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Nowhere-Zero Flows and Structures in Cubic Graphs

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Contents

Chapter 1

Background and Main Results

In comparison to other disciplines in mathematics, such as number theory or analysis, graph theory is a rather new branch in mathematics and was developed mostly in the 20th century. Despite its young history it is interacting with many fields in science such as physics, informatics, chemistry, neurology, genetics, etc. One essential reason for its wide applicability is that a discrete problem which - roughly speaking - can be illustrated by dots and lines can often be transformed into a graph theoretical problem and then attacked by graph theoretical tools. For instance, the problem of creating a good navigation-system, stable internet connections, cheap water or electricity supply systems are good examples which can be modeled as graph theoretical problems. However, we concentrate exclusively on theoretical problems.

This thesis is focusing on cubic graphs. Cubic graphs are 3-regular graphs, i.e., every vertex is adjacent with exactly three edges. At first glance, cubic graphs might seem to be very special graphs and thus may not deserve special attention. However, cubic graphs occur in nature, as for instance in the pattern of dry soil or in the pattern of the veins of someone's hand, etc. Structures which occur in nature are usually worth to look at. But there is also a pure mathematical reason why cubic graphs deserve special attention. Many graph theoretical problems and several conjectures on very general classes of graphs can be reduced to the case of cubic graphs. Later, we present some examples.

The thesis consists of two parts. The common thread of the thesis is the concept of a *flow*. We assume that the reader is familiar with the basic definitions in graph theory as presented, for instance, in [2].

A flow can most easily be defined on an oriented graph, i.e a graph where every edge $e = xy$ is replaced by an oriented edge $\overrightarrow{e} \in \{(x, y), (y, x)\}$. An oriented edge is called an arc. Denote an orientation of the graph G by $\overrightarrow{G}$ and the set of arcs by $A(\overrightarrow{G})$. For every arc $\overrightarrow{e} = (x, y)$, x is called the tail and y is called the head of $\overrightarrow{e}$. For every $v \in V(\overrightarrow{G})$, denote by $A^+(v)$ ($A^-(v)$) the set of arcs incident with v which have v as their tail (head). An integer flow of $\overrightarrow{G}$ is a mapping $f : A(\overrightarrow{G}) \mapsto \mathbb{Z}$ such that

$$\sum_{\overrightarrow{e} \in A^+(v)} f(\overrightarrow{e}) = \sum_{\overrightarrow{e} \in A^-(v)} f(\overrightarrow{e}) \quad \forall\, v \in V(\overrightarrow{G}) \ .$$

If $f(\overrightarrow{e}) \neq 0 \ \forall \overrightarrow{e} \in A(\overrightarrow{G})$ then f is called a nowhere-zero flow. If in addition $|f(\overrightarrow{e})| < k$, $\forall \overrightarrow{e} \in A(\overrightarrow{G})$ with $k \in \mathbb{N}$, then f is called a nowhere-zero k-flow of $\overrightarrow{G}$. Note that if one orientation of G has a nowhere-zero k-flow then every orientation of G has a nowhere-zero k-flow. Consequently, a graph G is said to have a nowhere-zero k-flow if one (and thus every) orientation of G has a nowhere-zero k-flow. Moreover the *flow-number* of G is the smallest possible $k \in \mathbb{N}$ such that G has a nowhere-zero k-flow. Note that a graph which has a *bridge* (a bridge is an edge which is not contained in any circuit) cannot have a nowhere-zero flow.

A famous (and still unsolved) conjecture in graph theory has been formulated by W.T. Tutte; it is called the *Nowhere-Zero 5-Flow Conjecture* (NZ5FC).

NZ5FC: *Every bridgeless graph has a nowhere-zero* 5*-flow.*

One can restrict this conjecture to cubic graphs; i.e., if the conjecture is true for bridgeless cubic graphs then it is generally true for bridgeless graphs. Seymour [27] proved that every bridgeless graph has a nowhere-zero 6-flow. It is also known that not every bridgeless graph has a nowhere-zero 4-flow (note that in cubic graphs, nowhere-zero 4-flows and 3-edge colorings are equivalent concepts). Such cubic graphs - usually some additional connectivity property is required - are called snarks. For instance, the Petersen graph is a snark and thus has no nowhere-zero 4-flow. Note that several conjectures on cubic graphs can be restricted to snarks, as for instance the NZF5C and the Cycle Double Cover Conjecture (CDCC) which is stated below.

One of the few approaches to the NZ5FC has been the *Bipartizing matching conjecture* (BMC) [10] by Fleischner. This conjecture is related to the "Cycle plus Triangles Theorem" (CPTT) [11], which is a solution to a problem of Erdös. Note that an edge-decomposition of a graph is a partition of the edges of a graph; also, a *triangle* is a circuit of length 3.

CPTT: Every 4-regular graph which can be edge-decomposed into a hamiltonian circuit and a set of triangles, has a vertex 3-coloring.

The BMC is also related to the following long-standing conjecture.

CDCC: *Every bridgeless graph G contains a set of circuits such that every edge of G is contained in exactly two circuits of this set.*

For our considerations we need some more concepts.

A *dominating circuit* in a graph G is a generalization of a hamiltonian circuit. It is a circuit C such that $E(G - V(C)) = \emptyset$. The CPTT implies, see [12], for every cubic graph G with dominating circuit C the existence of a certain type of matching with respect to C, namely: G has a matching M such that

1. $M \cap E(C) = \emptyset$;
2. $G - M$ has a nowhere-zero 3-flow and;
3. M covers $V(G) - V(C)$
(see [12]).

$G - M$ having a nowhere-zero 3-flow is equivalent to saying that $G - M$ is homeomorphic to a cubic bipartite graph. Consequently such a matching is called a *bipartizing matching* (BM). The following long-standing conjecture motivates further investigations of cubic graphs with dominating circuit.

Dominating Cycle Conjecture (DCC): *Every cyclically 4-edge connected cubic graph has a dominating circuit.*

Note that the DCC is equivalent to the following statements, see [8]:

1. The Matthews-Sumner conjecture: Every 4-connected claw-free graph is hamiltonian.

2. Thomassen's conjecture: Every 4-connected line graph is hamiltonian.

3. Every cubic cyclically 4-edge-connected non 3-edge colorable graph of girth at least five has a dominating circuit.

We are now able to state the BMC which has been the starting point of the thesis.

BMC: *Every cyclically 4-edge connected cubic non 3-edge colorable graph G with dominating circuit C has two edge-disjoint bipartizing matchings with respect to C.*

Note that the existence of a BM in a cubic graph G with dominating circuit C implies the existence of a nowhere-zero 6-flow of G. However, if G has two disjoint BMs with respect to C then G has a nowhere-zero 5-flow and a five cycle double cover, i.e., five 2-regular subgraphs such that every edge of G is contained in exactly two of them; see [10, 11] for details.

This is essentially the background and has been the motivation for the first part of the thesis which we describe now.

One main result is that the BMC is not true. We can even construct infinitely many counterexamples to this conjecture. As a consequence we also answer a problem posed in [7]. In determining the structure of a counterexample we achieve results of independent interest. To this end, we introduce a new intersection graph which generalizes the concept of a circle graph from hamiltonian circuits to dominating circuits. In terms of this new definition we obtain a reformulation of the BMC. This reformulation is surprisingly related to a type of problem, for which Gallai [23] has achieved the following classical result.

Gallai's Theorem: *The vertex set of every graph can be partitioned into two sets such that each set induces an eulerian subgraph.*

We extend Galai's theorem by characterizing the graphs for which the vertex set can be covered by two sets of vertices such that each set induces an anti-eulerian subgraph (i.e every vertex has odd degree). This theorem is important for the construction of certain counterexamples to the BMC. Finally we apply several non-standard graph theoretical constructions to obtain our counterexample. At the end of this chapter some natural modifications of the BMC are discussed and stated. Note that this first part of the thesis has already been published by the author, see [17].

Since the BMC is false we modify the definition of a BM and extend the concept of a BM to a so called *generalized BM* (gBM) which is not related to a dominating circuit: a gBM of a cubic graph G is a matching M of G such that every component of $G - M$ has a nowhere-zero 3-flow and an even number of 2-valent vertices. We show that every cubic bridgeless graph has a gBM by proving that the existence of a gBM implies the existence of a nowhere-zero 6-flow and vice versa. We further generalize the concept of a gBM and introduce the notion of *solving a cubic graph* which is an extension of an already existing concept called frames as follows (frames have been developed as an attempt to solve the CDCC, see [13, 14]).

Denote by $[F]$ the set of graphs which is the union of the set of 2-connected cubic graphs and C_2 (the circuit of length 2). We say that a set $S \subseteq [F]$ *solves* a cubic graph G if there is a matching M of G with the following properties.

1. Every component of $G - M$ is a subdivision of a graph of S.
2. Every component of $G - M$ has an even number of 2-valent vertices.

Moreover, we say that S solves a family $\mathcal{G}$ of cubic graphs if every graph of $\mathcal{G}$ is solved by S.

For instance, the family of 3-edge colorable cubic graphs is solved by $S := \{C_2\}$. We underline that only one element is necessary to solve this huge class of graphs. We pose the following question:

Which properties must a set $S_0 \subseteq [F]$ have such that every 3-connected cubic graph G is solved by S_0?

Note that such S_0, as described above, exists. For instance, the set which contains all 2-connected bipartite graphs of $[F]$ solves every 2-connected cubic graph.

It is conjectured in [14] that every 3-connected cubic graph can be solved by the set of Kotzig graphs and C_2 (Kotzig graphs, see [14], form a subfamily of the hamiltonian cubic graphs). We need the following notations.

Let H be a graph. Denote by $l(H)$ the smallest natural number such that the following is true: H contains a circuit C such that for every vertex $v \in V(H)$ there is a path of length at most $l(H)$ connecting v with (a vertex of) C. For instance, if H is hamiltonian then $l(H) = 0$. If H has a dominating circuit but no hamiltonian circuit then $l(H) = 1$. Let $S := \{G_1, G_2, ...\}$ be an infinite set of graphs such that for very $G_i \in S$ there is a $G_m \in S$ such that $l(G_i) < l(G_m)$ with $i, m \in \mathbb{N}$, then we write $l(S) := \infty$.

We prove that S_0 with the properties as stated above, must satisfy $l(S_0) = \infty$. This implies first of all that S_0 cannot be of finite order. Second, it implies that S_0 cannot contain hamiltonian graphs only. Hence we disprove the conjecture stated by Häggkwist and Markström in [14].

In the second part of the thesis we consider certain plane graphs. We call a plane graph a mosaic if it has only quadrangular and triangular faces. Hence mosaics form a generalization of triangulations and quadrangulations. We show that every mosaic corresponds to a unique cubic (not necessarily planar) graph. We transform nowhere-zero flow problems of arbitrary cubic graphs into vertex coloring problems of mosaics.

Note that flows have been introduced by W.T.Tutte as a generalization of coloring plane graphs. For instance, the questions whether a cubic plane graph has a face k-coloring (or its dual a vertex k-coloring) or a nowhere-zero k-flow, are equivalent. (The famous "Four Color Theorem" thus also states that every bridgeless cubic planar graph has a nowhere-zero 4-flow. A cubic graph is not planar in general and thus one cannot color its faces whereas one still can ask whether the graph has a nowhere-zero k-flow. Hence the NZ5FC is a natural extension of the already proved 4-color conjecture by extending the class of plane bridgeless cubic graphs to arbitrary bridgeless cubic graphs and by increasing the flow-number from 4 to 5.)

Moreover, we show that one can study snarks in the plane. In particular, we prove that every mosaic which has no proper vertex 4-coloring such that every quadrangular face Q is incident with an even number of differently colored vertices (i.e the four vertices incident with Q may have 4 different colors or 2 different colors) corresponds to a snark. This equivalence enables us to construct certain snarks. Among the snarks obtained by this approach there are even cyclically 6-edge connected ones. Note that they can also be obtained by different methods, see [20, 21, 24].

The usefulness of dealing with mosaics is also expressed by additional results.

First, we generalize a theorem on quadrangulations, i.e. mosaics without triangular faces, by Hoffmann and Kriegel, see [30, 16]. In particular, we characterize those mosaics which can be extended to an even triangulation, i.e. a triangulation where every vertex has even degree, by adding a diagonal-edge into every quadrangular face.

Second, we improve a theorem of Mohar [25], as follows. Let Q be a quadrangulation of the sphere with an arbitrary proper vertex 4-coloring $f : V(Q) \mapsto \{1, 2, 3, 4\}$. We prove that the number of quadrangular faces of Q whose vertices are colored $1, 2, 3, 4$ in clockwise order equals the number of quadrangular faces whose vertices are colored in counterclockwise order $1, 2, 3, 4$. Note that Q may contain quadrangular faces which don't have 4 different colors.

Finally, in the last chapter we pose open problems and state several new conjectures.

PART 1

Chapter 2

Bipartizing Matchings

The Nowhere Zero 5-Flow Conjecture (NZ5FC) states that the edges of every bridgeless graph can be oriented and assigned numbers from the set $\{1, 2, 3, 4\}$ such that for every vertex the sum of the incoming values equals the sum of the outcoming values. A proof of the NZ5FC can be restricted to bridgeless cubic graphs.

The *Bipartizing Matching Conjecture* (BMC) by H.Fleischner, [10], is an approach to this conjecture for cubic graphs with a dominating circuit; a *dominating circuit* in a graph is a cycle such that every edge of the graph is incident with a vertex of the cycle. In view of the Dominating Cycle Conjecture (DCC) (which states that every cyclically 4-edge-connected cubic graph has a dominating circuit) it is sufficient for a proof of the NZ5FC to focus on this type of graphs, since a minimal counterexample to the NZ5FC must be cyclically 6-edge-connected; see [22].

2.1 Definitions and basic results

For standard terminology we refer to [2, 5]. An *eulerian graph* is a graph in which every vertex has even degree (which includes the empty set). An *anti-eulerian graph* is a graph in which every vertex has odd degree. Therefore an anti-eulerian graph has always an even number of vertices.

In this chapter H denotes an arbitrary graph, G a cubic graph and D a dominating circuit. We write for G often (G, D), and for H, (H, D), in order to denote which dominating circuit in G, respectively H, we are referring to.

A *chord* vw of (G, D) is an edge with $v, w \in V(D)$ and $vw \notin E(D)$. Define $Q(G, D) := \{q_1, q_2, ..., q_k\} := V(G) \setminus V(D)$.

For $v \in V(H)$, E_v denotes the set of edges which are incident with v. For $F \subseteq V(H)$, let $\langle F \rangle_H$ denotes the induced subgraph of H by F. By suppressing a degree two vertex x we mean the deletion of x and the subsequent addition of a new edge joining the neighbours of x.

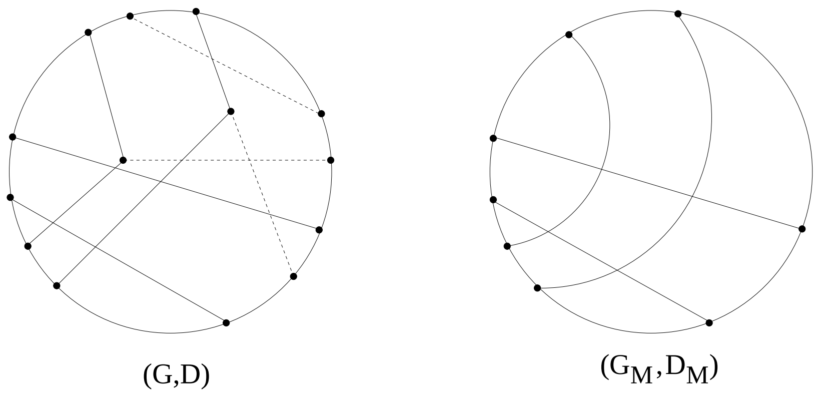


Figure 2.1: A graph (G, D) with a BM M which is illustrated by dashed drawn edges. (G_M, D_M) results from the removement of M and the suppression of the vertices in (G, D) which are incident with M.

Definition 2.1.1 *1. Let M be a matching in H. Then H_M is defined to be the graph which arises by deleting M from H and by suppressing possibly resulting vertices of degree two. In case H is cubic and $V(M) = V(H)$, define $H_M := \emptyset$.*
2. Let M be a matching in (H, D) with $E(M) \cap E(D) = \emptyset$. Then D_M is defined to be the cycle of $H - M$, arising from D by suppressing all vertices of D with degree two in $H - M$.

Definition 2.1.2 *A* bipartizing matching *(BM) of (G, D) is a matching $M \subseteq E(G) - E(D)$ such that G_M is bipartite and $E_{q_i} \cap M \neq \emptyset$ for $i = 1, 2, ..., k$. Define G_M to be bipartite if $V(G_M) = \emptyset$; see Remark 2.1.3 below.*

Remark 2.1.3 *Note that D may be a hamiltonian circuit. In this case $Q(G, D) = \emptyset$ and only the first requirement of Def. 2.1.2 must be fulfilled for M to be bipartizing.*

Every (G, D) has a BM which already implies that G has a nowhere zero 6-flow; see [10]. For an illustration of Definition 2.1.1, 2.1.2, see Figure 2.1.

Definition 2.1.4 *Define a snark to be a cubic cyclically* 4*-edge connected-graph which is not* 3*-edge-colorable.*

The Bipartizing Matching Conjecture: Let (G, D) be a snark. Then (G, D) has 2 disjoint bipartizing matchings.

Two disjoint bipartizing matchings (2dBMs) in (G, D) always imply a nowhere zero 5-flow and a 5-cycle double cover of G (independent on the cyclic edge connectivity of G); see [9].

2.2 Even and odd colorings

We want to transform 2dBMs into a coloring of a circle graph. To this end we need the following definitions and results.

