universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

„Solvability of partial differential equations with generalized complex coefficients “

Verfasserin

Gudrun Szewieczek

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag. rer. nat.)

Wien, im Juli 2012

Studienkennzahl lt. Studienblatt:	A 405
Studienrichtung lt. Studienblatt:	Mathematik
Betreuer:	ao. Univ.-Prof. Dr. Günther Hörmann

Abstract

The aim of this diploma thesis is to discuss the solvability of partial differential equations with generalized complex coefficients. Similar to the case of non generalized constant coefficients, we use the method of fundamental solutions. Since it turns out that in the Colombeau theoretic setting dual spaces provide an appropriate framework for them, we introduce these spaces in detail. A special focus is on basic functionals, which are determined by a net of distributions and play an important role in the context of partial differential operators.

As main results we obtain an adapted version of the Malgrange-Ehrenpreis-Theorem, which guarantees a fundamental solution, and based on this, equivalent assertions to the solvability for equations with compactly supported inhomogeneities.

The final part of the thesis begins with an extension of the convolution, which enlarges the class of possible inhomogeneities. In the following solutions for selected equations such as the Cauchy-Riemann and a generalized Schrödinger equation are presented.

Zusammenfassung

Diese Diplomarbeit diskutiert die Lösbarkeit von partiellen Differentialgleichungen mit verallgemeinerten komplexen Koeffizienten. Ähnlich wie im Fall von nicht verallgemeinerten konstanten Koeffizienten bedienen wir uns dabei dem Konzept der Fundamentallösungen. Da im Rahmen von Colombeau Algebren Dualräume den passenden theoretischen Hintergrund für dieses Vorgehen liefern, geben wir eine ausführliche Einführung in diese. Eine besondere Stellung nehmen dabei sogenannte Basisfunktionale ein, jene Funktionale die durch Netze von Distributionen bestimmt werden. Sie spielen im Weiteren eine wichtige Rolle für partielle Differentialgleichungen.

Die Hauptresultate der Arbeit sind eine verallgemeinerte Version des bekannten Malgrange-Ehrenpreis-Theorems und darauf aufbauend äquivalente Bedingungen zur Lösbarkeit von Gleichungen mit kompakt getragenen Inhomogenitäten.

Der letzte Teil der Arbeit ist anwendungsorientierter und beginnt mit einer Ausdehnung der Faltung, die unter bestimmten Bedingungen auch Inhomogenitäten mit nicht kompaktem Träger erlaubt. Im Folgenden werden dann einige Lösungen zu bekannten Differentialgleichungen, unter anderem die Cauchy-Riemann und eine Art der Schrödinger Gleichung, vorgestellt.

Thanks to all those who supported me so much,
in particular my advisor Günther Hörmann, my parents and Dimitrios Lenis.

Contents

0 Introduction

Partial differential equations with constant coefficients are undoubtedly an important and frequently discussed class within the theory of partial differential equations. In order to solve them in the distributional setting, one can use the method of fundamental solutions. The main idea of it is simple: Once we have found a fundamental solution, we can construct solutions for equations with inhomogeneities by convolution of the latter with the fundamental solution. The best known result in this field is the Malgrange-Ehrenpreis Theorem, which guarantees a fundamental solution for every differential operator with constant coefficients.

This diploma thesis treats these topics combined with Colombeau theory. More precisely, we investigate the situation of partial differential equations with coefficients in the ring of complex generalized numbers $\tilde{\mathbb{C}}$ and moreover, generalized inhomogeneities or generalized initial conditions.

It turns out that dual spaces of Colombeau algebras provide the appropriate framework for fundamental solutions. Elements in these spaces are $\tilde{\mathbb{C}}$-linear, continuous functionals from a Colombeau algebra to $\tilde{\mathbb{C}}$. In particular, we concentrate on the subspaces of so-called basic functionals, which are determined by nets of distributions and additionally fulfill a continuity-property. They enable us to solve partial differential equations on the level of representatives and therefore in the distributional setting. This strategy does not only help in applications as seen in the final part of the thesis, but also to prove a generalized version of the Malgrange-Ehrenpreis Theorem for an operator whose symbol is invertible at one point. Based on this, it is also possible to formulate equivalent assertions to the solvability of equations with compactly supported right hand sides. The result is obtained for dual spaces but also for Colombeau algebras, because a proof without the notion of fundamental solutions is possible.

As described above, the convolution has a central role in the theory of fundamental solutions. Therefore we discuss this operation in detail for a functional with compact support and an arbitrary basic functional, but also the extension to two functionals with non-compact support under special assumptions.

Finally we study several applications such as the Cauchy-Riemann operator and a generalized Schrödinger equation.

We follow several sources: for the theory of dual spaces we mainly use papers by Claudia Garetto ([Gar05a], [Gar05b]), for the results on existence also [Gar08] and in addition [Hör76] and [Hör04].

1 Topological issues and dual spaces

1.1 Colombeau algebras

In this introductory section we fix some notation and discuss the theoretic background of this thesis, namely algebras of generalized functions introduced by Colombeau. For details of the proofs and further results see for example [Gro01] or [Hör10].

Let Ω be always an open subset of $\mathbb{R}^n$. The basic objects of the theory are families $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. We call the factor algebra $\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$ the *Colombeau algebra*, where

$$\mathcal{E}_M(\Omega) := \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} \mid \forall K \Subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists p \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\}$$

and

$$\mathcal{N}(\Omega) := \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} \mid \forall K \Subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \forall q \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}.$$

Thus elements in $\mathcal{E}_M(\Omega)$ satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while elements in $\mathcal{N}(\Omega)$ vanish faster than any power of ε. We use the short-hand notation $(u_\varepsilon)_\varepsilon \in u$ which means that $(u_\varepsilon)_\varepsilon$ is a representative of u and furthermore denote the class by $[(u_\varepsilon)_\varepsilon]$

If $\Omega_1 \subseteq \Omega$ open and $u \in \mathcal{G}(\Omega)$, then the *restriction* $u|_{\Omega_1}$ is defined by $(u_\varepsilon|_{\Omega_1}) + \mathcal{N}(\Omega_1)$. Since $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras, we may define the *support of* $u \in \mathcal{G}(\Omega)$ by

$$\operatorname{supp}(u) := \Omega \setminus \{x \in \Omega \mid \exists \text{ open neighbourhood } W(x) \subseteq \Omega \text{ of } x :\ u|_{W(x)} = 0\}$$

and denote by $\mathcal{G}_c(\Omega)$ the *subalgebra of all compactly supported elements of* $\mathcal{G}(\Omega)$. Moreover, if $K \subseteq \mathbb{R}^n$ is a compact subset, abbreviated by $K \Subset \mathbb{R}^n$, the space $\mathcal{G}_K(\Omega)$ contains all elements of $\mathcal{G}(\Omega)$ having their supports in K.

The space of all compactly supported distributions $\mathcal{E}'(\Omega)$ is a subspace of $\mathcal{G}_c(\Omega)$ via the embedding

$$\iota_0 : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega), \quad \iota(w) = \Big[(w * (\varphi_\varepsilon)|_\Omega)_\varepsilon\Big], \tag{1.1}$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon) \text{ with } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ of integral one with all moments vanishing.} \tag{1.2}$$

Here $\mathcal{S}(\mathbb{R}^n)$ denotes the space of all rapidly decreasing functions, i.e. all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ that satisfy the estimate

$$\forall \alpha, \beta \in \mathbb{N}_0^n : \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty. \tag{1.3}$$

Using the sheaf property, it is possible to extend the embedding ι_0 in a unique way to an embedding of the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ which we denote by ι. We emphasize that the embeddings ι_0 and ι depend on the chosen function φ.
Note that also $\mathcal{C}^\infty(\Omega)$ becomes a subspace, the embedding even renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra.

Moreover, we consider the *ring of complex generalized numbers* defined as the factor space $\tilde{\mathbb{C}} := \mathcal{E}_M/\mathcal{N}$ with

$$\mathcal{E}_M := \{(\lambda_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]} \ : \ \exists N \in \mathbb{N} \ |\lambda_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\} \tag{1.4}$$

and

$$\mathcal{N} := \{(\lambda_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]} \ : \ \forall q \in \mathbb{N} \ |\lambda_\varepsilon| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}. \tag{1.5}$$

1.2 Topologies for algebras of generalized functions

Since we want to deal with continuous functionals between spaces of generalized functions, we need topologies on these spaces. For the construction of them our theoretic model is a topological vector space which leads us to the notion of a topological $\tilde{\mathbb{C}}$-module.

1.2.1 Topological $\tilde{\mathbb{C}}$-modules

Investigating $\tilde{\mathbb{C}}$-modules, valuations and the induced topology on them, in this section we follow mainly [Gar05a].

For convenience let us first recall the common definition of a *module over the ring* R: It is an abelian group $(\mathcal{G}, +)$ together with an operation $R \times \mathcal{G} \to \mathcal{G}$, $(c, g) \mapsto c.g$ such that $c_1 \cdot (c_2 \cdot g) = (c_1 \cdot c_2) \cdot g$, $(c_1 + c_2) \cdot g = c_1 \cdot g + c_2 \cdot g$ and $c \cdot (g_1 + g_2) = c \cdot g_1 + c \cdot g_2$.

In our context we choose the ring of complex generalized numbers $\tilde{\mathbb{C}}$ for the ring R.

Our following aim, the construction of a norm on a $\tilde{\mathbb{C}}$-module, requires a kind of absolute value on $\tilde{\mathbb{C}}$. For defining it we have to make a detour using a valuation on $\tilde{\mathbb{C}}$.

1.1 Definition. The function $\nu : \mathcal{E}_M \to (-\infty, +\infty]$

$$\nu((\lambda_\varepsilon)_\varepsilon) := \sup\{b \in \mathbb{R} : |\lambda_\varepsilon| = O(\varepsilon^b) \text{ as } \varepsilon \to 0\} \tag{1.6}$$

is called the *valuation on* $\mathcal{E}_M$.

As an easy consequence of this definition we gain the following properties:

1.2 Lemma. The valuation on $\mathcal{E}_M$ satisfies

(i) $\nu((\lambda_\varepsilon)_\varepsilon) = +\infty \;\Leftrightarrow\; (\lambda_\varepsilon)_\varepsilon \in \mathcal{N}$

(ii) $\nu((\lambda_\varepsilon)_\varepsilon(\mu_\varepsilon)_\varepsilon) \geq \nu((\lambda_\varepsilon)_\varepsilon) + \nu((\mu_\varepsilon)_\varepsilon)$

(ii)′ $\nu((\lambda_\varepsilon)_\varepsilon(\mu_\varepsilon)_\varepsilon) = \nu((\lambda_\varepsilon)_\varepsilon) + \nu((\mu_\varepsilon)_\varepsilon)$
if one or both terms are of the form $(c\varepsilon^b)_\varepsilon,\ c \in \mathbb{C},\ b \in \mathbb{R}$

(iii) $\nu((\lambda_\varepsilon)_\varepsilon + (\mu_\varepsilon)_\varepsilon) \geq \min\{\nu((\lambda_\varepsilon)_\varepsilon),\ \nu((\mu_\varepsilon)_\varepsilon)\}$

(iii)′ $\nu((\lambda_\varepsilon)_\varepsilon + (\mu_\varepsilon)_\varepsilon) = \min\{\nu((\lambda_\varepsilon)_\varepsilon),\ \nu((\mu_\varepsilon)_\varepsilon)\}$
if one or both terms are of the form $(c\varepsilon^b)_\varepsilon,\ c \in \mathbb{C},\ b \in \mathbb{R}$

(iv) $\nu((\lambda_\varepsilon)_\varepsilon) = \nu((\lambda'_\varepsilon)_\varepsilon)$ if $(\lambda_\varepsilon - \lambda'_\varepsilon)_\varepsilon \in \mathcal{N}$.

Lemma 1.2 (iv) shows that the following valuation is well-defined, i.e. it is independent of the choice of the representative.

1.3 Definition. For $\lambda = [(\lambda_\varepsilon)_\varepsilon] \in \tilde{\mathbb{C}}$ the *valuation* is defined as $\nu_{\tilde{\mathbb{C}}}(\lambda) := \nu((\lambda_\varepsilon)_\varepsilon)$ and furthermore the function $|.| : \tilde{\mathbb{C}} \to [0, +\infty)$ as $|\lambda| := \begin{cases} e^{-\nu_{\tilde{\mathbb{C}}}(\lambda)}, & \text{if } \nu_{\tilde{\mathbb{C}}}(\lambda) \neq \infty \\ 0, & \text{else.} \end{cases}$

As we will see later, $|.|$ is an ultra-pseudo-norm on $\tilde{\mathbb{C}}$ and can be used as an absolute value in $\tilde{\mathbb{C}}$. But before we pay attention to the construction of norms and topologies, we need to fix even more notions:

1.4 Definition. Let $\mathcal{G}$ be a $\tilde{\mathbb{C}}$-module.

(1) A *valuation* on $\mathcal{G}$ is a function $\nu : \mathcal{G} \to (-\infty, \infty]$ such that

(i) $\nu(0) = +\infty$

(ii) $\nu(\lambda u) \geq \nu_{\tilde{\mathbb{C}}}(\lambda) + \nu(u)$ for all $\lambda \in \tilde{\mathbb{C}}, u \in \mathcal{G}$

(ii)′ $\nu(\lambda u) = \nu_{\tilde{\mathbb{C}}}(\lambda) + \nu(u)$ for all $\lambda = [(c\varepsilon^a)_\varepsilon],\ c \in \mathbb{C},\ a \in \mathbb{R},\ u \in \mathcal{G}$

(iii) $\nu(u + v) \geq \min\{\nu(u),\ \nu(v)\}$.

(2) An *ultra-pseudo-seminorm* on $\mathcal{G}$ is a function $\mathcal{P} : \mathcal{G} \to [0,\ +\infty)$ such that

(i) $\mathcal{P}(0) = 0$

(ii) $\mathcal{P}(\lambda u) \leq |\lambda|\ \mathcal{P}(u)$ for all $\lambda \in \tilde{\mathbb{C}},\ u \in \mathcal{G}$

(ii)$'$ $\mathcal{P}(\lambda u) = |\lambda|\ \mathcal{P}(u)$ for all $\lambda = [(c\varepsilon^a)_\varepsilon],\ c \in \mathbb{C},\ a \in \mathbb{R},\ u \in \mathcal{G}$

(iii) $\mathcal{P}(u + v) \leq \max\{\mathcal{P}(u), \mathcal{P}(v)\}$.

(3) An *ultra-pseudo-norm* is an ultra-pseudo-seminorm $\mathcal{P}$ with the additional property

(i)$'$ $\mathcal{P}(u) = 0 \Rightarrow u = 0$.

1.5 Definition. Let $\mathcal{G}$ be a $\tilde{\mathbb{C}}$-module.

(1) A topology τ on $\mathcal{G}$ is said to be $\tilde{\mathbb{C}}$-*linear*, if

(i) $\mathcal{G} \times \mathcal{G} \to \mathcal{G} : (u, v) \to u + v$ and

(ii) $\tilde{\mathbb{C}} \times \mathcal{G} \to \mathcal{G} : (\lambda, v) \to \lambda v$

are continuous.

(2) A *topological* $\tilde{\mathbb{C}}$-*module* is a $\tilde{\mathbb{C}}$-module with a $\tilde{\mathbb{C}}$-linear topology.

(3) A subset B of $\mathcal{G}$ is $\tilde{\mathbb{C}}$-*convex* if $B + B \subseteq B$ and $[(\varepsilon^b)_\varepsilon]B \subseteq B$ for all $b \geq 0$.

(4) A *locally convex topological* $\tilde{\mathbb{C}}$-*module* is a topological $\tilde{\mathbb{C}}$-module which has a base of $\tilde{\mathbb{C}}$-convex neighbourhoods of the origin.

As in the theory of locally convex topological spaces, one can show the following theorem [Gar05a, Thm 1.10]:

1.6 Theorem. Let $\{\mathcal{P}_i\}_{i \in I}$ be a family of ultra-pseudo-seminorms on a $\tilde{\mathbb{C}}$-module $\mathcal{G}$. The topology induced by $\{\mathcal{P}_i\}$ on $\mathcal{G}$, i.e. the coarsest topology such that each ultra-pseudo-seminorm is continuous, induces the structure of a locally convex topological $\tilde{\mathbb{C}}$-module on $\mathcal{G}$.

1.2.2 Topologies for $\mathcal{G}_E$ and $\mathcal{G}_c$

The spaces we are interested in can be split into two groups: $\mathcal{G}(\Omega)$, $\mathcal{G}_{\mathcal{D}_K}(\Omega)$ and $\tilde{\mathbb{C}}$ have a similar structure, namely $\mathcal{G}_E$, while the algebra of the compactly supported functions, $\mathcal{G}_c$, is an inductive limit. Again our approach is based on [Gar05a].

Let us begin with the first group and consider the general way to construct spaces of generalized functions based on a locally convex topological vector space E.

1.7 Definition. Let E be a locally convex vector space with the topology induced by the familiy of seminorms $\{p_i\}_{i \in I}$. The elements of

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I\ \exists N \in \mathbb{N}\ :\ p_i(u_\varepsilon) = O(\varepsilon^{-N})\ as\ \varepsilon \to 0\} \tag{1.7}$$

and

$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} \mid \forall i \in I\ \forall q \in \mathbb{N}\ :\ p_i(u_\varepsilon) = O(\varepsilon^q)\ \textit{as}\ \varepsilon \to 0\} \tag{1.8}$$

are called *E-moderate* and *E-negligible*, respectively. The space of *generalized functions based on E* is then defined as the factor space $\mathcal{G}_E := \mathcal{M}_E/\mathcal{N}_E$.

Note that the space $\mathcal{G}_E$ is independent of the family of seminorms $\{p_i\}_{i\in I}$ and as in $\mathcal{G}(\Omega)$, there is a natural embedding of E into $\mathcal{G}_E$ via $f \mapsto [(f)_\varepsilon]$. Moreover, $\mathcal{G}_E$ becomes a $\tilde{\mathbb{C}}$-module if we define the product $\tilde{\mathbb{C}} \times\ \mathcal{G}_E \to \mathcal{G}_E$ of complex generalized numbers with elements of $\mathcal{G}_E$ by $([(\lambda_\varepsilon)_\varepsilon], [(u_\varepsilon)_\varepsilon]) \mapsto [(\lambda_\varepsilon u_\varepsilon)_\varepsilon]$.

