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Contents

Chapter 1

Introduction

Anticipating the consequences of one's action is a prerequisite for sensible choice. On grounds of hypothetical states of the world, alternatives can be evaluated and decisions can be taken. The quality of a choice, however, can only be evaluated ex post. When discrepancies between anticipated and actual consequences have been revealed by time, future expectations might be aligned. This observation supposedly holds true for the aggregate. The course of an economic system is fundamentally driven by expectations. Expectations, on the other hand, are driven by the course of the economy, if they are not formed in a completely arbitrary way.

Evans and Honkapohja (2001) note that systematic theories in which expectations play a role began as early as Henry Thornton's treatment of paper credit, published in 1802. In the 1930s, as Hoover (1992) remarks, thoughtful discussions about economic behaviour and about the role of expectations were prominent and the debates among Keynes, Hicks, Hayek and others were subtle and full of detail. However, from today's perspective a notable feature is that their analysis was mainly conducted in words. The formalisation of ideas was not attributed much relevance until Frisch (1933) and Tinbergen (1936) pioneered the extensive use of mathematics in conducting a formal theory. Their work is regarded as having marked the foundations of modern macroeconomics. From then on, ideas about reasonable behaviour were captured in mathematical terms.

In the 1950s, economists started to explore the microeconomic behaviour that was presumed to lie behind economic aggregates.[1] Relations, formerly assumed, were sought

[1] Important contributions in this respect were made by Modigliani and Brumberg (1954) and Friedman (1957) working on the consumption function as well as Baumol (1952) and Tobin (1956) who

to being explained by optimisation conditions. In a dynamic environment, optimal behaviour crucially rests on believes about upcoming states of the world. Thus, in course of this exploration, expectations were also paid more interest.

One of the most successful proposals at that time, on how believes should be modelled, was provided by Cagan (1956) and Friedman (1957).[2] They suggested that expectations are updated in light of the most recently observed forecast error and formulated the Adaptive Expectations Hypothesis (AEH). Despite being empirically tractable, the AEH turned out to be unattractive from a theoretical point of view since the algorithm required additional ad-hoc assumptions on a free parameter in order to provide a testable theory. Micro-foundations were already extensively employed and optimising behaviour regarded as being the grounds for individual choice. Expectations, however, being an integral part of individual choice had an arbitrary component in Cagan and Friedman's formulation.

Muth (1960) was aware of this problem and suggested to consistently expand the concept of optimising behaviour from the allocation of resources to that of the formation of expectations. He examined the AEH and showed that the algorithm could be used as an optimal estimator of some variable if and only if the variable under consideration was following an integrated moving average process of order (1,1). In this case, the supposedly free parameter of the algorithm was in fact uniquely determined by the structure of the underlying stochastic process. In any other instance, however, predictions based on adaptive expectations would be biased, implying that agents do not have access to all pertinent information or that they do not learn sufficiently from the past.

In 1961, Muth offered the Rational Expectations Hypothesis (REH) as an alternative. He observed that dynamic economic models use public expectations of variables as inputs and concurrently generate predictions of those same variables. He argued that agents should be expected to make use of the model if the prediction of the theory was substantially better than the public expectation; since rationality has been assumed in all other aspects of human behaviour. Each use of the model, however, would marginally exploit the model's information advantage. Thus, ultimately, informational differences should be depressed to zero and Muth concludes: " expectations of firms (or, more generally, the subjective probability distribution of outcomes) tend to be distributed, for the

provided micro-foundations for money demand equations, commonly used in Keynesian and Monetarist macroeconomics at that time.

[2]Ezekiel (1938) was the first to analyse the role of expectations in a simple dynamic model. His notion of naive expectations, however, overly simplified the information processing capacity of human beings.

same information set, about the prediction of the theory (or the "objective" probability distributions of outcomes)" (Muth, 1961, p. 3). Despite Muth's convincing argumentation, his insight was largely neglected for a decade until Lucas (1972b) worked out the implications of the REH for the Phillips curve. Hoover (1992) remarks that from this initial spark a great fire raced through macroeconomics, which did not consume anything completely but singed almost everything.[3]

In contemporary economic theory the REH is the predominant paradigm in expectation formation. Its popularity and widespread impact can be partly attributed to the fact that Rational Expectations (RE) are consistent with the principle of optimising behaviour, irrespective of the underlying stochastic process. Since the specific form of the forecast is, by definition, endogenously determined by the structure of the problem, RE are applicable to any dynamic environment. The theoretical convenience that different markets or systems do not have to be treated in completely different ways was already pointed out by Muth (1960). Sargent (1993) characterises RE by the two complementary requirements of individual rationality (i.e. that each agent is optimising an objective function subject to some constraints) and mutual consistency of perceptions (i.e. that the constraints perceived are mutually consistent). This definition most clearly reveals that RE constitute an equilibrium concept lying at the very heart of competitive (Walrasian) equilibria.

Note that RE crucially rest on the self-referential feature of economic models. Agents form expectations by conditioning on the true probability distribution of the system which in turn depends on these expectations. A rational expectations equilibrium (REE) is therefore characterised by a self-fulfilling prophecy; or put differently, by a fixed point in the self referential map linking subjective and objective dynamics.

Due to the strong informational and cognitive assumptions required by RE, Muth's hypothesis has also been subject to heavy critique. One of the most prominent objectors, Simon (1986), reckoned that the judgement that certain behaviour is rational, can only be reached by viewing the behaviour in the context of a set of givens. These include the situation in which the action takes place, the goals the action is aimed at realising and the computational possibilities which are available. Neoclassical economics mostly assumes that information processing capacities are unlimited, that values of individuals are given and consistent and most importantly, that an objective description of the

[3]The three great themes of new classical economics in the 1970s - Policy Ineffectiveness, Dynamic Consistency and the Lucas Critique - can be seen as a natural extension of concerns expressed in Lucas' article.

world exists. Given these assumptions, it is consistent to not distinguish between the real world and the decision maker's perception of it. Thus, economists can accurately predict the choices that will be made by individuals, entirely from their own knowledge of the world.

Social sciences, on the other hand, have developed a large body of evidence on the boundedness of computational resources and knowledge (see e.g. Kirchler, 2011), suggesting that constructivism is an integral part of our cognition. If this proposition is acknowledged, then one must consequently distinguish between the real world and subjective perceptions of it. This in turn pushes forward the need to reconsider the grounds rationality should be judged on. Economics has been inclined to view rationality in terms of the choices it produces. Given that information-processing capacities as well as time and patience are limited, the claim for an optimal result can hardly be maintained. In spite of this observation, the notion of rationality has not to be abandoned. Simon (1982) suggested to view rationality in terms of the processes it employs rather than the choices it produces. He called for a procedural theory which not only includes the processes of reasoning, but also the processes that generate the actor's subjective representation of the decision problem.

George Evans, Seppo Honkapohja and Thomas Sargent pioneered a strand of literature, known as adaptive learning, which reconciled the objections of Simon's "bounded rationality" with Muth's claim for a consistent economic theory (see e.g. Evans and Honkapohja, 2001; Sargent, 1993). In contrast to the REH, equivalence of subjective beliefs and actual dynamics is not imposed ex ante. Instead, agents are treated like econometricians who face estimation and inference problems and expectations are modelled as a projected image of the real world. By requiring that these projections generate forecast errors that are orthogonal to agents' forecasting models, rationality is still imposed; albeit in a procedural sense.[4]

This thesis reviews the implications of adaptive learning in the context of the New Keynesian model. In the presence of frictions, price setting today crucially depends upon beliefs about where prices are heading in the future, which in turn depends on the course of monetary policy and possibly on the past. The New Keynesian model, describing this interplay, therefore provides an excellent starting point to analyse the relevance and implications of bounded rationality on policy recommendations.

[4]Branch (2006) provides a comprehensive review of the adaptive learning literature, emphasising that beliefs which satisfy such a least squares orthogonality condition are in fact consistent with Muth's hypothesis.

The thesis is organised as follows: The first chapter presents a short historical review prior to a formal derivation of the New Keynesian model. The derivation will follow Preston (2006), who develops the model without imposing that expectations are rationale, ex ante. Upon substitution of some general expectations operator with the mathematical expectations operator, the framework will collapse to the standard two-equations reduced form representation of the New Keynesian model which is the standard workhorse of contemporary monetary theory.

The second chapter will be concerned with closing the model. One consequence of the Rational Expectations Revolution (see e.g. Hoover, 1992) was that authors started to treat the monetary authority symmetrically with the public: as dynamic optimisers. However, the maximum of a policy objective is characterised by a dynamic inconsistency. Kydland and Prescott (1977) were the first to argue that this should be anticipated by agents having RE. This line of reasoning gave rise to the distinction between two policy regimes: One where the central bank is assumed to optimise each period, and another one where the central bank can credibly commit to a rule. After having introduced these ideas, the corresponding policy rules will be derived in the second chapter.

The third chapter of this thesis is devoted to contrasting RE with adaptive learning. In spite of the REH's brilliance, some critical issues remain unacknowledged in its original formulation. In particular, a crucial question, intimately linked to the REH, is sought to being answered by the concept of adaptive learning: How can agents come to possess RE? Prior to establishing techniques in order to study whether adaptive learning serves as a justification for some REE, the concept of determinacy will be introduced. The chapter closes with a derivation of the reduced-form equations of the New Keynesian model under the assumption of bounded rationality.

The concepts, clarified in the preceding chapter, will then be applied to the New Keynesian model under optimal monetary policy in the fourth chapter. In course of reviewing results on Taylor-type rules, the connection between determinacy and E-stability will be analysed. Thereafter, the "expectations-based" interest rate rule, as proposed by Evans and Honkapohja (2003), will be analysed in detail.

The fifth chapter takes the assumption of bounded rationality on step further. The Restricted Perceptions Equilibrium (RPE), formalising the notion that agents are predicting by means of misspecified models, is introduced in the context of optimal discretionary policy. However, the form of misspecification in a RPE can easily be criticised as being ad hoc. While there only exists one way to form expectations in a rational way, there exist many ways to form suboptimal expectations. The Misspecification Equilibrium (ME), extending the concept of a RPE by endogenising the underparameterisation, will be presented as an interesting way to disintegrate the arbitrariness inherent to the RPE.

Chapter 2

The New Keynesian Model

"Keynes denies that there is an *Invisible Hand* channeling the self centered action of each individual to the social optimum. This is the sum and substance of his heresy." (Samuelson, 1949, p. 192). In contrast to what Say's Theorem postulated, Keynes (1936) did not believe that supply would automatically create its own demand, but that the economy could get trapped in an equilibrium where not all resources are used efficiently. Optimal actions of individuals, he argued, do not necessarily lead to optimal allocations in the aggregate and governmental intervention was therefore justified. Keynes ascribed the soaring unemployment during the Great Depression to a lack of aggregate demand. Due to fixed liquidity preferences, however, that he considered as being prevalent in times of heavy unemployment, Keynes argued that central banks could not efficiently intervene in the market equilibrium. Hicks (1939) captured Keynes' reasoning in the influential IS/LM model. By means of two simple relationships, one describing the money demand, the other relating output and savings decisions, he interpreted Keynes' general theory and therewith designed a framework which quickly became the dominant vehicle for macroeconomic analysis.

Friedman (1968) notes that the wide acceptance of Keynes' ideas meant that for some two decades monetary policy was believed to have been rendered obsolete. Central banks' main purpose was to keep interest rates at a low level in order to hold down interest payments in the government budget. One of the major shortcomings of the IS/LM model, however, was that it presumed a fixed, rigid price level and that it failed to distinguish between nominal and real terms. As a consequence of cheap money policies, inflation became a widespread phenomenon. Friedman notes that central banks were eventually forced to give up the pretence, that they could indefinitely keep the interest rate at a low level.

Friedman (1968) notes that in the 60s, it was generally agreed on that an expansory policy would tend to lower interest rates by making nominal cash balances higher than people actually desired. As a result, spending would be stimulated. Producers were believed to react to such an expansion by increasing output, the employed by working longer hours and the unemployed by taking on jobs. Phelps (1968) and Friedman (1968), however, argued on theoretical grounds that this so-called Phillips curve trade-off between unemployment and inflation would, if at all, hold in the short run.[1] In the long run, the neutrality of money would be maintained. Expectations played a crucial role in the Friedman-Phelps critique.[2]

Lucas (1972b) gave Friedman's insight an explicit theoretical foundation by combining imperfect information, competitive markets and perfectly flexible prices in an overlapping generations framework originally set up by Samuelson (1958). The basic idea of his model was that unpredicted variations in money generate price movements that agents may misinterpret as relative price movements. The decisive ingredient of Lucas' island model, however, was to assume that expectations are rational. This implied that any systematic course in monetary policy would be fully incorporated into the decision rules of agents. Only stochastic shocks to the money supply could cause output and employment to deviate from its trend. In this way, Lucas reconciled the important real effects of money in the short run, with its long run neutrality. Concurrently, he initiated the Rational Expectations Revolution in macroeconomics.

[1] Phillips (1958) analysed the British economy over the period 1861-1957 and observed an inverse relationship between *money wage changes* and the unemployment rate. As an explanation he offered the simple principle that prices should be expected to rise if demand for a commodity exceeds its supply. Moreover, the rise should be greater, the greater the excess in demand. Since wages are the price for labor services, the same principle should be expected to operate in the determination of money-wage growth rates. Phillips celebrated reasoning suggested that unemployment could be permanently decreased by expanding demand and hence by higher inflation.

[2] Friedman noted that product prices typically respond faster to unanticipated changes in nominal demand, than nominal wages do. Thus real wages would in fact decline if an expansion was taking place. But since employees had accommodated to a certain level of prices and inflation in the past and implicitly evaluated their wage on basis of these learned prices, they would be prone to misinterpret the nominal wage increase as being one of real terms. This illusion, Friedman argued, was the very reason for a decline in the rate of unemployment in the first place. Eventually, however, consumers would discover that not only wages, but the general price level had increased. Consequently, they would adapt their expectations about the future and request higher nominal wages. This would depress the employment rate and eventually, real wages as well as unemployment should return to their natural levels which were solely determined by real factors.

Hoover (1992) remarks that one consequence of this development was a paper where Lucas (1972a) outlined that the common practise for testing the natural rate hypothesis was flawed. A frequent test was to regress unemployment on inflation and to analyse whether the sum of coefficients on lagged inflation summed to zero. Lucas, however, showed that the Phillips curve was no structural relationship as Keynesians had regarded it, but that it was dependent on current policy rules. Thus, the long-run propensity was in general not zero, unless monetary policy was conducted in a purely random way.[3]

Another implication of Lucas' work was the policy ineffectiveness proposition, demonstrated by Sargent and Wallace (1976). They used a simple model that incorporated Lucas' basic assumptions to illustrate that a Phillips curve trade-off was only possible if the public could systematically be fooled; only unanticipated changes in the money supply would have a real effect on output. RE, however, uncovered the equilibrium to be independent of any policy rule. Thus, even in the short run, activist policy, as suggested by Keynes, would have no real effect.[4] Walsh (2010) remarks that empirical evidence suggests that the policy ineffectiveness hypothesis does not hold. Both anticipated and unanticipated changes in the money supply seem to affect output and the choice of policy rule is therefore not irrelevant for the behaviour of real economic activity.

A third consequence was the increased interest in policy rules instead of single policy actions. Since RE implied that any systematic part of policy would be incorporated into individual decision making, policy makers should account for those interactions in the design of policies. By specifying an objective function for the monetary authority and determining the values of the parameters in the policy rule that maximised the expected value of this objective function, Hoover (1992) remarks that economists started to treat policymakers symmetrically with the public; as maximisers. The policy rules themselves, however, were independent of the reactions of the public and treated as if they remained in place for all time. Kydland and Prescott (1977) pointed out that policymakers may be subject to additional constraints when choosing an optimal rule: absent enforcement, they argued, it may be optimal to deviate from the rule, once agents'

[3] In his critique, Lucas (1976, p.41) generalised this observation: "Given that the structure of an econometric model consists of optimal decision rules of economic agents, and that optimal rules vary systematically with the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models". Thus, rational expectations, being the consequence of optimising behaviour, implied that econometric policy evaluation was useless.

[4] It will be shown in the following that contemporaneous models are capable of explaining the empirical evidence of a short run trade-off between unemployment and inflation despite their assumption of rational expectations. This is due to the introduction of nominal price and wage rigidities

decisions were taken. RE implied that this temptation would be anticipated by agents. Hence, only a credible, time consistent rule would be incorporated into the decision process of agents.

The standard approach in contemporaneous monetary economics and monetary policy analysis has incorporated all of these historical developments. Since aggregate demand is of central importance in these models and since they suggest that fluctuations can and should be dampened by countercyclical policy, they are labeled *Keynesian*. One of the major differences between modern and traditional Keynesian theories, however, is that the latter regarded observed levels of employment, consumption and output as constraints on households and firms, while the former regard them as the outcome of dynamic optimisation decisions. Due to the use of dynamic optimisation techniques, these models are labeled *New Keynesian Models*.

As Clarida et al. (1999) point out, New Keynesian model have much of the empirical appeal of the traditional IS/LM models; yet they are grounded in dynamic general equilibrium theory which was pioneered in the real business cycle (RBC) theory that originated from the work of Kydland and Prescott (1982). Similar to RBC models, New Keynesian models ordinarily assume that agents are representative and that their expectations are rational. A key point of departure, however, is the explicit incorporation of frictions. Due to market imperfections like price or wage rigidities, informational frictions or portfolio frictions, equilibria in the New Keynesian model are not Pareto efficient and monetary policy is therefore justified.[5]

2.1 The Framework

The derivation of the New Keynesian model is following Preston (2006) who essentially uses a dynamic stochastic equilibrium model of Woodford (2003b). To simplify the exposition, Woodford's "cash-less limit" is assumed, i.e. the model abstracts from monetary frictions that would allow money to be held in equilibrium in spite of being dominated in rate of return.

The model consists of representative households i, maximising utility and a continuum of firms $j \in [0, 1]$, maximising profits. Following Dixit and Stiglitz (1977), the single good,

[5] It will be shown in the following, that in the limiting case of perfect price flexibility, the new Keynesian model boils down to a RBC model: Cycles are the consequence of optimisation and monetary policy is only affecting nominal variables and thus negligible.

commonly used in RBC models, is replaced by a continuum of differentiated products which are offered by monopolistically competitive firms. The consumption bundle is defined by

$$C_t^i = \left(\int C_t^i(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \tag{2.1}$$

where $C_t(j)$ denotes the demand for a differentiated good produced by firm j and $\theta > 1$ is the elasticity of substitution between any two goods. The corresponding price index P_t can be derived by assuming the artifice of a "bundler" (see e.g. Chari et al. (1996)). The bundler takes prices $\{P_t(j)|j \in [0,1]\}$ of consumption goods as given and chooses a combination of differentiated products as to minimise total costs of consumption $\int P(j)_t C(j)_t dj$ subject to the constraint of providing some level of the consumption bundle which is to be specified in the next section. Rearranging the first order condition to this problem and making use of the definition of the consumption bundle shows that the aggregated price index is given by

$$P_t = \left(\int P_t(j)^{1-\theta} dj \right)^{\frac{1}{1-\theta}}.$$

Although the assumption of monopolistic competition would suffice to justify monetary intervention, there was no scope for doing so. Any policy action would simultaneously change all prices, leaving total output unaffected. Therefore, nominal frictions are introduced by assuming that firms' ability to adjust prices are constrained. The specific model of price stickiness used here, is due to Calvo (1983) who assumes that opportunities to set prices arrive as an exogenous Poisson process: each period there is a constant probability $(1-\alpha)$ that prices may be adjusted. Demand curves and production technologies are assumed to be identical across firms. Therefore, all firms having the opportunity to set prices will choose to set the same price. Denoting the set of firms re-optimising in period t by $S(t) \subset [0,1]$, it holds that $P_t(j)^* = P_t^*$ for all $j \in S(t)$. The remaining fraction α is a random sample of firms which did not adjust prices. It follows that $P_t(j) = P_{t-1}(j)$ for all $j \in [0,1] \backslash S(t)$. Combining this insight with the definition of aggregate prices gives

$$P_t^{1-\theta} = (1-\alpha) P_t^{*(1-\theta)} + \alpha P_{t-1}^{1-\theta}. \tag{2.2}$$

In order to ease analytical handling, and in stark contrast to RBC models, capital is left out of the analysis. While endogenous variations in the capital stock traditionally play a key role in real business cycle models, Cogley and Nason (1995) showed that the response of investment to productivity shocks contributes little to the dynamics implied by real business cycle models. Moreover, Walsh (2010) reports empirical evidence of little correlation between output and capital stock at business cycle frequencies. Those results justify neglecting the capital stock.

Households' Optimisation Problem

There exists a single one-period risk-less non-monetary asset B_t in order to transfer wealth inter-temporally. Defining wealth at the beginning of period $t+1$ as $W_{t+1} = (1+i_t)B_t^i$, households' flow budget constraint is given by

$$\frac{W_{t+1}^i}{1+i_t} + P_t C_t^i \leqslant W_t^i + P_t Y_t^i. \tag{2.3}$$

This relation states that expenditures, represented by the left hand side, must not exceed available wealth, represented by the right hand side, in each period. Bond holdings yield a nominal interest rate i_t. Real income Y_t^i derives from wage $w_t(j)$ for labor supplied to firm j as well as from nominal profits $\Pi_t(j)$ from ownership of firms. Wages are paid on a competitive market and each households owns the same diversified portfolio of firms. Consequently, period nominal income is determined as

$$P_t Y_t^i = \int_0^1 \left[w_t(j) h_t^i(j) + \Pi_t(j) \right] dj,$$

where $h_t^i(j)$ denotes the number of hours household i worked in period t for good j, for each household i.

Households choose consumption bundles and labor supply in order to maximise future expected discounted utility

$$\mathrm{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[U\left(C_T^i; \xi_T\right) - \int_0^1 v\left(h_T^i(j); \xi_T\right) dj \right],$$

subject to their budget constraint (2.3). Preferences are defined over the composite consumption bundle C, a vector of aggregate preference shocks ξ and the labour supply $h^i(j)$. The discount factor is assumed to satisfy $\beta \in (0,1)$. The function $v(\centerdot)$ captures the disutility derived from work and is convex and increasing in the labor supply $h_t^i(j)$ for a given value of ξ_t. The function $U(\centerdot)$ is increasing and concave in the consumption bundle C_t, given some value of ξ_t.

The operator E_t^i denotes possibly non-rational subjective beliefs of household i about the probability distribution of the models' state variables. Expectations are assumed to be homogenous across households.

Given an initial wealth endowment, the theory of dynamic programming implies that households choose consumption bundles in order to satisfy a stochastic Euler equation

$$\frac{1}{1+i_t} = \beta \mathrm{E}_t^i \left[\frac{P_t}{P_{t+1}} \frac{U_c(C_{t+1}^i, \xi_{t+1})}{U_c(C_t^i, \xi_t)} \right] \tag{2.4a}$$

and a corresponding transversality condition

$$\lim_{k\to\infty} \mathbb{E}_t \beta^{t+k} U_c(C_{t+k}^i, \xi_{t+k}) \frac{B_{t+k}}{P_{t+k}} = 0, \tag{2.4b}$$

where the notation $\partial f(a,b)/\partial a = f_a(\cdot)$, for any function f is used. These two relations give necessary and sufficient conditions for an optimal, interior, inter-temporal allocation of consumption. The Euler equation can be justified by a variational argument. A marginal decrease in today's consumption by ΔC units could be used to increase the stock of bonds, yielding a real return of $P_t/P_{t+1}(1+i_t)\Delta C$ units in the next period. At an optimum, this postponement of consumption must not have an effect on lifetime utility. Taking account of the discount factor and uncertainty gives the Euler equation. Since there are no terminal conditions in the households optimisation problem, the transversality condition (2.4b) has to be imposed as to guarantee optimality. There might exist various paths of consumption, satisfying the Euler equation and in particular, there might be paths such that the present discounted real value of bonds is not approaching zero as time goes to infinity. Those paths, however, can't be optimal since bond holdings yield no utility.

Optimal intra-temporal decisions are obtained when the marginal rate of substitution between two commodities is equated to their marginal rate of transformation. The price of decreasing disutility derived from work is $w_t(j)$, while the price of increasing consumption is given by the aggregate price level P_t. Thus, for all $j \in [0,1]$ it must hold that

$$\frac{v_h(h_t^i(j), \xi_t)}{U_c(C_t^i, \xi_t)} = \frac{w_t(j)}{P_t}. \tag{2.5}$$

Furthermore, for the derivation of the optimal price setting rule, note that the consumption bundle is chosen optimally. Thus, demand for good j is obeying

$$C_t^i(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\theta} C_t^i, \tag{2.6}$$

which follows from maximising (2.1) with respect to $C_t(j)$ and subject to $C_t P_t = \int C_t(j) P_t(j) dj$.

The Consumption Rule

To derive a consumption rule, the Euler-equation is linearised around a steady-state which shall be characterised by $\xi_t = 0$ and $C_t = \bar{C}$ for all t. Inspection of (2.4a) implies that there exists a steady-state solution of the form $P_t/P_{t-1} = 1$ and $\bar{i}_t = \beta^{-1} - 1$ for all t, where the notation was introduced that $\bar{Z}$ denotes the steady state value of some variable Z. Furthermore, let $z = \ln(Z/\bar{Z})$ denote the log deviation of some variable Z around the deterministic steady state and redefine the nominal interest rate, correspondingly, as a percentage deviation $i_t \equiv \ln[(1 + i_t)/(1 + \bar{i})]$. Then, log-linearising the Euler equation (2.4a) shows that consumption decisions must approximately satisfy

$$c_t = \mathrm{E}_t^i c_{t+1} - \sigma(i_t - \mathrm{E}_t^i \pi_{t+1}) + g_t - \mathrm{E}_t^i g_{t+1},$$

where $\pi_t = \ln(P_t/P_{t-1})$ denotes inflation and the intertemporal elasticity of substitution σ and g_t are given by

$$\sigma \equiv -\frac{U_c(\bar{C}, \xi)}{U_{cc}(\bar{C}; \xi)\bar{C}} \quad \text{and} \quad g_t = -\frac{U_{C\xi}(\bar{C}, \xi)\xi_t}{U_{cc}(\bar{C}; \xi)\bar{C}}. \tag{2.7}$$

Under the assumption of RE, the linearised Euler-equation entailed all relevant information for optimal choice. Ordinarily, the life-time budget constraint is therefore not explicitly used in deriving a reduced-form system and only required to hold ex-post. However, if agents are not perfectly informed about the probability distribution of the economy, implying that next period's forecast does not depict an exhaustible description of state dynamics, the budget constraint provides an important source of information.