Definition 2.2.1 *An* even (odd) coloring *of a graph H is a map f from $V(H)$ into the set $\{a, b, c\}$ such that $f^{-1}(a) \cup f^{-1}(c)$ and $f^{-1}(b) \cup f^{-1}(c)$ both induce eulerian (anti-eulerian) subgraphs in H. Set $A := f^{-1}(a)$, $B := f^{-1}(b)$ and $C := f^{-1}(c)$. Hence f describes a covering of $V(H)$ by the induced eulerian (anti-eulerian) subgraphs $\langle A \cup C \rangle$ and $\langle B \cup C \rangle$.*

The next theorem was proved by Gallai and W.K. Chen; see [23].

Theorem 2.2.2 *H has an even coloring with $C = \emptyset$.*

Interestingly, not every H (with $|V(H)| > 1$) admits an odd coloring, neither with $C = \emptyset$ nor with $C \neq \emptyset$.

Theorem 2.2.3 *H has an odd coloring if and only if $\overline{H}$ has an even coloring such that $A \cup C$ and $B \cup C$ have both even cardinality.*

Proof: Let $X \in \{A, B\}$. Set $d(v) = d_{\langle X \cup C \rangle}(v)$ and $d(\overline{v}) = d_{\overline{\langle X \cup C \rangle}}(v)$. $|V(X \cup C)|$ is even, either because $X \cup C$ is anti-eulerian or by assumption. Moreover $d(v) + d(\overline{v}) + 1 = |V(X \cup C)|$. Now the theorem follows.

Example 2.2.4 *$\overline{C_{2n+1}}$ has no odd coloring.*

2.3 Generalizing the circle graph

We investigate the problem to find 2dBMs in (G, D) with a general definition of a circle graph which will be extended later. Let D be a hamiltonian circuit of H. (H does not need to be cubic.) We say two chords $c_1 := x_1x_2$, $c_2 := x_3x_4$ of (H, D) *intersect* if one of the following two conditions is fullfilled.
1. c_1 and c_2 are adjacent.
2. If we start at x_1 and pass through D, we pass x_i, $i = 1, 2, 3, 4$ in the order $x_1\, x_3\, x_2\, x_4$ or $x_1\, x_4\, x_2\, x_3$.

The *circle graph* of (H, D) is defined to be the intersection graph of its chords (w.r.t. D) and denoted by $(H, D)^c$ or in short H^c. A vertex in $(H, D)^c$ which corresponds to a chord r in (H, D) will be denoted by r^c.

Proposition 2.3.1 *[7] Let D be a hamiltonian circuit of G , then $(G, D)^c$ is eulerian if and only if G is bipartite. Hence a matching $M \subseteq E(G) - E(D)$ is a BM in (G, D) if and only if $(G_M, D_M)^c$ is eulerian.*

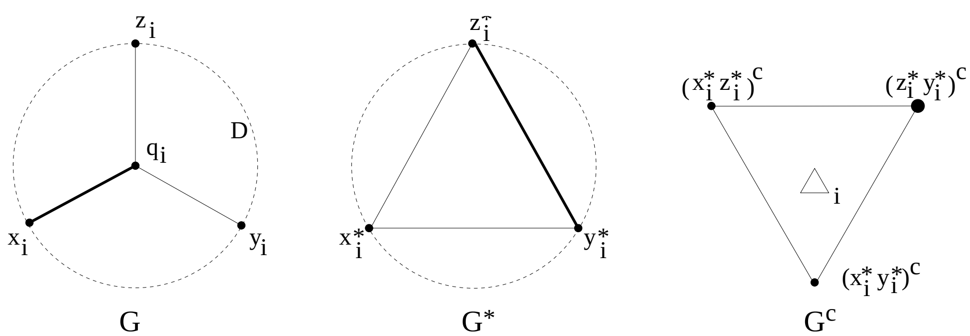


Figure 2.2: The transformation from G into G^* and G^c. The edges and vertex in bold face illustrate the map g on x_iq_i for $i = 1, 2, ..., k$.

The next proposition follows directly from Theorem 2.3.4 below and shows the connection between even colorings and BMs.

Proposition 2.3.2 *Let D be a hamiltonian circuit, then (G, D) has 2dBMs if and only if $(G, D)^c$ has an even coloring.*

Transform (G, D) into a graph with vertices of degree 3 or 4 by replacing q_i with a triangle containing the neighbours of q_i, $i = 1, 2, ..., k$. Call this graph $(G, D)^*$ or G^* if there is no other dominating circuit mentioned. A vertex $v \in V(G)$ which corresponds to a vertex in $V(G^*)$ will be denoted by v^*. We can now extend the definition of a circle graph from a hamiltonian circuit to a dominating circuit; set

$$(G, D)^c := ((G, D)^*)^c.$$

For $(G, D)^c$ we will often write G^c. The triangle of G^* resulting from the replacement of a $K_{1,3}$ with central vertex q_i coresponds to a triangle $\triangle_i$ in G^c, $i = 1, 2, ..., k$; see Figure 2.2. Set $T = \bigcup_{i=1}^k \triangle_i$.

Definition 2.3.3 *Call a map f from $V(G^c)$ into the set $\{a, b, c, o\}$ a $\triangle$-*coloring *w.r.to T if the following is true.*

1. Every triangle of T is colored properly with the colors $\{a, b, o\}$.
2. $f(v) \neq o$ $\forall v \in G^c - T$ and f restricted to $V(G^c) - f^{-1}(o)$ is an even coloring; set $A := f^{-1}(a)$, $B := f^{-1}(b)$ and $O := f^{-1}(o)$.

Note that if D is a chordless dominating circuit of G then $C = \emptyset$.

We shall prove the following:

Theorem 2.3.4 *(G, D) has $2dBMs$ if and only if $(G, D)^c$ has a $\triangle$-coloring.*

By the above theorem and since 2dBMs imply a nowhere zero 5-flow we obtain:

Corollary 2.3.5 *(G, D) has a nowhere zero 5-flow if $(G, D)^c$ has a $\triangle$-coloring.*

For the proof of Theorem 2.3.4 we need the following lemma below, see Figure 2.3.

Lemma 2.3.6 *Replace every chord vw in (G, D) by a $K_{1,3}$ as follows: Subdivide vw and an edge of D incident with w, calling the respective subdivision vertices q and w' and add qw'. Denote this graph by (G', D'). Then the following is true:*
1. (G, D) has 2dBMs if and only if (G', D') has 2dBMs.
2. $(G, D)^c$ has a $\triangle$-coloring if and only if $(G', D')^c$ has a $\triangle$-coloring.

Proof: Denote by R the subgraph induced by the chords of (G, D) and let R' denote the subgraph of (G', D') whose components are the corresponding $K_{1,3}$s.

1. Denote the 2dBMs in G by M_1, M_2 and in G' by M_1', M_2'. Let $t \in E(G-D)$ ($t \in E(G'-D')$) and set $\{j_1, j_2\} = \{1, 2\}$. Define M_{j_1} (M'_{j_1}) dependent on M'_{j_1} (M_{j_1}) as follows:

Case 1. $t \notin E(R)$ ($t \notin E(R')$). In this case $t \in M_{j_1}$ if and only if $t \in M'_{j_1}$.

Case 2. $t \in E(R)$ ($t \in E(R')$).

Assume $t \in \{vw, qv, qw, qw'\}$, see Figure 2.3.

(a) $vw \in M_{j_1}$ if and only if $vq \in M'_{j_1}$ and qw' or qw belongs to M'_{j_2}.

(b) $vw \notin M_{j_1}$ if and only if $qw \in M'_{j_1}$ and $qw' \in M_{j_2}$, or $qw \in M'_{j_2}$ and $qw' \in M'_{j_1}$.

This finishes the proof of statement 1.

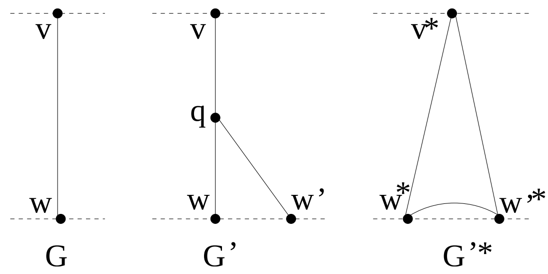


Figure 2.3:

2. Let f be a $\triangle$-coloring of G^c and f' a $\triangle$-coloring of G'^c. Let R^c be the set of vertices in G^c which correspond to the components R, and let R'^c be the set of vertices of the triangles in G'^c which correspond to R' in G'.

Case 1. $x \in V(G^c) - R^c$, ($x \in V(G'^c) - R'^c$). In this case we define

$$f(x) = f'(x).$$

Case 2. $x \in R^c$, ($x \in R'^c$).

Assume $x \in \{(v^*w^*)^c, (w^*w'^*)^c, (v^*w'^*)^c\}$. Set

$$f((v^*w^*)^c) = a \text{ if and only if } f'((w^*w'^*)^c) = b\,,$$

$$f((v^*w^*)^c) = b \text{ if and only if } f'((w^*w'^*)^c) = a$$

and

$$f((v^*w^*)^c) = c \text{ if and only if } f'((w^*w'^*)^c) = o\,;$$

see Figure 2.3. Now the lemma follows.

Proof of Theorem 2.3.4: By the last lemma we may assume that (G, D) has no chords. Thus $\{q_1, q_2, ..., q_k\} = V(G) - V(D)$ are the centers of the $K_{1,3}$'s.

Let M_1, M_2 be $2dBMs$ in (G, D). Using the notation of Figure 2.2, we define a bijection $g : \bigcup_{i=1}^{k} E_{q_i} \mapsto V(G^c)$ by setting

$$g(x_i q_i) = (z_i^* y_i^*)^c \,,$$

$$g(y_i q_i) = (z_i^* x_i^*)^c \,,$$

and

$$g(z_i q_i) = (x_i^* y_i^*)^c \,.$$

By the definition of g and since $M_1 \cap M_2 = \emptyset$ it follows that $g(M_1) \cap g(M_2) = \emptyset$. We show that the vertex set $g(M_i)$ for $i = 1, 2$ induces an eulerian subgraph in G^c. Then we obtain a $\triangle$-coloring by defining $A := g(M_1)$, $B := g(M_2)$, $C := \emptyset$ and $O := G - g(M_1) - g(M_2)$. Let $M \in \{M_1, M_2\}$. Since M is a BM, (G_M, D_M) is a bipartite graph. Hence $(G_M, D_M)^c$ is eulerian by Proposition 2.3.1. Since $V(g(M)) = V(G_M^c) \subset V(G^c)$ and $\langle g(M) \rangle_{G^c} = (G_M, D_M)^c$, the first part of the proof is finished.

Let f be a $\triangle$-coloring of $(G, D)^c$. We show that $g^{-1}(A)$ and $g^{-1}(B)$ are 2dBMs in (G, D). Let A^*, B^* and O^* be the sets of chords in $(G, D)^*$ which correspond to the sets of vertices A, B, and O in $(G, D)^c$. Since $M_1 := g^{-1}(A)$ and $M_2 := g^{-1}(B)$ are disjoint matchings - each covering all vertices of $G - V(D)$ - it suffices to show that (G_{M_i}, D_{M_i}) is bipartite for $i = 1, 2$. By Proposition 2.3.1 this is equivalent to showing that $(G_{M_i}, D_{M_i})^c$ is eulerian, $i = 1, 2$. By definition of g and the definition of the sets A, B, O (see Def.2.3.3) it follows, see Figure 2.2, that

$$(G_{M_1}, D_{M_1}) = ((G^* - B^*)_{O^*}, (D - B^*)_{O^*}) \,.$$

Since

$$(G_{M_1}, D_{M_1})^c = ((G^* - B^*)_{O^*}, (D - B^*)_{O^*})^c = \langle A \rangle_{G^c}$$

and $\langle A \rangle_{G^c}$ is eulerian by definition of a $\triangle$-coloring and because $C = \emptyset$, $g^{-1}(A)$ is a BM in G. The same arguments hold for B. This finishes the proof of the theorem.

2.4 Construction of a Counterexample

By Theorem 2.3.4 a counterexample to the BMC has no $\triangle$-coloring. Let (G, D) have chords. If we want to know whether G^c has a $\triangle$-coloring w.r.t

T, we can proceed in the following manner: we color the vertices of the triangles in T (each triangle must be colored by a, b, o), remove O from G^c and try to extend this coloring of $T - O$ to an even coloring of $G^c - O$. If we succeed, we obtain a $\triangle$-coloring, otherwise we start with a new coloring of T. If no proper coloring of T can be extended to an even coloring of $G^c - O$, G^c has no $\triangle$-coloring w.r.t T. We will always proceed in this way to show that G has no 2dBMs. This approach is very promising if $|Q(G, D)| = 1$ and $G^c - x_1 = G^c - y_1 = G^c - z_1$ with $\{x_1, y_1, z_1\} = V(\triangle_1)$. Then we have to consider only one coloring of $\triangle_1$ since $G^c - O$ is the same for every possible $\triangle$-coloring of G^c.

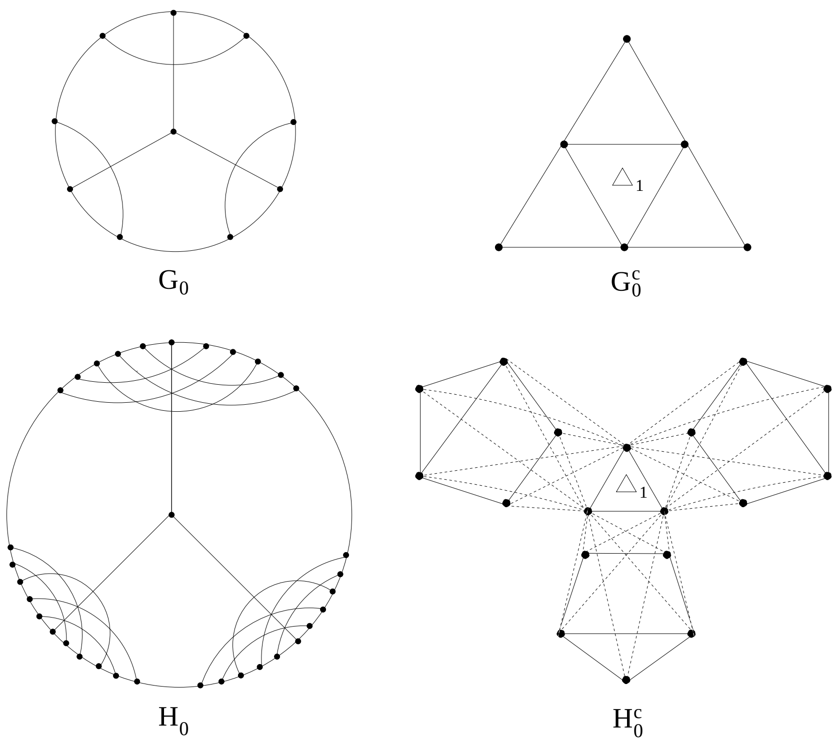


Figure 2.4: Two cyclically 3-edge-connected graphs without 2dBMs.

Definition 2.4.1 *Let H be a graph. Add a new edge xy with $x, y \notin V(H)$. Connect both vertices of the new edge to all vertices of H and denote this new graph by Hxy.*

Suppose Hxy is an induced subgraph of a graph H'' and suppose $\{x, y\}$ is a vertex cut of H''. Colour x with color a and y with color b. Then this coloring of x and y cannot be extended to an even coloring of Hxy and H'' if H has no odd coloring. This consideration implies the following theorem.

Theorem 2.4.2 *(G, D) has no $2dBMs$ if $(G, D)^c$ satisfies:*
1. $|Q(G, D)| = 1$.
2. For every pair of vertices $\{v^1, v^2\} \subset V(\triangle_1)$, $\{v^1, v^2\}$ is a vertex cut of G^c.
3. $G^c = H^1x_1y_1 \cup H^2x_1z_1 \cup H^3y_1z_1$.
4. H^i has no odd coloring, is connected and $H^i \cap \triangle_1 = \emptyset$ for $i = 1, 2, 3$.

As an application see the graph G_0 in Figure 2.4, already developed by H.Fleischner, where H^i for $i = 1, 2, 3$ corresponds to a single vertex. A more complicate graph H_0 is also illustrated there. For showing that H_0 has no 2dBMs one applies the last theorem and Theorem 2.2.3. We will transform G_0 into a counterexample of the BMC.

In order to make G_0 cyclically 4-edge-connected, we will apply the properties of a graph which are described in the next lemma.

2.5 Weak even colorings

Recall that an even coloring of H, where $X \in \{A, B\}$ requires by definition that $d_{\langle X \cup C \rangle}(v)$ is even for all $v \in X \cup C$. For the next lemma and theorem the following more general definition is useful.

Definition 2.5.1 *Call $f : V(H) \mapsto \{a, b, c\}$ a weak even coloring of H w.r.t. a set $S \subseteq V(H)$ if $d_{\langle X \cup C \rangle}(v)$ is even for all $v \in X \cup C - S$.*

Lemma 2.5.2 *Let v_1, v_2 be the vertices of the the graph H_1 of Figure 2.5 and let them be colored as illustrated. Then in both cases there exists no extension to a weak even coloring f of H_1 w.r.t. $S = \{v_1, v_2\}$.*

Suppose there is a weak even coloring of H_1 satisfying one of the colorings illustrated in Figure 2.5.

Case 1. $f(v_1) = a$, $f(v_2) = b$.
$v_3 \in A \cup C$ is impossible since otherwise $\langle A \cup C \rangle$ would contain v_1 as its unique odd vertex. Hence $v_3 \in B$, implying $v_4 \in B \cup C$; otherwise $v_5 \notin S$ would be an odd vertex of $\langle A \cup C \rangle$ or $\langle B \cup C \rangle$. Clearly, $d_{\langle B \cup C \rangle}(v_3)$ and $d_{\langle B \cup C \rangle}(v_4)$ have different parity, which is impossible and thus finishes this case.