Very similar to the previous case of $\tilde{\mathbb{C}}$, we can equip $\mathcal{G}_E$ with a valuation:

1.8 Definition. The function $\nu_{p_i} : \mathcal{M}_E \to (-\infty, +\infty]$

$$\nu_{p_i}((u_\varepsilon)_\varepsilon) := \sup\{b \in \mathbb{R} : p_i(u_\varepsilon) = O(\varepsilon^b) \text{ as } \varepsilon \to 0\} \tag{1.9}$$

is called the *p_i-valuation on $\mathcal{M}_E$*.

1.9 Lemma. The p_i-valuation on $\mathcal{M}_E$ has the following properties

(i) $\nu_{p_i}((u_\varepsilon)_\varepsilon) = +\infty$ for all $i \in I\ \Leftrightarrow\ (u_\varepsilon)_\varepsilon \in \mathcal{N}_E$

(ii) $\nu_{p_i}((\lambda_\varepsilon u_\varepsilon)_\varepsilon) \geq \nu((\lambda_\varepsilon)_\varepsilon) + \nu_{p_i}((u_\varepsilon)_\varepsilon)$ for all $(\lambda_\varepsilon)_\varepsilon \in \mathcal{E}_M$ and $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$

(ii)′ $\nu_{p_i}((\lambda_\varepsilon u_\varepsilon)_\varepsilon) = \nu((\lambda_\varepsilon)_\varepsilon) + \nu_{p_i}((u_\varepsilon)_\varepsilon)$ for all $(\lambda_\varepsilon)_\varepsilon = (c\varepsilon^b)_\varepsilon,\ c \in \mathbb{C},\ b \in \mathbb{R}$

(iii) $\nu_{p_i}((u_\varepsilon)_\varepsilon + (v_\varepsilon)_\varepsilon) \geq \min\{\nu_{p_i}((u_\varepsilon)_\varepsilon),\ \nu_{p_i}((v_\varepsilon)_\varepsilon)\}$

(iv) $\nu_{p_i}((u_\varepsilon)_\varepsilon) = \nu_{p_i}((u'_\varepsilon)_\varepsilon)$ if $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathcal{N}_E$.

Proof: (i) follows directly from the definition of $\mathcal{N}_E$.
(ii), (ii)′ and (iii) can be verified using the properties of the Landau symbol O.
(iv) If $(u_\varepsilon - u'_\varepsilon)_\varepsilon \in \mathcal{N}_E$, then we know that $\nu_{p_i}((u_\varepsilon - u'_\varepsilon)_\varepsilon) = +\infty$ and get

$$\nu_{p_i}((u_\varepsilon)_\varepsilon) = \nu_{p_i}((u_\varepsilon)_\varepsilon - (u'_\varepsilon)_\varepsilon + (u'_\varepsilon)_\varepsilon) \geq \min\{\nu_{p_i}((u_\varepsilon - u'_\varepsilon)_\varepsilon), \nu_{p_i}((u'_\varepsilon)_\varepsilon)\} = \nu_{p_i}((u'_\varepsilon)_\varepsilon).$$

Analogously, $\nu_{p_i}((u'_\varepsilon)_\varepsilon) \geq \nu_{p_i}((u_\varepsilon)_\varepsilon)$ holds. □

1.10 Definition. For a generalized function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_E$ the *p_i-valuation* is defined as $\nu_{p_i}(u) := \nu_{p_i}((u_\varepsilon)_\varepsilon)$.

Considering Lemma 1.9 we see that the p_i-valuation is well-defined, because it does not depend on the choice of the representative [(iv)] and, moreover, it is a valuation in the sense of Definition 1.4. Hence we have now the required background to endow $\mathcal{G}_E$ with a topology.

1.11 Proposition. The familiy $\{\mathcal{P}_i\}_{i\in I}$ defined as $\mathcal{P}_i(u) := e^{-\nu_{p_i}(u)}$ is a family of ultra-pseudo-seminorms on the $\tilde{\mathbb{C}}$-module $\mathcal{G}_E$. Hence, $\mathcal{G}_E$ endowed with the topology of these ultra-pseudo-seminorms $\{\mathcal{P}_i\}_{i\in I}$ is a locally convex topological $\tilde{\mathbb{C}}$-module.

Proof: The required properties of the ultra-pseudo-seminorms follow immediately from the properties (i)-(iii) of Lemma 1.9. Applying Theorem 1.6 yields the second assertion. □

We continue looking at several examples.

1.12 Example. (i) **Topology for $\tilde{\mathbb{C}}$.** Consider $E := \mathbb{C}$ with the usual complex absolute value $|.|$. In the same way as in 1.2.1 we get the ring of complex generalized numbers $\mathcal{G}_{\mathbb{C}} = \tilde{\mathbb{C}}$ with the ultra-pseudo-norm $\mathcal{P}(u) = e^{-\nu_{|.|}(u)} = e^{-\nu_{\tilde{\mathbb{C}}}(u)} = |u|$. That makes $\tilde{\mathbb{C}}$ a locally convex topological $\tilde{\mathbb{C}}$-module.

(ii) **Topology for $\mathcal{G}(\Omega)$.** Let $\Omega \subseteq \mathbb{R}^n$ be an open subset with a countable and exhausting sequence of compact subsets $K_0 \subseteq K_1 \subseteq ...$ of Ω.
Consider $E := \mathcal{C}^\infty(\Omega)$ with the topology induced by the family of seminorms

$$p_{K_i,j}(f) := \sup_{x\in K_i, |\alpha|\le j} |\partial^\alpha f(x)|. \tag{1.10}$$

Then it follows with Proposition 1.11 that $\mathcal{G}_E = \mathcal{G}(\Omega)$ is a locally convex topological $\tilde{\mathbb{C}}$-module with the family $\{\mathcal{P}_{K_i,j} := exp(-\nu_{p_{K_i,j}}(u))\}_{i\in I, j\in J}$ of ultra-pseudo-seminorms.

(iii) **Topology for $\mathcal{G}_{\mathcal{D}_K}(\Omega)$.** Let $K \subseteq \mathbb{R}^n$ be a compact subset, then $\mathcal{D}_K(\Omega)$ can be endowed with the topology of the family of seminorms

$$p_{K,j}(f) := \sup_{x\in K, |\alpha|\le j} |\partial^\alpha f(x)| \tag{1.11}$$

and in the same way as above, $\mathcal{G}_{\mathcal{D}_K}(\Omega)$ becomes a locally convex topological $\tilde{\mathbb{C}}$-module.

(iv) **Topology for $\mathcal{G}_K(\Omega)$.** The $\tilde{\mathbb{C}}$-module $\mathcal{G}_K(\Omega)$ is not of the form $\mathcal{G}_E$, but with the following two ideas we can construct a natural and useful topology on $\mathcal{G}_K(\Omega)$. Firstly, $\mathcal{G}_K(\Omega)$ is linearly embedded into $\mathcal{G}_{\mathcal{D}_{K'}}(\Omega)$, where $K' \Subset \Omega$ such that $K \subseteq (K')^\circ$: Choose a cut-off $\chi \in \mathcal{D}(\Omega)$ with $\text{supp}(\chi) \subseteq (K')^\circ$ and $\chi|_K \equiv 1$, then if $u \in \mathcal{G}_K(\Omega)$ and $(u_\varepsilon)_\varepsilon \in u$, we have that $(\chi \cdot u_\varepsilon)_\varepsilon \in u$ and $\text{supp}(\chi \cdot u_\varepsilon) \subseteq (K')^\circ$. So we get the embedding

$$\iota' : \mathcal{G}_K(\Omega) \hookrightarrow \mathcal{G}_{\mathcal{D}_{K'}}(\Omega),\ [(u_\varepsilon)_\varepsilon] \mapsto [(\chi \cdot u_\varepsilon)_\varepsilon]. \tag{1.12}$$

Secondly, use for $u \in \mathcal{G}_K(\Omega)$ the valuation of $\mathcal{G}_{\mathcal{D}_{K'}}(\Omega)$: In $\mathcal{G}_{\mathcal{D}_{K'}}(\Omega)$ we defined the $p_{K',n}$ -

valuation as

$$p_{K',n}(u_\varepsilon) = \sup\{b \in \mathbb{R} : \sup_{x\in K', |\alpha|\le n} |\partial^\alpha u_\varepsilon| = O(\varepsilon^b) \text{ as } \varepsilon \to 0\}.$$

Define now the valuation in $\mathcal{G}_K(\Omega)$ as

$$\nu_{K,n}(u) := \nu_{p_{K',n}}(u) \qquad \text{for all } u \in \mathcal{G}_K(\Omega). \tag{1.13}$$

This valuation is well-defined, because it does not depend on K': let K_1' and K_2' be two compact subsets of Ω with $K \subseteq (K_j')^\circ$ for $j = 1, 2$. Define $K_3 := K_1' \cap K_2'$, then K_3 is compact in $\mathbb{R}^n$ with $K \subseteq K_3^\circ$ and we can choose a cut-off $\chi \in \mathcal{D}(\Omega)$ with $\chi|_K \equiv 1$ and $\mathrm{supp}(\chi) \subseteq K_3^\circ$. If $(u_\varepsilon)_\varepsilon \in u \in \mathcal{G}_K(\Omega)$ is an arbitrary representative, then $(\chi u_\varepsilon)_\varepsilon \in u$ is also a representative with $\mathrm{supp}(\chi u_\varepsilon) \subseteq K_3^\circ$ for all $\varepsilon \in (0, 1]$. So we have for all $\varepsilon \in (0, 1]$

$$\begin{aligned} p_{K_1',n}(\chi u_\varepsilon) &= \sup_{x\in K_1', |\alpha|\le n} |\partial^\alpha \chi u_\varepsilon| = \sup_{x\in K_3^\circ, |\alpha|\le n} |\partial^\alpha \chi u_\varepsilon| = \\ &= \sup_{x\in K_2', |\alpha|\le n} |\partial^\alpha \chi u_\varepsilon| = p_{K_2',n}(\chi u_\varepsilon). \end{aligned}$$

Since the valuation is independent of the representatives [1.9 (iv)], it follows that $\nu_{K_1',n}(u) = \nu_{K_2',n}(u)$.

In summary, $\mathcal{G}_K(\Omega)$ with the topology induced by the ultra-pseudo-seminorms $\{\mathcal{P}_{\mathcal{G}_K(\Omega),n}(u) := e^{-\nu_{K,n}(u)}\}_{n\in\mathbb{N}}$ is a locally convex topological $\tilde{\mathbb{C}}$-module.

Unfortunately $\mathcal{G}_c(\Omega)$ is not of the form $\mathcal{G}_E$ and so we need another strategy for constructing its topology. We adopt the concept of inductive topologies for locally convex vector spaces:

1.13 Definition. Let $\mathcal{G}$ be a $\tilde{\mathbb{C}}$-module and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of $\tilde{\mathbb{C}}$-submodules of $\mathcal{G}$ such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ for all $n \in \mathbb{N}$ and $\mathcal{G} = \cup_{n\in\mathbb{N}}\mathcal{G}_n$. Assume that $\mathcal{G}_n$ is equipped with a locally convex $\tilde{\mathbb{C}}$ - linear topology τ_n such that the topology induced by τ_{n+1} on $\mathcal{G}_n$ is τ_n. Then $\mathcal{G}$ endowed with the inductive limit topology τ is called the *strict inductive limit* of the sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of locally convex topological $\tilde{\mathbb{C}}$-modules.

In [Gar05a, §1.3] was shown, that if $\mathcal{G}$ is a strict inductive limit, it is a locally convex topological $\tilde{\mathbb{C}}$-module. This allows us to construct the required topology as following:

1.14 Example (Topology for $\mathcal{G}_c(\Omega)$)**.**
Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of compact subsets of Ω such that $K_n \subseteq K_{n+1}$, then we have $\mathcal{G}_c(\Omega) = \bigcup_{n\in\mathbb{N}} \mathcal{G}_{K_n}(\Omega)$. Therefore $\mathcal{G}_c(\Omega)$ endowed with the strict inductive limit topology of the sequence $(\mathcal{G}_{K_n}(\Omega))_n$ is a locally convex topological $\tilde{\mathbb{C}}$-module.

1.3 Continuity issues

Let $(\mathcal{G}, \{\mathcal{P}_i\}_{i\in I})$ and $(\mathcal{H}, \{\mathcal{Q}_j\}_{j\in J})$ be locally convex topological $\tilde{\mathbb{C}}$-modules and consider a $\tilde{\mathbb{C}}$-linear map $T : \mathcal{G} \to \mathcal{H}$. It is continuous if and only if for all $j \in J$ there is a finite subset $I_0 \subseteq I$ and a constant $C > 0$ such that for all $u \in \mathcal{G}$

$$\mathcal{Q}_j(Tu) \leq C \max_{i\in I_0} \mathcal{P}_i(u). \tag{1.14}$$

In the particular case of a $\tilde{\mathbb{C}}$-linear map $T : \mathcal{G}_E \to \mathcal{G}_F$, where $(E, \{p_i\}_{i\in I})$ and $(F, \{q_j\}_{j\in J})$ are locally convex topological vector spaces, we obtain that T is continuous if and only if for all $j \in J$ there is a finite subset $I_0 \subseteq I$ and a constant $C > 0$ such that for all $u \in \mathcal{G}_E$

$$\nu_{q_j}(Tu) \geq -\log C + \min_{i\in I_0} \nu_{p_i}(u). \tag{1.15}$$

Also in the case of maps it is useful to be able to argue on the level of representatives:

1.15 Definition. Let $T : \mathcal{G}_E \to \mathcal{G}_F$ be a $\tilde{\mathbb{C}}$-linear map, then we say that T has a *representative* $t : E \to F$, if $Tu = [(tu_\varepsilon)_\varepsilon]$ for all $u \in \mathcal{G}_E$ and additionally, $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$ implies $(tu_\varepsilon)_\varepsilon \in \mathcal{M}_F$ and $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$ implies $(tu_\varepsilon)_\varepsilon \in \mathcal{N}_F$.

This approach enables us to construct $\tilde{\mathbb{C}}$-linear and continuous maps from $\mathcal{G}_E$ to $\mathcal{G}_F$:

1.16 Proposition. Let $t : (E, \{p_i\}_{i\in I}) \to (F, \{q_j\}_{j\in J})$ be a linear and continuous map of locally convex topological vector spaces that defines the map $T : \mathcal{G}_E \to \mathcal{G}_F$, $u \mapsto [(tu_\varepsilon)_\varepsilon]$. Then T is $\tilde{\mathbb{C}}$-linear, continuous and has t as a representative.

Proof: First of all T is well-defined, i.e. independent of the representative, since t is continuous. The $\tilde{\mathbb{C}}$-linearity of T follows directly from the linearity of t, since

$$\lambda T(u) = \lambda[(t(u_\varepsilon))_\varepsilon] = [(t(\lambda u_\varepsilon))_\varepsilon] = T(\lambda u_\varepsilon).$$

It remains to show the continuity of T, i.e. in terms of valuations

$$\nu_{q_j}(Tu) \geq -\log C + \min_{i\in I_0} \nu_{p_i}(u) \quad \text{for a finite subset } I_0 \subseteq I. \tag{1.16}$$

By the continuity of t we have $q_j(tu_\varepsilon) \leq C_1 \max_{i\in I_0} p_i(u_\varepsilon)$ for a constant $C_1 \geq 0$ and a finite subset $I_0 \subseteq I$ and hence gain that

$$\nu_{q_j}(Tu) = \nu_{q_j}([(tu_\varepsilon)_\varepsilon]) = \sup\{b \in \mathbb{R} : q_j(tu_\varepsilon) = O(\varepsilon^b)\} \geq$$

$$
\begin{aligned}
&\geq \sup\{b \in \mathbb{R} : C_1 \max_{i \in I_0} p_i(u_\varepsilon) = O(\varepsilon^b)\} \\
&= \min_{i \in I_0} \sup\{b \in \mathbb{R} : C_1 p_i(u_\varepsilon) = O(\varepsilon^b)\} \\
&= \min_{i \in I_0} \sup\{b \in \mathbb{R} : p_i(u_\varepsilon) = O(\varepsilon^b)\} = \min_{i \in I_0} \nu_{p_i}(u).
\end{aligned}
$$

Setting $C := 1$, yields (1.16). □

We emphasize that the converse is not true. Namely, a $\tilde{\mathbb{C}}$-linear, continuous map T with representative t does not guarantee the continuity of the representative. A counterexample can be found in [Gar05a, Remark 3.14 (iii)].

1.4 Dual spaces of Colombeau algebras

We investigate now the dual spaces of diverse topological $\tilde{\mathbb{C}}$-modules $\mathcal{G}$, based on [Gar05a] and [Gar05b].

Let $\mathcal{G}$ be a topological $\tilde{\mathbb{C}}$-module, then we denote its dual space by $\mathcal{L}(\mathcal{G}, \tilde{\mathbb{C}})$, i.e. the set of all $\tilde{\mathbb{C}}$-linear and continuous maps from $\mathcal{G}$ to $\tilde{\mathbb{C}}$. We are interested in the choices $\mathcal{G}(\Omega)$ and $\mathcal{G}_c(\Omega)$ for $\mathcal{G}$ which we investigate in the following.

1.4.1 The spaces $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$, $\mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$ and their basic subsets

Clearly we obtain the following inclusion:

$$\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}}) \subseteq \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}}). \tag{1.17}$$

Moreover, using the $\tilde{\mathbb{C}}$-linear continuous embedding $\iota_g : u \mapsto \Big(v \mapsto \int u(y)v(y)\,dy\Big)$, in [Gar05b, Theorem 3.1] the further inclusions were shown:

$$\mathcal{G}(\Omega) \subseteq \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}}), \tag{1.18}$$

$$\mathcal{G}_c(\Omega) \subseteq \mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}}). \tag{1.19}$$

Furthermore, distributions can also be interpreted as elements of $\mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$: Let $w \in \mathcal{D}'(\Omega)$ be an arbitrary distribution, i.e. the map $w : \mathcal{D}(\Omega) \to \mathbb{C}$ is linear and continuous. By Proposition 1.16 we obtain that $T : \mathcal{G}_{\mathcal{D}} \to \tilde{\mathbb{C}}$, $u \mapsto [(w(u_\varepsilon))_\varepsilon]$, is a $\tilde{\mathbb{C}}$-linear and continuous map and, accordingly, $T \in \mathcal{L}(\mathcal{G}_{\mathcal{D}}(\Omega), \tilde{\mathbb{C}})$. Since $\mathcal{G}_c$ can be continuously embedded into $\mathcal{G}_{\mathcal{D}}$, it follows that $\mathcal{L}(\mathcal{G}_{\mathcal{D}}(\Omega),\tilde{\mathbb{C}}) \subseteq \mathcal{L}(\mathcal{G}_c(\Omega),\tilde{\mathbb{C}})$. In summary,

$$\mathcal{D}'(\Omega) \hookrightarrow \mathcal{L}(\mathcal{G}_c(\Omega),\tilde{\mathbb{C}}), \quad w \mapsto \Big(u \mapsto [(w(u_\varepsilon))_\varepsilon]\Big) \qquad \text{for } u \in \mathcal{G}_c(\Omega) \tag{1.20}$$

is a continuous embedding, that we denote by ι_d.