In the appendix it is shown that the linearised Euler equation can be combined with the budget constraint as to obtain the optimal consumption rule

$$c_t^i = (1-\beta)\bar{\omega}_t^i + \mathrm{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)y_T^i - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1})\right], \qquad (2.8)$$

where $\bar{\omega} = W_t^i/(P_t\bar{Y})$ denotes the share of a household's real wealth as a fraction of steady-state income $\bar{Y}$. Preston (2005) emphasises the similarity of this decision rule to the predicted consumption allocation under the permanent income hypothesis. The first two terms capture the basic insight of the permanent income hypothesis that agents should consume a constant fraction of the expected future discounted wealth, given a constant real interest rate equal to $\beta^{-1} - 1$. The third term arises from expected fluctuations of the real interest rate which are either due to variations of the nominal interest rate or due to inflation while the final term captures stochastic disturbances of the economy and is due to the assumption of preference shocks.

Aggregate behaviour can be derived by integrating the optimal consumption rule (2.8) over households i. Note therefore that market clearing implies $\int \bar{\omega}_t^i \mathrm{d}i = 0$ and let $z_t = \int z_t^i di$ for any variable z. Specifically, let $\mathrm{E}_t = \int \mathrm{E}_t^i di$ denote the average expectations operator. Then, the aggregated consumption rule can be seen to follow

$$c_t = \mathrm{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)y_T^i - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1})\right].$$

Finally, noting that equilibrium requires $c_t = y_t$, aggregate demand shall be expressed in terms of the output gap $x_t = y_t - y_t^n$, where the natural rate of output y_t^n will be defined later. The current output gap can then be written as

$$x_t = \mathrm{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n\right], \qquad (2.9)$$

where $r_t^n \equiv \mathrm{E}_t(y_{t+1}^n - g_{t+1}) - (y_t^n - g_t)$ denotes the composite of exogenous disturbances in period t.

Firms' Optimisation Problem

There is a continuum of firms $j \in [0,1]$, producing diversified goods $Y_t(j)$ in period t, with the non-linear production function

$$Y_t(j) = A_t f(h_t(j)), \tag{2.10}$$

where A_t denotes aggregate stochastic productivity disturbances and $f(\centerdot)$ satisfies the Inada-conditions. Labor is hired on competitive markets and is paid a real wage of $w_t(j)/P_t$ for every unit hired in period t. Nominal profits for firm j in period t are therefor given by

$$\Pi_t(j) = Y_t P_t^{\theta} P_t(j)^{1-\theta} - w_t(j) f^{-1}(Y_t P_t^{\theta} P_t(j)^{-\theta}/A_t),$$

where the demand function (2.6) was used and $f^{-1}(\centerdot)$ denotes the inverse function of $f(\centerdot)$. When setting prices $P_t(j)$, firms are assumed to value future streams of income at the marginal rate of aggregate income in terms of the marginal value of an additional unit of income today. Thus, a unit of income in period T is valued by the stochastic discount factor

$$Q_{t,T} = \beta^{T-t} \frac{P_t}{P_T} \frac{U_c(Y_T, \xi_T)}{U_c(Y_t, \xi_t)}. \tag{2.11}$$

The Euler-equation unveils that the stochastic discount factor is equivalent to $1/(1+i_t)^t$, given that each household consumes the same amount. Note that all agents are assumed to have common beliefs and tastes and due to the assumption of owning the same diversified portfolios it follows that in equilibrium, each household receives the same income stream which is equal to aggregate income. This justifies the simplifying assumption that firms value future profits at the marginal value of aggregate income.

Firms' optimisation problem is to set a price $P_t(j)$ as to maximise the expected present discounted value of profits

$$\mathrm{E}_t^i \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[\Pi_t\right],$$

where α^{T-t} reflects the possibility that prices may not be adjusted in period T and therewith the relevance of this periods' profit. Differentiation with respect to $P_t(j)$ and substitution of (2.11) gives the first order condition

$$\mathrm{E}_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} U_c(Y_T;\xi_T) Y_T P_T^{\theta-1} \left[P_t(j)^* - \mu P_T \Phi_{t,T}\right] = 0, \tag{2.12}$$

where $\mu = \theta/(\theta-1)$ and $\Phi_{t,T}$ denotes real marginal costs in period T which are dependent, through firm-specific demand, upon price setting in period t. By combining households' supply of labor (2.5) with the production function (2.10) real marginal costs can be expressed as

$$\Phi(Y_t(j), Y_t, \tilde{\xi}_t) = \frac{\partial h_t(j)}{\partial Y_t(j)} \frac{w_t(j)}{P_t} = \frac{1}{f_h(f^{-1}(Y_t(j)/A_t))A_t} \frac{v_h(f^{-1}(Y_t(j)/A_t), \xi_t)}{U_c(Y_t, \xi_t)}. \tag{2.13}$$

This relation makes clear that individual production is dependent on firm specific as well as aggregate conditions. The stochastic disturbance vector $\tilde{\xi}$ summarises technology and taste shocks (A_t, ξ_t).

The Price Decision Rule

To begin with, consider the flexible price equilibrium ($\alpha = 0$) which will prove useful in the derivation of a linearised price decision rule. Optimal price setting (2.12) then implies the following standard result of a model of monopolistic competition:

$$\frac{P_t(j)^*}{P_t} = \frac{\theta}{\theta - 1} \Phi(Y_t(j), Y_t, \tilde{\xi}_t).$$

Firm j sets its price equal to a markup $\mu > 1$ over its nominal marginal costs $P_t\Phi$, where the size of the markup is determined by θ, the price elasticity of demand. As $\theta \to \infty$ the differentiated products become perfect substitutes and firm j's profits are depressed to zero.

Since the problem is symmetric it follows that $P_t(j) = P_t$ and consequently that $Y_t(j) = Y_t$ for all j and t. Combining this insight with (2.12) shows that marginal costs in a flexible price equilibrium are the same across firms and given by

$$\Phi^n \equiv \Phi(Y_t^n, Y_t^n, \tilde{\xi}_t) = \mu^{-1}, \tag{2.14}$$

where the output level Y_t^n satisfying this relation is, following Friedman, referred to as natural output. Natural output varies under fully flexible prices in accordance with aggregate taste and productivity shocks $\tilde{\xi}$ and is independent of any policy rule. Even in the absence of sticky prices this construct turns out to be a useful benchmark case as will become evident below. Woodford (2003b) notes that the complete irrelevance of monetary policy to the determination of real activity is a very special case. If one allows for real balance effects or endogenous capital accumulation, the flexible price equilibrium is affected by monetary policy. However, since these effects are not very large in quantitative terms, the conclusions from this simple model remain essentially correct.

The steady-state, around which the dynamics shall be linearised, is characterised by $\tilde{\xi}_t = 0$ and $Y_t = \bar{Y}$ for all t. Inspection of the price setting equation (2.12) indicates that there exists a solution with $P_t(j)/P_t = P_t/P_{t-1} = 1$, implying that marginal costs in the steady-state are the same across firms and given by $\bar{\Phi} = \Phi(\bar{Y}, \bar{Y}, 0) = \mu^{-1}$. Thus, the log deviation of firm j's marginal costs around the steady state-value μ^{-1} can be approximated by

$$\phi_{t,T}(j) = \omega y_T(j) + \sigma^{-1} y_T - (\omega + \sigma^{-1}) y_T^n,$$

where σ is the intertemporal elasticity of substitution, small letters, again, denote the log-deviation of the corresponding variable around its deterministic steady-state, so that e.g. $\phi_{t,T}(j) = \ln(\Phi_{t,T}(j)/\bar{\Phi})$ and ω is the elasticity of real marginal costs of firm j with respect to its own production

$$\omega \equiv \bar{Y} \left[\frac{v_{hh}(\bar{Y}, 0)}{v_h(\bar{Y}, 0)} - \frac{f''(\bar{Y})}{f'(\bar{Y})^2} \right]. \tag{2.15}$$

The double index on marginal costs emphasises that marginal costs in period T are dependent upon prices set in t. To work out this dependency explicitly, note that real marginal costs of producing aggregate output Y_t are given by

$$\phi_T = (\omega + \sigma^{-1})(y_T - y_T^n) = (\omega + \sigma^{-1}) x_T,$$

where the second equality implicitly defines the output gap $x_T = y_T - y_T^n$. To obtain a relation to prices, approximate (2.6) to get $y_T(j) = y_T - \theta(p_T(j) - p_T)$. This relation can be combined with the above approximations to show that the dependence of marginal supply costs upon a producer's level of output is ultimately due to optimal price setting in period t

$$\phi_{t,T}(j) = \phi_T - \omega\theta(p_t(j)^* - \sum_{\tau=t+1}^{T} \pi_\tau),$$

where $p_t(j)^* = \ln(P_t(j)^*/P_t)$ and $\pi_t = \ln(P_t/P_{t-1})$ denotes inflation. The term in brackets ($\ln(P_t(j)^*/P_T)$) shows that firm specific marginal costs differ from aggregate marginal costs to the extent that firm specific prices differ from aggregate prices. With this expression in hands, optimal price setting (2.12) can be approximated around the deterministic steady state as

$$\mathrm{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ p_t^* - \left[\phi_T - \omega\theta(p_t(j)^* - \sum_{\tau=t+1}^{T} \pi_\tau) + \sum_{\tau=t+1}^{T} \pi_\tau \right] \right\} = 0,$$

where the term in square brackets is the deviation of the log of $P_T/P_t\Phi(Y_T(j), Y_T, \tilde{\xi})$ from its steady state value of μ^{-1}. Rearrangement of this expression gives the following explicit solution for optimal price setting in t

$$p_t(j)^* = E_t^i \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{1-\alpha\beta}{1+\omega\theta} (\omega + \sigma^{-1}) x_T - \pi_T \right]. \tag{2.16}$$

Thus, the relative price chosen by firms who adjust their prices in period t is a function of future expected paths of the output gap and inflation. Log linearisation of equation (2.2) shows that aggregate price dynamics must approximately satisfy $\pi_t = (1-\alpha)/\alpha p_t^*$. Thus, upon integration over i, the above relation may be rewritten as

$$\pi_t = \kappa x_t + \mathrm{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa\alpha\beta x_{T+1} + (1-\alpha)\beta\pi_{T+1} \right] \tag{2.17}$$

where

$$\kappa = \frac{(1-\alpha)}{\alpha} \frac{(1-\alpha\beta)}{1+\omega\theta} (\omega + \sigma^{-1}) > 0$$

Equation (2.17) indicates that current inflation is a function of the current output gap and of future economic conditions. In contrast to traditional Phillips curves, the driving variable of this process is not the log-deviation of current output from trend output but the log deviation of real marginal costs from real marginal costs under flexible prices. Since it is anticipated that prices may not be adjusted every period, firms must account for an expected rise in the general price level already today. Note that κ is decreasing in β and α. Thus, the more weight agents are placing on the future and the more rigid prices are, the less sensitive is inflation to variations in current marginal costs.

2.2 Aggregate Dynamics

Under the assumption that expectations are rational, the aggregate relations describing the dynamics of the output gap (2.9) and inflation (2.17) can further be simplified by the law of iterated expectations, since agents, having full knowledge of the tastes and beliefs of other agents, are able to compute equilibrium probabilities and associated laws. This section is following Clarida et al. (1999) in analysing aggregate dynamics in the New Keynesian model.

Substitution of $\mathrm{E}_t = \mathbb{E}_t$, where $\mathbb{E}_t$ denotes the mathematical conditional expectation, in the price setting equation (2.17), applying the law of iterated expectations and quasi-differencing yields the New Keynesian Phillips curve

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t \pi_{t+1}, \tag{2.18}$$

relating inflation positively to the output gap. As Clarida et al. (1999) point out, equation (2.18) has the flavour of a traditional expectations augmented Phillips curve. However, it differs from this relation with respect to two key aspects. First, in contrast to the traditional Phillips curve, the coefficient on the output gap κ is restricted. In particular, it can be seen that κ is decreasing in α. Note that on average, prices are fixed for $1/(1-\alpha)$ periods. Thus, the longer prices are fixed on average, the less sensitive is inflation to movements in the output gap. Second, expected future inflation enters additively as opposed to expected current inflation implying that (2.18) can be iterated forward to give

$$\pi_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \kappa x_{t+i}. \tag{2.19}$$

In contrast to traditional Phillips curves there is no arbitrary dependence on lagged inflation. In the derivation of the model it was shown that the output gap is a measure of the deviation of actual marginal costs from marginal costs in a flexible price equilibrium. Thus, inflation depends entirely on current and expected economic conditions which are reflected by the output gap.

Most discussions of monetary policy give primary attention to two goals in terms of which alternative policies should be evaluated. Maintaining a low and stable rate of inflation is often regarded as the primary goal, central banks should be concerned with, while stabilisation of output depicts the second goal. In practise, Svensson (2010) notes, inflation targeting is never strict but always flexible in the sense that all inflation-targeting central banks not only aim at stabilising inflation but also put some weight on the stabilisation of economic activity. In section 2.3, grounds for these objectives will be discussed in more detail.

Relation (2.19) indicates that, in order for monetary policy to be consistent with stable prices, i.e. $\pi_t = 0$ for all t, the output gap must be closed at all times, i.e. output must be equal to natural output at all times. Woodford (2003b) reckons that the natural rate of output is exactly the level of output for which real marginal costs of supplying each good equal μ^{-1}, i.e. the reciprocal of the desired gross mark-up. The latter quantity is equal to marginal revenue for a firm adjusting its price in the case that all firms charge the same price. Therefore $Y_t = Y_t^n$ is the condition needed so that no firm wishes to charge a price different from the common price level and therewith guaranteeing zero inflation for all times.

The implementation of such a policy can be analysed by means of the New Keynesian IS curve

$$x_t = \mathbb{E}_t x_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1}) + r_t^n \tag{2.20}$$

which can be derived by applying the assumption of RE to expression (2.9). In contrast to the traditional IS curve, equation (2.20) expresses the dependence of current output on expected future output and the real interest rate.[6] Since individuals seek to smooth consumption, expectations of higher consumption next period lead to more output demand today. The negative effect of the real rate of interest on current output reflects inter-temporal substitution of consumption. Forward iteration of the New Keynesian IS curve yields

$$x_t = \mathbb{E}_t \sum_{i=0}^{\infty} \left[-\sigma(i_{t+i} - \pi_{t+1+i}) + r_{t+i}^n \right],$$

[6] According to the Fisher relation, the real interest rate equals the nominal rate of interest minus expected inflation, i.e. $r_t = i_t - \mathbb{E}_t \pi_{t+1}$ where r_t denotes the real interest rate. Thus the IS curve relates the output gap inversely to the real interest rate.

illustrating that current output not only depends on the current stochastic disturbance and the real rate of interest, but also on an infinite series of all future real rates of interest and stochastic disturbances. Substitution of $\pi_t = 0$ for all t, shows that the nominal interest rate $i_t = \sigma^{-1} r_t^n$ is consistent with price stability. Woodford (2003b) therefore interprets the stochastic disturbance $\sigma^{-1} r_t^n$ as deviations of the Wicksellian "natural rate of interest", from the value consistent with a zero inflation steady state.

Note that both policy objectives can be attained in this framework since the single policy instrument i_t succeeds in perfectly offsetting the single stochastic disturbance r_t^n. Due to the assumptions made, a constant price level eliminates the distortions resulting from price stickiness. The second best outcome $y_t = y_t^n$ for all times is then a consequence of $\pi_t = 0$ for all t.

2.3 Policy Objectives

Woodford (2003b) notes that there appears to be a fair amount of consensus that a desirable monetary policy is one that achieves a low expected value of a discounted loss function, where the losses are each period a weighted average of quadratic deviations of inflation from some target and output from potential output. While this general formulation is broadly accepted, there is ample space for discussions on details. Besides the exact weights that should be placed on output stabilisation and inflation stabilisation there exists also ambiguity on the measures representing these variables.

Hall and Mankiw (1997) proposed nominal income targeting, i.e. stabilising deviations of the price level from a deterministic trend, while Svensson (1997) suggested to stabilise deviations of the inflation rate from some target. An earlier alternative, money-growth targeting, has been abandoned since practical experience has consistently shown that the relation between money growth and inflation is too unstable and unreliable for money growth targeting to provide successful inflation stabilisation (Svensson, 2010). Similarly, there is the question of what output measure to stabilise. Should one stabilise deviations of output from potential output which varies in accordance with real disturbances, or should one stabilise deviations from a deterministic trend?

In order to be completely coherent in formulating the policy objective, Woodford (2003b) derives a loss function from a second-order Taylor series approximation to the level of expected utility of a representative household.[7] This utility-based welfare criterion

[7]Clarida et al. (1999) reckon that the approach of deriving a welfare criterion by means of the utility

not only provides justification for the general concern of price and output stabilisation but furthermore, provides an exact answer to the questions raised about the precise formulation of the appropriate loss function.

The reason for Woodford's resort to a quadratic approximation approach are twofold. The first is mathematical convenience. By using a quadratic expansion to the objective function and linear approximations to the structural equations, the nature of optimal policy can be analysed within a linear-quadratic optimal control framework which has been extensively studied. The second reason is comparability. Traditional literature on monetary policy evaluation almost always assumes a quadratic loss function. By deriving a similar objective from an optimising framework allows to discuss similarities and differences.

This section is following Woodford (2003b) in deriving the policy objective. The production function (2.10) together with market clearing implies that instantaneous utility of a representative household in period t can be expressed in terms of output as

$$U_t = U(Y_t; \xi_t) - \int_0^1 \tilde{v}(Y_t(j); \tilde{\xi}_t) dj, \tag{2.21}$$

where $\tilde{v}(y; \tilde{\xi}) \equiv v(f^{-1}(y/A), \xi)$ and Y_t is defined by the Dixit-Stiglitz aggregator (2.1). Note that the definition of $\tilde{v}(\cdot)$ implies that marginal costs may be written as

$$\Phi(Y_t(j), Y_t, \tilde{\xi}) = \frac{\tilde{v}_y(Y_t(j), \tilde{\xi})}{U_c(Y_t, \xi)}, \tag{2.22}$$

where $\tilde{\xi} = (A, \xi)$. Prior to seeking an approximation to instantaneous utility, the efficient level of output shall be characterised as to motivate grounds for policy intervention. The function U_t is strictly increasing and concave in $Y_t(j)$ and attaining a maximum at

$$\Phi(Y_t(j), Y_t, \tilde{\xi}) = \left[\frac{Y_t}{Y_t(j)}\right]^{1/\theta} \quad \text{for all } j. \tag{2.23}$$

This condition states that it is optimal to produce the same amount of each differentiated good, i.e. $Y_t(j) = Y_t$ for all j, and consequently that the marginal rate of transformation $v_h(h, \xi) U_y(Y_t, \xi)^{-1}$ is equated to the average marginal product of labour $A_t f'(h)$, at an

function has some major shortcomings. First, important effects like the uncertainty inflation generates for lifetime financial planning and for business planning seems not to be captured by this approach. Second, while the representative agent approach may work reasonably well to motivate behavioural relationships, it could be highly misleading for welfare analysis if insurance and credit markets are incomplete and some groups are suffering more in recessions than others.

optimum. However, the New Keynesian model, as introduced above, is characterised by two market distortions which cause a departure from these efficiency conditions. In order to analyse their implications separately assume for the moment that prices may be adjusted each period.

In a flexible price equilibrium it holds that $Y_t(j) = Y_t$ for all t. From (2.23) it then follows that $\Phi(Y_t^e, Y_t^e, \tilde{\xi}_t) = 1$, where Y_t^e denotes the efficient level of output. However, comparing this result to marginal costs under flexible prices (2.14) gives

$$\Phi(Y_t^n, Y_t^n, \tilde{\xi}) = \mu^{-1} < 1 = \Phi(Y_t^e, Y_t^e, \tilde{\xi}).$$

Since $\Phi(.)$ is increasing in its first argument, the inequality indicates that the natural rate of output is inefficiently low. The wedge between society's marginal costs of producing the consumption bundle and the household's marginal costs of acquiring it drives total output and hence employment below a socially desirable level. The reason for this inefficiency lies in the fact that each firm perceives the demand for its differentiated goods to be imperfectly elastic. The hereby implied market power leads to pricing above marginal costs.

To analyse a series of economies where distortions become progressively smaller, the parameter φ is introduced to summarise the overall distortion in the steady-state level of output which may be due to market power and possibly taxes, so that

$$\Phi(\bar{Y}, \bar{Y}, 0) = \frac{1-\tau}{\mu} \equiv 1 - \varphi, \tag{2.24}$$

where τ denotes taxes. Given that the distortion is small, i.e. the level of output in the steady state is nearly efficient, a log linear approximation of marginal costs around the efficient level of output (and in the absence of real disturbances) may be used to write

$$\ln(\bar{Y}/\bar{Y}^e) = -(\omega + \sigma^{-1})^{-1}\varphi. \tag{2.25}$$

To examine the effect of staggered price setting, suppose that e.g. an employment subsidy was placed so that $\Phi(\bar{Y}, \bar{Y}, 0) = 1$. If not all firms are able to adjust prices each period, then in general it holds that $P(j) \neq P(i)$ for some (i, j). Households maximisation behaviour requires in this case that consumption of more expensive goods is reduced in order to afford an increased consumption of cheaper products and thus $Y_t(j) \neq Y_t$ for some j. Due to the decreasing marginal utility of consumption, the gain derived from increasing one variety is not able to offset the disutility suffered from a proportional reduction of another variety. This in turn implies that the economy's average marginal product of labour does not equal the marginal rate of transformation. It is through this channel that price stability affects welfare, i.e. that it violates efficiency condition (2.23), even in the absence of monopoly power or distortionary taxes $\varphi = 0$.

To obtain more detailed results, households' utility shall now be approximated around a deterministic steady state with $Y_t(j) = \bar{Y}$ and $\tilde{\xi}_t = (0, 1)$ for all t and j. The first term of instantaneous utility (2.21) can be shown to obey

$$U(Y_t, \xi_t) \approx \bar{Y} U_c(\bar{Y}, 0) \left[y_t + \frac{1}{2}(1 - \sigma^{-1}) y_t^2 + \sigma^{-1} g_t y_t \right] + t.i.p., \tag{2.26}$$

where $t.i.p$ summarises constants and exogenous variables which are independent of policy and the notation $y_t = \ln(Y_t/\bar{Y})$ was used. Similarly, the second term in (2.21) may be approximated by

$$\tilde{v}(Y_t(j), \xi_t) \approx \bar{Y} U_c(\bar{Y}, 0) \left[(1 - \varphi) y_t(j) + \frac{1}{2}(1 + \omega) y_t(j)^2 - \omega q_t y_t(j) \right] + t.i.p., \tag{2.27}$$

where

$$q_t = -\frac{\tilde{v}_{y\xi}(\bar{Y}, 0)\xi_t}{\bar{Y} \tilde{v}_{yy}(\bar{Y}, 0)}$$

and the definition (2.24) was used together with (2.22) to replace $\tilde{v}_h(\bar{Y}, \tilde{xi})$ with $U_c(\bar{Y}, 0)(1 - \varphi)$. Furthermore, assuming that φ is close to zero, expression (2.27) was simplified by approximating $(1 - \varphi)z$ with z for z small.

Note that the Dixit-Stiglitz aggregator (2.1) can be approximated by

$$y_t = \mathbb{E} y_t(j) + \frac{1}{2}(1 - \theta^{-1}) \operatorname{var} y_t(j),$$

where $\mathbb{E} y_t(j) = \int y_t(j) dj$ denotes the mean of $y_t(j)$ and $\operatorname{var} y_t(j)$ denotes the corresponding variance. With this relation in hands, aggregate disutility derived from work can be approximated by integrating (2.27) over j and substituting for $\mathbb{E} y_t(j)$ and $\mathbb{E} y_t(j)^2$ to obtain

$$\int_0^1 \tilde{v}_h(Y_t(j); \xi_t) dj = \bar{Y} U_c(\bar{Y}, 0) \left[(1 - \varphi) y_t + \frac{1}{2}(1 + \omega) y_t^2 - \omega q_t y_t + \frac{1}{2}(\theta^{-1} + \omega) \operatorname{var} y_t(j) \right] + t.i.p.$$

Combining expressions this expression with (2.26) shows that instantaneous utility follows

$$U_t = -\frac{\bar{Y}U_c}{2}(\sigma^{-1} + \omega)\left[(x_t - x)^2 + \operatorname{var} y_t(j)\right] + t.i.p, \tag{2.28}$$

where $x \equiv \ln(\bar{Y}^e/\bar{Y})$ denotes the gap between the steady state efficient level of output and the actual steady state level of output as defined in (2.25).

Relation (2.28) constitutes a second order approximation to household's utility under the assumption that the natural rate of output is almost efficient ($\varphi \approx 0$). It is evident from this relation that monetary policy should be concerned with stabilising the output gap, rather than output relative to trend and such that actual output is at an efficient level. Moreover, expression (2.28) indicates that the dispersion of output levels across differentiated products matters. As mentioned earlier, this results from the concavity of the utility function together with the definition of the consumption bundle. Preference and technology disturbances $\tilde{\xi}$ matter only through their effect on the natural rate of output y_t^n which is captured in the definition of the output gap $x_t = y_t - y_t^n$.

The variability of output levels is grounded in the assumption of sticky prices. Therefore, a second order approximation to the demand curve (2.6) is used to rewrite the dispersion of output levels in terms of prices as

$$\operatorname{var} y_t(j) = \theta^2 \operatorname{var} p_t(j).$$

This implies that in addition to the stabilisation of the output gap, optimal monetary policy should also be concerned with reducing price dispersion. By making use of the Calvo-pricing assumption, it is shown in the appendix that price dispersion may be written in terms of inflation as

$$\operatorname{var} p_t(j) = \alpha \operatorname{var} p_{t-1}(j) + \frac{\alpha}{1-\alpha}\pi_t^2.$$

Thus, in the framework derived above, stabilisation of the general price level is a sufficient condition to achieve the policy objective of a minimal dispersion of output levels. If an environment is created in which firms, choosing an optimal price, have no incentive to set a price which deviates from the average of existing prices, the average of existing prices will remain the same and eventually, all prices will collapse to being the same value.

Substitution of this expression into the linearised utility function (2.28) shows that the normalised loss function is given by

$$\sum_{t=0}^{\infty} \beta^t U_t = -\Omega \sum_{t=0}^{\infty} \beta^t [\pi_t^2 + \lambda(x_t - x^*)] \tag{2.29}$$

where Ω summarises constants and terms which are independent of policy and the relative weight on the output gap stabilisation is given by $\lambda = \kappa/\theta$.

Chapter 3

Optimal Monetary Policy

The policy objectives of stabilising inflation at zero as well stabilising the deviation of output from an efficient level emerged from the analysis of the preceding chapter. In section 2.2 it was illustrated that there is no conflict in achieving these objectives simultaneously, in the basic framework, by holding the price level fixed. However, there are a number of caveats to this conclusion.

First of all, zero inflation may not be desirable. Schmitt-Grohé and Uribe (2010) note that the optimal long-run rate of inflation is governed by two sources of monetary non-neutrality. While one source is a nominal friction stemming from a demand for fiat money, the other source is given by the assumption of price-stickiness. The New Keynesian model, as derived above, concentrates on the role of sticky prices. Staggered price setting implies that higher inflation leads to higher price dispersion which causes an inefficient allocation of resources. The optimum of zero inflation directly follows. The role of money as a medium of exchanged, however, is neglected.