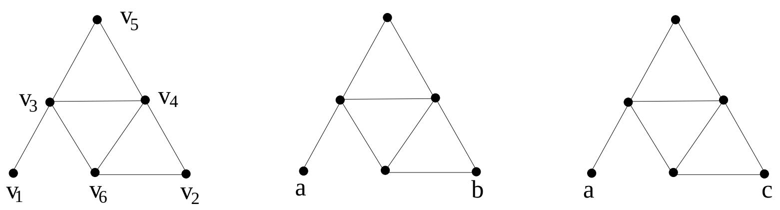


Figure 2.5: H_1 denotes the graph on the left hand side. The graph in the middle illustrates the coloring of case 1 and the graph on the right hand side the coloring of case 2.

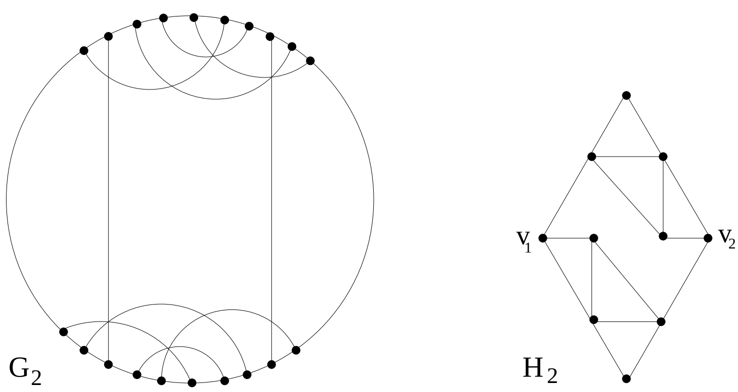


Figure 2.6:

Case 2. $f(v_1) = a$, $f(v_2) = c$.
Suppose $v_3 \in B$. We conclude as before $v_4 \in B \cup C$, and obtain again a contradiction because of the different parities of $d_{\langle B \cup C \rangle}(v_3)$ and $d_{\langle B \cup C \rangle}(v_4)$. Hence $v_3 \in A \cup C$ and $v_4 \in A \cup C$. Suppose $v_3 \in A$. Hence $v_6 \in B$ which implies $v_4 \in C$). Thus, independent of the coloring of v_5 at least one of $d_{\langle A \cup C \rangle}(v_4)$ and $d_{\langle B \cup C \rangle}(v_4)$ is odd. Hence $v_3 \in C$. $v_4 \in B \cup C$ implies that one of $d_{\langle B \cup C \rangle}(v_3)$ and $d_{\langle B \cup C \rangle}(v_4)$ is odd independent of the coloring of v_5, v_6. This contradiction implies $v_4 \in A$. $v_3 \in C$ and $v_4 \in A$ implies $v_5 \in A$ (otherwise, $d_{\langle B \cup C \rangle}(v_5) = 1$) and $v_6 \in B$ (otherwise $d_{\langle A \cup C \rangle}(v_6) = 3$). This results in the final contradiction $d_{\langle A \cup C \rangle}(v_3) = 3$. This finishes the proof.

We remark that for this lemma a and b can be exchanged. Observing that the preceding lemma can be applied to the more genreal situation where H_1 is an induced subgraph and $\{v_1, v_2\}$ is a vertex cut, we obtain the following

Theorem 2.5.3 *Let f be a weak even coloring of the graph H_2 of Figure 2.6, w.r.t. $S = \{v_1, v_2\}$; then $f(v_1) = f(v_2)$.*

Proof: By the last lemma , $f(v_1) = f(v_2)$ or $f(v_1) = c$ and $f(v_2) = a$ or b. By symmetry only $f(v_1) = f(v_2)$ can occur.

2.6 Graphs without disjoint bipartizing matchings

By the above considerations and the following observations we are able to construct cubic graphs without 2dBMs w.r.to a dominating circuit.
The graph H_2 in Figure 2.6 is the circle graph of (G_2, D). We insert two copies of $(G_2, D) - D$ to G_0, as shown in Figure 2.7 and denote this graph by J.

Lemma 2.6.1 (G, D) *is cyclically* 4*-edge-connected if* G^c *satisfies the following two conditions: 1.* G^c *is connected. 2.* $G^c - v$ *is connected for all* $v \in V(G^c) - \bigcup_{i=1}^{k} V(\triangle_i)$ *and* $G^c - v_i - w_i$ *is connected for all* v_i, w_i *in* $\triangle_i$, $i = 1, ..., k$.

Proof by contradiction. Suppose G^c satisfies condition 1 and 2 but $\lambda_c(G, D) < 4$. Since D is a dominating circuit, every cyclic edge cut Z of (G, D) satisfies $|Z \cap E(D)| > 1$. Thus we consider the following cases.

Case 1. $|Z| = 2$. Then $Z \subset E(D)$ by the preceding inequality. Let P_1, P_2 be the components of $D - Z$. Since $|Z| = 2$ every chord of G has both ends in P_i for some $i \in \{1, 2\}$ and the endvertices of every $K_{1,3}$ of $G - E(D)$ also lie in the same P_j for some $j \in \{1, 2\}$. Therefore, every chord of D in $(G, D)^*$ has both ends in P_1 or in P_2. For $i = 1, 2$ set $E_i^* := \{e \in E(G^*) - E(D) :$ e is incident with a vertex of $P_i\}$. Then no $e_1 \in E_1^*$ crosses any $e_2 \in E_2^*$. Therefore, $\{E_1^*, E_2^*\}$ corresponds to a vertex partition $\{V_1^c, V_2^c\}$ of G^c such that $v_1^c v_2^c \notin E(G^c)$ for every $v_1^c \in V_1^c$, $v_2^c \in V_2^c$. That is, G^c is not connected, contradicting the validity of condition 1.

Case 2. $|Z| = 3$. Since case 1 is already treated we may assume w.l.o.g. that $|Z|$ is minimal. Then we may write $Z = \{e_1, e_2, e_3\}$ with $e_1, e_2 \in E(D)$ and $e_3 \in E(G) - E(D)$. Then $Z^0 := Z - \{e_3\}$ is an edge cut of size 2 in $G^0 := G - e_3$.
(a) e_3 is a chord of G. Applying the considerations of case 1 to G^0, we obtain a partition $\{V_1^c, V_2^c\}$ of $G^c - e_3^c$ such that no edge of $G^c - e_3^c$ joins a $v_1^c \in V_1^c$ with a $v_2^c \in V_2^c$. That is $G^c - e_3^c$ is disconnected, which contradicts the first part of condition 2.
(b) e_3 is not a chord of G, i.e., it is incident to q_i of some $K_{1,3} \subset G - E(D)$. Set $K_{1,3} = \langle q_i, x_i, y_i, z_i \rangle$. W.l.o.g $e_3 = q_i z_i$. Then $Z^* := Z^0 \cup \{x_i^* z_i^*, y_i^* z_i^*\}$ is a cyclic edge cut of size 4 in $(G, D)^*$, and Z^0 is a cyclic edge cut (of size 2) in $G^* - \{x_i^* z_i^*, y_i^* z_i^*\}$. Consequently, by arguments similar to those of case 1 we conclude that $((G^* - x_i^* z_i^*)_{y_i^* z_i^*})^c$ is disconnected, i.e, $G^c - v_i - w_i$ is disconnected (where $v_i(w_i)$ corresponds to $(x_i^* z_i^*)^c$ $((y_i^* z_i^*)^c)$ in G^c). This contradicts the second part of condition 2 and finishes the proof of the lemma.

Theorem 2.6.2 *(J, D) is cyclically 4-edge-connected and has no 2dBMs.*

Proof: Consider the graph Y^c of Figure 2.8. There every vertex $v^c \in Y^c$ is illustrated without the upper index c. The dashed drawn edges form a minimal edge cut $\hat{E}$. Thus $Y^c - \hat{E}$ has precisely two components the larger of which is J^c. Set for reasons of simplicity $r_1^c := (x_1^* z_1^*)^c$, $r_2^c := (z_1^* y_1^*)^c$ and $r_3^c := (x_1^* y_1^*)^c$. Note that $\triangle_1$ is the only triangle of J^c which corresponds to (the unique) $K_{1,3}$ of (J, D). A straightforward check shows that J^c satisfies condition 1,2 of Lemma 2.6.1 implying that $\lambda_c(J) > 3$.

Suppose J^c has a $\triangle$-coloring f and suppose for $\triangle_1 \subset J^c$ that the coloring of $V(\triangle_1)$ is as indicated in Figure 2.8; i.e. $f(r_1^c) = a$, $f(r_2^c) = o$,$f(r_3^c) = b$.
By Theorem 2.5.3 and since α_1^c, α_2^c is a vertex cut, f must satisfy $f(\alpha_1^c) = f(\alpha_2^c)$, which yields the contradiction $d_{\langle A \cup C \rangle}(r_5^c) = 3$ or $d_{\langle B \cup C \rangle}(r_5^c) = 3$. The same contradiction is obtained if $f(r_1^c) = b$, $f(r_3^c) = a$.

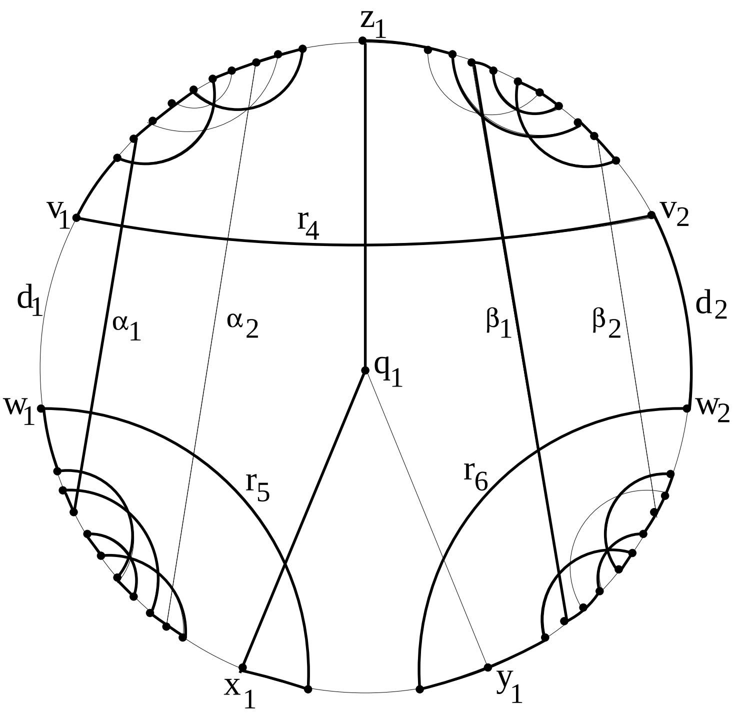


Figure 2.7: The graph J.

If $f(r_1^c) = o$, then by an argument symmetric to the case $f(r_2^c) = o$ (utilizing $f(\beta_1^c) = f(\beta_2^c)$) we obtain a contradiction w.r.t r_6^c.

Finally, if $f(r_3^c) = o$, then we obtain an analogouse contradiction w.r.t r_4^c by using both equations $f(\beta_1^c) = f(\beta_2^c)$, $f(\alpha_1^c) = f(\alpha_2^c)$.
Having obtained contradictions for all possible colorings of $\triangle_1$, we conclude that J^c has no $\triangle$-coloring, i.e., J has no 2dBMs. The theorem now follows.

2.7 Counterexamples to the BMC

Unfortunately, J is 3-edge-colorable because it is hamiltonian, see Figure 2.7. However, we extract the following from the second part of the proof of Theorem 2.6.2.

Proposition 2.7.1 *Suppose X^c is the circle graph of some cubic graph X such that $J^c \subset X^c$ and $d_{J^c}(r_i^c) = d_{X^c}(r_i^c)$, $i = 4, 5, 6$. Then there is no $\triangle$-coloring of X^c satisfying $f(\beta_1^c) = f(\beta_2^c)$, $f(\alpha_1^c) = f(\alpha_2^c)$.*

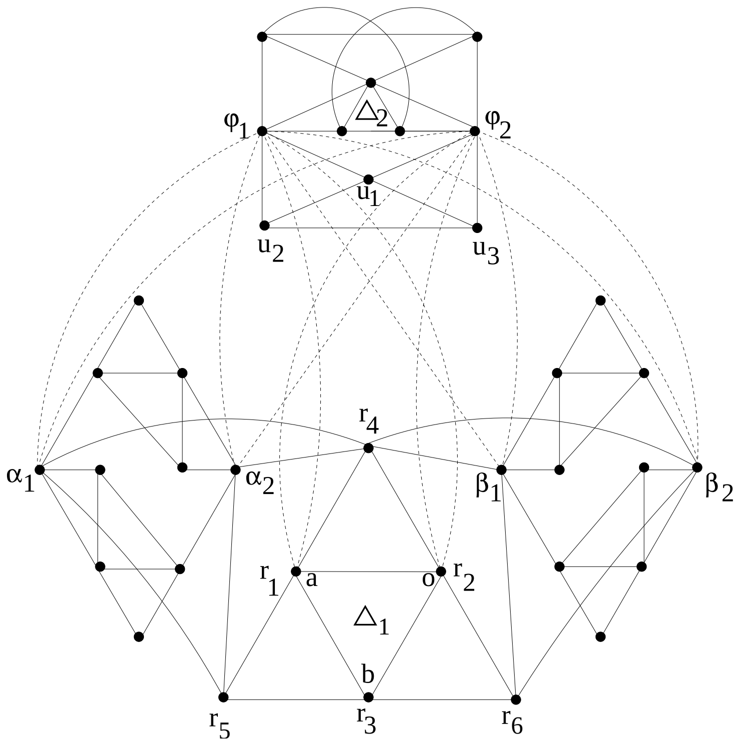


Figure 2.8: The graph Y^c and J^c.

Now we use Proposition 2.7.1 to construct a counterexample to the BMC. First, form

$$Y^- := (J - v_1w_1) \cup (P_{10} - l_1l_9) \cup \{v_1l_1, w_1l_9\}$$

where P_{10} is the Peterson graph (see Figure 2.9). Y^- is not 3-edge-colorable. Let P'_{10} be a copy of P_{10} and form the dot product [29] of Y^- with P'_{10} to obtain the final graph

$$Y := (Y^- - \{l_7l_3, v_2w_2\}) \cup (P'_{10} - \{l'_{10}, l'_9\}) \cup (\{l_3l'_3, l_7l'_6, l'_1v_2, l'_8w_2\})$$

; see Figure 2.9 (the edges in bold face illustrate the dominating circuit of (Y, D)) and Figure 2.10 which shows Y.

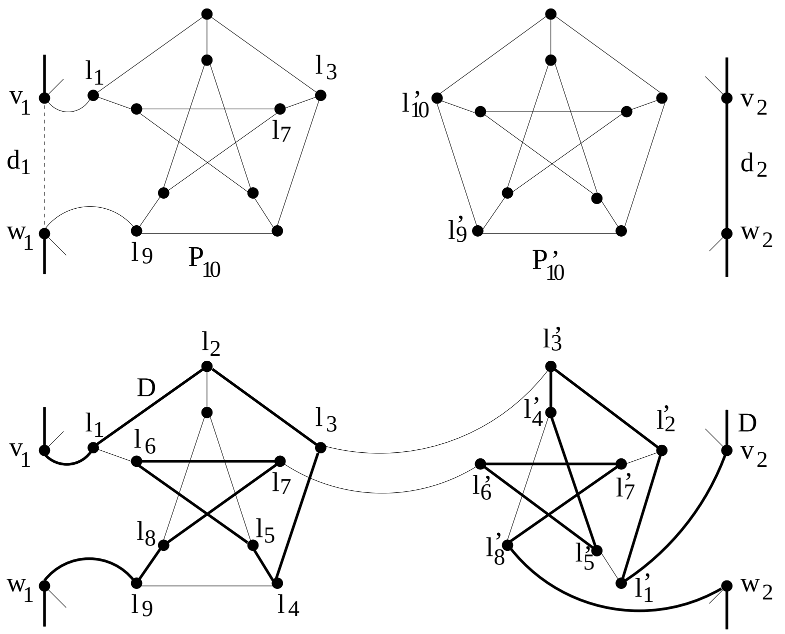


Figure 2.9: The transformation from the graph J into the graph Y.

Theorem 2.7.2 (Y, D) *is a counterexample to the BMC.*

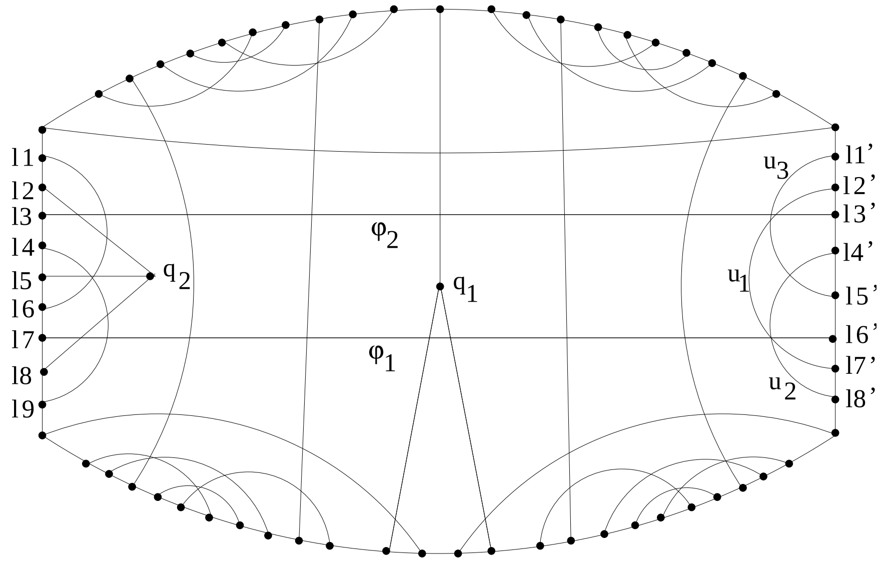


Figure 2.10: The graph (Y, D) without 2dBMs.

