DIPLOMARBEIT

Titel der Diplomarbeit

Investigation of Entanglement and Heisenberg's Uncertainty Relation in the Neutral Kaon System

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag.rer.nat.)

Verfasser:	Sasa Radic
Matrikel-Nummer:	0504304
Studienrichtung:	Physik
Betreuerin:	Priv. Doz. Dr. Beatrix C. Hiesmayr

Wien, 2011

Abstract

This thesis covers two distinct fields in physics: Particle Physics and the Foundations of Quantum Mechanics. The crossover is made by the neutral kaon system, which can be equipped with one of the most exciting attributes of quantum mechanics: Entanglement. Entanglement is a hot discussed topic as it leads to peculiar consequences and has been shown to have novel applications (e.g. quantum cryptography). It has been found in many different physical systems and possibly also in biological systems.

In this work we focus on a massive entangled system, the neutral kaon system, and discuss quantum information theoretic questions. This system is oscillating between its particle and antiparticle state (so called strangeness oscillation), and is decaying. Moreover, it violates the CP symmetry (charge symmetry C and parity symmetry P, nobel prize 1980), i.e. it proves that there is a difference between a world of matter and a world of antimatter. The origin of this symmetry violation is still a big open problem in Particle Physics. We present and discuss different frameworks to describe the phenomenology of the neutral kaon system - which is considerably different to stable systems - and apply them to analyze Bell inequalities and the Heisenberg uncertainty principle in the entropic version. In particular, we show that the CP violation introduces uncertainty to the dynamics.

Kurzfassung

Diese Diplomarbeit schneidet zwei Gebiete der Physik an: die Teilchenphysik und die Grundlagen der Quantenmechanik. Verbunden werden die Teilgebiete durch das neutrale Kaonensystem, welches eine der interessantesten und aufregendsten Eigenschaften der Quantenmechanik vorweisen kann: Verschränkung. Verschränkung ist ein heftig diskutiertes Thema, da es eigenartige Konsequenzen mit sich bringt sowie neuartige Verwendungen ermöglicht (z.B. Quantenkryptographie). Das Phänomen der Verschränkung konnte in verschiedenen Systemen beobachtet werden, was eventuell auch auf biologische Systeme ausgeweitet werden kann.

In dieser Arbeit legen wir den Schwerpunkt auf ein massives verschränktes System, das System der neutralen Kaonen, und untersuchen quanteninformationstheoretische Fragen. Das System oszilliert zwischen Teilchen- und Antiteilchen-Zustand (sogenannte Strangeness Oszillation). Weiters ist es ein zerfallendes System. Darüber hinaus verletzt es die CP Symmetrie (Ladungssymmetrie C und Parität P, Nobelpreis 1980), d.h. es beweist, dass es einen Unterschied zwischen der Welt der Materie und der Welt der Antimaterie gibt. Die Bestimmung des Ursprungs dieser Symmetrieverletzung stellt noch immer ein ungelöstes Problem in der Teilchenphysik dar. Wir werden verschiedene Herangehensweisen zur Beschreibung der Phänomenologie des neutralen Kaonen Systems - welches sich beträchlich von stabilen Systemen unterscheidet - vorstellen und diskutieren, um sie in den Bell Ungleichungen und in Heisenberg's Unschärferelation in der entropischen Version anzuwenden. Desweiteren zeigen wir, dass die CP Verletzung Unsicherheit in die Dynamik bringt.

Contents

1 Introduction

1.1 History

In the very beginning of the 20th century, "particle physics" had to deal with three different types of particles: the 1897 discovered electron [1], the proton, which was discovered in 1919 by Rutherford [2] and the photon, which was 1905 "invented" by Einstein [3] (with pioneer work by Planck [4]) in order to describe the photoelectric effect. It took a time until the physicist community accepted the photon concept and understood its implementation. However, after setting up quantum field theory, the photon was put in its place. In this way, the electromagnetic interaction, for example, was explained by a quantized field - namely in form of photons. This way of thinking was also used when a specific problem in nuclear physics arised: protons - the constituents of nucleus (along with neutrons) - have the same charge; how and why do they not fly apart? What was this "strong force" about that was holding the protons in the nucleus together?

In 1934, Yukawa postulated, similar to the quantized electromagnetic field, a theory, in which not photons were the quantized particles for the strong force field, but (due to the small range) a particle with a mass approximately 300 times the electrons mass, which he called meson [5] (from a Greek word standing for "middleweight" or "intermediate", while the electron is a lepton, which stands for "shortweight", baryons for "heavyweight"). After an exhausting search (at first one believed that the 1937 discovered muon was the Yukawa meson) the meson was finally found in 1947 [7]. Nowadays, it is called pion.

In the interim, physicists were engaged with other problems and questions: from the Dirac equation one obtained - when using the relativistic formula for energy $E^2 - \mathbf{p}^2c^2 = m^2c^4$ - two solutions. There was for every solution with positive energy ($E = +\sqrt{\mathbf{p}^2c^2 + m^2c^4}$) the corresponding negative energy ($E = -\sqrt{\mathbf{p}^2c^2 + m^2c^4}$). This could not be understood. However, Feynman, together with Stueckelberg formulated an interpretation of a new type of particles, carrying the negative energy: the antiparticles. According to them, every particle has its antiparticle, which carries the same properties apart from the opposite additive quantum numbers like charge, baryon number, lepton number etc., while the nonadditive quantum numbers like spin, mass, lifetime etc. remain the same. The antiproton, for example, has the same mass, spin and magnetic moment as the proton, but it's electric charge is given by -1e. The term "anti" is of course part of a convention; "particles" are those which can be identified as common matter, while antimatter can not be observed easily in nature.

So the new concept enlarged the particle-zoo by a factor of 2 (nearly - some particles are their own antiparticles). It took a while until the particle - antiparticle concept was accepted, and when the positron was discovered in 1931, no one doubted the solutions and interpretation, respectively, of the Dirac equation.

In this way, in 1947, one thought that the particle physics world was more or less undestood, since all the - at that time - known phenomena could be explained. Then in December 1947 something strange happened.

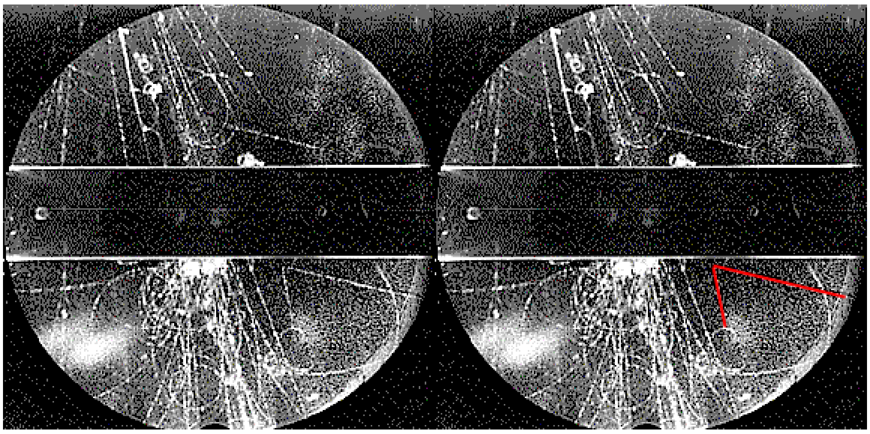


Figure 1.1: The first strange particle. Cosmic rays hit (from the upper left side) a 3cm thick lead slab, thereby producing a K^0, which then decays into a pair of two charged pions. By kind permission of the publisher "Nature"

Rochester and Butler published [8] the cloud chamber picture shown above (Fig. 1.1), where cosmic rays hit a lead slab, creating a neutral particle, which then produces the V-shaped decay into two particles. The authors showed then that these two charged particles were the pions π^+ and π^-, respectively. So there was without doubt a new particle, having at least twice the mass of a pion, which one names kaon nowadays. According to this, the decay of this particle has the following form:

$$K^0 \longrightarrow \pi^+ + \pi^- \tag{1.1}$$

Two years later, in 1949, Powel published another picture, showing the decay of a charged kaon, i.e.

$$K^+ \longrightarrow \pi^+ + \pi^+ + \pi^-, \tag{1.2}$$

and it took a little time until one found out that the two decaying particles ((1.1) and (1.2)) were different sorts of only one kind of particle, namely a charged and an uncharged version.

1.2 Kaon Properties

Although the investigation of kaons in this thesis is not yet profound, it is useful to bring in a table carrying the most important properties of kaons together. The appearance of the K-long and K-short, respectively, will be explained in the following sections. Yet it is important to show that there is more behind the concept than just one particle we call kaon.

Particle Name	Particle Symbol	Antiparticle Symbol	Quark content (particle)	Rest Mass (MeV/c^2)
Kaon	K^+	K^-	$u\bar{s}$	493.677 ± 0.016
Kaon	K^0	$\bar{K}^0$	$d\bar{s}$	497.614 ± 0.024
K-Short	K_S	Self	$\frac{d\bar{s}-s\bar{d}}{\sqrt{2}}$	-
K-Long	K_L	Self	$\frac{d\bar{s}+s\bar{d}}{\sqrt{2}}$	-

Table 1: Remarks: there is no definite lifetime for the eigenstates of the strong interaction, i.e. K^0 and $\bar{K}^0$. Furthermore, concerning the mass of K-Long and K-Short one can only declare a mass difference between them. However, it is important to stress that there is also a difference in lifetime between K-Short and K-Long (see below)

Particle Name	I^G	J^{PC}	S	C	B	Mean Lifetime (s)
Kaon K^+	1/2	0^-	1	0	0	$1.2380 \pm 0.0021 \times 10^{-8}$
Kaon K^0	1/2	0^-	1	0	0	-
K-Short K_S	1/2	0^-	-	0	0	$8.953 \pm 0.005 \times 10^{-11}$
K-Long K_L	1/2	0^-	-	0	0	$5.116 \pm 0.020 \times 10^{-8}$

Table 2: Further properties.

But what was so strange about the kaons?

While the production rate of kaons is only a few percent of the pion's production, kaons possess a long lifetime. The neutral pion π^0, for example, has a mean lifetime in order of $10^{-17}s$, while kaons surprise by having a mean lifetime in order of $10^{-8}s$, which in particle physics is a huge time. In experiments, therefore, the kaons can travel distances of centimeters (and even meters) in laboratory and thus can be measured directly (e.g. in a cloud chamber), while particles with a far smaller lifetime can not be observed easily.

In the next section we will describe the kaons using the quantum mechanical formalism.

2 The Quantum Mechanics of Kaons

After a short mathematical introduction we define kaon and antikaon states as the eigenvectors of the strangeness operator S. We discuss kaon production and their reactions. Furthermore we are going to introduce the K-short/K-long concept and compare kaons with photons. The most important and subtle part of the section will be a formalism of the decay property and therewith the time evolution, which is needed for a description of entanglement.

2.1 Basics

2.1.1 Mathematical Requirement

Since the mathematics in this thesis belongs more or less to the standard repertoire of quantum mechanics, only the most important concepts will be reviewed.

- Hilbert space: In plain words, a Hilbert space is an abstract vector space equipped with an inner product. For this thesis it is important to stress a postulate of quantum mechanics stating that the tensor product of two Hilbert spaces H_A and H_B factorizes the dimension of the new Hilbert space H_{tot}, i.e.

$$H_{tot} = H_A \otimes H_B \Rightarrow d_{tot} = d_A \cdot d_B, \tag{2.1}$$

  where d_A and d_B are the dimensions of the Hilbertspaces H_A and H_B, respectively.

- Notation of the Tensor Product: the proper notation (in the bracket form) has the form $|\psi\rangle \otimes |\phi\rangle$, or even $|\psi\rangle_A \otimes |\phi\rangle_B$, reminding that $|\psi\rangle_A$ belongs to the Hilbert space H_A, while $|\phi\rangle_B$ belongs to H_B. For convenience, one equivalently uses

$$|\psi_A\rangle\, |\phi_B\rangle\, ,\ |\psi\rangle\, |\phi\rangle\, , |\psi, \phi\rangle$$

  or even

$$|\psi\phi\rangle\, . \tag{2.2}$$

  In this thesis all notations will be used. The notation will sometimes be switched, if necessary, to point out important passages of equations.

- Direct Sum: when rolling out the decay of kaons, the mathematical operation of the direct sum will be required. It has the form

$$v_1 \oplus v_2, \tag{2.3}$$

  with $v_1 \in V_1$ and $v_2 \in V_2$, where V_1, V_2 are two vector spaces. It is important to stress that in our case V_1 and V_2 are two orhogonal subspaces of a greater Hilbert space with the dimension $d_{V_1} + d_{V_2}$.

- Bloch Sphere: when dealing with qubits, i.e.

$$|\psi\rangle = a\, |0\rangle + b\, |1\rangle\, , \tag{2.4}$$

  where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$, we may use the parametrization $a = \cos\frac{\theta}{2}, b = e^{i\phi}\sin\frac{\theta}{2}$, so that the vector $|\psi\rangle$ takes the form

$$|\psi\rangle = \cos\frac{\theta}{2}\, |0\rangle + e^{i\phi}\sin\frac{\theta}{2}\, |1\rangle\, , \tag{2.5}$$

giving us the ability to visualize the qubit within the Bloch sphere with the spherical coordinates $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

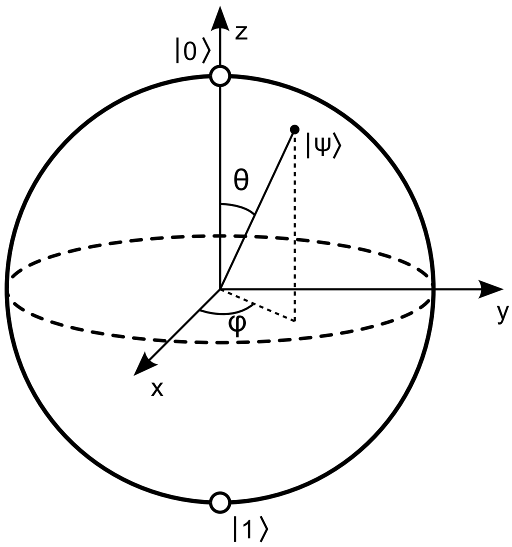


Figure 2.1: The Bloch sphere. Each point of the sphere represents a state. Pure states are located on the surface, whereas mixed states build the interior of the Bloch sphere.

After the small retrospection (additional terms should be evident from the context) we will start in introducing some common terms of particle physics.

2.1.2 Strangeness

Strangeness S is a quantum number introduced in order to describe the *strange* behavior of certain particles. It was suggested by Murray Gell-Mann [10] and Kazuhiko Nishijima [9] in the 50ies of the last century after numerous observations of the decay of kaons and other baryons containing the unknown strange quark at the time. In experiments, one noticed that these so called hyperons (baryons containing the strange quark) and strange mesons had a far greater lifetime than one would expect due to their mass. Gell-Mann and Nishijima therefore introduced a new quantity, the strangeness number S:

$$S = 2\left(Q - I_3 - \frac{B}{2}\right). \tag{2.6}$$

Here, B is the baryon number assigning that every baryon gets $B = 1$, while the antibaryon gets $B = -1$. Non-baryons - like bosons or fermions - get $B = 0$. Q is the electric charge. I_3 is the third component of the isospin introduced a few years before in order to describe the fact that strong force does not distinguish between protons and neutrons.

All experiments suggest that B and Q are exactly conserved. However, Gell-Mann and Nishijima supposed that I_3 was not conserved in all interactions. As we know now, the particles observed in Fig. 1.1 where produced by the strong force but decayed over the weak force: since the strangeness number S is conserved during the strong (and EM) interactions, but is not conserved for the weak interactions, the observed particles could not decay by the strong interactions. As a consequence of this, they could only decay by the weak interaction, which is much slower and thus provided the particles' uncommonly long lifetime. Below, we find a compilation of particles, which decay or interact by weak interaction:

			I_3				
B	S	I	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+1$
1	0	$\frac{1}{2}$		n		p	
1	-1	0			Λ^0		
0	0	1	π^-		π^0		π^+
0	+1	$\frac{1}{2}$		K^0		K^+	
0	-1	$\frac{1}{2}$		K^-		$\bar{K}^0$	
1	-1	1	Σ^-		Σ^0		Σ^+

Table 3: Particles which decay or interact by weak interaction

Nowadays the strangeness number S is defined by the strange (and antistrange) quarks, i.e.

$$S = -\left(n_s - \bar{n}_s\right), \tag{2.7}$$

with n_s standing for the number of the strange quarks and $\bar{n}_s$ for strange antiquarks.

2.1.3 CP symmetry & CP violation

CP is an operation composed of C (charge conjugation) and P (parity):

- Charge conjugation C: the operator C turns a particle into its antiparticle by simply changing the sign of its charge (electric, color,..), while leaving other properties, e.g. mass, energy, spin, momentum etc. unchanged.
- Parity P: the operator P causes a reflection in a spatial plane, thus $P\left|\Psi\left(\vec{r}\right)\right\rangle = \left|\Psi\left(-\vec{r}\right)\right\rangle$ - provided that there does exist a reflection.

In the first half of the previous century physicists assumed that physics is invariant under mirror inversion. This symmetry of parity seemed to be valid when considering early experiments, since they were concerned only with strong force, electro-magnetism and gravity. However, in particle decays (mostly weak force involved) one noticed slight "anomalies": some reactions did not occur as often as their mirror images. The parity violation had been found (1957, when observing cobalt-60 decay).
Appalled and surprised physicists soon picked a new "real" symmetry (what they believed to be a new symmetry): the combination of parity P and the particle-antiparticle swap C. Since C-symmetry and P-symmetry are (even maximally) individually violated, one thought that these two effects should neutalize each other. Thus, a particle exchanged into an antiparticle and observed through a mirror should underlie the same physics, differently stated: it shows the same behavior as the corresponding particle. This was in agreement with all experimental data and thus the problem was considered to be solved.
For that reason it was an even greater shock when one found in 1964 in experiments that the CP symmetry is violated, too, and this by particles, in which we do take great interest: the kaons.

In conclusion, it is important to stress that the CP violation is caused by the weak force, alternatively spoken when dealing with heavy quarks and decay. After the short introduction - CP violation will be investigated more precisely - we will start focussing on kaons and the *description* of kaons.

2.2 Describing Kaons

2.2.1 Definition of the strangeness and CP eigenstates

After introducing the strangeness we can define kaons being the eigenvectors of the strangeness operator S. More precisely, the strangeness quantum number +1, -1 serves to differ between the kaon K^0 and the antikaon $\bar{K}^0$, namely

$$\begin{aligned} S\left|K^0\right\rangle &= +\left|K^0\right\rangle \\ S\left|\bar{K}^0\right\rangle &= -\left|\bar{K}^0\right\rangle . \end{aligned} \tag{2.8}$$

The earlier introduced operation CP gives then

$$\begin{aligned} CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle , \\ CP\left|\bar{K}^0\right\rangle = -\left|K^0\right\rangle , \end{aligned} \tag{2.9}$$

showing straightforwardly that

$$\begin{aligned} \left|K_1^0\right\rangle = \frac{1}{\sqrt{2}}(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle), \\ \left|K_2^0\right\rangle = \frac{1}{\sqrt{2}}(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle) \end{aligned} \tag{2.10}$$

are the eigenstates of CP, i.e.

$$\begin{aligned} CP\left|K_1^0\right\rangle = \left|K_1^0\right\rangle , \\ CP\left|K_2^0\right\rangle = -\left|K_2^0\right\rangle . \end{aligned} \tag{2.11}$$

As mentioned before, CP-symmetry is violated in weak interactions. This will be further discussed as soon as the evolution of time can be formulated. Below there is a short overview:

kaon	Quark Composition	Strangeness	I_3
K^+	$u\bar{s}$	$+1$	$+\frac{1}{2}$
K^-	$\bar{u}s$	-1	$-\frac{1}{2}$
K^0	$d\bar{s}$	$+1$	$-\frac{1}{2}$
$\bar{K}^0$	$\bar{d}s$	11	$+\frac{1}{2}$

Table 4: Kaon overview

2.2.2 The Mass Eigenstates

As mentioned before, kaons decay in states, differing slightly in mass but immensely in lifetime. Therefore we will name the states the short lived states and the long lived states, defining them by

$$\left|K_S\right\rangle = \frac{1}{N}\left\{p\left|K^0\right\rangle - q\left|\bar{K}^0\right\rangle\right\} \tag{2.12}$$

and

$$|K_L\rangle = \frac{1}{N}\left\{p\left|K^0\right\rangle + q\left|\bar{K}^0\right\rangle\right\}. \tag{2.13}$$

Here we have $N^2 = |p|^2 + |q|^2$, containing the weights $p = 1+\epsilon$ and $q = 1-\epsilon$ with ϵ denoting the CP violating parameter. If we had assumed CPT violation (T stands for time reversal), then the CP violating parameters would be different for K_S and K_L (i.e. ϵ_S, ϵ_L) and thus we would find $\epsilon_S \neq \epsilon_L$. But since we suppose that physics is invariant under CPT, there is no difference in the CP violating parameters of the short lived and long lived states, respectively. Thus $\epsilon_S = \epsilon_L = \epsilon$. Furthermore, it was found $|\epsilon| \approx 10^{-3}$.

The table below shows the decay channels of K_S and K_L, respectively.

K_S decay modes	Fraction $\frac{\Gamma_i}{\Gamma_S}$	K_L decay modes	Fraction $\frac{\Gamma_i}{\Gamma_L}$
		$\pi^+\pi^-\pi^0$	$(12.55 \pm 0.20)\%$
		$\pi^0\pi^0\pi^0$	$(21.13 \pm 0.27)\%$
		$\pi^\pm\mu^\mp\nu_\mu$	$(27.18 \pm 0.25)\%$
		$\pi^\pm e^\mp\nu_e$	$(38.78 \pm 0.28)\%$
$\pi^+\pi^-$	$(68.61 \pm 0.28)\%$	$\pi^+\pi^-$	$(2.056 \pm 0.033)\cdot 10^{-3}\%$
$\pi^0\pi^0$	$(31.39 \pm 0.28)\%$	$\pi^0\pi^0$	$(9.27 \pm 0.19)\cdot 10^{-4}\%$

Table 5: Decay Channels of Kaons. One notices the small amount of the K_L decay into two pions

2.2.3 The Production and Common Reactions of Kaons

Since we can now describe kaons mathematically, it is important to show how kaons may be produced by "nature". Below we find some possible processes:

$$\begin{aligned}
&\pi^- + p \longrightarrow \Lambda + K^0 && E(\pi^-) \geq 0.91GeV \\
&\pi^+ + p \longrightarrow K^+ + \bar{K}^0 + p && E(\pi^+) \geq 1.50GeV \\
&\pi^- + p \longrightarrow \bar{\Lambda} + \bar{K}^0 + n + n && E(\pi^-) \geq 6.00GeV
\end{aligned} \tag{2.14}$$

One finds that the creation depends on the energy of the pions. The process which was observed in 1947 (see 1.1) was of the form

$$\pi^+ + n \longrightarrow K^+ + \Lambda \tag{2.15}$$

Fig 2.2.3 shows one possible way of producing kaons in a Feynman diagram.

Fig 2.2.3 shows the first process in (2.14). As usual, possible reactions obey the conservation laws (in strong interaction). Thus

$$\underset{S=1}{K^0} + \underset{S=0}{p} \longrightarrow \underset{S=0}{n} + \underset{S=1}{K^+} \tag{2.16}$$

is allowed ($S = 1$ on the left and right hand side), while

$$\underset{S=-1}{\bar{K}^0} + \underset{S=0}{p} \nrightarrow \underset{S=0}{n} + \underset{S=1}{K^+} \tag{2.17}$$

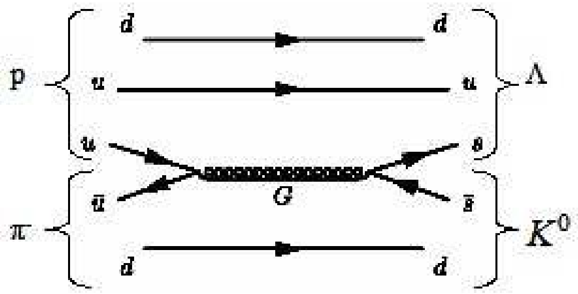


Figure 2.2: Creation of a Kaon

is forbidden ($S = -1 \neq S = 1$). Further reactions are given here:

$$\bar{K}^0 + p \longrightarrow \Lambda^0 + \pi$$
$$\bar{K}^0 + n \longrightarrow p + K^- \tag{2.18}$$

Note that the $\bar{K}^0$ state has two reaction channels with matter, while K^0 has only one. This assymetry leads to an effect known as regeneration (see

3 regener

. We will now study an interesting and even more important effect called strangeness oscillation.

3.0.4 Strangeness Oscillation

Since we operate with a decaying system, the mathematical description may not be easy. As a consequence one makes use of an approximation, the so called Wigner Weisskopf Approximation (WWA). Thus handling the decay, one can try an exponential ansatz:

$$|\psi(t)\rangle = e^{-i\left(m-i\frac{\Gamma}{2}\right)t} |\psi(0)\rangle . \tag{3.1}$$

Now the time evolution has the form

$$\langle\psi(t)|\,\psi(t)\rangle = e^{-\Gamma t}, \tag{3.2}$$

which leads to the desired form of exponential decreasing.

We now consider the effective Schrödinger equation guided by the non-Hermitian effective mass Hamiltonian $H = M - \frac{i}{2}\Gamma$:

$$i\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle \tag{3.3}$$

The eigenstates of this Hamiltonian are the short lived and long lived states discussed before, thus

$$H\left|K_{S,L}\right\rangle = \lambda_{S,L}\left|K_{S,L}\right\rangle, \tag{3.4}$$

with

$$\lambda_{S,L} = m_{S,L} - \frac{i}{2}\Gamma_{S,L}. \tag{3.5}$$

Using the schematic picture below, one can argue that since kaons decay into the same states as antikaons do, they can oscillate via the virtual decay products between particle and antiparticle before decaying. A more precice Feynman diagram will follow at the end of this section.

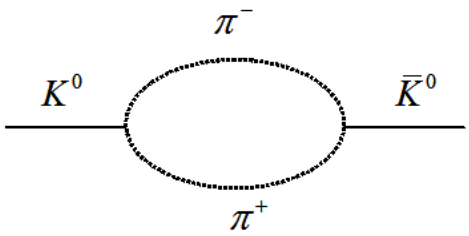


Figure 3.1: Strangeness Oscillation

In order to describe the oscillation mathematically, one can use the Weisskopf-Wigner approximation for the evolution of the decaying states, i.e.

$$\left|K_{S,L}(t)\right\rangle = e^{-i\lambda_{S,L}t}\left|K_{S,L}\right\rangle, \tag{3.6}$$

the corresponding time evolution for kaon and antikaon is then given by

$$\begin{aligned} \left|K^0(t)\right\rangle &= g_+(t)\left|K^0\right\rangle + \frac{q}{p}g_-(t)\left|\bar{K}^0\right\rangle, \\ \left|\bar{K}^0(t)\right\rangle &= \frac{p}{q}g_-(t)\left|K^0\right\rangle + g_+(t)\left|\bar{K}^0\right\rangle. \end{aligned} \tag{3.7}$$

p and q are the weights containing the CP violation parameter and the factors g_+, g_- are given by

$$g_\pm(t) = \frac{1}{2}\left[\pm e^{-i\lambda_S t} + e^{-i\lambda_L t}\right]. \tag{3.8}$$

Suppose at $t = 0$, one has a K^0. Now the probability for finding a K^0 at time t is given by

$$\begin{aligned} P\left(K^0, t; |K^0|\right) &= \left|\left\langle K^0\right| K^0(t)\rangle\right|^2 \\ &= \left|\left\langle K^0\right| \frac{N}{2p}\left\{e^{-i\lambda_S t}\left|K_S\right\rangle + e^{-i\lambda_L t}\left|K_L\right\rangle\right\}\right|^2 \\ &= \left|\frac{1}{2}e^{-i\lambda_S t} + \frac{1}{2}e^{-i\lambda_L t}\right|^2 \\ &= \frac{1}{4}\left\{e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-\Gamma t} Re\left\{e^{-i\Delta m t}\right\}\right\} \\ &= \frac{1}{4}\left\{e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-\Gamma t}\cos\left(\Delta m t\right)\right\}, \end{aligned} \tag{3.9}$$

with $\Delta m = m_L - m_S$ and $\Gamma = \frac{1}{2}(\Gamma_L + \Gamma_S)$. Since $\langle K_S | K_L \rangle = \delta$ (see following section), the effect of CP violation is cancelled out.

