrix}. \tag{4.10}$$

If h vanishes, ρ_{r_2} becomes a steady state of the generator and thus $\mathcal{H}_S$ cannot be attractive. Suppose instead $h \neq 0$. Then $\mathcal{L}\rho$ has a non trivial support on $\mathcal{H}_{R'} \oplus \mathcal{H}_{R'}^{\perp}$ and by Theorem 8 the subspace $\mathcal{H}_S$ attractive or, equivalently, the pure state $|s\rangle\langle s|$ is GAS.

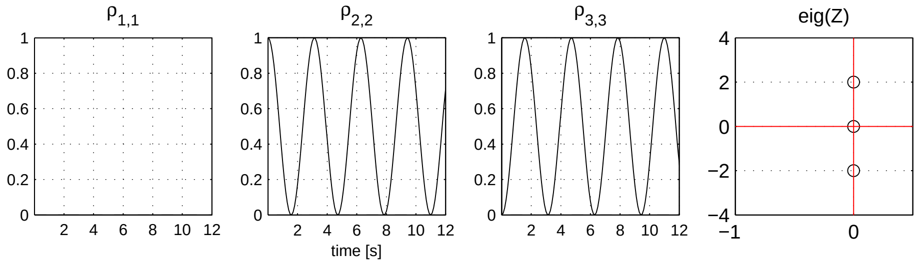
This time a non null Hamiltonian is required to ensure attractivity of $\mathcal{H}_S$. Indeed it is the Hamiltonian part of the generator "connecting" the subspaces $\mathcal{H}_{R'}$ and $\mathcal{H}_S$ (trough l). To see how this effects the modes of the generator we may exploit Theorem 12; this is possible since $\mathcal{H}_S$ is invariant. By recalling the definition of the Z matrix in (3.44) we find

$$Z = \begin{bmatrix} -|l|^2 & -ih & ih^* & 0 \\ -ih^* & -\frac{|l|^2}{2} & 0 & ih^* \\ ih & 0 & -\frac{|l|^2}{2} & -ih \\ 0 & ih & -ih^* & 0 \end{bmatrix} \tag{4.11}$$

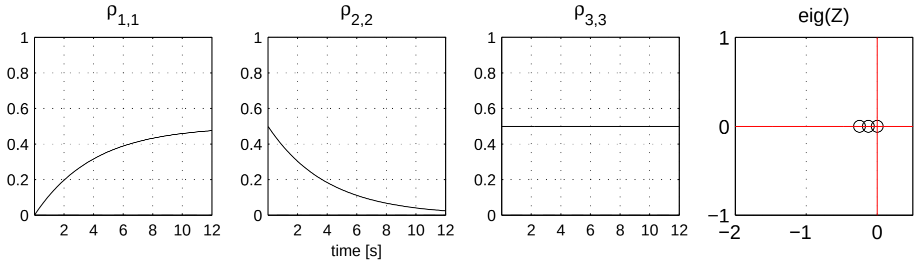
Its spectrum is given by:

$$\begin{aligned} \delta_Z(\lambda) = \Big\{ & -\frac{|l|^2}{2} - 2\sqrt{\frac{|l|^4}{16} - |h|^2}, \\ & -\frac{|l|^2}{2} + 2\sqrt{\frac{|l|^4}{16} - |h|^2}, \\ & -\frac{|l|^2}{2}, -\frac{|l|^2}{2}, \Big\}. \end{aligned} \tag{4.12}$$

Thus Z has real valued eigenvalues if $\frac{|l|^4}{16} - |h|^2 \geq 0$ and both real and complex conjugate eigenvalues otherwise. If we require $\lambda \neq 0 \; \forall \lambda \in \delta_Z(\lambda)$ we find that both the parameters h and l should be choosen in $\mathbb{C} \setminus \{0\}$, in agreement with the condition given by Theorem 8.



(a)



(b)

Figure 4.4: Density matrix trajectories and eigenvalues of Z: $\mathcal{L}_2(1,0)$, initial state ρ_{r_1} in (a); $\mathcal{L}_2(0,0.5)$, initial state ρ_m in (b).