Friedman (1969) emphasised that money balances represent a service to the public which is provided by the government at no cost. As to maximise public welfare, he suggested to equate the real rate of returns to money and other assets by either conducting a deflationary monetary policy or by paying interest on nominal money balances. Schmitt-Grohé and Uribe (2010) show that in monetary models where the demand for money constitutes the only nominal friction, optimal monetary policy implies inflation targets between -2% and -4%.

However, analysing monetary policy in an open economy introduces another channel through which inflation affects welfare. Countries whose currency is used abroad may have an incentive to increase inflation as a means to collect resources from foreign residents. This particularly provides a strong rationale for countries where the bulk of currency circulates abroad (e.g. USA). Schmitt-Grohé and Uribe (2010) analyse the optimal rate of inflation in an economy where the foreign demand for its currency is taken into account and show that the Friedman rule ceases to be Ramsey optimal. Calibrated versions of the model deliver optimal rates of inflation between 2% and 10%.

Schmitt-Grohé and Uribe conclude that models in which transactional demand for money is the sole source of nominal frictions fail to provide a compelling explanation for the magnitude of observed inflation targets which are concentrated at around 2% a year. While sticky price frictions, as incorporated in the New Keynesian model, bring the optimal rate of inflation much closer to observed inflation targets, the prediction of zero inflation still falls short of empirics.

An argument which has been proposed as an explanation to this gap is the zero lower bound on nominal interest rates. In order to implement optimal monetary policy, the nominal interest rate has to be adapted in response to the natural rate of interest which in turn is dependent upon real economic activity. However, it might turn out that at some time the natural rate of interest is negative, requiring the nominal interest rate to be negative too which is not possible under any policy (Woodford, 2003b).[1] Thus, zero inflation may not be a feasible objective.

As Christiano et al. (2009) emphasise, hitting the zero bound on nominal interest rates induces a deflationary mechanism which leads to increased volatility and therefore large welfare costs. A positive inflation target, on the other hand, eases the implementation of monetary policy by broadening the room for action. Coibion et al. (2012) derive the effects of non zero steady-state inflation on the loss function and show that hitting the zero bound is more costly, in their New Keynesian framework, than the welfare costs of constant positive inflation. Their main conclusion is that the optimal inflation rate is between 1% and 2%, or approximately the inflation targeting range used by the US Federal Reserve and the ECB.[2]

[1] McCallum (2011), on the other hand, questions the existence of a zero lower bound by drawing on modern technology institutions which could be designed so as to permit payment of negative nominal interest rates.

[2] Due to the difficulty of providing further monetary stimulus when the interest rate is at its zero lower bound, the appropriateness of the widely used 2% target has become subject to discussion recently. In light of the current economic crisis and the server constraints on monetary policy associated with a zero nominal interest rate, Olivier Blanchard, director of research of the International Monetary Fund, put forth the idea of raising the target to 4%.

To analyse the effects of this widely used inflation target, this chapter will make use of a slightly modified policy objective which is given by

$$L = -\frac{1}{2}\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left((\pi_{t+i} - \pi)^2 + \alpha(x_{t+i} - x)^2 \right), \tag{3.1}$$

where x allows for a possible deviation of socially optimal output from potential output and $\pi > 0$ is the target value for the inflation rate and the central banks "taste" parameter is from now on denoted by α. A policy with $\alpha = 0$ is called *strict inflation targeting*, while $\alpha > 0$ is referred to as *flexible inflation targeting*.

Another qualification to the observation that it suffices to hold the price level fixed in order to attain a welfare maximising allocation, is that this result is based on a framework where the flexible price equilibrium differed from the efficient allocation by a small *constant* factor only. However, if the inefficiency of the flexible price equilibrium becomes time varying, e.g. due to time varying market power or time varying distorting taxes, optimal pricing decisions will be altered and consequently, a constant price level will in general not minimise the variability of the gap between actual output and efficient output.

To counteract this shortcoming, a more general specification of the Phillips curve will be analysed. In particular, the Phillips curve shall be supplemented by a stochastic term. Optimal monetary policy is discussed in an environment where aggregate dynamics are given by

$$x_t = \mathbb{E}_t x_{t+1} - \sigma \left(i_t - \mathrm{E}_t \pi_{t+1} \right) + r_t, \tag{3.2a}$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{3.2b}$$

where u_t denotes a stochastic cost shock and the stochastic term r_t^n can in general arise from preference shocks, fluctuations in the natural rate of output or shocks to government purchases of goods (Walsh, 2010).[3] This chapter will refer to relation (3.2a) and (3.2b) as IS curve and Phillips curve, respectively and abstain from indicating the equation

[3] Clarida et al. (2002) suggest to introduce a stochastic term in the inflation equation by adding a stochastic wage mark-up. Erceg et al. (2000), on the other hand, provide theoretical foundations for a shifting Phillips curve by considering nominal wage stickiness. In recent models, Walsh (2010) notes, stochastic disturbances are introduced by assuming that individual firms face random variations in the price elasticity of demand, i.e. θ_t becomes time-varying.

counter. The shocks are assumed to follow stable AR(1) processes

$$\begin{aligned} r_t^n &= \mu r_{t-1} + \hat{r}_t \\ u_t &= \rho u_{t-1} + \hat{u}_t \end{aligned} \tag{3.3}$$

with $\mu, \rho \in (0, 1)$ and both $\hat{u}_t$ and $\hat{r}_t$ are i.i.d. random variables with zero mean and variance σ_u^2 and σ_r^2 respectively.

In the presence of cost push shocks, i.e. when the Phillips curve is supplemented with a stochastic term, the policy objectives of stabilising inflation and the output gap, can no longer mutually be achieved by holding the output gap closed at all times. Instead, it will be shown in the following that a shifting Phillips curve introduces a trade-off in achieving these policy objectives. Thus, the policy instrument i_t has to be designed such that the target variables are simultaneously controlled for.

An optimal policy is a feedback rule which relates the state of the economy, represented by (3.2), to the policy instrument i_t such that (3.1) is maximised. In the preceding section it was shown that equilibrium values of the target variables do not only depend on current policy measures, but also on expectations about future policy actions due to their effect on future marginal costs. Woodford therefore describes central banking as a management of expectations: Monetary policy uses the forward looking behaviour of the private sector as a tool for stabilisation.

3.1 Discretionary Policy and Commitment

In the literature two regimes of central banking are distinguished. The first corresponds to an environment where the monetary authority assumes that its policy choice effectively determines agents' expectations. The central bank therefore faces a dynamic optimisation problem of maximising (3.1) with respect to the economy's state variables, π_t and x_t for all t, subject to aggregate dynamics (3.2). Note that the IS curve is not directly dependent upon current values of inflation. Thus, the maximisation problem can be dealt with in two stages: In the first stage, the loss function is maximised with respect to inflation and the output gap and such that the Philips curve is satisfied for all times. The hereby obtained optimal paths of the target variables can then be used in the second stage where the IS curve is solved for the implied optimal nominal interest rate.

The first stage problem can be summarised by the Lagrangian

$$\mathcal{L} = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ -\frac{1}{2} \left[(\pi_{t+i} - \pi)^2 + \alpha(x_{t+i} - x)^2 \right] + \delta_{t+i} \left(\pi_{t+i} - \beta\pi_{t+1+i} - \kappa x_{t+i} - u_{t+i} \right) \right\},$$

where δ_{t+i} denotes the state contingent multiplier associated with the constraint at $t+i$. The first order conditions for an inter-temporal optimum, usually called the commitment solution, are given by

$$\alpha(x_{t+i} - x) + \delta_{t+i}\kappa = 0 \text{ for } i = 0, 1, 2, \ldots \tag{3.4a}$$

$$(\pi_{t+i} - \pi) + \delta_{t+i-1} - \delta_{t+i} = 0 \text{ for } i = 1, 2, \ldots \tag{3.4b}$$

and

$$(\pi_t - \pi) - \delta_t = 0. \tag{3.4c}$$

These conditions characterise optimal behaviour of the central bank. However, it is evident that the restriction on current inflation differs from the restrictions on future inflation. Combining (3.4a) with (3.4c) gives the optimal trade-off for current state variables to obey

$$(x_t - x) = -\frac{\kappa}{\alpha}(\pi_t - \pi). \tag{3.5}$$

Thus, if inflation is deviating from its target value of π in the current period, it is optimal to drive output in the opposite direction. The strength of the reaction depends on the gain of reduced inflation per unit of output lost κ and inversely on the relative weight placed on the output target α. For all future periods, the optimal plan can be obtained by combining (3.4a) and (3.4b) which gives

$$(x_{t+i} - x_{t+i-1}) = -\frac{\kappa}{\alpha}(\pi_{t+i} - \pi) \text{ for } i = 1, 2 \ldots. \tag{3.6}$$

In contrast to current period's optimality condition (3.5), this relation requires to adjust inflation in response to a *change* in the output gap. The target value for the output gap becomes irrelevant.

To interpret these conditions, suppose that a positive cost push shock has realised. An optimal reaction to this shock drives current output below its target value in order to counteract inflation through the Phillips curve. Relation (3.6) represents the supposedly credible threat that output will be further reduced as long as inflation remains above its target. The Phillips curve reveals that this threat has the immediate effect of dampening current inflation. Monetary policy under commitment therefore turns out to be more effective.

The strategy works since the public is assumed to integrate the policy rules into their own decision process. However, the above result also makes the time inconsistency, inherent to monetary policy under commitment, evident. Current decisions place a constraint on the future which is non-optimal when the future actually arrives. If the central bank was able to re-optimise in period $t+1$, it would choose to deviate from (3.6). In the aftermath of the Rational Expectations Revolution, Kydland and Prescott (1977) were the first to point out that agents should be expected to anticipate the central bank's incentive to deviate from its plan and consequently not consider (3.6) when taking their decisions. Credibility becomes a central issue in such an environment. One solution for creating a consistent plan, known as the timeless perspective solution, is to neglect the current constraint and fully commit to

$$\alpha(x_t - x_{t-1}) = -\kappa(\pi_t - \pi) \text{ for all } t. \tag{3.7}$$

This relation will depict the grounds for deriving optimal interest rate rules under commitment in the following section.

The second policy regime, being discussed in the literature, corresponds to an environment where the central bank is acting under discretion since it can't credibly manipulate agents' believes. The in general dynamic optimisation problem reduces to a sequence of static optimisation problems in this case. Each period, the central bank is maximising its objective function (3.1) subject to the Phillips curve, whilst treating expectations of future inflation as given. The first order condition to this problem takes the familiar form

$$\alpha(x_t - x) = -\kappa(\pi_t - \pi) \text{ for all } t. \tag{3.8}$$

Since the monetary authority has no incentive to change the plan (3.8) in an unexpected way, discretionary policy is said to be time consistent.

Taylor (1977) addressed the problem of distinguishing "rule like" from discretionary policy behaviour in practise. Recognising that no central bank would set its policy instrument according to some simple formula, Taylor suggested to characterise rule-like behaviour by being systematic. McCallum (1993), on the other hand, reckoned that central bankers who act under discretion would not characterise their behaviour as unsystematic. He therefore argued that optimising once as opposed to optimising each period should be regarded as an additional criterion for rule-like behaviour.[4]

While the distinction of these policy regimes is of interest in its own right, a considerably more important point concerns the feasibility of rule-like behaviour. While commitment will generate superior outcomes, its implementation requires to adopt instrument settings which differ from those, emerging if the central bank was able to re-optimise each period. Since there is no "commitment technology" to guarantee the implementation of interest rate settings, some authors (see e.g. Chari et al., 1989) consider central banks to be inevitably destined to behave in a discretionary fashion.

In the following, the implementation of equilibria, being associated with optimal behaviour of the central bank, will be discussed.

3.2 Expectations-Based Interest Rate Rules

One strategy for deriving an interest rate rule is to combine the assumption of RE with the method of undetermined coefficients in order to find a solution of the state variables which satisfies one of the first order conditions as well as the system of equations, describing the state dynamics, simultaneously. This practise will be illustrated below. Alternatively, the interest rate specification can be established whilst abstaining from the assumption that expectations are rational. Following Berardi and Duffy (2007) who retrace Evans and Honkapohja (2003), this approach will be illustrated here.

Consider the case of discretionary policy first. The Phillips curve can be combined with the first order condition for discretionary policy (3.8) in order to give a first order difference equation in the inflation rate

$$\pi_t = \frac{\kappa(\alpha x + \kappa\pi)}{\alpha + \kappa^2} + \frac{\alpha\beta}{\alpha + \kappa^2}\mathrm{E}_t\pi_{t+1} + \frac{\alpha}{\alpha + \kappa^2}u_t. \tag{3.9a}$$

[4]The work by Barro and Gordon (1983) is considered to having put an end to the notion that policy rules necessarily are linked to fixed settings for nominal interest rates. McCallum (1997) notes that this step served to separate the "rules vs. discretion" dichotomy from the issue of "activist vs. non-activist" policy behaviour and therewith, opened the door to possible interest in policy rules, on the part of central bankers. Recently, studies on policy rules have experienced an upsurge, partly due to the arrival of inflation targeting as "a leading candidate for the provision of a practical guideline for monetary policy", as remarked by McCallum.

Making use of the optimality condition (3.8) once more allows to express the output gap as function of inflation expectations and the cost shock

$$x_t = \frac{\alpha x + \kappa\pi}{\alpha + \kappa^2} - \frac{\kappa\beta}{\alpha + \kappa^2} \mathrm{E}_t \pi_{t+1} - \frac{\kappa}{\alpha + \kappa^2} u_t. \tag{3.9b}$$

Substitution of these relations into the New Keynesian IS curve and solving for i_t gives the relevant optimal policy rule to be

$$i_t = \delta_0 + \delta_1 \mathrm{E}_t \pi_{t+1} + \delta_2 \mathrm{E}_t x_{t+1} + \delta_3 r_t + \delta_4 u_t, \tag{3.10}$$

with coefficients

$$\delta_0 = -\frac{\kappa\bar{\pi} + \alpha\bar{x}}{(\alpha + \kappa^2)\sigma}, \quad \delta_1 = 1 + \frac{\kappa\beta}{(\alpha + \kappa^2)\sigma}, \quad \delta_2 = \delta_3 = \frac{1}{\sigma}, \quad \delta_4 = \frac{\kappa}{(\alpha + \kappa^2)\sigma}. \tag{3.11}$$

Evans and Honkapohja (2003) refer to relation (3.10) as the *expectations-based* optimal interest rate rule under discretion as it is assumed that the monetary authority conditions its policy on private sector forecasts and that it has ready access to such information.

Under commitment, a similar specification can be obtained by combining the optimality condition (3.7) with the Phillips curve to get the expectational difference equation

$$x_t = \frac{\kappa}{\alpha + \kappa^2} \left(\pi + \frac{\alpha}{\kappa} x_{t-1} - \beta \mathrm{E}_t \pi_{t+1} - u_t \right). \tag{3.12}$$

Substitution into the IS curve shows that the expectations-based optimal interest rate rule under commitment is given by

$$i_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 \mathrm{E}_t \pi_{t+1} + \phi_3 \mathrm{E}_t x_{t+1} + \phi_4 r_t + \phi_5 u_t \tag{3.13}$$

with parameters

$$\phi_0 = \frac{-\kappa\bar{\pi}}{\sigma(\alpha + \kappa^2)}, \quad \phi_1 = \frac{-\alpha}{\sigma(\alpha + \kappa^2)}, \quad \phi_2 = 1 + \frac{\kappa\beta}{\sigma(\alpha + \kappa^2)}, \quad \phi_3 = \phi_4 = \frac{1}{\sigma}, \quad \phi_5 = \frac{\kappa}{\sigma(\alpha + \kappa^2)}.$$

In chapter 5 it will be shown that these interest rate rules leave the corresponding REE determinate and learnable. The intuition for the learnability result is straight-forward: In contrast to the below derived *fundamentals-based* interest rate rules, the central bank directly responds to possibly non-rational expectations in (3.10) and (3.13). Deviations of subjective beliefs from RE are therefore efficiently counteracted.

Similar interest rate rules, being conditioned on current or future expectations, have been proposed by various authors; yet for different reasons. While some have been proposed on grounds of the observation that contemporaneous data is ordinarily not available (McCallum, 1997) (hence the practical consideration to rely on expected values), others have been put forward in order to counteract the long transmission mechanism of monetary policy (Svensson, 1999a).[5]

3.3 The Rational Expectations Equilibrium

After having specified the expectations-based form of interest rate rules, the standard practise in deriving the optimal policy instrument shall be retraced. With the assumption of RE in hands, an explicit solution to the set of expectational difference equations (3.9) and (3.12) can be obtained by the method of undetermined coefficients. Upon specification of the RE equilibrium process, the corresponding interest rate rule arises from the definition of the IS-curve.

Following McCallum (1981), the particular solution of the expectational difference equations shall be conditioned on a minimal set of predetermined variables. The hereby obtained *minimal state variable* (MSV) solution is unique by construction.[6] Problems associated with indeterminacy, i.e. an infinity of RE solutions, will be discussed in detail in section 4.2.

Discretionary Policy

Under discretionary policy, the expectations-based rule (3.10) implies that the MSV solution takes the form

$$\begin{aligned} x_t &= \bar{a}_x + \bar{c}_x u_t, \\ \pi_t &= \bar{a}_\pi + \bar{c}_\pi u_t, \end{aligned} \tag{3.14}$$

[5]Svensson (1999a) notes that monetary policy affects the demand side of the economy with a lag, via its effect on the short real interest rate. Aggregate demand then affects inflation with another lag, via the Phillips curve. The "expectations channel" in the Phillips curve, allows to affect inflation with a single lag. Svensson (1998) gives a simple example of a transmission mechanism. Aggregate demand, in this model, is affected with a one-year lag, inflation with a two-year lag.

[6]In fact, this one provision is not sufficient to yield a unique solution in all cases, as will be shown in section 4.3. In addition, the solution formulae must be valid for all admissible parameter values.

where the coefficients $(\bar{a}_x, \bar{a}_\pi)$ and $(\bar{c}_x, \bar{c}_\pi)$ are presently undetermined. Making use of the definition of the stochastic processes (3.3) shows that RE are given by

$$\begin{aligned}\mathbb{E}_t x_{t+1} &= \bar{a}_x + \bar{c}_x \rho u_t, \\ \mathbb{E}_t \pi_{t+1} &= \bar{a}_\pi + \bar{c}_\pi \rho u_t.\end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) into the expectational difference equations (3.9) and solving for the coefficients yields

$$\begin{aligned}\bar{a}_x &= \frac{(\alpha x + \kappa\pi)(1-\beta)}{\alpha(1-\beta)+\kappa^2}, \quad \bar{c}_x = -\frac{\kappa}{\alpha(1-\beta\rho)+\kappa^2}, \\ \bar{a}_\pi &= \frac{\kappa(\kappa\pi + \alpha x)}{\alpha(1-\beta)+\kappa^2}, \quad \bar{c}_\pi = \frac{\alpha}{\alpha(1-\beta\rho)+\kappa^2}.\end{aligned} \tag{3.16}$$

The REE of the economic system under discretionary optimal policy is fully specified by the relations (3.14) and (3.16) together with the specification of the exogenous process u_t.

Commitment

Under Commitment, the expectational difference equation (3.12) implies that current inflation and output realisations are dependent upon recent output gaps x_{t-1}. Since the cost shock u_t represents another predetermined variable in (3.12), a minimal set of state variables is given by $\{1, x_{t-1}, u_t\}$. Thus, a solution of the form

$$\begin{aligned}x_t &= \tilde{a}_x + \tilde{b}_x x_{t-1} + \tilde{c}_x u_t \\ \pi_t &= \tilde{a}_\pi + \tilde{b}_\pi x_{t-1} + \tilde{c}_\pi u_t\end{aligned} \tag{3.17}$$

is conjectured. By making use the definition of the stochastic process (3.3), one obtains RE to follow

$$\begin{aligned}\mathbb{E}_t x_{t+1} &= \tilde{a}_x(1+\tilde{b}_x) + \tilde{b}_x^2 x_{t-1} + \tilde{c}_x(\tilde{b}_x + \rho) u_t, \\ \mathbb{E}_t \pi_{t+1} &= \tilde{a}_\pi + \tilde{b}_\pi \tilde{a}_x + \tilde{b}_\pi \tilde{b}_x x_{t-1} + (\tilde{b}_\pi \tilde{c}_x + \tilde{c}_\pi \rho) u_t.\end{aligned} \tag{3.18}$$

Substitution of the conjectured solution (3.17) and the hereby implied expectations (3.18) into the Phillips curve gives

$$\begin{aligned} \tilde{a}_\pi &= \beta(\tilde{a}_\pi + \tilde{b}_\pi \tilde{a}_x) + \kappa \tilde{a}_x \\ \tilde{b}_\pi &= \beta \tilde{b}_\pi \tilde{b}_x + \kappa \tilde{b}_x, \\ \tilde{c}_\pi &= \beta(\tilde{b}_\pi \tilde{c}_x + \tilde{c}_\pi \rho) + \kappa \tilde{c}_x + 1, \end{aligned} \tag{3.19}$$

whereas substitution into the first order condition (3.7) implies

$$\begin{aligned} \tilde{a}_\pi &= \pi, \\ \tilde{b}_\pi &= -\alpha\kappa^{-1}(\tilde{b}_x - 1), \\ \tilde{c}_\pi &= -\alpha\kappa^{-1}\tilde{c}_x. \end{aligned} \tag{3.20}$$

These equations determine the presently unknown coefficients in (3.17). Combining the second equation in (3.19) and (3.20), respectively, gives the following quadratic in $\tilde{b}_x$

$$\beta \tilde{b}_x^2 - \gamma \tilde{b}_x + 1 = 0$$

where $\gamma = 1 + \beta + \kappa/\alpha$. The succession of signs suggests that the quadratic has two positive roots. Furthermore, note that their product is equal to 1 implying a reciprocal pair where one solution generates explosive time paths that eventually violate the transversality condition. Therefore, the only relevant solution for $\tilde{b}_x$ is given by

$$\tilde{b}_x = \frac{1}{2\beta}\left[\gamma - \left(\gamma^2 - 4\beta\right)^{1/2}\right].$$

The coefficient $\tilde{a}_x$ can be obtained by combining the first equation in (3.19) with the first equation in (3.20) while $\tilde{c}_x$ can be expressed in terms of $\tilde{b}_x$ by combining the third equations in these sets, respectively. This yields

$$\tilde{a}_x = \frac{\kappa\pi(1-\beta)}{\alpha\beta(1-\tilde{b}_x) + \kappa^2}, \quad \text{and} \quad \tilde{c}_x = -\frac{\kappa}{\alpha(1-\rho\beta) + \alpha\beta(1-\tilde{b}_x) + \kappa^2}.$$

The coefficients for the inflation process follow from (3.20).

Note that $0 < \tilde{b}_x < 1$, implying that even in the absence of a natural source of persistence (i.e., $\rho = 0$) there is inertia in the output process as well as in the inflation process. This result is a consequence of the monetary authority anticipating that its policy affects agents' expectations. Past movements in the output gap continue to affect current inflation with the effect of an improved short run trade-off in achieving the policy objectives.

3.4 Welfare Properties

As to reflect on the gains of commitment, the equilibrium under discretionary policy shall be considered first. The corresponding interest rate rule can be obtained by substituting the equilibrium process (3.14) and expectations (3.15) into the IS curve. Solving for i_t gives

$$i_t = \psi_0 + \psi_1 u_t + \psi_2 r_t \tag{3.21}$$

with parameters

$$\psi_0 = \bar{a}_\pi, \quad \psi_1 = \frac{(1-\rho)\kappa + \alpha\rho\sigma}{\sigma[\alpha(1-\beta\rho)+\kappa^2]}, \quad \psi_2 = \frac{1}{\sigma}.$$

Evans and Honkapohja (2003) refer to relation (3.21) as the fundamentals-based form of the optimal policy rule under discretionary policy since it depends only on current exogenous shocks. However, as Woodford (2003b) notes, there exist many equivalent ways of expressing optimal policy rules under the assumption of RE. In particular, the fundamentals-based interest rate rule can be rewritten, using $\mathbb{E}_t\pi_{t+1} = \bar{a} + \bar{c}\rho u_t$, in order to stress the role of expected inflation. This gives the optimal nominal interest rate rule

$$i_t = \gamma_0 + \gamma_\pi \mathbb{E}_t\pi_{t+1} + \gamma_r r_t \tag{3.22}$$

where

$$\gamma_0 = -\frac{(1-\rho)\kappa}{\rho\lambda\sigma}a_x, \quad \gamma_\pi = 1 + \frac{\kappa(1-\rho)}{\alpha\rho\sigma}, \quad \gamma_r = \frac{1}{\sigma}.$$

This formulation, as proposed by Clarida et al. (1999), affirms Taylor's intuition that the central bank should react with increasing the nominal interest rate more than one for one in response to expected inflation moving above the target value. The policy recommendation is easily justified by noting that an increased output gap would further amplify inflationary pressure through the Phillips curve. To counteract this process, the real interest rate has to be increased so that households are induced to decrease current consumption.

The fundamentals-based interest rate rule (3.21), on the other hand, highlights the response to current cost push shocks. A positive deviation of current inflation can only be counteracted by driving current output below its target. Note that this implies a trade-off, being inherent to counteracting cost push shocks, which will be addressed below. Clearly, by considering the functional form of expected inflation, the disparity between (3.21) and (3.22) collapses under RE. However, if one departs from this assumption, the rules are distinct and their properties differ as will be shown in chapter 5. In contrast to cost push shocks, there is no trade-off inherent to counteracting demand shocks. Irrespective of the representation, an optimally chosen interest rate rule will be designed such that demand shocks r_t are perfectly offset.

To obtain the fundamentals based interest rate rule, implementing the REE under commitment, the equilibrium process (3.17) and the implied RE (3.18) can be substituted into the IS curve. Solving for i_t gives

$$i_t = \psi_0 + \psi_1 x_{t-1} + \psi_2 u_t + \psi_3 r_t \tag{3.23}$$

with coefficients

$$\psi_0 = \frac{\tilde{a}_x \tilde{b}_x}{\sigma} + \tilde{a}_\pi + \bar{b}_\pi \tilde{a}_x, \quad \psi_1 = \bar{b}_x \left(\frac{\tilde{b}_x - 1}{\sigma} + \tilde{b}_\pi \right), \quad \psi_2 = c_x \left(\frac{\tilde{b}_x + \rho - 1}{\sigma} + \tilde{b}_\pi \right) + \tilde{c}_\pi \rho, \quad \psi_3 = \frac{1}{\sigma}. \tag{3.24}$$

As in the discretionary case, there are many equivalent representations of this interest rate rule. Consider again a form which highlights the role of expected inflation. Using the expression for $\mathrm{E}\pi_{t+1}$ in (3.18), the interest rate rule (3.23) can be rewritten as

$$i_t = \gamma_0 + \gamma_\pi \mathbb{E}_t \pi_{t+1} + \gamma_g r_t,$$

where

$$\gamma_0 = \frac{\tilde{a}_x \tilde{b}_x}{\sigma}, \quad \gamma_\pi = 1 - \frac{\kappa}{\sigma \alpha}, \quad \gamma_r = \frac{1}{\sigma}.$$

This representation suggests two things: First, the optimal response to a demand shock is independent of the policy regime. Like in an environment of discretion, optimal policy under commitment perfectly offsets any shock to the output gap. The optimal response to a cost push shock, however, crucially differs from a policy of discretion. Due to the efficient manipulation of beliefs, current output has to be reduced to a lesser extent than under discretion since a credible threat of further reducing the output gap has the immediate effect of reducing inflation through the "expectations channel" in the Phillips curve.