In the same manner one get the probability for the $\bar{K}^0$ by

$$P\left(K^0, t; |\bar{K}^0|\right) = \left|\langle \bar{K}^0 | K^0(t)\rangle\right|^2$$
$$= \frac{1}{4}\frac{|q|^2}{|p|^2}\left\{e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\Gamma t}\cos(\Delta m t)\right\} \tag{3.10}$$

What are these two results suppose to mean?

One finds out the beam oscillates with the frequency of $\Delta m/2\pi$. At times of the order of τ_S one can observe the oscillations without doubt, since $\Delta m \tau_S = 0.47$. Thus, studying a beam containing only K^0 mesons at the time $t = 0$, one will find after a certain time far from production source (thus the K_S mainly died out) the $\bar{K}^0$, even with same probability as finding the kaon K^0. This is because of its presence in the K_L.

Topping off, below a Feynman diagram showing the strangeness oscillation:

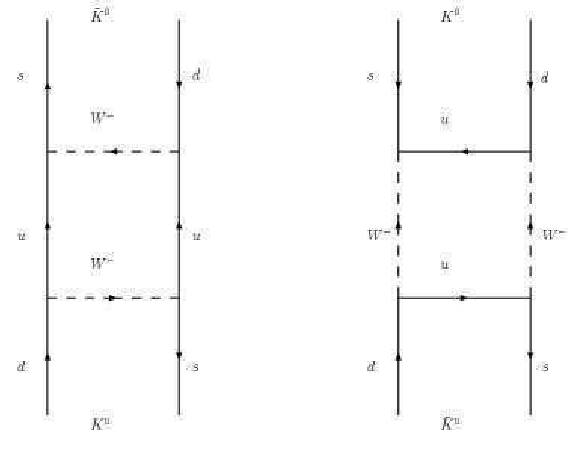


Figure 3.2: oscillation kaon-antikaon

As we can see in the picture above, one got used to describe particle oscillation within the QFT. It was no more mystic, but comprehensible, since particle oscillation is not an unusual phenomenon in QFT. Anyway, the more precise discussion of the important effect of CP violation will be presented in the next subsection.

3.0.5 And again: CP Violation

At first we define the charge assymetry and insert the results from (3.9), (3.10):

$$\delta(t) = \frac{P\left(K^0, t; |K^0|\right) - P\left(\bar{K}^0, t; |K^0|\right)}{P\left(K^0, t; |K^0|\right) + P\left(\bar{K}^0, t; |K^0|\right)} \tag{3.11}$$

$$\tag{3.12}$$

$$= \frac{\frac{|p|^2-|q|^2}{N^2}\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + 2\cos(\Delta m t)e^{-\Gamma t}}{\left(e^{-\Gamma_S t} + e^{-\Gamma_L t}\right) + \frac{|p|^2-|q|^2}{N^2} 2\cos(\Delta m t)e^{-\Gamma t}} \tag{3.13}$$

$$\tag{3.14}$$

$$= \frac{\cos(\Delta m t) + \delta\cosh(\frac{\Delta\Gamma}{2}t)}{\cosh(\frac{\Delta\Gamma}{2}t) + \delta\cos(\Delta m t)}, \tag{3.15}$$

$$\tag{3.16}$$

where $\delta = \frac{|p|^2-|q|^2}{|p|^2+|q|^2}$ and $\Delta\Gamma = \Gamma_L - \Gamma_S$. Expanding for small δ we get

$$\delta(t) = \frac{\cos(\Delta m t)}{\cosh(\Delta\Gamma/2t)} + \delta\left(1 - \frac{\cos^2(\Delta m t)}{\cosh^2(\Delta\Gamma/2t)}\right) + O(\delta^2), \tag{3.17}$$

where for small t one observes the strangeness oscillation. For large t, however, the $\frac{\cos}{\cosh}$ term tends to zero, thus leaving just δ. Experiments, however, also show that the assymetry does not vanish: considering the semileptonic decay, one compared the decay channels of the K_S, i.e.

$$\delta_l = \frac{\Gamma(K_L \longrightarrow \pi^- l^+ \nu_l) - \Gamma(K_L \longrightarrow \pi^+ l^- \bar{\nu}_l)}{\Gamma(K_L \longrightarrow \pi^- l^+ \nu_l) + \Gamma(K_L \longrightarrow \pi^+ l^- \bar{\nu}_l)} \tag{3.18}$$

and found that there is a charge assymmetry of the order 10^{-3} (i.e. $\delta_l \approx 10^{-3}$). In our formalism one obtains therefrom

$$\delta_l = \delta = \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} = \frac{2\Re\{\epsilon\}}{1 + |\epsilon|^2} \equiv \langle K_S | \, K_L \rangle, \tag{3.19}$$

wherewith we can define the electric charge in an absolute sense: positive charge is the charge of the lepton which is more often produced in the K_L decays!

3.0.6 Regeneration

Regeneration is the effect where one can "re"-generate a K_S from a pure K_L-beam by letting it react with matter [11] [12]. This is possible due to the fact that the K^0 and $\bar{K}^0$ components, which are present in the K_L, have different scattering amplitudes and thus show different reaction with matter. So starting with a K^0 beam, we have at the beginning both eigenstates K_S and K_L. After a certain time ($> 4.7\tau$) the K_S amount almost completely dies out by decaying into 2 pions, leaving back the K_L components. Hence, if one performed a measurement after this time, the probability of finding a K_S in the beam would converge towards zero. Now suppose that instead of measuring one places a block of matter into the beam (the strangeness is conserved in strong interactions). This block of matter then causes, due to the fact that the total cross sections on the nucleons p, n are greatier for $\bar{K}^0$

than for K^0, a generation of the K_S. Hence, after crossing the block we find the state

$$|\psi\rangle = \frac{1}{2}(f^0 + \bar{f}^0)\,|K_L\rangle + \frac{1}{2}(f^0 - \bar{f}^0)\,|K_S\rangle\,. \tag{3.20}$$

Since the scattering amplitudes f^0 (belonging to K^0) and $\bar{f}^0$ (belonging to $\bar{K}^0$) are not the same, $f^0 - \bar{f}^0 \neq 0$. Thus the second part of (3.20) is not vanishing. We call this term the regeneration term, while the first one is called the scattering term.

In this way we have shown that we can generate K_S from a K_L beam.

3.0.7 Kaons and Photons

Visualaizing the topic, it has become convenient to adopt the quasispin picture introduced by Lee and Wu [26]. In analogy to photons, having polarization directions V (vertically) and H (horizontally) or spin-1/2 particles, the two meson states $|K^0\rangle$ and $|\bar{K}^0\rangle$ may be defined as the quasi spin states up $|\uparrow\rangle$ and down $|\downarrow\rangle$. The operators discussed in (2.2) can then be expressed by linear combinations of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.21}$$

It is evident that the Pauli matrix σ_3 acts the same as the strangeness operator S, i.e.

$$S \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = \sigma_3 \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = \begin{pmatrix} K^0 \\ -\bar{K}^0 \end{pmatrix}. \tag{3.22}$$

In the same way one can identify the CP operator as $-\sigma_1$ with

$$CP \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = (-\sigma_1) \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = \begin{pmatrix} -K^0 \\ -\bar{K}^0 \end{pmatrix}. \tag{3.23}$$

One can go further and express the (effective) Hamiltonian by Pauli matrices in combination with the identity matrix, i.e.

$$H = M - \frac{i\Gamma}{2} = a\mathbb{1} + \vec{b}\cdot\vec{\sigma} = \begin{pmatrix} a + b_3 & b_1 - ib_2 \\ b_1 + ib_2 & a - b_3 \end{pmatrix} \tag{3.24}$$

with

$$\begin{aligned} b_1 &= b\cos\alpha, \\ b_2 &= b\sin\alpha, \\ b_3 &= 0. \end{aligned} \tag{3.25}$$

Furthermore, for diagonalizing the Hamiltonian, it is

$$a = \frac{\lambda_S + \lambda_L}{2}, b = \frac{\lambda_S - \lambda_L}{2}. \tag{3.26}$$

The phase α is related to the CP violation parameter ϵ, thus

$$e^{i\alpha} = \frac{1-\epsilon}{1-\epsilon}. \tag{3.27}$$

If we chose, in addition, $|K^0\rangle$ and $|\bar{K}^0\rangle$ to be $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively, the long and short lived states get the form

$$\begin{aligned} |K_S\rangle &= \frac{1}{N}\begin{pmatrix}p\\-q\end{pmatrix}, \\ |K_L\rangle &= \frac{1}{N}\begin{pmatrix}p\\q\end{pmatrix}. \end{aligned} \tag{3.28}$$

Now, since we can use the formulation shown in this section, it is convenient to compare the different kaon-representations with polarization directions of photons. For that, we conclude this section with a table carrying together kaons and photons:

Kaon	Isospin	Photons
$\|K^0\rangle$	$\|\uparrow\rangle_z$	$\|V\rangle$
$\|\bar{K}^0\rangle$	$\|\downarrow\rangle_z$	$\|H\rangle$
$\|K_1^0\rangle$	$\|\nwarrow\rangle$	$\|-45°\rangle = \frac{1}{\sqrt{2}}(\|V\rangle - \|H\rangle)$
$\|K_2^0\rangle$	$\|\nearrow\rangle$	$\|+45°\rangle = \frac{1}{\sqrt{2}}(\|V\rangle + \|H\rangle)$
$\|K_S\rangle$	$\|\rightarrow\rangle_y$	$\|L\rangle = \frac{1}{\sqrt{2}}(\|V\rangle - i\|H\rangle)$
$\|K_L\rangle$	$\|\leftarrow\rangle_y$	$\|R\rangle = \frac{1}{\sqrt{2}}(\|V\rangle + i\|H\rangle)$

Table 6: Analogy kaon - photon

3.1 Decay and Time Evolution

3.1.1 Describing the Decay

As discussed, the Hamiltonian H for the system is non-Hermitian. However, it can be split into a strangeness conserving part $H^{(0)}$ and a strangeness violating part, i.e. $H^{(2)}$. The latter is called $H^{(2)}$ due to the fact that the strangeness S is violated by $|\Delta 2|$. The Hamiltonian under investigation takes the form as discussed in 3.0.4, i.e.

$$H \equiv \begin{pmatrix} M_{11} - \frac{i}{2}\Gamma_{11} & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{21} - \frac{i}{2}\Gamma_{21} & M_{22} - \frac{i}{2}\Gamma_{22} \end{pmatrix} = \begin{pmatrix} \langle K^0| H^{(0)} |K^0\rangle & \langle K^0| H^{(2)} |\bar{K}^0\rangle \\ \langle \bar{K}^0| H^{(2)} |K^0\rangle & \langle \bar{K}^0| H^{(0)} |\bar{K}^0\rangle \end{pmatrix} \tag{3.29}$$

If we assume CPT-symmetry, we can set $M_{11} = M_{22} = M_0$. Furthermore $M_0 = m_{K^0} = m_{\bar{K}^0}$ and $\Gamma_{11} = \Gamma_{12} = \Gamma_0$. Since both M and Γ are Hermitian, we have $M_{21} = M_{12}^*$ and $\Gamma_{12} = \Gamma_{12}^*$. Thus our Hamiltonian H assumes the form

$$H = \begin{pmatrix} M_0 - \frac{i}{2}\Gamma_0 & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12}^* - \frac{i}{2}\Gamma_{12}^* & M_0 - \frac{i}{2}\Gamma_0. \end{pmatrix} \tag{3.30}$$

After diagonalizing the Hamiltonian we get the well-known eigenstates, i.e. the short lived $|K_S\rangle$ and the long lived $|K_L\rangle$:

$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}} \left\{ \frac{1}{\sqrt{2}} \left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) + \frac{\epsilon}{\sqrt{2}} \left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right) \right\},$$
$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}} \left\{ \frac{1}{\sqrt{2}} \left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right) + \frac{\epsilon}{\sqrt{2}} \left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right) \right\} \tag{3.31}$$

The parameter ϵ is given by

$$\epsilon = \frac{\sqrt{M_{12} - \frac{i}{2}\Gamma_{12}} - \sqrt{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}}{\sqrt{M_{12} - \frac{i}{2}\Gamma_{12}} + \sqrt{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}}. \tag{3.32}$$

We can also express $|K_S\rangle$ and $|K_L\rangle$ by the CP eigenstates discussed in 2.2:

$$|K_S\rangle = \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \left\{ \left|K_1^0\right\rangle + \epsilon \left|K_2^0\right\rangle \right\},$$
$$|K_L\rangle = \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \left\{ \left|K_2^0\right\rangle + \epsilon \left|K_1^0\right\rangle \right\}, \tag{3.33}$$

which assumes the form

$$|K_S\rangle = \frac{1}{N} \left\{ p\left|K^0\right\rangle - q\left|\bar{K}^0\right\rangle \right\},$$
$$|K_L\rangle = \frac{1}{N} \left\{ p\left|K^0\right\rangle + q\left|\bar{K}^0\right\rangle \right\}, \tag{3.34}$$

where

$$\frac{p}{q} = \frac{\sqrt{M_{12} - \frac{i}{2}\Gamma_{12}}}{\sqrt{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}} \qquad \text{and} \qquad N^2 = |p|^2 + |q|^2 \tag{3.35}$$

From (3.32) and (3.35) it follows that

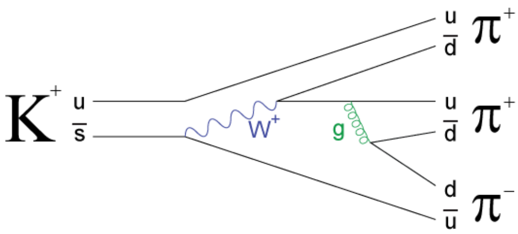
$$p = 1 + \epsilon \qquad \text{and} \qquad q = 1 - \epsilon. \tag{3.36}$$

Conservation of CP-symmetry would implicate $M_{12}, \Gamma_{12} \in \mathbb{R}$ and $\epsilon = 0$. From this follows

$$|K_{S,L}\rangle = |K_{1,2}\rangle, \tag{3.37}$$

i.e. implying that the mass eigenstates are equal to the CP eigenstates. As a consequence, there would not be any mixing between the CP eigenstates. Thus, CP-symmetry violation is necessary for the mixing!

The picture below shows the decay of the K^+:



3.1.2 Describing the Time Evolution

In this subsection, we again want to emphasize the time evolution of the system guided by the effective Hamiltonian and give a summary of the values that come out. Thus, we have the Schrödinger equation with the non hermitian effective mass Hamiltonian for the two state system, i.e.

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle, \tag{3.38}$$

which gives us the diagonalized mass eigenstates (using the Wigner-Weisskopf-Approximation):

$$\begin{aligned} |K_S(t)\rangle &= e^{-i\lambda_S t}|K_S\rangle, \\ |K_L(t)\rangle &= e^{-i\lambda_L t}|K_L\rangle. \end{aligned} \tag{3.39}$$

The explicit calculation then gives us

$$\begin{aligned} \lambda_S = m_S - \frac{i}{2}\Gamma_S = M_0 - \frac{i}{2}\Gamma_0 - \sqrt{(M_{12} - \frac{i}{2}\Gamma_{12})\cdot(M_{12}^* - \frac{i}{2}\Gamma_{12}^*)}, \\ \lambda_L = m_L - \frac{i}{2}\Gamma_S = M_0 - \frac{i}{2}\Gamma_0 + \sqrt{(M_{12} - \frac{i}{2}\Gamma_{12})\cdot(M_{12}^* - \frac{i}{2}\Gamma_{12}^*)} \end{aligned} \tag{3.40}$$

The differences in mass and decay, i.e. $\Delta m = m_L - m_S$ and $\Delta\Gamma = \Gamma_L - \Gamma_S$, we state by

$$\Delta m - \frac{i}{2}\Delta\Gamma = 2\sqrt{(M_{12} - \frac{i}{2}\Gamma_{12})\cdot(M_{12}^* - \frac{i}{2}\Gamma_{12}^*)}. \tag{3.41}$$

The second part of the equation we identify as

$$\sqrt{(M_{12} - \frac{i}{2}\Gamma_{12})\cdot(M_{12}^* - \frac{i}{2}\Gamma_{12}^*)} \equiv \frac{2pq}{C}, \tag{3.42}$$

where C must be a constant with dimension $[s/\hbar]$.

From the summary, i.e.

$$\begin{aligned} m_0 &= 500MeV \\ \Gamma_S &= \frac{1}{\tau_S} \approx 10^{10}\frac{1}{s} \\ \Gamma_L &= \frac{1}{\tau_L} \approx \frac{1}{600}\Gamma_S \\ \Delta m = m_L - m_S &\approx \frac{\Gamma_S}{2}, \end{aligned} \tag{3.43}$$

the immense lifetime difference of K_S and K_L, respectively, is again apparent.

3.1.3 Time Evolution Unitary

Now we will take a closer look to the time evolution of kaons. Kaons decay. Hence the decrease of the norm of the initial state $|K^0(t)\rangle$ must be incorporated to same amount by the norm of the final state $|f\rangle$, since we must suppose that the time evolution is unitary. Thus the time evolution of the K^0 meson (now including oscillation and decays) has the form

$$\left|K^0\right\rangle \longrightarrow a(t)\left|K^0\right\rangle + b(t)\left|\bar{K}^0\right\rangle + \sum_f c_f(t)\,|f\rangle\,, \tag{3.44}$$

with $a(t) = g_+(t)$ and $b(t) = \frac{q}{p}g_-(t)$.

Yet we will use a more convenient formalism. For the time evolution we will use an unitary operator U(t,0) acting on the eigenstates of the effective Hamiltonian, i.e. K_S and K_L. This has the form

$$U(t,0)\,|K_{S,L}\rangle = e^{-i\lambda_{S,L}t}\,|K_{S,L}\rangle + |\Omega_{S,L}(t)\rangle\,, \tag{3.45}$$

where $|\Omega_{S,L}(t)\rangle$ stays for all decay products. Here, it is important to stress that the $+$ in (3.45) is actually a direct sum $\oplus$, since the decayed states are not located in the same Hilbert space as the initial states, but in the othonormal Hilbert space. The direct sum was discussed in chapter 2.1.1.

Anyway, the use of the unitary operator U(t,0) gives us the opportunity to describe the evolution of time and thus we define

$$\left|K_{S/L}(t)\right\rangle \equiv U(t,0)\,|K_{S,L}\rangle\,. \tag{3.46}$$

The structure of this formalism, or, to be more precisely, the states $\Omega_{S,L}(t)$ allow to use the total Hilbert space, since not only the Hilbert space of the kaons themselves is of importance for entanglement, but also the Hilbert space of all other possible states, which can be reached by decay. One learned in particular that there might be some kind of entanglement, considering only the Hilbert space of the kaons, but investigating the whole Hilbert space, and thus enlarging the kaon's, the effects of entanglement may vanish. However, this formalism has not only advantages, of which more later. Let us return to the calculation of the time evolution.

Leaved off with (3.44), one can calculate the transition amplitudes of the decaying states by making use of the unitarity of U ($U^\dagger U = I$). Since

$$\langle K_S(t)|\, K_S(t)\rangle \equiv |K_S\rangle\, U^\dagger(t,0)U(t,0)\,|K_S\rangle = \langle K_S|\, K_S\rangle = 1, \tag{3.47}$$

one gets

$$1 = e^{-\Gamma_S t} + \langle \Omega_S(t)|\, \Omega_S(t)\rangle. \tag{3.48}$$

Here one assumed that

$$\langle K_S|\, \Omega_S(t)\rangle = 0. \tag{3.49}$$

Thus one gets

$$\begin{aligned} \langle \Omega_S(t)|\, \Omega_S(t)\rangle &= 1 - e^{-\Gamma_S t} \\ \langle \Omega_L(t)|\, \Omega_L(t)\rangle &= 1 - e^{-\Gamma_L t} \\ \langle \Omega_L(t)|\, \Omega_S(t)\rangle &= \langle K_L|\, K_S\rangle(1 - e^{i\Delta m t}e^{-\Gamma t}) \\ \left\langle K_{S/L}\right|\, \Omega_S(t)\rangle &= \left\langle K_{S/L}\right|\, \Omega_L(t)\rangle = 0. \end{aligned} \tag{3.50}$$

These results and the use of this unitary time evolution will be revived at the most important sector, namely when dealing with entanglement. It has become handy to introduce another idea when investigaten the decay: the open quantum formalism.

3.1.4 An Open Quantum Formalism

Another way to describe the decay (see 3.1.1) is given by the open quantum system formulation. In [52], [53] it has been shown that a master equation of the Lindblad type can describe a decaying system as well, whereupon the decay acts like a sort of decoherence and thus giving us another view to the problem. Therefore we will make use of this formalism in order to describe the kaon's decay.

We will start with the effective Schrödinger Equation in the Liouville-von Neumann form ($\hbar \equiv 1$):

$$\frac{d}{dt}\rho = -iH_{eff}\rho + i\rho H_{eff}^\dagger. \tag{3.51}$$

ρ is a 2 x 2 density matrix, while H_{eff} is non-Hermitian (see 3.1.1). From this we come to the time evolution of the short and long lived state:

$$\left|K_{S/L}(t)\right\rangle = e^{-im_{S/L}t}e^{-\frac{\Gamma_{S/L}}{2}t}\left|K_{S/L}\right\rangle, \tag{3.52}$$

where $m_{S/L}$ and $\Gamma_{S/L}$ are the masses and decay constants of the two states (see (3.43)). At first view, it becomes clear that for $t > 0$ the states (3.52) are no more normalized. Nevertheless, we start the new formalism by doubling the Hilbert space and thus incorporating the decay as dissipator of the new Hilbert space by a Lindblad operator. From the beginning: starting with the master equation in the Lindblad form, i.e.

$$\frac{d}{dt}\rho = -i\left[H,\rho\right] - D\left[\rho\right], \tag{3.53}$$

we have the dissipator given by

$$D\left[\rho\right] = \frac{1}{2}\sum_j (A_j^\dagger A_j\rho + \rho A_j^\dagger A_j - 2A_j\rho A_j^\dagger). \tag{3.54}$$

In this case, one doubles the two-dimensional Hilbert space to get a four-dimensional one, i.e.

$$H_{tot} = H_s \otimes H_f, \tag{3.55}$$

where H_s is responsible for the surviving part while H_f is for the decaying (final) part. In this Hilbert space H_{tot} lives the density matrix ρ, having the form

$$\rho = \begin{pmatrix} \rho_{ss} & \rho_{sf} \\ \rho_{fs} & \rho_{ff} \end{pmatrix}, \tag{3.56}$$

where the ρ's in (3.56) are 2 x 2-matrices themselves. Furthermore it is $\rho_{sf} = \rho_{fs}^\dagger$. Last but not least one extends the effective Hamiltonian relevant for the system to the total Hilbert space H_{tot} by

$$H = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.57}$$

Now, one decomposes the master equation into the components of the density matrix ρ from (3.56). Hence

$$\begin{aligned} \dot{\rho}_{ss} &= -i\left[H,\rho_{ss}\right] - \frac{1}{2}\left\{B^\dagger B, \rho_{ss}\right\} - \tilde{D}\left[\rho_{ss}\right], \\ \dot{\rho}_{sf} &= -iH\rho_{sf} - \frac{1}{2}B^\dagger B\rho_{sf} - \frac{1}{2}\sum_j A_j^\dagger A_j \rho_{sf}, \\ \dot{\rho}_{ff} &= B\rho_{ss}B^\dagger, \end{aligned} \tag{3.58}$$

where the A_0, A_j are Lindblad generators, whereas

$$A_0 = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \qquad \text{with} \qquad B : H_s \to H_f, \tag{3.59}$$

is incorporating the decay, while

$$A_j = \begin{pmatrix} A_j & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \qquad j \neq 0 \tag{3.60}$$

is responsible for the survive components of the density matrix ρ. Furthermore it is $B^\dagger B = \Gamma$ (Γ being the decay matrix of H_{eff}) and

$$\tilde{D}\left[\rho_{ss}\right] = \frac{1}{2}\sum_{j=1}(A_j^\dagger A_j \rho + \rho A_j^\dagger A_j - 2A_j \rho A_j^\dagger) \tag{3.61}$$

stands for any decoherence or dissipation which may emerge.

But what do we have now? Well, the master equation (3.58) replaces the Schrödinger equation completely, while we can say more about the density matrix (and thus about the decay):

1. the time evolution ρ_{ss} is independent of ρ_{sf}, ρ_{fs} and ρ_{ff}.
2. since ρ_{sf} and ρ_{fs} decouple from ρ_{ss}, we can w.l.o.g. set them to be zero, without changing any physics.
3. the initial condition $\rho_{ff}(0) = 0$, which is reasonable, implies that the time evolution only depends on ρ_{ss} and thus has the form

$$\rho_{ff}(t) = B \int dt' \rho_{ss}(t') B^\dagger, \tag{3.62}$$

which evidently shows the form we do expect concerning particle decay. Furthermore one can see that the decay is somehow concerned with decoherence and thus these two subjects belong together.

We will now start in introducing the "entangled side of life".

4 EPR Bell Entanglement

In this section we give the definition of entanglement and try to treat it with both, the so called local realistic theories (LRT) and quantum theory. We will find that one can derive an inequality (namely the Bell inequality) under a certain assumption, which is called Bell's locality assumption. This inequality predicts a certain result – better said a bound - when regarding the expectation values (or probabilities). Furthermore we will study this Bell inequality and for that compare the predictions of QM and LRT, finding that there is a measurable difference, which will be resolved by the experiment determining the winner: quantum mechanics.

4.1 Entanglement

It was Schrödinger himself who set the ball rolling in 1935 when studying the Schrödinger equation in a two particle system [13]. Anyhow, if Ψ_1 and Ψ_2 are solutions of the Schrödinger equation, then is any linear combination of Ψ_1 and Ψ_2 a solution, too. This holds for a multi-particle system, too. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be Hilbert basis of a two level system. Now the linear combination

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle\right) \tag{4.1}$$

is a solution of the Schrödinger equation, too.