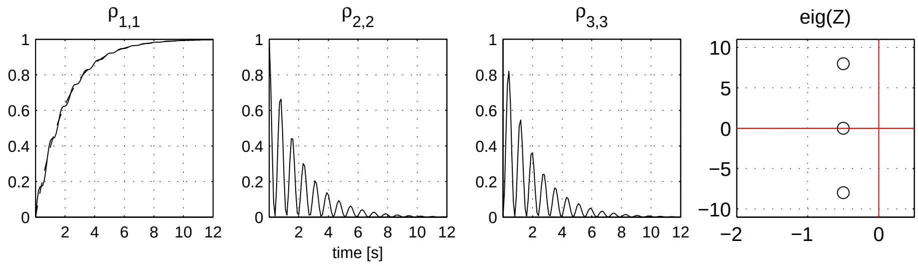
Selected evolutions of the the density matrix for different choices of the the parameters h and l are commented in the following. We choose initial states such to have support on $\mathcal{H}_R$ alone; let $\rho_{r_1} = |r_1\rangle\langle r_1|$, $\rho_{r_2} = |r_2\rangle\langle r_2|$ and $\rho_m = \frac{1}{2}(\rho_1 + \rho_2)$.

In Figure 4.4 we graph two evolutions relative to generators that cannot stabilize $\mathcal{H}_S$. By choosing $h \neq 0$ and $l = 0$ we "turn off" the dissipative part of the master equation. In this case $\mathcal{H}_{R'} = \mathcal{H}_R$, and any initial state in $\mathfrak{I}_R(\mathcal{H}_I)$ is constrained to "stay" in $\mathcal{H}_R$ evolving under Hamiltonian dynamics only (Figure 4.4(a)).

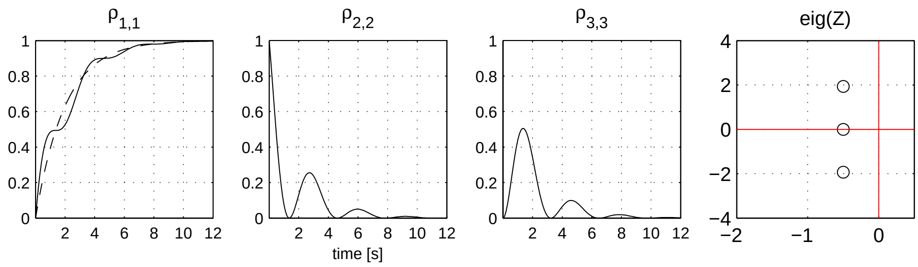
In Figure 4.4(b), instead, we turn-off the Hamiltonian part of the evolution. The one-dimensional subspace $\mathcal{H}_{R'}$ is invariant and as such any initial state ρ with non trivial support on it cannot converge to $\mathcal{H}_S$. Indeed we have that the quantity $\mathrm{tr}(\Pi_{R'}\rho\Pi_{R'}^\dagger)$ is a constant with respect to time.

An example of attractive dynamics is given in Figure 4.5 where we compare the different evolutions generated by $\mathcal{L}_2(h \neq 0, l = 1)$ when scaling the factor h constraining to $\frac{|l|^4}{16} - |h|^2 < 0$. By recalling (4.12) we see that this does not affect the asymptotic behavior of the dynamics which is dominated by l.

Notice how the slower the Hamiltonian dynamics the more sensible becomes the "ringing" effect appearing in the evolution of $\rho_{11}(t) = \Pi_S\rho(t)\Pi_S^\dagger$. While it vanishes asymptotically with time it slows initial convergence and



(a)



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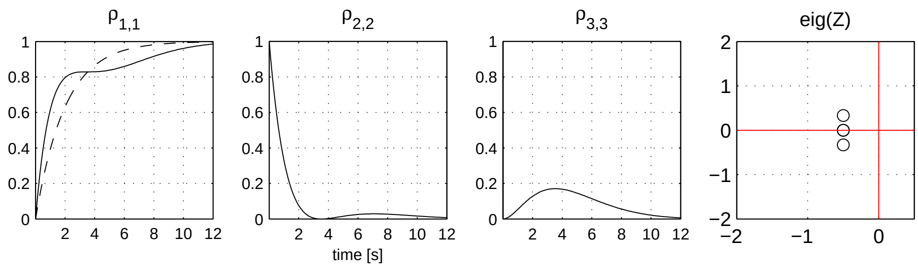
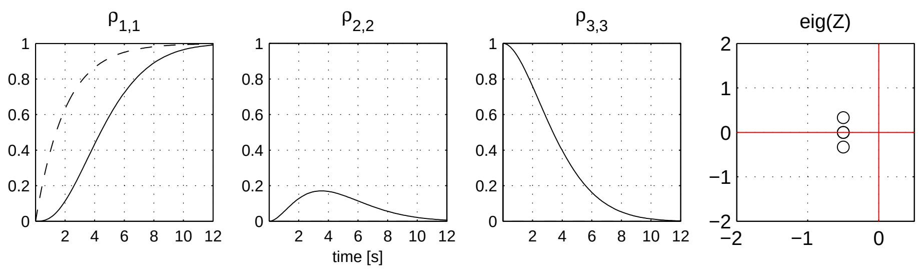