Proof: Y is, by construction, not 3-edge-colorable. Since it is straightforward to verify the validity of condition 1,2 of Lemma 2.6.1 for the graph Y^c (see Figure 2.8), we conclude that $\lambda_c(Y) > 3$. Looking again at Y^c in Figure 2.8 we observe that it contains two copies of H_2 (see Figure 2.6) which have only α_1^c and α_2^c, β_1^c and β_2^c respectively, in common with the rest of Y^c. Therefore, any $\triangle$-coloring of Y^c -if such coloring exists- must satisfy $f(\alpha_1^c) = f(\alpha_2^c)$, $f(\beta_1^c) = f(\beta_2^c)$ (Theorem 2.5.3). However, $d_{Y^c}(r_i^c) = d_{J^c}(r_i^c)$ for $i = 4, 5, 6$ implies that Y^c has no $\triangle$-coloring at all (Proposition 2.7.1). Therefore Y has no 2dBMs. The theorem now follows.

Remark 2.7.3 *Note that we can construct infinitely many counterexamples to the BMC with the aid of Prop. 2.7.1 (by extending Y). Moreover we can transform H_0, see Fig. 2.4 or some other (G, D) without 2dBMs into a counterexample of the BMC with the preceding methods which we used for the transformation of G_0 into Y.*

Y has no 2dBMs w.r.t. one given dominating circuit. Figure 2.11 shows Y with another dominating circuit D_1 and 2dBMs of (Y, D_1). The edges marked by an a from one BM. The other BM consists of the chords which are not marked at all and the edge (incident to q) which is marked by b. (One can see straightforward that each described BM is indeed a BM by verifying that Y_{BM} is bipartite.)

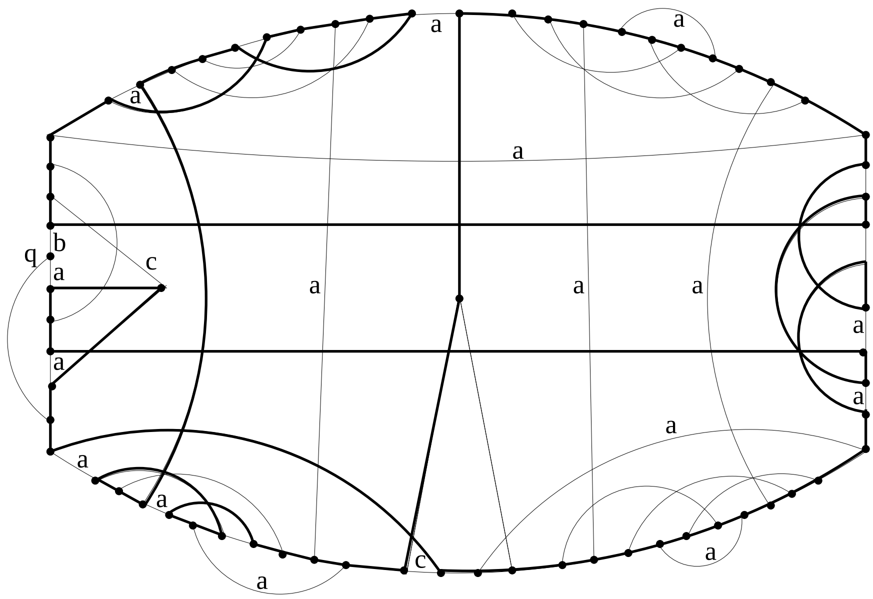


Figure 2.11: The graph (Y, D_1) with 2dBMs. The edges in bold face show D_1.

We suggest the following modified BMC:

Conjecture 2.7.4 *Every cubic cyclically* 4*-edge-connected graph* G *has at least one dominating circuit* D *such that* (G, D) *has 2dBMs.*

In contrast to the BMC we do not demand (G, D) to be a snark. We believe that if there exists a counterexample to Conjecture 2.7.4 then this counterexample can also be transformed to a snark, see Def. 5.1.2.

The truth of Conjecture 2.7.4 and the truth of the DCC would still prove the NZ5FC and the CDCC, see [9, 10]. However little is known about the existence and properties of dominating circuit in cubic cyclically 4-edge-connected graphs, which makes it hard to hope for a proof of the said conjectures via the DCC and the modified BMC.

We state a more promising conjecture than the above one which is very close to the BMC and which would imply that the CDCC holds for all cubic graphs which have a dominating circuit. We need to generalize some definitions.

Definition 2.7.5 *Define $\mathcal{W}$ to be the class of graphs with the following property. Every graph W of $\mathcal{W}$ is simple, connected, loopless and has a 2-factor T_W consisting only of triangles $\triangle_i$, $i = 1, 2, ..., |V(W)|/3$, which are connected to each other in the following manner. Every vertex of $V(\triangle_i)$ is adjacent to an even number of vertices of $V(\triangle_j)$, $1 \leq i, j \leq |V(W)|/3$. Only two configurations satisfy these conditions which are both illustrated in Figure 2.12.*

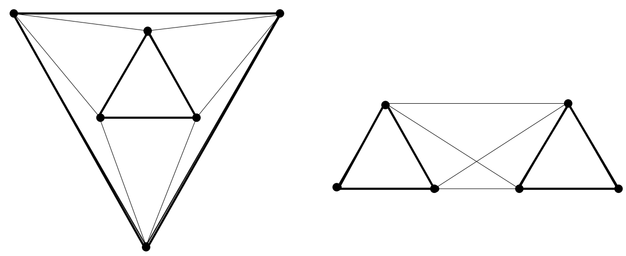


Figure 2.12: The triangles $\triangle_i$ and $\triangle_j$ are illustrated in bold face.

Definition 2.7.6 *Call a coloring f of $V(W)$ with $W \in \mathcal{W}$ with colors $\{a, o\}$ a BM-coloring if exactly one vertex of every triangle in T_W obtains color a (i.e two vertices of every triangle obtain color o) and $< f^{-1}(a) >$ is eulerian.*

Note that a BM-coloring of $(G, D)^c$ with (G, D) having no chords corresponds to a BM of (G, D) and vice versa.

Conjecture 2.7.7 *Let $W \in \mathcal{W}$. Then W has a BM-coloring with respect to T_W.*

We extend Def.2.3.3 in the following manner.

Definition 2.7.8 *Let H be an arbitrary graph with a 2-factor T_H consisting only of triangles. A coloring $f : V(H) \mapsto \{a, b, o\}$ is called a $\triangle$-coloring of H if every triangle of T_H is colored properly and $< f^{-1}(a) >$ and $< f^{-1}(b) >$ are eulerian.*

Conjecture 2.7.9 *Let $W \in \mathcal{W}$. Then T_W can be decomposed into the subgraph T_r consisting of red triangles and the subgraph T_b consisting of blue triangles such that $< V(T_r) >$ ($< V(T_b) >$) has a $\triangle$-coloring w.r.to T_r (T_b).*

Theorem 2.7.10 *If Conjecture 2.7.9 holds, then every (G, D) has a CDC.*

Proof: We may assume that (G, D) has no chords. Consider $(G, D)^c$ and $T_{(G,D)^c}$, then $(G, D)^c$ has by assuming the truth of Conjecture 2.7.9, a decomposition into T_r and T_b (with the described properties above). Denote the cubic graph which is obtained by deleting the $K_{1,3}$'s which correspond to T_b (T_r) by $(G, D)_r$ ($(G, D)_b$). Since $< T_r >$ and $< T_b >$ have a $\triangle$-coloring, $(G, D)_r$ and $(G, D)_b$ have 2dBMs. Since the existence of 2dBMs of a cubic graph G_1 with dominating circuit D_1 gives rise to a CDC which contains D_1, i.e. the circuit itself is element of the CDC of G_1, the union of such CDCs of $(G, D)_r$ and $(G, D)_b$ form a CDC of (G, D), which finishes the proof.

Chapter 3

Solving Cubic Graphs

3.1 B-sets

A bipartizing matching (BM) has been defined as a matching with a given restriction with respect to a dominating circuit D. We generalize this definition in the following manner:

Definition 3.1.1 *A generalized BM (gBM) in a cubic graph G is a matching M such that every component of $G - M$ has a nowhere-zero 3-flow and an even number of 2-valent vertices.*

Note that the deletion of a BM from the cubic graph (G, D) leads to a graph which consists of one component only which contains an even number of 2-valent vertices.

We apply several times the following well-known results, see [2].

Theorem 3.1.2 *A cubic graph G has a nowhere-zero 3-flow if and only if G is bipartite.*

Theorem 3.1.3 *A cubic graph G has a nowhere-zero 4-flow if and only if G is 3-edge colorable.*

Definition 3.1.4 *We say that a set S of graphs solves a cubic graph G or G is solvable by S if there is a matching M such that every component of $G-M$ is a subdivision of an element of S and has an even number of 2-valent vertices. Moreover we say that the graph H solves G if $S := \{H\}$ solves G.*

Definition 3.1.5 *Denote by* $[B]$ *the set of graphs which is the union of the set of* 2*-connected bipartite cubic graphs and* C_2 *(the circuit of length* 2*). We call a subset of* $[B]$*, a* B*-set.*

Proposition 3.1.6 *A cubic graph* G *has a gBM if and only if it is solvable by a* B*-set.*

Proof: A gBM induces a B-set and vice versa.

See Figure 3.1 for an example of the preceding definitions. The gBM is illustrated by dashed lines.

Note that every even circuit of length greater than two can be regarded as a subdivision of C_2.

Proposition 3.1.7 *A cubic graph* G *has a nowhere zero* 4*-flow if and only if* G *is solvable by* C_2*.*

Proof: If G has a nowhere zero 4-flow then G has a 3-edge coloring. The perfect matching M consisting of the edges of one color of the 3-edge coloring of G forms a gBM since $G-M$ consists only of even circuits. Hence C_2 solves G. If G is solvable by C_2, then M is a perfect matching and every circuit of $G-M$ is even. Hence G has a 3-edge coloring and thus a nowhere zero 4-flow.

Hence the question of finding a small B-set for cubic graphs with nowhere-zero 4-flow is not very interesting. However the definitions above motivate to search for a general structure in cubic graphs which do not have an even 2-factor. We pose the following questions:

1. Is every bridgeless cubic graph solvable by some B-set?

2. Is there a finite B-set which solves every cubic bridgeless graph?

3. Is there a "nice" infinite B-set which solves every cubic bridgeless graph?

Since for cubic graphs with nowhere-zero 4-flow we need only one bipartite graph to solve them, one might suspect that we do not need all bipartite graphs to solve the remaining bridgeless cubic graphs (see question 2. and 3.).

We need the following known lemma [27] for the next theorem which as a consequence will answer question 1. above in the affirmative..

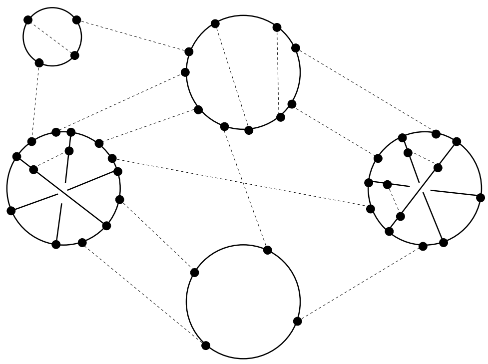


Figure 3.1: A cubic graph solved by the B-set $S := \{K_{3,3}, C_2\}$.

Lemma 3.1.8 *A cubic graph G has a nowhere-zero 6-flow if and only if there is a 2-flow f_2 and a 3-flow f_3 of G with $sup(f_2) \cup sup(f_3) = E(G)$.*

Recall that *sup* denotes the support of a flow.

Theorem 3.1.9 *A cubic graph G has a generalized BM if and only if G has a nowhere-zero 6-flow.*

Proof: Let G have a gBM which is denoted by M. Since every component of $G - M$ is a subdivision of a bipartite graph or an even circuit itself, $G - M$ has a nowhere-zero 3-flow denoted by g. Define a 3-flow, f_3, of G by setting

$$f_3(e) := g(e) \ \ \forall e \in G - M$$

and

$$f_3(e) := 0 \ \ \forall e \in M\,.$$

We construct a 2-flow of G by finding a cycle which contains M in order to apply the above Lemma. Denote the components of $G - M$ by $L_1, L_2, ..., L_k$, $k \in \mathbb{N}$.

Moreover denote the subset of edges of M where each edge has only one endvertex in L_i, $i \in \{1, 2, ..., k\}$ by M', and denote the set of endvertices of M' in L_i by $V'(L_i)$, $i = 1, 2, ..., k$.

Since $V'(L_i)$ has even order (by definition of a gBM), say l_i, we can partition $V'(L_i)$ into disjoint pairs, say $\{v_1^i, w_1^i\}, \{v_2^i, w_2^i\}, ..., \{v_{l_i/2}^i, w_{l_i/2}^i\}$. Since L_i is connected, there is a path from v_j^i to w_j^i which we denote by P_j^i with $j \in \{1, 2, ..., l_i/2\}$. Set $P^i := P_1^i \triangle P_2^i ... \triangle P_{l_i/2}^i$, $i = 1, 2, ..., k$. Then

$$C := M' \cup P^1 \cup P^2 ... \cup P^k$$

is a cycle in G containing M'.

Set $M_i := \{e \in M - M' \mid \text{both endvertices of } e \text{ are in } L_i\}$ and set $|M_i| = m_i$, $i \in \{1, 2, ..., k\}$. Let $e \in M_i$. Then $L_i \cup e$ has a circuit which contains e which we denote by C_e^i.

Define the cycle $C^i := C^i_{e^1} \triangle C^i_{e^2} ... \triangle C^i_{e^{m_i}}$ where $M_i = \{e^1, e^2, ..., e^{m_i}\}$. Hence

$$C' := C^1 \cup C^2 ... \cup C^k$$

is a cycle in G which contains $M - M'$.

Since C' and M' are edge-disjoint and since C and $M - M'$ are edge-disjoint, $C \triangle C'$ is a cycle in G containing M. Define a 2-flow, f_2, of G by setting

$$f_2(e) := 1 \ \ \forall e \in C \triangle C'$$

and

$$f_2(e) := 0 \ \ \forall e \notin C \triangle C' .$$

Since $E(G) - M \subset sup(f_3)$, $M \subset sup(f_2)$ and thus $sup(f_2) \cup sup(f_3) = E(G)$, G has a nowhere-zero 6-flow by the previous Lemma.

Let G have a nowhere-zero 6-flow. By the previous Lemma, G has a 2-flow g_2 and a 3-flow g_3 with $sup(g_2) \cup sup(g_3) = E(G)$. Since G is cubic, $sup(g_2)$ is a cycle which we denote by C. Moreover, set $H := sup(g_3)$.
Consider $H \subset G$. Then $V(H) = V(G)$ since $E(H) \cup E(C) = E(G)$ and since C cannot contain all three edges incident with one vertex.
Set $M := E(G) - E(H)$, then M is a matching in G; (otherwise there would be two edges e_1, e_2 in M incident with the vertex, say v_0). Since $V(H) = V(G)$, the remaining edge e_3 incident with v_0 must be contained in $E(H)$. But in this case H is not bridgeless (v_0 has degree 1 in H) which is a contradiction to the fact that H has a nowhere-zero 3-flow.

We claim that M is a gBM. Since $G - M = H$ and since every component H' of H has a nowhere-zero 3-flow and is thus a subdivision of a bipartite cubic graph or a circuit, it remains to show that H' has an even number of 2-valent vertices. Suppose by contradiction that a component of $G - M$, say L, has an odd number of 2-valent vertices. Then it is also incident with an odd number of edges of M which have only one endvertex in L. Let M'_L denote this set of edges. Since $E(C) \cup E(H) = E(G)$ and since M'_L is edge-disjoint from $E(H)$, they must be contained in $E(C)$. Hence M'_L forms an odd edge-cut in G, which is a contradiction since an odd edge-cut cannot be contained in a cycle. Hence the theorem is proven.

Theorem 3.1.10 *Every bridgeless cubic graph is solvable by a B-set.*

Proof: Since every bridgeless cubic graph has a nowhere-zero 6-flow, see [27], the theorem follows by applying the previous theorem.

Thus the original question 1. has been answered. Denote the Petersen graph by P_{10}.

Proposition 3.1.11 *Let* Θ_i, $i = 1, 2, 3$ *be the three graphs illustrated in Figure 3.2. Then each* Θ_i, $i \in \{1, 2, 3\}$ *solves* P_{10} *and no other B-set of order one, solves* P_{10}.

Proof: Each component of $P_{10} - gBM$ (by Definition 3.1.1) is 2-connected. Thus it must contain at least one circuit. P_{10} contains only two disjoint 2-connected subgraphs, namely two circuits of length 5. Since a circuit of length 5 cannot be a component of $P_{10} - gBM$ (by Definition 3.1.1), $P_{10} - gBM$ is connected. Since C_2 is not solving P_{10} (since P_{10} is not 3-edge-colorable), P_{10} can only be solved by a cubic graph which we denote by H.

Since P_{10} is not bipartite, $|V(H)| \leq 8$. Suppose that $|V(H)| = 8$, then there must be an edge e_0 such that $P_{10} - e_0$ has a nowhere-zero 3-flow. Since this is not the case which is easy to verify since P_{10} is edge-transitive, $|V(H)| \leq 6$.

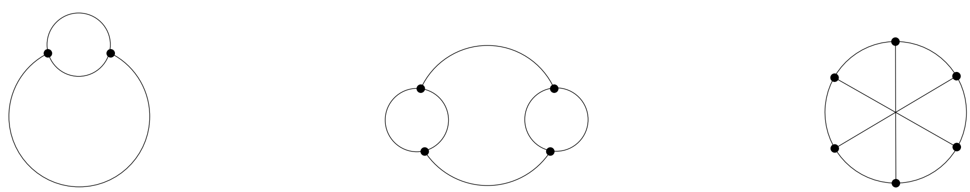


Figure 3.2: Θ_i, $i = 1, 2, 3$.