This is a strong statement: if the left subsystem A (measured by Alice) detects the state $|\uparrow\rangle$, then the right subsystem (Bob) has no other choice than detecting $|\downarrow\rangle$ (supposed they have the same orientation, which is important) and vice versa. Thus the two states of the system - which are spatially separated - seem to have some kind of an invisible connection (correlation); Schrödinger called states of the form (4.1) - which is by the way one of the Bell states - entangled states (verschränkte Zustände). Before giving the mathematical definition of entanglement in the next section, we will conclude this section with some (familiar) states in physics, which are known to be entangled [47]:

$$\begin{aligned}
\left|\Psi^-\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|H\rangle \otimes |V\rangle - |V\rangle \otimes |H\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle \otimes \left|\bar{K}^0\right\rangle - \left|\bar{K}^0\right\rangle \otimes \left|K^0\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\left|B^0\right\rangle \otimes \left|\bar{B}^0\right\rangle - \left|\bar{B}^0\right\rangle \otimes \left|B^0\right\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|I\rangle \otimes |\Uparrow\rangle - |II\rangle \otimes |\Downarrow\rangle\right)
\end{aligned} \tag{4.2}$$

The states (4.2), all share the same scheme: a two state system, which is entangled. Either it is a spin-$\frac{1}{2}$ system, a photon system, where H/V stands for vertically/horizontally polarized, a meson-antimeson system, or even a single neutron which is travelling through a two-way interferometer (path I and path II), implicating that the spin depends on which path was taken.

4.2 Mathematical View of Entanglement

A bipartite **pure** state $|\Psi\rangle$ is called separable iff it can be written as a single tensor product, namely

$$|\Psi\rangle = \left|\Psi^A\right\rangle \otimes \left|\Psi^B\right\rangle. \tag{4.3}$$

Every nonseperable (pure) state is called entangled. Such a state has the form

$$|\Psi\rangle = \sum_i \sum_j c_{ij} \left|\Psi_i^A\right\rangle \otimes \left|\Psi_j^B\right\rangle, \tag{4.4}$$

with at least two components of c_{ij} not vanishing. Here, the $|\Psi_i\rangle$ build an ONB, i.e. a basis of the Hilbert space with pairwise orthogonal and to the lenght 1 normalized elements.

More complications arise if we consider mixed states: for mixed states it is not possible to write down explicitly the form of entangled states. Hence we call a state entangled iff it cannot be written in the form of a separable states, namely

$$\rho = \sum_{i=1} p_i \rho_i^A \otimes \rho_i^B, \tag{4.5}$$

where p_i stands for the probabilities, so that $p_i \geq 0$ and $\sum_i p_i = 1$.
Since a more detailed analysis of this problem would go beyond the scope of this thesis, it is only important to add that precisely because of the impossibility of writing down a unified form of entangled mixed states, it can be enormously difficult to find out whether a state is entangled or separable [28] [32] [33] [34] [35].

4.3 The EPR Scenario

"Spooky action at distance." That was the phrase Einstein used, when writing about entanglement [14]. His expression is an indication for that he was not really satisfied with the whole construct (of interpretation) that was build around quantum mechanics. On the one hand there was the statistical and even non-deterministic and random ("God does not play dice") face of quantum mechanics and on the other hand some kind of action at distance which he had been able to "eliminate" in gravitation theory some years ago. Now by use of the uncertainty relation of Heisenberg [15] - which he tried to outsmart by cleverer and cleverer gedankenexperiments - he wanted to use the non classical properties of quantum mechanics to show once and for all that the grand new theory, quantum mechanics, was incomplete and thus simply spoken: false.
We will start with the classical (in the sense of popular) Heisenberg's uncertainty principle, namely using position and momentum (here $\hbar \neq 1$):

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \tag{4.6}$$

or alternatively written

$$\langle \Delta x \rangle_\psi \langle \Delta p \rangle_\psi \geq \frac{\hbar}{2}, \tag{4.7}$$

Now the concept of the EPR paradox is about completeness, reality and locality of quantum theory [17]. EPR wanted to show that the general quantum mechanical idea, that is to say that a system only comes to have a proper state when s.o. measured it, was wrong, since a value of a parameter should be real, no matter if there is a measurement or not. Thus the nondeterministic character should drop out. On the other side, however, Einstein could not get used to the concept of nonlocality (keyword: spooky action at a distance), so that - if procurable - the violation of Heisenberg's uncertainty relation would show the incompleteness of quantum mechanics.

Therefore EPR created following gedankenexperiment. Let us consider two particles emitted by a source and flying apart from each other with opposite momenta of equal (but random) magnitude. Then we measure the momentum of particle 1 while leaving particle 2 as it is. The knowledge of the momentum of particle 1 (now the momentum has changed due to the measurement itself, but this is no longer of importance) allows us to predict the momentum of particle 2: the momentum vector just changed the sign (supposed, they are same sort of particles). Now we can perform a measurement of the position of particle 2 and get now both kinds of informations of the complementary observables, i.e. position and momentum. With this simple gedankenexperiment one just either violated the uncertainty relation or showed that quantum mechanics cannot be complete, since the experiment shows that both observables, position and momentum, are in fact part of one reality while quantum mechanics cannot predict both at one time. This, however, turned out not to be true.

Before resolving this it is advantageous to introduce another configuration of the experiment. Since we deal with particle physics, too, let us have a pion decaying (among others) into an electron and positron. Thus the emitted two particles have opposite spins (in every direction), i.e. one has $|\uparrow\rangle$ and the other has $|\downarrow\rangle$. Now since the spin components in the x_1, x_2 and x_3 direction are complementary to each other, (and due to the uncertainty relation) one cannot measure one without disturbing the others. Thus, if one measured the spin of the electron's x_1-axis, one can definitely tell what spin-state the positron is in (concerning x_1-axis). If we measure the positrons x_2-axis, we can even say the positron has spin about two axes, which contradicts to our concept of spin.

In short: some arguments used in the above paragraph have to be wrong. At first: Heisenberg's uncertainty relation holds for 2 observables, which can be measured for identically preparated systems, not for two different particles. Niels Bohr argued [18] that since the set-up was arranged in a way such that it was no more possible to repeat the *complementary* experiment (i.e. to switch the order of measurements) and thus the two observables have to be complementary and not part of one reality. The Copenhagen interpretation of QM, however, solves the seeming paradox by negating that the measurement on particle 1 has anything to do with a determination of any property of particle 2.

Since most of this belongs to the category "interpretation", we have to start in searching for physical descriptions:

4.4 Hidden Variables

One could say of course, there is no correlation between the particles and therefore one could try to explain entanglement and nonlocality by intrinsic properties of the particles. It would be appropriate to introduce so called hidden variables which assign the particle for every event or measurement a proper value. For example, let us consider the photon case where we are able to measure the polarizations vertical V or horizontal H of two entangled photons. Now every photon has the intrinsic information (by hidden variables) what value to show if measured e.g. in y direction. Thus if photon A shows $|H\rangle$, photon B shows exactly the opposite (as arranged at the beginning), namely $|V\rangle$, guaranteeing the (no more) spooky behavior of entanglement. It is instantly clear that the two photons must have an infinite number of "arrangements", since we can measure any arbitrary angle (i.e. choose any direction). For every angle (direction) Θ the measurement outcomes must be anticorrelated. But what if we took different angles? Let us consider the situation where we measure Θ for photon A and Φ for photon B, with $\Theta \neq \Phi$. This is going to be investigated in the following section.

4.5 The Bell inequality

4.5.1 Set Theoretical Derivation

Consider the following scenario. Alice (A) and Bob (B) set up two Stern-Gerlach apparatuses, as illustrated in the Figure below:

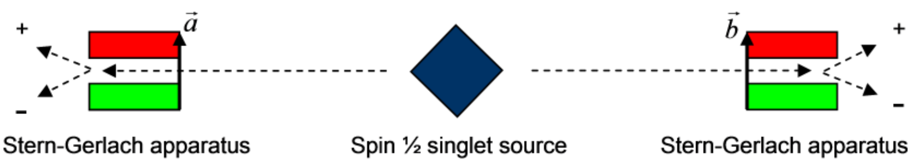


Figure 4.1: Scheme

The source (in the middle) produces two particles which fly to Alice and Bob, respectively, whereupon they can make independently measurements. We denote the possible outcomes of the measurements with + or −, respectively. Next, Alice and Bob have the ability to measure in one out of three directions each, i.e. $\vec{a}, \vec{b}$ and $\vec{c}$. Investigating the possible combinations of measurements, it is easy to see that there are 8 possible combinations by frequency N_i, namely [47]

As a direct consequence, the relation

$$N_3 + N_4 \leq N_2 + N_3 + N_4 + N_7. \tag{4.8}$$

holds. The probability P for Alice to obtain the result + if her measurement setting was $\vec{a}$ and Bob measuring + in direction $\vec{b}$ is given by

$$P(\vec{a}+, \vec{b}+) = \frac{N_3 + N_4}{\sum N_i}. \tag{4.9}$$

Frequency	Alice	Bob
N_1	$(\vec{a}+, \vec{b}+, \vec{c}+)$	$(\vec{a}-, \vec{b}-, \vec{c}-)$
N_2	$(\vec{a}+, \vec{b}+, \vec{c}-)$	$(\vec{a}-, \vec{b}-, \vec{c}+)$
N_3	$(\vec{a}+, \vec{b}-, \vec{c}+)$	$(\vec{a}-, \vec{b}+, \vec{c}-)$
N_4	$(\vec{a}+, \vec{b}-, \vec{c}-)$	$(\vec{a}-, \vec{b}+, \vec{c}+)$
N_5	$(\vec{a}-, \vec{b}+, \vec{c}+)$	$(\vec{a}+, \vec{b}-, \vec{c}-)$
N_6	$(\vec{a}-, \vec{b}+, \vec{c}-)$	$(\vec{a}+, \vec{b}-, \vec{c}+)$
N_7	$(\vec{a}-, \vec{b}-, \vec{c}+)$	$(\vec{a}+, \vec{b}+, \vec{c}-)$
N_8	$(\vec{a}-, \vec{b}-, \vec{c}-)$	$(\vec{a}+, \vec{b}+, \vec{c}+)$

Table 7: The possible measurements by Alice and Bob

In the same manner one can find

$$P(\vec{a}+, \vec{c}+) = \frac{N_2 + N_4}{\sum N_i}, \tag{4.10}$$

$$P(\vec{c}+, \vec{b}+) = \frac{N_3 + N_7}{\sum N_i}. \tag{4.11}$$

Using the probabilities and inserting them into (4.8) one then obtains

$$P(\vec{a}+, \vec{b}+) \leq P(\vec{a}+, \vec{c}+) + P(\vec{c}+, \vec{b}+). \tag{4.12}$$

This is a Bell inequality!

What does quantum mechanics say regarding this inequality? Let us take the Bell state $|\psi^-\rangle$. In the $\{|0\rangle, |1\rangle\}$-notation it has the form

$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}} \left\{|01\rangle - |10\rangle\right\}. \tag{4.13}$$

The probability for Alice finding "+" at $\vec{a}$ and "+" at $\vec{b}$ is given by

$$\begin{aligned} P(\vec{a}+, \vec{b}+) = & \quad Tr\left(\left|\vec{a+}\right\rangle\left\langle\vec{a+}\right| \otimes \left|\vec{b+}\right\rangle\left\langle\vec{b+}\right| \left|\psi^-\right\rangle\left\langle\psi^-\right|\right) \\ = & \quad \left\|\left|\vec{a+}\right\rangle\left\langle\vec{a+}\right| \otimes \left|\vec{b+}\right\rangle\left\langle\vec{b+}\right| \left|\Psi^-\right\rangle\right\|^2 \end{aligned} \tag{4.14}$$

We take a parametrization, i.e.

$$\begin{aligned} |\vec{a}+\rangle &= \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle \\ |\vec{a}-\rangle &= -\sin\alpha\,|0\rangle + \cos\alpha\,|1\rangle \\ \left|\vec{b}+\right\rangle &= \cos\beta\,|0\rangle + \sin\beta\,|1\rangle \\ \left|\vec{b}+\right\rangle &= -\sin\beta\,|0\rangle + \cos\beta\,|1\rangle \end{aligned} \tag{4.15}$$

$$\tag{4.16}$$

Strictly speaking, the phase $e^{i\phi}$ is missing, but since the BI can be violated even without it, it can be neglected so far. However, it comes useful visualizing the common picture referred to the parametrization. It has the form

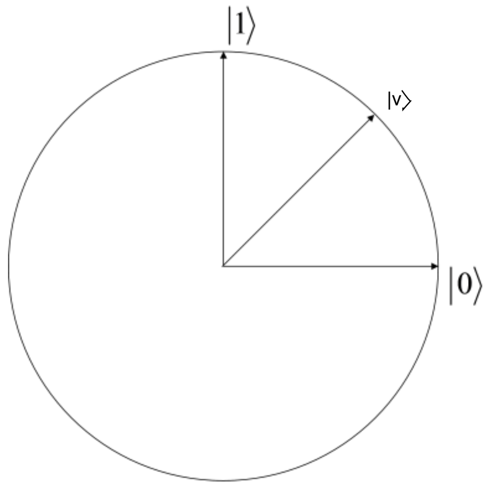


Figure 4.2: Parametrization with an arbitrary vector $|v\rangle$

Now inserting the parametrization into (4.14), the probability assumes the form

$$\begin{aligned} P(\vec{a}+,\vec{b}+) &= \frac{1}{2}\left\|\cos\alpha\sin\beta\,|\vec{a}+\rangle\left|\vec{b}+\right\rangle - \sin\alpha\cos\beta\,|\vec{a}+\rangle\left|\vec{b}+\right\rangle\right\|^2 \\ &= \frac{1}{2}\left|\cos\alpha\sin\beta - \sin\alpha\cos\beta\right|^2 \\ &= \frac{1}{2}\cos^2(\alpha-\beta) \\ &= \frac{1}{4}\left(1+\cos\phi_{ab}\right), \end{aligned} \tag{4.17}$$

where ϕ_{ab} is the angle between $\vec{a}$ and $\vec{b}$.

Using this result and inserting it into the Bell inequality (4.12), one gets

$$\frac{1}{4}\left(1+\cos\phi_{ab}\right) \leq \frac{1}{4}\left(1+\cos\phi_{ac}\right) + \frac{1}{4}\left(1+\cos\phi_{cb}\right), \tag{4.18}$$

which leads to the simplified inequality

$$\cos\phi_{ab} \leq 1 + \cos\phi_{ac} + \cos\phi_{cb}. \tag{4.19}$$

Let

$$\phi_{ac} = \phi_{cb} = 2\phi, \phi = \frac{\pi}{2}, \phi_{ab} = \phi, \tag{4.20}$$

then the inequality becomes

$$0 \leq 1 - 1 - 1 = -1, \tag{4.21}$$

which shows a contradiction! Thus, the Bell inequality may be violated by quantum mechanical predictions while the assumptions of locality *and* realism lead to probabilities which have to satisfy the BI.

4.5.2 Original Derivation by J. Bell

As introduced in the original paper [20], let us consider a system of two spin-1/2 particles in the so called singlet state, i.e.

$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}\left\{\left|+\right\rangle\left|-\right\rangle - \left|-\right\rangle\left|+\right\rangle\right\}, \tag{4.22}$$

where the notation has been changed again to suggest that $|\pm\rangle$ are the states for the spin components $\pm\hbar/2$ in z-direction. Measuring a spin then corresponds to multiplying the Pauli matrices to the direction of the measurement, i.e.

$$\vec{\sigma}\cdot\vec{a}. \tag{4.23}$$

If the pauli matrices are in the $|\pm\rangle$ basis, (4.23) has the form

$$\begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}. \tag{4.24}$$

We use this to derive the (quantum mechanical) expectation value E^{QM} for both spin measurements:

$$\begin{aligned} E^{QM}(\vec{a},\vec{b}) = \left\langle\psi^-\right|\vec{\sigma}\cdot\vec{a}\otimes\vec{\sigma}\cdot\vec{b}\left|\psi^-\right\rangle = \frac{1}{2}[&\langle+|\,\vec{\sigma}\cdot\vec{a}\,|+\rangle\langle-|\,\vec{\sigma}\cdot\vec{b}\,|-\rangle \\ &- \langle+|\,\vec{\sigma}\cdot\vec{a}\,|-\rangle\langle-|\,\vec{\sigma}\cdot\vec{b}\,|+\rangle \\ &- \langle-|\,\vec{\sigma}\cdot\vec{a}\,|+\rangle\langle+|\,\vec{\sigma}\cdot\vec{b}\,|-\rangle \\ &+ \langle-|\,\vec{\sigma}\cdot\vec{a}\,|-\rangle\langle-|\,\vec{\sigma}\cdot\vec{b}\,|-\rangle] \\ &= \frac{1}{2}[-a_3b_3 - (a_1 - ia_2)(b_1 + ib_2) - (a_1 + ia_2)(b_1 - ib_2) - a_3b_3] \\ &= -[a_1b_1 + a_2b_2 + a_3b_3] = -\vec{a}\cdot\vec{b} \end{aligned} \tag{4.25}$$

Every "complete" - to use the same term as EPR did - theory should be able to reproduce this result (which is predicted by quantum mechanics already). To "complete up" quantum mechanics, one introduces a parameter λ, which may be an arbitrary mathematical object. Now the result of a spin measurement depends on the angle $\vec{a}$ and λ. W.l.o.g. we let λ be real. The results of measuring the observables (A,B) are denoted as above

$$A(\vec{a},\lambda) = \pm 1, B(\vec{b},\lambda) = \pm 1. \tag{4.26}$$

So A,B are functions which only depend on $\vec{a}$ ($\vec{b}$) and λ. Furthermore we introduce an arbitrary probability distribution $\rho(\lambda)$ to get the expectation value E(a,b), namely by

$$E(a,b) = \int d\lambda\rho(\lambda)A(\vec{a},\lambda)\cdot B(\vec{b},\lambda), \tag{4.27}$$

which is normalized by $\int d\lambda\rho(\lambda) = 1$. It is important to stress that the dot "$\cdot$" represents "Bell's locality assumption", i.e. that obsevable A does not depend on the choice for B's observable and vice verca. From (4.25) it is obvious that

$$\begin{aligned} E(a,a) = -E(a,-a) = -1 \\ E(a,a^n) = 0. \end{aligned} \tag{4.28}$$

From this John S. Bell constructed an inequality which can be violated using the quantum mechanical expectation value E^{QM} [20]. (4.27) cannot be lower than -1, and this can be achieved with $\vec{a} = \vec{b}$ only iff

$$A(\vec{a}, \lambda) = -B(\vec{b}, \lambda). \tag{4.29}$$

Assuming this, we can rewrite (4.27) into

$$E(\vec{a}, \vec{b}) = -\int d\lambda \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda) \tag{4.30}$$

Considering the difference of the following expectation values, we get

$$\begin{aligned} E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c}) &= -\int d\lambda \rho(\lambda)[A(\vec{a}, \lambda)A(\vec{b}, \lambda) - A(\vec{a}, \lambda)A(\vec{c}, \lambda)] \\ &= \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda)[A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1]. \end{aligned} \tag{4.31}$$

Here, we used (4.26) and therewith $A(\vec{a}, \lambda)^2 = B(\vec{b}, \lambda)^2 = 1$. Furthermore,

$$\begin{aligned} \left| E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c}) \right| &= \left| \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda)[A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1] \right| \\ &\leq \int d\lambda \left| \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda)[A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1] \right| \\ &= \int d\lambda \rho(\lambda) \left| [A(\vec{b}, \lambda)A(\vec{c}, \lambda) - 1] \right| \\ &= \int d\lambda \rho(\lambda)[1 - A(\vec{b}, \lambda)A(\vec{c}, \lambda)] \end{aligned} \tag{4.32}$$

Explicitly written, (4.32) has the form

$$\left| E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c}) \right| \leq 1 + E(\vec{b}, \vec{c}), \tag{4.33}$$

the Bell inequality. We derived it using only Bell's locality assumption. Hence, all local realistic theories have to satisfy this inequality. We will again ask Quantum Mechanics. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors and let

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \frac{1}{\sqrt{2}}, \qquad \vec{a} \cdot \vec{c} = 0. \tag{4.34}$$

Then our Bell inequality (4.33) reads

$$1 - \frac{1}{\sqrt{2}} \geq \frac{1}{\sqrt{2}}, \tag{4.35}$$

which is, indeed, a contradiction.

4.5.3 The CHSH - Bell inequality

Since the CHSH inequality (Clause, Horne, Shimony, Holt) is more adapted to reality and thus easier to implement experimentally, it will be presented as well. One of the most different aspects in contrast to the original BI is that the CHSH inequality allows non-perfect situations; thus, the average measurement A, B must not be 1/-1, i.e. $|A|, |B| \leq 1$, since not every particle has to be detected [22]. Now the expectation value depends on arbitrary variables n, m. It is given by

$$E(n,m) = \int d\lambda \rho(\lambda) A(n,\lambda) \cdot B(m,\lambda). \tag{4.36}$$

Again we have $\int d\lambda \rho(\lambda) = 1$ and presume Bell's locality assumption. Subtracting, i.e.

$$\begin{aligned} E(n,m) - E(n,m') &= \int d\lambda \rho(\lambda) \left\{ A(n,\lambda)B(m,\lambda) - A(n,\lambda)B(m',\lambda) \right\} \\ &= \int d\lambda \rho(\lambda) A(n,\lambda)B(m,\lambda) \left\{ 1 \pm A(n,\lambda)B(m',\lambda) \right\} \\ &- \int d\lambda \rho(\lambda) A(n,\lambda)B(m',\lambda) \left\{ 1 \pm A(n',\lambda)B(m,\lambda) \right\}, \end{aligned} \tag{4.37}$$

adding a term that will be subtracted again and turning the minus in (4.37) into a plus one can certainly claim that

$$\begin{aligned} |E(n,m) - E(n,m')| \leq &\int d\lambda \rho(\lambda) \left\{ 1 \pm A(n',\lambda)B(m',\lambda) \right\} \\ + &\int d\lambda \rho(\lambda) \left\{ 1 \pm A(n',\lambda)B(m,\lambda) \right\}. \end{aligned} \tag{4.38}$$

holds. Rewritten, we can bring it into a more appealing form, namely

$$|E(n,m) - E(n,m')| \leq 2 + E(n',m') + E(n',m), \tag{4.39}$$

and by defining a correlation function S, we get

$$S(n,m,n',m') = |E(n,m) - E(n,m')| + E(n',m') + E(n',m) \leq 2, \tag{4.40}$$

or, without using absolute values,

$$-2 \leq E(n,m) - E(n,m') + E(n',m') + E(n',m) \leq 2. \tag{4.41}$$

Again, we will compare this with quantum mechanics. This time, we will use another formalism to get the answer. That is because we will use the same formalism when dealing with entangled kaons, see section 5.

However, the quantum mechanical expectation value has the form

$$\begin{aligned} E^{QM}(\vec{n},\vec{m}) &= \left\langle \psi^- \right| \vec{n}\vec{\sigma} \otimes \vec{m}\vec{\sigma} \left| \psi^- \right\rangle \\ &= Tr\left(\vec{n}\vec{\sigma} \otimes \vec{m}\vec{\sigma} \left| \psi^- \right\rangle \left\langle \psi^- \right| \right) \\ &= -n_1 m_1 - n_2 m_2 - n_3 m_3 = -\vec{n}\vec{m} = -\cos\phi_{nm}. \end{aligned} \tag{4.42}$$

Thus the correlation function S has the form

$$S = |-\cos\phi_{nm} + \cos\phi_{nm'}| + |-\cos\phi_{n'm} - \cos\phi_{n'm'}| \leq 2 \tag{4.43}$$

The left hand side of (4.43) turns to be for the arrangement of directions illustrated in the picture below:

$$
\begin{aligned}
S &= \left|-\cos\frac{\pi}{4} + \cos\frac{3\pi}{4}\right| + \left|-\cos\frac{\pi}{4} - \cos\frac{\pi}{4}\right| \\
&= \left|-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right| + \left|-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right| \\
&= 2\sqrt{2} \approx= 2.8,
\end{aligned}
\tag{4.44}
$$

which certaintly is a contradiction, again!

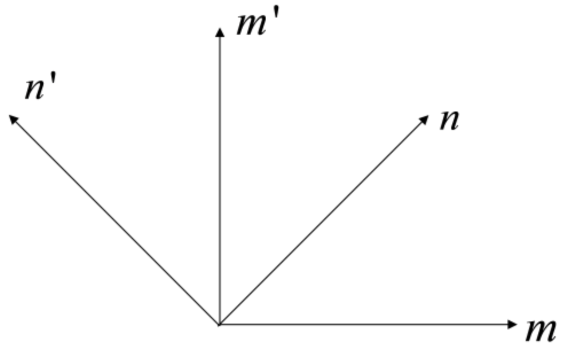


Figure 4.3: Possible Arrangement of the Directions n, n', m, m'

So there is obviously a measurable difference between quantum mechanics and all local realistic theories. Thus an experiment can determine which of the two theories describes the world correctly (actually, which describes the world incorrectly).

5 Kaons and Entanglement

The main part of the thesis. After formulating the time evolution for entangled kaons, we will set up a Bell inequality and investigate it in the kaon system. After discussing the problems arising through the common formalism we will introduce a new formalism which will be recapitulated as well.

5.1 Preparing

We will now be concerned with states having the form as introduced in section 3, i.e. entangled states:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle \right). \tag{5.1}$$

Applying to kaons we get

$$|\Psi\rangle = \frac{1}{N_{SL}\sqrt{2}} \left(|K_S\rangle \otimes |K_L\rangle - |K_L\rangle \otimes |K_S\rangle \right). \tag{5.2}$$

Actually, (5.2) has this particular form just at the beginning where $t = 0$. Since kaons decay, we have to use a proper time evolution. Thus the $|\Psi\rangle$ is strictly speaking a $|\psi(t=0)\rangle$. Furthermore the normalization $N_{SL} = \frac{N^2}{2pq}$ comes from the $K_S K_L$ basis. Choosing the strangeness-basis, we get

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}} \left(\left|K^0\right\rangle \otimes \left|\bar{K}^0\right\rangle - \left|\bar{K}^0\right\rangle \otimes \left|K^0\right\rangle \right), \tag{5.3}$$

which is the practically produced state in accelerator facilities.

5.1.1 Time Evolution of Entangled Kaons

As introduced in section 3.1.3, we will now make use of the unitary operator $U(t,0)$ which acts (to repeat) as follows:

$$U(t,0)\,|K_{S,L}\rangle = e^{-i\lambda_{S,L}t}\,|K_{S,L}\rangle + |\Omega_{S,L}(t)\rangle. \tag{5.4}$$

Note again that (5.4) is in fact a direct sum (see 2.1.1). For the time evolution of (5.2) we need to compose the time evolutions for K_S and K_L, respectively. This gets us to

$$U(t,0) = U_l(t,0) \otimes U_r(t,0), \tag{5.5}$$

where $U_l(t,0), U_r(t,0)$, belonging to the left and right (space of the) meson, respectively, act like (5.4) in the ordinary way. Furthermore they fulfill the composition law, i.e.

$$U(t_2,0) = U_l(t_2,t_1) \cdot U_r(t_1,0). \tag{5.6}$$

Applying the unitary time evolution onto the the initial state (5.2), we get at time t_r:

$$|\psi(t_r)\rangle = U(t_r, 0)\,|\psi(t=0)\rangle = U_l(t_r, 0) \otimes U_r(t_r, 0)\,|\psi(t=0)\rangle\,. \tag{5.7}$$

Now we are able to compute the probabilities whether the state is in a certain quasispin (on the left and right hand side, in each case) or not. These quasispin states are called $|k_n\rangle_l$ and $|k_n\rangle_r$, respectively. For calculation of the probabilities we use the projection operators $P_{l,r}(k_n)$ that act onto the quasispin states, and the projection operators $Q_{l,r}(k_n)$ that act onto the orthogonal states, which are quoted below:

$$\begin{aligned} P_l(k_n) &= |k_n\rangle_l\,\langle k_n|_l\,, \\ P_r(k_n) &= |k_n\rangle_r\,\langle k_n|_r \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} Q_l(k_n) &= 1 - P_l(k_n), \\ Q_r(k_n) &= 1 - P_r(k_n). \end{aligned} \tag{5.9}$$

After using (5.7) in order to describe the time evolution, will now perform a measurement at $|k_m\rangle$ at t_r on the right hand side. For this we have to project - according to QM - onto the state. Hence the new state becomes

$$\left|\tilde{\psi}(t_r)\right\rangle = P_r(k_m)\,|\psi(t_r)\rangle\,. \tag{5.10}$$

As long as we do not measure on the left hand side, (5.10) represents the current state. When applying a measurement on the left hand side, the state derives to

$$\left|\tilde{\psi}(t_l, t_r)\right\rangle = P_l(k_n) U_l(t_l, t_r) P_r(k_m)\,|\psi(t_r)\rangle\,. \tag{5.11}$$

Anyway, moving together P_l, P_r and U_l, U_r and applying them onto $|\psi(t=0)\rangle$, we get a state called $|\Psi\rangle$, i.e.

$$|\Psi(t_l, t_r)\rangle = P_l(k_n) P_r(k_m) U_l(t_l, t_r) U_r(t_l, t_r)\,|\psi(t=0)\rangle\,, \tag{5.12}$$

which can be seen as a factorization of eigentime t_l and t_r, respectively. Now there is a theorem [46] which claims that these two states, i.e. (5.11) and (5.12) have the same norms. Hence

$$\left\|\left|\tilde{\psi}(t_l, t_r)\right\rangle\right\|^2 = \||\psi(t_l, t_r)\rangle\|^2\,. \tag{5.13}$$

The results from this chapter are important for the following ones, especially when constructing a Bell inequality.