(c)



(d)

Figure 4.5: Density matrix evolution (asymptotic bound in dashed lines) and eigenvalues of Z: $\mathcal{L}_2(4,1)$, exp. bound $z_0 = -0.5$, initial state ρ_{r_1} in (a); $\mathcal{L}_2(1,1)$, exp. bound $z_0 = -0.5$, initial state ρ_{r_1} in (b); $\mathcal{L}_2(0.3,1)$, exp. bound $z_0 = -0.5$, initial state ρ_{r_1} in (c); $\mathcal{L}_2(0.3,1)$, exp. bound $z_0 = -0.5$, initial state ρ_{r_2} in (d).

ultimately acts as unwanted noise superposed on $\rho_{11}(t)$. As a rough measure of the period of the "ripples" we might consider the quantity:

$$r_0 = \frac{2\pi}{\min_\lambda |(\delta_\lambda(Z)|}. \tag{4.13}$$

In the attractive case this last quantity may be used to give an approximation of the trajectory of $\rho_{(1,1)}(t)$ by means of

$$\rho_{11}(t_0) + (1 - \rho_{11}(t_0))e^{(z_0+2\pi r_0 i)t}. \tag{4.14}$$

Unfortunately, this last relation gives a very rough approximation. Indeed the convergence dynamics depend both on the initial state and on the exact placement of the complex eigenvalues of Z, i.e. algebraic multiplicity and position of the "dominant" poles. In Figure 4.5(c) and 4.5(d), we confront evolutions from two different initial states for the same generator, $\mathcal{L}(0.3, 1)$. Notably, when we choose the initial state ρ_{r_2}, initial convergence lags sensibly due to the slow Hamiltonian dynamics which are necessary to break $\mathcal{H}_{R'}$'s invariance.

From these result we could be induced to conclude that "fast" Hamiltonian dynamics are to be preferred over "slower" Hamiltonian dynamics, but this is not always case. As we will see in our third example, there is a distinction to be done.

We conclude our analysis of $\mathcal{L}_2$ with one last interesting case. We graph the trajectories generated by the QDS with fixed h when varying l such that all eigenvalues of Z are real (Figure 4.6). This shows graphically the worst-case nature of the bound given in Corollary 3, that is there might exists states with asymptotically faster convergence but not the converse (Figure 4.6(c) and 4.6(c)).

In simple toy examples like these first two QDS, the control task reduces to the placement of the eigenvalue of Z by means of the parameters according to the constraints that might be imposed on such parameters. However, as we shall see in the next section, even low-dimensional real physical systems may have many enough degrees of freedom to make difficult a complete characterization of the dynamics.

4.3 Parameterized physical system

The last example QDS we analyze is physically motivated. Consider a four-level open quantum system with three degenerate stable ground states, $|i\rangle_{i=1..3}$, coupled to an unstable excited state $|e\rangle$ trough external lasers with coupling constants Ω_i, $i = 1..3$ respectively. The Hamiltonian of such a