There is exactly one simple bipartite cubic graph of order 6, namely $K_{3,3}$ which solves P_{10}, see Figure 3.3 on the right side. Consider the case that H is bipartite, not simple and of order 6. There are exactly two graphs which fulfill these conditions. Both of them contain two disjoint C_2's. Since every C_2 in them would correspond to a circuit of length at least 5 in P_{10}, the order of P_{10} would be greater than 10. Hence $K_{3,3}$ is the only graph of order 6 which solves P_{10}.

The remaining cubic bipartite graphs of order at most 4 solve both P_{10} (see Figure 3.3) which finishes the proof.

Corollary 3.1.12 *Every B-set which solves every bridgeless cubic graph must contain at least one* Θ_i, $i \in \{1, 2, 3\}$.

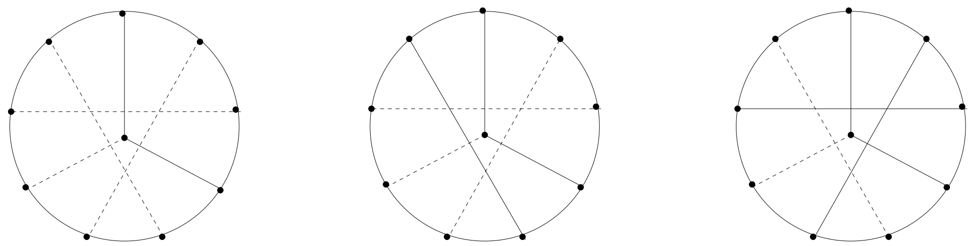


Figure 3.3: The Petersen graph solved by Θ_i, $i = 1, 2, 3$. The gBM is illustrated by dashed lines.

Corollary 3.1.13 *For every gBM M of P_{10}, $P_{10} - M$ is connected.*

The next three theorems do not only hold for B-sets but also for F-sets (see the next section) which form a generalization of B-sets. Therefore their proofs are contained in the next section.

Theorem 3.1.14 *For every finite B-set S there are infinitely many 3-connected cubic graphs which are not solvable by S.*

Theorem 3.1.15 *Let S be the B-set which contains all hamiltonian graphs of $[B]$. Then there are infinitely many 3-connected cubic graphs which are not solvable by S.*

For the understanding of the theorem below we need the following definitions.

Definition 3.1.16 *The length of a path α is denoted by $|\alpha|$. Let G be a graph and H_i, $i = 1, 2$ a subgraph of G or a subset of $V(G)$. Denote by $\mathcal{P}(H_1, H_2)$ the set of all paths with connect a vertex of H_1 with a vertex of H_2. Then the distance $d_G(H_1, H_2)$ or in short $d(H_1, H_2) := \min_{\alpha \in \mathcal{P}(H_1,H_2)} |\alpha|$.*

Definition 3.1.17 *Let G be a graph. Denote by $U(G)$ the set of all circuits of G. Define $l(G) := \min_{C \in U(G)} \{ \max_{v \in V(G)} d(C, v)\}$. Let S be a set of graphs. Define $l(S) := \max_{G \in S} l(G)$ if this maximum exists; otherwise set $l(S) := \infty$.*

Theorem 3.1.18 *Let S be a B-set which solves every 3-connected cubic graphs G, then $l(S) = \infty$.*

3.2 F-Sets

The concept of a frame has been introduced by Goddyn [13] and investigated by Häggkwist and Markström, see [14]. We extend the definition of a frame to F-sets.

Definition 3.2.1 *A matching M of a cubic graph G is called an f-matching if every component of $G - M$ is 2-connected and has an even number of 2-valent vertices.*

Definition 3.2.2 *Denote by $[F]$ the set of graphs which is the union of the set of 2-connected cubic graphs and C_2 (the circuit of length 2). We call a subset of $[F]$ an F-set.*

Note that by definition every B-set is an F-set and every gBM is an f-matching whereas the converse needs not to be true. As an analogue of Proposition 3.1.6 we have the following.

Proposition 3.2.3 *A cubic graph G has an f-matching M if and only if G is solvable by an F-set.*

Note that $M = \emptyset$ may also be an f-matching and that a cubic graph which contains a bridge has no f-matching and is thus not solvable. We need the following lemmas for proving the generalized versions of Theorems 3.1.14, 3.1.15 and 3.1.18.

Notation. Set $P := P_{10} - z$, $z \in V(P_{10})$ where P_{10} denotes the Petersen graph.

Lemma 3.2.4 *For every f-matching M of P_{10}, $P_{10} - M$ is 2-connected.*

Proof: P_{10} contains exactly two disjoint 2-connected subgraphs, namely two circuits of length 5. The definition of an f-matching thus implies that $P_{10}-M$ is connected.

Lemma 3.2.5 *Suppose a cubic graph G has a minimal 3-edge cut E_0. Then for every f-matching M of G, $|M \cap E_0| \in \{0, 1\}$.*

Proof: Suppose $|M \cap E_0| = 3$. Let L be a component of $G - E_0$ which contains an odd number of 2-valent vertices. Then $L - (M - E_0)$ contains at least one component which has an odd number of 2-valent vertices, in contradiction to Def. 3.2.1.

Suppose $|M \cap E_0| = 2$. Then one edge of E_0 is a bridge in a component of $G - M$ which contradicts Def. 3.2.1. Hence the proof is finished.

Lemma 3.2.6 *Let $E_0 := \{e_1, e_2, e_3\}$ be a minimal 3-edge cut in a bridgeless cubic graph G such that P is one component of $G - E_0$. Then for every f-matching M of G the following is true.*

1. Consider $P \subseteq G$ as a graph and M restricted to P. Then $P - M$ is connected.

2. $G - M$ contains a 3-valent vertex within $V(P)$, i.e. at least one vertex of $P \subseteq G$ is not matched by M.

Proof: Let $W_0 := \{w_1, w_2, w_3\}$ denote the set of the 2-valent vertices of P and let e_i be incident with w_i, $i = 1, 2, 3$. Since E_0 is a 3-edge cut and by Lemma 3.2.5, $|M \cap E_0| \in \{0, 1\}$.

Proof of the first statement:

Case 1. $|M \cap E_0| = 0$.
All w_i's are contained in the same component, say L, of $P - M$ since otherwise one component of $G - M$ would have e_i, $i \in \{1, 2, 3\}$ as a bridge in contradiction to Def. 3.2.1. Suppose by contradiction that $v' \in V(P)$ is in a different component than L, say L'. Since $V(L') \cap W_0 = \emptyset$ and by Def. 3.2.1, $L' \subseteq P$ must be 2-connected and must therefore contain a circuit. There is exactly one circuit C' in P which contains no vertex of W_0, see Figure 3.4 in which case w_i, $i = 1, 2, 3$ is a bridge in $G - M$ contradicting Def. 3.2.1. Hence $P - M$ is connected.

Case 2. $|M \cap E_0| = 1$. Let w.l.o.g. $M \cap E_0 = e_3$.
Then w_1 and w_2 are contained in the same component, say L, of $P - M$ (otherwise one component of $G - M$ has a bridge consisting of e_i, $i \in \{1, 2\}$). Suppose by contradiction that $v' \in V(P - M)$ is in a different component than L, say L'. Since e_3 is matched and $w_i \in L$, $i = 1, 2$ there is in $G - M$ no to L vertex disjoint path connecting $V(L')$ with a vertex $v \notin P - M$. Hence Def. 3.2.1 implies that $L' \subseteq P - M$ is 2-connected. Hence L' contains a circuit C'. Since $L \subseteq P$ is connected it contains a path β (which is vertex-disjoint with C') connecting w_1 with w_2. P_{10} is obtained from $P \cup E_0$ by identifying the three endvertices of e_i, $i = 1, 2, 3$ (which are not in P). Then β corresponds to a circuit in P_{10}. Since P_{10} contains only two disjoint circuits L' can only be a circuit of length 5 which contradicts Def. 3.2.1.

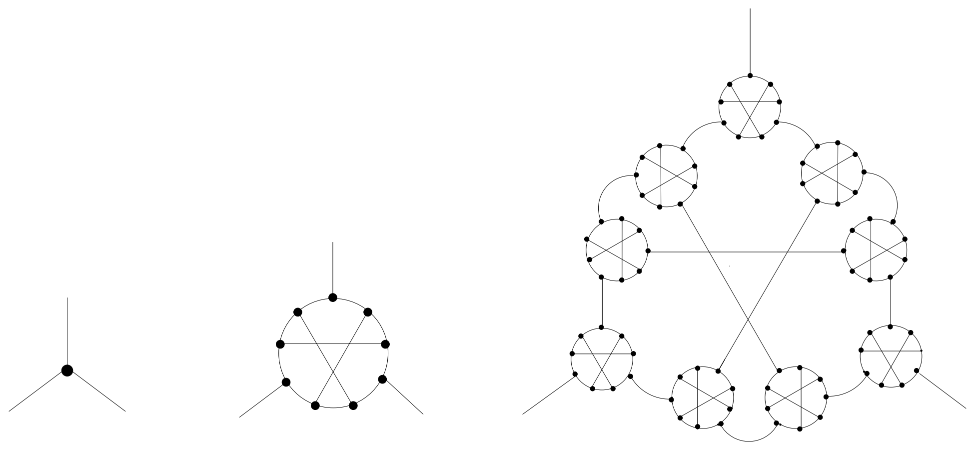
Proof of the second statement:

Suppose by contradiction that every vertex of P is matched by M. Since $|V(P)|$ is odd, $|E_0 \cap M| = 1$. Such matching M covering $V(P)$ corresponds to a perfect matching of P_{10}. Hence, $P - M$ would consist of a path and a circuit of length 5 which contradicts Def. 3.2.1.

Lemma 3.2.7 *Let G, E_0 and P be as in the previous lemma. Let α be a path in G which passes through P, i.e. $< E(\alpha) \cap E(P) >$ is a path in P and $|E(\alpha) \cap E_0| = 2$. Then for every f-matching with $E(\alpha) \cap E(M) = \emptyset$ the following is true: $G - M$ contains a 3-valent vertex within $V(\alpha) \cap V(P)$, i.e. at least one vertex of $V(\alpha) \cap V(P)$ is not matched by M.*

Proof: Suppose by contradiction that every vertex of $V(\alpha) \cap V(P)$ is matched by M. Then $\alpha \cap P$ is a component of $P - M$ and thus by the previous lemma the only component of $P - M$. Since $\alpha \cap P$ contains no 3-valent we obtain a contradiction to the previous lemma which finishes the proof.

Definition 3.2.8 *Let G be a graph with $d(v) \leq 3 \forall v \in V(G)$. A vertex v in G is replaced by P by deleting v, adding P and connecting each former neighbor of v to one distinct 2-valent vertex of P. We call this operation a P-inflation of G at v. Moreover, $G^0, G^1, G^2, ..., G^k$ with $k \in \mathbb{N}$ and $G^0 := G$, is defined to be the sequence of graphs where G^i, $i \in \{1, 2, ..., k\}$ results from G^{i-1} by applying the P-inflation to every vertex in G^{i-1}; see Figure 3.4 for an example.*



G $\quad$ G^1 $\quad$ G^2

Figure 3.4: A vertex in a cubic graph G and the corresponding P-inflations in G^1 and G^2.

Definition 3.2.9 *Let $X_1, X_2, ..., X_m$, $m \in \mathbb{N}$, be vertex disjoint subgraphs in a loopless graph G. Then $G/X_1, X_2, ..., X_m$ denotes the graph which is obtained from G by contracting every X_i, $i = 1, 2, ..., m$ to a distinct vertex and by removing all loops arising by this contraction procedure.*

Lemma 3.2.10 *Let G be a loopless 2-edge connected graph and X_i, $i = 1, 2, ..., m$ defined as above. Then $G/X_1, X_2, ..., X_m$ is also 2-edge connected.*

Proof: The lemma follows from the the observation that the contraction of an edge in G or the identification of two of its vertices does not increase the numbers of bridges in G.

Lemma 3.2.11 *For every f-matching M of P_{10}^k, $k \in \mathbb{N}$, $P_{10}^k - M$ is 2-connected.*

Proof: Write Q for P_{10}^k and Q' for P_{10}^{k-1}, $k > 0$.

$M \subseteq E(Q)$ corresponds to an edge set $M' \subset E(Q')$, where $|M'| \leq |M|$ since M may also contain edges in copies of P's which correspond to vertices of Q'. By Lemma 3.2.5, at most one edge of M is incident with a 2-valent vertex of a copy of $P \subset Q$. Hence M' is a matching of Q'.

We show that M' is an f-matching of Q'.

Let $\dot{P}_1, \dot{P}_2, ..., \dot{P}_z$, $z \in \mathbb{N}$ be the disjoint subgraphs in Q isomorphic to P. Then

$$Q' - M' = (Q - M)/\dot{P}_1 - M, \dot{P}_2 - M, ..., \dot{P}_z - M \ .$$

Therefore and by Lemma 3.2.10 and since every component of $Q - M$ is, by the choice of M, 2-connected, every component of $Q' - M'$ is 2-connected.

Thus it remains to show that every component of $Q' - M'$, say L', has an even number of 2-valent vertices. Since $\dot{P}_i - M$, $i = 1, 2, ..., z$ is connected (by Lemma 3.2.6), every component, say L, of $Q - M$ corresponds to a unique component, say L', of $Q' - M'$. Therefore and since $M - M'$ is contained in the edge set of the subgraphs (of Q) isomorphic to P and since they satisfy by Lemma 3.2.6 that $\dot{P}_i - M$ is connected, the number of all 2-valent vertices in L and L' have the same parity. Therefore and since L has an even number of 2-valent vertices by assumption, M' is an f-matching of Q'.

The previous considerations imply that $Q - M$ and $Q' - M'$ have the same number of components. Hence, by induction on k and by Lemma 3.2.4, the proof is finished.

Theorem 3.2.12 *For every finite F-set S, there are infinitely many 3-connected cubic graphs which are not solvable by S.*

Proof: Let M be an f-matching of P_{10}^k, $k \in \mathbb{N}$. $P_{10}^k - M$ is 2-connected by Lemma 3.2.11. Moreover, by Lemma 3.2.6 every induced subgraph of P_{10}^k which is isomorphic to P has at least one vertex which is 3-valent in $P_{10}^k - M$. Therefore P_{10}^k is not solvable by S if the number of disjoint copies of P in P_{10}^k is greater than the maximum order of all graphs in S. Hence if k is large enough, P_{10}^k is not solvable by S which finishes the proof.

Definition 3.2.13 *Let G be a cubic graph and H a graph where every component is either a 2-connected cubic graph or a C_2. Then H is called a frame of G if and only if there is an f-matching M of G such that $G - M$ is a subdivision of H.*

Note that if H is a frame of G, then every component of H is in $[F]$ and the set of the components of H solves G. It is conjectured in [14] that every 3-connected cubic graph can be solved by certain hamiltonian graphs. With the help of Theorem 3.2.17 we disprove this conjecture. To achieve this we need some more notation and a lemma.

Definition 3.2.14 *Let G be a graph and H_i, $i = 1, 2$, a subgraph of G or a subset of $V(G)$ and α a path in G. Denote by $p(\alpha)$ the number of copies of P in G with which α has a non-empty vertex-intersection. Moreover set $p(H_1, H_2) := \min_{\alpha \in A}\{p(\alpha)\}$ where A is the set of all paths which connect H_1 with H_2.*

Definition 3.2.15 *Denote by $a_k := \max_{v \in P^k} p(W_k, v)$ where W_k denotes the set of the three 2-valent vertices of P^k.*

Lemma 3.2.16 *Let a_k be defined as above, then $a_k < a_{k+1}$ and $a_0 = 1$.*

Proof: For $k = 0$, $a_0 = 1$ since $P^0 = P$ and $p(\alpha) = 1$ for every path α in P^0.

Let α_n be a minimal path w.r.to $p(\alpha_n)$ connecting W_n with a vertex $x_n \in P^n$ such that $p(\alpha_n) = a_n$. Every copy of P in P^{n+1} corresponds to a vertex in P^n. Denote by x_{n+1} an arbitrary vertex in a copy of P in P^{n+1} which corresponds to $x_n \in V(P^n)$. Then there is a path α_{n+1} in P^{n+1} connecting W_{n+1} with x_{n+1} such that

$$p(\alpha_{n+1}) \leq a_{n+1}\,. \qquad (1)$$

The path α_{n+1} corresponds to a path β_n in P^n with

$$|\beta_n| + 1 = p(\alpha_{n+1})\,. \qquad (2)$$

Since β_n connects W_n with x_n and since α_n is minimal,

$$p(\alpha_n) \leq p(\beta_n)\,. \qquad (3)$$

Since one endvertex of β_n is a 2-valent vertex of P^n, not every vertex of β_n can be in a distinct copy of P in P^n. Hence

$$p(\beta_n) < |\beta_n| + 1\,. \qquad (4)$$

Thus, by (1), (2), (3) and (4), $a_{n+1} > a_n$.

Theorem 3.2.17 *Let S be an F-set which solves every 3-connected cubic graphs G, then $l(S) = \infty$.*

Proof: Denote by $T(k)$ the set of all frames of P_{10}^k, $k > 0$. Lemma 3.2.11 implies that every element of $T(k)$ is cubic and 2-connected. We show that

$$\min_{T \in T(k)} \{l(T)\} \to \infty$$

as k tends to ∞, which proves the theorem since S contains at least one $T \in T(k)$ for every k.

Set $Q := P_{10}^k$. Consider the ten disjoint induced subgraphs, say X_i, $i = 1, 2, ..., 10$ of Q isomorphic to P^{k-1} or v^k (by regarding v as the trivial graph consisting of one vertex). Then each of them contains three 2-valent vertices and they satisfy

$$Q/X_1, X_2, ..., X_{10} = P_{10}\ . \qquad (2)$$

Let M be an f-matching of Q. Denote the 2-connected cubic graph which is homeomorphic to $Q - M$ by $\overline{Q}(k)$. Suppose that M is chosen in such a way that $l(\overline{Q}(k))$ is minimal.

A subgraph (vertex set) of $\overline{Q}(k)$ is denoted by $\overline{H}$, say, and the corresponding graph (vertex set) in Q and $Q - M$ by H, and vice versa. Moreover, denote the set of 3-valent vertices of a graph J by $V_3(J)$.