By analysing the unconditional standard deviations of inflation and the output gap, and their response to the policy parameter α, the welfare properties of different policy regimes and the trade-off implied by the model can nicely be illustrated, as pointed out by Clarida et al. (1999). The MSV solution of the REE under discretion (3.14) implies

$$\sigma_{dx} = \frac{\kappa}{\alpha(1-\beta\rho)+\kappa^2}\sigma_u, \quad \sigma_{d\pi} = \frac{\alpha}{\alpha(1-\beta\rho)+\kappa^2}\sigma_u, \tag{3.25}$$

where σ_u denotes the standard deviation of the cost shock u_t and $\sigma_{d\pi}$ and σ_{dx} denote the standard deviations of inflation and output under discretionary policy. Partial derivatives with respect to the "taste" parameter α unveil the ambivalent effect of a shift in policy preferences:

$$\frac{\partial \sigma_{dx}}{\partial \alpha} = -\frac{\kappa(1-\beta\rho)}{[\alpha(1-\beta\rho)+\kappa^2]^2}\sigma_u^2 < 0, \qquad \frac{\partial \sigma_{d\pi}}{\partial \alpha} = \frac{\kappa^2}{[\alpha(1-\beta\rho)+\kappa^2]^2}\sigma_u^2 > 0.$$

The more attention is paid to output stabilisation, i.e. as α is increasing, the lower output volatility becomes in equilibrium. Lower output volatility, on the other hand, can only be achieved at the cost of increasing inflation volatility. Due to this trade-off there is in general ($\alpha > 0$) gradual convergence towards the inflation target

$$\lim_{i\to\infty} \mathbb{E}_t \pi_{t+i} = \lim_{i\to\infty} (\bar{a}_\pi + \bar{c}_\pi \rho^i u_t) = \bar{a} > \pi.$$

However, the actual target π is not reached for any $\alpha x > 0$. A result which is known as the inflationary bias under discretionary policy (see e.g. Clarida et al., 1999). This stands in stark contrast to a policy under commitment where the first order conditions were shown to be independent of the output target. As a consequence, the inflation target is asymptotically approached

$$\lim_{i\to\infty} \mathbb{E}_t \pi_{t+i} = \lim_{i\to\infty} (\tilde{a}_\pi + \tilde{c}_\pi \rho^i u_t) = \pi.$$

Furthermore, the MSV solution (3.17) implies that the unconditional standard deviations of the output gap and inflation, denoted by σ_{cx} and $\sigma_{c\pi}$ respectively, are given by

$$\sigma_{cx} = \frac{\kappa}{\alpha\left(1-\beta\rho\right)+\alpha\beta\left(1-b_x\right)+\kappa^2}\sigma_u, \quad \sigma_{c\pi} = \frac{\alpha}{\alpha\left(1-\beta\rho\right)+\alpha\beta\left(1-b_x\right)+\kappa^2}\sigma_u. \tag{3.26}$$

By comparing (3.26) to equilibrium deviations of the target variables under discretion (3.25) it immediately follows that the economy can be stabilised to a higher degree if the central bank is able to credibly commit to a rule.

Chapter 4

Learning and Rational Expectations

The REH presupposes a great deal of knowledge on the part of economic agents. Considering its complexity, this applies particularly strong to the New Keynesian model. The derivation of aggregate dynamics not only relied on both optimal price setting of firms as well as optimal consumption decisions by households, but also on the assumption that these considerations are mutually available to all agents. Thus, rationality is not merely a property of the individual, but of the economic model.

In fact, the REH constitutes a solution concept, rather than a behavioural notion. Yet, the hypothesis does not provide sufficient grounds with this respect. Arrow (1986) notes that the powerful implications of the REH derive from the conjunction with other basic concepts of neoclassical theory, like equilibrium, competition or completeness of markets. Arrow reckons that in some parts of the literature it seems to be asserted that economic theory must be based on rationality, as a matter of principle. However, most macroeconomic theories rely at least partly on concepts other than that of rationality. The price rigidities in the New Keynesian model constitute a prominent example with this respect. Friedman's loose arguments on "shoe leather" which substitute for a true derivation of the demand for money, is another.

Arrow (1986) points out that economic theory defines rationality in terms of the choices it produces. In light of computational restrictions, however, it appears disproportionate to claim an optimal outcome. Theoretical work presupposes that agents know all parameter values in order to solve their optimisation problem. In applied work, these parameter values have to be estimated before the model can be tested. Arrow therefore suggests to redefine rationality in terms of the processes it employs, rather than the choices it produces. With this definition in hands, rationality, or more specifically, its intersections with neoclassical theory, do not have to be abandoned, but are rather supplemented by a behavioural approach.

Evans and Honkapohja (2001) take up on this point. They argue that it seems to be more natural to assume the same limitations in knowledge and computation capacity on the part of agents, as economists are facing in everyday work and propose adaptive learning as a minimal deviation from the concept of rationality: Agents are assumed to form expectations on grounds of econometric models whose parameters are updated as new data becomes available.

On the one hand, adaptive learning can be considered as a plausible alternative to the REH. While agents are not assumed to have optimal expectations ex ante, they are assumed to employ an optimal technique in order to form expectations. Thus, they are rational in a procedural sense. Given their limited information set, containing a history of data points, they form expectations such that forecasting errors are orthogonal to their model. On the other hand, adaptive learning provides a powerful backup for the REH. This becomes apparent on grounds of the following observation. Even if all the structural assumptions needed for the derivation of a REE are made, Arrow (1986) lines out that one critical question remains: How can equilibrium be reached? The attainment of equilibrium is in need of some disequilibrium process. But what should be regarded as rational in the presence of disequilibrium? Adaptive learning can in some cases provide a justification for the REH. The continuous updating of expectations may asymptotically lead expectations to be rational. Thus, even if not coincidence of an objective and a subjective world is presupposed, expectations can turn out to be self-fulfilling.

This chapter starts with an introduction to the idea of adaptive learning. Thereafter, its intersections with, and applications to RE will be discussed. In course of this discussion the concepts of determinacy and E-stability will be introduced. The chapter concludes with a justification for applying the presented concepts to the reduced-form New Keynesian model.

4.1 Adaptive Learning

Upon implementation of some interest rate rule, the New Keynesian model can in general be represented by

$$\begin{aligned} y_t &= A + B\mathrm{E}_t y_{t+1} + C y_{t-1} + D w_t, \\ w_t &= F w_{t-1} + \epsilon_t \end{aligned} \tag{4.1}$$

where $y_t = (x_t, \pi_t)'$ denotes the vector of endogenous variables and $w_t = (r_t, u_t)'$ is assumed to follow a stationary VAR, so that ϵ_t is white noise and both eigenvalues of F lie inside the unit circle. To close the model, assumptions on the expectation formation mechanism, i.e. on the definition of $\mathrm{E}_t y_{t+1}$, are needed. The approach followed in this thesis is to model bounded rationality by assuming that agents' expectations are based on econometric models, i.e. expectations are assumed to follow some linear function whose parameters are updated each period by using Least Squares estimation.

Upon departure from rationality, care must be taken in distinguishing subjective from objective descriptions of the economy. Following Evans and Honkapohja (2001), agents' (homogeneous) belief of the world will be referred to as the Perceived Law of Motion (PLM) while the dynamics implied by their behaviour will be referred to as the Actual Law of Motion (ALM). A priori, no coincidence of these dynamics will be assumed. However, in order to enable asymptotic convergence of perceived and actual dynamics, it will be assumed that agents condition their forecasts on the correct information set in the following sense: Given that system (4.1) has a unique dynamically stable solution under the assumption of RE, this solution will be a linear combination of $\{1, y_{t-1}, w_t\}$.[1] While the exact nature of the solution is not common knowledge, agents are assumed to understand the structure of the economy qualitatively such that their PLM takes the form

$$y_t = a + b y_{t-1} + c w_t. \tag{4.2}$$

The coefficient matrices a, b and c, having dimension 2×1, 2×2 and 2×2 respectively,

[1] The assumption that agents are basing their expectations on the correct information set is hypothesised for convenience and as to check for the robustness of some REE. In light of computational restrictions, however, it appears plausible that agents belief formation is based on an information set, whose dimensionality is strictly less than the dimensionality of actual dynamics in a REE. However, if beliefs are not retracing the structure of the MSV solution, the REE can be attained by no learning rule whatsoever. Chapter 6 discusses the Restricted Perceptions Equilibrium as a consistent alternative to the REE, for the case when the PLM does not admit convergence to the latter.

are estimated in period T by running Least Squares on the set $\{y_t, 1, y_{t-1}, w_t\}_{t=0}^{T-1}$. This formulation is following the convention to not consider current variables for updating coefficient estimates in order to avoid a simultaneity complication in the learning dynamics.

A crucial issue under adaptive learning is the so-called "dating of expectations". Depending on how the sequence of events is formulated, the information set being available to agents at the time expectations are formed differs. While current exogenous variables are mostly treated as observable, assumptions on the availability of current endogenous variables are not consistent. Bullard and Mitra (2002) considered various Taylor-type rules under the assumption that current endogenous variables are available so that forecasts are formed according to

$$\mathbb{E}_t y_{t+1} = a + b y_t + cF w_t,$$

where F is assumed to be known. Since the structural equations of the New Keynesian framework imply that endogenous variables and expectations of next period's variables are simultaneously determined, this formulation might seem plausible. However, as McCallum (2008) points out, the specification of information sets, being available for expectation formation in period t, is a completely different matter from the specification of what period's expectations influence the determination of current variables.[2] Evans and Honkapohja (2006) analysed optimal monetary policy under commitment under the assumption that current endogenous variables are not available for expectation formation. This leads to the forecasting rule

$$\mathrm{E}_t y_{t+1} = a + b(a + b y_{t-1} + c w_t) + cF w_t.$$

Chapter 5 will outline that assumptions with respect to the available information set, crucially affect the stability properties of REE under learning when the reduced form system exhibits persistence. Under discretionary optimal policy, however, as analysed by Evans and Honkapohja (2003), the relevant information set for the MSV solution does not contain endogenous variables of the model. Expectational stability of this solution is therefore independent of whether the information set includes y_t or not.

[2] To underline the lack of any necessary connection between these assumption, consider a model which both includes $\mathrm{E}_{t-1} y_t$ as well as $\mathrm{E}_t y_{t+1}$. What should be regarded as the appropriate information set for learning? While the prior expectations operator suggests that y_t should not be available for learning, the second operator indicates the contrary.

This work will treat time-t endogenous variables as not being available on the following grounds. As mentioned before, under adaptive learning a simultaneity complication would occur if current endogenous variables were available for estimating the coefficients in the linear model. For consistency then, it seems more natural to not include those variables in the information set available for forming expectations. Furthermore, McCallum (1981) points out that data on contemporaneous economic measures is hardly available for policy makers. It seems appropriate to presume the same limitations in knowledge on part of the households. The sequence of events is therefore assumed as follows: Upon determination of time t parameter estimates (a_t, b_t, c_t), the vector of exogenous shocks w_t realises. Agents then apply their model to current data and form conditional expectations according to

$$\mathrm{E}_t y_{t+1} = a_t + b_t(a_t + b_t y_{t-1} + c_t w_t) + c_t F w_t. \tag{4.3}$$

Note that this formulation presumes knowledge on the stochastic process $\{w_t\}$ in order to predict w_{t+1}. Equivalently, subjective belief formation could be modelled by explicitly treating the persistence parameter matrix F as an estimate. Although this formulation has some appeal due to its consistency, the main conclusions would not be altered (see e.g. Evans and Honkapohja, 2008). For convenience then, the literature is followed by treating exogenous shock processes as known.

The implied temporary equilibrium can be obtained by inserting subjective expectations (4.3) into the reduced-form system (4.1). This shows that the ALM in period t is following

$$y_t = A + B\left(a_t + b_t(a_t + b_t y_{t-1} + c w_t) + c_t F w_t\right) + C y_{t-1} + D w_t.$$

This relation implicitly defines a mapping from the PLM to the ALM. Let this mapping be denoted by $T : \Theta \to \mathbb{R}^m$ where $\Theta \subset \mathbb{R}^n$ is the set of parameter estimates with n corresponding to the dimensionality of the information set, the PLM is based on and where $\mathbb{R}^m$ is the set of implied dynamics. Thus, an element of Θ is the matrix $\theta = (a, b, c)$ and the corresponding T-map is given by

$$T(a, b, c) = \big(A + B(\mathrm{I} + b)a, Bb^2 + C, B(bc + cF) + D\big). \tag{4.4}$$

In terms of this mapping, the temporary equilibrium in period t can be rewritten compactly as

$$y_t = T(\theta_t)z_t, \tag{4.5}$$

where $z_t = (1, y_{t-1}, w_t)'$. From this formulation it becomes apparent that each observation of the economy is generated by a different model, or more precisely, by a model with time varying parameters. Since this fact is not captured by the PLM (6.8), agents are basing their parameter estimates on a misspecified model which implies that one central assumption needed for consistency of the Least-Squares estimate is violated. However, Evans and Honkapohja (2001) argue that expectations formed in this manner might still be regarded as reasonable, if not fully rational, on the following grounds. First, Bray and Savin (1986) show that in many cases the temporary misspecification during transition to the REE would not be detectable by good econometric practise. Furthermore, asymptotically, the misspecification becomes vanishingly small if the REE is in fact learnable.

4.2 Determinacy

The previous section introduced adaptive learning under the assumption that agents correctly understand the structure of the economy. However, if the structure of the economy gives rise to an infinity of solutions, agents may not be able to coordinate on a particular one. Indeterminacy, i.e. local non-uniqueness of equilibrium, is widely considered a non-desirable feature of economic models.[3][4] If an infinity of solutions satisfy the structural equations, implying that the objective distribution of the model is not determined, the definition of RE is evidently meaningless. Moreover, if the equilibrium path is not saddle point stable there is potential for non-fundamental variables, i.e. variables having no inherent effect on the economy, to matter, only because agents believe that they do.[5] Woodford (2003a) reckons that one does not have to be certain

[3] McCallum (2003) notes that the term "indeterminacy" became first prominent due to Patinkin (1949), writing about an alleged logical inconsistency in classical monetary theory. Since households, firms as well as the central bank were assumed to only care about real variables, the price level did not appear in these models and was consequently not determined. This type of indeterminacy, to which McCallum refers to as "nominal-indeterminacy" is very different in character from the multiplicity of stable solutions which are considered here.

[4] Gandolfo (1971) reckons that this feature has also been exploited to explain macroeconomic phenomena by means of sunspots, self-fulfilling prophecies and animal spirits.

[5] Singular points in dynamic systems are said to be saddle point stable, if there exists exactly one way to choose initial conditions such that the resulting dynamics are asymptotically stationary. Apart

that an indeterminate economy will settle in such a bad equilibrium to nonetheless prefer rules which give people a clear reason to not have such expectations, i.e. to prevent indeterminacy through a suitably designed policy.

Furthermore, non-uniqueness also poses a challenge to the adaptive learning approach. Which equilibrium point should serve as a grounds for the PLM? Honkapohja and Mitra (2004) investigate "non-fundamental" solutions (non-MSV solutions) in the New Keynesian model and show that these are also attainable under adaptive learning. The next chapter will discuss the link between E-stability and determinacy in detail. For the moment it suffices to emphasise that determinacy is a desirable property, also from the perspective of adaptive learning.

The Blanchard and Kahn (1980) technique to analyse determinacy shall now be introduced for the New Keynesian model which is represented by system (4.1). If the economic system exhibits persistence, i.e. $C \neq 0$, this system can be rewritten in first order form. Let therefore the $(i,j)^{\text{th}}$ entry of any matrix G be denoted by g_{ij} and note that under all interest rate specifications considered so far, the second column of C in (4.1) was zero. By defining $x_t^L \equiv x_{t-1}$ and assuming that $c_{11} \neq 0$, system (4.1) can be rewritten as

$$\begin{pmatrix} x_t \\ \pi_t \\ x_t^L \end{pmatrix} = \begin{pmatrix} 1 & 0 & -c_{11} \\ 0 & 1 & -c_{21} \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{E}_t x_{t+1} \\ \mathbb{E}_t \pi_{t+1} \\ x_{t+1}^L \end{pmatrix} + other, \tag{4.6}$$

where *other* summarises terms being irrelevant for the present analysis. Note that REH requires that there are no systematic forecasting errors, implying that $\mathbb{E}_t z_{t+1} = z_{t+1} + \epsilon_{t+1}$ with $\mathbb{E}_t \epsilon_{t+1} = 0$ for any variable z. Furthermore, forecast errors are supposed to be uncorrelated over time so that $\mathbb{E}[\epsilon_t \epsilon_{t+i}] = 0$ for all $i \neq 0$. Thus, system (4.6) can be rewritten as a first order difference equation with a stochastic mean zero driving process. These systems can easily be solved.

A fundamental notion for investigating determinacy of RE models is the distinction between predetermined and jump variables. The following definition is provided by Buiter (1982, pp. 6): "X_t is predetermined if and only if X_t is not a function of expectations, formed at t, of future endogenous and/or exogenous variables. P_t is non predetermined if and only if P_t is a function of expectations, formed at t, of future endogenous and/or ex-

from Rational Expectations models, optimal control problems also "endowed" with the right amount of freedom, to study this "conditional stability" (see e.g. Gandolfo, 1971).

ogenous variables."[6] Less formally this can be summarised by noting that predetermined variables are fixed throughout a period, while jump variables are determined within a period.

To find unique trajectories of a first order 3×3 system of difference equations, three boundary conditions are needed in order to determine the arbitrary coefficients of the homogeneous solution. In the case of predetermined variables, these usually take the form of initial conditions. Jump variables, however, are characterised by the very absence of "historically" given values. Requiring the homogeneous part of the difference system converge to zero as $t \to \infty$, i.e. that the solution be dynamically stable, constitutes a typical remedy for imposing constraints on these variables.

Blanchard and Kahn (1980) showed that Rational Expectations models in first order form, like system (4.6), are uniquely determined if and only if the number of unstable eigenvalues of the Jacobian matrix, given in this case by

$$J = \begin{pmatrix} 1 & 0 & -c_{11} \\ 0 & 1 & -c_{21} \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.7}$$

coincide with the number of predetermined variables.[7] Given that all other eigenvalues are located inside the unit circle, the initial conditions of non-predetermined variables are uniquely determined by the requirement that the system be dynamically stable.

If, on the other hand, the number of eigenvalues outside the unit circle exceeds the number of predetermined variables, there exist infinitely many initial conditions, jump variables can move to, without generating explosive dynamics. Since the notion of "agents' expectation" does not have a precise meaning when there is "too much" stability, determinacy is considered a necessary condition for the consistency of REE.

The matrix J, given by (4.7), will serve as a grounds for evaluating determinacy under various interest rate specifications in chapter 5. From the derivation of the New Keynesian model it is evident that both output and inflation are jump variables. Given

[6]The use of variables in this definition implicitly discloses that jump variables, in a wide sense, have the dimension of prices.

[7]The notion of unstable eigenvalues is a little misleading in this context. Note that system (4.6) is forward looking for which reason "unstable eigenvalues" of J, i.e. eigenvalues, strictly lying outside the unit circle, will lead to "unconditionally" (independent of initial conditions) stable dynamics. This can be seen by noting that their reciprocal, relevant for the recursively formulated difference system, is strictly inside the unit circle.

appropriate expectations, these variables can instantaneously take on any possible value. The lagged output gap x_t^L, on the other hand, constitutes a predetermined variable in period t. Thus monetary policy will be considered desirable if exactly two eigenvalues of J lie inside the unit circle.

If the interest rate specification implies that current values of output and inflation are independent of its lagged values, i.e. all elements of C are zero, determinacy can be directly evaluated on the grounds of matrix B. If both eigenvalues of B lie inside the unit circle there exists a unique way to iterate (4.1) forward, i.e. to obtain a unique bounded RE solution.

4.3 E-Stability

Bullard (2006) notes that several key papers in the 1980's explored adaptive learning as a resolution for the following problem: How should agents come to possess RE if they initially do not possess of detailed information on the economic situation they find themselves in? As has been repeatedly argued (see e.g. McCallum, 2008), agents must ultimately learn about the nature of the economy from data which is generated by the economy itself. This section will introduce the "E-stability principle", thoroughly discussed in Evans and Honkapohja (2001), as a general tool for analysing whether some REE can be attained under adaptive learning. In other words, whether adaptive learning serves as a justification for the REE of interest.

Learnability of a REE is investigated under the assumption that agents know the "qualitative" structure of the economy, implying that forecasts are based on the correct information set. Furthermore, it is assumed that agents collect an ever increasing amount of information while the structure of the economy remains unchanged. These assumptions are clearly biased towards finding the RE process. In addition, the theorems used to analyse convergence under adaptive learning are local in nature. Bullard (2006) therefore reckons that this setup in fact represent a "minimal deviation from Rational Expectations" and McCallum (2008) argues that if a proposed REE is not learnable in this setup, it appears implausible that it could prevail in practise. Thus, Least-Squares learnability is regarded a compelling necessary condition for a REE to be considered plausible.

Besides providing a behavioural justification for some REE, adaptive learning can also be used as a selection criterion. In general, the dependence of current variables on future conditions implies the possibility of a multiplicity of REE, as was discussed in the preceding section. While indeterminacy is mostly considered as an undesirable feature of the economy, some authors disregard indeterminacy as a theoretical curiosity which only poses the challenge to the theorist to select the correct equilibrium (see e.g. McCallum, 2003). Evans and Honkapohja (2001) adopt the perspective that one should focus on those solutions which are robust to small forecast errors made by agents initially, i.e. on equilibria which are stable under learning. Thus, they argue that indeterminacy should not be considered a problem if all solutions except one turn out to be unstable under learning.

This section will concentrate on learnability of the MSV solution of system (4.1). It is evident that the relevant determinants of y_t include a constant, y_{t-1} and w_t. Therefore, the MSV solution to this system takes the form

$$y_t = \bar{a} + \bar{b}y_{t-1} + \bar{c}w_t, \tag{4.8}$$

where $(\bar{a}, \bar{b}, \bar{c})$ are currently undetermined coefficient matrices. Rational Expectations are given by

$$\mathbb{E}_t y_{t+1} = \bar{a} + \bar{b}y_t + \bar{c}Fw_t.$$

Inserting these relations into (4.1) shows that the coefficients in (4.8) and therewith the MSV solutions of a REE must satisfy

$$(\mathrm{I} - B\bar{b} - B)\bar{a} = A, \tag{4.9a}$$

$$B\bar{b}^2 - \bar{b} + C = 0, \tag{4.9b}$$

$$(\mathrm{I} - B\bar{b} - BF)\bar{c} = D. \tag{4.9c}$$

Note that even when attention is restricted to stationary solutions, the matrix quadratic (4.9b) can in general have multiple solutions lying inside the unit circle. This implies that the REE is indeterminate and gives rise to the existence of sunspot solutions. However, McCallum (2003) emphasises that the MSV solution is in fact unique if the additional provision is satisfied that the solution is valid for all admissible parameter values. For the case of (4.9b), the "correct" MSV solution is the one whose value equals zero when all elements of C equal zero. The relevant matrix $\bar{b}$ will normally coincide with the one whose eigenvalues are the smallest in modulus, as noted by McCallum (2003).

To analyse learnability of this MSV solution, reconsider the T-map (4.4) which was introduced as a mapping from beliefs to outcomes. It is reproduced here for convenience

$$T(a,b,c) = \left(A + B(\mathrm{I}+b)a, Bb^2 + C, B(bc + cF) + D\right).$$

Initially, Evans (1985, 1986) investigated a discrete version of the E-stability principle.[8] He considered several iterations from the PLM to the ALM and imagined that parameter estimates are updated after each iteration such that the specification of the PLM holds. Letting $\theta(n)$ denote parameter estimates, where n indexes iterations, the T-map can be used together with the definition of the PLM to obtain the dynamics

$$\theta(n+1) = T(\theta(n)).$$

If $\theta(n) \to \bar{\theta}$ over time, the analysed REE is said to be iteratively E-stable. Note that this concept is "eductive" in spirit since it investigates whether the coordination of expectations on some REE can be attained by a mental process of reasoning. Evans (1985) showed that in several controversial and prominent examples, the MSV solution turns out to be iteratively E-stable. For the New Keynesian model, the discrete dynamics of the parameter estimates can be seen to obey

$$\begin{aligned} a(n+1) =& A + B[\mathrm{I} + b(n)]a(n) \\ b(n+1) =& Bb(n)^2 + C \\ c(n+1) =& B\left[b(n)c(n) + c(n)F\right] + D. \end{aligned} \tag{4.10}$$

Building upon prior work by Marcet and Sargent (1989), Evans (1989) then turned to a continuous representation of the above equations. Appendix A illustrates that this continuous version is intimately related to an adaptive learning process which is modelled as taking place in real time. Under various specifications of this process, the asymptotic dynamics of the parameter estimates can be analysed with the E-stability principle which advises to consider the differential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta,$$

where τ denotes notional time. Note that the parameters $\bar{\theta}$ satisfy $\bar{\theta} = T(\theta)$ due to the assumption that the PLM nests the RE solution.[9] If the REE solution associated with

[8] The E-stability principle was first mentioned by Evans (1983).

[9] This nicely illustrates the self-fulfilling prophecy inherent to the concept of REE: the REE is a fixed point in the mapping from beliefs to outcomes.

the parameters $\bar{\theta}$ is locally asymptotically stable under this differential equation, then the REE equilibrium is expectationally stable under Least Squares and other related learning rules. Specifically, in the New Keynesian framework, this leads to the matrix differential equations

$$\begin{aligned} \frac{da}{d\tau} &= A + B(\mathrm{I} + b)a - a, \\ \frac{db}{d\tau} &= Bb^2 + C - b, \\ \frac{dc}{d\tau} &= Bbc + BcF + D - c, \end{aligned} \tag{4.11}$$

which can be regarded as a conversion of equations (4.10) to a continuous form, appropriate as the iteration interval approaches zero. Since the condition for the discrete system (4.10) to be stable is that all roots of $DT(\bar{\theta})$ lie inside unit circle, while the continuous system (4.11) is stable given that $DT(\theta) - \mathrm{I} < 0$, iterative E-stability is evidently a stricter condition than E-stability. In order to evaluate E-stability, system (4.11) has to be linearised (since the second equation is non-linear) and vectorised. Evans and Honkapohja show that these stability conditions can be stated in terms of the derivative matrices

$$\begin{aligned} DT_a(\bar{a}, \bar{b}) &= B(\mathrm{I} + \bar{b}), \\ DT_b(\bar{b}) &= \bar{b}' \otimes B + \mathrm{I} \otimes B\bar{b}, \\ DT_c(\bar{b}, \bar{c}) &= F' \otimes B + \mathrm{I} \otimes B\bar{b}, \end{aligned} \tag{4.12}$$

where $\otimes$ denotes the Kronecker product. Proposition 10.5. in Evans and Honkapohja (2001) implies local convergence of adaptive learning in the New Keynesian framework, if and only if all eigenvalues of $DT_a - \mathrm{I}, DT_b - \mathrm{I}$ and $DT_c - \mathrm{I}$ have negative real parts. Chapter 5 will analyse E-stability of different equilibria, being associated with different interest rate rules, by means of derivative matrices similar to (4.11). Note however, that the relevant matrices for analysing stability are mostly of lower dimension than seems to be suggested by this expression. This results from the fact that the MSV solution is in all examined cases, among other variables, a function of u_t only and not of the entire vector w_t.