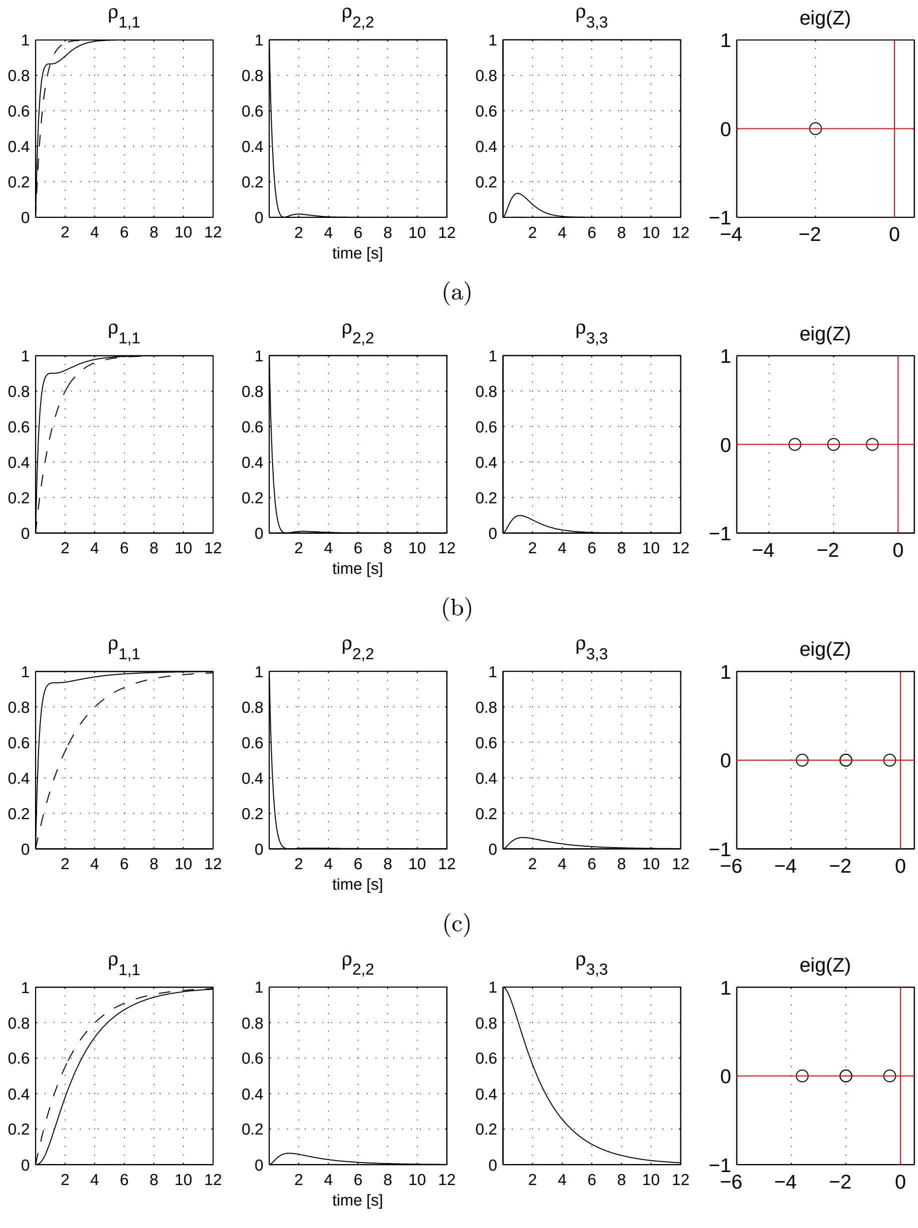


Figure 4.6: Density matrix evolution with asymptotic bound in dashed lines and eigenvalues of Z: $\mathcal{L}_2(1,2)$, exp. bound $z_0 = -2$, initial state ρ_{r_1} in (a); $\mathcal{L}_2(0.8,2)$, exp. bound $z_0 = -0.8$, initial state ρ_{r_1} in (b); $\mathcal{L}_2(0.6,2)$, exp. bound $z_0 = -0.4$, initial state ρ_{r_1} in (c); $\mathcal{L}_2(0.6,2)$, exp. bound $z_0 = -0.4$, initial state ρ_{r_2} in (d).