Let $\overline{C}$ be a circuit of $\overline{Q}(k)$ such that $\max\limits_{v \in \overline{Q}(k)} d_{\overline{Q}(k)}(\overline{C}, v) = l(\overline{Q}(k))$. C does not pass through every X_i since otherwise (2) would imply that P_{10} is hamiltonian. Let us denote one distinct X_i for which $V(X_i) \cap V(C)) = \emptyset$ by X.

That is, $\overline{V}_3(X-M)\cap V(\overline{C})=\emptyset$. Set

$$b_k := \max_{x\in\overline{V}_3(X-M)} d_{\overline{Q}(k)}\left(\overline{C},x\right) .$$

By definition of $l(\overline{Q}(k))$ and b_k, we have $b_k \leq l(\overline{Q}(k))$.

Denote by W_{k-1} the vertex set in X corresponding to W_{k-1} in $X = P^{k-1}$ (for W_{k-1}, see Definition 3.2.15). By Lemma 3.2.7, every path $\overline{\alpha}$ in $\overline{X}$ contains a 3-valent vertex in $\overline{X}$ for every copy of P which α passes through in X. Since, by Lemma 3.2.6, every copy of $P \subseteq X$ contains a 3-valent vertex of $\overline{X}$, $b_k \geq \max_{v\in P^{k-1}} p(W_{k-1},v)-2 = a_{k-1}-2$. Since $l(\overline{Q}(k) \geq b_k \geq a_{k-1}-2$ and since by Lemma 3.2.16, a_k tends to infinity, the proof is finished.

Corollary 3.2.18 *Let S be the F-set consisting of all hamiltonian graphs of $[F]$, then S does not solve every 3-connected cubic graph.*

Proof: Since $l(G)=0$ for every $G\in S$ we have $l(S)=0$. The Corollary now follows by the preceding theorem.

PART 2

Chapter 4

Mosaics

For standard terminology which is not defined here, we refer to [2, 5]. The graphs which we consider are of finite order and may have multiple edges and loops if not otherwise stated. We say a vertex v is incident with a face F if v is a vertex of the facial cycle of F. A quadrangle is a face with a facial cycle consisting of 4 distinct edges and 4 distinct vertices. A quadrangulation is a plane graph where every face is a quadrangle. When we speak of a coloring of a quadrangle, we always mean a coloring of the vertices of the quadrangle. A triangle is a face with a facial cycle consisting of 3 distinct edges and 3 distinct vertices. A snark is defined to be a non-3-edge-colorable cubic graph. All colorings which are considered are proper colorings if not otherwise stated. A trail is a walk where no edge appears more than once. An orientation of a graph G is denoted by $\overrightarrow{G}$. Note that we also regard $\emptyset$ as a cubic graph.

4.1 Wild maps and mosaics

We introduce a generalization of *C-simple* 4*-regular maps*, which we call "wild maps". C-simple 4-regular maps are plane 4-regular graphs which are constructed by superposition of simple closed curves (tangencies are not allowed); for details see [6, 19].

A *wild map* of a cubic graph G is, roughly speaking, obtained by drawing G in the plane with possible crossings, by adding some closed curves and transforming the crossings (which need to be all transversal) into additional vertices. For an illustration, see Figure 4.1.

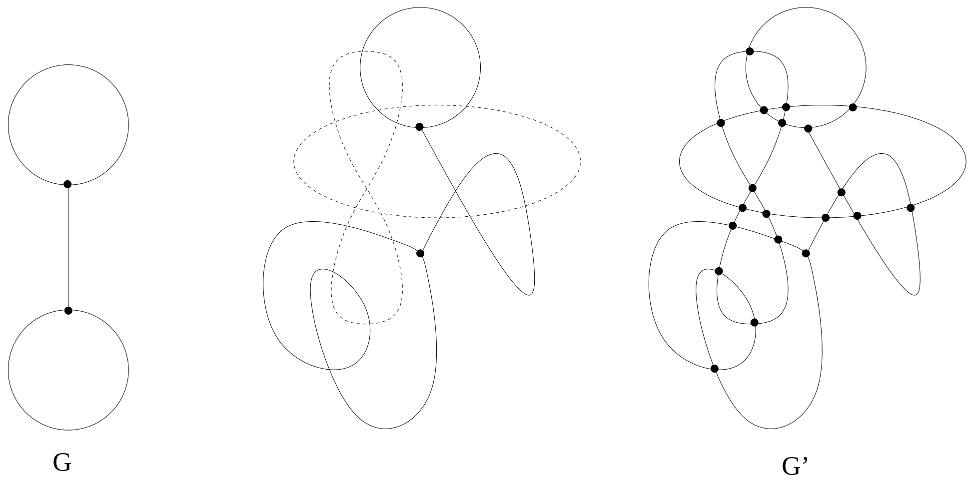


Figure 4.1: Producing a wild map G' (on the right side) of a cubic graph G (on the left side). The mt-trails which contain no 3-valent vertices are illustrated (in the middle) by dashed lines.

To be more precise we make the following definitions.

Definition 4.1.1 *Let $\mathcal{H}_{34}$ denote the class of graphs where every member, say H_{34}, is a plane connected graph which satisfies:*
1. Every vertex is 3- *or* 4*-valent.*
2. Every vertex v is incident with $d(v)$ distinct faces.

Moreover we call the 4*-valent vertices in H_{34} cross-vertices and we set*
$V_i := \{v \in V(H_{34}) \,|\, d(v) = i,\ i \in \{3, 4\}\}$.

Definition 4.1.2 *Call two edges e_1, e_2 which are incident with a vertex v of a graph in $\mathcal{H}_{34}$, opposite if no face incident with v contains e_1 and e_2 in its facial cycle.*

Hence, for $H_{34} \in \mathcal{H}_{34}$, no two edges incident with a 3-valent vertex are opposite and every cross-vertex contains 2 disjoint pairs of opposite edges. Call a trail in H_{34} *transversal* if all pairs of consecutive edges in it are opposite. Then the maximal transversal trails, abbreviated by *mt-trails*, decompose uniquely $E(H_{34})$ by passing through every cross-vertex in a transversal manner. Let us call two (not necessarily distinct) vertices $v, w \in V_3 \subseteq V(H_{34})$ to be *trans-adjacent* if there is an mt-trail S such that $S \cap V_3 = \{v, w\}$. Note that for every mt-trail S, $|S \cap V_3| \in \{0, 1, 2\}$.

Definition 4.1.3 *Call $H_{34} \in \mathcal{H}_{34}$ a wild map of a cubic graph G if there is a bijection between $V(G)$ and $V_3 \subseteq V(H_{34})$ such that two (not necessarily*

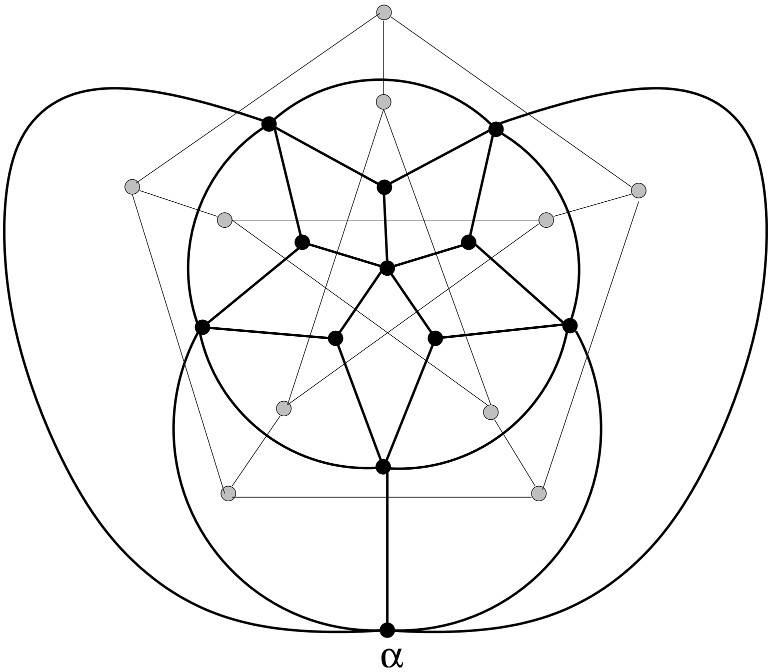


Figure 4.2: A mosaic X, illustrated in bold face, of the Petersen graph P_{10}.

distinct) vertices in G are adjacent if and only if the corresponding vertices in V_3 are trans-adjacent. Then we denote H_{34} by G'.

From the above definition it follows that every mt-trail in G' which contains a 3-valent vertex corresponds to an edge in G. Note that condition 2 in Definition 4.1.1 implies that G' is 2-connected and loopless whereas G may even be disconnected.

Lemma 4.1.4 *Every $H_{34} \in \mathcal{H}_{34}$ is a wild map of a unique cubic graph G.*

Proof: Form the unique cubic graph G from $V(H_{34})$ by letting $V(G) = V_3 \subseteq V(H_{34})$ and by connecting two vertices by an edge if and only if they are trans-adjacent.

Definition 4.1.5 *A mosaic is a connected plane graph X where every face is a triangle or a quadrangle.*

Definition 4.1.6 *Let G' be a wild map of a cubic graph G. Then the dual graph of G' is a mosaic which we call, in particular, a mosaic of G.*

See Figure 4.2 for an illustration of a mosaic X of the Petersen graph P_{10}. There X is constructed from the classical drawing of P_{10}, by transforming the cross-vertices into vertices to obtain a wild map P'_{10} and by forming the dual graph of P'_{10}.

Since the dual graph of a mosaic is a graph contained in $\mathcal{H}_{34}$ and since this graph is by Lemma 4.1.4 a wild map of a unique cubic graph, we obtain the following proposition.

Proposition 4.1.7 *Every mosaic corresponds to a unique cubic graph.*

Hence every mosaic can be regarded as a mosaic of a special cubic graph; see Definition 4.1.6.

Since there are infinitely many wild maps of a cubic graph G there are also infinitely many mosaics of G. Note that a mosaic of $G = \emptyset$, is a quadrangulation since a wild map of $G = \emptyset$ is a plane 4-regular graph.

4.2 Nowhere-zero flows and mosaics

Nowhere-zero integer flows can be regarded as a natural extension of coloring plane graphs according to Tutte's Theorem [28] which states that every nowhere-zero k-flow of a plane graph G implies a proper face k-coloring of G and vice versa. The next theorem states nowhere-zero flow problems of possibly non planar graphs as special planar problems.

Definition 4.2.1 *We define that the cubic graph $G = \emptyset$ has a 3-edge coloring and a nowhere-zero A-flow for every abelian group A of order $k > 1$.*

Theorem 4.2.2 *Let X be a mosaic of a cubic graph G and let $(A, +)$ be an additive abelian group of order $k > 1$. Then G has a nowhere-zero A-flow if and only if X has a vertex coloring $q : V(X) \mapsto A$ such that every quadrangle satisfies*

$$q(x_1) + q(x_3) = q(x_2) + q(x_4) \tag{4.2.1}$$

where x_1, x_2, x_3, x_4 denotes the vertices of the quadrangle in cyclic order.

Proof: We first prove the second part of the theorem, i.e. we assume that X has the above vertex coloring. If X is a quadrangulation, then the cor-

responding cubic graph $G = \emptyset$ and has by definition a nowhere-zero A-flow. Hence we suppose that X is a mosaic which contains triangles.
The dual graph of X is a wild map of G and will be denoted by G'. The coloring q of $V(X)$ induces by duality a proper face coloring of G' which we denote by q'. Let us orient every mt-trail in G' according to one of its two possible traversals . Since the set of mt-trails decompose $E(G')$, we obtain an orientation of G' which we denote by $\overrightarrow{G'}$. Since every mt-trail which contains a 3-valent vertex corresponds to an edge in G, $\overrightarrow{G'}$ also induces an orientation of G which we denote by $\overrightarrow{G}$.

Assign to every arc $\vec{e}$ of $\overrightarrow{G'}$ the A-flow value,

$$f(\vec{e}) := q'(F) - q'(\hat{F}) \ ,$$

where F and $\hat{F}$ are the faces of $\overrightarrow{G'}$ incident with $\vec{e}$ (i.e $\vec{e}$ is an arc of the facial cycle of F and $\hat{F}$), and F $(\hat{F})$ is on the left (right) side of $\vec{e}$. Then, f is a nowhere-zero A-flow of $\overrightarrow{G'}$.

By applying the definition of f towards opposite (Definition 4.1.2) arcs $\vec{e_1}$, $\vec{e_2}$ which are incident with a cross-vertex $v \in V(\overrightarrow{G'})$ and by using the dual version of equation (4.2.1) (for the four faces incident with v), it follows that $f(\vec{e_1}) = f(\vec{e_2})$. Hence all arcs in an mt-trail in $\overrightarrow{G'}$ have the same flow value. Assign to every arc in $\overrightarrow{G}$ the flow value of the corresponding oriented mt-trail in $\overrightarrow{G'}$. Then we obtain a nowhere-zero A-flow of $\overrightarrow{G}$ which finishes the second part of the proof.

We prove the first part of the theorem. Let g be a nowhere-zero A-flow of $\vec{G}$, where $\vec{G}$ denotes a fixed orientation of G. G' denotes the wild map of G which is also the dual graph of X.
Every mt-trail in G' which contains a 3-valent vertex corresponds to an edge in G. Assign to every such mt-trail in G' the corresponding orientation and g-flow value of the corresponding arc in $\vec{G}$. To every mt-trail S which does not correspond to an edge in G, i.e. S is an eulerian subgraph of G', assign an orientation and an arbitrary element of $A - \{0\}$ to S; this covers the case $G = \emptyset$.
Then all these assignments together induce an orientation of G' which we denote by $\overrightarrow{G'}$ and a nowhere-zero A-flow of $\overrightarrow{G'}$ which we call f.

We define a coloring q' of the faces of $\overrightarrow{G'}$, using colors from A by the following common coloring algorithm: Let one face in $\overrightarrow{G'}$ be arbitrarily precolored. Set

$$q'(\hat{F}) := q'(F) - f(\vec{e})$$

where F and $\hat{F}$ are the faces incident with $\vec{e}$, and F ($\hat{F}$) is on the left (right) side of $\vec{e}$. Then this algorithm executed step by step leads to a coloring of all faces of G'. For proving that q' is well defined, see the arguments used in the proof of Theorem 1.4.5 in [29]. By definition of q' and since opposite arcs have the same f-values by definition of $\overrightarrow{G'}$, the coloring q' induces by duality a vertex coloring of the mosaic X for which equation (4.2.1) is fulfilled for q' (in place of q). Hence the proof is finished.

Chapter 5

Applications Of Mosaics

5.1 Mosaics of snarks

Recall that in our terminology a snark is a cubic graph which is not 3-edge colorable. Let q be a vertex coloring of a mosaic X. We say that a quadrangle Q in X is colored with exactly k different colors if and only if $|\{q(v) \mid v \in V(Q)\}| = k$.

Definition 5.1.1 *A vertex 4-coloring of a plane graph is called a q_4-coloring if no quadrangle is colored by exactly three different colors.*

Thus, in a q_4-coloring every quadrangle is colored with exactly 2 or exactly 4 distinct colors. We obtain the following theorem concerning q_4-colorings.

Theorem 5.1.2 *A mosaic X has a q_4-coloring if and only if the corresponding cubic graph G is 3-edge colorable.*

Proof: First suppose that G is 3-edge colorable, then G has a nowhere-zero 4-flow and thus, in general, a nowhere-zero A-flow, where A is an arbitrary abelian group of order 4, see [29, 28]. Then we obtain, by applying Theorem 4.2.2, and by setting $A := \mathbb{Z}_2 \times \mathbb{Z}_2$, a coloring q of X satisfying equation (4.2.1). Let f be a bijection from $\mathbb{Z}_2 \times \mathbb{Z}_2$ into $\{1, 2, 3, 4\}$.

We claim that $f(q)$ is a q_4-coloring of X (using colors $1, 2, 3, 4$). Suppose not, then there is a quadrangle Q which is colored by exactly three different colors. Hence two vertices of Q which are non-adjacent in Q, must be colored with the same q-values which sum up to $(0, 0)$, whereas the two remaining vertices must be colored differently which contradicts equation (4.2.1) and thus finishes the first part of the proof.

Now let a q_4-coloring of X be given, using colors $1, 2, 3, 4$. Let f be as above. Then $q := f^{-1}(q_4)$ satisfies equation (4.2.1) in Theorem 4.2.2 and thus, G has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow and therefore a nowhere-zero 4-flow. Hence G is 3-edge colorable which finishes the proof.

Definition 5.1.3 *Let X^0 be a plane graph where every face except the face of infinite area which we denote by F_∞ is a triangle or quadrangle; i.e. F_∞ may also be a triangle or a quadrangle but need not to be. Let $\partial(X^0)$ denote the facial cycle of F_∞ which we name the outer cycle. Let X be the mosaic which is obtained from X^0 by adding one new vertex called α to F_∞ and by connecting α to all vertices of $\partial(X^0)$. Let G be the corresponding cubic graph of X (Proposition 4.1.7) then X^0 is called an open mosaic of G. Moreover G is called the corresponding cubic graph of X^0.*

For instance $X^0 := X - \alpha$ in Figure 4.2 is an open mosaic of P_{10} and thus P_{10} is the corresponding cubic graph of X and X^0.

Corollary 5.1.4 *Let X^0 be an open mosaic of a snark G. Then for every q_4-coloring of X^0, the outer cycle $\partial(X^0)$ contains all four colors.*

Proof: Suppose there is a q_4-coloring of X^0 using less than 4 colors for $\partial(X^0)$, then this coloring can be extended to a q_4-coloring of the mosaic X. Hence, by Theorem 5.1.2 we obtain a 3-edge coloring of G and thus a contradiction to the above assumption that G is a snark.