5.1.2 The $\{Y, Y\}$, $\{N, N\}$, $\{Y, N\}$ and $\{N, Y\}$ probabilities

Armed with the results above we can now calculate the quantum mechanical probabilities for finding certain quasispins at certain times on the left and right hand side. For this, we call every event, where on the left hand side at time t_l a k_n has been measured, $\mathcal{P}_n(Y, t_l)$. If, however, on the left hand side at time t_l the k_n has *not* been measured, we call the event $\mathcal{P}_n(N, t_l)$. Thus, the probabilities for measuring on both sides have the form (procedure shown in (5.1.1) explicitly):

$$
\begin{aligned}
\mathcal{P}_{n,m}(Y, t_l; Y, t_r) &= \|P_l(k_n)P_r(k_m)U_l(t_l, 0)U_r(t_r, 0)\,|\psi(t=0)\rangle\|^2 \\
\mathcal{P}_{n,m}(N, t_l; N, t_r) &= \|Q_l(k_n)Q_r(k_m)U_l(t_l, 0)U_r(t_r, 0)\,|\psi(t=0)\rangle\|^2 \\
\mathcal{P}_{n,m}(Y, t_l; N, t_r) &= \|P_l(k_n)Q_r(k_m)U_l(t_l, 0)U_r(t_r, 0)\,|\psi(t=0)\rangle\|^2 , \\
\mathcal{P}_{n,m}(N, t_l; Y, t_r) &= \|Q_l(k_n)N_r(k_m)U_l(t_l, 0)U_r(t_r, 0)\,|\psi(t=0)\rangle\|^2 , \qquad (5.14)
\end{aligned}
$$

with

$$
\mathcal{P}_{n,m}(Y, t_l; Y, t_r) + \mathcal{P}_{n,m}(N, t_l; N, t_r) + \mathcal{P}_{n,m}(Y, t_l; N, t_r) + \mathcal{P}_{n,m}(N, t_l; Y, t_r) = 1. \qquad (5.15)
$$

The latter is a consequence of the claimed unitary transformation. To get a brief glimpse of the calculating procedure it is useful to calculate the probabilities.
Let us derive the probability $\mathcal{P}_{\bar{K}^0,\bar{K}^0}(Y, t_l; Y, t_r)$, i.e. the probability of finding (or not) a $\bar{K}^0$ on the left hand side and finding a $\bar{K}^0$ on the right hand side at time t_l and t_r. The projection operators $P(\bar{K}^0)$ act on the short lived and long lived states $|K_S\rangle$ and $|K_L\rangle$, respectively, as this:

$$
P(\bar{K}^0)\,|K_S\rangle = |\bar{K}^0\rangle \langle \bar{K}^0|\,|K_S\rangle = \frac{-q}{N}\,|\bar{K}^0\rangle \qquad (5.16)
$$

$$
P(\bar{K}^0)\,|K_L\rangle = |\bar{K}^0\rangle \langle \bar{K}^0|\,|K_L\rangle = \frac{q}{N}\,|\bar{K}^0\rangle . \qquad (5.17)
$$

From this we get

$$
\begin{aligned}
&P_{\bar{K}^0,\bar{K}^0}(Y, t_l; Y, t_r) = \left\|P_l(\bar{K}^0)P_r(\bar{K}^0)U_l(t_l, 0)U_r(t_r, 0)\,|\psi(0)\rangle\right\|^2 \\
&= \frac{N^4}{8\,|p|^2\,|q|^2}\|P_l(\bar{K}^0)P_r(\bar{K}^0) \\
&\{(e^{-i\lambda_S t_l}\,|K_S\rangle_l + |\Omega_S(t_l)\rangle_l) \otimes (e^{-i\lambda_L t_r}\,|K_L\rangle_r + |\Omega_L(t_r)\rangle_r) \\
&- (e^{-i\lambda_L t_l}\,|K_S\rangle_l + |\Omega_L(t_l)\rangle_l) \otimes (e^{-i\lambda_S t_r}\,|K_S\rangle_r + |\Omega_S(t_r)\rangle_r)\}\|^2 \\
&= \frac{N^4}{8\,|p|^2\,|q|^2}\left\|\frac{-q\cdot q}{N\cdot N}\left\{e^{-i\lambda_S t_l - i\lambda_L t_r} - e^{-i\lambda_L t_l - i\lambda_S t_r}\right\}|\bar{K}^0\rangle_l\,|\bar{K}^0\rangle_r\right\|^2 \\
&= \frac{|q|^2}{8\,|p|^2}\left\{e^{-\Gamma_S t_l - \Gamma_L t_r} + e^{-\Gamma_L t_l - \Gamma_S t_r} - 2\cos(\Delta m \Delta t)\cdot e^{-\Gamma(t_l+t_r)}\right\} , \qquad (5.18)
\end{aligned}
$$

which can also be expressed via the charge asymmetry parameter, i.e.

$$
\begin{aligned}
&P_{\bar{K}^0,\bar{K}^0}(Y, t_l; Y, t_r) = \\
&\frac{1-\delta}{8(1+\delta)}\left\{e^{-\Gamma_S t_l - \Gamma_L t_r} + e^{-\Gamma_L t_l - \Gamma_S t_r} - 2\cos(\Delta m \Delta t)\cdot e^{-\Gamma(t_l+t_r)}\right\} \qquad (5.19)
\end{aligned}
$$

5.2 The Bell inequality

We are now prepared to formulate a Bell inequality for the kaon system. As introduced in the previous chapter, we must assume Bell's locality hypothesis. That is, the correlation function $O(k_n, t_a; k_m, t_b)$ must be equal to the procut of the two observables O_l (left hand side) and O_r (right hand side). Thus, applied to kaons the hypothesis assumes the form

$$O(k_n, t_a; k_m, t_b) = O_l(k_n, t_a) \cdot O_r(k_m, t_b). \tag{5.20}$$

That is to say that there is no mutual dependence on the measurements of the two sides: what Alice is measuring on her left hand side does not influence Bob's measurement on the right hand side.

Following the derivations in section 4.5.3 and assuming (5.20), one gets the relation

$$\begin{aligned} &|O(k_n, t_a; k_m, t_b) - O(k_n, t_a; k_{m'}, t_d)| \\ &+ |O(k_{n'}, t_c; k_{m'}, t_d) + O(k_{n'}, t_c; k_m, t_b)| = 2, \end{aligned} \tag{5.21}$$

where $k_n, k_m, k_{n'}$ and $k_{m'}$ are arbitrary quasispin eigenstates and t_a, t_b, t_c, t_d are four times. That is to say that Alice can choose among two settings, i.e. (k_n, t_a) and $(k_{n'}, t_c)$, while Bob can choose between (k_m, t_b) and $(k_{m'}, t_d)$. It is important to stress that (5.21) is not an inequality, since it contains pure states. This can be shown when using all combinations of the values $\{-1\}$ and $\{1\}$ for the observables and thus building the correlation function (5.20). To get an inequality we can obtain the average M of the correlation function O when considering N identical measurements and thus denoting with O_i the value of the correlation function made in the ith experiment:

$$M(k_n, t_a; k_m, t_b) = \frac{1}{N} \sum_{i=1}^{N} O_i(k_n, t_a; k_m, t_b) \tag{5.22}$$

One notices that now $|M| \leq 1$. Performing the same operations when constructing the generalized Bell-CHSH inequality in chapter 3, we finally get

$$\begin{aligned} &|M(k_n, t_a; k_m, t_b) - M(k_n, t_a; k_{m'}, t_d)| \\ &+ |M(k_{n'}, t_c; k_{m'}, t_d) - M(k_{n'}, t_c; k_m, t_b)| \leq 2. \end{aligned} \tag{5.23}$$

Thus, (5.23) is a CHSH inequality equal to the spin-$\frac{1}{2}$ or photon case, as long as we identify $M(k_n, t_a; k_m, t_b) \equiv M(n, m)$.

5.3 The CHSH inequality

Using the quantum mechanical probabilities introduced in chapter 5.1.2 we can reexpress the expectation value, i.e.

$$M^{QM}(k_n, t_a; k_m, t_b) = P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b) \\ -P_{n,m}(Y, t_a; N, t_b) - P_{n,m}(N, t_a; Y, t_b). \quad (5.24)$$

This form is valid for both, quantum mechanics and local realistic theories. Using

$$P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b) \\ +P_{n,m}(Y, t_a; N, t_b) + P_{n,m}(N, t_a; Y, t_b) = 1, \quad (5.25)$$

we finally get

$$M(k_n, t_a; k_m, t_b) = -1 + 2\left\{P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b)\right\}. \quad (5.26)$$

Now comes the point: setting (5.24) into the CHSH inequality, i.e.

$$|M(n, m) - M(n, m')| \leq 2 \pm |M(n', m') + M(n', m)| \quad (5.27)$$

we finally arrive to

$$|P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b) - P_{n,m'}(Y, t_a; Y, t_d) - P_{n,m'}(N, t_a; N, t_d)| \\ \leq 1 \pm \{-1 + P_{n',m}(Y, t_c; Y, t_b) + P_{n',m}(N, t_c; N, t_b) \\ +P_{n',m'}(Y, t_c; Y, t_d) + P_{n',m'}(N, t_c; N, t_d)\}, \quad (5.28)$$

or rewritten (and expressed as (5.23)):

$$S(k_n, k_m, k_{n'}, k_{m'}; t_a, t_b, t_c, t_d) = \\ \|P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b) - P_{n,m'}(Y, t_a; Y, t_d) - P_{n,m'}(N, t_a; N, t_d)\| \\ +\| - 1 + P_{n',m}(Y, t_c; Y, t_b) + P_{n',m}(N, t_c; N, t_b) + P_{n',m'}(Y, t_c; Y, t_d) + P_{n',m'}(N, t_c; N, t_d)\| \leq 1. \quad (5.29)$$

This is a CHSH inequality which can now be tested.

5.4 Results and Problems

Since the files with the whole results are posted in the APPENDIX we will just consider the way of proceeding.

At first we start with the most general state (in the $K_S K_L$-basis), i.e.

$$|\psi\rangle = r_1 e^{i\varphi_1} |K_S K_S\rangle + r_2 e^{i\varphi_2} |K_S K_L\rangle + r_3 e^{i\varphi_3} |K_L K_S\rangle + r_4 e^{i\varphi_4} |K_L K_L\rangle. \quad (5.30)$$

However, we do not use the direct method (as introduced in 5.1.1 by calculating the norm) for calculating the $\{Y, Y\}$, $\{N, N\}$, $\{Y, N\}$ and $\{N, Y\}$ probabilities. We trace over the product of the density matrix created by the most general state (5.30) and a matrix standing for the particular probabilities, which we will call

PYY, PYN, PNY and PNN, respectively. The reason for this way of proceeding is that we can set both the density matrix and the matrix standing for the particular probabilities into the most general form and thus just by varying over angles having the capability to choose the quasi-spin states which we want. Anyway, the $\{Y, Y\}$, $\{N, N\}$, $\{Y, N\}$ and $\{N, Y\}$ probabilities are now accessible over

$$\begin{aligned}
\{Y, Y\} &= Tr\left[PYY \cdot \rho(t_l, r_r)\right] \\
\{N, N\} &= Tr\left[PNN \cdot \rho(t_l, r_r)\right] \\
\{Y, N\} &= Tr\left[PYN \cdot \rho(t_l, r_r)\right] \\
\{N, Y\} &= Tr\left[PNY \cdot \rho(t_l, r_r)\right].
\end{aligned} \tag{5.31}$$

Now, equipped with these details, we construct the expectation value introduced in (5.24), namely:

$$\begin{aligned}
M^{QM}(k_n, t_a; k_m, t_b) = &P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b) \\
&-P_{n,m}(Y, t_a; N, t_b) - P_{n,m}(N, t_a; Y, t_b).
\end{aligned} \tag{5.32}$$

5.5 A New Formalism: Schrödinger Picture → Heisenberg Picture

To be able to compute the values for the Bell inequality more easily, we have to change the formalism. It would be convenient if we could get a factorization inside of the expectation value, to get it from a form like (5.24), i.e.

$$\begin{aligned}
M(k_n, t_a; k_m, t_b) = &P_{n,m}(Y, t_a; Y, t_b) + P_{n,m}(N, t_a; N, t_b) \\
&-P_{n,m}(Y, t_a; N, t_b) - P_{n,m}(N, t_a; Y, t_b)
\end{aligned} \tag{5.33}$$

into

$$E = Tr\left(O_{eff} \left|\psi\right\rangle \left\langle\psi\right|\right), \tag{5.34}$$

where we could insert all information like time evolution into the effective operator O_{eff} while ψ can set to be the most general state. What we want to perform is the switch from the Schrödinger picture into the Heisenberg picture. By this, we can avoid the expanding of the Hilbert space (as introduced in (3.1.3), while the mathematical computation gets simpler.

Again, our prospect of the expectation value E is of the form:

Is the state in the quasispin k_n at time t_n (Yes) or is it not (No)?

As in (5.24), this has the following form:

$$\begin{aligned}
E(k_n, t_n) &= P(\text{Yes} : k_n, t_n) - P(\text{No} : k_n, t_n) \\
&\underset{P(\text{No}:k_n,t_n)+P(\text{Yes}:k_n,t_n)=1}{=} 2\,P(\text{Yes} : k_n, t_n) - 1.
\end{aligned} \tag{5.35}$$

To switch from the Schrödinger picture to the Heisenberg picture, we commence writing down the probabilities via the trace over the products of a quasispin k_n and a general initial state ρ, i.e.

$$\begin{aligned}
P(\text{Yes} : k_n, t_n) &= Tr(\begin{pmatrix} |k_n\rangle\langle k_n| & 0 \\ 0 & 0 \end{pmatrix} \rho(t_n)) \\
&= \rho_{SS} \cdot \cos^2 \frac{\alpha_n}{2} e^{-\Gamma_S t_n} + \rho_{LL} \cdot \sin^2 \frac{\alpha_n}{2} e^{-\Gamma_L t_n} \\
&+ \rho_{SL} \cdot \cos \frac{\alpha_n}{2} \sin \frac{\alpha_n}{2}\, e^{i(\phi_n - t_n)} \cdot e^{-\Gamma t_n} \\
&+ (\rho_{SL} \cdot \cos \frac{\alpha_n}{2} \sin \frac{\alpha_n}{2}\, e^{i(\phi_n - t_n)} \cdot e^{-\Gamma t_n})^* \,. \qquad (5.36)
\end{aligned}$$

Again, we use the parametrizations

$$|k_n\rangle = \cos \tfrac{\alpha_n}{2} |K_S\rangle + \sin \tfrac{\alpha_n}{2} \cdot e^{i\phi_n} |K_L\rangle \,, \qquad (5.37)$$

while $\rho(t)$ is derived from the master equation in section 3.1.4, see (3.53). Also a convenient re-scaling was used, i.e.

$$\Delta m := 1 \quad \text{and} \quad \Gamma_i := \frac{\Gamma_i}{\Delta m}. \qquad (5.38)$$

Now we extract the time dependence from the initial state ρ and insert it into a time dependent effective operator with dimensions 2 x 2. The expectation value now gets into

$$E(k_n, t_n) = Tr(O^{eff}(\alpha_n, \phi_n, t_n)\, \rho), \qquad (5.39)$$

which is the desired result. In this way, we have found an effective operator in the Heisenberg picture (for the complete form of O^{eff} see the APPENDIX), which allows to handle with general decaying systems and which has besides the computational and interpretative advantage a conceptual one, since we can easy generalize the description of multipartite systems. This is made simply by using the usual tensor product structure, i.e.

$$\begin{aligned}
& E(k_{n_1}, t_{n_1}; k_{n_1}, t_{n_1}; \ldots ; k_{n_k}, t_{n_k}) \qquad (5.40) \\
= \;& Tr(O^{eff}(\alpha_{n_1}, \phi_{n_1}, t_{n_1}) \otimes O^{eff}(\alpha_{n_2}, \phi_{n_2}, t_{n_2}) \otimes \cdots \otimes O^{eff}(\alpha_{n_k}, \phi_{n_k}, t_{n_k})\, \rho) \,.
\end{aligned}$$

Equipped with this we can start testing Bell inequalities for arbitrary choices.

5.6 Results

Using the effective operator framework we can rather rewrite the Bell-CHSH inequality in a witness type, i.e using the Bell operator

$$\mathbf{Bell}^{eff} = O_n^{eff} \otimes (O_m^{eff} - O_{m'}^{eff}) + O_{n'}^{eff} \otimes (O_m^{eff} + O_{m'}^{eff}), \qquad (5.41)$$

in the sense that any local realistic hidden parameter theory has to satisfy

$$|Tr(\mathbf{Bell}^{eff}\rho)| \;\; \leq \;\; 2\,. \tag{5.42}$$

Furthermore the effective operator framework allows us to analyze the operator O_{eff} itself for the purpose of examining the local behavior of the whole kaon system. Details hereto are given in the paper [40].

Let us come back to the Bell-CHSH inequality in a witness type. For this we will decompose the effective operator O_{eff} into the Pauli matrices σ, i.e.

$$O^{eff}(\alpha_n, \phi_n, t_n) = -n_0(\alpha_n, t_n) \quad + \vec{n}(\alpha_n, \phi_n, t_n)\vec{\sigma} \tag{5.43}$$

with

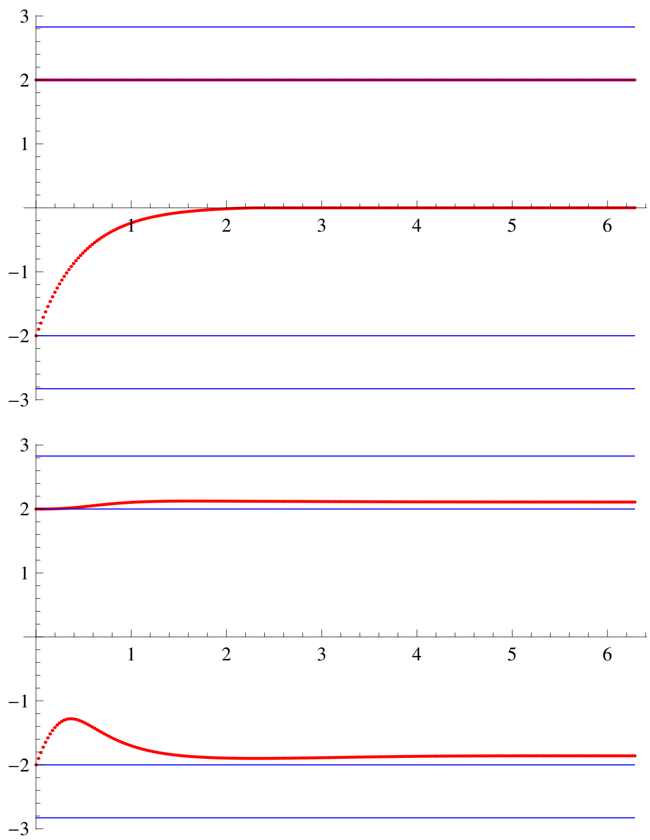
$$\Delta\Gamma = \frac{\Gamma_L - \Gamma_S}{2}, \tag{5.44}$$

$$\vec{n}(\alpha_n, \phi_n, t_n) = e^{-\Gamma t_n} \begin{pmatrix} \cos(t_n + \phi_n)\sin(\alpha_n) \\ \sin(t_n + \phi_n)\sin(\alpha_n) \\ \sinh(\Delta\Gamma t_n) + \cosh(\Delta\Gamma t_n)\cos\alpha_n \end{pmatrix}, \tag{5.45}$$

while

$$n_0(\alpha_n, t_n) = 1 - |\vec{n}(\alpha_n, \phi_n, t_n)|. \tag{5.46}$$

After forming the Bell operator (5.41) we let MATHEMATICA find the minimal and maximal eigenvalues of $\mathbf{Bell}^{eff}$ and plot the evolution after varying in time. Fig. 5.1 shows the results.



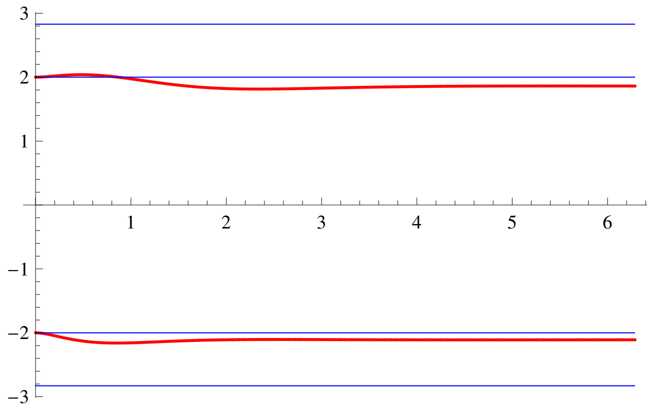


Figure 5.1: Violations of the Bell inequality by the strangeness eigenstates (i.e. $\alpha_n = \alpha_m = \alpha_{nn} = \alpha_{mm} = \pi/2$) at $\{t_n = t_m = t_{n'} = t_{m'} = t\}$, (b) $\{t_n = 0, t_m = t, t_{n'} = t, t_{m'} = 0\}$ and (c) $\{t_n = t, t_m = 0, t_{n'} = 0, t_{m'} = t\}$.

One can see that the new formalism opens up new possibilities of approaching and examining the kaon-entanglement question. Since we can now handle arbitrary Bell inequalities, it has become possible to analyze and investigate the $Bell^{eff}$ itself. Thus it has become a goal to find a proper Bell inequality which can be violated and tested in accelerator facilities.

5.7 The Entropic Uncertainty Relation

The last part of the thesis will be the discussion of the entropic uncertainty relation (introduced by D. Deutsch [36], improved in Ref. [37] and proven by Ref. [38]). It is given by

$$H(O_n^{eff}) + H(O_m^{eff}) \geq -2\log_2\left(\max_{i,j}\{|\langle\chi_n^i|\chi_m^j\rangle|\}\right) \tag{5.47}$$

where

$$H(O_n^{eff}) = -p(n)\log_2 p(n) - (1-p(n))\log_2(1-p(n)) \tag{5.48}$$

is the binary entropy for a certain prepared state ρ. In case of pure states, the $p(n)$'s are given by

$$p(n) = |\langle\chi_n|\psi\rangle|^2, \tag{5.49}$$

i.e. are the probability distribution associated with the measurement of O_n^{eff} for ψ (provided, O_n^{eff} is $2x2$) . One finds that the maximal value of the right hand side of (5.47) is obtained for

$$|\langle\chi_n|\chi_m\rangle| = \frac{1}{\sqrt{2}}. \tag{5.50}$$

In this case the two observables are called complementary to each other (priorly, their eigenvalues have to be nondegenerate). The detailed discussion can be found at [40].

We will now discuss what is learnt by finding a certain quasispin $|k_n\rangle$ at a certain time t_n or not which can correspond to a certain decay channel, compared to the situation to find a k_m at the creation point $t_m = 0$ or not. If, for example, Alice prepares a certain state ψ and sends it to Bob. What Alice wants is to minimize her uncertainty concerning Bob's measurement result. Bob will now carry out one of the two measurements O_n^{eff}, O_m^{eff} and announce his choice to Alice (n or m). The result Alice can get is bounded by the equation (5.47).

In paper [40] the right hand side of the entropic uncertainty relation (5.47) was investigated. One found that for unstable systems, the right hand side, given by

$$\max\left\{\langle\chi_m^1|\chi_n^1\rangle, \langle\chi_m^1|\chi_n^2\rangle, \langle\chi_m^2|\chi_n^1\rangle, \langle\chi_m^2|\chi_n^2\rangle\right\}, \tag{5.51}$$

has $|\chi_n^1\rangle = |\chi(\alpha_n, \phi_n, t_n)\rangle$ and $|\chi_n^2\rangle = |\chi(\alpha_n+\pi, \phi_n+2t_n, -t_n)\rangle$ being the eigenvectors of the effective operators or - in other words - the quasispin propagating forward or backward in time, respectively. Below, Fig. 5.2 shows the results.

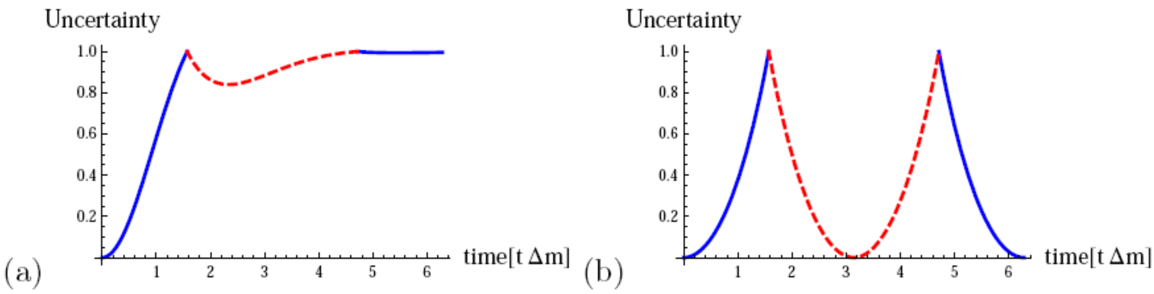


Figure 5.2: The lower bound of the entropic quantum uncertainty inequality (5.47) is plotted in case of a strangeness event at $t = 0$ compared to a strangeness event at a later time. Thus the observables are given by $A = O_{eff}(\frac{\pi}{2}, \phi_n, 0)$ and $B = O_{eff}(\frac{\pi}{2}, \phi_m, t)$ with $\phi_n = \phi_m = 0$. The solid blue line shows when the eigenvectors both propagating forward in time or both propagating backward in time overlap maximally, whereas the red dashed line shows the case when forward and backward propagating quasipins overlap. While figure (a) depicts the kaon system, figure (b) shows the case of a slow decaying system or all other meson systems (i.e . B_d, B_s,..), that is $\Delta\Gamma = 0$.

In the APPENDIX, the investigation of the left hand side of (5.47) can be found. When plotting one of the entropies (5.48), one finds for strangeness events that this matches with what we have learned from Fig. 5.2. This is shown below in Fig. 5.3.