This embedding is in general not the same as

$$\mathcal{D}'(\Omega) \overset{\iota}{\hookrightarrow} \mathcal{G}(\Omega) \overset{\iota_g}{\hookrightarrow} \mathcal{L}(\mathcal{G}_c(\Omega),\tilde{\mathbb{C}}), \tag{1.21}$$

where the distributions are primarily embedded in $\mathcal{G}(\Omega)$ and then in $\mathcal{L}(\mathcal{G}_c(\Omega),\tilde{\mathbb{C}})$. Consider for example the delta-distribution $\delta \in \mathcal{D}'(\mathbb{R}^n)$, then δ is represented in $\mathcal{G}(\mathbb{R}^n)$ by a model delta net $[(\rho_\varepsilon)_\varepsilon]$, i.e. $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$, where $\rho \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\rho = 1$ and for all $\alpha \in \mathbb{N}^n, |\alpha| \geq 1$ the moment conditions $\int_{\mathbb{R}^n} x^\alpha \rho(x)dx = 0$ hold. Hence we obtain

$$\delta \overset{\iota}{\hookrightarrow} [(\rho_\varepsilon)_\varepsilon] \overset{\iota_g}{\hookrightarrow} \left([(u_\varepsilon)_\varepsilon] \mapsto \Big[\Big(\int \rho_\varepsilon u_\varepsilon dx\Big)_\varepsilon\Big]\right), \tag{1.22}$$

whereas embedding $\delta \in \mathcal{D}'$ directly to $\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n),\tilde{\mathbb{C}})$ yields

$$\delta \overset{\iota_d}{\hookrightarrow} \Big[\Big(u_\varepsilon \mapsto \delta(u_\varepsilon) = u_\varepsilon(0)\Big)_\varepsilon\Big]. \tag{1.23}$$

Note that nevertheless these two different embedded elements are associated to each other, i.e. they coincide in the limit.

Let $\Omega' \subseteq \Omega$ be an open subset, then the *restriction of T to Ω'* is defined in the obvious way: $T|_{\Omega'} : \mathcal{G}_c(\Omega') \to \tilde{\mathbb{C}}$, $u \mapsto Tu$ and clearly $T|_{\Omega'} \in \mathcal{L}(\mathcal{G}_c(\Omega'),\tilde{\mathbb{C}})$. This leads to the definition of the *support of T*:

$$\operatorname{supp}(T) := \Omega \setminus Z(T)$$

$$\text{with } Z(t) := \{x \in \Omega \mid \exists \text{ V(open)} \subseteq \Omega, x \in V \text{ such that } T|_V = 0\}.$$

The following proposition shows that we may recognize elements in $\mathcal{L}(\mathcal{G}(\Omega),\tilde{\mathbb{C}})$ by their compact supports.

1.17 Proposition. A functional $T \in \mathcal{L}(\mathcal{G}_c(\Omega),\tilde{\mathbb{C}})$ belongs to $\mathcal{L}(\mathcal{G}(\Omega),\tilde{\mathbb{C}})$ if and only if $\operatorname{supp}(T)$ is a compact subset of $\mathbb{R}^n$.

Proof: Assume first that $\operatorname{supp}(T)$ is a compact subset of Ω and let $\chi \in \mathcal{C}_c^\infty(\Omega)$ be a cut-off with $\chi = 1$ on a neighbourhood of $\operatorname{supp}(T)$. Since for all $u \in \mathcal{G}(\Omega)$ it follows that $\chi u \in \mathcal{G}_c(\Omega)$, we can construct the following $\tilde{\mathbb{C}}$-linear map T' on $\mathcal{G}(\Omega)$

$$T' : \mathcal{G}(\Omega) \to \tilde{\mathbb{C}},\ u \mapsto T(\chi u). \tag{1.24}$$

The restriction $T'|_{\mathcal{G}_c(\Omega)} = T$, because $(1-\chi)u \in \mathcal{G}_c(\Omega \setminus \mathrm{supp}(T))$ and therefore

$$T'(u) - T(u) = T((1-\chi)u) = 0. \tag{1.25}$$

It remains to show that T' is continuous: Using the notation of Example 1.12 we obtain by the continuity of T for $K := \mathrm{supp}(\chi)$ and $\chi u \in \mathcal{G}_K(\Omega)$ that

$$|T'(u)|_{\tilde{\mathbb{C}}} \le C\ \mathcal{P}_{\mathcal{G}_K(\Omega),m}(\chi u). \tag{1.26}$$

By the Leibniz rule, $\mathcal{P}_{\mathcal{G}_K(\Omega),m}(\chi .)$ can be bounded by some ultra-pseudo-seminorms which determine the sharp topology on $\mathcal{G}(\Omega)$ and therefore T' is a continuous, $\tilde{\mathbb{C}}$-linear map from $\mathcal{G}(\Omega)$ into $\tilde{\mathbb{C}}$.

Conversely, let T be an element of $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ and assume that its support is not a compact subset of Ω. If $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ is an exhausting sequence of compact sets of Ω, then for all $n \in \mathbb{N}$, we have that $\mathrm{supp}(T) \cup (\Omega \setminus K_n) \neq \emptyset$. This means that there is a sequence $(u_n)_n \in \mathcal{G}_c(\Omega)^{n\in\mathbb{N}}$ such that $\mathrm{supp}(u_n) \subseteq \Omega \setminus K_n$ and $T(u_n) \neq 0$. Denoting the valuation of $T(u_n)$ by a_n, the generalized function $v_n := [(\varepsilon^{-a_n})_\varepsilon]u_n$ has its support contained in $\Omega \setminus K_n$ and

$$|T(v_n)|_{\tilde{\mathbb{C}}} = \exp(a_n)\ |T(u_n)|_{\tilde{\mathbb{C}}} = 1. \tag{1.27}$$

Contrary, we show that the sequence $(v_n)_n$ converges to 0 in $\mathcal{G}(\Omega)$, which gives a contradiction to the continuity of T on $\mathcal{G}(\Omega)$. Indeed, since for all $K \subseteq \Omega$ there exists a parameter $n_0 \in \mathbb{N}$ such that $K \subseteq K_{n_0}$ and from $K_{n_0} \subseteq \Omega \setminus \mathrm{supp}(v_n)$ for all $n \ge n_0$ we obtain that

$$\sup_{x\in K, |\alpha|\le m} |\partial^\alpha v_{n,\varepsilon}(x)| \le \sup_{x\in K_{n_0}, |\alpha|\le m} |\partial^\alpha v_{n,\varepsilon}(x)| = O(\varepsilon^q) \quad (q \in \mathbb{N}). \tag{1.28}$$

Hence, $(v_n)_n$ is a zero-sequence in $\mathcal{G}(\Omega)$. □

Since for $w \in \mathcal{D}'(\Omega)$ we have $\mathrm{supp}(w) = \mathrm{supp}(\iota_d(w))$, it follows that

$$\mathcal{E}'(\Omega) \hookrightarrow \mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}}),$$
$$\mathcal{D}'(\Omega) \setminus \mathcal{E}'(\Omega) \hookrightarrow \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}}).$$

In the following we focus on special elements of $\mathcal{L}(\mathcal{G}_E, \tilde{\mathbb{C}})$, namely, those that can be represented by a net of continuous linear maps from E to $\mathbb{C}$ fulfilling a continuity-property:

1.18 Definition. Let $(E, \{p_i\}_{i\in I})$ be a locally convex topological vector space, then we call T $\in \mathcal{L}(\mathcal{G}_E, \tilde{\mathbb{C}})$ *basic*, if it is of the form $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$, where $(T_\varepsilon)_\varepsilon$ is a net of continuous linear maps from E to $\mathbb{C}$ satisfying the following property

$$\exists I_0 \subseteq I \text{ finite } \exists N \in \mathbb{N}\ \exists \eta \in (0,1]\ \forall u \in E\ \forall \varepsilon \in (0,\eta]: \\ |T_\varepsilon(u)| \le \varepsilon^{-N} \max_{i\in I_0}\ p_i(u). \tag{1.29}$$

We denote the space of all basic functionals by $\mathcal{L}_b(\mathcal{G}_E, \tilde{\mathbb{C}})$.
We emphasize, that $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$ holds for all representatives $(u_\varepsilon)_\varepsilon$ of u, because by (1.29) we obtain for $(u_\varepsilon)_\varepsilon \in \mathcal{M}_E$ that $(T_\varepsilon u_\varepsilon)_\varepsilon \in \mathcal{M}_\mathbb{C}$ and analogously for $(u_\varepsilon)_\varepsilon \in \mathcal{N}_E$ that $(T_\varepsilon u_\varepsilon)_\varepsilon \in \mathcal{N}_\mathbb{C}$.

As we know from Example 1.12 (ii), setting $E := \mathcal{C}^\infty(\Omega)$ gives the definition for the case $\mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$. Accordingly, the continuous linear maps from E to $\mathbb{C}$ are distributions in $\mathcal{E}'(\Omega)$ and more precisely we get: A Functional $T \in \mathcal{L}(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$ is called *basic*, if it is of the form $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$, where $(T_\varepsilon)_\varepsilon$ is a net of distributions in $\mathcal{E}'(\Omega)$ satisfying the following property:

$$\exists K \Subset \Omega\ \exists j \in \mathbb{N}\ \exists N \in \mathbb{N}\ \exists \eta \in (0,1]\ \forall u \in \mathcal{C}^\infty(\Omega)\ \forall \varepsilon \in (0,\eta]: \\ |T_\varepsilon(u)| \le \varepsilon^{-N} \sup_{x\in K, |\alpha|\le j} |\partial^\alpha u(x)|. \tag{1.30}$$

Since $\mathcal{G}_c(\Omega)$ is not of the form $\mathcal{G}_E$, we have to slightly modify the preceding definition regarding its inductive structure:

1.19 Definition. Let $E = \operatorname{span}(\bigcup_{\gamma\in\Gamma} \iota_\gamma(E_\gamma))$, $\iota_\gamma : E_\gamma \to E$ be the inductive limit of the locally convex topological vector spaces $(E_\gamma, \{p_{i,\gamma}\}_{i\in I_\gamma})_{\gamma\in\Gamma}$. Let $\mathcal{G} = \tilde{\mathbb{C}}\text{-span}(\bigcup_{\gamma\in\Gamma} \iota_\gamma(\mathcal{G}_{E_\gamma})) \subseteq \mathcal{G}_E$ be the inductive limit of the locally topological $\tilde{\mathbb{C}}$-modules $(\mathcal{G}_{E_\gamma})_{\gamma\in\Gamma}$. Then we call T $\in \mathcal{L}(\mathcal{G}, \tilde{\mathbb{C}})$ *basic*, if it is of the form $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$, where $(T_\varepsilon)_\varepsilon$ is a net of continuous linear maps from E to $\mathbb{C}$ satisfying the following property

$$\forall \gamma \in \Gamma\ \exists I_{0,\gamma} \subseteq I_\gamma \text{ finite } \exists N \in \mathbb{N}\ \exists \eta \in (0,1]\ \forall u \in E_\gamma\ \forall \varepsilon \in (0,\eta]: \\ |T_\varepsilon \iota_\gamma(u)| \le \varepsilon^{-N} \max_{i\in I_{0,\gamma}}\ p_{i,\gamma}(u). \tag{1.31}$$

Again the space of all basic functionals is denoted by $\mathcal{L}_b(\mathcal{G}_E, \tilde{\mathbb{C}})$ and as in Definition 1.18 $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$ holds for every representative.

We are now also able to name the basic functionals of $\mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$. By example 1.14, $\mathcal{G}_c(\Omega)$ is an inductive limit and the previous definition gives: A functional $T \in \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$ is called *basic*, if it is of the form $Tu = [(T_\varepsilon u_\varepsilon)_\varepsilon]$, where $(T_\varepsilon)_\varepsilon$ is a net of distributions in $\mathcal{D}'(\Omega)$

satisfying the following property:

$$\forall K \Subset \Omega \ \exists j \in \mathbb{N} \ \exists N \in \mathbb{N} \ \exists \eta \in (0,1] \ \forall u \in \mathcal{D}_K(\Omega) \ \forall \varepsilon \in (0,\eta] : \\ |T_\varepsilon(u)| \leq \varepsilon^{-N} \sup_{x \in K, |\alpha| \leq j} |\partial^\alpha u(x)|. \tag{1.32}$$

Note that every constant net $(T)_\varepsilon$ with $T \in \mathcal{D}'(\Omega)$ defines a basic functional in $\mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$. Indeed, by the seminorm-estimate for distributions [Hör09, Theorem 1.26] , we obtain that $\forall K \Subset \Omega \ \exists C > 0 \ \exists m \in \mathbb{N} \ \forall u \in \mathcal{D}_K(\Omega)$

$$|T(u)| \ \leq \ C \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha u(x)| \ \leq \ C_1 \sup_{x \in K, |\alpha| \leq m} |\partial^\alpha u(x)|. \tag{1.33}$$

Since the constant $C_1 > 0$ does not depend on ε, there are parameters $N \in \mathbb{N}$ and $\eta \in (0,1]$ such that property (1.32) is fulfilled.

A similar argument holds for a constant net $(T)_\varepsilon$ with $T \in \mathcal{E}'(\Omega)$ that always determines an element in $\mathcal{L}_b(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$. This can be seen using the seminorm-estimate for compactly supported distributions [Hör09, Theorem 1.62], i.e. $\exists K \Subset \Omega \ \exists C > 0 \ \exists m \in \mathbb{N} \ \forall u \in \mathcal{C}^\infty(\Omega)$

$$|T(u)| \leq C \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha u(x)|. \tag{1.34}$$

In particular we obtain that embedded distributions are basic elements in $\mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$ and, respectively, compactly supported distributions are elements in $\mathcal{L}_b(\mathcal{G}(\Omega), \tilde{\mathbb{C}})$.

Moreover, a net $(T_\varepsilon)_\varepsilon \in \mathcal{D}'(\Omega)^{(0,1]}$ that converges in $\mathcal{D}'(\Omega)$, defines a basic functional. Let $(T_\varepsilon)_\varepsilon$ be a convergent net in $\mathcal{D}'(\Omega)$ and $K \Subset \Omega$, then by the seminorm-estimate for distributions we obtain for every $\varepsilon \in (0,1]$ that $\exists C_\varepsilon > 0 \ \exists m_\varepsilon \in \mathbb{N} \ \forall u \in \mathcal{D}_K(\Omega)$

$$|T_\varepsilon(u)| \ \leq \ C_\varepsilon \sum_{|\alpha| \leq m_\varepsilon} \sup_{x \in K} |\partial^\alpha u(x)|, \tag{1.35}$$

which means that the set $\{T_\varepsilon | \varepsilon \in (0,1]\}$ is pointwise bounded on $\mathcal{D}_K(\Omega)$. By the principle of uniform boundedness (cf. [Hör09, Theorem 1.44], [Sch66, Chapter III, 4.2 Corollary]) this set is also strongly bounded. Hence there are constants $C > 0$ and $m \in \mathbb{N}$ independent of ε, such that for all $\varepsilon \in (0,1]$

$$|T_\varepsilon(u)| \ \leq \ C \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha u(x)| \tag{1.36}$$

and we obtain the required property (1.32) as in the case of a constant net.
Therefore, a net of distributions that does not define a basic element, is not convergent in $\mathcal{D}'$. But there are also divergent nets of distributions that define a basic functional, e.g.

consider the net $(\frac{1}{\varepsilon}H)_\varepsilon$. Since for a test function $u \in \mathcal{D}(\Omega)$ with support contained in $[0,\infty)$ and $\int u(t)dt = 1$ we have that

$$\Big\langle \frac{1}{\varepsilon}H, u \Big\rangle = \int_0^\infty \frac{1}{\varepsilon}u(t)dt = \frac{1}{\varepsilon} \tag{1.37}$$

and therefore the net does not converge for $\varepsilon \to 0$.
On the other hand, for all $u \in \mathcal{D}_K$ the following estimate

$$\Big|\frac{1}{\varepsilon}H(u)\Big| \ \leq \ \varepsilon^{-1} \sup_{x\in K} |u(x)||K| \tag{1.38}$$

holds and therefore the net defines a basic element.

For the framework of these $\tilde{\mathbb{C}}$-linear, continuous functionals, we need some common operations, whose definitions are similar to that for distributions:

1.20 Definition. Let $T \in \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$, $v \in \mathcal{G}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$ be a multiindex.
(i) The *derivative of* T is the $\tilde{\mathbb{C}}$-linear map $\partial^\alpha : \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}}) \to \mathcal{L}(\mathcal{G}_c(\Omega), \tilde{\mathbb{C}})$ defined by

$$\partial^\alpha T(u) := (-1)^{|\alpha|} T(\partial^\alpha u). \tag{1.39}$$

(ii) We define the *multiplication*, $\mathcal{G}(\mathbb{R}^n) \times \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}}) \to \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, by

$$vT(u) = T(vu) \quad \text{ for all } u \in \mathcal{G}_c(\mathbb{R}^n). \tag{1.40}$$

If in particular, $v \in \mathcal{G}_c(\mathbb{R}^n)$, it follows that vT has compact support and therefore $vT \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$.

Basic functionals are stable under these two operations: First note that basic functionals remain basic when multiplied with a generalized function, which follows directly from the definition.
Moreover, if the net $(T_\varepsilon)_\varepsilon$ defines a basic functional in $\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, then for all multiindices $\alpha \in \mathbb{N}^n$, the net $(\partial^\alpha T_\varepsilon)_\varepsilon$ is again basic. Indeed, for

$$|\partial^\alpha T_\varepsilon(u)| = |T_\varepsilon(\partial^\alpha u)| \tag{1.41}$$

the required estimate (1.32) holds, since for $u \in \mathcal{D}_K$ the derivative $\partial^\alpha u \in \mathcal{D}_K$.

1.4.2 Convolution for elements of $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ with $\mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$

Next, we want to extend the common convolution to the Colombeau theoretic setting (cf. [Fol95], [Gar06]). For this purpose we have some preparatory observations:

1.21 Definition. A subset W of $\mathbb{R}^n \times \mathbb{R}^n$ is called *proper* if $\pi_1^{-1}(B) \cap W$ and $\pi_2^{-1}(B) \cap W$ are compact in $\mathbb{R}^n \times \mathbb{R}^n$ for all compact subsets B of $\mathbb{R}^n$. Here, π_1 and π_2 are the projection maps from $\mathbb{R}^n \times \mathbb{R}^n$ onto the first and second factors, respectively.