We have the following more general corollary (of Theorem 4.2.2) which is analogous to Corollary 5.1.4

Corollary 5.1.5 *Let X^0 be an open mosaic of a cubic graph G without a nowhere-zero $\mathbb{Z}_k$-flow, $k > 1$. Let $q : V(X^0) \mapsto \mathbb{Z}_k - \{0\}$ be a vertex coloring of X^0 which satisfies for every quadrangle in X^0*

$$q(x_1) + q(x_3) = q(x_2) + q(x_4) \tag{5.1.1}$$

where x_1, x_2, x_3, x_4 denote the vertices of the quadrangle in cyclic order. Then the outer cycle $\partial(X^0)$ contains all k colors.

Proof: The proof is analogous to the proof of Corollary 5.1.4, applying Theorem 4.2.2 in place of Theorem 5.1.2.

We demonstrate in Figure 5.1 some constructions of snarks which follow easily from Corollary 5.1.4.

Remark 5.1.6 *Note that the so constructed snarks in Figure 5.1 can be constructed by different methods, see [20, 21, 24]. (If the reader is mainly interested in snarks, choose $k = 4$ in the next theorem and its proof. Moreover replace q by q_4 and apply Corollary 5.1.4 instead of Corollary 5.1.5.)*

Description of Figure 5.1: on the top, the Blanusa snark is shown together with one corresponding (overlapping) open mosaic of it. (The open mosaic of the Blanusa snark is constructed by two open mosaics of the Petersen graph; see Figure 4.2.) Below are three open mosaics denoted by X_1, X_2, X_3 illustrated. The outer boundary of X_j, $j \in \{1, 2, 3\}$ is illustrated by dotted lines. S_i, $i = 1, 2, ..., 7$ is a subgraph of the respective X_j and an open mosaic itself. The outer boundary of S_i is illustrated by the circuit around the character S_i which consists of a path (illustrated by dotted lines) and a path of length $j + 1$ illustrated by j edges in bold face; with one exception: $\partial(S_1)$ in X_3 consists of 6 edges which are illustrated in bold face. Finally, Y is a subgraph of X_2 and an open mosaic itself with $E(\partial(Y)) = \{e_1, e_2, ..., e_5\}$.

Theorem 5.1.7 *Let the corresponding cubic graph of S_i, $i = 1, 2, ..., 7$ in Figure 5.1 have no nowhere-zero k-flow, $k > 1$. Then the corresponding cubic graph G of $X^0 \in \{X_1, X_2, X_3\}$ has no nowhere-zero k-flow.*

Proof: $X^0 \in \{X_1, X_2, X_3\}$. Suppose by contradiction that G has a nowhere-zero k-flow. Thus X (Definition 5.1.3) has a coloring q using colors $1, 2, ..., k$, satisfying equation (4.2.1) for $A = \mathbb{Z}_k$ (Theorem 4.2.2). Moreover, without loss of generality, suppose that α (Definition 5.1.3) has color k.

1. $X^0 = X_1$. Since $\partial(S_i)$, $i = 1, 2$ has by Corollary 5.1.5 a vertex of color k and since only one vertex v_i in $\partial(S_i)$ is not adjacent to α, v_i must have color k. Since v_1 and v_2 are adjacent, X has no coloring q. Hence by Theorem 4.2.2, G has no nowhere-zero k-flow.

2. $X^0 = X_2$. By Corollary 5.1.5 and since α has color k, one endvertex of every edge e_j of S_j has color k, $j = 1, 2, ..., 5$. Since $|\partial(Y)|$ is odd, no coloring q of X is possible. (Note that Y need not to be an open mosaic of a snark.) Hence by Theorem 4.2.2, G has no nowhere-zero k-flow.

3. $X^0 = X_3$. By Corollary 5.1.5 one vertex w of $\partial(S_1)$ has color k. Then there is a unique S_i, $i \in \{2, 3, ..., 7\}$ which has 3 vertices adjacent to w. Since α and w have color k and every vertex of $\partial(S_i)$ is adjacent to α or w, $\partial(S_i)$

cannot contain color k which contradicts Corollary 5.1.5. Hence there is no coloring q of X. Hence by Theorem 4.2.2, G has no nowhere-zero k-flow.

Remark 5.1.8 *(The preceding constructions of snarks by mosaics using Corollary 5.1.4 follow the idea of constructing from one illusionary counterexample to the* 4*-color problem another one.) Note that the Blanusa snark is obtained by X_1 as indicated in Figure 5.1. Moreover X_1 can correspond to a cyclically* 4*, X_2 to a cyclically* 5 *and X_3 to a cyclically* 6*-edge-connected snark. This depends on the structure of the S_i's and Y in Figure 5.1.*

5.2 Even triangulations

Definition 5.2.1 *A vertex* 3*-coloring of a mosaic X is called a q_3-coloring if every quadrangle of X is colored with all three colors.*

Theorem 5.2.2 *A mosaic X has a q_3-coloring if and only if the corresponding cubic graph G has a nowhere zero* 3*-flow.*

Proof: Suppose G has a nowhere-zero 3-flow, then G has a nowhere-zero $\mathbb{Z}_3$ flow. By setting $A := \mathbb{Z}_3$ and applying Theorem 4.2.2 we obtain a coloring q of X satisfying equation (4.2.1). Let f be a bijection from $\mathbb{Z}_3$ into $\{1, 2, 3\}$.

We claim that $f(q)$ is a q_3-coloring of X, using colors $1, 2, 3$. Suppose not, then there is a quadrangle in X which vertices are colored with exactly two distinct colors $a, b \in \mathbb{Z}_3$, satisfying $a + a \equiv b + b \pmod 3$, $a \neq b$ (by equation (4.2.1) of Theorem 4.2.2) which is impossible and thus finishes the first part of the proof.

Conversely, let a q_3-coloring of X be given, using colors $1, 2, 3$. Then $q := f^{-1}(q_3)$ satisfies equation (4.2.1) in Theorem 4.2.2 since $2a \equiv b + c \pmod 3$ where $\{a, b, c\} = \mathbb{Z}_3$. Hence by Theorem 4.2.2, G has a nowhere-zero 3-flow.

Corollary 5.2.3 *A mosaic X can be extended to an even triangulation, by adding one diagonal-edge into every quadrangle, if and only if the corresponding cubic graph G has a nowhere-zero* 3*-flow.*

Proof: Let X be given, and extendable to an even triangulation T as stated above, then $\chi(T) = 3$, see [15]. Let f be a 3-coloring of T. Then f induces a q_3-coloring of X. Thus, by Theorem 5.2.2 the corresponding cubic graph G

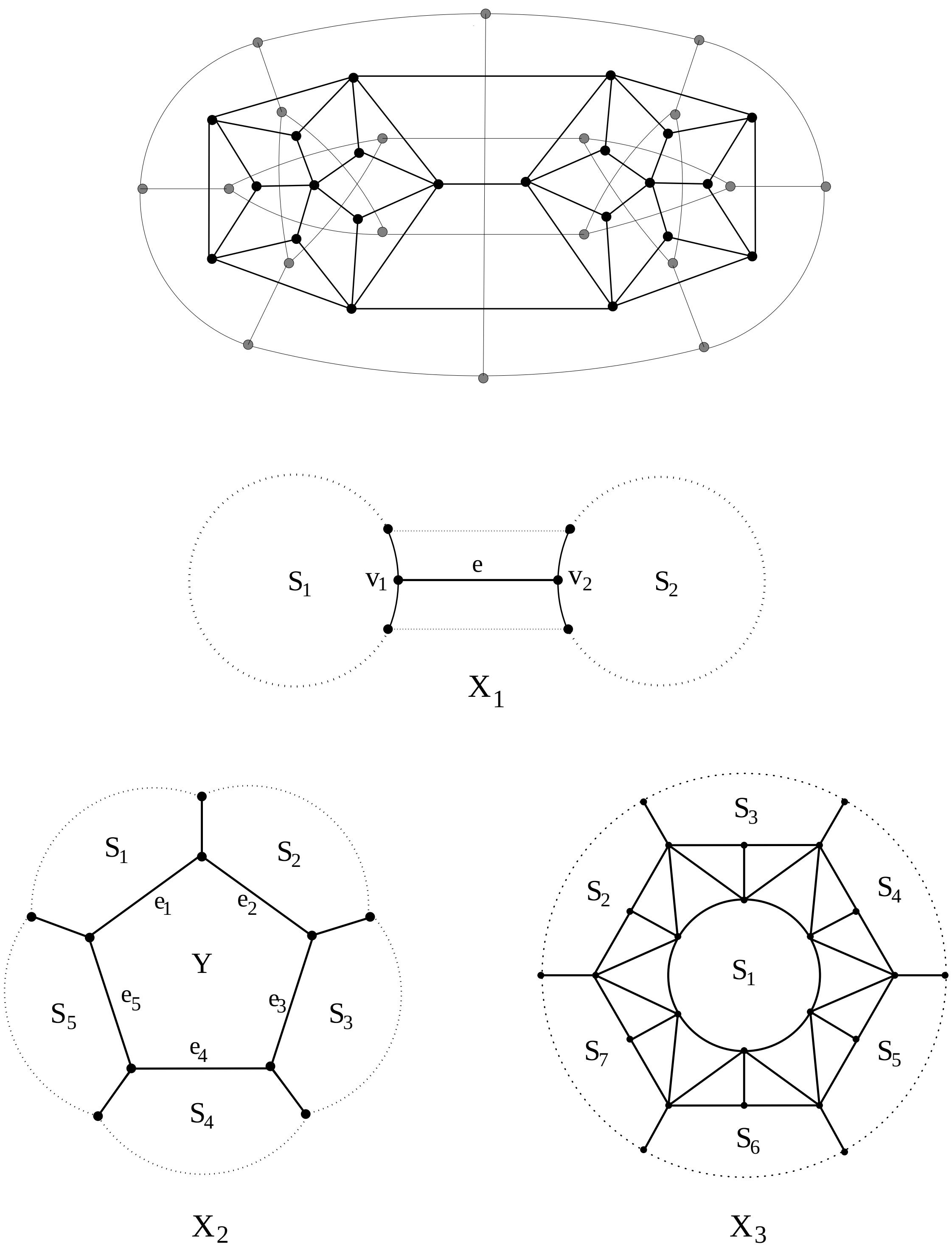


Figure 5.1: See Theorem 5.1.7 and the description above.

has a nowhere zero 3-flow.
Let G have a nowhere zero 3-flow, then X has a q_3-coloring by Theorem 5.2.2. Hence, by connecting in every quadrangle of X the two differently colored, non adjacent vertices by an edge, one obtains a 3-colorable and thus even triangulation, which finishes the proof.

Thus we obtain for $G = \emptyset$ the following known result, see [16, 30].

Corollary 5.2.4 *Every quadrangulation can be extended to an even triangulation by adding one diagonal-edge into every quadrangle.*

Chapter 6

Quadrangulations

Since we showed that every quadrangulation has a q_3-coloring (Theorem 5.2.2 for the case $G = \emptyset$), and thus a 3-coloring such that every quadrangle is colored with all three colors, one might ask whether the same is true for four colors. We call a colored quadrangle where all vertices have distinct colors "multicolored" and a vertex 4-coloring where all quadrangles are multicolored a "multicoloring". We will apply the following definition.

Let us associate with $H_{34} \in \mathcal{H}_{34}$ (see Definition 4.1.1) an *intersection graph* which we denote by $I(H_{34})$ where the mt-trails of H_{34} are the vertices and two vertices are joined by an edge if and only if the corresponding mt-trails in H_{34} have a cross-vertex in common. We apply the following dual version of a theorem by Berman and Shank, see [1].

Theorem 6.0.5 *Let X be a quadrangulation of the sphere, then X has a multicoloring if and only if the dual graph of X, denoted by X^* satisfies:*
1. Every mt-trail of X^* is a circuit.
2. $\chi(I(X^*)) < 4$.

Consider a multicolored quadrangle Q (with colors from $\{1, 2, 3, 4\}$). Passing through clockwisely (anticlockwisely) the facial cycle of Q one encounters the colors in a certain order which we call clockwise (anticlockwise) color order. Note that Q has one of the following six clockwise color orders where every two of them with reverse clockwise color order are presented in a column:

$$(1, 2, 3, 4) \quad (1, 2, 4, 3) \quad (1, 3, 2, 4)$$

$$(1, 4, 3, 2) \quad (1, 3, 4, 2) \quad (1, 4, 2, 3).$$

See also Figure 6.2.

Definition 6.0.6 *Define* $\{a, b, c, d\} := \{1, 2, 3, 4\}$. *Let* X *be a quadrangulation of the sphere and* $f : V(X) \mapsto \{1, 2, 3, 4\}$ *a coloring of* X. *Then* $T^{+}(a, b, c, d)$ $(T^{-}(a, b, c, d))$ *denotes the set of multicolored quadrangles of* X *with clockwise (anticlockwise) color order* (a, b, c, d).

Then $T^{+}(a, b, c, d) = T^{+}(b, c, d, a) = T^{+}(c, d, a, b) = T^{+}(d, a, b, c)$ and $T^{+}(a, b, c, d) = T^{-}(d, c, b, a)$.

Mohar proved in [25] that $|T^{+}(a, b, c, d)|$ and $|T^{-}(a, b, c, d)|$ have the same parity.

Theorem 6.0.7 *For every quadrangulation* X *of the sphere and for every vertex* 4*-coloring* f *of* X, $|T^{+}(a, b, c, d)|$ *equals* $|T^{-}(a, b, c, d)|$.

For an illustration of the theorem see Figure 6.2. For the proof of the above theorem, we will apply the following lemma.

Lemma 6.0.8 *Let* H *be the plane graph constructed from two circuits* C^1, C^2 *of even length* k, *where* C^1 *is inside* C^2 *and every vertex* $v_i \in C^1$ *is adjacent to* $w_i \in C^2$ *where* $v_1, v_2, ..., v_k$ $(w_1, w_2, ..., w_k)$ *denote the vertices of* $C^1(C^2)$ *in clockwise order,* $i = 1, 2..., k$. *Let* $g : V(H) \mapsto \{1, 2, 3, 4\}$ *be a vertex* 4*-coloring such that every quadrangle* Q_i *with vertices* $w_i, w_{i+1}, v_i, v_{i+1}$ *(taking indices mod* k*) is multicolored. Then the number of* Q_i *with clockwise color order* (a, b, c, d) *equals the number of* Q_i *with anticlockwise color order* (a, b, c, d).

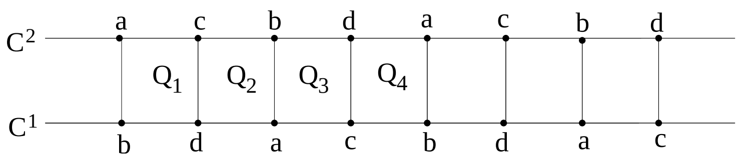


Figure 6.1: A unique multicoloring of H for $k = 8$.

Proof: For $k = 2$ the statement is true. Suppose by contradiction that there is a minimal counterexample Z. Since the Q_i's are multicolored

$$\{g(v_i), g(w_i)\} = \{g(v_{i+2}), g(w_{i+2})\} \tag{6.0.1}$$

for $i = 1, ..., k$ (taking indices mod k). Suppose $g(v_i) = g(v_{i+2})$ (and thus $g(w_i) = g(w_{i+2})$ by (6.0.1)). Delete v_{i+1} and w_{i+1} in H and identify v_i with

v_{i+2} and w_i with w_{i+2}. Then this graph would be a smaller counterexample than Z. Hence

$$g(v_i) \neq g(v_{i+2}), \;\; g(w_i) \neq g(w_{i+2}) \tag{6.0.2}$$

must be fulfilled for Z. Then by (6.0.1) and (6.0.2) we obtain a unique coloring of Z (ignoring permutations), see Figure 6.1 where k must be divisible by 4 and $g(v_i) = g(v_{i+4})$ and $g(w_i) = g(w_{i+4})$.

Moreover, by the unique coloring of Z, Q_i and Q_{i+2}, as well as Q_{i+1} and Q_{i+3} have reverse clockwise color orders. Therefore and since $k \equiv 0 \pmod 4$ the proof is finished.

Proof of Theorem 6.0.7:
Suppose there is a counterexample Y. We construct a multicolored counterexample by applying the following two operations which either increase $|T^+(a,b,c,d)|$ and $|T^-(a,b,c,d)|$ to the same amount or leave both numbers unchanged:

1. Introduce into every quadrangle Q, colored with exactly 3 distinct colors a diagonal-edge e connecting the two differently colored vertices. Insert a 2-valent vertex into e colored with the non-used color in Q.

2. Introduce in every quadrangle Q colored with only 2 distinct colors a diagonal-edge h. Insert a 2-valent vertex into h with one in Q non-used color, and reapply the first operation.

Let X, f be such a counterexample, where X is the quadrangulation and f its multicoloring. Then the dual graph of X, denoted by X^* is a plane 4-regular graph. By Theorem 6.0.5 every mt-trail in X^* is a circuit. Denote the set of mt-trails in X^* by C. Note that every circuit $C_j \in C$, has even length since all consecutive edges in C_j are opposite for $j = 1, 2, ..., |C|$. Let us consider the face coloring of X^* induced by f.

Analogously to $T^+(a,b,c,d)$ we define $U_B^+(a,b,c,d)$ with $B \subseteq V(X^*)$. Then $U_B^+(a,b,c,d)$ ($U_B^-(a,b,c,d)$) denotes the subset of vertices of B such that for every vertex v in it the four faces around v are colored in clockwise (anticlockwise) order (a,b,c,d). We obtain the following statement:

By duality every quadrangle of $T^+(a,b,c,d)$ in X corresponds to a vertex in $U^+_{V(X^*)}(a,b,c,d)$ and vice versa. Moreover $|T^+(a,b,c,d)| = |U^+_{V(X^*)}(a,b,c,d)|$. The analog holds for $T^-(a,b,c,d)$ and $U^-_{V(X^*)}(a,b,c,d)$.