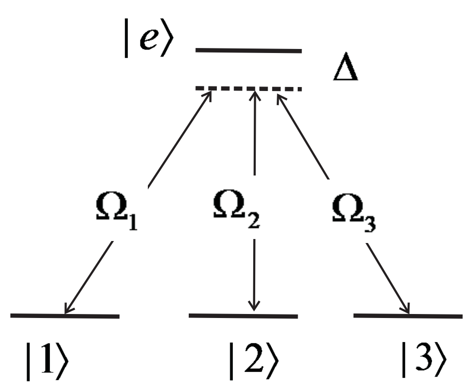


Figure 4.7: System's atomic configurations.

system has the form

$$H_0 = \Delta|e\rangle\langle e| + \sum_{i=1}^{3} \Omega_i \big(|e\rangle\langle i| + |i\rangle\langle e|\big) \tag{4.15}$$

where Δ is the real parameter denoting detuning, that is the difference between the atomic $|i\rangle \to |e\rangle$ transition frequency (Figure 4.7). Following [20] we parameterize the factors Ω_j in spherical coordinates by means of

$$\begin{cases} \Omega_1 = \Omega \cdot \sin\theta \cdot \cos\phi \\ \Omega_2 = \Omega \cdot \sin\theta \cdot \sin\phi \\ \Omega_3 = \Omega \cdot \cos\theta \end{cases} \tag{4.16}$$

where the real parameters $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$ represent elevation and azimuth respectively.

The process of the excited state decaying to the stable states is Markovian and characterized by decaying rates γ_i, $i = 1..3$. The noise operators for the Lindblad master equation governing the QDS are given by the atomic rising operators $L_i = \sqrt{\gamma_i}|e\rangle\langle i|$. We want to show that the decoherence-free-subspace $\mathcal{H}_{DFS}$ of the just defined generator and spanned by the orthonormal vectors

$$\begin{cases} |d_1\rangle = -\sin\phi|1\rangle + \cos\phi|2\rangle \\ |d_2\rangle = \cos\theta(\cos\phi|1\rangle + \sin\phi|2\rangle) - \sin\theta|3\rangle \end{cases} \wedge \quad \begin{matrix} \theta \neq k\pi \\ \phi \neq \dfrac{\pi}{2} + k\pi \end{matrix} \tag{4.17}$$

is an attractive subspace for the QDS dynamics[3].

In order to do this we first apply a basis transformation mapping operators acting on the decomposition $|e\rangle \oplus |1\rangle \oplus |2\rangle \oplus |3\rangle$ to operators acting on

[3]The constraints on θ and ϕ are necessary for the DFS to exists and for the set $\{|d_1\rangle, |d_2\rangle\}$ to be a well defined orthonormal basis of $\mathcal{H}_{DFS}$.

$\mathcal{H}_I = \mathcal{H}_{DFS} \oplus \mathcal{H}_R$. An orthonormal basis $\{|r_1\rangle, |r_2\rangle\}$ of $\mathcal{H}_R$ can be built by exploiting the Graham-Schmidt method. We choose $|r_1\rangle = |e\rangle$.

In the new basis, we find by explicit calculation:

$$\begin{aligned}
\hat{H} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta & \Omega' \\ 0 & 0 & \Omega' & 0 \end{bmatrix} \\
\hat{L}_1 &= \begin{bmatrix} 0 & 0 & -\sqrt{\gamma_1}\sin\phi & 0 \\ 0 & 0 & \sqrt{\gamma_1}\cos\phi\cos\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_1}|\cos\phi\sin\theta| & 0 \end{bmatrix} \\
\hat{L}_2 &= \begin{bmatrix} 0 & 0 & \sqrt{\gamma_2}\cos\phi & 0 \\ 0 & 0 & \sqrt{\gamma_2}\cos\theta\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_2}\,\mathrm{sgn}(\cos\phi)\sin\phi|\sin\theta| & 0 \end{bmatrix} \\
\hat{L}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\gamma_3}\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_3}\,\mathrm{sgn}(\cos\phi\sin\theta)\cos\theta & 0 \end{bmatrix}
\end{aligned} \tag{4.18}$$

where $\Omega' = \Omega\,\mathrm{sgn}(\sin\theta\cos\phi)$ and invariance of $\mathcal{H}_{DFS}$ is straightforward to verify by means of Corollary 9.