Because we will use "proper" related with the support, the following observation is quite helpful : A generalized function on $\mathbb{R}^n \times \mathbb{R}^n$ with a proper support has a representative $(v_\varepsilon)_\varepsilon$ such that every v_ε has proper support. More precisely, let $u \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$ with proper support and choose a cut-off $\chi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi|_{\mathrm{supp}(u)} \equiv 1$ and $\mathrm{supp}(\chi)$ proper, whose existence is shown in [Fol95, Prop 8.14]. If $(u_\varepsilon)_\varepsilon$ is a representative of $u \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$, then $(\chi.u_\varepsilon)_\varepsilon$ is also a representative and the support of $(\chi.u_\varepsilon)$ is proper:
By assumption, for all compact subsets B of $\mathbb{R}^n$ we have $\pi_1^{-1}(B) \cap \mathrm{supp}(\chi)$ and $\pi_2^{-1}(B) \cap \mathrm{supp}(\chi)$ are compact in $\mathbb{R}^n \times \mathbb{R}^n$, and so in particular closed. Furthermore, since the closed set $\mathrm{supp}(\chi.u_\varepsilon)$ is a subset of $\mathrm{supp}(\chi)$, we know that $\pi_1^{-1}(B) \cap \mathrm{supp}(\chi.u_\varepsilon)$ is a closed subset of $\pi_1^{-1}(B) \cap \mathrm{supp}(\chi)$ and therefore compact in $\mathbb{R}^n \times \mathbb{R}^n$. Doing the same for the set $\pi_2^{-1}(B) \cap \mathrm{supp}(\chi.u_\varepsilon)$, we can conclude that $\mathrm{supp}(\chi.u_\varepsilon)$ is proper.

In the following we have to distinguish exactly between fixed and variable parameters. For this we establish the following notations: a fixed variable is always denoted by a superscript star and $u^{(1)}(x^*) := u(x^*, .)$ for $u \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$. More precisely, $u^{(1)}(x^*) = [(u_\varepsilon(x^*, .))_\varepsilon]$ depends on the second parameter, while the first parameter is fixed.

1.22 Lemma. Suppose $u \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$ with proper support, then

$$u^{(1)}(x^*) = u(x^*, .) \in \mathcal{G}_c(\mathbb{R}^n) \quad \text{for every fixed } x^* \in \mathbb{R}^n. \tag{1.42}$$

Proof: Choosing a cut-off $\chi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi|_{\mathrm{supp}(u)} \equiv 1$ and $\mathrm{supp}(\chi)$ proper, we have to show that there is a $K \Subset \mathbb{R}^n$ such that $\mathrm{supp}((\chi.u_\varepsilon)^{(1)}(x^*)) \subseteq K$ for a fixed point $x^* \in \mathbb{R}^n$.
For this purpose, we define the set $K := \pi_2(\mathrm{supp}(\chi.u_\varepsilon) \cap \pi_1^{-1}(x^*))$, which fulfills the following requirements: On the one hand it is compact in $\mathbb{R}^n$, because $\mathrm{supp}(\chi.u_\varepsilon) \cap \pi_1^{-1}(x^*)$ is compact by the preceding observation and, on the other hand, if $z \in \mathbb{R}^n \setminus K$, it follows that $(x^*, z) \in \pi_1^{-1}(x^*)$ and therefore $(x^*, z) \notin \mathrm{supp}(\chi.u_\varepsilon)$. In particular, this means that $z \notin (\mathrm{supp}((\chi.u_\varepsilon)^{(1)}(x^*)))^\circ$. □

We are now interested in finding out what happens, if we apply a functional to such an element of the form $u^{(1)}(x^*) \in \mathcal{G}_c(\mathbb{R}^n)$. In the setting of functionals we also introduce a special notation: for a basic functional T we denote by $T(u^{(1)}(.))$ the class of $(x \mapsto T_\varepsilon(u_\varepsilon^{(1)}(x)))_\varepsilon$ and prove in the following proposition that this gives a well-defined generalized function.

1.23 Proposition. (i) Let $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $u \in \mathcal{G}_c(\mathbb{R}^n \times \mathbb{R}^n)$ then $T(u^{(1)}(.))$ is a well-defined element of $\mathcal{G}_c(\mathbb{R}^n)$.
(ii) Let $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $u \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$ with proper support then $T(u^{(1)}(.)) \in \mathcal{G}(\mathbb{R}^n)$.

Proof: (i) Let $u \in \mathcal{G}_c(\mathbb{R}^n \times \mathbb{R}^n)$ and $(u_\varepsilon)_\varepsilon \in u$, then there are subsets $K_1, K_2 \Subset \mathbb{R}^n$ such that $\mathrm{supp}(u_\varepsilon) \subseteq K_1 \times K_2$ for all $\varepsilon \in (0,1]$. Obviously, $u^{(1)}(x^*) \in \mathcal{G}_c(\mathbb{R}^n)$ for a fixed point $x^* \in \mathbb{R}^n$ and $u_\varepsilon^{(1)}(x^*) \in \mathcal{D}_{K_2}(\mathbb{R}^n)$. Since T is a basic functional, we obtain:

$$\exists (T_\varepsilon)_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)^{(0,1]}\ \exists N \in \mathbb{N}\ \exists j \in \mathbb{N}\ \exists \eta \in (0,1]\ \forall \varepsilon \in (0,\eta] :$$
$$|T_\varepsilon(u_\varepsilon^{(1)}(x^*))| \le \varepsilon^{-N} \sup_{y \in K_2, |\beta| \le j} |\partial_y^\beta u_\varepsilon^{(1)}(x^*)|. \tag{1.43}$$

Since $x^* \in \mathbb{R}^n$ is arbitrary, (1.43) is true for all $x^* \in \mathbb{R}^n$ and it follows that $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{c,M}(\mathbb{R}^n \times \mathbb{R}^n)$ implies $(x \mapsto T_\varepsilon(u_\varepsilon^{(1)}(x)))_\varepsilon \in \mathcal{E}_{c,M}(\mathbb{R}^n)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{N}_c(\mathbb{R}^n \times \mathbb{R}^n)$ implies $(x \mapsto T_\varepsilon(u_\varepsilon^{(1)}(x)))_\varepsilon \in \mathcal{N}_c(\mathbb{R}^n)$. Let us emphasize that $T(u^{(1)}(.)) = [(x \mapsto T_\varepsilon(u_\varepsilon^{(1)}(x)))_\varepsilon]$ depends on the parameter x which is no longer fixed.
It remains to show that $T(u^{(1)}(.))$ is independent of the choice of the net $(T_\varepsilon)_\varepsilon$ which determines T. Suppose $(T'_\varepsilon)_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)^{(0,1]}$ is another net defining T and $(x_\varepsilon)_\varepsilon \in \tilde{x}$ a generalized point in $\tilde{\mathbb{R}}^n_c$. Since $u^{(1)}(\tilde{x}) := [(u_\varepsilon(x_\varepsilon, .))_\varepsilon] \in \mathcal{G}_c(\mathbb{R}^n)$ we have that $((T_\varepsilon - T'_\varepsilon)(u_\varepsilon(x_\varepsilon, .))) \in \mathcal{N}$ and therefore $((T_\varepsilon - T'_\varepsilon)(u_\varepsilon(x, .))) \in \mathcal{N}_c(\mathbb{R}^n)$.

(ii) Once more we choose a cut-off $\chi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi|_{\mathrm{supp}(u)} \equiv 1$ and proper support. Lemma 1.22 guarantees that $u^{(1)}(x^*) \in \mathcal{G}_c(\mathbb{R}^n)$ and so $T(u^{(1)}(x^*))$ is a well-defined element for all $x^* \in \mathbb{R}^n$. Since it is basic, there is a net of distributions $(T_\varepsilon)_\varepsilon$ such that $T(u^{(1)}(x^*)) = [(T_\varepsilon(u_\varepsilon^{(1)}(x^*)))_\varepsilon]$. For all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ we obtain therefore

$$\psi(x)T(u^{(1)}(.)) = [(\psi(x)T_\varepsilon(\chi^{(1)}(x)u_\varepsilon^{(1)}(x)))_\varepsilon] \in \mathcal{G}_c(\mathbb{R}^n). \tag{1.44}$$

Our aim is now to choose a family of cut-off functions $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so that (1.44) becomes a coherent family of generalized functions. Let $(\Omega_\lambda)_{\lambda \in \Lambda}$ be a locally finite open covering of $\mathbb{R}^n$ and $(\psi_\lambda)_{\lambda \in \Lambda}$ be a family in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\psi_\lambda|_{K_\lambda} \equiv 1$ with $K_\lambda := \pi_2(\mathrm{supp}(u) \cap \pi_1^{-1}(\overline{\Omega_\lambda}))$. The latter is compact since the support of u is proper. These choices yield the coherent family $(\psi_\lambda(x)T(u^{(1)}(.))|_{\Omega_\lambda})_{\lambda \in \Lambda}$ of generalized functions in $\mathcal{G}(\mathbb{R}^n)$ and therefore by the sheaf property of $\mathcal{G}(\mathbb{R}^n)$ the assertion follows. $\square$

Since the definition of the convolution makes use of generalized functions of the form $u(x-y)$, we investigate them in the next lemma:

1.24 Lemma. (i) If $u \in \mathcal{G}(\mathbb{R}^n)$ then $u(x-y) \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$.
(ii) If $u \in \mathcal{G}_c(\mathbb{R}^n)$ then $u(x-y) \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$ and its support is proper.

Proof: (i) Let $(u_\varepsilon)_\varepsilon$ be a representative of $u \in \mathcal{G}(\mathbb{R}^n)$, then we have to show that $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ implies $(u_\varepsilon(x-y))_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n \times \mathbb{R}^n)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^n)$ implies $(u_\varepsilon(x-y))_\varepsilon \in \mathcal{N}(\mathbb{R}^n \times \mathbb{R}^n)$.
If $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$, then we know that

$$\forall K \Subset \mathbb{R}^n \ \forall \alpha \in \mathbb{N}_0^n \ \exists p \geq 0 : \sup_{z \in K} |\partial_z^\alpha u_\varepsilon(z)| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0 \tag{1.45}$$

and we have to show that

$$\forall K_1, K_2 \Subset \mathbb{R}^n \ \forall (\beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \ \exists q \geq 0 : \sup_{x \in K_1, y \in K_2} |\partial_x^\beta \partial_y^\gamma u_\varepsilon(x-y)| = O(\varepsilon^{-q}) \text{ as } \varepsilon \to 0.$$

Since $\sup_{x \in K_1, y \in K_2} |\partial_x^\beta \partial_y^\gamma u_\varepsilon(x-y)| \leq \sup_{z \in K_1 - K_2} |\partial_z^{\beta+\gamma} u_\varepsilon(z)|$, follows by (1.45) with $K := K_1 - K_2$ compact and $\alpha := \beta + \gamma$ the required property. Analogously we gain $(u_\varepsilon(x-y))_\varepsilon \in \mathcal{N}(\mathbb{R}^n \times \mathbb{R}^n)$ for $(u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^n)$.

(ii) By (i) we know that $u(x-y) \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$ and it remains to show that $u(x-y)$ has proper support, i.e. $\forall B \Subset \mathbb{R}^n : P_1 := \operatorname{supp}(u(x-y)) \cap \pi_1^{-1}(B)$ and $P_2 := \operatorname{supp}(u(x-y)) \cap \pi_2^{-1}(B)$ are compact in $\mathbb{R}^n \times \mathbb{R}^n$.
Let $u \in \mathcal{G}_c(\mathbb{R}^n)$, then there is a $K \Subset \mathbb{R}^n$ with $u \in \mathcal{G}_K(\mathbb{R}^n)$. Since $\operatorname{supp}(u(x^* - .)) = x^* - \operatorname{supp}(u)$ for a fixed $x^* \in \mathbb{R}^n$, $u(x^* - .) \in \mathcal{G}_{B-K}(\mathbb{R}^n)$ for all $x^* \in B$. Accordingly we obtain

$$P_1 = \operatorname{supp}(u(x-y)) \cap (B \times \mathbb{R}^n) \subseteq \bigcup_{x^* \in B} \operatorname{supp}(u(x^* - .) \subseteq B - K. \tag{1.46}$$

Clearly, $B - K$ is compact and $\operatorname{supp}(u(x-y)) \cap (B \times \mathbb{R}^n)$ a closed subset which yields that P_1 is compact since we operate in a metric space. In the same way we can deduce that P_2 is compact and therefore the support is proper. □

Assembling the previous results we can state the following definition:

1.25 Definition. The *convolution* $* : \mathcal{G}_c(\mathbb{R}^n) \times \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}}) \to \mathcal{G}(\mathbb{R}^n)$ is defined by

$$(u, T) \mapsto u * T(x) = T(u(x - .)). \tag{1.47}$$

This is well-defined: Using Lemma 1.24(ii) on $u \in \mathcal{G}_c(\mathbb{R}^n)$, we obtain that $u(x-y) \in \mathcal{G}(\mathbb{R}^n \times \mathbb{R}^n)$ with proper support. Therefore the assumptions of Proposition 1.23(ii) are

fulfilled and we obtain $u * T \in \mathcal{G}(\mathbb{R}^n)$.
Moreover, one can show that the convolution is separately continuous. For the extension of the convolution of two functionals, we are in particular interested in the continuity with respect to the first parameter:

1.26 Proposition. The $\tilde{\mathbb{C}}$-bilinear map

$$*_1 : \mathcal{G}_c(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n) : u \mapsto u * T \tag{1.48}$$

is continuous for a fixed $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$.

Proof: As we observed in Example 1.14 $\mathcal{G}_c(\mathbb{R}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{G}_{K_m}(\mathbb{R}^n)$ for $(K_m)_m$ an exhausting sequence of compact subsets of $\mathbb{R}^n$ such that $K_m \subseteq K_{m+1}$. Therefore the continuity of $*_1$ is equivalent to the continuity of $(*_1 \circ \iota_m)$ for all $m \in \mathbb{N}$ and $\iota_m : \mathcal{G}_{K_m} \hookrightarrow \mathcal{G}_c$.
Accordingly, we have to show for all $m \in \mathbb{N}$ the following continuity-property: $\forall L \Subset \mathbb{R}^n\ \forall l \in \mathbb{N}^n\ \exists K' \Subset \mathbb{R}^n\ \exists k \in \mathbb{N}^n\ \exists c > 0$ such that $\forall u \in \mathcal{G}_{K_m}(\mathbb{R}^n)$ and $(u_\varepsilon)_\varepsilon \in u$

$$p_{L,l}(*_1(u)) = \sup_{x\in L, |\alpha|\le l} |\partial_x^\alpha T(u_\varepsilon(x - .))| \ \le\ c \cdot p_{K',k}(u) = c \cdot \sup_{z\in K', |\beta|\le k} |\partial_z^\beta u_\varepsilon(z)|. \tag{1.49}$$

Let $K \Subset \mathbb{R}^n$ be an arbitrary element of $(K_m)_m$ and $L \Subset \mathbb{R}^n$. Since $(u_\varepsilon)_\varepsilon \in u \in \mathcal{G}_c(\mathbb{R}^n)$, there is a $K' \Subset \mathbb{R}^n$ such that $K \subseteq (K')^\circ$ and $\mathrm{supp}(u_\varepsilon) \subseteq K'$ for all $\varepsilon \in (0, 1]$. Moreover, if $u \in \mathcal{G}_K(\mathbb{R}^n)$ then $u(x^* - .) \in \mathcal{G}_{L-K}(\mathbb{R}^n)$ for all $x^* \in L$ because $\mathrm{supp}(u(x^* - .)) = x^* - \mathrm{supp}(u)$. Furthermore, for the basic functional T there is a net $(T_\varepsilon)_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)^{(0,1]}$ such that $Tu = [(T_\varepsilon(u_\varepsilon))_\varepsilon]$ satisfies the property (1.32). So we finally obtain that $\exists\ j, N \in \mathbb{N}$ such that the estimate

$$\begin{aligned}
\sup_{x\in L, |\alpha|\le l} |\partial_x^\alpha T_\varepsilon(u_\varepsilon(x - .))| &= \sup_{x\in L, |\alpha|\le l} |T_\varepsilon(\partial_x^\alpha u_\varepsilon(x - .))| \\
&\le \varepsilon^{-N} \sup_{x\in L, |\alpha|\le l}\ \sup_{y\in L-K', |\beta|\le j} |\partial_x^\alpha \partial_y^\beta (u_\varepsilon(x - y))| \\
&\le \varepsilon^{-N} \sup_{z\in K', |\gamma|\le l+j} |\partial_z^\gamma (u_\varepsilon(z))|
\end{aligned}$$

holds for all ε small enough. This leads to the required property (1.49), if we set $k := l + j$ and $c := \varepsilon^{-N}$. □

By the previous results, we are now able to define the convolution of two functionals:

1.27 Definition. Let S be in $\mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$. The *convolution* $S * T \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is defined by

$$S * T(u) = S_x(T_y(u(x + y))), \tag{1.50}$$

in which the index denotes the variable parameter of the functional.

This definition is meaningful, because (1.50) can be rephrased using the continuous map $\tilde{.} : \mathcal{G}_c(\mathbb{R}^n) \to \mathcal{G}_c(\mathbb{R}^n),\ \tilde{v}(y) := v(-y)$:

$$S_x((\tilde{u} * T_y)\tilde{\ }) = S_x(T_y(u(x+y))). \tag{1.51}$$

Moreover, considering Proposition 1.26, we obtain that the map $\mathcal{G}_c(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n) : u \mapsto (\tilde{u} * T)\tilde{\ }$ is continuous. Since S is a $\tilde{\mathbb{C}}$-linear and continuous functional, the composition $S_x((\tilde{u} * T_y)\tilde{\ }) = S * T \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$.

1.28 Proposition. If $S \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, then

$$\operatorname{supp}(S * T) \subseteq \operatorname{supp}(S) + \operatorname{supp}(T). \tag{1.52}$$

Proof: Set $A := \operatorname{supp}(T)$ and $B := \operatorname{supp}(S)$, then the set $A + B$ is a closed subset of $\mathbb{R}^n$. Let $V := \mathbb{R}^n \setminus (A + B)$ and $u \in \mathcal{G}_c(V)$. It follows that $(S * T)(u) = 0$, because $\operatorname{supp}(u(x+y)) \subseteq \{(x,y) : x + y \in V\}$. □

There is one more situation in which we are interested in, namely the case of two basic functionals. This is explained in the following corollary.

1.29 Corollary. Let S be in $\mathcal{L}_b(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, then the convolution $S * T$ is in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$.