Before proceeding to the analysis of E-stability and determinacy under optimal monetary policy, a justification for working with the reduced-form equations, as were derived under the assumption of RE, shall be presented.

4.4 A Justification for Reduced-Form Learning

Until recently, the learning literature has focused exclusively on whether, by using estimated models, agents can learn to forecast optimally. The connection between forecast and agent level decision making, however, has been neglected. Evans and McGough (2006) note that the standard procedure, as applied to DSGE models, was ordinarily as follows: (1) Rationality is assumed in deriving conditions that capture optimising behaviour of agents, (2) relations are aggregated and market clearing is imposed, (3) in order to obtain a reduced-form system of linear difference equations, the aggregate relations are simplified and linearised. Only then, bounded rationality is imposed by exchanging the RE operator by some other operator, capturing the boundedly rational behaviour. This approach, referred to as "reduced-form learning" by Evans and McGough (2006), may well be criticised for it is not precise about the "actions" taken given some expectation and whether these are consistent with equilibrium.

This section follows Branch et al. (2010) in contrasting different learning dynamics. Preston (2006) will be retraced in outlining a critique to the Euler-equation learning approach and Honkapohja et al. (2011) will be followed in justifying the analysis of bounded rationality on grounds of the reduced-form equations of the New Keynesian model.

The ad-hoc nature of reduced-form learning was first addressed by Honkapohja et al. (2011) and Evans and Honkapohja (2006). In order to provide a behavioural foundation for adaptive learning, they identify agents as 2-period planners. Agents take decisions today, conditional on their expectation of tomorrow, as to equate marginal utility with marginal loss. That is, the Euler-equation

$$C_t^i = \mathrm{E}^i C_{t+1}^i - \sigma \left(i_t - \mathrm{E}^i \pi_{t+1} \right) + g_t - \mathrm{E}_t^i g_{t+1}, \tag{4.13}$$

is considered as the behavioural primitive. Note the peculiarity inherent to interpreting (4.13) as a decision rule when agents are boundedly rational: Although C_{t+1}^i will be determined by individuals themselves, forecasts of this quantity are needed in order to decide on the level of consumption today. However, Evans and Honkapohja (2006) assume that $\mathrm{E}^i C_{t+1}^i$ is based on forecasts of x_{t+1} and show that "Euler-equation learning" provides a justification for analysing the New Keynesian model by means of reduced-form learning. That is, they show that the reduced-form equations of the New Keynesian model can be obtained by assuming that (4.13) depicts a behavioural rule.[10]

Since the REE analysis of the New Keynesian model suggests that once the true probability-laws are known, only one period ahead expectations matter for aggregate dynamics, the Euler-equation learning approach appears intuitive. However, Preston (2005) reckons that after having departed from RE, agents can't be assumed to make use of the information that other agents' consumption decision satisfy an Euler-equation in deciding what to do themselves.[11] Preston's crucial conclusion from analysing optimal decision rules of boundedly rational agents is that agents have to make long-horizon forecasts. Drawing on prior work by Marcet and Sargent (1989) he proposes "infinite-horizon learning" as an alternative to Euler-equation learning.

In chapter 2, Preston's approach in deriving the consumption rule for the New Keynesian model has been illustrated. The consumption rule (2.8) is rewritten here for convenience

$$C_t^i = (1-\beta)\bar{\omega}_t^i + \mathrm{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)Y_T - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1}) \right]. \tag{4.14}$$

Behaviour, represented by this decision rule, is assumed to follow a linear combination of infinitely many forecasts which results from the fact that infinite-horizon learning makes elaborate use of the individual budget constraint.[12] Preston (2005) claims that conditioning decisions on an infinite horizon is irreducible if agents are indeed optimisers. In particular, he asserts that Euler-equation learning is not arbitrarily close to being optimal. This follows from the observation that the behavioural rule (4.13) does not take account of initial wealth endowments whatsoever. Thus, Euler-equation learning will lead to systematic under-consumption for households with $\bar{\omega}_t^i > 0$ and over-consumption for households with $\bar{\omega}_t^i < 0$.

Even if initial wealth endowments are constrained to being zero, Euler-equation learning

[10]In order to establish equivalence, Evans and Honkapohja (2003) assume identical households, homogeneous forecasts, and most importantly knowledge on the relation $C_t^i = Y_t = x_t + Y_t^n$ for all i and t such that agents form forecasts $\mathrm{E}_t^i C_{t+1}^i = E_t^i x_{t+1} + Y_{t+1}^n$. In deriving the Phillips curve it is assumed that agents realise that the deviation of optimal prices to aggregate prices, denoted by $\hat{p}^i$, is a linear function of $\mathrm{E}_t^i \hat{p}^i, x_t$ and $\mathrm{E}_t^i \pi_{t+1}$. Since $\hat{p}^i = \hat{p}$ and there is a proportional relationship between $\hat{p}$ and inflation, firms will rewrite $\hat{p}^i$ as a linear function of expected inflation and current output.

[11]Honkapohja (2003) responds to this critique that such knowledge is in fact not presupposed when deriving aggregate relations by means of Euler-equation learning.

[12]Although the Euler-equation learning approach does not make use of the budget constraint at all, Honkapohja (2003) notes that it is not necessarily inconsistent with it. If the economy converges to a REE, the transversality condition must hold ex post also under Euler-equation learning.

is not necessarily optimal. Iterating the optimal decision rule (4.14) one period ahead and taking expectations at time t gives the households' expected optimal choice for period $t+1$ as

$$\mathrm{E}_t^i C_{t+1}^i = \mathrm{E}_t^i \sum_{T=t+1}^{\infty} \beta^{T-t} \left[(1-\beta) Y_T - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1}) \right]. \tag{4.15}$$

The individual Euler-equation constitutes an optimal decision rule if and only if $\mathrm{E}_t^i C_{t+1}^i$ in (4.13) coincides with the optimal forecast given above. Since this optimal decision rule is a particular linear combination of forecasts of the state variables, there is in general no reason for forecasts of C_{t+1}^i, constructed from past observations of aggregate disturbances, to coincide with (4.15).

Preston (2005) concludes that such suboptimal behaviour is a manifestation of a general point: Forecasting $\mathrm{E}_t^i C_{t+1}$ from $\mathrm{E}_t^i x_{t+1}$ is internally inconsistent with household optimisation. It represents a forecast of future consumption which differs from what the household expects to be optimal, given its current forecast of future income, inflation and interest rates. Thus, forecasts relying on aggregate measures represent a less sophisticated approach to forecasting since they fail to make use of information that the agent necessarily possesses.

However, Honkapohja et al. (2011) note that once the REE is approached, both learning mechanisms lead to the same forecast. Moreover, they demonstrate that the Euler-equation analysis is consistent with the infinite-horizon analysis in the New Keynesian model. This justification shall be retraced here. Thus, it will be shown that the aggregated reduced-form system

$$x_t = \mathrm{E}_t x_{t+1} - \sigma(i_t - \mathrm{E}_t \pi_{t+1}) + r_t^n,$$
$$\pi_t = \kappa x_t + \beta \mathrm{E}_t \pi_{t+1} + u_t,$$

can be derived from Preston's micro-founded model of boundedly rational agents. The most central assumption with this respect is that the law of iterated expectations holds at the individual level. That is for any variable z it holds that

$$\mathrm{E}_t^i \mathrm{E}_{t+s}^i z = \mathrm{E}_t^i z \text{ for} s = 0, 1, \ldots. \tag{4.16}$$

This is a standard assumption for agents making forecasts from linear laws of motion estimated by Recursive Least Squares, which will be assumed in the following. Assume that agents are basing their expectation on the same information set and are using the same estimation procedure. Given that each agent i has identical parameter estimates it follows that the forecast of each agent is the same, that is $\mathrm{E}_t^i = \mathrm{E}$ for all i. Note that there is no need for any single agent to make this inference when forming the forecast needed for his decision making. Every agent i forms his forecast independently from the other agents in the economy and uses this forecast in the optimal consumption rule (4.14). It follows from (4.16) and the assumption of homogenous expectations that $C_t^i = C_t$ for all i.

In order to establish consistency of infinite horizon learning and Euler-equation learning in this particular context, assume that g_t, g_{t+1}, Y_t^n and Y_{t+1}^n are known at time t. Taking quasi-differences of the optimal consumption rule (4.14), advancing the relation by one time period, taking expectations of both sides and applying the law of iterated expectations yields

$$C_t^i = \beta \mathrm{E}_t^i C_{t+1}^i + (1-\beta)(x_t + Y_t^n) - \beta\sigma(i_t - \mathrm{E}_t^i \pi_{t+1}) + \beta(g_t - g_{t+1}), \tag{4.17}$$

where additionally, the definition of the output gap was used. In order to implement this behavioural rule, agents must forecast their own consumption decision. Market clearing and the representative agent assumption imply $C_t^i = Y_t$ for all t and i. With this regard, Honkapohja et al. (2011) assume that agents observe the equality $C_t^i = Y_t$ from historical data and are furthermore making use of the relation $Y_t = x_t + Y_t^n$ as to base their forecast on aggregate output.[13] Thus, individual expectations on consumption decisions are given by

$$\mathrm{E}_t^i C_{t+1}^i = \mathrm{E}_t^i x_{t+1} + Y_{t+1}^n, \tag{4.18}$$

Substituting this into (4.17) and making use of the relation $r_t^n = g_t - g_{t+1} + Y_{t+1}^n - Y_t^n$ gives the behavioural equation

$$C_t^i = \beta \mathrm{E}_t^i x_{t+1}^i + (1-\beta)x_t + Y_t^n - \beta\sigma(i_t - \mathrm{E}_t^i \pi_{t+1}) + \beta r_t^n. \tag{4.19}$$

Finally, from market clearing $C_t^i = Y_t = x_t + Y_t^n$ and using $\mathrm{E}_t^i x_{t+1} = \mathrm{E}_t x_{t+1}$ and

[13]This critical assumption has been subject to critique by Preston (2005) since he regards market clearing conditions as belonging to the set of REE conditions that agents are attempting to learn.

$E_t^i \pi_{t+1} = E_t \pi_{t+1}$ the aggregate Euler-equation

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1}) + r_t^n,$$

is obtained. The derivation of the Phillips curve, from Preston's optimal pricing rule, is analogous and omitted here.

The Euler-equation learning approach and the infinite-horizon learning approach, clearly, depict two contrasting positions of how expectation formation, other than rational, can be modelled. Euler-equation learning relies on the subjective optimality margin only one period ahead. Under RE this behaviour is optimal since forecasts of tomorrow contain all the relevant information needed to take optimal decisions, in the New Keynesian framework, today. Out of equilibrium, however, the quality of this learning rule is less clear. Infinite-horizon learning, on the other hand, incorporates the subjective budget constraint into the decision process, implying that agents' choice satisfies their subjective Euler-equation not only one period ahead, but at all iterations. Branch et al. (2010) point out that this can be interpreted as assuming that agents solve each period their dynamic optimisation problem.

While the consistency of infinite-horizon learning with the micro-foundations certainly has appeal, Branch et al. (2010) note on a number of drawbacks of this approach: First, agents are required to make forecasts at all horizons. This stands in stark contrast to applied econometricians who ordinarily face finite horizons problems. Second, agents are assumed to be sophisticated enough as to solve their infinite-horizon dynamic programming problem. And most crucially, third, agent's behaviour is based on the assumption that their beliefs are correct. In a model of adaptive learning, however, forecasts are updated by some estimation procedure and agents should be assumed to recognise that their beliefs will change over time. Branch et al. (2010) point out that it is no longer obvious that agents' optimal decision will be determined by the fully optimal solution to their dynamic programming problem, given this current belief. While this critique also applies to short horizons, the conclusions are more dramatic in the case of infinite-horizon learning since this approach places considerable weight on distant forecasts.[14]

Although both approaches are consistent in the New Keynesian model, they are not identical, and lead to different paths of learning dynamics. Preston (2005) investigates variants of monetary policies that respond to one-step ahead forecasts of inflation and

[14]Branch et al. (2010) introduce "finite-horizon learning" where decisions are based on N-step ahead Euler-equations in order to generalise existing learning mechanisms.

the output gap, similar to the investigation of forward looking Taylor-type rules as considered by Bullard and Mitra (2002). While agents in Preston's analysis are required to forecast all future paths of nominal interest rates, these forecasts are not required in Bullard and Mitra's article. Forecasts of inflation are not relevant for individual Euler-equations. Honkapohja et al. (2011) point out that the difference in these setups can also be interpreted by means of central bank transparency. That is, Preston's analysis can be explained by the assumption that agents do not know the exact policy rule. Greater central bank transparency, on the other hand, allows agents to infer on this information in Bullard and Mitra (2002). Depending on whether this knowledge is assumed or not, Preston concludes that learning dynamics can be different between the infinite-horizon and the Euler-equation approach in the one case, while they are exactly the same in the other. A recent paper by Bullard and Eusepi (2009) isolates conditions under which the Euler equation and infinite horizon approaches to learning yield the same E-stability conditions.

The appropriateness of the learning assumptions can supposedly be assessed only by addressing the ultimately empirical question of the nature of individual behaviour outside equilibrium. In a stable or transparent regime one might expect that agents are using short term forecasts. If the economy is constantly changing, on the other hand, behaviour might be better described by a sophisticated learning approach. This thesis will work out the implications of adaptive learning under the assumption that agents are making one period forecasts only.

Chapter 5

Properties of Interest Rate Rules

Earlier work has evaluated the desirability of interest rate rules almost exclusively, on whether their implementation implies the existence of a unique dynamically stable REE. Only then, agents' expectation are considered to have a precise meaning. Recently, the learnability criterion has been put forward as another necessary condition for the plausibility of some REE (see e.g. McCallum, 2008).[1] This criterion imposes the behavioural requirement that some REE may be attainable, in an out-of-equilibrium process, after the economy was perturbed.

The relevance of these concepts is subject to heavy debates in the literature. McCallum (2003) advocates the view that indeterminacy is rather a problem with one's understanding of the model and not a problem with the economy which is represented by the model. He argues that any well formulated model should provide a unique prediction and regards indeterminacy as only posing the challenge to distinguish the right prediction from the others. In a series of papers, McCallum suggests to view the MSV solution as the only relevant equilibrium prediction and to regard the other solutions as theoretical curiosities. McCallum interprets results from the learning literature, showing that there exist REE which are indeterminate but where the MSV solution is nonetheless E-stable, as supporting his perspective and as emphasising the relevance of the MSV

[1] LS-learnability, that is asymptotic convergence to some REE under adaptive learning, under the assumptions that 1. the PLM is correctly specified, 2. the structure of the economy remains unchanged and 3. agents are estimating with an efficient technique is not considered a sufficient condition for the plausibility of a REE for obvious reasons.

solution.

An opposing stance is taken by Woodford (2003a) who reckons that multiple equilibria genuinely represent alternative possible outcomes for the economy in question. On grounds of earlier work (Woodford, 1990), illustrating that adaptive learning dynamics do not necessarily converge to the MSV solution, but may as well converge to stationary sunspot equilibria, Woodford strongly rejects McCallum's view that adaptive learning supports the validity of the MSV principle. Instead, he interprets results from the learning literature as confirming the main point of the determinacy analysis. He points out that both E-stability and determinacy obtain, "when outcomes are not overly sensitive to expectations, in a way that allows a perturbation of expectations to change outcomes to an extent that justifies the alternative expectations."(Woodford, 2003a, pp. 1179).

This chapter will summarise results on determinacy and E-stability for various interest rate rules. In course of sketching some results for Taylor-type rules, the link between E-stability and determinacy will be clarified. Thereafter, optimal monetary policy is evaluated in light of uniqueness and learnability.

5.1 Taylor-Type Rules

The seminal paper by Bullard and Mitra (2002) gave an early example that conditions for determinacy and E-stability to obtain can indeed coincide. As a baseline specification, Bullard and Mitra considered interest rate rules of the form

$$i_t = \varphi_\pi \pi_t + \varphi_x x_t, \tag{5.1}$$

where φ_π and φ_x are non-negative parameters which are not both equal to zero. Note that this specification of monetary policy is a simple feedback rule of the type discussed by Taylor (1993).[2] Bullard and Mitra show that the equilibrium implemented by this interest rate rule is determinate, given that

$$\varphi_\pi + \frac{1-\beta}{\kappa}\varphi_x > 1. \tag{5.2}$$

Clearly, this is a version of the Taylor principle, stressing that a desirable monetary policy should react sufficiently strong in response to inflation movements. Bullard (2006)

[2] In fact, Taylor considered inflation rules of the form $i_t = r + \pi_t + \varphi_\pi(\pi_t - \pi) + \varphi_x x_t$ where π denotes the inflation target and r is the average real interest rate.

provides the following justification for this result: The Phillips curve implies that a permanent increase in inflation goes along with an increased output gap of $(1-\beta)/\kappa$ percentage points. In light of the interest rate specification (5.1), the left hand side of (5.2) can be interpreted as the required adjustment of the nominal interest rate in response to an increase in permanent inflation. Thus, the conclusion of analysing determinacy for an arbitrary interest rate specification is similar to the result obtained from analysing optimal monetary policy in chapter 3: The response of the interest rate should be more than one-for-one to variations in expected inflation.

A key result of Bullard and Mitra (2002) is that the Taylor principle (5.2) is a necessary and sufficient condition for the REE associated with the feedback rule (5.1) to be expectationally stable. This seems to support Woodford's view of a tight connection between E-stability and determinacy. However, the authors also gave examples where these conditions did not coincide. They considered forward-looking versions of the Taylor rule, taking the form

$$i_t = \varphi_\pi \mathrm{E}_t \pi_{t+1} + \varphi_x \mathrm{E}_t x_{t+1}, \tag{5.3}$$

where expectations of the monetary authority can be interpreted as being based on Least Squares estimations. Bernanke and Woodford (1997) discussed policy rules of this form, known as inflation forecast targeting, as a means to overcome the problems associated with the long lag between change in policy and change in the inflation rate.[3]

Bullard and Mitra (2002) illustrate that the necessary and sufficient conditions for uniqueness of equilibrium are given by

$$\varphi_x < \sigma^{-1}(1+\beta^{-1}), \qquad \kappa(\varphi_\pi - 1) + (1+\beta)\varphi_x < 2\sigma^{-1}(1+\beta),$$

and

$$\varphi_\pi + \frac{(1-\beta)}{\kappa}\varphi_x > 1. \tag{5.4}$$

In contrast to other specifications of the Taylor rule, values assigned to φ_x are of primary importance for determining uniqueness in this case. In particular, an aggressive response to output movements leads to indeterminacy while specifications of the form $\varphi_\pi > 1$ and

[3] Svensson (1999b) emphasised that inflation forecast targeting can be seen as an optimal intermediate-targeting rule since it provides the central bank with a way to implement first order conditions for an optimum, while outsiders are given the opportunity to verify that these first order conditions are implemented.

φ_x sufficiently small, leave the resulting equilibrium determinate.

It turns out that relation (5.4) proves to be necessary as well as sufficient for the MSV solution to be stable under adaptive learning. Thus, determinacy implies E-stability of the MSV solution in this case and Bullard (2006) reckons that it seems to be a good advice, both from the point of view of uniqueness of equilibrium as well as from the point of view of attainability of that equilibrium, that policymakers adopt the Taylor principle in selecting a particular policy rule.

However, even if policy makers adopt this recommendation there is ample space for believe driven fluctuations and the analysis by Bullard and Mitra (2002) leaves open the question whether these solutions are stable under learning. Honkapohja and Mitra (2004) close this gap by analysing the stability of non-fundamental solutions in the New Keynesian model. They show that there exist E-stable sunspot solutions for plausible parameter specifications under forward looking Taylor-type rules and interpret this result as strengthening the worries of Bernanke and Woodford (1997).[4]

With this result, Honkapohja and Mitra (2004) contribute to the discussion on the relevance of determinacy by shifting attention to the learnability criterion. While they do not consider sunspot solutions as theoretical curiosities, Honkapohja and Mitra argue that the mere existence of these solutions should neither circumvent the associated policy measure. Instead, monetary policy should focus on the learnability criterion and be concerned with designing policy rules such that non-fundamental (non-MSV) solutions are unstable under learning while the fundamental solution is stable under learning.

The results summarised until so far indicate that there exists a close connection between the concepts of determinacy and E-stability. In particular, determinacy turned out to be a sufficient condition for E-stability in each case. This poses the question why one should not neglect the learnability criterion and concentrate on determinacy instead? This view is in fact strengthened by McCallum (2007) who demonstrates for a broad class of models that determinacy indeed implies E-stability. However, the crucial assumption underlying his analysis is that agents have information on contemporaneous endogenous variables available in their learning process. Extending his analysis to learning specifications where agents have information on lagged variables available only, McCallum (2008)

[4] Bernanke and Woodford (1997) argue against inflation targeting on grounds of two sets of reasons: First, inflation forecast targeting provides broad scope for indeterminacy, implying the potential of arbitrary volatility in the inflation process. Second, if monetary policy is successful, the signal-to-noise ratio in the inflation forecast is likely to become small. In the limit of perfect stabilisation, this implies that there is no incentive for the private sector to gather information since the inflation forecast becomes uninformative in this case.

concludes that the strong link between determinacy and E-stability does not pertain in this environment.

While Bullard and Mitra (2002) analysed various Taylor-type rules under the baseline specification that agents have contemporaneous data available in their learning process, they also checked for robustness of their results by considering the case that agents have only lagged data available. In particular, they considered an interest rate rule of the form

$$r_t = \varphi_x x_{t-1} + \varphi_\pi \pi_{t-1},$$

and restricted, for consistency, agents' information set to variables as of time $t-1$. Analytical results for characterising E-stability could not be obtained under this specification. Nevertheless, computations showed that there exist determinate equilibria which are not attainable under adaptive learning. Note that the "lagged data" specification differs from the aforementioned specifications since it induces persistence into the reduced-form model. Consequently, assumptions on the information set being available to form forecasts matter in this case, while they are irrelevant for the other specifications.

5.2 Optimal Policy Rules

This section follows Evans and Honkapohja (2003) and Evans and Honkapohja (2006) in analysing determinacy and E-stability under optimal monetary policy. While the discussion primarily serves to further clarify the link between these concepts, it will also serve as to apply the techniques which were presented in chapter 4. It was sketched that Preston's New Keynesian model with boundedly rational agents can be represented by the aggregate, reduced-form equations

$$\begin{aligned} x_t &= \mathrm{E}x_{t+1} - \sigma(i_t - \mathrm{E}_t\pi_{t+1}) + r_t, \\ \pi_t &= \kappa x_t + \beta \mathrm{E}_t \pi_{t+1} + u_t, \end{aligned} \tag{5.5}$$

where the stochastic disturbances r_t and u_t are both following stationary AR(1) processes and E_t denotes aggregated, non-rational expectations. The derivation of this dynamic system was enabled by the assumptions that subjective expectations of individuals obey the law of iterated expectations, that forecasts are homogenous and that agents possess knowledge of market clearing conditions. Given these assumptions, the Euler-equation learning approach provides a behavioural foundation for obtaining the above relations.

To close the model, the nominal interest rate i_t has to be specified. In chapter 3, it was shown that the implementation of optimal monetary policy is consistent with various interest rate specifications. In particular, expectations-based interest rate rules were distinguished from fundamentals-based rules. To avoid redundancy, this chapter will focus on two specifications: The fundamentals based optimal interest rate rule under commitment

$$i_t = \psi_0 + \psi_1 x_{t-1} + \psi_2 u_t + \psi_3 r_t \tag{5.6}$$

with coefficients given in (3.24), and the expectations-based optimal interest rate rule under discretion

$$i_t = \delta_0 + \delta_1 \mathrm{E}_t \pi_{t+1} + \delta_2 \mathrm{E}_t x_{t+1} + \delta_3 r_t + \delta_4 u_t, \tag{5.7}$$

where the coefficients can be found in (3.11). Note that these rules are structurally similar to Bullard and Mitra's "lagged data" specification and their forward-looking Taylor rule. Upon substitution of these interest rate specifications into the structural model (5.2), the New Keynesian model can be represented by

$$\begin{aligned} y_t &= A + B\mathrm{E}_t y_{t+1} + C y_{t-1} + D w_t, \\ w_t &= F w_{t-1} + \epsilon_t, \end{aligned} \tag{5.8}$$

where $y_t = (x_t, \pi_t)'$, $w_t = (r_t, u_t)'$, both eigenvalues of F lie inside the unit circle and ϵ_t is a white noise vector. The properties associated with the expectations-based interest rate rules will be analysed by means of the techniques which were established in chapter 4 and in terms of the implied matrices A, B, C, D in (5.8).

Fundamentals-Based rule

The fundamentals-based rule under commitment provides an example in the New Keynesian framework that determinacy does not necessarily imply E-stability. Under the fundamentals-based rule (5.6), system (5.8) is characterised by

$$A = \begin{pmatrix} -\sigma\psi_0 \\ -\kappa\sigma\psi_0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \sigma \\ \kappa & \beta + \kappa\sigma \end{pmatrix}, \quad C = \begin{pmatrix} -\sigma\psi_1 & 0 \\ -\kappa\sigma\psi_1 & 0 \end{pmatrix} \quad , D = \begin{pmatrix} 0 & -\sigma\psi_2 \\ 0 & 1 - \kappa\sigma\psi_2 \end{pmatrix}.$$

By retracing the steps outlined in section 4.2, the system is rewritten in first order form to obtain the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta & \kappa \\ (\sigma\psi_1)^{-1} & \psi_1^{-1} & -(\sigma\psi_1)^{-1} \end{pmatrix},$$

which governs determinacy. Numerical calculations by Evans and Honkapohja (2006) show that exactly two eigenvalues of J lie outside the unit circle for α small, while for larger values of α, only one root lies outside the unit circle. Thus, there exist parameter calibrations such that optimal monetary policy under commitment can be implemented by the fundamentals-based policy rule without implying indeterminacy.