Recall the definition of $\mathcal{H}_{R'}$ in (3.15). It is not difficult to see that no choice of parameters ϕ and θ satisfies the relation $\ker(\hat{L}_{P,k}) = \mathcal{H}_R \ \forall\, k$. Furthermore beside the trivial cases when all the γ_i or Ω vanish there is only one more in which $\mathcal{H}_R$ becames invariant, namely $\gamma_2 = \gamma_3 = 0 \wedge (\theta, \phi) = (\pi/2, \pi)$. For all other licit choices of the parameters we have:

$$\mathcal{H}_{R'} = \bigcap_{k=1} \ker(\hat{L}_{P,k}) = \mathrm{span}(|r_2\rangle). \tag{4.19}$$

Now let $\rho_{r_2} = |r_2\rangle\langle r_2|$. Applying the generator to ρ_{r_2} we find

$$\mathcal{L}\rho_{r_2} = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega' \\ 0 & 0 & -\Omega' & 0 \end{bmatrix}. \tag{4.20}$$

If $\Omega > 0$, $\mathcal{L}\rho$ has non trivial support on $\mathcal{H}_{R'}^{\perp}$ and by Theorem 8, the subspace $\mathcal{H}_{DFS}$ is attractive. Thus Hamiltonian dynamics are fundamental to ensure attractivity of the DFS in this case.

We may consider to conduct a modal analysis of the convergence dynamics by exploiting Theorem 12 as we did before. This time, however, the associated Z matrix has the nontrivial structure

$$Z = \begin{bmatrix} -\sum_i \gamma_i & -\Omega' i & \Omega' i & 0 \\ -\Omega' i & -\sum_i \frac{\gamma_i}{2} + \Delta i & 0 & \Omega' i \\ \Omega' i & 0 & -\sum_i \frac{\gamma_i}{2} - \Delta i & -\Omega' i \\ \sum_i \gamma_i \frac{\Omega_i^2}{\Omega^2} & \Omega' i & -\Omega' i & 0 \end{bmatrix}. \tag{4.21}$$

It helps slightly to consider its reduced form Z' defined in (3.54):

$$Z' = \begin{bmatrix} -\sum_i \gamma_i & 2\Omega' i & 0 \\ \Omega' i & -\sum_i \frac{\gamma_i}{2} - \Delta i & -\Omega' i \\ \sum_i \gamma_i \frac{\Omega_i^2}{\Omega^2} & -2\Omega' i & 0 \end{bmatrix}, \tag{4.22}$$

with characteristic polynomial:

$$\begin{aligned} \Delta_{Z'}(s) = (s + \sum_i \gamma_i)((s + \sum_i \frac{\gamma_i}{2} + \Delta i)s + 2\Omega^2) \\ + 2\Omega^2 s - 2\sum_i \gamma_i \Omega_i^2. \end{aligned} \tag{4.23}$$

By evaluating $\Delta_{Z'}(s)$ in $s = 0$ we have:

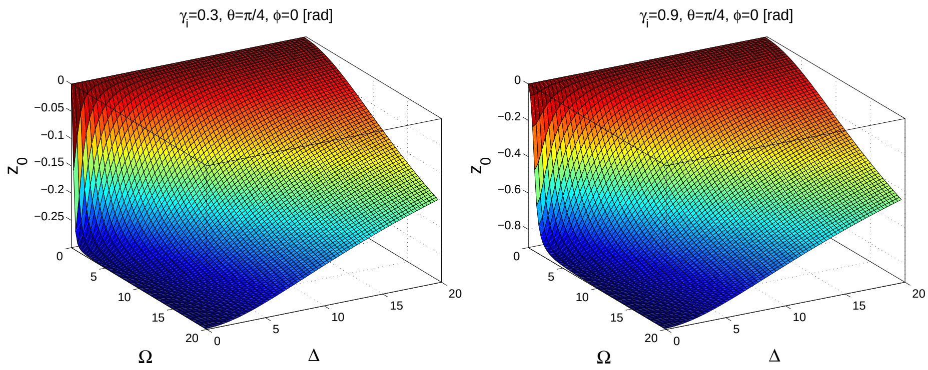
$$\Delta_{Z'}(0) = 2\sum_i \gamma_i(\Omega^2 - \Omega_i^2). \tag{4.24}$$

By recalling (4.16) we find this last quantity vanishes if either all of the γ_i vanish, or if $\Omega = 0$ or if $\gamma_2 = \gamma_3 = 0 \wedge (\theta, \phi) = (\pi/2, \pi)$. For all other licit choices of the parameters $\mathcal{H}_{DFS}$ is attractive by Theorem12, in agreement with Theorem 8.