Proof: Since S is a basic functional, likewise $S(T(u(x+y)))$ is basic, and combining this with the observations above, yields $S * T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$. □

As in the case of distributions, $\iota_d(\delta)$ is the neutral element of the convolution of functionals. This follows immediately from the definitions of the convolution.
Moreover, let us emphasize that different from the distributional convolution, in the Colombeau theoretic setting of dual spaces convolution is not commutative. More precisely, if for $S \in \mathcal{L}_1$ and $T \in \mathcal{L}_2$ the convolution $(\mathcal{L}_1 \times \mathcal{L}_2) \to \mathcal{L}_1 * \mathcal{L}_2,\ (S,T) \mapsto S * T$ is defined, then in general the convolution $(\mathcal{L}_2 \times \mathcal{L}_1) \to \mathcal{L}_2 * \mathcal{L}_1,\ (T,S) \mapsto T * S$ may not be defined. But in the context of partial differential operators this does not raise problems.

2 Solvability of partial differential operators with constant coefficients in $\tilde{\mathbb{C}}$

In this chapter we discuss the solvability of partial differential operators with constant coefficients in $\tilde{\mathbb{C}}$ that operate in dual spaces. After some observations about the symbol and the clarification of the meaning of a fundamental solution in dual spaces in 2.1, our aim is to extend the well-known Malgrange-Ehrenpreis Theorem for functionals. For this we use the fundamental solution for the distributional case constructed in [Hör76, (3.1.18)]. Since in dual spaces the fundamental solutions have an ε-dependence via the symbol, they provide a representative for a functional that solves the differential operator.
The initial point of our considerations in 2.3 is the following solvability result shown in [Hör04, Corollary 7.9]:

2.1 Theorem. Let P(D) be a differential operator with constant Colombeau coefficients, then the following two properties are equivalent:
(i) $\forall f \in \mathcal{G}_c(\mathbb{R}^n) : \ \exists u \in \mathcal{G}(\mathbb{R}^n) : \ P(D)u = f$
(ii) $\tilde{P}^2$ is invertible at some point in $\mathbb{R}^n$.

We extend this equivalence for the case of functionals, using a generalized Malgrange-Ehrenpreis Theorem, the convolution from the previous chapter and various properties of the symbol.

2.1 Basics and preparatory observations

Let P(D) be a differential operator of order m with constant Colombeau coefficients, i.e. coefficients in $\tilde{\mathbb{C}}$. For the symbol $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$, we construct a representative $(P_\varepsilon)_\varepsilon$ of P by choosing representatives of the coefficients of P. Note that for $(P_\varepsilon)_\varepsilon$ we obtain a net of polynomials.
Moreover, similar as in the classical case, we consider the function $\widetilde{P}^2 : \widetilde{\mathbb{R}}^n \to \widetilde{\mathbb{R}}$ defined by

$$\widetilde{P}^2(\xi) = \sum_{|\alpha| \le m} \partial^\alpha P \cdot \overline{\partial^\alpha P}, \tag{2.1}$$

or alternatively in terms of representatives

$$\widetilde{P}^2_\varepsilon(\xi) = \sum_{|\alpha|\leq m} |\partial_\xi^\alpha P_\varepsilon(\xi)|^2. \tag{2.2}$$

This notation is useful, since later we mainly work with $\widetilde{P} := \sqrt{\widetilde{P}^2}$ instead of $\widetilde{P}^2$.

In the following we investigate different spaces of functions and distributions that provide an appropriate background for the desired results.
By $\mathcal{K}$ we denote the set of all *tempered weight functions*, i.e. all positive functions k on $\mathbb{R}^n$ such that for some constants $C > 0$ and $N \in \mathbb{N}$ we have

$$k(\xi + \eta) \leq (1 + C|\xi|)^N k(\eta) \qquad \text{for all } \xi, \eta \in \mathbb{R}^n. \tag{2.3}$$

Furthermore, for $k \in \mathcal{K}$ and $p \in [1, \infty]$ we consider the space $\mathcal{B}_{p,k}(\mathbb{R}^n)$ that consists of all temperate distributions $w \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|w\|_{p,k} := (2\pi)^{-\frac{n}{p}} \|k\hat{w}\|_{L_p} < \infty, \tag{2.4}$$

and respectively, if $p = \infty$,

$$\|w\|_{\infty,k} := \text{ess sup}|k(\xi)\hat{w}(\xi)| < \infty, \tag{2.5}$$

where $\hat{w}$ denotes the Fourier transformation of w.

As shown in [Hör83, Theorem 10.1.7], $\mathcal{B}_{p,k}$ with the norm $\|w\|_{p,k}$ is a Banach space, for $p < \infty$ the space of all test functions $\mathcal{D}$ is dense in $\mathcal{B}_{p,k}$ and

$$\mathcal{S} \subset \mathcal{B}_{p,k} \subset \mathcal{S}'. \tag{2.6}$$

The situation of the dual spaces of $\mathcal{B}_{p,k}$ is explained in the next proposition proved in [Hör83, Theorem 10.1.14].

2.2 Proposition. If L is a continuous linear form on $\mathcal{B}_{p,k}$, $p \in [1, \infty)$, we have for some $v \in \mathcal{B}_{p',\frac{1}{k}}$ with $1/p + 1/p' = 1$ and $p' \in [1, \infty]$,

$$L(u) = \check{v}(u) \qquad \text{for u} \in \mathcal{S} \text{ and } \check{v}(u) := v(-u). \tag{2.7}$$

The norm of this linear form is $\|v\|_{p',\frac{1}{k}}$ and hence $\mathcal{B}_{p',\frac{1}{k}}$ is the dual space of $\mathcal{B}_{p,k}$.

For later purposes let us also state the next proposition, whose proof can be found in [Hör76, (2.1.7) and Theorem 2.2.5].

2.3 Proposition. (i) If we define for $k \in \mathcal{K}$ the function

$$M_k(\xi) := \sup_{\eta \in \mathbb{R}^n} \frac{k(\xi+\eta)}{k(\eta)}, \quad \xi \in \mathbb{R}^n, \tag{2.8}$$

then M_k is the smallest function such that $k(\xi+\eta) \leq M_k(\xi)k(\eta)$ and $M_k \in \mathcal{K}$.

(ii) Let $u \in \mathcal{B}_{p,k}$ and $\varphi \in \mathcal{S}$, then it follows that $\varphi u \in \mathcal{B}_{p,k}$ and, moreover, we have the following estimate

$$\|\varphi u\|_{p,k} \leq \|\varphi\|_{1,M_k} \|u\|_{p,k}. \tag{2.9}$$

The spaces $\mathcal{B}_{p,k}$ provide an appropriate framework for our setting, because as we see in the next lemma, the above defined functions $\widetilde{P}_\varepsilon$ are tempered weight functions for all $\varepsilon \in (0,1]$:

2.4 Lemma. For the representative $(P_\varepsilon)_\varepsilon$ there is a constant $C > 0$ depending only on the dimension n and the order m of the differential operator such that the inequality

$$\widetilde{P}_\varepsilon(\xi+\eta) \leq (1+C|\xi|)^m \widetilde{P}_\varepsilon(\eta) \tag{2.10}$$

holds for all $\xi, \eta \in \mathbb{R}^n$ and all $\varepsilon \in (0,1]$.

Proof: Since P_ε is a polynomial of order m, from Taylor's formula follows

$$|\partial^\alpha P_\varepsilon(\xi+\eta)| \leq |\partial^\alpha P_\varepsilon(\eta)| + \sum_{|\beta|=1}^{m-|\alpha|} \frac{1}{\beta!} |\partial^\beta \partial^\alpha P_\varepsilon(\eta)||\xi|^{|\beta|} \quad \text{for all } \xi, \eta \in \mathbb{R}^n. \tag{2.11}$$

Therefore there is a constant $C > 0$, such that for all $\varepsilon \in (0,1]$ the asserted inequality (2.10) holds. We emphasize that the constant C is independent of the parameter ε. □

This leads us to the space $\mathcal{B}_{\infty,\widetilde{P}_\varepsilon}(\mathbb{R}^n)$ that turns out to be a meaningful framework for the existence of a solution of P(D).

But before we can continue with an existence result, we have to clarify the meaning of the concept of fundamental solutions in dual spaces of Colombeau elements. Since we need the Dirac delta function in the dual space, we recall from (1.20) that the embedding of $w \in \mathcal{D}'(\mathbb{R}^n)$ into $\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is given by $\iota_d(w)(u) = [(w(u_\varepsilon))_\varepsilon]$. This allows us to give a similar definition of the fundamental solution as in the classical case:

2.5 Definition. Let P(D) be a partial differential operator with constant Colombeau coefficients, then $E \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is called a *fundamental solution of P(D)* if $P(D)E = \iota_d(\delta)$ in $\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$.

Analogous to the classical setting, a solution for the inhomogeneous case $P(D)U = F$ can be found by convolution:

2.6 Proposition. Let P(D) be as above, $S \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ then

$$P(D)(S * T) = P(D)S * T = S * P(D)T. \tag{2.12}$$

Hence, if $F \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $E \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is a fundamental solution of P(D), then $U := F * E$ is a solution for $P(D)U = F$.

Proof: Let $u \in \mathcal{G}_c(\mathbb{R}^n)$ and denote the transpose of the partial differential operator by ${}^tP(D)$, then we have

$$\begin{aligned} P(D)S * T(u) &= (P(D)S)_x(T_y(u(x+y))) = S_x({}^tP(D)(T_y(u(x+y))) \\ &= S_x((P(D)T)_y(u)(x+y)) = S * P(D)T(u) \\ &= S_x(T_y({}^tP(D)u(x+y))) = (S*T)({}^tP(D)u) \\ &= P(D)(S*T)(u). \end{aligned}$$

Now suppose that $E \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is a fundamental solution and $F \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$, then we obtain by

$$P(D)U = P(D)(F * E) = F * P(D)E = F * \iota_d(\delta) = F \tag{2.13}$$

that $U := F * E$ is a solution of the inhomogeneous equation $P(D)U = F$. □

We emphasize that for using the concept of fundamental solutions as known from distribution theory in the Colombeau theoretic setting, it is essential to work in dual spaces. In the Colombeau algebra $\mathcal{G}$ the crucial property that the embedded delta-distribution is the neutral element of the convolution is not fulfilled. More precisely, choose a Colombeau-mollifier $\varrho \in \mathcal{S}(\mathbb{R})$, such that $\hat{\varrho} \geq 0$, $\hat{\varrho}$ is symmetric and $\|\hat{\varrho}\|_{L^1} \neq \frac{1}{2\pi}\|\hat{\varrho}\|^2_{L^2}$, then the corresponding model delta net $\varrho_\varepsilon(x) := \frac{1}{\varepsilon}\varrho(\frac{x}{\varrho})$, $(\varepsilon \in (0,1])$ is a representative of the delta distribution in $\mathcal{G}(\mathbb{R})$, but

$$(\varrho_\varepsilon * \varrho_\varepsilon - \varrho_\varepsilon)_\varepsilon \notin \mathcal{N}(\mathbb{R}) \tag{2.14}$$

and hence $\iota(\delta) * \iota(\delta) \neq \iota(\delta)$. Here ι means the embedding constructed by the above defined model delta net $(\varrho_\varepsilon)_\varepsilon$.

Indeeed, noting that $\varrho(-z) = \overline{\varrho(z)}$ and $\int_{\mathbb{R}} \varrho(z)dz = 1$, we obtain

$$\begin{aligned} |(\varrho_\varepsilon * \varrho_\varepsilon - \varrho_\varepsilon)(0)| &= \Big| \int_{\mathbb{R}} \varrho_\varepsilon(y)\Big(\varrho_\varepsilon(0-y) - \varrho_\varepsilon(0)\Big)dy\Big| \\ &= \Big|\frac{1}{\varepsilon}\int_{\mathbb{R}} \varrho(z)\Big(\varrho(-z) - \varrho(0)\Big)dz\Big| \end{aligned}$$

$$= \Big|\frac{1}{\varepsilon}\int_{\mathbb{R}} |\varrho(z)|^2 dz - \frac{1}{\varepsilon}\varrho(0)\int_{\mathbb{R}} \varrho(z)dz\Big|$$

$$= \frac{1}{\varepsilon}\left|\frac{1}{2\pi}\|\hat{\varrho}\|^2_{L^2} - \|\hat{\varrho}\|_{L^1}\right| \to +\infty \quad \text{as } \varepsilon \to 0$$

and therefore, since $\{0\}$ is a compact subset, we can conclude that $(\varrho_\varepsilon * \varrho_\varepsilon - \varrho_\varepsilon)_\varepsilon \notin \mathcal{N}(\mathbb{R})$.

2.2 A version of the Malgrange-Ehrenpreis Theorem for fundamental solutions in the space $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$

Our aim in this section is the proof of the following theorem, shown in [Gar08] and stemming from the Malgrange-Ehrenpreis Theorem in [Hör76].

2.7 Theorem. Let $P(D)$ be a differential operator with coefficients in $\tilde{\mathbb{C}}$ such that $\tilde{P}(\xi)$ is invertible at some $\xi_0 \in \mathbb{R}^n$, then there exists a fundamental solution $E \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$. More precisely, for every $c > 0$ and every representative $(P_\varepsilon)_\varepsilon$ of P exists a fundamental solution E given by a net of distributions $(E_\varepsilon)_\varepsilon$ such that for all $\varepsilon \in (0, 1]$

$$\frac{E_\varepsilon}{\cosh(\mathrm{c}|\mathrm{x}|)} \in \mathcal{B}_{\infty,\tilde{P}_\varepsilon}(\mathbb{R}^n) \quad \text{and} \quad \left\|\frac{E_\varepsilon}{\cosh(\mathrm{c}|\mathrm{x}|)}\right\|_{\infty,\tilde{P}_\varepsilon} \leq C_0, \tag{2.15}$$

where C_0 is a constant depending only on c, the dimension n and the order m.

We postpone the proof of the Theorem to the end of this section and discuss some results for fundamental solutions on the level of representatives before. The plan is to begin with investigating the symbol P in detail or rather more precisely, its representative $(P_\varepsilon)_\varepsilon$ that is a net of polynomials. Based on these properties we show the existence of a fundamental solution for $P_\varepsilon(D)$ and use this for the proof of the above Theorem.

For convenience let us denote the set of all polynomials on $\mathbb{R}^n$ with coefficients in $\mathbb{C}$ of degree less than m +1 by Π_m and moreover, for $p \in \Pi_m$ we define

$$\tilde{p}(\xi) := \sum_{|\alpha|\leq m} |\partial^\alpha p(\xi)|^2. \tag{2.16}$$

The following three lemmas offer estimates from below and above for $\tilde{p}$ and $p \in \Pi_m$.

2.8 Lemma. Let $A \subseteq \mathbb{R}^n$ be a bounded subset such that no polynomial of Π_m vanishes in A without vanishing identically. Then there exists a positive constant C_1 depending on A, such that for all $p \in \Pi_m$ and $\xi \in \mathbb{R}^n$ the inequality

$$C_1\tilde{p}(\xi) \le \sup_{\theta\in A} |p(\xi+\theta)| \tag{2.17}$$

holds.

Proof: Note that by assumption the set A has to contain at least m+1 elements, otherwise there would be at least one polynomial that vanishes on A. Since we look for an upper bound we can therefore assume without loss of generality that A has exactly m+1 elements.
Let $p \in \Pi_m$ and consider the m+1 equations $p(\theta) := a_\theta$ $(\theta \in A)$. By Lagrange's interpolation formula we then gain a representation of p, namely

$$p(\eta) = \sum_{\theta\in A} p(\theta)R_\theta(\eta), \tag{2.18}$$

where R_θ is the element of Π_m that is equal to 1 at θ and vanishes elsewhere in A. Replacing $p(\eta)$ by $p(\xi+\eta)$ in (2.18), leads to

$$p(\xi+\eta) = \sum_{\theta\in A} p(\xi+\theta)R_\theta(\eta). \tag{2.19}$$

By differentiation with respect to η we obtain

$$\partial^\alpha p(\xi) = \sum_{\theta\in A} p(\xi+\theta)\partial^\alpha R_\theta(0) \tag{2.20}$$

and conclude by summing up and using the triangle inequality that the required property

$$C_1\tilde{p}(\xi) \le \sup_{\theta\in A} |p(\xi+\theta)| \tag{2.21}$$

is valid. □

2.9 Lemma. Let $p \in \Pi_m$, then the following estimate holds:

$$\sup_{0\le k\le m} \inf_{|z|=k/m} |p(z)| \ge (4m+1)^{-m}|p(1)|, \tag{2.22}$$

where k takes integral values.

Proof: Let $z_1, ..., z_\mu$ be the zeros of p, then there is a constant C $\neq 0$ such that

$$p(z) = C\prod_{j=1}^{\mu}(z-z_j). \tag{2.23}$$

Note that the number of zeros μ is less or equal than m. Therefore there is a constant r :=k/m where $0 \leq k \leq m$, $k \in \mathbb{N}$, such that $\left| r - |z_j| \right| \geq 1/(2m)$ for all $j = 1, ..., \mu$. For $|z| = r$ we have

$$|p(z)| \;\geq\; |C| \prod_{j=1}^{\mu} \left| |z| - |z_j| \right| \;\geq\; |p(1)| \prod_{j=1}^{\mu} \frac{\left| r - |z_j| \right|}{1 + |z_j|} \tag{2.24}$$

since $|C| \geq |p(1)|$ and $1 + |z_j| \geq 1$. If we show that for every $j = 1, ..., \mu$

$$\frac{\left| r - |z_j| \right|}{1 + |z_j|} \;\geq\; \frac{1}{4m + 1}, \tag{2.25}$$

the proof is finished.
If $|z_j| < 1 + 1/(2m)$, the inequality (2.25) follows immediately. In the other case, i.e. if $|z_j| \geq 1 + 1/(2m)$, from the first derivative it follows that the function $\frac{|r-|z_j||}{1+|z_j|}$ is increasing in $|z_j|$ and therefore

$$\frac{\left| r - |z_j| \right|}{1 + |z_j|} \geq \frac{|r - (1 + 1/(2m))|}{2 + 1/(2m)} = \frac{1 - r + 1/(2m)}{2 + 1/(2m)} \geq \frac{1/(2m)}{2 + 1/(2m)} = \frac{1}{4m + 1}.$$

□

2.10 Lemma. Let $A \subseteq \mathbb{R}^n$ be a bounded subset such that no polynomial of Π_m vanishes in A without vanishing identically and set $A' := \{k\theta/m : \; 0 \leq k \leq m, k \in \mathbb{N} \text{ and } \theta \in A\}$. Then there is a constant $C_2 > 0$ such that

$$\tilde{p}(\xi) \leq C_2 \sup_{\theta \in A'} \inf_{|z|=1} |p(\xi + z\theta)| \tag{2.26}$$

for all $p \in \Pi_m$ and $\xi \in \mathbb{R}^n$.