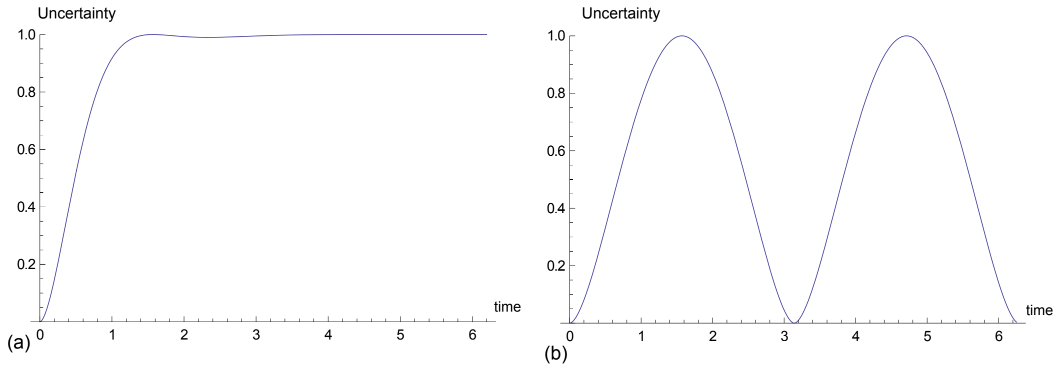


Figure 5.3: Since the entropy tends to zero at t=0, figure (a) and (b) correspond to the same questions as in Fig.5.2. Again, (a) depicts the kaon system while (b) shows the case of a slow decaying system or all other meson systems.

6 Summary, Outlook and Conclusion

Physicists still agonize over the one question: *how does the world work?* On the one hand it seems we have come quite near to an answer, waiting for a positive reply from the LHC in Genf, verifying that we have understood the concept of mass by finding the famous Higgs particle. On the other hand, however, we are in our infancy when trying to combine the two great theories of physics, quantum theory and theory of relativity. But what we can admit at any rate is: there is progress. Not least because of our understanding that we got from the kaon system: on one side it brought us into the concept of strangeness, while helping us to comprehend and analyze the important CP violation on the other side. Of course, there is even more to say about kaons, for example the concept of entanglement. This is the main topic of this diploma thesis.

After introducing the history of the detection of kaons in chapter I, we started with the recapitulation of basic mathematics in order to describe kaons in a proper way in chapter II. After pointing out crucial effects like strangeness oscillation and regeneration, we learned about the important formalism of describing the time evolution of kaons (including the decay property). Investigating the decay property, we also learned that decay can act as some kind of decoherence. In chapter III, however, we started rolling out the concept of entanglement. Here we could set up an inequality (the Bell inequality), allowing us to determine that local realistic theories (LRT) are not compatible with the existing world around us. In chapter IV we then had to describe the time evolution for entangled kaons for setting up a Bell inequality sensitive to kaon properties. In this way we were able to investigate the entanglemend of kaons. We realized that we had to change the formalism to be capable of handling the decay of kaons when dealing with Bell inequalities.

Armed with this new effective formalism, we are able to make more precise investigations of the kaon entanglement, especially by investigating the Bell inequality (operator) in the whitness type. Moreover, using this formalism we are able to handle arbitrary Bell inequalities and therefore we are searching for a suitable and useful Bell inequality when regarding the difficult accelerator experiments. Hence, it seems to be only a matter of time before finding a proper Bell inequality which can be violated by (relative) simple experimental set up. Last but not least we discussed Heisenberg's uncertainty relation adapted to kaons.

A Effective Operator Formalism

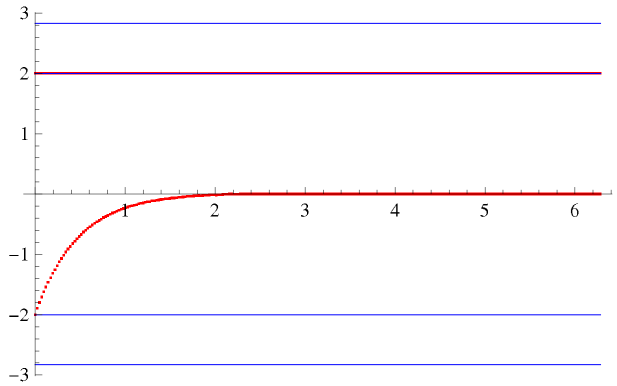
Construction and Testing of the effective Operator

```
σ[0] = IdentityMatrix[2];
σ[1] = {{0, 1}, {1, 0}};
σ[2] = {{0, -I}, {I, 0}};
σ[3] = {{1, 0}, {0, -1}};
bn[0] = 1 / 2 (e^(-tn ΓL) + e^(-tn ΓS) - 2 + (-e^(-tn ΓL) + e^(-tn ΓS)) Cos[αn]);
bn[1] = Cos[tn + ϕn] Sin[αn] e^(-1/2 (tn) (ΓL+ΓS));
bn[2] = Sin[αn] Sin[tn + ϕn] e^(-1/2 (tn) (ΓL+ΓS));
bn[3] = 1 / 2 (-e^(-tn ΓL) + e^(-tn ΓS) + (e^(-tn ΓL) + e^(-tn ΓS)) Cos[αn]);
Op[αn_, ϕn_, tn_] = Sum[bn[i] σ[i], {i, 0, 3}] // Simplify
ρ = {{ r1^2,  r1 r2 e^(i (ϕ1-ϕ2)),  r1 r3 e^(i (ϕ1-ϕ3)),  r1 r4 e^(i (ϕ1-ϕ4))},
    { r1 r2 e^(i (-ϕ1+ϕ2)), r2^2, r2 r3 e^(i (ϕ2-ϕ3)),  r2 r4 e^(i (ϕ2-ϕ4))},
    { r1 r3 e^(i (-ϕ1+ϕ3)),  r2 r3 e^(i (-ϕ2+ϕ3)),  r3^2,  r3 r4 e^(i (ϕ3-ϕ4))},
    {r1 r4 e^(i (-ϕ1+ϕ4)), r2 r4 e^(i (-ϕ2+ϕ4)), r3 r4 e^(i (-ϕ3+ϕ4)),  r4^2}};
bell = KroneckerProduct[Op[αn, ϕn, tn], Op[αm, ϕm, tm] - Op[αmm, ϕmm, tmm]] +
    KroneckerProduct[Op[αnn, ϕnn, tnn], Op[αm, ϕm, tm] + Op[αmm, ϕmm, tmm]];
```

$$\left\{\left\{e^{\,\mathrm{tn}\,\Gamma S}\left(1 - e^{\mathrm{tn}\,\Gamma S} + \mathrm{Cos}[\alpha n]\right),\ e^{\,\frac{1}{2}\mathrm{tn}\,(\Gamma L+\Gamma S)}\,\mathrm{Sin}[\alpha n]\,(\mathrm{Cos}[\mathrm{tn}+\phi n] - i\,\mathrm{Sin}[\mathrm{tn}+\phi n])\right\},\ \left\{e^{\,\frac{1}{2}\mathrm{tn}\,(\Gamma L+\Gamma S)}\,\mathrm{Sin}[\alpha n]\,(\mathrm{Cos}[\mathrm{tn}+\phi n] + i\,\mathrm{Sin}[\mathrm{tn}+\phi n]),\ -e^{\,\mathrm{tn}\,\Gamma L}\left(-1 + e^{\mathrm{tn}\,\Gamma L} + \mathrm{Cos}[\alpha n]\right)\right\}\right\}$$

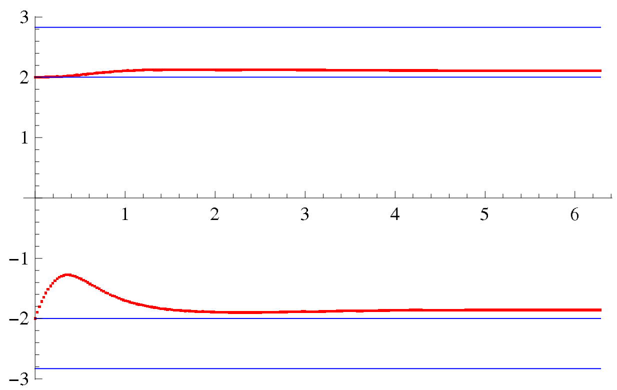
```
ΓS = 1.119 / 0.5300;
ΓL = 0.00193 / 0.5300 ;
tn = tmm = t;
tnn = tm = t;
αn = αm = αnn = αmm = Pi / 2;
ϕn = ϕm = ϕnn = ϕmm = 0;
tmax = 2 Pi;
tstep = Pi / 32 / 4;
Clear[i]
Do[test1 = Re[Eigenvalues[bell]]; listbellmax[t] = Max[test1];
 listbellmin[t] = Min[test1], {t, 0, tmax, tstep}]
Clear[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm,
 tnn, tmm, r1, r2, r3, r4, ϕ1, ϕ2, ϕ3, ϕ4, ΓS, ΓL]
q1 = ListPlot[Table[{t, listbellmax[t]}, {t, 0, tmax, tstep}],
    PlotStyle → {Red, PointSize[0.005]}];
q2 = ListPlot[Table[{t, listbellmin[t]}, {t, 0, tmax, tstep}],
    PlotStyle → {Red, PointSize[0.005]}];
bellgrenzen = Plot[{2 Sqrt[2], -2 Sqrt[2], 2, -2}, {t, 0, tmax}, PlotStyle -> {Blue}];
```

```
Show[q1, q2, bellgrenzen]
```



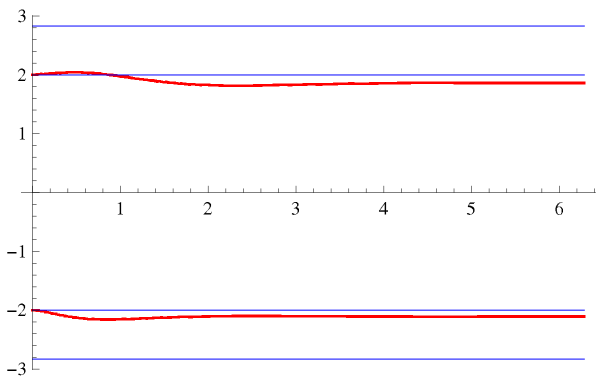

```
ΓS = 1.119 / 0.5300;
ΓL = 0.00193 / 0.5300 ;
tn = tmm = 0;
tnn = tm = t;
αn = αm = αnn = αmm = Pi / 2;
ϕn = ϕm = ϕnn = ϕmm = 0;
tmax = 2 Pi;
tstep = Pi / 32 / 4;
Clear[i]
Do[test1 = Re[Eigenvalues[bell]]; listbellmax[t] = Max[test1];
 listbellmin[t] = Min[test1], {t, 0, tmax, tstep}]
Clear[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm,
 tnn, tmm, r1, r2, r3, r4, ϕ1, ϕ2, ϕ3, ϕ4, ΓS, ΓL]
q1 = ListPlot[Table[{t, listbellmax[t]}, {t, 0, tmax, tstep}],
   PlotStyle → {Red, PointSize[0.005]}];
q2 = ListPlot[Table[{t, listbellmin[t]}, {t, 0, tmax, tstep}],
   PlotStyle → {Red, PointSize[0.005]}];
bellgrenzen = Plot[{2 Sqrt[2], -2 Sqrt[2], 2, -2}, {t, 0, tmax}, PlotStyle -> {Blue}];
```

```
Show[q1, q2, bellgrenzen]
```



```
ΓS = 1.119 / 0.5300;
ΓL = 0.00193 / 0.5300 ;
tn = tmm = t;
tnn = tm = 0;
αn = αm = αnn = αmm = Pi / 2;
ϕn = ϕm = ϕnn = ϕmm = 0;
tmax = 2 Pi;
tstep = Pi / 32 / 4;
Clear[i]
Do[test1 = Re[Eigenvalues[bell]]; listbellmax[t] = Max[test1];
 listbellmin[t] = Min[test1], {t, 0, tmax, tstep}]
Clear[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm,
 tnn, tmm, r1, r2, r3, r4, ϕ1, ϕ2, ϕ3, ϕ4, ΓS, ΓL]
q1 = ListPlot[Table[{t, listbellmax[t]}, {t, 0, tmax, tstep}],
   PlotStyle → {Red, PointSize[0.005]}];
q2 = ListPlot[Table[{t, listbellmin[t]}, {t, 0, tmax, tstep}],
   PlotStyle → {Red, PointSize[0.005]}];
bellgrenzen = Plot[{2 Sqrt[2], -2 Sqrt[2], 2, -2}, {t, 0, tmax}, PlotStyle -> {Blue}];

Show[q1, q2, bellgrenzen]
```



B Former Formalism

Preparation:

```
PYY = {{Cos[αn / 2] ^2 Cos[αm / 2] ^2, Cos[αn / 2] ^2 Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] ^2,
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
     0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {Cos[αn / 2] ^2 Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]), Cos[αn / 2] ^2 Sin[αm / 2] ^2,
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Sin[αm / 2] ^2, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0}, {Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] ^2,
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
     Sin[αn / 2] ^2 Cos[αm / 2] ^2, Sin[αn / 2] ^2 Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
     0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Sin[αm / 2] ^2,
     Sin[αn / 2] ^2 Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
     Sin[αn / 2] ^2 Sin[αm / 2] ^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}};
PYN = {{Cos[αn / 2] ^2 (1 - Cos[αm / 2] ^2), -Cos[αn / 2] ^2 Cos[αm / 2] Sin[αm / 2]
      (Cos[ϕm] - I Sin[ϕm]), Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) (1 - Cos[αm / 2] ^2),
     -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
     0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {-Cos[αn / 2] ^2 Cos[αm / 2]
      Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]), Cos[αn / 2] ^2 (1 - Sin[αm / 2] ^2),
     -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) (1 - Sin[αm / 2] ^2), 0, 0, 0, 0, 0, 0,
     0, 0, 0, 0, 0, 0}, {Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) (1 - Cos[αm / 2] ^2),
     -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
     Sin[αn / 2] ^2 (1 - Cos[αm / 2] ^2), -Sin[αn / 2] ^2 Cos[αm / 2]
      Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {-Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) (1 - Sin[αm / 2] ^2),
     -Sin[αn / 2] ^2 Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
     Sin[αn / 2] ^2 (1 - Sin[αm / 2] ^2), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, Cos[αn / 2] ^2, 0, Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]),
     0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, Cos[αn / 2] ^2, 0,
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]), 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0,
     Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]), 0, Sin[αn / 2] ^2, 0, 0, 0, 0, 0, 0, 0, 0, 0},
    {0, 0, 0, 0, 0, Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]), 0, Sin[αn / 2] ^2,
```

```
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}};
PNY = {{(1 - Cos[αn / 2]^2) Cos[αm / 2]^2, (1 - Cos[αn / 2]^2) Cos[αm / 2] Sin[αm / 2]
     (Cos[ϕm] - I Sin[ϕm]), -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2]^2,
   -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {(1 - Cos[αn / 2]^2) Cos[αm / 2]
    Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]), (1 - Cos[αn / 2]^2) Sin[αm / 2]^2,
   -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
   -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] - I Sin[ϕn]) Sin[αm / 2]^2, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0}, {-Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2]^2,
   -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]),
   (1 - Sin[αn / 2]^2) Cos[αm / 2]^2, (1 - Sin[αn / 2]^2) Cos[αm / 2]
    Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]), 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {-Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
   -Cos[αn / 2] Sin[αn / 2] (Cos[ϕn] + I Sin[ϕn]) Sin[αm / 2]^2,
   (1 - Sin[αn / 2]^2) Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
   (1 - Sin[αn / 2]^2) Sin[αm / 2]^2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0,
   Cos[αm / 2]^2, Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]), 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
   Sin[αm / 2]^2, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   Cos[αm / 2]^2, Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] - I Sin[ϕm]), 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, Cos[αm / 2] Sin[αm / 2] (Cos[ϕm] + I Sin[ϕm]),
   Sin[αm / 2]^2, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}};
PNN = IdentityMatrix[16] - PYY - PYN - PNY;

rho[t1_, t2_] =
```

$$\Big\{\Big\{e^{(-t1-t2)\,\Gamma S}\,r1^2,\ e^{i\,(mL-mS)\,t2}\,e^{-\frac{t2\,\Gamma L}{2}-t1\,\Gamma S-\frac{t2\,\Gamma S}{2}}\,r1\,r2\,e^{i\,(\phi1-\phi2)},\ e^{i\,(mL-mS)\,t1}\,e^{-\frac{t1\,\Gamma L}{2}-\frac{t1\,\Gamma S}{2}-t2\,\Gamma S}\,r1\,r3\,e^{i\,(\phi1-\phi3)},$$

$$e^{\frac{1}{2}\,i\,(t1+t2)\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\,r1\,r4\,e^{i\,(\phi1-\phi4)},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{e^{-i\,mL\,t2+i\,mS\,t2-\frac{t2\,\Gamma L}{2}-t1\,\Gamma S-\frac{t2\,\Gamma S}{2}}\,r1\,r2\,e^{i\,(-\phi1+\phi2)},\ e^{-t2\,\Gamma L-t1\,\Gamma S}\,r2^2,\ e^{i\,mL\,(t1-t2)-i\,mS\,(t1-t2)-\frac{1}{2}\,(t1+t2)\,(\Gamma L+\Gamma S)}$$

$$r2\,r3\,e^{i\,(\phi2-\phi3)},\ e^{i\,mL\,t1-i\,mS\,t1-\frac{t1\,\Gamma L}{2}-t2\,\Gamma L-\frac{t1\,\Gamma S}{2}}\,r2\,r4\,e^{i\,(\phi2-\phi4)},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{e^{-i\,mL\,t1+i\,mS\,t1-\frac{t1\,\Gamma L}{2}-\frac{t1\,\Gamma S}{2}-t2\,\Gamma S}\,r1\,r3\,e^{i\,(-\phi1+\phi3)},\ e^{-i\,mL\,(t1-t2)+i\,mS\,(t1-t2)-\frac{1}{2}\,(t1+t2)\,(\Gamma L+\Gamma S)}\,r2\,r3\,e^{i\,(-\phi2+\phi3)},$$

$$e^{-t1\,\Gamma L-t2\,\Gamma S}\,r3^2,\ e^{i\,mL\,t2-i\,mS\,t2-t1\,\Gamma L-\frac{t2\,\Gamma L}{2}-\frac{t2\,\Gamma S}{2}}\,r3\,r4\,e^{i\,(\phi 3-\phi 4)},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{e^{-\frac{1}{2}\,(t1+t2)\,(2\,i\,mL-2\,i\,mS+\Gamma L+\Gamma S)}\,r1\,r4\,e^{i\,(-\phi 1+\phi 4)},\ e^{-i\,mL\,t1+i\,mS\,t1-\frac{t1\,\Gamma L}{2}-t2\,\Gamma L-\frac{t1\,\Gamma S}{2}}\,r2\,r4\,e^{i\,(-\phi 2+\phi 4)},$$

$$e^{-i\,mL\,t2+i\,mS\,t2-t1\,\Gamma L-\frac{t2\,\Gamma L}{2}-\frac{t2\,\Gamma S}{2}}\,r3\,r4\,e^{i\,(-\phi 3+\phi 4)},\ e^{(-t1-t2)\,\Gamma L}\,r4^2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ e^{(-t1-t2)\,\Gamma S}\left(-1+e^{t2\,\Gamma S}\right)r1^2,\ e^{-t1\,\Gamma S}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r2\,e^{i\,(\phi 1-\phi 2)},$$

$$e^{\frac{1}{2}\,i\,t1\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\left(1-e^{-t2\,\Gamma S}\right)r1\,r3\,e^{i\,(\phi 1-\phi 3)},$$

$$e^{\frac{1}{2}\,i\,t1\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r4\,e^{i\,(\phi 1-\phi 4)},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ e^{-t1\,\Gamma S}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r2\,e^{i\,(-\phi 1+\phi 2)},\ e^{-t1\,\Gamma S}\left(1-e^{-t2\,\Gamma L}\right)r2^2,$$

$$e^{\frac{1}{2}\,i\,t1\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r2\,r3\,e^{i\,(\phi 2-\phi 3)},$$

$$e^{\frac{1}{2}\,i\,t1\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\left(1-e^{-t2\,\Gamma L}\right)r2\,r4\,e^{i\,(\phi 2-\phi 4)},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ e^{\frac{1}{2}\,t1\,(-2\,i\,mL+2\,i\,mS-\Gamma L-\Gamma S)}\left(1-e^{-t2\,\Gamma S}\right)r1\,r3\,e^{i\,(-\phi 1+\phi 3)},$$

$$e^{-\frac{1}{2}\,t1\,(2\,i\,mL-2\,i\,mS+\Gamma L+\Gamma S)}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r2\,r3\,e^{i\,(\phi 3-\phi 2)},\ e^{-t1\,\Gamma L}\left(1-e^{-t2\,\Gamma S}\right)r3^2,$$

$$e^{-t1\,\Gamma L}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r3\,r4\,e^{i\,(\phi 3-\phi 4)},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ e^{-\frac{1}{2}\,t1\,(2\,i\,mL-2\,i\,mS+\Gamma L+\Gamma S)}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r4\,e^{i\,(-\phi 1+\phi 4)},$$

$$e^{\frac{1}{2}\,t1\,(-2\,i\,mL+2\,i\,mS-\Gamma L-\Gamma S)}\left(1-e^{-t2\,\Gamma L}\right)r2\,r4\,e^{i\,(\phi 4-\phi 2)},\ e^{-t1\,\Gamma L}\sqrt{1-e^{-t2\,\Gamma L}}\ \sqrt{1-e^{-t2\,\Gamma S}}\ r3\,r4\,e^{i\,(\phi 4-\phi 3)},$$

$$e^{(-t1-t2)\,\Gamma L}\left(-1+e^{t2\,\Gamma L}\right)r4^2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ e^{(-t1-t2)\,\Gamma S}\left(-1+e^{t1\,\Gamma S}\right)r1^2,$$

$$e^{\frac{1}{2}\,i\,t2\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\left(1-e^{-t1\,\Gamma S}\right)r1\,r2\,e^{i\,(\phi 1-\phi 2)},\ e^{-t2\,\Gamma S}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r1\,r3\,e^{i\,(\phi 1-\phi 3)},$$

$$e^{\frac{1}{2}\,i\,t2\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r1\,r4\,e^{i\,(\phi 1-\phi 4)},\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ e^{\frac{1}{2}\,t2\,(-2\,i\,mL+2\,i\,mS-\Gamma L-\Gamma S)}\left(1-e^{-t1\,\Gamma S}\right)r1\,r2\,e^{i\,(\phi 2-\phi 1)},$$

$$e^{-t2\,\Gamma L}\left(1-e^{-t1\,\Gamma S}\right)r2^2,\ e^{-\frac{1}{2}\,t2\,(2\,i\,mL-2\,i\,mS+\Gamma L+\Gamma S)}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r2\,r3\,e^{i\,(\phi 2-\phi 3)},$$

$$e^{-t2\,\Gamma L}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r2\,r4\,e^{i\,(\phi 2-\phi 4)},\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ e^{-t2\,\Gamma S}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r1\,r3\,e^{i\,(\phi 3-\phi 1)},$$

$$e^{\frac{1}{2}\,i\,t2\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r2\,r3\,e^{i\,(\phi 3-\phi 2)},\ e^{-t2\,\Gamma S}\left(1-e^{-t1\,\Gamma L}\right)r3^2,$$

$$e^{\frac{1}{2}\,i\,t2\,(2\,mL-2\,mS+i\,(\Gamma L+\Gamma S))}\left(1-e^{-t1\,\Gamma L}\right)r3\,r4\,e^{i\,(\phi 3-\phi 4)},\ 0,\ 0,\ 0,\ 0\Big\},$$

$$\Big\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ e^{-\frac{1}{2}\,t2\,(2\,i\,mL-2\,i\,mS+\Gamma L+\Gamma S)}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r1\,r4\,e^{i\,(\phi 4-\phi 1)},$$

$$e^{-t2\,\Gamma L}\sqrt{1-e^{-t1\,\Gamma L}}\ \sqrt{1-e^{-t1\,\Gamma S}}\ r2\,r4\,e^{i\,(\phi 4-\phi 2)},\ e^{\frac{1}{2}\,t2\,(-2\,i\,mL+2\,i\,mS-\Gamma L-\Gamma S)}\left(1-e^{-t1\,\Gamma L}\right)r3\,r4\,e^{i\,(\phi 4-\phi 3)},$$

$$e^{(-t1-t2)\,\Gamma L}\left(-1+e^{t1\,\Gamma L}\right) r4^2,\ 0,\ 0,\ 0,\ 0\},\ \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,$$

$$\left(1-e^{-t1\,\Gamma S}\right)\left(1-e^{-t2\,\Gamma S}\right) r1^2,\ \sqrt{1-e^{-t2\,\Gamma L}}\left(1-e^{-t1\,\Gamma S}\right)\sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r2\,e^{i(\phi1-\phi2)},$$

$$\sqrt{1-e^{-t1\,\Gamma L}}\sqrt{1-e^{-t1\,\Gamma S}}\left(1-e^{-t2\,\Gamma S}\right) r1\,r3\,e^{i(\phi1-\phi3)},$$

$$\sqrt{1-e^{-t1\,\Gamma L}}\sqrt{1-e^{-t2\,\Gamma L}}\sqrt{1-e^{-t1\,\Gamma S}}\sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r4\,e^{i(\phi1-\phi4)}\},$$

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \sqrt{1-e^{-t2\,\Gamma L}}\left(1-e^{-t1\,\Gamma S}\right)\sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r2\,e^{i(\phi2-\phi1)},$$

$$\left(1-e^{-t2\,\Gamma L}\right)\left(1-e^{-t1\,\Gamma S}\right) r2^2,\ \sqrt{1-e^{-t1\,\Gamma L}}\sqrt{1-e^{-t2\,\Gamma L}}\sqrt{1-e^{-t1\,\Gamma S}}\sqrt{1-e^{-t2\,\Gamma S}}\ r2\,r3\,e^{i(\phi2-\phi3)},$$

$$\sqrt{1-e^{-t1\,\Gamma L}}\left(1-e^{-t2\,\Gamma L}\right)\sqrt{1-e^{-t1\,\Gamma S}}\ r2\,r4\,e^{i(\phi2-\phi4)}\},$$

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \sqrt{1-e^{-t1\,\Gamma L}}\sqrt{1-e^{-t1\,\Gamma S}}\left(1-e^{-t2\,\Gamma S}\right) r1\,r3\,e^{i(\phi3-\phi1)},$$

$$\sqrt{1-e^{-t1\,\Gamma L}}\sqrt{1-e^{-t2\,\Gamma L}}\sqrt{1-e^{-t1\,\Gamma S}}\sqrt{1-e^{-t2\,\Gamma S}}\ r2\,r3\,e^{i(\phi3-\phi2)},$$

$$\left(1-e^{-t1\,\Gamma L}\right)\left(1-e^{-t2\,\Gamma S}\right) r3^2,\ \left(1-e^{-t1\,\Gamma L}\right)\sqrt{1-e^{-t2\,\Gamma L}}\sqrt{1-e^{-t2\,\Gamma S}}\ r3\,r4\,e^{i(\phi3-\phi4)}\},$$