We were not able to factorize $\Delta_{Z'}(s)$ and thus to give a full parameterized characterization of the dynamics. Indeed this might not be possible symbolically. However graphical methods still have some insight into the generated attractive dynamics to offer.

In Figure 4.8 we graph the value of z_0 as defined in Corollary 3 as a function of the parameters Δ and Ω for some fixed values of the γ_i, θ and ϕ. This reveals an interesting behavior related to the Hamiltonian dynamics. Low coupling Ω as well as high detuning Δ, dramatically slow convergence independently from the γ_i. Indeed we do expect that an higher coupling gives rise to faster convergence, this is true in particular for states with non trivial support on $\mathcal{H}_{R'}$, since Hamiltonian dynamics depending on Ω itself are fundamental to break invariance of the subspace. On the other hand the dynamics related to detuning might not be so intuitive. Recall the form of $\hat{H}$ given in (4.18); its spectrum is easily found to be

$$\delta_{\hat{H}}(\lambda) = \{\lambda_{1,2} = 0, \lambda_3 = \frac{\Delta + \sqrt{\Delta^2 + 4\Omega^2}}{2}, \lambda_4 = \frac{\Delta - \sqrt{\Delta^2 + 4\Omega^2}}{2}\} \tag{4.25}$$



(a) (b)

Figure 4.8: Convergece bound z_0 as a function of the parameters Δ and Ω, with fixed: $\theta = \pi/4$, $\phi = 0$. $\gamma_i \equiv 0.3$ in (a) and $\gamma_i \equiv 0.9$ in (b).

In in the limit of $\Omega \to 0$ we find $\lambda_4 \to 0$, and the same holds for $\Delta \to \infty$. Furthermore the corresponding eigenvector tends to $|r_2\rangle$ in each of the two limits. Thus detuning does act concurrently, in a sense, to the coupling dynamics by attenuating them in a much similar way than if Ω itself was decreased.

In Figure 4.9 we the value of scale ϕ while keeping γ_i and θ fixed as before, to show how dramatic this two parameters effect the dynamics. As ϕ tends to the critical value of π, at which Ω_2 vanishes and the same happens for some of the places in the matrices $\hat{L}_{P,k}$, convergence dynamics are greatly slowed down for any choice of Δ and Ω. This marks a very interesting point. Parameters which do not affect the *existence* of the attractive subspace like Δ or almost don't, like θ and ϕ, might effect the dynamics in a way that "practically inhibits" convergence.

At last we observe that in these selected cases and by means of more extensive numerical simulations, we found that as long the two parameters Ω and Δ remain non "critical", that is they do not bound convergence, and the γ_i are chosen such that $\gamma_i = \gamma\ \forall i$, the slowest real mode in the dynamics approaches γ, uncovering a non trivial symmetry in the QDS. Furthermore for non critical Ω and Δ the value z_0 is linear with respect to the γ_i (Figure 4.10).