Let us consider an mt-trail $C_j \in C$. Denote the even length of C_j by k. Every vertex of C_j is incident with four differently colored faces since f is a multicoloring. Every edge of C_j is incident with 2 distinct colored faces. Passing through clockwisely all edges of C_j we thus obtain two color sequences along C_j of same length k. We can regard the two color sequences as vertex colorings of C^1 and C^2 in Lemma 6.0.8. We obtain by Lemma 6.0.8 for $j = 1, 2, ..., |C|$ that

$$|U^+_{V(C_j)}(a,b,c,d)| = |U^-_{V(C_j)}(a,b,c,d)| \ .$$

Since C decomposes $E(X^*)$ and since every vertex of X^* is contained in exactly two circuits of C we obtain that

$$\sum_{C_j \in C} |U^+_{V(C_j)}(a,b,c,d)| = 2\ |U^+_{V(X^*)}(a,b,c,d)|$$

and that

$$\sum_{C_j \in C} |U^-_{V(C_j)}(a,b,c,d)| = 2\ |U^-_{V(X^*)}(a,b,c,d)| \ .$$

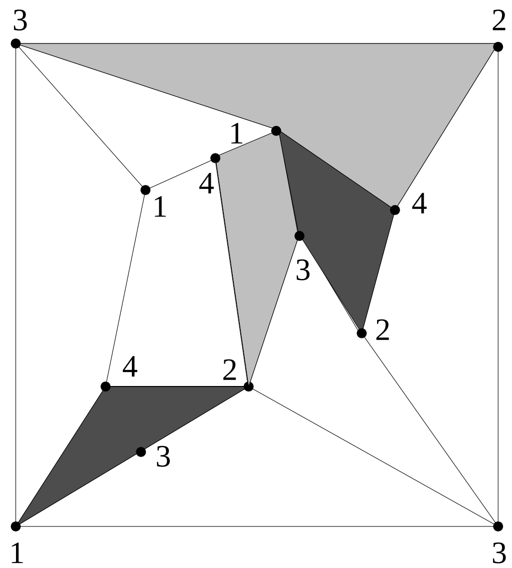


Figure 6.2: Multicolored quadrangles of the same clockwise color order are illustrated with the same type of gray, see Theorem 6.0.7.

By the three above equations it follows that

$$|U^{+}_{V(X^*)}(a,b,c,d)| = |U^{-}_{V(X^*)}(a,b,c,d)| \ .$$

By duality we obtain that $|T^{+}(a,b,c,d)| = |T^{-}(a,b,c,d)|$ which finishes the proof.

6.1 The cyclic chromatic number

By Theorem 6.0.5, not every quadrangulation has a vertex 4-coloring in which all vertices in every quadrangle obtain different colors. Hence we ask how many colors do we need to assure such a coloring. We apply the following well-known definition.

Definition 6.1.1 *Let G be a plane graph. The cyclic-chromatic number of G, denoted by $\chi_c(G)$, is defined as the minimum number of colors which is needed to color the vertices of G in such a way that vertices which are incident with the same face are colored differently.*

We cite from [4]: "A graph is said to be 1-embeddable in a surface S if it can be embedded in S so that each of its edges is crossed by at most one other edge [26]." A graph 1-embeddable in the plane is called 1-planar. We apply the following theorem, see [3, 4].

Theorem 6.1.2 *Every 1-planar graph has a vertex 6-coloring.*

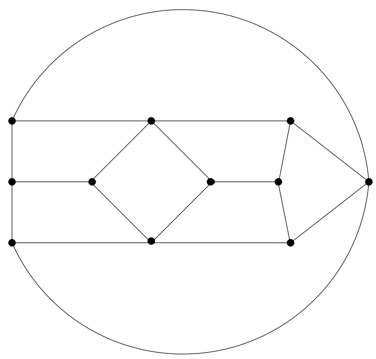


Figure 6.3: The quadrangulation X_0.

Denote by $\mathcal{Q}$ the set of all quadrangulations.

Theorem 6.1.3 *Let $X \in \mathcal{Q}$. Then $4 \leq \chi_c(X) \leq 6$ and both bounds are sharp.*

Proof: Let $X \in \mathcal{Q}$. Denote by X^+ the graph which results from X by adding into every quadrangle of X both chords. Then X^+ is 1-planar and $\chi_c(X) = \chi(X^+)$. Since X^+ contains several subgraphs isomorphic to K_4, $\chi_c(X) \geq 4$ follows. By the previous theorem $\chi_c(X) \leq 6$. It can be verified straightforwardly that X_0 which is illustrated in Figure 6.3 satisfies $\chi_c(X_0) = 6$. Hence the proof is finished.

Chapter 7

Open Problems and Conjectures

1. Solving Cubic Graphs.

For the understanding of this part, the reader needs Definition 2.1.2, 3.1.4, 3.1.5, 3.2.13. The following conjecture is an approach to the dominating cycle conjecture (DCC).

Conjecture 7.0.4 *Every cyclically* 4*-edge connected cubic graph has a hamiltonian bipartite cubic graph as a frame.*

Note that if Conjecture 7.0.4 is false, then also the DCC is false. This follows from the fact that every (G, D) has a BM.

The knowledge that a certain structure occurs in every 2-connected cubic graph could lead to several new results in graph theory. We need the following definitions.

Definition 7.0.5 *A graph is called a cubic parallel series graph if it can be constructed from a* C_2 *by an alternate sequence of the following two operations.*
1. Connect the two adjacent 2*-valent vertices by an edge (i.e. double an edge).*
2. Introduce two subdivision vertices into an arbitrary edge.

Denote the set of all cubic parallel series graphs by S'*.*

We conjecture the following.

Conjecture 7.0.6 *Every 2-connected cubic graph is solvable by* $S := \{C_2\} \cup S'$.

Conjecture 7.0.7 *Every 3-connected cubic graph has a cubic parallel series graph as a frame.*

Note that the truth of one of the above two conjectures would imply a new proof of the 6-flow theorem, see Theorem 3.1.9 and Proposition 3.1.6.

Problem 7.0.1 *Denote by* $\mathcal{M}_G$ *the set of all 2-connected cubic graphs which can be embedded on a given surface* $\mathcal{M}$. *For instance, let* $\mathcal{M}$ *be the torus. Construct a B-set* S *which solves every graph of* $\mathcal{M}_G$ *and which is minimal with respect to* $|S|$ *or* $l(S)$. *Note that if* $\mathcal{M}$ *is the sphere, every graph of* $\mathcal{M}_G$ *is solved by* $S = \{C_2\}$.

2. Spanning Trees in Cubic Graphs.

The truth of the following conjecture would provide more information about the structure of cubic graphs.

Conjecture 7.0.8 *Every connected cubic graph can be edge-decomposed into a spanning tree, a set of disjoint circuits and a matching.*

3. Bipartizing Matchings and Generalized Circle Graphs.

The investigation of the BMC led to several new questions. For the understanding of this part, the reader needs several definitions from the last two pages of the second chapter.

We know that if a cubic graph has a chordless dominating circuit, then its circle graph has a BM-coloring. Note that $\mathcal{W}$ contains all these circle graphs. We suspect the following.

Conjecture 2.7.7 Let $W \in \mathcal{W}$. Then W has a BM-coloring with respect to T_W.

The truth of the next conjecture would imply the truth of the CDCC for cubic graphs with dominating circuits.

Conjecture 2.7.9 Let $W \in \mathcal{W}$. Then T_W can be decomposed into the subgraph T_r consisting of red triangles and the subgraph T_b consisting of blue triangles such that $< V(T_r) >$ ($< V(T_b) >$) has a $\triangle$-coloring w.r.to T_r (T_b).

4. Cycle Double Covers of Cubic Graphs.

A *cycle* in a cubic graph G is a 2-regular subgraph of G. A 5-cycle double cover of G is a set of five cycles of G such that every edge of G is contained in exactly two of these cycles. For a survey on cycle double covers, see [29]. The BMC and frames have been approaches to the CDCC. The conjecture below is motivated by the following theorem, see [18].

Theorem 7.0.9 *Let G be 2-connected cubic graph and C a cycle in G. Then G has a 5-cycle double cover $R := \{C_1, C_2, ..., C_5\}$ such that $C \subseteq C_i$, $i \in \{1, 2, ..., 5\}$ if and only if there is a matching M of G with the following two properties.*

1. $G - M$ has a nowhere-zero 4-flow.
2. G contains two cycles C^1 and C^2 with $C \subseteq C^1$ such that $E(C^1) \cap E(C^2) = M$.

We conjecture that the following stronger version of the CDCC is true, see [18].

Conjecture 7.0.10 *Let C be a circuit in a 2-connected cubic graph G. Then there is a 5-cycle double cover of G such that C is a subgraph of one of these five cycles.*

Note that this conjecture is a combination of the 5-cycle double cover conjecture and the strong cycle double cover conjecture, see [29].

Chapter 8

Appendix

8.1 Deutsche Zusammenfassung

Im ersten Teil der Dissertation widerlegen wir die Bipartizing Matching Conjecture (BMC). Die BMC steht im Zusammenhang mit zentralen Vermutungen in der Graphentheorie wie zum Beispiel der Cycle Double Cover Conjecture (CDCC) und der Nowhere-Zero 5-Flow Conjecture. Die BMC besagt, dass zu jedem zyklisch 4-fach zusammenhängenden und nicht 3-kantenfärbbaren kubischen Graphen G mit dominierendem Kreis C zwei kanten-disjunkte Matchings M_1 und M_2 existieren (welche jeweils bipartizing matchings heissen), sodass für $i = 1, 2$ gilt:

1. $V(G) - V(C) \subseteq V(M_i)$.
2. $M_i \cap E(C) = \emptyset$.
3. $G - M_i$ hat einen nowhere-zero 3-flow.

Wir konstruieren unendlich viele Gegenbeispiele zu dieser Vermutung.

Wir verallgemeinern das Konzept eines bipartizing matchings und das Konzept eines frames. Wir führen den Begriff des Lösens eines kubischen Graphen ein: Sei $[F]$ die Vereinigung der Menge der 2-fach zusammenhängenden kubischen Graphen mit dem Kreis der Länge 2. Wir sagen eine Teilmenge S von $[F]$ löst einen kubischen Graphen G genau dann, wenn es ein Matching M gibt, sodass folgendes gilt:

1. Jede Komponente von $G - M$ ist ein Unterteilung eines Elementes aus S.
2. Jede Komponente von $G - M$ hat eine gerade Anzahl von 2-valenten Knoten.

Wir stellen die folgende Frage: Welche Eigenschaften muss eine Teilmenge S_0 von $[F]$ haben, sodass sie jeden 3-fach zusammenhängenden kubischen Graphen löst?

Wir benötigen die folgende Definition: Sei G ein Graph, dann bezeichnet $l(G)$ die kleinste natürliche Zahl, sodass folgendes gilt: G enthält einen Kreis C sodass die Distanz jedes Knotens aus G zu C kleiner oder gleich $l(G)$ ist. Sei H eine Menge von Graphen, dann bezeichnet $l(H)$ das Supremum über alle l-Werte der Elemente von H.

Wir erhalten folgende Antwort zur vorigen Frage: $l(S_0) = \infty$. Das heißt insbesondere, dass S_0 unendlich viele Elemente besitzen muss und nicht alle davon einen hamiltonschen oder dominierenden Kreis besitzen können. Dieses Resultat widerlegt einige Vermutungen zum Thema frames. Jene hätten die Lösbarkeit jedes 3-fach zusammenhängenden kubischen Graphen durch spezielle hamiltonsche Graphen aus $[F]$ impliziert.

Weiters zeigen wir, dass die Menge aller bipartiten Graphen aus $[F]$ jeden 2-fach zusammenhängenden kubischen Graphen löst. Dieses Resultat steht in engem Zusammenhang mit 6-flows.

Im zweiten Teil der Dissertation untersuchen wir spezielle planare Graphen, sogenannte Mosaiks. Sie zeichnen sich dadurch aus, nur Länder mit Kreislänge 3 oder 4 zu besitzen. Jedem Mosaik kann genau ein kubischer Graph zugeordnet werden, der nicht notwendigerweise planar ist. Wir transformieren Fragestellungen zu flows in kubischen Graphen in Knotenfärbungsprobleme von Mosaiks. Wir erhalten dadurch die folgenden Resultate.

1. Jedes Mosaik, das keine gültige 4-Knotenfärbung derart besitzt, sodass jedes Land mit Kreislänge 4 eine gerade Anzahl von verschiedenen Farben hat, entspricht einem nicht 3-kantenfärbbaren kubischen Graphen. Wir erhalten dadurch neue Konstruktionsmöglichkeiten von snarks.

2. Wir erweitern ein Resultat über Quadrangulierungen der Ebene und charakterisieren jene Mosaics, die durch Hinzufügen einer Diagonalkante in jedes Quadrates, d.h. eines Landes mit Kreislänge 4, in eine eulersche Triangulierung der Ebene transformiert werden können.

3. Wir verschärfen ein Resultat über 4-Knotenfärbungen von planaren Quadrangulierungen. Wir zeigen, dass für jede gültige 4-Knotenfärbung (mit den Farben 1, 2, 3, 4) einer planaren Quadrangulierung das folgende gilt: Die Anzahl der Länder, deren Kreise im Uhrzeigersinn 1,2,3,4 gefärbt sind, ist gleich der Anzahl der Länder deren Kreise gegen den Uhrzeigersinn 1,2,3,4

gefärbt sind.

Abschließend stellen wir neue Vermutungen auf.

8.2 Abstract

In the fist part of the thesis we disprove the Bipartizing Matching Conjecture (BMC). The BMC is related to central conjectures in graph theory such as the Nowhere-Zero 5-Flow Conjecture and the Cycle Double Cover Conjecture.

BMC: Every cyclically 4-edge connected non-3-edge-colorable cubic graph G with dominating circuit C contains two edge disjoint matchings (which are called bipartizing matchings) M_i, $i = 1, 2$ such that for $i = 1, 2$,

1. $V(G) - V(C) \subseteq V(M_i)$.
2. $M_i \cap E(C) = \emptyset$.
3. $G - M_i$ has a nowhere-zero 3-flow.

We construct infinitely many counterexamples to this conjecture. We generalize the concept of a bipartizing matching and a frame and introduce the notion of solving a cubic graph.

Denote by $[F]$ the set of graphs which is the union of the set of 2-connected cubic graphs and the circuit of length 2. We say that a set $S \subseteq [F]$ *solves* a cubic graph G if there exists a matching M of G with the following properties.

1. Every component of $G - M$ is a subdivision of a graph of S.
2. Every component of $G - M$ has an even number of 2-valent vertices.

We pose the following question: *Which properties must a set $S_0 \subseteq [F]$ have such that every 3-connected cubic graph G is solved by S_0?*

We apply the following definition. Let H be a graph. Denote by $l(H)$ the smallest natural number such that the following is true: H contains a circuit C such that for every vertex $v \in V(H)$ there is a path of length at most $l(H)$ connecting v with (a vertex of) C. Let S be a set of graphs, then $l(S)$ denotes the supremum over all l-values of the elements of S.

We obtain the following answer to the above question: $l(S_0) = \infty$. This implies that S_0 cannot be of finite order and cannot contain hamiltonian graphs only. We disprove several conjectures on the topic of frames by applying this result.

In the second part of the thesis we consider certain plane graphs. We call a plane graph a mosaic if it has only quadrangular and triangular faces. Hence mosaics form a generalization of triangulations and quadrangulations. We show that every mosaic corresponds to a unique cubic (not necessarily planar) graph. We transform nowhere-zero flow problems of arbitrary cubic graphs into vertex coloring problems of mosaics. We obtain the following results:

1. Every mosaic which has no proper vertex 4-coloring such that every quadrangular face Q is incident with an even number of differently colored vertices, corresponds to a snark. This result leads to new constructions of snarks.

2. We characterize those mosaics which can be extended to an even triangulation, i.e. a triangulation where every vertex has even degree, by adding a diagonal-edge into every quadrangular face. This result is a generalization of a theorem about quadrangulations.

3. Let Q be a quadrangulation of the sphere with an arbitrary proper vertex 4-coloring $f : V(Q) \mapsto \{1, 2, 3, 4\}$. We prove that the number of quadrangular faces of Q whose vertices are colored $1, 2, 3, 4$ in clockwise order equals the number of quadrangular faces whose vertices are colored in counterclockwise order $1, 2, 3, 4$. Note that Q may contain quadrangular faces which don't have 4 different colors. This result is a sharpening of a theorem about quadrangulations.

Finally, we pose open problems and state several new conjectures.

8.3 Curriculum Vitae

Born in Vienna, on the 13th of April 1977.
Elementary school in Vienna.
Secondary school BG 13 in Vienna. Matura, 12.6.1995.

1995-2003: Study of Mathematics and two years study of Physics at the University of Vienna. Diploma thesis "Colouring the Vertices of Digraphs". Supervisor: Herbert Fleischner.

2004: Travels through Europe, in particular France.

"Workshop Cycles and Colourings", Slovakia.

2005-2008: Project-assistant in the FWF-project (P18383): Construction, Study and Application of Snarks.

Presentation of my own results in conferences and workshops:

ÖMG/JSMF conference, Slovakia.

Sixth Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications (Prague, Czech Republic).

21nd British Combinatorial Conference (University of Reading, England).

2008-2011: Project-assistant in the FWF-project (P20543): Cycle Double Covers, Bipartite Minors and Snarks.

Presentation of my own results in conferences and workshops:

5th Workshop on the Matthews-Sumner Conjecture and Related Problems (Domazlice, Slovakia).

22nd British Combinatorial Conference (University of St Andrews, Scotland).

Graph Theory seminar (Bratislava, Slovakia).

Conference "Algebraic Graph Theory" dedicated to Sabidussi's eightieth birthday (Dubrovnik, Croatia).

8th French Combinatorial Conference (University Paris Sud, France).

6th Workshop on the Matthews-Sumner Conjecture and Related Problems (Domazlice, Slovakia).

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