Proof: Applying Lemma 2.9 to the polynomial $p(\xi + z\theta)$ in z, gives the inequality

$$(4m + 1)^m \sup_{0 \leq k \leq m} \inf_{|z|=k/m} |p(\xi + z\theta)| \geq |p(\xi + \theta)|. \tag{2.27}$$

Taking the supremum gives

$$(4m + 1)^m \sup_{\theta \in A} \sup_{0 \leq k \leq m} \inf_{|z|=k/m} |p(\xi + z\theta)| \geq \sup_{\theta \in A} |p(\xi + \theta)| \tag{2.28}$$

and we can use Lemma 2.8 to obtain a constant $C_1 > 0$ such that

$$C_1 \tilde{p}(\xi) \leq (4m + 1)^m \sup_{\theta \in A} \sup_{0 \leq k \leq m} \inf_{|z|=k/m} |p(\xi + z\theta)| = (4m + 1)^m \sup_{\theta \in A'} \inf_{|z|=1} |p(\xi + z\theta)|.$$

Choosing $C_2 := (4m + 1)^m / C_1 > 0$, concludes the proof. □

If we apply Lemma 2.10 to the net $(P_\varepsilon)_\varepsilon \in \Pi_m^{(0,1]}$, we obtain a constant $C > 0$ such that the estimate

$$\widetilde{P_\varepsilon}(\xi) \le C \sup_{\theta \in A'} \inf_{|z|=1} |P_\varepsilon(\xi + z\theta)| \tag{2.29}$$

holds for all $\xi \in \mathbb{R}^n$ and for all $\varepsilon \in (0, 1]$.

Although the following proposition only deals with fundamental solutions in the distributional setting, it plays an important role in the proof of our version of the Malgrange-Ehrenpreis Theorem. So let us recall some notation:
A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a *fundamental solution* for the differential operator P(D) with constant coefficients, if $P(D)E = \delta$. Introducing $\check{E}(u) := E(-u) = E * u(0)$ for $u \in \mathcal{D}(\mathbb{R}^n)$, we obtain that E is a fundamental solution if

$$\check{E}(P(D)u) = u(0) \quad \text{ for all } u \in \mathcal{D}(\mathbb{R}^n). \tag{2.30}$$

The latter follows from

$$u(0) = \check{E}(P(D)u) = E * P(D)u(0) = P(D)E * u(0) \tag{2.31}$$

and the observation that δ is the neutral element for the convolution of distributions.

Now we have the required background for the following existence result:

2.11 Proposition. Let A' be a finite subset of the ball $|\xi| < c$ for an arbitrary constant $c > 0$, such that (2.29) is valid for every net $(P_\varepsilon)_\varepsilon \in \Pi_m^{(0,1]}$. Let us fix a net $(P_\varepsilon)_\varepsilon$ and let $\varphi_{\theta,\varepsilon}$ $(\theta \in A', \varepsilon \in (0, 1])$ be measurable functions on $\mathbb{R}^n$ such that $\varphi_{\theta,\varepsilon} \ge 0$, $\sum_{\theta \in A'} \varphi_{\theta,\varepsilon} = 1$ and

$$\varphi_{\theta,\varepsilon}(\xi) > 0 \quad \Rightarrow \quad \widetilde{P_\varepsilon}(\xi) \le C \inf_{|z|=1} |P_\varepsilon(\xi + z\theta)|. \tag{2.32}$$

Then the formula

$$\check{E}_\varepsilon(u) = (2\pi)^{-n} \sum_{\theta \in A'} \int_{\mathbb{R}^n} \varphi_{\theta,\varepsilon}(\xi) \, \frac{1}{2\pi i} \int_{|z|=1} \frac{\hat{u}(\xi + z\theta)}{P_\varepsilon(\xi + z\theta) \cdot z} \, dz \, d\xi \qquad (u \in \mathcal{D}(\mathbb{R}^n)) \tag{2.33}$$

defines a fundamental solution $E_\varepsilon(u) := \check{E}_\varepsilon * u(0)$ of $P_\varepsilon(D)$ such that $E_\varepsilon / \cosh(c|x|) \in \mathcal{B}_{\infty,\widetilde{P_\varepsilon}}(\mathbb{R}^n)$. In particular, there exists a constant C_0 depending only on n, m and c such that

$$\left\| \frac{E_\varepsilon}{\cosh(c|x|)} \right\|_{\infty, \tilde{P}_\varepsilon} \le C_0 \tag{2.34}$$

for all $\varepsilon \in (0, 1]$.

Proof: To begin with, we show the existence of the required family $(\varphi_{\theta,\varepsilon})_{\theta\in A',\varepsilon\in(0,1]}$: for every $\varepsilon \in (0,1]$, let $q_\varepsilon(\xi)$ be the number of elements $\theta \in A'$ such that

$$\widetilde{P_\varepsilon}(\xi) \leq C \inf_{|z|=1} |P_\varepsilon(\xi + z\theta)| \tag{2.35}$$

is fulfilled. Lemma 2.10 guarantees that $q_\varepsilon(\xi) \geq 1$ and so we can define the required functions by

$$\varphi_{\theta,\varepsilon}(\xi) := \begin{cases} 1/q_\varepsilon(\xi), & \text{if (2.35) is valid,} \\ 0, & \text{else.} \end{cases} \tag{2.36}$$

These functions play an important role for the well-definedness of the integral in (2.33): if $\varphi_{\theta,\varepsilon} > 0$, we know by (2.32) that $\widetilde{P_\varepsilon}(\xi) \leq C \inf_{|z|=1} |P_\varepsilon(\xi + z\theta)|$ and hence in particular

$$0 < C \inf_{|z|=1} |P_\varepsilon(\xi + z\theta)|. \tag{2.37}$$

Moreover, since $\hat{u} \in \mathcal{S} \subseteq L^1$ and the functions $\varphi_{\theta,\varepsilon}$ are bounded, the integral is convergent.

Next we want to show that $E_\varepsilon/\cosh(c|x|) \in \mathcal{B}_{\infty,\widetilde{P_\varepsilon}}(\mathbb{R}^n)$: by property (2.32) and $0 \leq \phi_{\theta,\varepsilon} \leq 1$ we obtain for u $\in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned} |\check{E}_\varepsilon(u)| &\leq C(2\pi)^{-n-1} \sum_{\theta\in A'} \int\limits_{|z|=1} dz \int\limits_{\mathbb{R}^n} \frac{|\hat{u}(\xi + z\theta)|}{\widetilde{P_\varepsilon}(\xi)}\, d\xi \\ &= C(2\pi)^{-n-1} \sum_{\theta\in A'} \int\limits_{|z|=1} dz \int\limits_{\mathbb{R}^n} \frac{1}{\widetilde{P_\varepsilon}(\xi)} |(e^{-i\langle \xi, z\theta\rangle} u(\xi))\hat{}\ | d\xi \\ &= C \frac{1}{2\pi} \sum_{\theta\in A'} \int\limits_{|z|=1} dz \left\| e^{-i\langle .,z\theta\rangle} u(.) \right\|_{1,1/\tilde{P}_\varepsilon}. \end{aligned} \tag{2.38}$$

We claim that for an arbitrary $c > 0$ the estimate

$$\left\| e^{-i\langle .,z\theta\rangle} u \right\|_{1,1/\tilde{P}_\varepsilon} \leq \left\| \frac{e^{-i\langle .,z\theta\rangle}}{\cosh(c|.|)} \right\|_{1,M_{1/\tilde{P}_\varepsilon}} \|\cosh(c|.|)u(.)\|_{1,1/\tilde{P}_\varepsilon}$$

holds.

Since $1/\cosh(c|x|) \in \mathcal{S}$ and $e^{-i\langle x,z\theta\rangle}$ is a moderate function, it follows that the product $x \mapsto \frac{e^{-i\langle x,z\theta\rangle}}{\cosh(c|x|)}$ is an element of $\mathcal{S}$. By (2.6), it therefore follows that $\frac{e^{-i\langle .,z\theta\rangle}}{\cosh(c|.|)} \in \mathcal{B}_{1,M_{1/\tilde{P}_\varepsilon}}$ and we can deduce from Proposition 2.3(ii) that for an arbitrary $c > 0$

$$\left\| e^{-i\langle .,z\theta\rangle} u \right\|_{1,1/\tilde{P}_\varepsilon} = \left\| \frac{e^{-i\langle .,z\theta\rangle}}{\cosh(c|.|)} \cosh(c|.|)u(.) \right\|_{1,1/\tilde{P}_\varepsilon} \leq \left\| \frac{e^{-i\langle .,z\theta\rangle}}{\cosh(c|.|)} \right\|_{1,M_{1/\tilde{P}_\varepsilon}} \|\cosh(c|.|)u(.)\|_{1,1/\tilde{P}_\varepsilon}$$

is true and the claim is proven.

Considering that $\frac{e^{-i\langle .,z\theta\rangle}}{\cosh(c|.|)} \in \mathcal{B}_{1,M_{1/\tilde{P}_\varepsilon}}$ is bounded by a constant, this gives in summary that there is a constant $C_3 > 0$ such that

$$|\check{E}_\varepsilon(u)| \leq C_3 \left\|\cosh(c|.|)u(.)\right\|_{1,1/\tilde{P}_\varepsilon}. \tag{2.39}$$

Setting $E_{\varepsilon,c} := E_\varepsilon/\cosh(\mathrm{c}|\mathrm{x}|)$, yields therefore for $u \in \mathcal{D}(\mathbb{R}^n)$

$$|\check{E}_{\varepsilon,c}(u)| \leq C_3 \left\|u\right\|_{1,1/\tilde{P}_\varepsilon} \tag{2.40}$$

and that guarantees the continuity of the linear form $\check{E}_{\varepsilon,c}$ on $(\mathcal{D}, \|.\|_{1,1/\tilde{P}_\varepsilon})$. Since $\mathcal{D}$ is dense in $\mathcal{B}_{1,1/\tilde{P}_\varepsilon}$, $\check{E}_{\varepsilon,c}$ defines a continuous linear form on $\mathcal{B}_{1,1/\tilde{P}_\varepsilon}$ and we can apply Proposition 2.2. This gives the existence of a $v \in \mathcal{B}_{\infty,\widetilde{P}_\varepsilon}$ such that

$$\check{E}_{\varepsilon,c}(u) = \check{v}(u) \quad \text{for } u \in \mathcal{S} \tag{2.41}$$

which is equivalent to $E_{\varepsilon,c}(u) = v(u)$. Since $v \in \mathcal{B}_{\infty,\widetilde{P}_\varepsilon}$, we therefore obtain that

$$\|E_{\varepsilon,c}\|_{\infty,\widetilde{P}_\varepsilon} = \|v\|_{\infty,\widetilde{P}_\varepsilon} < \infty \tag{2.42}$$

and this shows that $E_\varepsilon/\cosh(c|x|) \in \mathcal{B}_{\infty,\widetilde{P}_\varepsilon}(\mathbb{R}^n)$.

It remains to show, that E_ε is indeed a fundamental solution: from (2.30) we know that this is equivalent to $\check{E}_\varepsilon(P_\varepsilon(D)u) = u(0)$ for $u \in \mathcal{D}$. Setting $w_\varepsilon := P_\varepsilon(D)v$ where $v \in \mathcal{D}(\mathbb{R}^n)$, we have $\hat{w}_\varepsilon(\zeta) = P_\varepsilon(\zeta)\hat{v}(\zeta)$ and

$$\begin{aligned}
\check{E}(P_\varepsilon(D)v) &= (2\pi)^{-n} \sum_{\theta\in A'} \int_{\mathbb{R}^n} \varphi_{\theta,\varepsilon}(\xi)\ \frac{1}{2\pi i} \int_{|z|=1} \frac{P_\varepsilon(\xi+z\theta)\hat{v}(\xi+z\theta)}{P_\varepsilon(\xi+z\theta)\cdot z}\ dz\ d\xi \\
&= (2\pi)^{-n} \sum_{\theta\in A'} \int_{\mathbb{R}^n} \varphi_{\theta,\varepsilon}(\xi)\ \hat{v}(\xi)\ d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} 1\cdot\hat{v}(\xi)\ d\xi \ = \ v(0).
\end{aligned}$$

In the last two steps we used the assumption that $\sum_{\theta\in A'} \varphi_{\theta,\varepsilon} = 1$ and Fourier's inversion formula. This completes the proof. □

In the proof of Theorem 2.7 we use the above constructed fundamental solutions $\check{E}_\varepsilon$ and prove that the net $(\check{E}_\varepsilon)_\varepsilon$ provide a fundamental solution for $P(D)$.

But before we have to investigate one more property of $\tilde{P}^2_\varepsilon$, namely an estimate from below:

2.12 Lemma. If there is a $\xi_0 \in \mathbb{R}^n$ such that $\tilde{P}^2(\xi_0)$ is invertible in $\tilde{\mathbb{C}}$ then $\tilde{P}^2$ is invertible in $\mathcal{G}(\mathbb{R}^n)$ and $\tilde{P} := \sqrt{\tilde{P}^2}$ is a well-defined Colombeau function. More precisely, there are constants $d > 0, N \geq 0$ and an $\varepsilon_0 \in (0,1]$ such that

$$\tilde{P}_\varepsilon^2(\xi) \geq \varepsilon^N (1 + d|\xi_0 - \xi|)^{-2m} \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and } \varepsilon \in (0, \varepsilon_0]. \tag{2.43}$$

Proof: By Lemma 2.4 there is a constant $d > 0$ such that for all $\xi, \eta \in \mathbb{R}^n$ and $\varepsilon \in (0,1]$

$$\tilde{P}_\varepsilon^2(\xi + \eta) \leq (1 + d|\eta|)^{2m} \tilde{P}_\varepsilon^2(\xi). \tag{2.44}$$

Since $\tilde{P}_\varepsilon^2(\xi_0)$ is invertible, it is strictly nonzero, i.e. for some $N > 0$ and $\varepsilon_0 \in (0,1]$ is $\varepsilon^N \leq \tilde{P}_\varepsilon^2(\xi_0)$ when $\varepsilon \in (0, \varepsilon_0)$.

Substituting $\eta = \xi_0 - \xi$ in (2.44) yields for all $\xi \in \mathbb{R}^n$ and $\varepsilon \in (0, \varepsilon_0)$

$$\varepsilon^N \leq \tilde{P}_\varepsilon^2(\xi_0) \leq (1 + d|\xi_0 - \xi|)^{2m} \tilde{P}_\varepsilon^2(\xi) \tag{2.45}$$

and therefore shows (2.43) and the smoothness of $\tilde{P}_\varepsilon$. By (2.43) we can furthermore deduce that $\tilde{P}^2$ is invertible as a generalized function on $\mathbb{R}^n$. □

Finally, we have all required results to prove a version of the Malgrange-Ehrenpreis Theorem for fundamental solutions in the space $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ as stated in Theorem 2.7.

Proof of Theorem 2.7: Let us fix a representative $(P_\varepsilon)_\varepsilon$ of P.
For the existence of a fundamental solution $E \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ for the differential operator P(D), we show that the net $\check{E} := (\check{E}_\varepsilon)_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)^{(0,1]}$ given by (2.33) determines a fundamental solution in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$.

So first we have to prove that the net $\check{E}(u) = [(\check{E}_\varepsilon(u_\varepsilon))_\varepsilon]$, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_c(\mathbb{R}^n)$, satisfies property (1.32), i.e.

$$\begin{gathered}\forall K \Subset \mathbb{R}^n \ \exists j \in \mathbb{N} \ \exists N \in \mathbb{N} \ \exists \eta \in (0,1] \ \forall u \in \mathcal{D}_K(\mathbb{R}^n) \ \forall \varepsilon \in (0, \eta] : \\ |\check{E}_\varepsilon(u)| \leq \varepsilon^{-N} \sup_{x \in K, |\alpha| \leq j} |\partial^\alpha u(x)|. \end{gathered} \tag{2.46}$$

By assumption, $\tilde{P}$ is invertible at some point $\xi_0 \in \mathbb{R}^n$ and therefore with Lemma 2.12 there are constants $d > 0, N \geq 0$ and an $\varepsilon_0 \in (0,1]$ such that

$$\tilde{P}_\varepsilon^2(\xi) \geq \varepsilon^N (1 + d|\xi_0 - \xi|)^{-2m} \qquad \text{for all } \xi \in \mathbb{R}^n \text{ and } \varepsilon \in (0, \varepsilon_0]. \tag{2.47}$$

With (2.38) this yields

$$\begin{aligned}|\check{E}_\varepsilon(u)| &\leq C(2\pi)^{-n-1}\sum_{\theta\in A'}\int_{|z|=1}\int_{\mathbb{R}^n}\frac{|\hat{u}(\xi+z\theta)|}{\varepsilon^N(1+C_1|\xi_0-\xi|)^{-m}}\,d\xi\;dz\\ &\leq C'\varepsilon^{-N}\sum_{\theta\in A'}\int_{|z|=1}\int_{\mathbb{R}^n}(1+|\xi|)^m|\hat{u}(\xi+z\theta)|\;d\xi\;dz\end{aligned}$$

and since $\hat{u}(\xi+z\theta)\in\mathcal{S}$ it follows that

$$|\check{E}_\varepsilon(u)|\leq C''\varepsilon^{-N}\sup_{y\in K,|\beta|\leq m+n+1}|\partial^\beta u(y)| \tag{2.48}$$

for all $u\in\mathcal{D}_K(\mathbb{R}^n)$ and ε small enough. This shows that $\check{E}\in\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n),\tilde{\mathbb{C}})$.
By construction and Proposition 2.11, we furthermore have that $\check{E}(u)=E*u(0)$ fulfills $P(D)E=\iota_d(\delta)$, hence is a fundamental solution and fulfills the properties (2.15). □

2.3 Solvability and its equivalences

To begin with, we state the following proposition about the invertibility of $\tilde{P}^2$ that is shown in [Hör04, Theorem 7.8] for generalized functions. Since the main part of the proof is based on properties of the symbol that are independent of the space of the solution, it can be easily transferred to functionals.

2.13 Proposition. Let $F\in\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n),\tilde{\mathbb{C}})$ be such that $F(u_0)$ is invertible in $\tilde{\mathbb{C}}$ for some $u_0\in\mathcal{G}_c(\mathbb{R}^n)$. If $P(D)U=F$ is solvable with $U\in\mathcal{L}(\mathcal{G}(\mathbb{R}^n),\tilde{\mathbb{C}})$, then $\tilde{P}^2$ is invertible in $\mathcal{G}(\mathbb{R}^n)$.