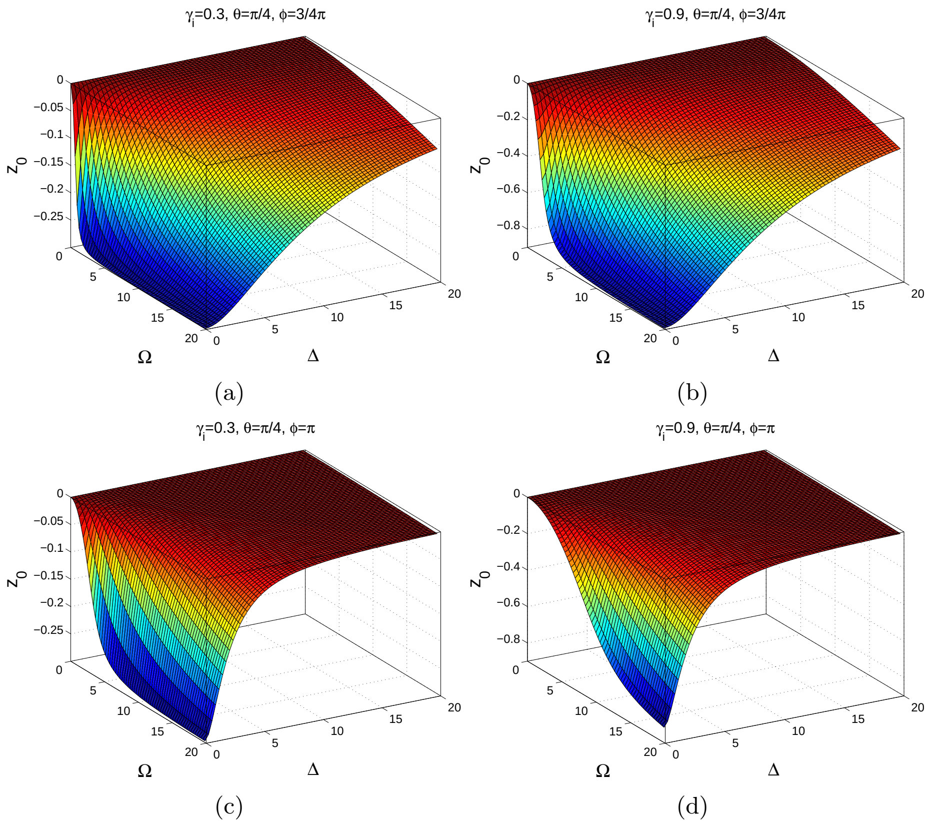


Figure 4.9: Convergece bound z_0 as a function of the parameters Δ and Ω. We fixed: $\gamma_i \equiv 0.3$ and $\theta = \pi/4$ and $\phi = 3/4\pi$ in (a); $\gamma_i \equiv 0.9$ and $\theta = \pi/4$ and $\phi = 3/4\pi$ in (b); $\gamma_i \equiv 0.3$, $\theta = \pi/4$ and $\phi = \pi$ in (c); $\gamma_i \equiv 0.9$, $\theta = \pi/4$ and $\phi = \pi$ in (d).

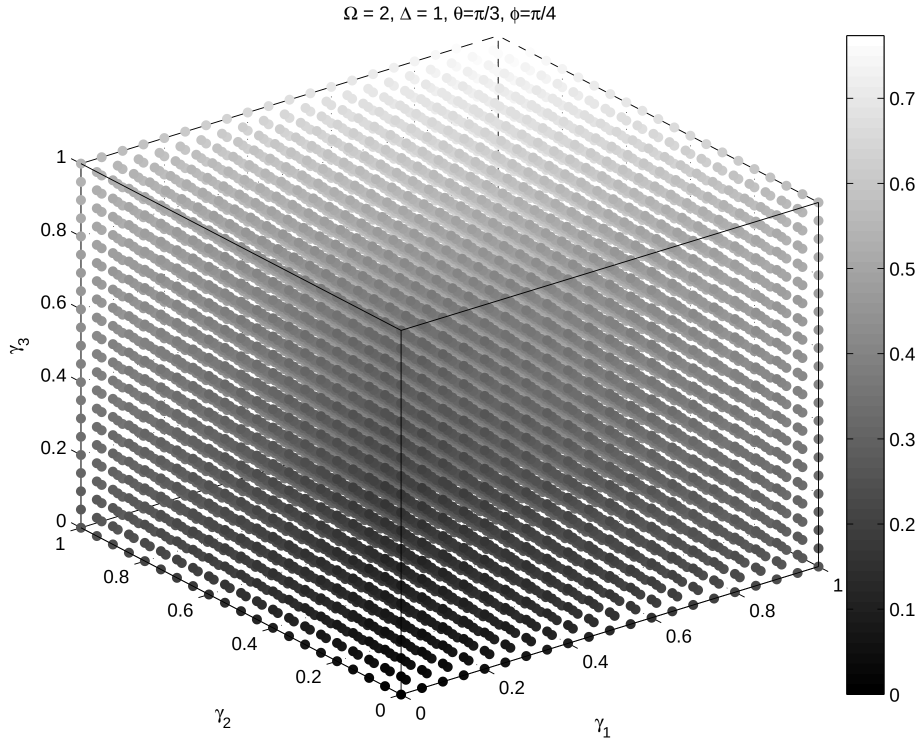


Figure 4.10: Absolute value of z_0 as a function of the decaying rates γ_i. We used $\Omega = 2$, $\Delta = 1$, $\theta = \pi/3$ and $\phi = \pi/4$.

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