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \sqrt{1-e^{-t1\,\Gamma L}}\sqrt{1-e^{-t2\,\Gamma L}}\sqrt{1-e^{-t1\,\Gamma S}}$$

$$\sqrt{1-e^{-t2\,\Gamma S}}\ r1\,r4\,e^{i(\phi4-\phi1)},\ \sqrt{1-e^{-t1\,\Gamma L}}\left(1-e^{-t2\,\Gamma L}\right)\sqrt{1-e^{-t1\,\Gamma S}}\ r2\,r4\,e^{i(\phi4-\phi2)},$$

$$\left(1-e^{-t1\,\Gamma L}\right)\sqrt{1-e^{-t2\,\Gamma L}}\sqrt{1-e^{-t2\,\Gamma S}}\ r3\,r4\,e^{i(\phi4-\phi3)},\ \left(1-e^{-t1\,\Gamma L}\right)\left(1-e^{-t2\,\Gamma L}\right) r4^2\}\};$$

```
probYN[t1_, t2_] = Tr[PYN.rho[t1, t2]];
probNY[t1_, t2_] = Tr[PNY.rho[t1, t2]];
probYY[t1_, t2_] = Tr[PYY.rho[t1, t2]];
probNN[t1_, t2_] = Tr[PNN.rho[t1, t2]];
Erwart[αn_, ϕn_, αm_, ϕm_, t1_, t2_, r1_, r2_, r3_, r4_] =
  probYY[t1, t2] + probNN[t1, t2] - (probYN[t1, t2] + probNY[t1, t2]);

Tr[(PYY + PYN + PNY + PNN).rho[t1, t2]] // Simplify
```

$$r1^2+r2^2+r3^2+r4^2$$

Bell Inequality and Numerical Results:

```
bell[αn_, ϕn_, αm_, ϕm_, αnn_, ϕnn_, αmm_, ϕmm_, tn_, tm_, tnn_, tmm_] =
  Re[Erwart[αn, ϕn, αm, ϕm, tn, tm, r1, r2, r3, r4] -
    Erwart[αn, ϕn, αmm, ϕmm, tn, tmm, r1, r2, r3, r4] + Erwart[αnn, ϕnn, αm, ϕm, tnn,
     tm, r1, r2, r3, r4] + Erwart[αnn, ϕnn, αmm, ϕmm, tnn, tmm, r1, r2, r3, r4]];
r1 = r4 = 0; r2 = 1 / Sqrt[2]; r3 = 1 / Sqrt[2];
ΓS = 1.119;
ΓL = 0.00193;
mS = 0;
mL = 0.5300;
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, 0, 0, 0, 0]},
 {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, 0, 0, 0, 0]},
 {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[-Pi / 4, 0, -Pi / 4, 0, -Pi / 4, 0, -Pi / 4, 0, tn, tm, tnn, tmm],
  tn ≥ 0, tm ≥ 0, tnn ≥ 0, tmm ≥ 0}, {tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[-Pi / 4, 0, -Pi / 4, 0, -Pi / 4, 0, -Pi / 4, 0, tn, tm, tnn, tmm],
  tn ≥ 0, tm ≥ 0, tnn ≥ 0, tmm ≥ 0}, {tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[Pi / 4, 0, -Pi / 4, 0, Pi / 4, 0, -Pi / 4, 0, tn, tm, tnn, tmm],
  tn ≥ 0, tm ≥ 0, tnn ≥ 0, tmm ≥ 0}, {tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[Pi / 4, 0, -Pi / 4, 0, Pi / 4, 0, -Pi / 4, 0, tn, tm, tnn, tmm],
  tn ≥ 0, tm ≥ 0, tnn ≥ 0, tmm ≥ 0}, {tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[αn, 0, αm, 0, αnn, 0, αmm, 0, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, αm, αnn, αmm, tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[αn, 0, αm, 0, αnn, 0, αmm, 0, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, αm, αnn, αmm, tn, tm, tnn, tmm, ϕ1, ϕ2, ϕ3, ϕ4}]
Clear[r1, r2, r3, r4, ΓS, ΓL, mS, mL]
```

```
{2.82843, {αn → 0.669616, ϕn → 0.126693, αm → 2.01225,
  ϕm → 1.34584, αnn → 1.75161, ϕnn → 1.46446, αmm → -0.820887, ϕmm → -0.68073,
  ϕ1 → 0.541355, ϕ2 → 0.814294, ϕ3 → 0.441008, ϕ4 → -0.128178}}

{-2.82843, {αn → 1.57488, ϕn → -0.963385, αm → -0.820775,
  ϕm → -0.518142, αnn → -0.253301, ϕnn → 0.623194, αmm → 0.812849,
  ϕmm → -0.0271023, ϕ1 → 1.31262, ϕ2 → 0.594168, ϕ3 → -0.0956626, ϕ4 → -0.146547}}

{1.99992, {tn → 5720.15, tm → 4541.03, tnn → 4630.79,
  tmm → 685.913, ϕ1 → 1753.98, ϕ2 → 6540.63, ϕ3 → -929.626, ϕ4 → 2.61411}}

{-2., {tn → 0.0563912, tm → 5.86338 × 10^13, tnn → 5.86338 × 10^13,
  tmm → 1.07665 × 10^11, ϕ1 → 3.63355, ϕ2 → -1.88061, ϕ3 → -5.02221, ϕ4 → -2.71169}}

{1.9578, {tn → 5.86338 × 10^13, tm → 4.91895, tnn → 4.91692,
  tmm → 5.86338 × 10^13, ϕ1 → -1.65247, ϕ2 → 6.76853, ϕ3 → 0.485273, ϕ4 → 2.60932}}

{-2., {tn → 2.02011, tm → 5.86338 × 10^13, tnn → 5.86338 × 10^13,
  tmm → 5.86338 × 10^13, ϕ1 → 4.5855, ϕ2 → 2.46775, ϕ3 → 2.46775, ϕ4 → -0.708762}}

{1.99869, {αn → 0.0463396, ϕn → -1.22567, αm → -1.74498, ϕm → -0.124889, αnn → 0.00103989,
  ϕnn → 0.896235, αmm → -0.00907351, ϕmm → 0.309512, tn → 5.86338 × 10^13, tm → 5.79385,
  tnn → 6.07964, tmm → 1.13059, ϕ1 → -1.6113, ϕ2 → 1.88677, ϕ3 → 0.946515, ϕ4 → 2.7567}}
```

```
{1.99869, {αn → 0.0463395, ϕn → -1.22567, αm → -1.74498, ϕm → -0.124889, αnn → 0.00103992,
  ϕnn → 0.896234, αmm → -0.00907357, ϕmm → 0.309512, tn → 5.86338 × 10^13, tm → 5.79385,
  tnn → 6.07964, tmm → 1.13058, ϕ1 → -1.6113, ϕ2 → 1.88677, ϕ3 → 0.946516, ϕ4 → 2.7567}}

{-2., {αn → -0.0000149581, ϕn → 0.759629, αm → -8.79606 × 10^-6,
  ϕm → -0.421392, αnn → -0.0000148648, ϕnn → -1.03383, αmm → -0.0000198258,
  ϕmm → -1.3445, tn → 2.03135, tm → 5.86338 × 10^13, tnn → 5.86338 × 10^13,
  tmm → 0.569454, ϕ1 → -1.23409, ϕ2 → 0.259331, ϕ3 → -2.3368, ϕ4 → -0.640589}}

{1.99771, {αn → -3.44768 × 10^-6, αm → -2.99244 × 10^-6, αnn → -3.09101 × 10^-6,
  αmm → -3.54069 × 10^-6, tn → 0.410887, tm → 6.93058, tnn → 6.46951,
  tmm → 0.279009, ϕ1 → 1.1932, ϕ2 → 2.3759, ϕ3 → 0.741889, ϕ4 → 0.178908}}

{-2., {αn → 0.000023857, αm → -0.0000509073, αnn → 0.0000484447,
  αmm → -0.0000181588, tn → 0.987017, tm → 5.86338 × 10^13, tnn → 5.86338 × 10^13,
  tmm → 1.58327, ϕ1 → -1.72344, ϕ2 → -0.0279759, ϕ3 → -0.102102, ϕ4 → 0.05913}}
```

```
bell[αn_, ϕn_, αm_, ϕm_, αnn_, ϕnn_, αmm_, ϕmm_, tn_, tm_, tnn_, tmm_] =
  Re[Erwart[αn, ϕn, αm, ϕm, tn, tm, r1, r2, r3, r4] -
    Erwart[αn, ϕn, αmm, ϕmm, tn, tmm, r1, r2, r3, r4] + Erwart[αnn, ϕnn, αm, ϕm, tnn,
     tm, r1, r2, r3, r4] + Erwart[αnn, ϕnn, αmm, ϕmm, tnn, tmm, r1, r2, r3, r4]];
r1 = Cos[p1]; r2 = Sin[p1] Cos[p2]; r3 = Sin[p1] Sin[p2] Cos[p3];
r4 = Sin[p1] Sin[p2] Sin[p3];
ΓS = 1.119;
ΓL = 0.00193;
mS = 0;
mL = 0.5300 ;
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, 0, 0, 0, 0]},
 {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, 0, 0, 0, 0]},
 {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0, tnn ≥ 0,
  tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0, tnn ≥ 0,
  tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}]
NMaximize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm],
  tn ≥ 0, tm ≥ 0, tnn ≥ 0, tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm,
  tn, tm, tnn, tmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}, Method → "NelderMead"]
NMinimize[{bell[αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm, tn, tm, tnn, tmm],
  tn ≥ 0, tm ≥ 0, tnn ≥ 0, tmm ≥ 0}, {αn, ϕn, αm, ϕm, αnn, ϕnn, αmm, ϕmm,
  tn, tm, tnn, tmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}, Method → "NelderMead"]
NMaximize[{bell[αn, 0, αm, 0, αnn, 0, αmm, 0, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, αm, αnn, αmm, tn, tm, tnn, tmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}]
NMinimize[{bell[αn, 0, αm, 0, αnn, 0, αmm, 0, tn, tm, tnn, tmm], tn ≥ 0, tm ≥ 0,
  tnn ≥ 0, tmm ≥ 0}, {αn, αm, αnn, αmm, tn, tm, tnn, tmm, p1, p2, p3, ϕ1, ϕ2, ϕ3, ϕ4}]
Clear[r1, r2, r3, r4, ΓS, ΓL, mS, mL]
```

```
{2.82843, {αn → -0.466249, ϕn → 0.27904, αm → -0.795344, ϕm → -0.375786, αnn → 1.27422,
  ϕnn → -0.639183, αmm → 0.860282, ϕmm → 0.191288, p1 → -0.935395, p2 → -1.07287,
  p3 → 0.996062, ϕ1 → 0.0560232, ϕ2 → 0.240611, ϕ3 → -0.294791, ϕ4 → -0.110203}}

{-2.82843, {αn → -1.59359, ϕn → 1.1544, αm → 0.819201, ϕm → -1.08674, αnn → 0.0345987,
  ϕnn → -1.13544, αmm → -0.754845, ϕmm → -0.972736, p1 → -1.60033, p2 → -0.785835,
  p3 → -0.0417712, ϕ1 → -0.611773, ϕ2 → 0.0394318, ϕ3 → -0.913906, ϕ4 → -0.262695}}

{2., {αn → -0.534066, ϕn → 0.311486, αm → -0.375366, ϕm → -0.275506,
  αnn → -0.845673, ϕnn → -0.00458672, αmm → -1.28956, ϕmm → 1.03036, tn → 2.10046,
  tm → 2.8619, tnn → 4.20518, tmm → 0.398238, p1 → 0.836989, p2 → 0.283299,
  p3 → 2.32486, ϕ1 → 0.556404, ϕ2 → 1.79773, ϕ3 → -0.360707, ϕ4 → -2.26155}}
```

C Heisenberg's Uncertainty Relation

Uncertainty Relation

```
n[αn_, ϕn_, tn_] =
  Exp[-(ΓL + ΓS) / 2 tn] (                    Cos[tn + ϕn] Sin[αn]                      );
                         (                    Sin[tn + ϕn] Sin[αn]                      )
                         ( Sinh[(ΓL - ΓS) / 2 tn] + Cosh[(ΓL - ΓS) / 2 tn] Cos[αn]      )
```

```
KS = (1
      0);

KL = (0
      1);
```

```
χn[αn_, ϕn_, tn_] =
  (Cos[αn / 2] Exp[-ΓS tn / 2] KS + Sin[αn / 2] Exp[I (tn + ϕn)] Exp[-ΓL tn / 2] KL) /
   Norm[(Cos[αn / 2] Exp[-ΓS tn / 2] KS + Sin[αn / 2] Exp[I (tn + ϕn)] Exp[-ΓL tn / 2] KL)];

χχfast = χn[αn, ϕn, tn].χn[αn, ϕn, tn]†;

χχ =
  Simplify[χχfast, Assumptions -> Element[{tn, tm, ΓS, ΓL, 1 / Sqrt[N], αn, ϕn}, Reals]];

rho = (r11   r12
       r12  1 - r11);
```

pn=Norm^2;

```
pn[αn_, ϕn_, tn_] = Re[Tr[χχ.rho]];
```

Funktion H

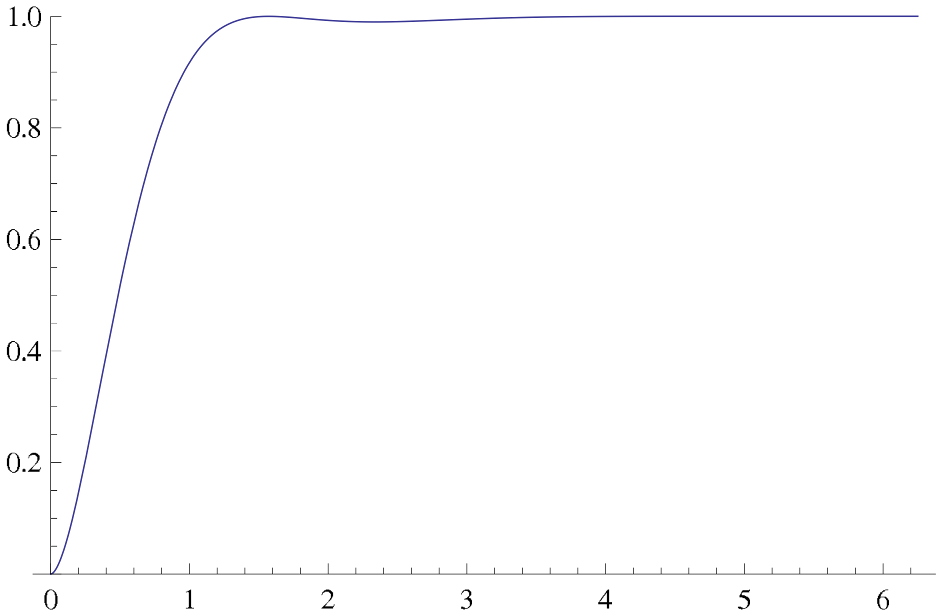
```
H[αn_, ϕn_, tn_] =
  -pn[αn, ϕn, tn] Log[2, pn[αn, ϕn, tn]] - (1 - pn[αn, ϕn, tn]) Log[2, 1 - pn[αn, ϕn, tn]];

H[αm_, ϕm_, tn_] =
  -pn[αm, ϕm, tn] Log[2, pn[αm, ϕm, tn]] - (1 - pn[αm, ϕm, tn]) Log[2, 1 - pn[αm, ϕm, tn]];
```

bel. Werte eingesetzt:

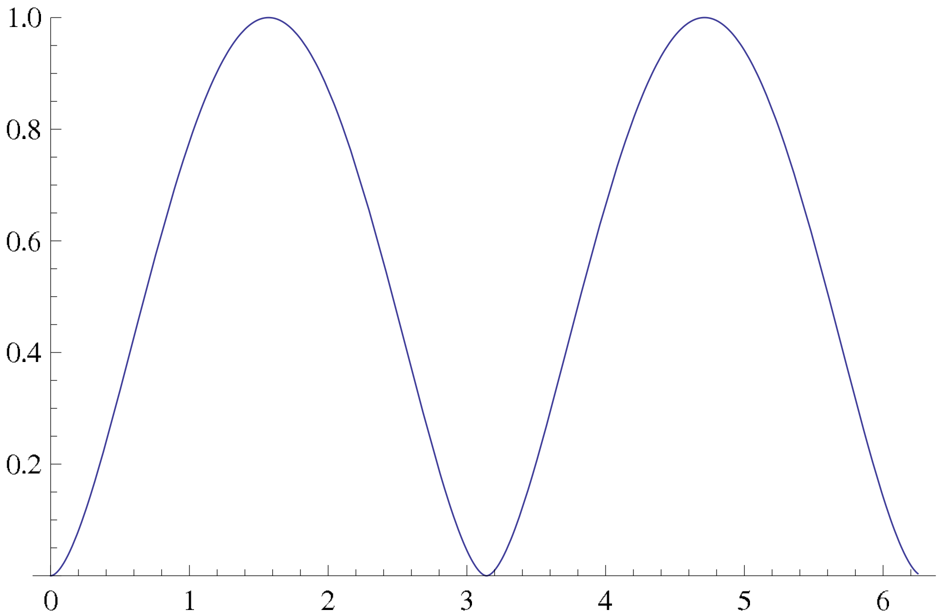
```
ΓS = 1.119 / 0.5300;
ΓL = 0.00193 / 0.5300;
αn = Pi / 2;
ϕn = 0;
r11 = r22 = r12 = 1 / 2;
```

```
Plot[H[αn, ϕn, tn], {tn, 0, 6.25}, PlotRange → {0, 1}]
```




```
ΓS = 1.119 / 0.5300;
ΓL = 1.119 / 0.5300;
αn = Pi / 2;
ϕn = 0;
r11 = r22 = r12 = 1 / 2;
```

```
Plot[H[αn, ϕn, tn], {tn, 0, 6.25}, PlotRange → {0, 1}]
```



References

[1] J.J. Thomson
Cathode Rays
Philosophical Magazine 44: 293 (1897)

[2] E. Rutherford
Radiations from Radioactive Substances
Philosophical Magazine, 4:1-23, 1902 (1919)

[3] A. Einstein
Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt
Annalen der Physik 17: 132–148 (1905)

[4] M. Planck
On the Law of Distribution of Energy in the Normal Spectrum
Annalen der Physik 4: 553–563 (1901)

[5] H. Yukawa
Proc. Phys.-Math. Soc. Jap. 17 (t935), 48 [completed-October 1934; presented at the meeting of the physico-mathematieal society–t7. 11. 1934]

[6] V. Mukherji
A history of the meson theory of nuclear forces
Archive for History of Exact Sciences 13: 27-102 (1974)

[7] G. Occhialini and C. Powell
Nuclear disintegrations produced by slow charged particles of small mass Nature 159, 186–190 and 160, 453–456 (1947)

[8] G. D. ROCHESTER and C. C. BUTLER
Evidence for the Existence of New Unstable Elementary Particles Nature 160, 855-857 (1947)

[9] T. Nakano and K. Nishijima
Charge independence for v-particles
prog. theor. phys., osaka 10 (1953), 581

[10] M. Gell-Mann
Isotopic spin and new unstable particles
phys. rev. 92 (1953), 833

[11] A. Pais and O. Piccioni
Note on the Decay and Absorption of the Θ^0
Phys. Rev. 100, 5 (1955)

[12] V. L. Telegdi
K_S*-Regeneration: An Interplay Of Nuclear And Particle Physics*
AIP Conference Proceedings, Volume 26, pp. 289-302 (1975)

[13] E. Schrödinger
The Present Situation in Quantum Mechanics
Naturw. 23, 807, 823, 844 (1935).

[14] A. Einstein
Letter from Einstein to Max Born, 3 March 1947. The Born-Einstein Letters; Correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955, Walker, New York, 1971
cited in Quantum Entanglement and Communication Complexity (1998), by M. P. Hobson et. al., p.1/13

[15] W. Heisenberg
Physikalische Prinzipien der Quantentheorie
S. Hirzel Verlag, Leipzig 1930.

[16] W. Heisenberg
Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik Zeitschrift für Physik. 43, Nr. 3, 1927, S. 172–198

[17] A. Einstein, B. Podolsky and N. Rosen
Can quantum-mechanical description of physical reality be considered complete?
Phys. Rev. 47, 777 (1935)

[18] N. Bohr
Can Quantum-Mechanical Description of Physical Realitiy be Considered Complete?
Physical Review, 48 (1935), p. 700.

[19] D. Bohm
A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I
Physical Review. 85, no. 2, 1952, p. 166–179

[20] J. S. Bell
On the Einstein-Podolsky-Rosen paradox
Physics. 1, Nr. 3, 1964, S. 195-200

[21] J.S. Bell
On the problem of hidden variables in quantum mechanics
Reviews of Modern Phys. 38, 447 (1966)

[22] J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt
Proposed Experiment to Test Local Hidden-Variable Theories
Physical Review Letters. 23, Nr. 15, (1969)

[23] A. Aspect, J. Dalibard and G. Roger
Experimental test of Bell's inequalities using time-varying analyzers
Phys. Rev. Lett. 49, 1804 (1982)

[24] J. Audretsch
Entangled systems
Wiley-VCH (2007)

[25] D. Griffith
Einführung in die Elementarteilchenphysik
Akademie Verlag, Berlin (1996)

[26] T.D. Lee, C.S. Wu
Ann. Rev. Nucl. Sci. 16, 511 (1966)

[27] R.A. Bertlmann
Verschränkung, Dekohärenz und Geometrie von Quantensystemen
Lecture (2009)

[28] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki
Quantum entanglement
Overview (2007)

[29] R. Bertlmann and P. Krammer
Bloch vectors for qudits
J. Phys. A: Math.Theor. 41, 235303 (2008)

[30] M. Horodecki, P. Horodecki, and R. Horodecki
Separability of mixed states: necessary and sufficient conditions
Phys. Lett. A 223, 1 (1996)

[31] M. Horodecki and P. Horodecki
Reduction criterion of separability and limits for a class of distillation protocols
Phys. Rev. A 59, 4206 (1999)

[32] Ch. Spengler, M. Huber and B.C. Hiesmayr
A composite parameterization of unitary groups, density matrices and subspaces
J. Phys. A: Math. Theor. 43, 385306 (2010)

[33] Ch. Spengler, M. Huber and B.C. Hiesmayr
Composite parameterization and Haar measure for all unitary and special unitary groups
arXiv:1103.3408v1 [math-ph] (2011)

[34] A. Gabriel, B.C. Hiesmayr, M. Huber
Criterion for k-separability in mixed multipartite systems
arXiv:1002.2953v1 [quant-ph] (2011)

[35] M. Huber, H. Schimpf, A. Gabriel, Ch. Spengler, D. Bruß, B.C. Hiesmayr
Experimentally implementable criteria revealing substructures of genuine multipartite entanglement
arXiv:1011.4087v2 [quant-ph] (2010)

[36] D. Deutsch
Uncertainty in quantum measurements
Phys. Rev. Lett. 50, 631 (1983)

[37] K. Kraus
Complementary observables and uncertainty relations Phys. Rev. D 35, 3070 (1987)

[38] H. Maassen and J.B.M. Uffink
Generalized Entropy Uncertainty Relation Phys. Rev. Lett. 60, 1103 (1988)

[39] Andreas Gabriel, Marcus Huber, Sasa Radic, Beatrix C. Hiesmayr *A Computable Criterion for Partial Entanglement in Continuous Variable Quantum Systems*
arXiv:1101.0972 [quant-ph] (2011)

[40] A. Di Domenico, A. Gabriel, B.C. Hiesmayr, F. Hipp, M. Huber, G. Krizek, K. Mühlbacher, S. Radic, Ch. Spengler, L. Theussl
Heisenberg's Uncertainty Relation and Bell Inequalities in High Energy Physics
arXiv:1101.4517 [quant-ph, hep-th] (2011)

[41] R. A. Bertlmann
Entanglement, Bell Inequalities and Decoherence in Particle Physics
lecture notes, http://arxiv.org/abs/quant-ph/0410028v2

[42] R.A. Bertlmann and Beatrix C. Hiesmayr
Bell inequalities for entangled kaons and their unitary time evolution
Phys. Rev. A 63, 062112 (2001)

[43] R.A. Bertlmann, W. Grimus, and B.C. Hiesmayr
Bell inequality and CP violation in the neutral kaon system
Phys. Lett. A 289, 21 (2001)

[44] A. Bramon, G. Garbarino and B.H. Hiesmayr
Kaonic Quantum 2: Erasing the Present, changing the Past
Frascati Physics Neutral Kaon Interferometry at the Φ factory, 17 (2006)

[45] A. Bramon, G. Garbarino and B.C. Hiesmayr
Quantum Mechanics with Neutral Kaons
Acta Phys. Polon. B, Vol. 38, 2763 (2007)

[46] B.C. Hiesmayr
The Puzzling Story of the Neutral Kaon System
Thesis University of Vienna (2002)

[47] B.C. Hiesmayr
Testing Quantum Foundations in Particle Physics
lecture notes (2010)

[48] G. Amelino-Camelia et.al
Physics with the KLOE-2 experiment at the upgraded DAPHNE
European Physical Journal C 68, Number 3-4, 619 (2010)

[49] N.E. Mavromatos
Decoherence and CPT Violation in a Singy Model of Space-Time Foam
arXive:0906.2712

[50] R.A. Bertlmann, K. Durstberger and B.C. Hiesmayr
Decoherence of entangled kaons and its connection to entanglement measures
Phys. Rev. A 68, 012111 (2003)

[51] B.C. Hiesmayr and M. Huber
Bohr's complementarity relation and the violation of the CP symmetry in high energy physics
Physics Letters A 372, 3608 (2008)

[52] H. P. Breuer and F. Petruccione
The theory of open quantum systems
Oxford University Press (2002)

[53] G. Lindblad
On the generators of quantum dynamical semigroups
Comm. Math. Phys. 48, 119 (1976)

Acknowledgements

Ich möchte diese Diplomarbeit meinem kürzlich verstorbenen Großvater widmen. Er hatte sich sehnlichst gewünscht, meinen Abschluss der Physik mitzuerleben.

Bedanken möchte ich mich in erster Linie bei meiner Betreuerin, Beatrix C. Hiesmayr, bei der ich jederzeit und ohne schlechtes Gewissen Rat finden konnte. Auch möchte ich mich bei den Kollegen meiner Gruppe bedanken. Dabei möchte ich insbesondere betonen, dass die Chemie hier immer eine besondere war, da das Zwischenmenschliche stets über dem Eigennützigen gestellt wurde.

Zu guter Letzt bedanke ich mich bei all jenen, die mir bei dieser Diplomarbeit Hinweise gegeben und Korrektur gelesen haben, sowie bei meinen Eltern und meinem Bruder, die mich seit jeher unterstützt haben.

UWThPh-2011-10

A Computable Criterion for Partial Entanglement in Continuous Variable Quantum Systems

Andreas Gabriel,* Marcus Huber,† and Sasa Radic‡
University of Vienna, Faculty of Physics, Boltzmanngasse 5, 1090 Vienna, Austria

Beatrix C. Hiesmayr§
Research Center for Quantum Information, Institute of Physics,
Slovak Academy of Sciences, Dúbravská cesta 9, 84511 Bratislava, Slovakia and
University of Vienna, Faculty of Physics, Boltzmanngasse 5, 1090 Vienna, Austria.

A general and computable criterion for k-(in)separability in continuous multipartite quantum systems is presented. The criterion can be experimentally implemented with a finite and comparatively low number of local observables. We discuss in detail how the detection quality can be optimised.

I. INTRODUCTION

Quantum entanglement has been studied quite intensely over the last few decades, resulting in a rather wide understanding of simple entangled systems (i.e. bipartite systems of low dimensions, especially two qubit systems; for an overview, see e.g. Ref. [1]). However, still many puzzling features and open questions are revealed in more general systems, such as multipartite systems or systems of high (or, in particular, infinite) dimensions.