Proof: Suppose that $\tilde{P}^2$ is not invertible. Then, as observed in Lemma 2.12, $\tilde{P}^2(0)$ cannot be invertible and therefore is a zero-divisor [Hör10, Theorem 5.9]. Hence we may choose a representative $(b_\varepsilon)_\varepsilon$ of $\tilde{P}^2(0)$, which vanishes on a zero sequence of ε-values. This means that there is a sequence $(\nu_k)_k\in(0,1]^{\mathbb{N}}$ with $\nu_k\to 0$ as $\varepsilon\to\infty$ such that $b_{\nu_k}=0$ for $k\in\mathbb{N}$. Let us define a generalized number $c\in\tilde{\mathbb{R}}$ by the representative

$$c_\varepsilon=\begin{cases}1, & \text{if } \varepsilon=\nu_k \text{ for some } k\in\mathbb{N},\\ 0, & \text{else.}\end{cases} \tag{2.49}$$

Then $c\neq 0$, but $c\cdot a_\alpha=0$ for $|\alpha|\leq m$: first note that $\tilde{P}^2_\varepsilon(0)$ is of the form

$$\tilde{P}^2_\varepsilon(0)=\sum_{|\alpha|\leq m}|\partial^\alpha P_\varepsilon(0)|^2=\sum_{|\alpha|\leq m}(\alpha!)^2|a^\varepsilon_\alpha|^2, \tag{2.50}$$

where all terms are nonnegative. Since $(b_\varepsilon)_\varepsilon$ is a representative of $\tilde{P}^2(0)$, we have for all q that $|b_\varepsilon - \tilde{P}^2_\varepsilon(0)| = O(\varepsilon^q)$ $(\varepsilon \to 0)$ and can therefore deduce that $|a_\alpha^{\nu_k}| = O(\nu_k^q)$ $(k \to \infty)$ for all q. Therefore it follows that $c \cdot a_\alpha = 0$ for $|\alpha| \leq m$.
Since $P(D)U = F$ and $F(u_0)$ is invertible and therefore no zero-divisor, we obtain

$$0 \neq c \cdot F(u_0) = c \cdot P(D)U(u_0) = \sum_\alpha c a_\alpha D^\alpha u(x_0) = 0, \tag{2.51}$$

which produces a contradiction. □

Now the following theorem can be shown combining several previous results.

2.14 Theorem. Let P(D) be a partial differential operator with constant coefficients in $\tilde{\mathbb{C}}$, then the following are equivalent:
(i) $\tilde{P}^2$ is invertible at some point in $\mathbb{R}^n$.
(ii) For all $f \in \mathcal{G}_c(\mathbb{R}^n)$ there is a $u \in \mathcal{G}(\mathbb{R}^n)$ such that $P(D)u = f$.
(iii) For all $F \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ there is a $U \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ such that $P(D)U = F$.
(iv) For all $F \in \mathcal{L}_b(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ there is a $U \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ such that $P(D)U = F$.

Proof: Since in all three cases (ii)-(iv) there is certainly at least one inhomogeneity that is invertible at one point, we can apply Proposition 2.13 and hence proved that $\tilde{P}^2$ is invertible at some point in $\mathbb{R}^n$.
For the other equivalences, suppose (i) is true and let $E \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ be a fundamental solution of P(D) whose existence follows from Theorem 2.7.
For assertion (ii) our aim is to show that $u := f * E \in \mathcal{G}(\mathbb{R}^n)$ is a solution for $P(D)u = f$. In fact, Definition 1.25 guarantees that $f * E$ is a generalized function in $\mathcal{G}(\mathbb{R}^n)$. Moreover, since by (1.19) $f \in \mathcal{G}_c(\mathbb{R}^n) \subseteq \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$, Proposition 2.6 yields that $u := f * E$ is indeed a fundamental solution.
For (iii) assume that $F \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and prove that $U := F * E \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ solves $P(D)U = F$. In this case, by Definition 1.27 we have that $F * E \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ and by the virtue of Proposition 2.6 it follows that $F * E$ is the desired solution.
Likewise we can prove (iv), however, that $U := F * E$ is a functional in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ follows from Corollary 1.29. □

3 Extension of the convolution and several applications for solutions in dual spaces

In this chapter we study specific types of operators with Colombeau coefficients such as a Cauchy-Riemann operator and a Schrödinger equation with generalized initial data.
Before we discuss these partial differential equations in detail, we investigate once more the convolution and enlarge the class of functionals for which the operation is defined. In the following this also enables us to consider certain non-compactly supported inhomogeneities in partial differential equations and to construct solutions by convolving them with special fundamental solutions.

3.1 Convolution of two functionals in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$

As seen in Chapter 2 we have a guaranteed fundamental solution in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ for a large class of differential operators $P(D)$. By convolution with an inhomogeneity $F \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ this fundamental solution provides a solution for $P(D)U = F$. Since so far we only have a definition for the convolution of elements in $\mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ with $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, inhomogeneities with non-compact supports can not be considered. These are for example distributions in $\mathcal{D}'(\mathbb{R}^n) \setminus \mathcal{E}'(\mathbb{R}^n)$ like the Heaviside-distribution. Therefore an extended convolution would be really helpful and can be achieved similarly to the concept used in the distributional setting, as for example carried out in [Fri98, 5.3] and [Hör09, 4.3]. The important additional assumption is that the map

$$\operatorname{supp}(S) \times \operatorname{supp}(T) \to \mathbb{R}^n,\ (x, y) \mapsto x + y \tag{3.1}$$

is proper.

This property introduced in the following is different from the definition of proper in 1.21, where it describes a property for sets in $\mathbb{R}^n \times \mathbb{R}^n$.

3.1 Definition. Let $A \subseteq \mathbb{R}^n$ be a closed subset and $f : A \to \mathbb{R}^m$ be continuous. The function f is called proper, if for every compact subset $K \Subset \mathbb{R}^m$ the inverse image $f^{-1}(K)$ is compact in $\mathbb{R}^n$.

It can be shown that f is proper if and only if

$$\forall \eta > 0 \; \exists \gamma > 0 \; \forall x \in A: \; |f(x)| \le \eta \;\Rightarrow\; |x| \le \gamma. \tag{3.2}$$

The extension of convolution is achieved by the following theorem, in which we choose an appropriate cut-off-function and reduce the definition to cases already known.

3.2 Theorem. Let $S \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ such that the map

$$\operatorname{supp}(S) \times \operatorname{supp}(T) \to \mathbb{R}^n, \; (x, y) \mapsto x + y \;\text{ is proper,} \tag{3.3}$$

i.e. $\forall \eta > 0 \; \exists \gamma > 0$: $|x + y| \le \eta \Rightarrow \max(|x|, |y|) \le \gamma$.
For every $\eta > 0$ let $\chi_\eta \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off such that $\chi_\eta = 1$ on a neighbourhood of $\overline{B_\gamma(0)}$, where $\gamma > 0$ is chosen as above. Then the convolution $S * T \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is defined for every $u \in \mathcal{G}_c(\mathbb{R}^n)$ with $\operatorname{supp}(u) \subseteq B_\eta(0)$ by

$$S * T(u) := ((\chi_\eta S) * T)(u) = (\chi_\eta S)_x \Big(T_y(u(x + y)) \Big), \tag{3.4}$$

where the terms on the right-hand side mean convolution as in Definition 1.27 applied to $\chi_\eta S \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$. This convolution is well-defined.

Proof: We have to show that the restriction $((\chi_\eta S) * T)|_{B_\eta(0)}$ is independent of the choice of the cut-off. For this, let $\chi^1_\eta \in \mathcal{D}(\mathbb{R}^n)$ be another cut-off that is equal to 1 on a neighbourhood of $\overline{B_\gamma(0)}$. It follows that $\operatorname{supp}((\chi^1_\eta - \chi_\eta)S) \cap \overline{B_\gamma(0)} = \emptyset$ and furthermore

$$\operatorname{supp}((\chi^1_\eta - \chi_\eta) S * T) \cap \overline{B_\eta(0)} = \emptyset \tag{3.5}$$

holds: let $z \in \mathbb{R}^n$ with $|z| \le \eta$ and $z \in \operatorname{supp}((\chi^1_\eta - \chi_\eta) S * T)$. Since $(\chi^1_\eta - \chi_\eta) S \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, we know by Proposition 1.28 that

$$z \in \operatorname{supp}((\chi^1_\eta - \chi_\eta) S * T) \subseteq \operatorname{supp}((\chi^1_\eta - \chi_\eta) S) + \operatorname{supp}(T) \subset \operatorname{supp}(S) + \operatorname{supp}(T).$$

Hence $z = x + y$ with $x \in \operatorname{supp}((\chi^1_\eta - \chi_\eta) S)$ and $y \in \operatorname{supp}(T)$ and (3.3) implies that $x, y \in \overline{B_\gamma(0)}$. This gives a contradiction, because $x \in \operatorname{supp}((\chi^1_\eta - \chi_\eta) S) \cap \overline{B_\gamma(0)} = \emptyset$. Therefore $(\chi^1_\eta - \chi_\eta) S * T|_{B_\eta(0)} = 0$ and we obtain that the following holds on $B_\eta(0)$

$$(\chi^1_\eta S) * T = (\chi_\eta S) * T + ((\chi^1_\eta - \chi_\eta) S) * T = (\chi_\eta S) * T. \qquad \square$$

The restriction of the convolution defined above to $S \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ coincides with the Definition 1.27 for elements in $\mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$ and $\mathcal{L}_b(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$:
Since S has compact support, there is a constant $R > 0$ such that $\operatorname{supp}(S) \subseteq \overline{B_R(0)}$. If

$x \in \operatorname{supp}(S)$, $y \in \operatorname{supp}(T)$ and $|x+y| < \eta$, then it follows that

$$|y| \leq |x+y| + |x| \leq \eta + R =: \gamma \tag{3.6}$$

and therefore the map $\operatorname{supp}(S) \times \operatorname{supp}(T) \to \mathbb{R}^n$, $(x,y) \mapsto x+y$, is proper. In this case the cut-off χ_η in Theorem 3.2 is equal to 1 on a neighbourhood of $\overline{B_{\eta+R}(0)}$. Therefore we have for all $\eta > 0$, that $\chi_\eta S = S$ and

$$(\chi_\eta S) * T(u) = S * T(u). \tag{3.7}$$

This shows that the two definitions coincide for $S \in \mathcal{L}(\mathcal{G}(\mathbb{R}^n), \tilde{\mathbb{C}})$.

The convolution defined in Theorem 3.2 fulfills properties analogous to such proved for the convolution in Definition 1.27: $\operatorname{supp}(S*T) \subseteq \operatorname{supp}(S) + \operatorname{supp}(T)$ and $P(D)(S*T) = (P(D)S)*T = S*(P(D)T)$.

In general it is not clear whether a fundamental solution and an inhomogeneity considered in a partial differential equation will fulfill the property required in Theorem 3.2, but there are classes of functionals that guarantee this property.

3.3 Example. Consider the space

$$\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}}) := \{T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}}) : \exists a \in \mathbb{R} \text{ such that } \operatorname{supp}(T) \subseteq [a, \infty)\}. \tag{3.8}$$

Clearly, the embedded Heaviside-function $\iota_d(H) \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ is an element of $\mathcal{L}_b^+$.

We show that two elements of $\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ fulfill property (3.3), hence can be convolved according to (3.4). Indeed, let S and $T \in \mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ with $\operatorname{supp}(S) \subseteq [a_1, \infty)$ and $\operatorname{supp}(T) \subseteq [a_2, \infty)$, then we have to show, that

$$[a_1, \infty) \times [a_2, \infty) \to \mathbb{R}, \ (x,y) \mapsto x+y \tag{3.9}$$

is proper, i.e. $\forall \eta > 0 \ \exists \gamma > 0$: $|x+y| \leq \eta \Rightarrow \max(|x|, |y|) \leq \gamma$.
Let $\eta > 0$ be arbitrary, $x \in [a_1, \infty)$ and $y \in [a_2, \infty)$ with $|x+y| \leq \eta$. For $a := \min(a_1, a_2)$ we obtain $\max(|x|, |y|) \leq \eta + 2|a|$ and therefore the addition is proper on these sets.

Furthermore, since $\operatorname{supp}(S*T) \subseteq \operatorname{supp}(S) + \operatorname{supp}(T)$, we may conclude that $S*T$ remains in $\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ and therefore $\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ is a convolution algebra.

Similarly, the set

$$\mathcal{L}_b^-(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}}) := \{T \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}}) : \exists a \in \mathbb{R} \text{ such that } \operatorname{supp}(T) \subseteq [-\infty, a)\} \tag{3.10}$$

is a convolution algebra.

This example seems to extend the space of possible inhomogeneities, but the theorem on existence of a fundamental solution for a partial differential operator with constant Colombeau coefficients guarantees only a fundamental solution in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ and contains no information on the support of the solution. Nevertheless, there are operators with fundamental solutions in $\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ as shown in [Gar08, Chapter 2.3]:

3.4 Definition. A partial differential operator with constant Colombeau coefficients, defined on $\mathbb{R}^n$, is called an evolution operator with respect to $H_n := \{x \in \mathbb{R}^n : x_n \geq 0\}$ if it has a fundamental solution $E \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$ whose support is contained in H_n.

Since $\mathcal{L}_b^+$ is a convolution algebra with $\iota_d(\delta)$ as a neutral element, we obtain solutions for this kind of equations also for inhomogeneities $\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$.

3.2 Solutions for selected differential operators in dual spaces

Finally we discuss some applications where a solution in $\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ can be constructed by convolving the fundamental solution with the inhomogeneity.

3.2.1 A simple ordinary differential equation

We begin with the following ordinary differential operator

$$L_a = \frac{d}{dx} - a, \tag{3.11}$$

where $a \in \tilde{\mathbb{C}}$ (cf. [Gar08, 4.1.1]).

In the following we first investigate the operator

$$L := \frac{d}{dx} \quad \text{in} \ \ \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}}) \tag{3.12}$$

and consider the particular case, where $L = 0$. We show that every solution $T \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ is of the form $T = \lambda \in \tilde{\mathbb{C}}$: Let T be in $\mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ such that $\frac{d}{dx}T = 0$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\int \varphi = 1$, then every $u \in \mathcal{G}_c(\mathbb{R})$ can be written as

$$u(x) = \left(u(x) - \int u(s)ds \ \varphi(x) \right) + \int u(s)ds \ \varphi(x). \tag{3.13}$$

Furthermore

$$u(x) - \int u(s)ds\ \varphi(x) = \frac{d}{dx}\Big[- \int_x^\infty \Big(u(t) - \Big(\int u(s)\ ds\Big)\varphi(t)\Big)\ dt \Big] := v(x) \in \mathcal{G}_c(\mathbb{R}^n)$$

and hence by (3.13)

$$T(u) = T(v) + T\Big(\int u(s)ds\ \varphi(x)\Big) = \int u(s)ds\ T(\varphi). \tag{3.14}$$

By this, one can conclude that all fundamental solutions of the operator L are of the form

$$E = \iota_d(H) + \lambda \quad (\lambda \in \tilde{\mathbb{C}}). \tag{3.15}$$

Let $T' \in \mathcal{L}(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$ be another fundamental solution of L, then $L(T' - \iota_d(H)) = 0$, since $\iota_d(H)$ is a fundamental solution. By the above we have that $T' - \iota_d(H) \in \tilde{\mathbb{C}}$ and therefore the claim is proven.

To solve an inhomogeneous equation, note that $\mathrm{supp}(\iota_d(H)) = \mathrm{supp}(H) = [0, \infty)$ and therefore $\iota_d(H) \in \mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$. Hence a solution can be found by convolution not only for right-hand sides in $\mathcal{L}(\mathcal{G}(\mathbb{R}), \tilde{\mathbb{C}})$, but also in $\mathcal{L}_b^+(\mathcal{G}_c(\mathbb{R}), \tilde{\mathbb{C}})$.

Next we investigate the similar operator

$$L_a = \frac{d}{dx} - a \tag{3.16}$$

for a generalized complex number $a \in \tilde{\mathbb{C}}$.

For the particular case, where $a \in \tilde{\mathbb{C}}$ has a real part of log-type, i.e. $\exists \varepsilon_0 > 0$ such that $|a_\varepsilon| \leq \log \frac{1}{\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_0)$, we can describe the fundamental solution in detail. Namely, all fundamental solutions of L_a are then of the form

$$E = \iota_d(H)e^{ax} + \lambda e^{ax}, \tag{3.17}$$

where $\lambda \in \tilde{\mathbb{C}}$: by the condition of $a \in \tilde{\mathbb{C}}$, the function e^{ax} is a well-defined element in $\mathcal{G}(\mathbb{R})$. Let E be a fundamental solution of L_a, then

$$\begin{aligned}\frac{d}{dx}\Big(e^{-ax}E\Big)(u) &= -e^{-ax}E(u') = -E(e^{-ax}u') \\ &= -E((e^{-ax}u)' + ae^{-ax}u) = \frac{d}{dx}E(e^{-ax}u) - aE(e^{-ax}u) \\ &= \iota_d(\delta)(u),\end{aligned}$$

and, accordingly, $e^{-ax}E$ is a fundamental solution of $L = \frac{d}{dx}$. By (3.15) it follows that $e^{-ax}E = \iota_d(H) + \lambda$ for some $\lambda \in \tilde{\mathbb{C}}$ and therefore $E = \iota_d(H)e^{ax} + \lambda e^{ax}$.

3.2.2 A general way of constructing a fundamental solution

Before investigating further examples, let us discuss once more the possible approach to finding a fundamental solution, where we solve the equation on the level of representatives.

In the following, let P(D) be a differential operator with coefficients in $\tilde{\mathbb{C}}$ such that $\tilde{P}(\xi)$ is invertible at some $\xi_0 \in \mathbb{R}^n$. Then, by Theorem 2.7, there exists a fundamental solution in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$. Let $(P_\varepsilon)_\varepsilon$ be a representative of P and $E_\varepsilon \in \mathcal{D}'(\mathbb{R}^n)$ a fundamental solution for $P_\varepsilon(D)$ for all $\varepsilon \in (0,1]$ in the distributional sense. If $E := [(E_\varepsilon)_\varepsilon] \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^n), \tilde{\mathbb{C}})$, then E is a fundamental solution for $P(D)$ in the dual space: Since $P(D)E = \iota_d(\delta)$ means on the level of representatives $P_\varepsilon(D)E_\varepsilon(u_\varepsilon) = u_\varepsilon(0)$ for $(u_\varepsilon)_\varepsilon \in u$ in $\mathcal{G}_c(\mathbb{R}^n)$ and $u_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$, the latter is fulfilled by the distributional fundamental solution.