However, application of the E-stability principle shows that the resulting equilibrium leads to instability under learning for all structural parameter values. This can be seen by considering the following linearised differential equation

$$DT_a - \mathrm{I} = \begin{pmatrix} \tilde{b}_x + \sigma\tilde{b}_\pi & \sigma \\ \kappa(\tilde{b}_x + 1) + (\beta + \kappa\sigma)\tilde{b}_\pi & (\beta + \kappa\sigma) - 1 \end{pmatrix}.$$

Expectational stability requires that all eigenvalues of $DT_a - \mathrm{I}$ are negative, which is equivalent with requiring that $tr(DT_a - \mathrm{I}) < 0$ and $det(DT_a - \mathrm{I}) > 0$. However, the determinant is given by

$$(\beta - 1)\tilde{b}_x - \sigma\tilde{b}_\pi - \kappa\sigma,$$

which is negative for all parameter values.[5] Since this necessary condition for stability failed, the eigenvalues of $DT_b - \mathrm{I}$ and $DT_c - \mathrm{I}$ do not have to be evaluated.

As a partial intuition for this result, Evans and Honkapohja provide the following observation. Suppose that all coefficients in the PLM are held constant at their equilibrium values. The T-map for the coefficient a_π, the constant in the perceived inflation process,

[5] This can be seen by noting that κ, σ are positive, $0 < \beta < 1$ and $\tilde{b}_x, \tilde{b}_\pi$ are also positive.

then becomes

$$T(a_\pi) = -\kappa\sigma\psi_0 + \left[\kappa(1+\tilde{b}_x) + (\beta+\kappa\sigma)\tilde{b}_\pi\right]\tilde{a}_x + (\beta+\kappa\sigma)a_\pi.$$

In economic terms this relation can be explained by interpreting a positive deviation of a_π from its equilibrium value $\tilde{a}_\pi$ as an exogenous shock to inflation expectations. From the Phillips curve it follows that this has the immediate effect of increasing current inflation by β times the shock. Through its effect on the real interest rate, the shock induces households to increase consumption by σ times the shock which in turn increases current inflation indirectly by $\kappa\sigma$ times the shock, again through the Phillips curve.

Thus, if policy makers concentrate on fundamental shocks and are not concerned with counteracting non-rational expectations, the above reasoning implies that a shock to inflation expectations raises current inflation by $(\beta+\kappa\sigma)$ times the shock. Reconsidering the iterative E-stability approach, the above relation implies $a_\pi(n+1) = constant + (\beta+\kappa\sigma)a_\pi(n)$. Since $(\beta+\kappa\sigma)$ is typically greater than one, a positive deviation of a_π from $\tilde{a}_\pi$ will initiate explosive dynamics. However, this story only provides a partial intuition since expectational instability holds for all possible parameter values and in particular for $(\beta+\kappa\sigma) < 1$.

Evans and Honkapohja emphasise that this result should be taken seriously by policy makers. One should not assume automatically that some REE will be attained only because it is locally unique. A similar result is obtained for the fundamentals-based optimal policy rule (3.21) under discretionary policy by Evans and Honkapohja (2003). In contrast to optimal policy under commitment, however, the equilibrium is also indeterminate in this case.

Expectations-Based Rule

To overcome the problems associated with the fundamentals based policy rule, Evans and Honkapohja (2003, 2006) propose to base interest rate specifications on private sector forecasts. Substitution of the expectations-based interest rate rule under discretionary policy (3.10) into the structural equations (5.2) implies the following matrices of system

(5.8)

$$A = \begin{pmatrix} -\sigma\delta_0 \\ -\sigma\delta_0\kappa \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\beta\kappa(\kappa^2+\alpha)^{-1} \\ 0 & \beta\alpha(\kappa^2+\alpha)^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -\sigma\delta_4 \\ 0 & 1-\sigma\delta_4\kappa \end{pmatrix}. \tag{5.9}$$

Since there is no persistence under discretionary policy, i.e. all elements of C are zero, determinacy is governed by the matrix B. It can easily be verified that both eigenvalues associated with this matrix are located inside the unit circle for which reason the resulting REE is locally unique.[6]

To analyse E-stability of the MSV-solution, consider the independent linear subsystems for a and c, with slope coefficients

$$DT_a - \mathrm{I} = \begin{pmatrix} -1 & -\beta\kappa(\kappa^2+\alpha)^{-1} \\ 0 & \beta\alpha(\kappa^2+\alpha)^{-1}-1 \end{pmatrix} = B - \mathrm{I}$$

$$DT_c - \mathrm{I} = \begin{pmatrix} -1 & -\beta\kappa(\kappa^2+\alpha)^{-1}\rho \\ 0 & \beta\alpha(\kappa^2+\alpha)^{-1}\rho-1 \end{pmatrix} = \rho B - \mathrm{I}.$$

E-stability obtains if the eigenvalues of both $DT_a - \mathrm{I}$ and $DT_c - \mathrm{I}$ are negative. Given that the eigenvalues of B are inside the unit circle it immediately follows that the REE, implemented by the expectations-based optimal interest rate rule, is stable under learning since $\rho < 1$. Evans and Honkapohja (2006) show that these desirable properties also hold for the expectations-based rule under commitment.

An intuition for the increased stability can be obtained by reconsidering the definition of the interest rate rule. A positive deviation of a_π from its equilibrium value and the hereby implied increased inflation expectations are directly counteracted by the monetary authority. The nominal interest rate is lowered more than one for one in order to decrease the output gap by $\kappa\beta/(\alpha+\kappa^2)$ times the shock. This in turn has the effect of lowering current inflation from β (the direct effect) to $\alpha\beta/(\alpha+\kappa^2)$, through the Phillips curve.

Note that the condition for determinacy is again more stringent in this case. In fact, it turns out that this relation is systematic. Bullard and Eusepi (2009) consider a general New Keynesian model and show that if the policy rule includes expectations of variables for the next period, then determinacy implies E-stability in the case of Euler-equation

[6] The eigenvalues are given by $0 < \beta\alpha(\kappa^2+\alpha)^{-1} < 1$ and 0.

learning. Under infinite-horizon learning, however, this connection breaks down.

Chapter 6

Restricted Perceptions Equilibria

The preceding chapter outlined the desireable properties of expectations-based interest rate rules: Given that the economy is close to the REE, these interest rate specifications endorse private sector forecasts, being based on econometric models, to become self-fulfilling. Moreover, there exists a unique way of forming expectations if the central bank conditions its reaction on private sector expectations, i.e. the REE associated with these rules is determinate. However, a central prerequisite for learning this equilibrium process is that the structure of the economy is understood qualitatively. In the preceding chapter this assumption was made for convenience and in order to check for the robustness of different equilibria.

This chapter will analyse the implications when the assumption of bounded rationality is taken one step further. Professional forecasters limit the number of variables in their statistical models due to degrees of freedom restrictions or due to computational costs. In light of scarcity and costliness of information, Branch (2006) argues that underparameterisation is a reasonable approach to expectation formation. The concept of RE, however, has to be abandoned if this assertion is to be incorporated into a formal theory. Given that the PLM is of lower dimensionality than the implied dynamics, it is impossible for agents' beliefs to coincide with the actual distribution of the model.

This incapability of agents' perception to reflect actual dynamics might at first appear implausible; especially in light of the REH and the hereby intimately linked optimising assumptions that are prevalent throughout economics. However, Branch (2006) proposes to reconsider the following assertions made by Muth: "(1) Information is scarce, and the economic system generally does not waste it. (2) The way expectations are formed depends specifically on the structure of the relevant system describing the economy. (3) A "public prediction," in the sense of Grunberg and Modigliani [14], will have no substantial effect on the operation of the economic system (unless it is based on inside information)."(Muth, 1961, pp. 316)[1]

Branch (2006) convincingly argues that adaptive learning in general, and in particular by means of underparameterised models, is consistent with these assertions, given that agents' forecasting model satisfies an orthogonality condition. He interprets a self-referential economy as consisting of agents who only understand the economy so far as their own subjective model. In order to be consistent with Muth's hypothesis, these models must reflect, in some dimension, the true dynamics of the economy. Branch argues that if agents' beliefs within their forecasting model are not contradicted by actual outcomes, then agents can not be freely disposing of useful information. Furthermore, there is also correspondence between beliefs and outcomes under adaptive learning since beliefs are supported by the structure of the economy. Branch concludes that Muth's hypothesis is satisfied whenever agents' forecasting errors are uncorrelated with their forecasting model. Then, the forecasting models are consistent with the economy in the sense that agents can not tell that their model is distinct from the underlying stochastic process.

The notion that agents' beliefs come from misspecified forecasting models but agents are unable to detect their misspecification was first used by Evans and Honkapohja (2001) and defined as a Restricted Perceptions Equilibrium (RPE). In the following, this equilibrium notion is illustrated in the New Keynesian model under optimal discretionary policy. The E-stability of one particular RPE will be discussed and conclusions for the value of central banks's transparency will be drawn. Thereafter, an extension to the RPE, the so-called Misspecification Equilibrium (ME), will be presented.

6.1 Central Bank's Transparency

Recently, Berardi and Duffy (2007) pointed out the importance of the RPE in light of central banks' transparency. High-quality monetary policy reports as well as an explicit

[1][14] refers to: Grunberg, E. and F. Modigliani (1954): "The Predictability of Social Events," Journal of Political Economy, 62:465-478

announcement of policy objectives are widely considered essential for establishing and maintaining credibility. Under adaptive learning, it may reasonably be argued that such policy measures help to form a correctly specified PLM. This section is following Berardi and Duffy (2007) in discussing the advantages and drawbacks of central banks' transparency when agents are learning.

Inflation targeting is regarded as one of the most important and successful monetary policy strategies in place (see e.g. Svensson, 2010). Around 25 industrialised and non-industrialised countries have adopted this strategy, which is, inter alia, characterised by a high degree of transparency and accountability.[2] In view of traditional central banks' behaviour, where policy objectives and policy decisions have been subject to considerable secrecy, the claim for transparency is exceptional. However, the claim can be retraced by drawing on contemporaneous monetary models. With the incorporation of micro-foundations, aggregate movements have been explained by private sector forecasts and central banking, consequently, became understood as a management of expectations. Transparency, being intimately linked to credibility, clearly depicts a valuable measure with this respect.

Under adaptive learning, the value of central bank's transparency has been interpreted in various ways. Orphanides and Williams (2005) concentrated on the increased informational supply transparency depicts. They show that agents, using a correctly specified model but a truncated sample of data, find it easier to learn the REE if the central bank announces its inflation target. Convergence to equilibrium is therefore faster. Berardi and Duffy (2007), on the other hand, understand the value of central bank transparency as not only providing better information in a quantitative sense, but as providing more information qualitatively.

Specifically, they argue that central banks are in the position of providing the public with the correct forecast. In the New Keynesian model, this forecast not only depends on whether optimal policy is conducted under discretion or under commitment, but moreover on the specific policy targets. As the central bank is assumed to hold this information, it is in the position to inform the public about the relevant structure and therewith, to provide the correct PLM.

[2] Svensson (2010) notes that inflation targeting is characterised by (1) an announced numerical inflation target (usually around 2%) (2) an implementation of monetary policy that gives a major role to an inflation forecast ("inflation forecast targeting") and (3) a high degree of transparency and accountability.

Restricted Perceptions under Discretionary Policy

The preceding chapter indicated that, upon implementation of the expectations-based interest rate rule under discretion, the economy can be represented by the system

$$\begin{aligned} y_t &= A + B\mathrm{E}_t y_{t+1} + D w_t \\ w_t &= F w_{t-1} + \epsilon_t \end{aligned} \tag{6.1}$$

where, as usual, $y_t = (x_t, \pi_t)'$, $w_t = (r_t, u_t)'$ and the matrices A, B and D are given in (5.9). In analysing the stability of the MSV solution (which takes the form $y_t = a + cu_t$) it was implicitly assumed that agents anticipate nonzero target values x and π. If the central bank is not explicit about these targets, agents might as well consider solutions of the form

$$y_t = cu_t \tag{6.2}$$

where $y_t = (x_t, \pi_t)'$ and $c = (c_x, c_\pi)$. With policy objectives $x = \pi = 0$, all elements of A are zero for which reason the MSV solution for the system (6.1) was given by (6.2). Berardi and Duffy (2007) rationalise the assumption that agents omit the constant term of the MSV solution on two grounds. First, if the private sector is unaware of the existence of a central bank, it might choose to exclude constant components from its model as the inclusion of these terms is only necessary because there is a monetary policy. As a second, more compelling reason, they suggest that the private sector is aware of the existence of the monetary authority, but falsely assumes that both x and π equal zero.

Under the assumption that private forecasts are based on (6.2) and therefore given by $\mathrm{E}_t y_{t+1} = c\rho u_t$, the implied T-map takes the form

$$T\begin{pmatrix} c_x \\ c_\pi \end{pmatrix} = \left(\begin{pmatrix} (\kappa\pi + \alpha x)(\alpha + \kappa^2)^{-1} \\ \kappa(\kappa\pi + \alpha x)(\alpha + \kappa^2)^{-1} \end{pmatrix}, \begin{pmatrix} -\kappa(\beta\hat{c}_\pi\rho - 1)(\alpha + \kappa^2)^{-1} \\ \alpha(\beta\hat{c}_\pi\rho + 1)(\alpha + \kappa^2)^{-1} \end{pmatrix} \right). \tag{6.3}$$

Clearly, the REE no longer depicts an attainable fixed point of this map. Since the estimated PLM (6.2) can not possibly converge to the REE, Evans and Honkapohja (2003) refer to models of this form as "misspecified models". It was pointed out that subjective models are "misspecified" in general since the PLM does not capture the fact that the parameters of the ALM are time-varying. However, in contrast to misspecifications of the form (6.2), the misspecification of a correctly specified PLM becomes vanishingly small if the economy convergence to the REE.

Nevertheless, Branch (2006) argues that forecasts based on the model (6.2) are consistent with Muth's assertions if the parameter vector c is formed as the optimal linear projection of the state $y_t = T(c)(1, u_t)'$ on the subjective model space u_t. That is, the parameter c must satisfy the Least-Squares orthogonality condition

$$\mathbb{E}u_t\left(y_t - \hat{c}u_t\right) = 0.$$

Then, in equilibrium, agent's beliefs are consistent with the actual process in the sense that their forecasting errors are undetectable within their perceived model. This condition implies that in a RPE, the parameter vector c has to obey

$$\hat{c} = \begin{pmatrix} \hat{c}_x \\ \hat{c}_\pi \end{pmatrix} = \begin{pmatrix} \kappa(\alpha + \kappa^2 - \alpha\beta\rho)^{-1} \\ \alpha(\alpha + \kappa^2 - \alpha\beta\rho)^{-1} \end{pmatrix} = \bar{c}, \tag{6.4}$$

where $\bar{c}$ denotes the REE value of the vector c.[3] Thus, agents correctly perceive the one dimension of reality which they are adhering to. Furthermore, within their model, they are predicting optimally. Consequently, it may well be argued that the RPE is fully rational, given that agents are ignorant of their restricted perceptions. Combining the T-map (6.3) with (6.4) gives the dynamics in a RPE to obey

$$\begin{pmatrix} x_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} (\kappa\pi + \alpha x)(\alpha + \kappa^2)^{-1} \\ \kappa(\kappa\pi + \alpha x)(\alpha + \kappa^2)^{-1} \end{pmatrix} + \begin{pmatrix} \kappa(\alpha(1 - \beta\rho) + \kappa^2)^{-1} \\ \alpha(\alpha(1 - \beta\rho) + \kappa^2)^{-1} \end{pmatrix} u_t. \tag{6.5}$$

The next section investigates the question whether this equilibrium point is attainable under adaptive learning.

E-stability of the Restricted Perceptions Equilibrium

When the PLM is underparameterised, the E-stability principle does not apply in general. Instead, the associated ODE, governing the asymptotic dynamics of the system, has to be derived explicitly since the misspecification can alter the stability condition for the resulting equilibrium (Evans and Honkapohja, 2001). Consider therefore the recursive representation of the Least-Squares estimate for c. Appendix A shows that upon the transformation $R_t = S_{t-1}$, with R_t being the second moment of u_t, the Recursive

[3] For equilibrium values of the coefficients under RE see (3.16)

Least Squares algorithm takes the form

$$\begin{aligned} c_t =& c_{t-1} + t^{-1} S_{t-1}^{-1} u_{t-1} \left[T(c_{t-1})(1, u_{t-1})' - c_{t-1} u_{t-1} \right]', \\ S_t =& S_{t-1} + t^{-1} \frac{t}{t+1} (u_t u_t - S_{t-1}). \end{aligned} \tag{6.6}$$

For establishing the associated ODE, the steps outlined in section A.3 are followed. Consider the asymptotic mean

$$h(\Phi) \equiv \lim_{t\to\infty} \mathbb{E} \begin{pmatrix} S_{t-1}^{-1} u_{t-1} \left[T(c_{t-1})(1, u_{t-1}) - c_{t-1} u_{t-1} \right]' \\ \frac{t}{t+1} (u_t u_t - S_{t-1}) \end{pmatrix}$$

with $\Phi = (c_t, S_t)'$ which governs the dynamics of the recursive algorithm, at an equilibrium point (i.e. at a fixed point of (6.6)), as $t \to \infty$. Taking expectations and limits gives

$$\begin{aligned} \frac{dc}{d\tau} =& S^{-1} \frac{\sigma_u^2}{1-\rho^2} (T_2(\bar{c}) - \bar{c}) \\ \frac{dS}{d\tau} =& \frac{\sigma_u^2}{1-\rho^2} - S, \end{aligned}$$

where T_2 denotes the second column of the T-map (6.3). Since S converges globally to $\sigma_u/(1-\rho^2)$ the asymptotic dynamics for the vector c are determined by the smaller differential equation

$$h(c) = \frac{dc}{d\tau} = \begin{pmatrix} -\kappa(\beta\bar{c}_\pi\rho - 1)/(\alpha + \kappa^2) - \bar{c}_x \\ \alpha(\beta\bar{c}_\pi\rho - 1)/(\alpha + \kappa^2) - \bar{c}_\pi \end{pmatrix}.$$

Note that stability of the misspecified model is governed by the E-stability principle in this case. The linearised ODE for the parameter c takes the already familiar form

$$\frac{d}{dc} h(\bar{c}) = \begin{pmatrix} -1 & -\beta\kappa(\kappa^2 + \alpha)^{-1}\rho \\ 0 & \beta\alpha(\kappa^2 + \alpha)^{-1}\rho - 1 \end{pmatrix} = DT_c - \mathrm{I},$$

with associated eigenvalues -1 and $\beta\alpha\rho/(\kappa^2 + \alpha) - 1$. Since both of these eigenvalues are strictly negative, the RPE (6.5) is stable under adaptive learning.

The Value of Transparency

The preceding analysis illustrated that both the REE as well as the RPE are expectationally stable under the expectations-based interest rate rule. While the variance of these equilibria is identical, their means differ. It turns out that the equilibria, indicated by the superscript, satisfy the simple relation

$$\begin{pmatrix} x_t^{REE} \\ \pi_t^{REE} \end{pmatrix} = \begin{pmatrix} x_t^{RPE} \\ \pi_t^{RPE} \end{pmatrix} + \begin{pmatrix} 0 & -\beta\kappa(\kappa^2+\alpha)^{-1} \\ 0 & \beta\alpha(\kappa^2+\alpha)^{-1} \end{pmatrix} \begin{pmatrix} \bar{a}_x \\ \bar{a}_\pi \end{pmatrix}, \tag{6.7}$$

where the REE vector $\bar{a}$ is given in (3.16). Since $\bar{a}_\pi > 0$, relation (6.7) implies that $x_t^{REE} < x_t^{RPE}$ while $\pi_t^{REE} > \pi_t^{RPE}$. However, the relative welfare properties of these equilibria is not directly apparent from this relation. In order to assess the magnitude of period loss, further restrictions in the analysis are needed.

The response to cost push shocks is equivalent in these equilibria since $\hat{c} = \bar{c}$. For analytical convenience then, it is assumed that $u_t = 0$ for all t. Consider first the case that the central bank's target values for inflation and the output gap are given by $\pi = 0$, $x > 0$. Relation (6.7) implies in this case

$$\begin{aligned} x_t^{REE} &= \frac{\alpha(1-\beta)}{\alpha(1-\beta)+\kappa^2}x < \frac{\alpha}{\alpha+\kappa^2}x = x_t^{RPE} < x, \\ \pi_t^{REE} &= \frac{\alpha\kappa}{\alpha(1-\beta)+\kappa^2}x > \frac{\alpha\kappa}{\alpha+\kappa^2}x = \pi_t^{RPE} > 0. \end{aligned}$$

This relation shows that inflation is persistently above its target value while the output gap can not be increased efficiently. A result which is often referred to as the "inflationary bias problem" (see e.g. Clarida et al., 1999). From the above ordering it is immediate that the REE yields unambiguously lower welfare than the RPE. In particular, by drawing on the definition of the loss function (3.1), one obtains $L_t^{REE} > L_t^{RPE}$ for any value of α. Thus, the central bank could benefit from being intransparent about its non-zero target value x.

However, non-zero inflation targets have been adopted by most central banks either explicitly or implicitly, as Berardi and Duffy (2007) note.[4] Given $\pi > 0$ and $x = 0$,

[4] The Federal Reserve was a prominent example of an implicit inflation targeting bank until Fed's Chairman Ben Bernanke made the 2% inflation target explicit on January 25th, 2012.

relation (6.7) implies

$$0 < x_t^{REE} = \frac{\kappa(1-\beta)}{\alpha(1-\beta)+\kappa^2}\pi < \frac{\kappa}{\alpha+\kappa^2}\pi = x_t^{RPE},$$
$$\pi > \pi_t^{REE} = \frac{\kappa^2}{\alpha(1-\beta)+\kappa^2}\pi > \frac{\kappa^2}{\alpha+\kappa^2}\pi = \pi_t^{RPE}$$

and it follows from the definition of the loss function that $L_t^{REE} < L_t^{RPE}$ for all values of α. Thus, central banks, pursuing a policy which stabilises output around its natural level and inflation around $\pi > 0$, profit from being transparent about their policy objectives.

Berardi and Duffy (2007) use different calibrations in simulating the model numerically for the case when u_t is not constrained to being zero. Their main finding is that the REE yields unambiguously higher welfare given that the inflation target is sufficiently large and that the output target is sufficiently small. Specifically, if

$$\pi > \pi_t^{REE} > \pi_t^{RPE},$$
$$y_t^{RPE} > y_t^{REE} > y,$$

for all t, then the loss for the central bank under RE is always smaller, implying a positive value of transparency.

If the central bank is acting under commitment, an announcement of the policy targets is not sufficient for the attainment of RE. Instead, agents also have to be informed about the relevance of lagged variables. Clearly, there exists an abundance of underparameterised equilibria in this environment. This raises the question which equilibrium, or more specifically, which underparameterised PLM should be thought of as the most plausible. The next section will introduce the concept of a Misspecification Equilibrium as to deal with this ambiguity.

However, for optimal monetary policy under commitment, Berardi and Duffy (2007) show that, irrespective of the calibration and of the particular misspecification, the REE always yields higher welfare. This result is intuitive since the gains from commitment are grounded in an efficient management of expectations. If the private sector is boundedly rational and running the risk of adopting a misspecified forecasting rule, the central bank can enhance the management of expectations by being transparent.

6.2 Extending Restricted Perceptions: Misspecifiation Equilibria

The idea of modelling agents' expectations on grounds of underparameterised forecasting models is appealing due to its apparent proximity to reality. From a theoretical perspective, however, the concept is highly unattractive. The power of the REH can be attributed to the universality of its applicability. Irrespective of the particular model, there exists exactly one way of forming rational expectation (provided the model is determinate). Restricted perceptions, on the other hand, are by definition non-unique and hence arbitrary. Instead of having neglected the constant term in the MSV solution, agents could as well have been assumed to neglect the dependence on u_t in the example above.

Branch and Evans (2006, 2007) resolve this weakness of the equilibrium concept by endogenising the underparameterisation. They consider the set of all underparameterised forecasting rules, for some model, and argue that an equilibrium concept most consistent with Muth's hypothesis will have agents only choose the best performing statistical models from this set. The idea of a Misspecification Equilibrium (ME) shall be presented in the afore discussed environment.

In the New Keynesian model under discretionary policy the set of all under parameterised forecasting rules is given by the elements

$$\begin{aligned} PLM_1 &: y_t = a, \\ PLM_2 &: y_t = cu_t, \end{aligned} \tag{6.8}$$

where $a = (a_x, a_\pi)'$ and c is defined equivalently. Restricted to using underparameterised forecasts, agents can either choose to base their expectations on the shock u_t, or they can choose to form expectations on basis of the sample mean. Branch and McGough (2009) derive a New Keynesian model under the assumption that agents' expectations are heterogenous. The reduced form equations of their model are equivalent to the reduced form equations of this thesis. Letting n denote the fraction of agents using PLM_1, aggregate expectations in their model obey $\mathrm{E}z = n\mathrm{E}^1 z + (1-n)\mathrm{E}^2 z$, for any state variable z.

Substituting in the diverse beliefs along with the expectations-based interest rate rule under discretion, the New Keynesian model can be written in the form

$$y_t = A + B(an + (1-n)c\rho u_t) + Dw_t, \tag{6.9}$$

where the matrices A, B, D can be found in (5.9). Although agents are assumed to base their forecasts on misspecified models, they are required to forecast in a statistically optimal manner. Thus, their models have to satisfy the orthogonality conditions

$$\begin{aligned}\mathbb{E}u_t(y_t - \hat{c}u_t) &= 0,\\ \mathbb{E}(y_t - \hat{a}) &= 0,\end{aligned}$$

implying that the coefficient vectors $\hat{a}$ and $\hat{c}$ are, in equilibrium, constrained to

$$\begin{aligned}\hat{a}_x(n) &= \frac{(\alpha x + \kappa\pi)(1-n\beta)}{\alpha(1-n\beta)+\kappa^2}, \quad \hat{c}_x(n) = -\frac{\kappa}{\alpha(1-\beta\rho(1-n))+\kappa^2}\\ \hat{a}_\pi(n) &= \frac{\kappa(\alpha x + \kappa\pi)}{\alpha(1-n\beta)+\kappa^2}, \quad \hat{c}_\pi(n) = \frac{\alpha}{\alpha(1-\beta\rho(1-n))+\kappa^2}.\end{aligned} \tag{6.10}$$

Note the proximity of these coefficients to the corresponding coefficients in a REE. It holds that $\hat{c}(0) = \bar{c}$ and $\hat{a}(1) = \bar{a}$, where the bar denotes REE values. Given a proportion n of agents using forecasting model PLM_1, it follows from combining (6.10) and (6.9) that the RPE is uniquely determined by

$$y_t(n) = \hat{a}(n) + \hat{c}(n)u_t. \tag{6.11}$$

The RPE analysed in the preceding section is a particular case of this set of RPEs, corresponding to the case $n = 0$. This equilibrium was shown to be expectationally stable. If, on the other hand, all agents are basing their prediction on PLM_1, i.e. $n = 1$, the unconditional mean of the RPE would coincide with the unconditional mean of the corresponding REE. By retracing the steps outlined in the previous section, the linearised differential equation, governing the asymptotic dynamics of the parameter vector $a(n)$, for the case $n = 1$, can be seen to obey

$$\frac{d}{da}h(\bar{a}) = \begin{pmatrix} -1 & -\beta\kappa/(\alpha+\kappa^2) \\ 0 & \alpha\beta/(\alpha+\kappa^2)-1 \end{pmatrix}.$$

Since both eigenvalues of this matrix are negative, it follows that the RPE (6.11) is stable under adaptive learning for $n = 1$. Moreover, it immediately follows from (6.10) that the unconditional standard deviation of $y(1)$ is smaller than the corresponding standard deviation of the REE. Combining this insight with the fact that the unconditional mean of these equilibria coincide, it follows that $y(1)$ is associated with an unambiguously smaller loss for each period.