In multipartite entangled systems (which are of grave importance to technological applications of quantum information theory, such as quantum secret sharing [2] or quantum computation [3]), complications arise (among others) due to the multiple different forms in which a multipartite state can be entangled (see e.g. [4, 5]). In particular, while a bipartite state is either entangled or separable, a multipartite state can be partially entangled (as investigated e.g. in Refs [6, 7]), as opposed to genuinely multipartite entangled (see e.g. [8–12]).

In infinite-dimensional (continuous) systems, problems arise because certain notions of finite-dimensional systems are no longer met (see e.g. [13, 14]). Nevertheless, many concepts have been generalised from the finite-dimensional case to the continuous one during the last decade, most noteworthy the PPT-criterion [15] and the scheme of quantum teleportation [16] (which has also already been experimentally verified [17]).

Rather seldomly are systems considered which contain both these sets of difficulties (e.g. [18, 19], although multipartite continuous quantum systems do hold the possibility for certain applications, such as certain kinds of teleportation networks [20].

In this letter, we use a general framework (which was introduced in Refs. [21, 22] for finite-dimensional systems) to formulate a criterion for partial separability (k-separability) of arbitrary states of a continuous variable multipartite system, thus proving implicitly, that the framework also works perfectly well in infinite-dimensional systems.

The article is organised as follows. In section II, the basic definitions and terminology will be reviewed, such that in section III we can present our criterion for continuous variable k-separability, which is the main result of this letter. In section IV, a guideline to the application of the criterion will be given (such that its detection power can be optimally used). The criterion and its application are then demonstrated in two exemplary cases in section V. Finally, in section VI we show how the criterion can also be implemented experimentally.

*Electronic address: Andreas.Gabriel@univie.ac.at
†Electronic address: Marcus.Huber@univie.ac.at
‡Electronic address: Sasa.Radic@univie.ac.at
§Electronic address: Beatrix.Hiesmayr@univie.ac.at

II. BASIC DEFINITIONS

In order to formulate our result, we firstly need to define the concept of k-separability for continuous variable systems. A general n-partite pure quantum state can be written in the form

$$|\Psi\rangle = \int_{-\infty}^{\infty} d^n x \; \Psi(x_1, x_2, \cdots, x_n) \; |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle \; . \tag{1}$$

It is called k-separable ($k \leq n$) iff its distribution function factorises into k factors, i.e.

$$\Psi(x_1, x_2, \cdots, x_n) = \Psi_1(\{x_{i_1}\}) \cdot \Psi_2(\{x_{i_2}\}) \cdot \cdots \cdot \Psi_k(\{x_{i_k}\}) \tag{2}$$

where $\{x_{i_j}\}$ denotes a set of coordinates, i.e. corresponds to one or several particles. That is, a pure state is called k-separable iff there is a k-partition $(\{x_{i_1}\}|\{x_{i_2}\}|\cdots|\{x_{i_k}\})$ with respect to which it is separable. If $k = n$ then the state is called fully separable, i.e. there is no entanglement in the multipartite system. If $k = 2$ the state is called biseparable, if it is not biseparable, it is called genuinely multipartite entangled. Genuine multipartite entanglement is found to be a key ingredient for many quantum algorithms, see e.g. Ref. [23]. In finite-dimensional systems, there exist different inequivalent classes of genuine multipartite entanglement, e.g. the GHZ-class, the W-class or the Dicke-class. Such substructures are only known for very special and rather simple systems (see e.g. [4, 24, 25]) and it is not known how this generalises for continuous variable systems.

Example: A tripartite pure state can be either fully separable, i.e. $k = n$ with the 3-partition $(x_1|x_2|x_3)$, or 2–separable (biseparable, i.e. k=2) with one of the three possible partitions $(x_1 x_2|x_3)$, $(x_1 x_3|x_2)$ or $(x_1|x_2 x_3)$, or genuinely multipartite entangled ($k = 1$) with the partition $(x_1 x_2 x_3)$.

For mixed states, we can extend this definition in a straightforward way. A mixed n-partite quantum state has the general form

$$\rho = \int_{-\infty}^{\infty} d^n x \; d^n x' \; \rho(x_1, x_1', x_2, x_2', \ldots, x_n, x_n') \; |x_1\rangle\langle x_1'| \otimes |x_2\rangle\langle x_2'| \otimes \cdots |x_n\rangle\langle x_n'| \tag{3}$$

It is called k-separable, iff it has a decomposition into k-separable pure states, i.e. iff it can be written as a convex combination of pure k-separable states:

$$\rho = \int d\alpha \; p_\alpha \; |\Psi_\alpha\rangle\langle\Psi_\alpha| \tag{4}$$

where p_α is a probability distribution (i.e. $p_\alpha \geq 0$ and $\int_{-\infty}^{\infty} d\alpha \; p_\alpha = 1$) and $|\Psi_\alpha\rangle$ is k-separable for all α.
Note that a k-separable mixed state may not be separable with respect to any partition, since the pure states in its decomposition may be separable with respect to different k-partitions (i.e. $|\Psi_\alpha\rangle$ may split into different partitions for different α). The concept of k-separability is of high impoertance to quantum information theory, since many of its applications rely on specific kinds of states (in particular on genuinely multipartite entangled states).

III. CRITERION FOR K-SEPARABILITY

In Refs. [21, 22], a framework for constructing very general separability criteria for finite Hilbert spaces was introduced. We now extend this framework such that we can apply it to continuous quantum systems. The main result of this paper is an inequality, given in the following theorem, which is satisfied for all k–separable states, such that any violation implies that the state under investigation is not k–separable.

Theorem: All k-separable states ρ satisfy the inequality

$$\sqrt{\langle\Phi|\rho^{\otimes 2} P_{total}|\Phi\rangle} - \sum_{\{\alpha\}} \left(\prod_{i=1}^{k} \langle\Phi|P_{\alpha_i}^\dagger \rho^{\otimes 2} P_{\alpha_i}|\Phi\rangle \right)^{\frac{1}{2k}} \leq 0 \tag{$*$}$$

where $|\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$ is an arbitrary fully separable state, P_{α_i} is a permutation operator permuting the α_i-th elements of $|\phi_1\rangle$ and $|\phi_2\rangle$, P_{total} wholly permutes the $|\phi_i\rangle$ and the sum runs over all k-partitions $\{\alpha\}$.

Proof: Firstly, observe that the left hand side of the inequality $(*)$ is a convex quantity. Thus, its validity for mixed states follows from its validity for pure states. To prove the latter, let us assume w.l.o.g. that a given pure state $\rho = |\Psi\rangle\langle\Psi|$ is separable with respect to the partition α'. Therefore, the permutation operators corresponding to this partition do not change the two copy state

$$P_{\alpha'_i}|\Psi\rangle^{\otimes 2} = P^\dagger_{\alpha'_i}|\Psi\rangle^{\otimes 2} = |\Psi\rangle^{\otimes 2} \;\forall\; i\;. \tag{5}$$

Furthermore, note that the total permutation acts as

$$P_{total}|\phi_1\rangle \otimes |\phi_2\rangle = |\phi_2\rangle \otimes |\phi_1\rangle \tag{6}$$

then the inequality reads

$$|\langle\phi_1|\rho|\phi_2\rangle| - \sqrt{\langle\phi_1|\rho|\phi_1\rangle\langle\phi_2|\rho|\phi_2\rangle} - \sum_{\{\alpha\neq\alpha'\}}\left(\prod_{i=1}^{k}\langle\Phi|P^\dagger_{\alpha_i}\rho^{\otimes 2}P_{\alpha_i}|\Phi\rangle\right)^{\frac{1}{2k}} \leq 0\;. \tag{7}$$

It follows from the Cauchy-Schwarz inequality that the first two terms cancel (because the pure state ρ is a projector). Since the remaining sum has strictly nonnegative terms with an overall negative sign, the whole inequality has to be satisfied for ρ which are separable with respect to the partition α'. □

IV. OPTIMISING THE DETECTION QUALITY

Evidently, the detection quality of inequality $(*)$ strongly depends on the choice of the separable state $|\Phi\rangle$. Unlike in the case of discrete systems, numerical optimisation is quite out of the question, as the number of optimisation parameters would be infinite. However, there is a quite intuitive way of choosing $|\Phi\rangle$ effectively. Given the state ρ in question, $|\Phi\rangle$ should satisfy the following conditions:

C1: $|\Phi\rangle$ has to be fully separable, i.e. $|\Phi\rangle = \bigotimes_{i=1}^n |\phi_{1i}\rangle \otimes \bigotimes_{i=1}^n |\phi_{2i}\rangle$.

C2: $|\phi_1\rangle$ and $|\phi_2\rangle$ should be orthogonal in each subsystem, i.e. $\langle\phi_{1i}|\phi_{2i}\rangle = 0 \;\forall\; i$.

C3: Each $|\phi_{ji}\rangle$ should be sharp, i.e. $|\phi_{ij}\rangle = \int_{-\infty}^{\infty} dx\; \delta(x - x_{ij})|x\rangle = |x_{ij}\rangle$ for some x_{ij}.

C4: $|\phi_1\rangle$ and $|\phi_2\rangle$ should be chosen such that $|\langle\phi_1|\rho|\phi_2\rangle|$ is maximal.

Let us illustrate the background of these conditions.
Condition $\mathcal{C}1$ is necessary for the criterion (ineq. $(*)$) to remain valid for all k-separable states and thus is rather a technical requirement.
Condition $\mathcal{C}2$ guarantees that the permutation operators used in the criterion $(*)$ have maximal impact, such that a maximal violation of the inequality can be achieved. Each pair $|\phi_{ji}\rangle$ $(j = 1, 2)$ is responsible for detecting entanglement in the i-th subsystem. Thus, non-orthonormality of this pair of vectors means non-optimal detection of entanglement in this subsystem.
Condition $\mathcal{C}3$ stems from the fact, that any average is always lower than (or equal to) its highest contribution. Any distribution containing more than one element leads to an averaging in the scalar products in $(*)$, which can never increase violation of the inequality, but will in general decrease it.
Condition $\mathcal{C}4$ incorporates the dependance of $|\Phi\rangle$ on the investigated state and thus most of the subtleties involved. The first term in ineq. $(*)$ (and the only positive one) is the absolute value of the scalar product $\langle\phi_1|\rho|\phi_2\rangle$. Evidently, for the inequality to be violated, this product needs to be as big as possible. This corresponds to chosing $|\phi_1\rangle$ and $|\phi_2\rangle$ as two of the main contributing vectors of the investigated state.
Although this procedure might not be unique (i.e. might not unambiguously lead to a definite choice of $|\Phi\rangle$), it reveals the optimal structure of $|\Phi\rangle$ and thus drastically reduces the number of optimisation parameters to a finite set, which can be optimised numerically (e.g. by means of the method introduced in [26]) or even analytically. This will be demonstrated in the next section.

V. EXAMPLES

Consider the family of tripartite states (i.e. n=3)

$$|\omega\rangle = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} d^3x \; e^{-\frac{x_1^2}{2\sigma}} e^{-\frac{(x_1-x_2)^2+(x_1-x_3)^2}{2\epsilon}} \\ (|x_1-\Delta\rangle\otimes|x_2\rangle\otimes|x_3+\Delta\rangle + |x_1\rangle\otimes|x_2+\Delta\rangle\otimes|x_3-\Delta\rangle + |x_1+\Delta\rangle\otimes|x_2-\Delta\rangle\otimes|x_3\rangle + \\ |x_1+\Delta\rangle\otimes|x_2\rangle\otimes|x_3-\Delta\rangle + |x_1\rangle\otimes|x_2-\Delta\rangle\otimes|x_3+\Delta\rangle + |x_1-\Delta\rangle\otimes|x_2+\Delta\rangle\otimes|x_3\rangle) \tag{8}$$

where

$$N = 2\pi^{\frac{3}{2}}\epsilon\sqrt{\sigma}(e^{-\frac{2\Delta^2}{\epsilon}} + 2e^{-\frac{\Delta^2}{2\epsilon}} + 2e^{-\frac{\Delta^2(\epsilon+5\sigma)}{\epsilon\sigma}} + 4e^{-\frac{\Delta^2(\epsilon+5\sigma)}{4\epsilon\sigma}} + 2e^{-\frac{\Delta^2(\epsilon+9\sigma)}{4\epsilon\sigma}} + 2e^{-\frac{\Delta^2(\epsilon+18\sigma)}{4\epsilon\sigma}} + 2\sqrt{2}e^{\frac{\sigma-2\Delta(\Delta\epsilon+6\sigma)}{8\epsilon\sigma}}) \tag{9}$$

is a normalisation constant and all parameters σ, ϵ and Δ are larger than or equal to zero. Note that

$$\lim_{\epsilon\to 0}(2\pi\epsilon)^{-\frac{1}{2}}e^{-\frac{x^2}{2\epsilon}} = \delta(x) \tag{10}$$

is the Dirac delta function, such that in this limit, for $\Delta = 0$ the state $|\omega\rangle$ is a generalisation of the GHZ-state to continuous systems:

$$\lim_{\epsilon\to 0}|\omega\rangle_{\Delta=0} = \frac{1}{\sqrt{N}}\int_{-\infty}^{\infty} dx \; e^{-\frac{x^2}{2\sigma}}|x\rangle\otimes|x\rangle\otimes|x\rangle \tag{11}$$

while for $\Delta > 0$ it is a generalisation to the W-state, since it is genuinely multipartite entangled and has entangled reduced density matrices.
In the following we are interested in the introduced tripartite states mixed with Gaussian distributed noise

$$\rho_{mix} = (2\pi\delta)^{-\frac{3}{2}}\int_{-\infty}^{\infty} d^3x \; e^{-\frac{x_1^2+x_2^2+x_3^2}{2\delta}} \; |x_1\rangle\langle x_1|\otimes|x_2\rangle\langle x_2|\otimes|x_3\rangle\langle x_3| \tag{12}$$

i.e. the state under investigation is

$$\rho = p\;|\omega\rangle\langle\omega| + (1-p)\;\rho_{mix}\;. \tag{13}$$

A. GHZ-like state ($\Delta = 0$)

In the case $\Delta = 0$, the state $|\omega\rangle$ assumes the comparatively simple form

$$|\omega\rangle = \frac{1}{\sqrt{N}}\int_{-\infty}^{\infty} d^3x \; e^{-\frac{x_1^2}{2\sigma}} e^{-\frac{(x_1-x_2)^2+(x_1-x_3)^2}{2\epsilon}}|x_1\rangle\otimes|x_2\rangle\otimes|x_3\rangle \tag{14}$$

As this state's distribution function has its peak at $x_1 = x_2 = x_3 = 0$ and is correlated such that all three coordinates are most likely to be very close to each other (or even equal for $\epsilon \to 0$), $|\Phi\rangle$ is best chosen to be of the form

$$\begin{aligned} |\Phi\rangle \;&=\; |\phi_1\rangle\otimes|\phi_2\rangle \qquad \text{with} \\ &|\phi_1\rangle = |x_0\rangle\otimes|x_0\rangle\otimes|x_0\rangle \\ &|\phi_2\rangle = |-x_0\rangle\otimes|-x_0\rangle\otimes|-x_0\rangle \end{aligned} \tag{15}$$

for some $x_0 \neq 0$ (due to condition $\mathcal{C}2$). The optimal value for x_0 can be obtained from analytical or numerical optimisation. Since for $\sigma \to 0$ this state becomes separable (as can be seen directly by substituting eq. (10) in eq. (14)), we are only interested in $\sigma > 0$ and thus can, without loss of generality, define $\sigma = 1$ and measure all lengths in units of $\sqrt{\sigma}$.
In order to estimate the best value for x_0, we first assume a pure state, i.e. $p = 1$ and investigate the detection behaviour of the criterion $(*)$ for different values of the two remaining parameters ϵ and x_0 (as illustrated in Fig. 1). It can be seen that the optimal choice of x_0 depends on ϵ only slightly and is best chosen in the range $0.7 \leq x_0 \leq 1.2$. In the further study of this state, we will chose $x_0 = 1$, such that only parameters of the state remain.
Now, we can investigate the mixed state case (i.e. $p < 1$). In Fig. 2 the detection range of our criterion $(*)$ for $k = 2$ and $k = 3$ is illustrated .

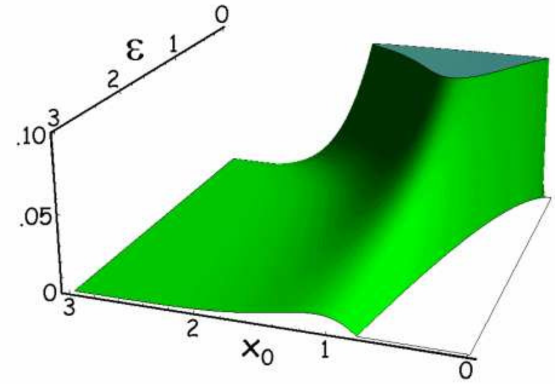


FIG. 1: (Colour online) Illustration of the detection parameter space of the criterion (∗) for $k = 2$ and the state $|\omega\rangle$, Eq. (14), with $p = 1$ and $\Delta = 0$. The state is detected to be genuinely tripartite entangled wherever the graph is above zero (i.e. in the green plotted region). It can be seen that the optimal choice for x_0 is between 0.7 and 1.2, depending only slightly on the value of ϵ.

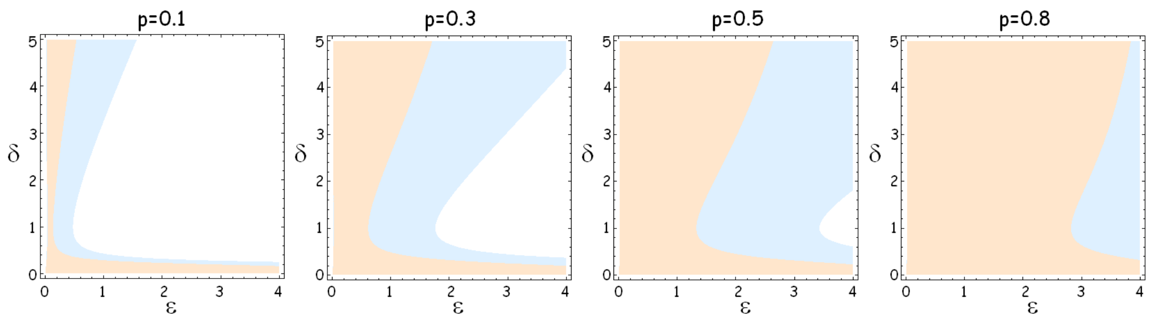


FIG. 2: (Colour online) Illustration of the detection quality of ineq. (∗) for $k = 2$ (red) and $k = 3$ (blue) for the state (14) for $\sigma = 1, x_0 = 1$ and different values of p, ϵ and δ. For $p = 1$ the whole state space is detected to be entangled ($k \leq 2$) and to be genuinely multipartite entangled ($k = 1$) for $\epsilon < 4.648$. For lower values of p, still very large portions of the state space are detected to be genuinely multipartite entangled (red regions, $k = 2$) or at least entangled (red and blue regions ($k \geq 2$)).

B. W-like state ($\Delta = 1$)

Let us now investigate a more general state, namely the case $\Delta \neq 0$. Without loss of generality, we can set $\Delta = 1$ and thus measure all lengths in units of Δ. The optimal choice of $|\Phi\rangle$ of course has to be a generalisation of the one used in the previous section, which coincides with it for $\Delta = 0$. The choice is quite straightforward, since the state (8) contains six terms. Combination of any two of them leads to the desired high-magnitude off-diagonal density matrix element, the only restrictions being conditions $\mathcal{C}1$-$\mathcal{C}3$. This leads to

$$\begin{aligned} |\Phi\rangle \ &= \ |\phi_1\rangle \otimes |\phi_2\rangle \qquad \text{with} \\ & \quad |\phi_1\rangle = |x_0 + \Delta\rangle \otimes |x_0\rangle \otimes |x_0 - \Delta\rangle \\ & \quad |\phi_2\rangle = |-x_0 - \Delta\rangle \otimes |-x_0 + \Delta\rangle \otimes |-x_0\rangle \end{aligned} \tag{16}$$

In this rather complicated case, numerical optimisation is necessary for achieving optimal detection quality, since the optimal x_0 depends strongly on the other parameters (more than in the previous case with $\Delta = 0$). However, even without numerical optimisation and using only two different choices of x_0 (namely $x_0 = 0$ and $x_0 = 1.5$) the detection range of our criterion (∗) is very wide, as illustrated in Fig. 3.

C. Non-Gaussian States

The above two examples belong to the class of Gaussian states, which have always been the main focus of research in the field of continuous variable entanglement (see e.g. Refs. [27–30]). However, recently questions regarding non-

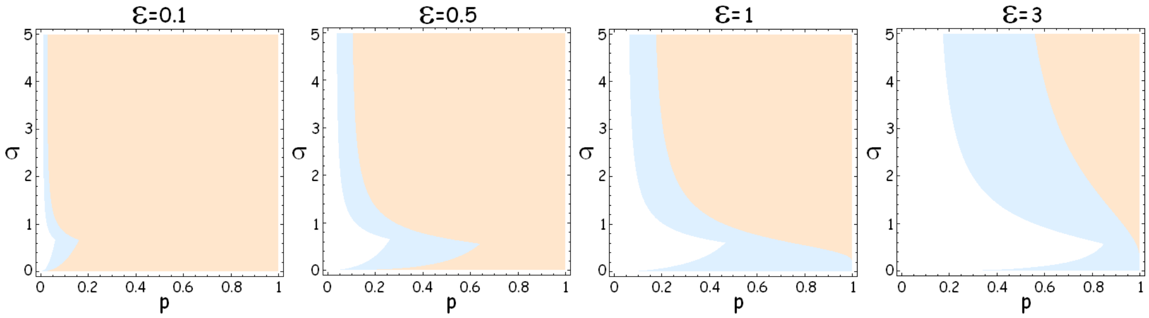


FIG. 3: (Colour online) Detection quality of the inequality (∗) for the family of the tripartite states (8) with $\Delta = 1, \delta = 1$ and $x_0 = \{0, 1.5\}$ (where $x_0 = 0$ detects better if $\sigma < 1$, even for very small p, but ceases to detect for large ϵ; and $x_0 = 1.5$ detects better for $\sigma > 1$ and still detects a considerable amount of entanglement for large ϵ). The red area is detected to be genuinely multipartite entangled (2-inseparable) and the blue and red areas are detected to be entangled (3-inseparable).

Gaussian entangled states have attracted a lot of interest within the scientific community. Since the criterion presented in this paper is - unlike most other criteria for continuous variable entanglement - not tailored specifically for Gaussian states, we will also show its detection quality for non-Gaussian states.
Consider the state

$$\rho = p|\omega\rangle\langle\omega| + (1-p)\rho_{mix} \tag{17}$$

where

$$|\omega\rangle = \frac{1}{N_1}\int d^3x \Theta(\beta - |x_1|)\Theta(\epsilon - |x_1 - x_2|)\Theta(\epsilon - |x_1 - x_3|)|x_1\rangle|x_2\rangle|x_3\rangle \tag{18}$$

$$\rho_{mix} = \frac{1}{N_2}\int d^3x \Theta(\delta - |x_1|)\Theta(\delta - |x_2|)\Theta(\delta - |x_3|)|x_1\rangle\langle x_1| \otimes |x_2\rangle\langle x_2| \otimes |x_3\rangle\langle x_3| \tag{19}$$

with

$$N_1 = 8\epsilon^2\beta \quad N_2 = 8\delta^3 \quad \beta, \epsilon, \delta > 0 \tag{20}$$

where $\Theta(x)$ is the Heaviside-function. This state is a non-Gaussian modification of the GHZ-like state (14).
Using (15) and chosing x_0 appropriately, the criterion (∗) reads

$$0 \leq \begin{cases} \frac{p}{8\epsilon^2\beta} & \text{if } \frac{\epsilon}{2} < \beta \text{ and } \delta < \beta \\ \frac{p}{8\epsilon^2\beta} - \frac{1-p}{8\delta^3} & \text{if } \frac{\epsilon}{2} < \beta \leq \delta \\ 0 & \text{else} \end{cases} \tag{21}$$

where the first condition is always positive for $p > 0$ and thus indicates genuine multipartite entanglement, the yield of the second condition depends on the parameters and the third condition can never be violated. In particular, the first condition detects a large portion of the state space to be genuinely multipartite entangled already for infinitesimal $p > 0$.

Since the state discussed above is only a rather simple example and not very close to experimental realisation, let us illustate the detection quality of our criterion for another non-Gaussian state, which is more likely to be implemented experimentally [31, 32], namely

$$|\omega\rangle = a_1a_2a_3\frac{1}{\sqrt{N}}\int_{-\infty}^{\infty} d^3x\ e^{-\frac{x_1^2}{2\sigma}} e^{-\frac{(x_1-x_2)^2+(x_1-x_3)^2}{2\epsilon}}|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \tag{22}$$

where the a_i are annihilation operators and

$$N = \frac{1}{8}\pi^{3/2}\epsilon\sigma^{3/2}(\epsilon^2 + 6\epsilon\sigma + 15\sigma^2) \tag{23}$$

mixed with Gaussian noise, as in (13) with (12). This state also represents a modification of (14) and can therefore be detected by the same choice of $|\Phi\rangle$.
Without loss of generality, we set $\sigma = 1$ and thus measure all lengths in units of $\sqrt{\sigma}$. For $\delta \leq \frac{3\sigma}{2}$, the whole state space with $p > 0$ is detected to be genuinely multipartite entangled, independantly of ϵ. For $\delta > \frac{3\sigma}{2}$, still large areas of the state space are detected, as illustrated in Fig. 4.

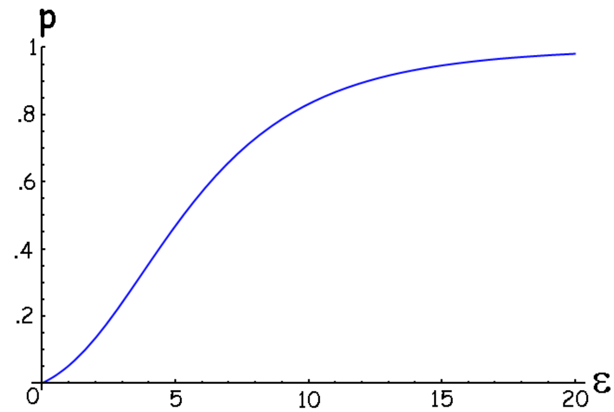


FIG. 4: (Colour online) Illustration of the detection parameter space of the criterion $(*)$ for $k = 2$ and the state ρ, Eq. (13) where $|\omega\rangle$ as in (22), with $\sigma = 1$ and $\delta = 3$. The curve indicates the critical proportion p, i.e. the detection threshold for genuine multipartite entanglement.

VI. EXPERIMENTAL IMPLEMENTATION

Since it is quite important for multipartite entanglement criteria to not only work in theory but also to be implementable experimentally (without resorting to a full state tomography, since this would mean an infinite number of required measurements), let us illustrate how this can be done for our criterion, ineq. $(*)$, for the pure example state from section V A, the GHZ–like state (14). Generalisation to other states is straightforward, but might be rather cumbersome.