The constructed net $(E_\varepsilon)_\varepsilon$ is not automatically basic. In general there are nets that are solutions on the level of representatives but are not basic. Consider for example the operator $L = \frac{d}{dx}$, then the net provided by $E_\varepsilon := H + c_\varepsilon$ with $[(c_\varepsilon)_\varepsilon] \notin \tilde{\mathbb{C}}$ is a distributional fundamental solution for $P_\varepsilon(D)$, but $(E_\varepsilon)_\varepsilon$ does not define a basic functional, since for $u \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(u) \subseteq [0,\infty)$ and $\int u(x)dx = 1$ we have

$$|(H + c_\varepsilon)(u)| = |1 + c_\varepsilon| \geq \varepsilon^{-N} \tag{3.18}$$

for infinitely many ε as $\varepsilon \to 0$and all $N \in \mathbb{N}$ and hence this net does not define a basic functional.

In the very special case, where $P(D)U = F$ is solvable and the solution E_ε for $P_\varepsilon(D)U_\varepsilon = F_\varepsilon$ for every $\varepsilon \in (0,1]$ is uniquely determined, the constructed net $[(E_\varepsilon)_\varepsilon]$ is automatically basic, since it is the only one and a basic solution is guaranteed. Unique solutions are rare, but as shown in [Hör90, Theorem 7.3.2] there are operators that fulfill this property:

3.5 Theorem. If $f \in \mathcal{E}'(\mathbb{R}^n)$ then the equation $P(D)u = f$ has a solution $u \in \mathcal{E}'(\mathbb{R}^n)$ if and only if $\hat{f}(\zeta)/P(\zeta)$ is an entire function. In this case the solution is uniquely determined.

3.2.3 A generalized Cauchy-Riemann operator

Consider the operator

$$P(D) = \partial_t + ic\partial_x \quad \text{on } \mathbb{R}\times]0,\infty[\ , \text{ where } c \in \tilde{\mathbb{R}} \text{ strictly positive }. \tag{3.19}$$

Fixing a representative $(c_\varepsilon)_\varepsilon$ of c, we find for all $\varepsilon \in (0,1]$ a distributional fundamental solution for $P_\varepsilon(D) := \partial_t + ic_\varepsilon\partial_x$ and show that the functional, determined by these solutions, is basic.
A fundamental solution $E_\varepsilon(x,t)$ for $P_\varepsilon(D)$ can be obtained by modifying the fundamental solution $S := \frac{1}{2\pi(t+ix)}$ for the Cauchy-Riemann-Operator $\partial_t + i\partial_x$:
defining $\tilde{E}_\varepsilon(y,t) := E_\varepsilon(c_\varepsilon y, t)$, we obtain

$$\partial_t\tilde{E}_\varepsilon(y,t) + i\partial_y\tilde{E}_\varepsilon(y,t) = \partial_2 E_\varepsilon(c_\varepsilon y,t) + ic_\varepsilon\partial_1 E_\varepsilon(c_\varepsilon y,t) = \delta(c_\varepsilon y,t) = \frac{1}{c_\varepsilon}\delta \tag{3.20}$$

and hence $c_\varepsilon\tilde{E}_\varepsilon$ is a fundamental solution. Setting $\tilde{E}_\varepsilon := \frac{1}{c_\varepsilon}S$, we can conclude that

$$E_\varepsilon(x,t) = \tilde{E}_\varepsilon(\frac{1}{c_\varepsilon}x,t) = \frac{1}{c_\varepsilon}S(\frac{1}{c_\varepsilon}x,t) = \frac{1}{2\pi(c_\varepsilon t + ix)} \tag{3.21}$$

is the desired fundamental solution for $P_\varepsilon(D)$. To show that $(E_\varepsilon)_\varepsilon$ is a basic functional, we use the following coordinate transformation

$$\frac{1}{2\pi}\int_{\mathbb{R}^n}\frac{1}{c_\varepsilon t + ix}u(x,t)d(x,t) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}(\cos\theta - i\sin\theta)\ u(c_\varepsilon r\sin\theta, r\cos\theta)dr\ d\theta$$

and therefore obtain for $K \Subset \mathbb{R}^n$ and $u \in \mathcal{D}_K(\mathbb{R}^n)$

$$\begin{aligned}
|E_\varepsilon(u)| &\le \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}|(\cos\theta - i\sin\theta)\ u(c_\varepsilon r\sin\theta, r\cos\theta)|dr\ d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi}\int_0^{C(K)\frac{1}{c_\varepsilon}}|u(c_\varepsilon r\sin\theta, r\cos\theta)|dr\ d\theta \\
&\le C(K)\frac{1}{c_\varepsilon}\sup_{(x,t)\in K}|u(x,t)| \le \varepsilon^{-N}\sup_{(x,t)\in K}|u(x,t)|.
\end{aligned}$$

Accordingly, $E := [(E_\varepsilon)_\varepsilon] \in \mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^2), \tilde{\mathbb{C}})$ is a basic fundamental solution in the sense of Definition 2.5.

3.2.4 A generalized Schrödinger equation

In the following we investigate a Schrödinger equation with a constant generalized coefficient and a vanishing potential of the form

$$\partial_t u - ic\partial_x^2 u = 0 \qquad \text{for } c \in \tilde{\mathbb{R}} \text{ strictly nonzero.} \tag{3.22}$$

Again we begin by finding a distributional fundamental solution E_ε of $L_\varepsilon := \partial_t - ic_\varepsilon \partial_x^2$ by modifying $S := \frac{1}{\sqrt{4\pi it}} \exp(-\frac{x^2}{4it})$, a standard fundamental solution of the operator $\partial_t - i\partial_x^2$. Similar to Example 3.2.3 we obtain for $t > 0$ that

$$E_\varepsilon(x,t) = \frac{1}{\sqrt{4c_\varepsilon \pi it}} \exp\Big(- \frac{x^2}{4c_\varepsilon it}\Big). \tag{3.23}$$

The functional defined by $(E_\varepsilon)_\varepsilon$ is a fundamental solution in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R} \times (0,\infty)), \tilde{\mathbb{C}})$ for L: For a given subset $K \Subset \mathbb{R} \times (0,\infty)$ there are $j \in \mathbb{N}$ and $N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_K$ and ε sufficiently small the estimate

$$|E_\varepsilon(\varphi)| \ \leq \ \int_{\mathbb{R}^2} \frac{1}{|\sqrt{4c_\varepsilon \pi it}|} \ |\varphi(x,t)| \ d(x,t) \leq \varepsilon^{-N} \sup_{x \in K, |\alpha| \leq j} |\partial^\alpha \varphi(x,t)|$$

holds and therefore it is basic.

Based on this fundamental solution, one can solve the Cauchy problem

$$\begin{aligned} &\partial_t U - ic\partial_x^2 U = 0 \qquad \text{for } t > 0 \text{ and } c \in \tilde{\mathbb{R}} \text{ strictly nonzero} \\ &U(x,0) = u_0(x), \qquad u_0 \in \mathcal{G}_c(\mathbb{R}), \end{aligned} \tag{3.24}$$

by solving the problem

$$\begin{aligned} &\partial_t \tilde{U} - ic\partial_x^2 \tilde{U} = F \qquad \text{in } \mathbb{R}^2, \\ &\text{where } \tilde{U} = [(\tilde{u}_\varepsilon)_\varepsilon] \text{ with } \operatorname{supp}(\tilde{u}_\varepsilon) \subseteq \mathbb{R} \times [0,\infty[\\ &\text{and } F := [(f_\varepsilon)_\varepsilon] \text{ with } f_\varepsilon(x,t) := u_{0,\varepsilon}(x) \otimes \delta_0(t) \in \mathcal{E}'(\mathbb{R}^2). \end{aligned} \tag{3.25}$$

To solve this problem by convolution, it has to be guaranteed that the inhomogeneity $F := [(f_\varepsilon)_\varepsilon]$ is an element in $\mathcal{L}_b(\mathcal{G}(\mathbb{R}^2), \tilde{\mathbb{C}})$. This is always the case: For $(u_{0,\varepsilon})_\varepsilon \in u_0$ there is a compact subset $K' \Subset \mathbb{R}$ such that $u_{0,\varepsilon} \in \mathcal{D}_{K'}(\mathbb{R})$ for all $\varepsilon \in (0,1]$ and we have that

$$\Bigg[\Bigg(\int_{\mathbb{R}} u_{0,\varepsilon}(x) dx \Bigg)_\varepsilon \Bigg] \in \tilde{\mathbb{C}}, \tag{3.26}$$

i.e. there is a $N \in \mathbb{N}$ such that $\left| \int_{\mathbb{R}} u_{0,\varepsilon}(x)dx \right| = O(\varepsilon^{-N})$ and therefore there is a constant $C > 0$ and $\eta \in (0,1]$ such that for all $\varepsilon \in (0,\eta]$

$$\left| \int_{\mathbb{R}} u_{0,\varepsilon}(x)dx \right| \leq C\varepsilon^{-N}. \tag{3.27}$$

To show that the net $[(u_{0,\varepsilon} \otimes \delta_0)_\varepsilon]$ determines an element in $\mathcal{L}_b(\mathcal{G}(\mathbb{R}^2), \tilde{\mathbb{C}})$, estimate (1.30) has to be satiesfied: By (3.27) we gain for all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and ε sufficiently small that

$$\begin{aligned} |\langle u_{0,\varepsilon} \otimes \delta_0, \varphi\rangle| &= |\langle u_{0,\varepsilon}(x), \varphi(x,0)\rangle| = \left| \int_{K'} u_{0,\varepsilon}(x)\varphi(x,0)dx \right| \\ &\leq \sup_{x\in K'} |\varphi(x,0)| \left| \int_{K'} u_{0,\varepsilon}(x)dx \right| \;\leq\; C\varepsilon^{-N} \sup_{x\in K'} |\varphi(x,0)|. \end{aligned}$$

Hence $F := [(u_{0,\varepsilon} \otimes \delta_0)_\varepsilon]$ is indeed a functional in $\mathcal{L}_b(\mathcal{G}(\mathbb{R}^2), \tilde{\mathbb{C}})$ and therefore

$$\partial_t \tilde{U} - ic\partial_x^2 \tilde{U} = F \tag{3.28}$$

is solved by $\tilde{U} := F * E$, where $E = [(E_\varepsilon)_\varepsilon]$ is the fundamental solution from above.

Setting $u_\varepsilon := \tilde{u}_\varepsilon|_{\mathbb{R}\times]0,\infty[}$ we obtain a solution for the Cauchy problem (3.24): For $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we obtain

$$\langle u_0(x) \otimes \delta_0(t), \varphi(x,t)\rangle\Big|_{t>0} = 0, \tag{3.29}$$

because $\operatorname{supp}(u_0(x) \otimes \delta_0(t)) = \operatorname{supp}(u_0(x)) \times \{0\}$ and hence, for $t > 0$, $\operatorname{supp}(u_0(x) \otimes \delta_0(t)) \cap \operatorname{supp}(\varphi) = \emptyset$. Therefore $\partial_t u_\varepsilon - ic_\varepsilon \partial_x^2 u_\varepsilon = 0$ for $t > 0$.

It remains to show that the initial condition is fulfilled. Note that for $t > 0$ the solution $\tilde{u}_\varepsilon \in \mathcal{C}(]0,\infty[, \mathcal{D}'(\mathbb{R}))$, because we have

$$\begin{aligned} \langle \tilde{u}_\varepsilon, \varphi\rangle &= \langle E_\varepsilon * f_\varepsilon, \varphi\rangle \\ &= \langle (E_\varepsilon(.,t) * u_{0,\varepsilon})(x), \varphi(x,t)\rangle = \int_0^\infty \langle (E_\varepsilon(.,t) * u_{0,\varepsilon})(x), \varphi(x,t)\rangle dt, \end{aligned}$$

where the convolution is to be understood with respect to x.

That $\tilde{u}_\varepsilon(.,t)$ converges to $u_{0,\varepsilon}$ as $t \to 0$, is easier seen on the Fourier-transformed side. By [Hör90, Theorem 7.6.1] the Fourier transformation of E_ε with respect to x is given by

$$\hat{E}_\varepsilon(\xi,t) = \exp(-c_\varepsilon it\xi^2) \tag{3.30}$$

and therefore $\hat{E}_\varepsilon(\xi,t) \to 1$ as $t \to 0$. Hence we obtain that $E_\varepsilon(.,t)$ converges to the delta-distribution as t goes to 0 and in summary, the initial condition is satisfied and the Cauchy problem (3.24) is indeed solved.

We conclude this section by remarking that this type of Schrödinger equation is subject of various recent papers and is applicable to several topics of research (cf. [Hoo08]).
Focusing on the initial data, it is discussed in detail in [Hör11], that data which are square roots of probability measures, hence clearly are no functions in the classical sense, are of particular interest.
More precisely, in quantum mechanics the square of the modulus of a solution to the standard Schrödinger equation is usually interpreted as a probability density. Therefore the question arises to turn this around and consider a generalized initial data which represents a square root of a given arbitrary probability measure. In [Hör11, Proposition 2.1] it is shown that for a probability measure μ on $\mathbb{R}$, there is an element $\phi \in \mathcal{G}(\mathbb{R})$, such that ϕ^2 is associated with μ. This means that for any representative (ϕ_ε) of ϕ, $\phi_\varepsilon \to \mu$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0$ and is denoted by $\phi^2 \approx \mu$.
To avoid problems with the convolution, we additionally assume that the probability measure μ has compact support and hence $\phi \in \mathcal{G}_c(\mathbb{R})$.
In summary, we obtain the Cauchy problem

$$\begin{aligned} &\partial_t U - ic\partial_x^2 U = 0 \qquad \text{for } c \in \tilde{\mathbb{R}} \text{ strictly nonzero} \\ &U(x,0) = \phi, \end{aligned} \tag{3.31}$$

that can also be solved in $\mathcal{L}_b(\mathcal{G}_c(\mathbb{R}^2), \tilde{\mathbb{C}})$ by the method presented above.

The Schrödinger equation with this kind of coefficients is also discussed in recent papers on semiclassical quantum dynamics (cf. [Spa12], [Mar10], [Mar12]). The authors rescale all physical parameters such that only one semi-classical parameter $0 < \varepsilon \leq 1$ remains and investigate the following problem

$$\begin{aligned} &i\varepsilon\partial_t u_\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2 u_\varepsilon - V(x)u_\varepsilon = 0, \\ &u_\varepsilon(x,0) = u_{0,\varepsilon}(x) \in L^2(\mathbb{R}). \end{aligned} \tag{3.32}$$

This problem is yet not considered in the framework of generalized functions, although this would be possible: remaining in the case of constant coefficients we neglect the potential and obtain

$$\partial_t u_\varepsilon - i\frac{\varepsilon}{2}\partial_x^2 u_\varepsilon = 0, \tag{3.33}$$

which is exactly an equation of the type considered above, since $[(\frac{\varepsilon}{2})_\varepsilon] \in \tilde{\mathbb{R}}$ and is strictly nonzero.

Bibliography

[Fri98] F.G. Friedlander and M. Joshi: Introduction to the theory of distribution. *Cambridge University Press, second edition, 1998*

[Fol95] G.B. Folland: Introduction to partial differential equations. *Princeton University Press, second edition, 1995*

[Gar05a] C. Garetto: Topological structures in Colombeau algebras: Topological $\tilde{\mathbb{C}}$ -modules and duality theory. *Acta. Appl. Math. 88/1, 81-123, 2005*

[Gar05b] C. Garetto: Topological structures in Colombeau Algebras: Investigation of the Duals of $\mathcal{G}_c(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^n)$. *Monatsh. Math. 146/3, 203-226, 2005*

[Gar05c] C. Garetto, T. Gramchev and M. Oberguggenberger: Pseudodifferential operators with generalized symbols and regularity theory. *Electron. J. Diff. Eqns. No. 116, 1-43, 2005*

[Gar06] C. Garetto: Microlocal analysis in the dual of a Colombeau algebra: generalized wave front sets and noncharacteristic regularity. *New York J. Math. 12, 275-318, 2006*

[Gar08] C. Garetto: Fundamental solutions in the Colombeau framework: applications to solvability and regularity theory. *Acta. Appl. Math. 102, 281-318, 2008*

[Gro01] M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer: Geometric Theory of Generalized Functions with Applications to General Relativity. *Mathematics and its Applications 537, Kluwer, 2001*

[Hoo08] M. de Hoop, G. Hörmann and M. Oberguggenberger: Evolution systems for paraxial wave equations of Schrödinger-type with non-smooth coefficients. *Jour. Diff. Equs. 245, 1413-1432, 2008*

[Hör76] L. Hörmander: Linear partial differential operators. *Springer-Verlag, Berlin, 1976*

[Hör90] L. Hörmander: The Analysis of Linear Partial Differential Operators, volume I. *Springer-Verlag, second edition, 1990*

[Hör83] L. Hörmander: The Analysis of Linear Partial Differential Operators, volume II. *Springer-Verlag, 1983*

[Hör04] G. Hörmann and M. Oberguggenberger: Elliptic regularity and solvability for partial differential equations with Colombeau coefficients. *Electr. Jour. Diff. Equ., Vol. 2004, 1-30*

[Hör09] G. Hörmann and R. Steinbauer: Theory of distributions. *Lecture notes, University of Vienna, 2009*

[Hör10] G. Hörmann: Ausgewählte Kapitel aus Moderne Analysis. *Lecture notes, University of Vienna, 2010*

[Hör11] G. Hörmann: The Cauchy problem for Schrödinger-type partial differential operators with generalized functions in the principal part and as data. *Monatsh. Math. 163, 445-460, 2011*

[Mar10] P. Markowich, T. Paul and C. Sparber: Bohmian measures and their classical limit. *J. Funct. Anal. 259, 1542-1576, 2010*

[Mar12] P. Markowich, T. Paul and C. Sparber: On the dynamics of Bohmian measures. *to appear in Archive Ration. Mech. Anal., 2012*

[Sch66] H. H. Schaefer: Topological Vector Spaces. *New York, 1966*

[Spa12] C. Sparber: Semiclassical Quantum Dynamics and Bohmian Trajectories. *Internat. Math. Nachrichten Nr. 219, 1-11, 2012*

Curriculum vitae

Name	Gudrun Szewieczek
Date of Birth	22.11.1985
Place of Birth	Linz, Austria
Nationality	Austria

1992 - 1996	Primary school, Leonding
1996 - 2000	Grammar school, Khevenhüllerstraße, Linz
2000 - 2005	Highschool Stifterstraße, Linz
06/2005	Matura passed with distinction
since 10/2005	**Diploma studies of Mathematics**, University of Vienna
06/2008	1st diploma passed with distinction
2008 - 2012	Tutor, University of Vienna
09/2009	Scholarship, MathMods summer school in Alba Adriatica, Italy
since 10/2005	**Instrumental studies**, University of Music & Performing Arts Vienna
07/2006	1st diploma passed with distinction
03/2011	2nd diploma passed with distinction