However, instead of analysing various proposals for some RPE, the idea of a ME is to determine the proportion n endogenously. This is achieved by assigning a fitness measure to each available predictor. Following Branch and Evans (2011) it is assumed that agents seek to minimise their forecast mean square error. Thus, forecasting model PLM_j, $j = 1, 2$, is ranked according to

$$\mathrm{EU^j} = -\mathbb{E}(y_{t+1} - \mathrm{E}^j y_{t+1})'(y_{t+1} - \mathrm{E}^j y_{t+1}), \tag{6.12}$$

where E^j denotes forecasts based on model j. In the framework of the New Keynesian model this metric seems appropriate since one period ahead forecasts are needed in the structural equations. Furthermore, if agents were conditioning their forecasts on "full" information, the hereby implied rational expectations were assigned the maximal fitness under this metric. Branch and Evans (2007) therefore argue that an ordering implied by the mean square error preserves the structure of rational expectations when agents are restricted to a limited information set.

The endogenous value n is assumed to depend on the relative forecast performance. Following Brock and Hommes (1997) the map from predictor fitness to predictor choice is a multinomial logit (MNL) map, taking the form

$$n = \frac{\exp(\lambda EU^1)}{\exp(\lambda EU^1) + \exp(\lambda EU^2)}.$$

The parameter λ, called the "intensity of choice", determines the sensitivity of agents' reaction to changes in the forecasting success. While Brock and Hommes (1997) concentrate on large but finite values of λ, Branch and Evans (2006) focus on the case when $\lambda \to \infty$. They argue that finite values of λ parameterise deviations from full utility maximisation whereas agents are only selecting the best-performing predictor as $\lambda \to \infty$. Furthermore, the latter specification is analytically convenient. Branch and Evans (2006) show that fixed points of the MNL map can be easily characterised for infinite λ. Letting $F : [0, 1] \to \mathbb{R}$ with $F(n) = \mathrm{EU}^1 - \mathrm{EU}^2$, the MNL can be rewritten as

$$n = \frac{1}{2}(\tanh(\lambda F(n)) + 1) \equiv T_\lambda(n).$$

On grounds of this map it is possible to give the following definition: A Misspecification Equilibrium (ME) is a distribution of agents n^* such that $T_\lambda(n^*) = n^*$. Note that $T_\lambda(n) : [0,1] \to [0,1]$ is a continuous and well-defined function provided that an RPE exists.

In a ME the parameters of the forecasting models, the distribution of agents using the diverse forecasting models as well as the stochastic process for the state variables are jointly determined. From the definition of T_λ it is evident that the number and the nature of ME depend on the properties of $F(n)$. Specifically, Branch and Evans (2011) provide the following proposition:

Proposition 1 *Let $N^*_\lambda = \{n^* | n^* = T_\lambda(n^*)\}$ denote the set of MEs. In the limit of large λ, N^* has one of the following properties:*

1. *If $F(0) < 0$ and $F(1) < 0$ then $n^* = 0 \in N^*$ (Condition PLM2)*
2. *If $F(0) > 0$ and $F(1) > 0$ then $n^* = 1 \in N^*$ (Condition PLM1)*
3. *If $F(0) < 0$ and $F(1) > 0$ then $\{0, \hat{n}, 1\} \subset N^*$, where $\hat{n} \in (0,1)$ (Condition ME)*
4. *If $F(0) > 0$ and $F(1) < 0$ then $n^* = \hat{n} \in N^*$, where $\hat{n} \in (0,1)$ (Condition H)*

In general, F is not necessarily monotonic for which reason there may exist more equilibria than those stated in the above proposition. However, when condition PLM2 applies, no agent has an incentive to base his forecast on another model when all agents are using PLM_2. Condition PLM1 is the analog in favour of PLM_1. In contrast to these conditions, there is potential for heterogenous use of forecasting models under condition ME and condition H. Condition ME characterises a sufficient condition for the existence of multiple equilibria. Both $n^* = 0$ as well as $n^* = 1$ are potential equilibria in this case. Furthermore, there also exists an interior equilibrium $\hat{n}$. However, Branch and Evans (2006) show that this equilibrium is not stable under adaptive learning when F is monotonic since then $T'(F(\hat{n}) > 1$ and any deviation of $\hat{n}$ results in better forecasting performance of either PLM_1 or PLM_2. When F is monotonic, condition H implies that $F(n)$ crosses zero from above. The ME with Intrinsic Heterogeneity is thus stable under adaptive learning in this case (Branch and Evans, 2006).

In the New Keynesian framework under the expectations-based interest rate rule, the function F is given by

$$F(n) = \langle \hat{a}(n), \hat{a}(n) \rangle - \langle \hat{c}(n), \hat{c}(n) \rangle \frac{\rho^2}{1-\rho^2} \sigma_u^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. From (6.10) it follows that

$$\begin{aligned} F(0) \gtrless 0 \text{ iff } (\alpha x + \kappa\pi)^2 \Omega_1 \gtrless \frac{\rho^2}{1-\rho^2}\sigma_u^2, \\ F(1) \gtrless 0 \text{ iff } (\alpha x + \kappa\pi)^2 \Omega_2 \gtrless \frac{\rho^2}{1-\rho^2}\sigma_u^2. \end{aligned} \tag{6.13}$$

where

$$\Omega_1 = \frac{(1+\kappa^2)[\alpha(1-\beta\rho)+\kappa^2]^2}{(\alpha+\kappa^2)^2(\alpha^2+\kappa^2)} \quad \text{and} \quad \Omega_2 = \frac{[(1-\beta)^2+\kappa^2](\alpha+\kappa^2)^2}{[\alpha(1-\beta)+\kappa^2]^2(\alpha^2+\kappa^2)}.$$

Each condition of proposition 1 can be satisfied for suitable choice of the structural parameters. In particular, as $\sigma_u^2 \to \infty$, forecasts based on model PLM_2 are more accurate, implying that $n^* = 0$. The same is true for the case when the central bank has target values $x = \pi = 0$. If, on the other hand, the stochastic process of the shock comprises little information, i.e. $\sigma_u^2 \to 0$ or $\rho \to 0$, forecasts based on the sample mean are more efficient, implying that $n^* = 1$.

To gain some intuition on how intrinsic heterogeneity may arise, consider the the case when the central bank is following a strict inflation targeting policy (i.e. $\alpha = 0$). In this case, it holds that $\Omega_1 > \Omega_2$, implying that we have condition PLM1, PLM2 or H, depending on the parameters of the model. Condition ME can not arise in this case. The mean square forecast error associated with PLM_2 can be seen to obey

$$-\text{EU}^2 = \frac{\pi[(1-n\beta)^2+\kappa^2]}{\kappa^2} + \sigma_u^2,$$

in this case. Thus, the accuracy of this model is decreasing in its relative usage. This follows from the observation that the unconditional mean of the output gap, which agents are neglecting under PLM_2, is increasing in $(1-n)$. Under PLM_1, on the other hand, expectations are constant and the MSE is given by

$$-\text{EU}^1 = \frac{1}{1-\rho^2}\sigma_u^2.$$

This expression is more rapidly increasing in the variance σ_u^2, compared to $-\text{EU}^2$ and therefore continuously performing worse as $\sigma_u^2 \to \infty$. However, note that the error associated with PLM_1 is independent of its relative usage since private sector expectations only have an effect on the response to shocks. The long-run mean of inflation and the output gap is not affected by expectations.

Suppose first that all agents are basing their forecast on PLM_1. Whether $n = 1$ depicts a misspecification equilibrium depends on whether the zero-mass agent has an incentive to deviate from the consensus forecast. Whenever

$$F(1) = \frac{\pi^2[(1-\beta)^2 + \kappa^2]}{\kappa^2} - \frac{\rho^2}{1-\rho^2}\sigma^2 < 0, \tag{6.14}$$

PLM_1 is suboptimal and adopting model PLM_2 will result in more accurate predictions. Thus, $n = 1$ is no equilibrium point and agents will start to make use of the alternative model. However, as mentioned before, the absolute performance of PLM_2 is decreasing in its relative usage. Since F is monotonic, there exists an $\hat{n} \in (0, 1)$ such that

$$F(\hat{n}) = \frac{\pi^2[(1-\hat{n}\beta)^2 + \kappa^2]}{\kappa^2} - \frac{\rho^2}{1-\rho^2}\sigma^2 = 0.$$

This illustrates that intrinsic heterogeneity may arise under optimal discretionary policy. Moreover, the ME associated with $\hat{n}$ is unique, provided that

$$\frac{\pi^2(1+\kappa^2)}{\kappa^2} > \frac{\rho^2}{1-\rho^2}\sigma_u^2 > \frac{\pi^2[(1-\beta)^2 + \kappa^2]}{\kappa^2}.$$

Note that these inequalities are satisfied for a typical calibration of the model, as proposed by Woodford (2003b): $\sigma = 1/0.157$, $\kappa = 0.024$, $\beta = 0.99$. The exogenous disturbance is, following Branch and Evans (2011), specified by $\rho = .5$, $\sigma_u^2 = .25$ and the inflation target is assumed to be $\pi = 0.02$. The equilibrium proportion, relying on forecasts based on PLM_1, is then given by $n^* \approx 0.66$.

6.3 Concluding Remarks

This thesis analysed adaptive learning from two perspectives: On the one hand, the continuous updating of expectations was presented as a plausible, procedurally rational way, to justify the attainment of RE. With this regard, E-stability was presented as a necessary condition for a REE to be considered plausible given that economic agents do not posses of unlimited information processing capacity and knowledge. On the other hand, the role of adaptive learning as a meaningful alternative to RE was investigated. Given that subjective forecasting models satisfy an orthogonality condition, Muth's assertions for a consistent theory of expectation formation are met while the critique stemming from the bounded rationality literature is simultaneously accounted for. In particular, the adaptive learning approach abstains from confounding an objective description of reality with subjective perceptions of it. This certainly depicts a desirable feature.

However, if adaptive learning is regarded as an alternative to RE, rather than a plausibility check, then no rationale prevents the assumption of misspecified perceptions. While Muth's assertions still continue to hold in this environment, the theoretical attractiveness of the concept is decreased immensely. After all, the REH initiated a revolution due to its universality of applicability. Adaptive learning, on the other hand, involves an arbitrary component if the assumption of bounded rationality is taken one step further. On grounds of this observation it might be tempting to choose the unambiguous approach and assume the optimum rather than hypothesising about the right magnitude of deviation from it.

The ME depicts an ambitious remedy to the arbitrariness inherent to the concept of restricted perceptions. By endogenising the particular form of underparameterisation, Muth's claim for a consistent theory, being based on the principle of optimality, is effectively accounted for. Yet, adaptive learning by means of restricted models, remains, by definition, a supplemental concept rather than an alternative to RE.

Despite adaptive learning's deficiency to impair the dominance of RE, the concept clearly entails valuable ideas for policy recommendation. Consider one of the main implications of the E-stability analysis, presented in this thesis: Optimal monetary policy should be conditioned on private sector expectations rather than fundamental measures. If the path of the economy is to be stabilised, possible deviations of the public's prediction from the "rational" prediction must be accounted for. Adaptive learning, therefore, not only provides an intuitive explanation for the importance of reacting to possibly non-rational beliefs, but moreover provides a theoretical justification for the widespread policy practise of inflation forecast targeting.

Appendix A

Stochastic Approximation

The thesis introduced the idea of modelling agents as econometricians. Each period as new data points enter the information set, parameter estimates of the PLM are updated by Least Squares estimation. Whether the resulting dynamics asymptotically converge to some equilibrium point can be studied by results of the stochastic approximation literature. This chapter will concentrate on stability results for Least-Squares estimation. However, a general formulation of recursive algorithms as well as conditions for convergence shall also be presented.

A.1 The Robbins-Monro Algorithm and Stochastic Gradient Learning

To introduce ideas on this issue, Robbins and Monro (1951) are followed in this section, who were among the first to study stochastic approximation methods. They considered the problem of finding a value for a vector x that solves the equation

$$M(x) = \alpha, \tag{A.1}$$

where $M(x)$ denotes the expected value at level x of the response to a certain experiment and α is some constant. A stochastic generalisation of this problem can be attained by assuming that for each value x there is a random variable $Y = Y(x)$ with cumulative distribution function $P(Y(x) \leq y) = H(y|x)$, so that

$$M(x) = \int_{-\infty}^{\infty} y \mathrm{d}H(y|x).$$

While neither the exact nature of $M(x)$ nor that of $H(y|x)$ is known, it is assumed that successive observations on Y at levels $x_1, x_2 \ldots$, denoted by y_t with $P(y_t \leq y|x_t) = H(y|x_t)$, can be made. Note that this problem is closely related to the regression problem where under the assumption that $M(x)$ is of known form, the parameters $\theta = (\beta_1, \beta_2, \ldots \beta_n)$ in

$$\mathbb{E}(Y|x) = \theta x, \tag{A.2}$$

are estimated by Least-Squares on basis of observations $\{y_t, x_t\}_{t=1}^n$. However, instead of estimating the parameter vector θ of $M(x)$, under the assumption that the conditional expectation is a linear function of x, the problem in this setup is to estimate a vector x such that (A.1) holds.[1]

Robbins and Monro (1951) assume therefore that equation (A.1) has a unique solution, that $M(x)$ is non-decreasing in x and that there exist finite constants α and φ such that

$$M(x) \leq \alpha \text{ for } x < \varphi, \qquad M(x) \geq \alpha \text{ for } x > \varphi. \tag{A.3}$$

Then, given a sequence $\{\gamma_t\}$ of non-increasing positive scalars, satisfying

$$\sum_{t=1}^{\infty} \gamma_t = \infty \qquad \text{and} \qquad \sum_{t=1}^{\infty} \gamma_t^2 < \infty, \tag{A.4}$$

Robbins and Monro prove that irrespective of the initial condition x_1, the scheme

$$x_{t+1} = x_t + \gamma_t(\alpha - y_t), \tag{A.5}$$

determines a non-stationary Markov chain which converges to the solution of (A.1) in the mean square sense, i.e. $\mathbb{E}(\varphi - x_t)^2 \to 0$ as $t \to \infty$.

The functioning of this algorithm is straight-forward to analyse: Any deviation of the outcome from the target value results in an adjustment of the input. Assumption (A.3),

[1] Kiefer and Wolfowitz (1952) took up on this point and further developed the Robbins-Monroe algorithm (A.5), as to find the maximum of a regression function $M(x)$, whose form is unknown.

directs each successive input towards the right direction. The speed of adaption is governed by the "gain" sequence $\{\gamma_t\}$. Convergence to the solution φ can only take place if the noise produced by the random variable $(\alpha - y_t)$, is paid less and less attention, i.e. $\gamma_t \to 0$ as $t \to \infty$. Concurrently, assumption (A.4) guarantees that the gain sequences does not decline too rapidly as to prevent convergence to a non-equilibrium point.

Robbins and Monro assumed that the function $M(x)$ can be evaluated for any admissible candidate as to find a vector x which satisfies (A.1) and proved that (A.5) succeeds in doing so. Conversly, it is possible to use algorithm (A.5) in order to find a parameter vector θ, under the assumption that the sample $\{x_t\}$ is generated by nature, as Sargent (1993) notes. To see this, let y_t follow the stochastic process

$$y_t = \bar{\theta} x_{t-1} + \epsilon_t, \tag{A.6}$$

where $\bar{\theta} = (\beta_1, \beta_2)$, $x'_{t-1} = (1, w_{t-1})$ and ϵ_t is an iid error term with zero mean. Assume that agents correctly perceive that y_t depends on x_{t-1} while they are not aware of the specific form. Furthermore, suppose that agents are only concerned about minimising their next period's forecast error. Then, the vector $\theta = (\hat{\beta}_1, \hat{\beta}_2)$ will be determined such that the first order condition for a minimum of $\mathbb{E}[(y_{t+1} - \theta x_t)^2]$, given by

$$\mathbb{E}\left[-x_t(y_{t+1} - \theta x_t)'\right] = 0,$$

is satisfied each period t. Clearly, this condition amounts to finding the roots of $M(\theta) = \alpha$, for $\alpha = 0$. Note that $\mathrm{d}/\mathrm{d}\theta M(x) = xx'$ is positive semidefinite and thus non-decreasing in θ. Given that the variance of Y is finite, algorithm (A.5) can be applied to find the unique root $\bar{\theta}$.

In period t, the latest observed forecast error, denoted by e_{t-1}, is given by

$$e_{t-1} = -x_{t-1}(y_t - \theta'_{t-1} x_{t-1}).$$

Inserting this expression into the Robbins-Monro algorithm (A.5) yields the following learning rule

$$\theta_t = \theta_{t-1} + \gamma_t \left[x_{t-1}(y_t - \theta'_{t-1} x_{t-1})\right]. \tag{A.7}$$

Evans and Honkapohja (1998b) refer to the learning rule (A.7) as Stochastic Gradient Learning. Note that the algorithm adjusts θ each period in the direction estimated to decrease the squared forecast error most rapidly. In contrast to Least Squares, however, higher order moments of the distribution are neglected for which reason (A.7) is a gradient algorithm rather than a Newton-type algorithm as noted by Evans and Honkapohja (2001).

As a particularly simple example, consider the case $\bar{\theta} = (1, 0)$. If agents correctly adapt their PLM and choose an initial estimate $x_0 = 0$ and a gain sequence $\gamma_t = t^{-1}$, then (A.7) reduces to the recursive formulation of the sample mean.

It is worth emphasising that agents forecast did not have an effect on the distribution of the variable agents were trying to predict in this example. This stands in stark contrast to self-referential processes, like the New Keynesian model, where expectations feed back into the system and thereby affect the distribution of observations. Consequently, the results of Robbins and Monro (1951), despite being useful for illustration, are not general enough as to analyse learning dynamics in economic models and in particular for analysing Least Squares convergence.

A.2 Recursive Least Squares

The idea of econometric learning has been introduced in chapter 4 without explicitly formulating the algorithm, agents are using as to update their parameter estimates. This section will introduce Recursive Least Squares and therewith specify the dynamics of the model. Defining the dynamics can be regarded as a preparatory step to analysing convergence under Least Squares learning which will be discussed thereafter.

Under adaptive learning, the PLM is time dependent and takes the form

$$y_t = a_t + b_t y_{t-1} + c_t w_t$$

Agents update their 2×5 coefficients matrix $\theta_T = (a_T, b_T, c_T)$ in period T, by running Least Squares on $\{y_t, 1, y_{t-1}, w_t\}_{t=0}^{T-1}$. This formulation is following the convention in the learning literature to update coefficient estimates in period t by using information available as of time $t-1$. Including current data would create a simultaneity complication since the system would determine time t variables at the same time that agents are using time t variables to form expectations.

Let Y denote the $T \times 2$ matrix with i'th row y_i' and X denote the $T \times 5$ matrix given by $X = (x_0, x_1 \dots x_{T-1})'$, where $x_t = (1, y'_{t-1}, w'_t)$.[2] The coefficient matrix θ minimising the sum of squared residuals

$$(Y - X\theta')'(Y - X\theta'),$$

is given by the Least Squares formula

$$\theta' = \left(X'X\right)^{-1} X'Y.$$

Equivalently, this solution can be expressed in terms of the vectors x_t and y_t:

$$\theta'_T = \left(\sum_{t=0}^{T-1} x_t x'_t\right)^{-1} \left(\sum_{t=0}^{T-1} x_t y'_t\right).$$

To establish a recursive formulation of the estimate in period t, let the matrix of second moments in period $t-1$ be denoted by R_{t-1} and given by

$$R_{t-1} = \frac{1}{t-1} \sum_{i=0}^{t-2} x_i x'_i,$$

so that R_t can be written recursively as

$$R_t = R_{t-1} + t^{-1}(x_{t-1}x'_{t-1} - R_{t-1}). \tag{A.8a}$$

Substitution of this expression into the Least Squares solution shows that the coefficient matrix θ_t can be expressed as

$$\theta'_t = \theta'_{t-1} + t^{-1} R_t^{-1} x_{t-1} \left(y_{t-1} - \theta_{t-1} x_{t-1}\right)'. \tag{A.8b}$$

Given some sequence of the state variables x_t, the dynamics of the parameter estimates θ are fully described by the recursive stochastic system (A.8). By incorporating information on the most recent forecast error $(y_{t-1} - \theta_{t-1} x_{t-1})$, the parameter matrix is updated each period into the direction which minimises the sum of squared errors. The speed of these incremental steps is governed by the matrix of second moments as well as by time.

[2] The assumption that observations x_0 are available is made in order to express the recursive formulation conveniently.

Upon substitution of the T-map, the parameter estimate can be expressed as

$$\theta'_t = \theta'_{t-1} + t^{-1} R_t^{-1} x_{t-1} x'_{t-1} \left(T(\theta_{t-1}) - \theta_{t-1}\right)'.$$

implying that the algorithm has a fixed point at $T(\theta) = \theta$, i.e. the REE.

A.3 General Recursive Algorithms and the ODE

Recursive Least Squares can be represented by the following quite general formulation of a recursive algorithm, as presented by Ljung (1977) or Evans and Honkapohja (2001)

$$\theta_t = \theta_{t-1} + \gamma_t \mathcal{H}(\theta_{t-1}, X_t), \tag{A.9a}$$

$$X_t = A(\theta_{t-1}) X_{t-1} + B(\theta_{t-1}) W_t. \tag{A.9b}$$

Here, $\theta_t \in \mathbb{R}^n$ is a vector of parameters which are the subject of interest and $X_t \in \mathbb{R}^m$ is a vector of observations which causes θ_t to be updated to take new information into account. In economic models (A.9a) corresponds to the learning dynamics while (A.9b) corresponds to the (conditionally linear) state dynamics. $A(\centerdot)$ and $B(\centerdot)$ are matrix-valued functions and W_t is an error vector.[3][4]

Basic Results

To begin with, the basic results from the work of Evans and Honkapohja (2001) are summarised here. Evans and Honkapohja consider an open set $D \subset \mathbb{R}^n$ around some equilibrium point of interest and assume the following

A.1 The sequence $\{\gamma_t\}$ is positive, nonstochastic, nonincreasing and satisfies

$$\sum_{t=1}^{\infty} \gamma_t = \infty \qquad \text{and} \qquad \sum_{t=1}^{\infty} \gamma_t^2 < \infty$$

[3] In fact, both Ljung (1977) and Evans and Honkapohja (2001) analysed more general formulations of the algorithm. Ljung considered the case where the function Q is possibly time dependent and Evans and Honkapohja considered the case when there is a second-order term present in (A.9a). For motivating the associated differential equation, these assumptions are not necessary and therefore omitted.

[4] The Robbins-Monro algorithm can be seen to be a particular case when $A(\centerdot) = 0$.

A.2 For any compact $Q \subset D$, there exist C and q such that for all $\theta \in Q$

$$|\mathcal{H}(\theta, x)| \leq C(1 + |x|^q)$$

A.3 For any compact $Q \subset D$, the function $\mathcal{H}(\theta, x)$ satisfies for all $\theta, \theta' \in Q$, $x_1, x_2 \in \mathbb{R}^m$ and some constants L_1, L_2

$$\begin{aligned} (i) \qquad & |\partial\mathcal{H}(\theta, x_1)/\partial x - \partial\mathcal{H}(\theta, x_2)/\partial x| \leq L_1|x_1 - x_2|, \\ (ii) \qquad & |\mathcal{H}(\theta, 0) - \mathcal{H}(\theta', 0)| \leq L_2|\theta - \theta'|, \\ (iii) \qquad & |\partial\mathcal{H}(\theta, x)/\partial x - \partial\mathcal{H}(\theta', x)/\partial x| \leq L_2|\theta - \theta'|, \end{aligned}$$

As discussed in course of the Robbins-Monro algorithm, assumption A.1 is required as to ensure convergence by neglecting noise asymptotically as well as to prevent convergence to non-equilibrium points. Assumption A.2 imposes polynomial bounds on $\mathcal{H}(\theta, x)$ and assumption A.3 assures that $\mathcal{H}(\theta, x)$ is twice continuously differentiable with bounded second derivatives on every Q.

For the state dynamics Evans and Honkapohja (2001) impose the following assumptions:

B.1 W_t is iid with finite absolute moments.

B.2 For any compact subset $Q \subset D$ it holds that

$$\sup_{\theta \in Q} |B(\theta)| \leq M \qquad \text{and} \qquad \sup_{\theta \in Q} |A(\theta)| < 1,$$

for some matrix norm $|\centerdot|$ and a constant M. Furthermore $A(\theta)$ and $B(\theta)$ satisfy Lipschitz conditions on Q.

For assumption B.2, Evans and Honkapohja remark that the condition on $A(\theta)$ is a little stronger than asymptotic stationarity. However, if at some point θ^* the spectral radius satisfies $r(A(\theta < 1))$, then the condition on $A(\theta)$ holds in a neighbourhood of θ^*.

Algorithm (A.9), being a time-variant, stochastic, nonlinear difference equation, appears fairly complex to analyse. However, it turns out that under assumptions A and B, the limiting behaviour of θ_t can be described by an associated differential equation which is derived as follows.

Some equilibrium point of interest θ is chosen and the corresponding state dynamics are defined by

$$\bar{X}_t(\theta) = A(\theta)\bar{X}_{t-1}(\theta) + B(\theta)W_t. \tag{A.10}$$

Given these state dynamics, the asymptotic influence on the parameter vector θ is analysed by considering

$$h(\theta) \equiv \lim_{t\to\infty} \mathbb{E}[\mathcal{H}(\theta, \bar{X}_t(\theta)], \tag{A.11}$$

i.e. the asymptotic mean of $\mathcal{H}(\theta, \bar{X}_t(\theta))$. The associated ordinary differential equation (ODE), determining the limiting behaviour of the parameter estimate θ_t around θ, is then given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = h(\theta). \tag{A.12}$$

The essence of the work on stochastic approximation is that only locally stable equilibrium points of the associated differential equation (A.12) are possible convergence points of the recursive algorithm. Conversely, the recursive algorithm (A.9) can not converge to fixed points of (A.12) which are not locally stable. These conclusions are summarised in Theorems 6.4 and 6.5 in Evans and Honkapohja (2001).

Consider an equilibrium point θ^* with $h(\theta^*) = 0$. Then, θ^* is a possible convergence point for the recursive algorithm (A.9) if and only if the linear differential equation,

$$H(\theta^*) = \frac{\mathrm{d}}{\mathrm{d}\theta} h(\theta^*)$$

obtained from linearisation of (A.12) around θ^*, has all eigenvalues within the unit-circle (see Ljung, 1977). The formal proof of these results is omitted here and the interested reader shall be referred to Ljung (1977), Evans and Honkapohja (1998a) or Evans and Honkapohja (2001). However, a heuristic justification, following Ljung (1977) and Evans and Honkapohja (2001) shall be presented.

Heuristic Justification

Solve the linear difference equation (A.9b) to see that observation X_t depends on all previous estimates

$$X_t = \sum_{j=1}^{t} \left(\prod_{i=j+1}^{t} A(\theta_{i-1}) \right) B(\theta_{j-1}) W_j. \tag{A.13}$$

Clearly, the estimates can only converge to some equilibrium point if the process X is exponentially stable. Given assumption B.2, the first terms in (A.13) will be very small and for some M

$$X_t \approx \sum_{j=t-M}^{t} \left(\prod_{i=j+1}^{t} A(\theta_{i-1}) \right) B(\theta_{j-1}) W_j.$$

However, as t increases, assumption A.1 and (A.9a) imply that the difference $\theta_t - \theta_{t-1}$ becomes smaller, such that for some t sufficiently large and $t \geq k \geq t - 2M$, it holds that $\theta_t \approx \theta_k$. Thus,

$$X_k \approx \sum_{j=1}^{k} A(\theta_t)^{k-j} B(\theta_t) W_j \equiv \bar{X}(k; \theta_t),$$

for $t \geq k \geq t - M$. This implies that $\mathcal{H}(\theta_{k-1}, X_k) \approx \mathcal{H}(\theta_t, \bar{X})(k, \theta_t)$. Consequently, the learning dynamics (A.9a) approximately follow

$$\theta_{t+s} \approx \theta_t + \sum_{k=t+1}^{t+s} \gamma_k \mathcal{H}(\theta_t, \bar{X}(k; \theta_t)).$$

Now define $h(\theta_t) = \mathbb{E}\mathcal{H}\left(\theta_t; \bar{X}(k; \theta_t)\right)$ so that $\mathcal{H}(\theta_t, \bar{X}(k; \theta_t)) = h(\theta_t) + w_k$ where w_k is a random variable with zero mean. With this definition, the above expression can be simplified to

$$\theta_{t+s} \approx \theta_t + h(\theta_t) \sum_{k=t+1}^{t+s} \gamma_k + \sum_{k=t+1}^{t+s} \gamma_k w_k,$$

$$\approx \theta_t + h(\theta_t) \sum_{k=t+1}^{t+s} \gamma_k \tag{A.14}$$

where the second line follows since the last term is a zero mean random variable which is dominated by the second term. This expression suggests that the limiting behaviour of the sequence $\{\theta_t\}$ can be described by the difference equation

$$\theta(\tau + \Delta\tau) = \theta(\tau) + \Delta\tau h(\theta(\tau)),$$

with $\Delta\tau$ corresponding to $\sum_{t+1}^{t+s} \gamma_k$. Ljung (1977) then suggests to interpret this expression as a way of solving the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\theta(\tau) = h(\theta(\tau)) \tag{A.15}$$

where the fictitious time τ relates to the original time t by $\tau_t = \sum_{k=1}^{t} \gamma_k$.[5] As Evans and Honkapohja (2001) note, the rigorous proof consists of finding and verifying the conditions of validity for the approximately equal signs in this heuristic justification.

A.4 Application to Learning Rules

To illustrate the implications of this result, consider first the Stochastic Gradient learning rule (A.7), of the preceding section which is restated here for convenience

$$\theta_t = \theta_{t-1} + \gamma_t(y_t - \theta_{t-1}).$$

The estimate θ is single valued and the process y_t is assumed to follow $y_t = a + \epsilon_t$ where ϵ is iid with zero mean. Note first that $\mathcal{H}(\theta_{t-1}, X_t)$ is given by $(x_t - \theta_{t-1})$, in this example and the state variable X_t is independent of θ. Hence, the results by Robbins and Monro (1951) are sufficient as to guarantee convergence to the "root" a. However, consider the expected value of $\mathcal{H}$, evaluated at the asymptotic stationary distribution of θ. This

[5]Note that (A.14) could also have been related to

$$\theta_{t+1} - \theta_t = \gamma_{t+1} h(\theta_{t+1}).$$

However, since the differential equation is time-invariant, it is easier to handle.

yields the associated differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\theta = a - \theta,$$

whose solution is

$$\theta(t) = a + e^{-t}(\theta(0) - a),$$

which converges to a for every initial condition $\theta(0)$. Since the objective is quadratic and the distribution of y_t was assumed to be invariant with respect to θ, there is global convergence to the mean a.

Now consider Recursive Least Squares in the New Keynesian model. As to apply the results of the stochastic approximation literature, the learning dynamics (A.8) have to be converted into standard form. Evans and Honkapohja (2001) stress that it is therefore necessary to make a timing change in the equation governing R_t. Thus, set $S_{t-1} = R_t$, so that the dynamics of the parameter estimates are described by the system

$$\begin{aligned} \theta_t' =& \theta_{t-1}' + t^{-1} S_{t-1}^{-1} x_{t-1} x_{t-1}' \left[T(\theta_{t-1}) - \theta_{t-1} \right]' \\ S_t =& S_{t-1} + t^{-1} \frac{t}{t+1} (x_t x_t' - S_{t-1}). \end{aligned} \tag{A.16}$$

Evans and Honkapohja (2001) show that this system can be put into standard form (A.9).

To derive the associated differential equation follow the steps which were outlined previously. Consider a REE corresponding to a fixed point $\bar{\theta} = (\bar{a}, \bar{b}, \bar{c})$ of $T(\theta)$ and assume that the REE of interest is asymptotically stationary, i.e. the eigenvalues of $\bar{b}$ are strictly inside the unit circle.

Define the state dynamics for a particular point θ as $x_t(\theta)' = (1, y_{t-1}(\theta)', w_t')$ with $y_{t-1}(\theta) = T(\theta)x_{t-1}$. Then $x_t(\theta)$ is a stationary process for all θ sufficiently close to $\bar{\theta}$.

Next, define $M_x(\theta) = \mathbb{E}[x_t(\theta)x_t(\theta)']$ and assume that $\bar{S} = \mathbb{E}[x_t(\bar{\theta})x_t(\bar{\theta})']$ is positive definite. Taking expectations and limits based on equations (A.16) yields the associated differential equations

$$\frac{\mathrm{d}\theta'}{\mathrm{d}\tau} = S^{-1} M_x(\theta) \left[T(\theta) - \theta \right],$$
$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = M_x(\theta) - S.$$

From linearising this system at $(\bar{\theta}', \bar{S})$, Evans and Honkapohja (2001) show that the system is locally stable, provided the eigenvalues of $DT(\bar{\theta})$ have real parts less than 1. They remark that heuristically, this is evident since stability is governed by the E-stability principle

$$\frac{\mathrm{d}\theta'}{\mathrm{d}\tau} = T(\theta) - \theta.$$

when $S \rightarrow M_x(\theta)$.

Appendix B

Derivations

B.1 The Consumption Rule

Forward iteration of the budget constraint (2.3) gives

$$W_t \geq \sum_{j=1}^{\infty} R_{t,t+j} P_{t+j} \left[C_{t+j^i} - Y^i_{t+j} \right],$$

where $R_{t,t+j}$ is given by

$$R_{t,t+j} = \prod_{s=1}^{j} \left(\frac{1}{1 + i_{t+s-1}} \right).$$

and the No-Ponzi constraint $\lim_{j\to\infty} R_{t,t+j} W_{t+j+1} = 0$, was used. Letting $\bar{\omega} = W^i_t/(P_t\bar{Y})$ denote the share of a household's real wealth as a fraction of steady state income $\bar{Y}$ the life-time budget constraint can be linearised as

$$\mathrm{E}^i_t \sum_{T=t}^{\infty} \beta^{T-t} c^i_T = \bar{\omega}^i_t + \mathrm{E}^i_t \sum_{T=t}^{\infty} \beta^{T-t} y^i_T,$$

Backwards iteration of the linearised Euler equation from date T to t and taking expectations gives

$$\mathrm{E}^i_t c^i_T = c^i_t - g_t + \mathrm{E}^i_t \left[g_T + \sigma \sum_{T=t}^{T-1} (i_t - \pi_{t+1}) \right],$$

which can be substituted into the linearised budget constraint to yield the optimal consumption rule

$$c_t^i = (1-\beta)\bar{\omega}_t^i + \mathrm{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) y_T^i - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1}) \right]. \tag{B.1}$$

B.2 Price Dispersion

This section follows Woodford (2003b) in deriving an approximation to the variability of prices. Calvo pricing implies that the distribution of prices $\{P_t(j)\}$ in period t consists of α times the distribution of prices last period, plus an atom of size $(1-\alpha)$ at the price P_t^*. Letting $\hat{P}_t$ denote the expected value of $\ln P_t(j)$, price variance may be written as

$$\begin{aligned}
\operatorname{var} \ln P_t(j) &= \operatorname{var}[\ln P_t(j) - \hat{P}_{t-1}] \\
&= \mathbb{E}[(\ln P_t(j) - \hat{P}_{t-1})^2] - \mathbb{E}(\ln P_t(j) - \hat{P}_{t-1})^2 \\
&= \alpha \mathbb{E}[(\ln P_{t-1}(j) - \hat{P}_{t-1})^2] + (1-\alpha)(\ln P_t^* - \hat{P}_{t-1})^2 - (\hat{P}_t - \hat{P}_{t-1})^2 \\
&= \alpha \operatorname{var} \ln P_{t-1}(j) + \frac{\alpha}{1-\alpha}(\hat{P}_t - \hat{P}_{t-1})^2,
\end{aligned}$$

where the last line uses

$$\begin{aligned}
\hat{P}_t - \hat{P}_{t-1} &= \mathbb{E}[\ln P_t(j) - \hat{P}_{t-1}] \\
&= \alpha \mathbb{E}[\ln P_{t-1}(j) - \hat{P}_{t-1}] + (1-\alpha)(\ln P_t^* - \hat{P}_{t-1}) \\
&= (1-\alpha)(\ln P_t^* - \hat{P}_{t-1}),
\end{aligned}$$

By substitution of the log linear approximation $\hat{P}_t = \ln P_t$, price dispersion may further be simplified to

$$\operatorname{var} \ln P_t(j) = \alpha \operatorname{var} \ln P_{t-1}(j) + \frac{\alpha}{1-\alpha}\pi_t^2. \tag{B.2}$$

Integrating forward yields

$$\operatorname{var} \ln P_t(j) = \alpha^{t+1} \operatorname{var} P_{-1}(j) + \sum_{s=0}^{t} \alpha^{t-s} \frac{\alpha}{1-\alpha}\pi_s^2,$$

where $\operatorname{var} \ln P_{-1}(j)$ denotes some initial price dispersion which is independent of policy. Thus, taking the discounted value of these terms over all periods $t > 0$, one obtains

$$\sum_{t=0}^{\infty} \beta^t \operatorname{var} \ln P_t(j) = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} \sum_{t=0}^{\infty} \beta^t \pi_t^2 + t.i.p. \tag{B.3}$$

By substituting this expression into the linearised utility function (2.28) one finally arrives at the normalised quadratic loss function

$$\sum_{t=0}^{\infty} \beta^t U_t = -\Omega \sum_{t=0}^{\infty} \beta^t [\pi_t^2 + \lambda(x_t - x^*)] \tag{B.4}$$

where the relative weight on the output gap stabilisation is given by $\lambda = \kappa/\theta$.

Bibliography

Arrow, K. J. (1986). Rationality of self and others in an economic system. *The Journal of Business 59*(4), S385–S399.

Barro, R. J. and D. Gordon (1983). A positive theory of monetary policy in a natural rate model. *Journal of Political Economy 91*, 589–610.

Baumol, W. (1952). The transactions demand for cash: An inventory theoretic approach. *Quarterly Journal of Economics 66*, 545–556.

Berardi, M. and J. Duffy (2007). The value of central banking when agents are learning. *European Journal of Political Economy 23*(1), 9–29.

Bernanke, B. and M. Woodford (1997). Inflation forecasts and monetary policy. *Journal of Money, Credit and Banking 29*(4), 653–684.

Blanchard, O. J. and C. M. Kahn (1980). The solution of linear difference models under rational expectations. *Econometrica 48*(5), 1305–1312.

Branch, W. and G. Evans (2011). Monetary policy and heterogeneous expectations. *Journal of Economic Theory 47*, 365–393.

Branch, W., G. Evans, and B. McGough (2010). Finite horizon learning. Working Paper 2010-15, University of Oregon Economics Department.

Branch, W. and G. W. Evans (2006). Intrinsic heterogeneity in expectation formation. *Journal of Economic Theory 127*(1), 264–295.

Branch, W. A. (2006). Restricted perceptions equilibria and learning in macroeconomics. In D. Colander (Ed.), *Post Walrasian Macroeconomics*. Middlebury College, Vermont.

Branch, W. A. and G. W. Evans (2007). Model uncertainty and endogenous volatility. *Review of Economic Dynamics 10*(2), 207–237.

Branch, W. A. and B. McGough (2009). A new keynesian model with heterogenous expectations. *Journal of Economic Dynamics and Control 33*(5), 1036–1051.

Bray, M. and N. Savin (1986). Rational expectations equilibria, learning, and model specification. *Econometrica 54*(5), 1129–1160.

Brock, W. A. and C. H. Hommes (1997). A rational route to randomness. *Econometrica 65*(5), 1059–1160.

Buiter, W. H. (1982). Predetermined and non-predetermined variables in rational expectations models. Technical Paper 21, National Bureau of Economic Research.

Bullard, J. (2006). The learnability criterion and monetary policy. *Federal Reserve Bank of St. Louis Review 88*(3), 203–217.

Bullard, J. and S. Eusepi (2009). When does determinacy imply expectational stability? Working Paper 2008-007B, Federal Reserve Bank of St. Louis.

Bullard, J. and K. Mitra (2002). Learning about monetary policy rules. *Journal of Monetary Economics 49*(6), 1105–1129.

Cagan, P. D. (1956). The monetary dynamics of inflation. In M. Friedman (Ed.), *Studies in the Quantity Theory of Money.* Chicago: University of Chicago Press.

Calvo, G. A. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics 12*(3), 993–998.

Chari, V. P., P. Kehoe, and E. M. Grattan (1996). Sticky price models of the business cycle: Can the contract multiplier solve the persistence problem? Staff report, Federal Reserve Minneapolis.

Chari, V. V., P. Kehoe, and E. Prescott (1989). Time consistency and policy. In *Modern Business cycle theory.* Cambridge, MA: Harvard Univ. Press.

Christiano, L., M. Eichenbaum, and S. Rebelo (2009). When is the government spending multiplier large? Working Paper 15394, National Bureau of Economic Research.

Clarida, R., J. Gali, and M. Gertler (1999). The science of monetary policy: A new keynesian perspective. Working Paper 7147, National Bureau of Economic Research.

Clarida, R., J. Gali, and M. Gertler (2002). A simple framework for international monetary policy analysis. *Journal of Monetary Economics 49*(5), 877–904.

Cogley, T. and J. Nason (1995). Output dynamics in a real business cycle model. *American Economic Review 85*(3), 492–511.

Coibion, O., Y. Gorodnichenko, and J. Wieland (2012). The optimal inflation rate in new keynesian models: Should central banks raise their in light of the zlb? *Review of Economic Studies*.

Dixit, A. K. and J. Stiglitz (1977). Monopolistic competition and optimum product diversity. *American Economic Review 67*(3), 297–308.

Erceg, C. J., D. W. Henderson, and A. T. Levin (2000). Optimal monetary policy with staggered wage and price contracts. *Journal of Monetary Economics 46*(2), 281–313.

Evans, G. W. (1983). The stability of rational expectations in macroeconomic models. In *Individual Forecasting and Aggregate Outcomes*. R. Frydman AND E. E. Phelps.

Evans, G. W. (1985). Expectational stability and the multiple equilibrium problem in linear rational expectations models. *Quarterly Journal of Economics 100*, 1217–1233.

Evans, G. W. (1986). Selection criteria for models with non-uniqueness. *Journal of Monetary Economics 18*, 147–157.

Evans, G. W. (1989). The fragility of sunspots and bubbles. *Journal of Monetary Economics 34*, 297–317.

Evans, G. W. and S. Honkapohja (1998a). Economic dynamics with learning: New stability results. *The Review of Economic Studies 65*(1), 23–44.

Evans, G. W. and S. Honkapohja (1998b). Stochastic gradient learning in the cobweb model. *Economic Letters 61*, 333–337.

Evans, G. W. and S. Honkapohja (2001). *Learning and Expectations in Macroeconomics*. Princeton University Press.

Evans, G. W. and S. Honkapohja (2003). Expectations and the stability problem for optimal monetary policies. *Review of Economic Studies 70*, 807–824.

Evans, G. W. and S. Honkapohja (2006). Monetary policy, expectations and commitment. *Scandinavian Journal of Economics 108*(1), 15–38.

Evans, G. W. and S. Honkapohja (2008). Expectations, learning and monetary policy: An overview of recent research. Working Paper CDMA08/02, Center for Dynamic Macroeconomic Anaylsis.

Evans, G. W. and B. McGough (2006). Learning to optimize. preliminary.

Ezekiel, M. (1938). The cobweb theorem. *Quarterly Journal of Economics 52*(2), 255–280.

Friedman, M. (1957). *Theory of the consumption function*. Princeton University Press.

Friedman, M. (1968). The role of monetary policy. *American Economic Review 58*, 1–17.

Friedman, M. (1969). The optimum quantity of money. In *The Optimum Quantity of Money and Other Essays*. Chicago: Aldine Publishing Co.

Frisch, R. (1933). Propagation problems and impulse problems in dynamic economics.

Gandolfo, G. (1971). *Economic Dynamics*. Springer Verlag.

Hall, R. E. and G. N. Mankiw (1997). Nominal income targeting. In G. N. Mankiw (Ed.), *Monetary Policy*. University of Chicago Press.

Hicks, J. R. (1939). *Value and Capital*. Oxford: Oxford University Press.

Honkapohja, S. (2003). Discussion of preston, "learning about monetary policy rules when long-horizon expectations matter". Working Paper 2003-19, Federal Reserve Bank of Atlanta.

Honkapohja, S. and K. Mitra (2004). Are non-fundamental equilibria learnable in models of monetary policy. *Journal of Monetary Economics 51*, 1743–1770.

Honkapohja, S., K. Mitra, and G. W. Evans (2011). Notes on agents' behavioral rules under adaptive learning and studies of monetary policy. Working Paper 1102, Center for Dynamic Macroeconomic Anaylsis.

Hoover, K. D. (1992). The rational expectations revolution: An assessment. *The Cato Journal 12*(1), 81–106.

Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money*. Macmillan, London.

Kiefer, J. and J. Wolfowitz (1952). Stochastic estimation of the maximum of a regression function. *The Annals of Mathematical Statistics 23*(3), 462–466.

Kirchler, E. (2011). *Wirtschaftspsychologie*. Hogrefe.

Kydland, F. and E. Prescott (1977). Rules rather than discretion: The inconsistency of optimal plans. *Journal of Political Economy 85*(3), 473–492.

Kydland, F. E. and E. Prescott (1982). Time to build and aggregate fluctuations. *Econometrica 50*(6), 1345–1370.

Ljung, L. (1977, August). Analysis of recursive stochastic algorithms. *IEEE Transactions on Automatic Control AC 22*(4), 551–575.

Lucas, R. E. (1972a). Econometric testing of the natrual rate hypothesis. In O. Eckstein (Ed.), *The Econometrics of Price Determination*. Washington, D.C.: Federal Reserve System.

Lucas, R. E. (1972b). Expectations and the neutrality of money. *Journal of Economic Theory 4*(2), 103–124.

Lucas, R. E. (1976). Econometric policy evaluation - a critique. *Carnegie-Rochester Conference Series on Public Policy 1*(1), 19–46.

Marcet, A. and T. Sargent (1989). Convergence of least-sqares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory 48*, 337–368.

McCallum, B. T. (1981). On non-uniqueness in rational expectations models - an attempt at perspective. Working Paper 0684, National Bureau of Economic Research.

McCallum, B. T. (1993). Discretion versus policy rules in practise: Two critical point. a comment.

McCallum, B. T. (1997). Issues in the design of monetary policy rules. Working Paper 6016, National Bureau of Economic Research.

McCallum, B. T. (2003). Multiple solution indeterminacies in monetary policy analysis. Working Paper 9837, National Bureau of Economic Research.

McCallum, B. T. (2007). E-stablity vis-à-vis determinacy. results for a broad class of linear rational expectations models. *Journal of Economic Dynamics and Control 31*(4), 1376–1391.

McCallum, B. T. (2008). Determinacy, learnability, and plausibility in monetary policy analysis: Additional results. Working Paper 502, Central Bank of Chile.

McCallum, B. T. (2011). Should central banks raise their inflation targets? some relevant issues. Working Paper 17005, National Bureau of Economic Research.

Modigliani, F. and R. Brumberg (1954). Utility analysis and the consumption function: an interpretation of cross-section data. In K. K. Kurihara (Ed.), *Post Keynesian Economics*, pp. 388–436. New Brunswick, NJ: Rutgers University Press.

Muth, J. (1960, June). Optimal properties of exponentially weighted forecasts. *Journal of the American Statistical Association 55*(290), 299–306.

Muth, J. (1961). Rational expectations and the theory of price movements. *Econometrica 29*(3), 315–335.

Orphanides, A. and J. Williams (2005). Imperfect knoledge, inflation expectations, and monetary policy. In B. Bernanke and M. Woodford (Eds.), *The Inflation-Targeting Debate*. University of Chicago Press.

Patinkin, D. (1949). The indeterminacy of absolute prices in classical economic theory. *Econometrica 17*, 1–27.

Phelps, E. S. (1968). Money-wage dynamics and labor-market equilibrium. *Journal of Political Economy 76*, 678–711.

Phillips, W. (1958, November). The relation between unemployment and the rate of change of money wage rates in the united kingdom, 1861-1957. *Economica 25*(100), 283–99.

Preston, B. (2005). Adaptive learning in infinite horizon decision problems. unpublished.

Preston, B. (2006). Learning about monetary policy rules when long-horizon expectations matter. *International Journal of Central Banking 1*(2).

Robbins, H. and S. Monro (1951). A stochastic approximation method. *Annals of Mathematical Statistics 22*, 400–407.

Samuelson, P. A. (1949). Lord keynes and the general theory. *Econometrica 14*, 187–200.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the contrivance of money. *Journal of Political Economy 66*, 467–482.

Sargent, T. J. (1993). *Bounded Rationality in Macroeconomics.* Oxford University Press Inc., New York.

Sargent, T. J. and N. Wallace (1976). Rational expectations and the theory of economic policy. *Journal of Monetary Economics 2*(2), 169–183.

Schmitt-Grohé, S. and M. Uribe (2010). The optimal rate of inflation. Working Paper 16054, National Bureau of Economic Research.

Simon, H. A. (1982). *Models of Bounded Rationality.* MIT Press.

Simon, H. A. (1986, Oct). Rationality in psychology and economics. *The Journal of Business 59*(4), 209–224.

Svensson, L. E. (1997). Inflation forecast targeting: Implementing and monitoring inflation targets. *European Economic Review 41*, 1111–1147.

Svensson, L. E. (1998). Inflation targeting as a monetary policy rule. Working Paper 6790, National Bureau of Economic Research.

Svensson, L. E. (1999a). Inflation targeting as a monetary policy rule. *Journal of Monetary Economics 43*, 607–654.

Svensson, L. E. (1999b). Price stability as a target for monetary policy: Defining and maintaining price stability. Seminar Paper 673, Institute for International Economic Studies.

Svensson, L. E. (2010). Inflation targeting. Working paper, National Bureau of Economic Research.

Taylor, J. B. (1977, September). Conditions for unique solutions in stochastic macroeconomic models with rational expectations. *Econometrica 45*(6), 1377–1385.

Taylor, J. B. (1993). Discretion versus policy rules in practise. In *Carnegie-Rochester Conference Series on Public Policy*, Volume 39, pp. 195–214. North-Holland.

Tinbergen, J. (1936). An economic policy for 1936. In L. Klaassen, L. Koyck, and H. Witteveen (Eds.), *Jan Tinbergen, Selected Papers*, pp. 37–84. North-Holland.

Tobin, J. (1956). The interest elasticity of the transactions demand for cash. *Review of Economics and Statistics 38*(3), 241–247.

Walsh, C. E. (2010). *Monetary Theory and Policy* (3rd ed.). The MIT Press.

Woodford, M. (1990). Learning to believe in sunspots. *Econometrica 58*, 277–307.

Woodford, M. (2003a). Comment on: Multiple-solution indeterminacies in monetary policy analysis. *Journal of Monetary Economics 50*, 1177–1188.

Woodford, M. (2003b). *Interest and Prices.* Princeton University Press.

Abstract (German)

Gegenstand dieser Arbeit ist die Analyse von optimaler Geldpolitik, unter der Annahme, dass Erwartungen des privaten Sektors auf ökonometrischen Modellen beruhen. Adaptives Lernens wird als minimale, wenngleich notwendige Abweichung vom Konzept der Rationalen Erwartungen (Muth, 1960) dargestellt. Die Implikationen dieser Annahme werden für das "New Keynesian" Modell analysiert. Bedingungen werden charakterisiert unter welchen die kontinuierliche Aktualisierung der ökonometrischen Modelle und der damit verbundenen subjektiven Erwartungen zu einem Gleichgewicht Rationaler Erwartungen (REE) führen kann. Diese Bedingungen werden als notwendige Voraussetzung für die Plausibilität eines REE präsentiert und der Forderung nach Determiniertheit des Gleichgewichts gegenübergestellt. Aufgrund dieser Untersuchungen wird argumentiert, dass die Orientierung einer optimalen Geldpolitik an subjektiven Erwartungen, der Stabilität des Gleichgewichts förderlich ist. Abschließend wird die Relevanz einer transparenten Geldpolitik besprochen und in diesem Zusammenhang das Konzept des Gleichgewichts eingeschränkter Wahrnehmungen (Restricted Perceptions Equilibrium) erläutert.

Abstract (English)

This thesis investigates optimal monetary policy under the assumption that private sector expectations are based on econometric models. Adaptive learning is presented as a minimal deviation from the concept of Rational Expectations (Muth, 1960). Its implications for the dynamics of the New Keynesian model are analysed. Conditions are characterised under which the continuous updating of econometric models and the hereby associated subjective beliefs may asymptotically lead to a Rational Expectations Equilibrium (REE). These conditions, which are evaluated as necessary for the plausibility of a REE, are compared with the requirement of determinacy. Based on these considerations it is argued that the orientation of optimal monetary policy on subjective expectations is conducive for the stability of equilibrium. Furthermore, in course of discussing the relevance of central bank‘s transparency, the concept of a Restricted Perceptions Equilibrium is introduced.

Curriculum Vitae
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