While the criterion detects optimally for sharp $|\Phi\rangle$, such states cannot be measured physically, since detectors always have a finite size. We will thus assume

$$|\Phi\rangle = |\alpha\rangle|\alpha\rangle|\alpha\rangle|\beta\rangle|\beta\rangle|\beta\rangle \tag{24}$$

with

$$\begin{aligned} |\alpha\rangle &= \frac{1}{\sqrt{\xi}} \int dx\ \Theta\left(x - x_0 + \frac{\xi}{2}\right) \Theta\left(x_0 + \frac{\xi}{2} - x\right) |x\rangle \\ |\beta\rangle &= \frac{1}{\sqrt{\xi}} \int dx\ \Theta\left(x + x_0 + \frac{\xi}{2}\right) \Theta\left(-x_0 + \frac{\xi}{2} - x\right) |x\rangle \end{aligned} \tag{25}$$

where $\Theta(x)$ is the Heaviside distribution and ξ is the size of the detector, for example a charge-coupled device (CCD). The $(2^n - 1)$ scalar products needed for computation of ineq. $(*)$ thus take forms like

$$\begin{aligned} \langle\phi_1|\rho|\phi_2\rangle &= \left(\frac{\pi\epsilon}{2\xi^{\frac{3}{2}}}\right)^2 \quad \int_{x_0-\frac{\xi}{2}}^{x_0+\frac{\xi}{2}} dx\ e^{-\frac{x^2}{2\sigma}} \left(\mathrm{Erf}\left(\frac{2x_0 - 2x + \xi}{2\sqrt{2\epsilon}}\right) + \mathrm{Erf}\left(\frac{2x_0 - 2x - \xi}{2\sqrt{2\epsilon}}\right)\right)^2 \cdot \\ &\quad \int_{x_0-\frac{\xi}{2}}^{x_0+\frac{\xi}{2}} dy\ e^{-\frac{y^2}{2\sigma}} \left(\mathrm{Erf}\left(\frac{2x_0 - 2y + \xi}{2\sqrt{2\epsilon}}\right) + \mathrm{Erf}\left(\frac{2x_0 - 2y - \xi}{2\sqrt{2\epsilon}}\right)\right)^2 \\ \langle\Phi|P_1^\dagger\rho P_1|\Phi\rangle &= \left(\frac{\pi\epsilon}{2\xi^{\frac{3}{2}}}\right)^2 \quad \left(\int_{-x_0-\frac{\xi}{2}}^{-x_0+\frac{\xi}{2}} dx\ e^{-\frac{x^2}{2\sigma}} \left(\mathrm{Erf}\left(\frac{2x_0 - 2x + \xi}{2\sqrt{2\epsilon}}\right) + \mathrm{Erf}\left(\frac{2x_0 - 2x - \xi}{2\sqrt{2\epsilon}}\right)\right)^2\right)^2 \end{aligned} \tag{26}$$

where

$$\mathrm{Erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a dx\ e^{-x^2} \tag{27}$$

is the Gaussian error function. These integrals can easily be computed numerically, given the parameters used in the experimental setup, which allows for a simple prediction of measurement outcomes. We now explicitly show how to write ineq. $(*)$ in terms of expectation values of local observables in the exemplary three particle case. To that end let us first define the following local observables constructed from the finite sized detectors

$$\begin{aligned}\sigma_x &= |\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha| \\ \sigma_y &= i|\alpha\rangle\langle\beta| - i|\beta\rangle\langle\alpha| \\ \sigma_z &= |\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta|\,,\end{aligned} \tag{28}$$

which are the Pauli operators of the two dimensional subspace spanned by $|\alpha\rangle$ and $|\beta\rangle$. Using the short hand notation

$$ijk := \langle\sigma_i \otimes \sigma_j \otimes \sigma_j\rangle_\rho\,, \tag{29}$$

where $\sigma_1 := \mathbb{1}$, we can explicitly rewrite ineq.($*$) for $k = 2$ and $n = 3$ as

$$\begin{aligned}&\frac{1}{8}|xxx - yyx - yxy - xyy + i(yyy - xxy - xyx - yxx)| - \\ &\frac{1}{8}(\sqrt{(111 + zz1 - z1z - 1zz + 11z - 1z1 - z11 + zzz)(111 + zz1 - z1z - 1zz - 11z + 1z1 + z11 - zzz)} + \\ &\sqrt{(111 - zz1 + z1z - 1zz + 11z - 1z1 + z11 - zzz)(111 - zz1 + z1z - 1zz - 11z + 1z1 - z11 + zzz)} + \\ &\sqrt{(111 - zz1 - z1z + 1zz + 11z + 1z1 - z11 - zzz)(111 - zz1 - z1z + 1zz - 11z - 1z1 + z11 + zzz)}) \leq 0\,.\end{aligned} \tag{30}$$

It is also possible to decompose the inequalities in terms of local expectation values for larger n or different k in a straightforward way. This however yields rather cumbersome expressions, which is why they are not presented then here in full detail.

Experimental measurement uncertainties can be estimated by means of the Gaussian law of error propagation, which states that the measurement uncertainty Ξ of a function f of several arguments x_i is given by

$$\Xi = \sqrt{\sum_i \left(\frac{\partial f}{\partial x_i}\zeta_i\right)} \tag{31}$$

where ζ_i are the respective measurement uncertainties of the x_i.
We will assume that all expectation values $x_i = \langle\alpha_i|\rho|\alpha_i\rangle$ underlie the same relative uncertainty (i.e. $\zeta_i/\langle\alpha_i|\rho|\alpha_i\rangle = \zeta$ is independant of i), such that only two uncertainty parameters remain, namely ζ and o, the latter being the absolute uncertainty of the first term in ineq. ($*$). Now, the measurement uncertainty Ξ of the whole inequality is given by

$$\Xi^2 = o^2 + \sum_{\alpha,i}\left(\frac{1}{2k}\prod_{j=1}^{2k}\frac{(x_j)^(1/2k)}{x_i}\zeta_i\right)^2 = o^2 + \frac{1}{4k^2}\sum_{\alpha,i}\prod_{j=1}^{2k}(x_j)^{1/k}\zeta^2 \leq o^2 + \frac{\zeta^2\gamma}{8k^3} \tag{32}$$

where the ineqality follows from the fct that a geometric mean is maximal whenever all its factors are equal, and

$$\gamma = \sum_{l=1}^{k}\frac{(-1)^{k-i}i^{n-1}}{(i-1)!(k-i)!} \tag{33}$$

is the number of all k-partitions of an n-partite system.
In our above examples (i.e. if $n = 3$ and $k = 2$ or $k = 3$), the second term in (32) is much smaller than the first, such that the measurement uncertainty of the complete expression ($*$) is approximately equal to the uncertainty of its first term: $\Xi \approx o$, which makes this kind of experimental uncertainty easy to deal with.

Another kind of complication that is to be expected in experimental realisations (e.g. in quantum optics) are imperfect detectors. These correspond to nonperfect projective measurements, i.e. each scalar product $\langle\alpha|\rho|\beta\rangle$ is multiplied by some factor $0 \leq \tau \leq 1$. Since the whole inequality is now linear in τ, this does not alter the violation or nonviolation of the inequality, but reduces the magnitude of violation linearly.

VII. SUMMARY

We present a criterion for k-separability in multipartite continuous variable systems. It is an inequality which is satisfied for all k–separable states, i.e. any violation implies that the state is not k–separable. The criterion

particularly allows to distinguish between biseparable states (which can only be used in few applications) and genuinely multipartite entangled ones (which are a basic building piece for several applications of quantum information theory which go beyond the potential of classical systems). We show how the inequality can be optimised by chosing an appropriate state $|\Phi\rangle$ (for which we give four explicit conditions) and thus being left with a reduced (finite) number of optimisation parameters, which can be computed. We analyse two different families of states, which may be considered to be generalisations of the most famous genuinely multipartite states in finite quantum information theory, the GHZ-type entangled states and the W-type entangled states. Our criterion easily detects a large parameter space of entangled states when mixed with Gaussian noise. Since no comparable criteria exist, it can not be said how tight these detection thresholds are.
Moreover, we explicitly show how the developed criterion for k-separability in multipartite continuous variable systems can be rewritten by local expectations values, thus how this criterion can be experimentally implemented.
In summary, we presented a computable criterion for detecting k-inseparability (and particularly genuine multipartite entanglement) in continuous variable systems which can be experimentally implemented by finitely many local observables.

VIII. ACKNOWLEDGEMENTS

The authors would like to thank Gerardo Adesso and Paul Erker for productive and useful discussions. Andreas Gabriel and Marcus Huber gratefully acknowledge the Austrian Fund project FWF-P21947N16. Andreas Gabriel is supported by the University of Vienna's research grant. Beatrix C. Hiesmayr acknowledges the EU project QESSENCE.

[1] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009)
[2] S. Schauer, M. Huber and B.C. Hiesmayr, Phys. Rev. A 82, 062311 (2010)
[3] R. Raussendorf and H.-J. Briegel, Phys. Rev. Lett. 86, 5188 (2001)
[4] M. Huber, H. Schimpf, A. Gabriel, Ch. Spengler, D. Bruß and B.C. Hiesmayr, Phys. Rev. A 83, 022328 (2011)
[5] M. Huber, N. Friis, A. Gabriel, Ch. Spengler and B.C. Hiesmayr, "Lorentz invariance of entanglement classes in multipartite systems", arXiv: 1011.3374
[6] M. Seevinck and J. Uffink, Phys. Rev. A 78, 032101 (2008)
[7] M. Huber, P. Erker, H. Schimpf, A. Gabriel and B.C. Hiesmayr, "Experimentally feasible set of criteria detecting genuine multipartite entanglement in n-qubit Dicke states and in higher dimensional systems", arXiv: 1011.4579
[8] M. Horodecki, P. Horodecki and R. Horodecki, Phys.Lett. A 283, 1 (2001)
[9] P. Krammer, H. Kampermann, D. Bruß, R.A. Bertlmann, L.C. Kwek and C. Macchiavello, Phys. Rev. Lett. 103, 100502 (2009)
[10] J.-D. Bancal, N. Brunner, N. Gisin, and Y.-C. Liang, Phys. Rev. Lett. 106, 020405 (2011)
[11] D. Collins, N. Gisin, S. Popescu, D. Roberts and V. Scarani, Phys. Rev. Lett. 88, 170405 (2002)
[12] O. Gühne and M. Seevinck, New J. Phys. 12, 053002 (2010)
[13] G. Adesso and F. Illuminati, J. Phys. A 40, 7821 (2007)
[14] P. Horodecki and M. Lewenstein, Phys. Rev. Lett. 85, 2657 (2000)
[15] R. Simon, Phys. Rev. Lett. 84, 2726 (2000)
[16] S.L. Braunstein and H.J. Kimble, Phys. Rev. Lett. 80, 869 (1998)
[17] N. Takei, H. Yonezawa, T. Aoki and A. Furusawa, Phys. Rev. Lett. 94, 220502 (2005).
[18] G. Giedke, B. Kraus, M. Lewenstein and J.I. Cirac, Phys. Rev. A 64, 052303 (2001)
[19] P. Hyllus and J. Eisert, New Journal of Physics 8, 51 (2006)
[20] P. van Loock and S.L. Braunstein, Phys. Rev. Lett. 84, 3482 (2000)
[21] A. Gabriel, B.C. Hiesmayr and M. Huber, Quantum Information and Computation (QIC), 10, No. 9 & 10, 829 (2010)
[22] M. Huber, F. Mintert, A. Gabriel and B.C. Hiesmayr, Phys. Rev. Lett. 104, 210501 (2010)
[23] D. Bruß and C. Macchiavello, "Multipartite entanglement in quantum algorithms", arXiv:1007.4179
[24] W. Dür, G. Vidal and J.I. Cirac, Phys. Rev. A 62, 062314 (2000)
[25] F. Verstraete, J. Dehaene, B. De Moor and H. Verschelde, Phys. Rev. A, 65: 052112 (2002)
[26] Ch. Spengler, M. Huber and B.C. Hiesmayr, J. Phys. A: Math. Theor. 43, 385306 (2010)
[27] G. Giedke, B. Kraus, M. Lewenstein and J.I. Cirac, Phys. Rev. Lett. 87, 167904 (2001)
[28] G. Adesso, A. Serafini and F. Illuminati, Phys. Rev. A 73, 032345 (2006)
[29] G. Adesso and F. Illuminati, Phys. Rev. A 78, 042310 (2008)
[30] P. van Loock and A. Furusawa, Phys. Rev. A 67, 052315 (2003)
[31] G. Adesso, Phys. Rev. A 79, 022315 (2009)

[32] A. Kitagawa, M. Takeoka, M. Sasaki and A. Chefles, Phys. Rev. A 73, 042310 (2006)

Heisenberg's Uncertainty Relation and Bell Inequalities in High Energy Physics

An effective formalism for unstable two-state systems*

Antonio Di Domenico[1],† Andreas Gabriel[2],‡ Beatrix C. Hiesmayr[2,3],§ Florian Hipp[2],¶ Marcus Huber[2],**
Gerd Krizek[2],†† Karoline Mühlbacher[2],‡‡ Sasa Radic[2],§§ Christoph Spengler[2],¶¶ and Lukas Theussl[3]***
[1] *Università degli Studi di Roma, La Sapienza, Piazzale Aldo Moro 5, 00185 Rome, Italy*
[2] *University of Vienna, Faculty of Physics, Boltzmanngasse 5, 1090 Vienna, Austria. and*
[3] *Research Center for Quantum Information, Institute of Physics,*
Slovak Academy of Sciences, Dùbravskà cesta 9, 84511 Bratislava, Slovakia

An effective formalism is developed to handle decaying two-state systems. Herewith, observables of such systems can be described by a single operator in the Heisenberg picture. This allows for using the usual framework in quantum information theory and, hence, to enlighten the quantum feature of such systems compared to non–decaying systems. We apply it to systems in high energy physics, i.e. to oscillating meson–antimeson systems. In particular, we discuss the entropic Heisenberg uncertainty relation for observables measured at different times at accelerator facilities including the effect of $\mathcal{CP}$ violation, i.e. the imbalance of matter and antimatter. An operator–form of Bell inequalities for systems in high energy physics is presented, i.e. a Bell–witness operator, which allows for simple analysis of unstable systems.

PACS numbers:

I. INTRODUCTION

The theoretical framework introduced in this paper can be applied in general to a broad variety unstable systems, however, the focus is on meson-antimeson systems and their information theoretic interpretations of certain quantum features of single and bipartite (entangled) systems. In particular, we discuss meson-antimeson systems, e.g. the neutral K–meson or B–meson system, which are very suitable to discuss various quantum foundation issues (see e.g. Refs [1–23]). Neutral kaons are popular research objects in Particle Physics as they were the first system that was found to violate the $\mathcal{CP}$ symmetry ($\mathcal{C}\ldots$charge conjugation; $\mathcal{P}\ldots$ parity), i.e. the imbalance of matter and antimatter. They are also well suited to investigate a possible violation of the $\mathcal{CPT}$ symmetry ($\mathcal{T}\ldots$ time reversal); see e.g. Refs. [21, 22].

Neutral meson–antimeson systems are oscillating and decaying two-state systems and can also described as bipartite entangled systems opening the unique possibility to test various aspects of quantum mechanics also for systems not consisting of ordinary matter and light.

The purpose of this paper is twofold. Firstly to enlighten that these systems provide different insights into quantum theory which are not available in other quantum systems via exploring e.g. the Heisenberg uncertainty relation in its entropic formulation or Bell inequalities which prove that there are correlations stronger than those obtainable in classical physics. Secondly, we introduce a comprehensive and simple mathematical framework which is close to the usual framework to handle stable systems and, therefore, allows for developing novel tools and potential applications.

In Section II we introduce how the time evolution of neutral kaons are usually obtained. In Section III we discuss

---

*Beatrix C. Hiesmayr wants to thank Gregor Weihs for the invitation to give a talk at the conference "Advances in Quantum Theory" (Växjö, Sweden, June 2010) and Andrei Khrennikov for inviting her to organize a session "Fundamentals of QM tested in High Energy Physics" in the forthcoming conference "Foundations of Probability and Physics-6" (Växjö, Sweden, 13-16 June 2010), which initiated this work.
†Electronic address: antonio.didomenico@roma1.infn.it
‡Electronic address: Andreas.Gabriel@univie.ac.at
§Electronic address: Beatrix.Hiesmayr@univie.ac.at (Corresponding author)
¶Electronic address: Florian.Hipp@univie.ac.at
**Electronic address: Marcus.Huber@univie.ac.at
††Electronic address: Gerd.Krizek@univie.ac.at
‡‡Electronic address: Karoline.Muehlbacher@univie.ac.at
§§Electronic address: Sasa.Radic@univie.ac.at
¶¶Electronic address: Christoph.Spengler@univie.ac.at
***Electronic address: Lukas.Theussl@savba.sk

what kind of questions can be raised to the quantum system at accelerator facilities and what is measured at such facilities. In particular, we outline that there are two different measurement procedures not available to other quantum systems. Then the effective formulation of the observables corresponding to a certain question raised to the quantum system is introduced (Section IV), which is our main result. Then we analyze different measurement settings and their uncertainty (Section V). In particular, we show that $\mathcal{CP}$ violation introduces an uncertainty in the observables of the mass eigenstates and, herewith, in the dynamics. Last but not least we proceed to bipartite entangled systems and present the generalized Bell–CHSH inequality for meson–antimesons systems [28] in a witness form (Section VI). This allows to derive the maximal and minimal bound of the Bell inequality by simply computing the eigenvalues of the effective Bell operator, i.e. without relaying on optimizations over all possible initial states.

II. THE DYNAMICS OF DECAYING AND OSCILLATING SYSTEMS

The phenomenology of oscillation and decay of meson-antimeson systems can be described by nonrelativistic quantum mechanics effectively, because the dynamics are rather depending on the observable hadrons than on the more fundamental quarks. A quantum field theoretical calculation showing negligible corrections can e.g. be found in Refs. [22, 23].

Neutral meson M_0 are bound states of quarks and antiquarks. As numerous experiments have revealed the particle state M_0 and the antiparticle state $\bar{M}_0$ can decay into the same final states, thus the system has to be handled as a two state system similar to spin $\frac{1}{2}$ systems. In addition to being a decaying system these massive particles show the phenomenon of flavor oscillation, i.e. an oscillation between matter and antimatter occurs. If e.g. a neutral meson is produced at time $t = 0$ the probability to find an antimeson at a later time is nonzero.

The most general time evolution for the two state system $M^0 - \bar{M}^0$ including all its decays is given by an infinite–dimensional vector in Hilbert space

$$|\tilde{\psi}(t)\rangle \;=\; a(t)|M^0\rangle + b(t)|\bar{M}^0\rangle + c(t)|f_1\rangle + d(t)|f_2\rangle + \dots \tag{1}$$

where f_i denote all decay products and the state is a solution of the Schrödinger equation ($\hbar \equiv 1$)

$$\frac{d}{dt}|\tilde{\psi(t)}\rangle \;=\; -i\hat{H}|\tilde{\psi(t)}\rangle \tag{2}$$

where $\hat{H}$ is an infinite-dimensional Hamiltonian operator. There is no method known how to solve this infinite set of coupled differential equations affected by strong dynamics. The usual procedure is based on restricting to the time evolution of the components of the flavour eigenstates, $a(t)$ and $b(t)$. Then one uses the Wigner-Weisskopf approximation and can write down an effective Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle \;=\; -i\, H|\psi(t)\rangle \tag{3}$$

where $|\psi\rangle$ is a two dimensional state vector and H is a non-hermitian Hamiltonian. Any non-hermitian Hamiltonian can be written as a sum of two hermitian operators M, Γ, i.e. $H = M + \frac{i}{2}\Gamma$, where M is the mass-operator, covering the unitary part of the evolution and the operator Γ describes the decay property. The eigenvectors and eigenvalues of the effective Schrödinger equation, we denote by

$$H\;|M_i\rangle \;=\; \lambda_i\;|M_i\rangle \tag{4}$$

with $\lambda_i = m_i + \frac{i}{2}\Gamma_i$. For neutral kaons the first solution (with the lower mass) is denoted by K_S, the short lived state, and the second eigenvector by K_L, the long lived state, as there is a huge difference between the two decay constants $\Gamma_S \simeq 600\Gamma_L$.

Certainly, the state vector is not normalized for times $t > 0$ due to the non- hermitian part of the dynamics. Different strategies have been developed to cope with that. We present here one which is based on the open quantum formalism, i.e. we show that the effect of decay is a kind of decoherence.

In quantum information theory and in experiments one often has to deal with situations where the system under investigation unavoidable interacts with the environment which is in general inaccessible. In this case only the joint system evolves according to the Schrödinger equation, it is unitary. The dynamics of the system of interest then is given by ignoring all degrees of freedom of the environment, by tracing them out. Such systems are called open quantum systems and under certain assumptions they may be described by a so called master equation.

In Ref. [24] the authors showed that systems with non-hermitian Hamiltonians generally can be described by a master equation. As time evolves the kaon interacts with an environment which causes the decay. In our case the environment plays the role as the QCD vacuum in quantum field theory, but has not to be modeled, only the generators have to be defined describing the effect of the interaction. In particular the time evolution of neutral kaons can be described by the master equation (found by Lindblad [25] and, independently, by Gorini, Kossakowski and Sudarshan [26])

$$\frac{d}{dt}\rho \;=\; -i[\mathcal{H},\rho] - \mathcal{D}[\rho] \tag{5}$$

where the dissipator under the assumption of complete positivity and Markovian dynamics has the well known general form $\mathcal{D}[\rho] = \frac{1}{2}\sum_j(\mathcal{A}_j^\dagger\mathcal{A}_j\rho + \rho\mathcal{A}_j^\dagger\mathcal{A}_j - 2\mathcal{A}_j\rho\mathcal{A}_j^\dagger)$ with $\mathcal{A}_j$ are the generators. The density matrix ρ lives on $\mathbf{H}_{tot} = \mathbf{H}_s \bigoplus \mathbf{H}_f$ where s/f denotes "surviving" and "decaying" or "final" components, and has the following decomposition

$$\rho = \begin{pmatrix} \rho_{ss} & \rho_{sf} \\ \rho_{sf}^\dagger & \rho_{ff} \end{pmatrix} \tag{6}$$

where ρ_{ij} with $i,j = s,f$ denote 2×2 matrices. The Hamiltonian $\mathcal{H}$ is the mass matrix M of the effective Hamiltonian H extended to the total Hilbert space $\mathbf{H}_{tot}$ and Γ of H_{eff} defines a Lindblad operator by $\Gamma = A^\dagger A$, i.e.

$$\mathcal{H} = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}\;,\; \mathcal{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{with} \quad A : \mathbf{H}_s \to \mathbf{H}_f\,.$$

Rewriting the master equation for ρ, Eq. (6), on $\mathbf{H}_{tot}$

$$\dot\rho_{ss} \;=\; -i[H,\rho_{ss}] - \frac{1}{2}\{A^\dagger A, \rho_{ss}\}\;, \tag{7}$$

$$\dot\rho_{sf} \;=\; -iH\rho_{sf} - \frac{1}{2}A^\dagger A\,\rho_{sf}\;, \tag{8}$$

$$\dot\rho_{ff} \;=\; A\,\rho_{ss}\,A^\dagger\,, \tag{9}$$

we notice that the master equation describes the original effective Schrödinger equation (3) but with properly normalized states (see Ref. [24]). By construction the time evolution of ρ_{ss} is independent of ρ_{sf}, ρ_{fs} and ρ_{ff}. Further ρ_{sf}, ρ_{fs} completely decouples from ρ_{ss} and thus can without loss of generality be chosen to be zero since they are not physical and can never be measured. With the initial condition $\rho_{ff}(0) = 0$ the time evolution is *solely* determined by ρ_{ss}—as expected for a spontaneous decay process—and formally given by integrating the components of Eq. (9). It proves that the decay is Markovian and moreover completely positive.

Explicitly, the time evolution of a neutral kaon is given in the lifetime basis, $\{K_S, K_L\}$, by (

$$\rho_{ij} = \langle K_i|\rho|K_j\rangle$$

, $\rho_{SS} + \rho_{LL} = 1$):

$$\rho(t) = \begin{pmatrix} e^{-\Gamma_S t}\rho_{SS} & e^{-i\Delta m t-\Gamma t}\rho_{SL} & 0 & 0 \\ e^{i\Delta m t-\Gamma t}\rho_{SL}^* & e^{-\Gamma_L t}\rho_{LL} & 0 & 0 \\ 0 & 0 & (1-e^{-\Gamma_L t})\rho_{LL} & 0 \\ 0 & 0 & 0 & (1-e^{-\Gamma_S t})\rho_{SS} \end{pmatrix}\,. \tag{10}$$

Note that, formally, one also obtains off–diagonal contributions in the ρ_{ff} component, but as they cannot be measured we set them to zero without loss of generality.

The extension to bipartite systems is straightforward, i.e. by

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathcal{H}\otimes\mathbb{1} + \mathbb{1}\otimes\mathcal{H} \\ A_0 &\longrightarrow A_0\otimes\mathbb{1} + \mathbb{1}\otimes A_0 \end{aligned} \tag{11}$$

but we will not need to use that as our introduced effective formalism for single particles (Section IV) generalizes simply for any multipartite systems, i.e. as in the usual way by simple tensor products.

III. WHAT CAN BE MEASURED AT ACCELERATOR FACILITIES?

There are obviously two different questions that *in principle* can be raised to the quantum system evolving in time:

- Are you a certain quasispin $|k_n\rangle$ at a certain time t_n or not?
- Or: Are you a certain quasispin $|k_n\rangle$ or its orthogonal state $|k_n^\perp\rangle$ ($\langle k_n^\perp|k_n\rangle = 0$) at a certain time t_n?

where we denote by a quasispin k_n any superposition of the mass eigenstates which are the solutions of the effective Schrödinger equation (3).

For non-decaying systems these questions are equivalent, but for decaying systems the second one means that you ignore all cases in which the neutral kaons decayed before the measurement, thus one does not take all information available into account. For studying certain quantum properties of these systems neglecting this kind of information is of no importance, however, e.g. if one is interested to show that there exists no explanation in terms of local hidden parameters for bipartite entangled decaying states, one is not allowed to selected only the surviving pairs, because one would not test the whole ensemble (consult Refs [27–29] for more details).

Let us here also remark on what is meant by a measurement at a certain time t_n. Indeed, one does not measure time, but a certain final decay product or an interaction taking place at a certain position, point in space, in the detector. To be more precise one detects often only secondary reaction products and with the energy-momentum signature reconstructs the final states. Knowing the production point and thus the distance traveled as well as the momentum one can infer the proper time passed between production and decay or interaction.

There are in principle two different options which are denoted as an *active* measurement procedure and a *passive* measurement procedure, for reasons which may become clear in a moment, how to obtain the quasispin content of neutral mesons. This is a remarkable difference and gives raise to two further options of quantum erasure [31, 32] proving the very concept of a quantum eraser, i.e. sorting events to different available information. This kaonic quantum eraser is also in the future work programme of the upgraded KLOE detector which will start in 2011 (for a detailed program see Ref. [1]).

For neutral kaons there exist two physical alternative bases. The first basis is the strangeness eigenstate basis $\{|K^0\rangle, |\bar{K}^0\rangle\}$. It can be measured by inserting along the kaon trajectory a piece of ordinary matter. Due to strangeness conservation of the strong interactions the incoming state is projected either onto K^0 by $K^0 p \to K^+ n$ or onto $\bar{K}^0$ by $\bar{K}^0 p \to \Lambda\pi^+$, $\bar{K}^0 n \to \Lambda\pi^0$ or $\bar{K}^0 n \to K^- p$. Here nucleonic matter plays the same role as a two channel analyzer for polarized photon beams.

Alternatively, the strangeness content of neutral kaons can be determined by observing their semileptonic decay modes (see Eq.(27)). Obviously, the experimenter has no control over the kaon decay process, neither of the mode nor of the time. The experimenter can only sort at the end of the day all observed events in proper decay modes and time intervals. We call this procedure opposite to the *active* measurement procedure described above a *passive* measurement procedure of strangeness.

The second basis $\{K_S, K_L\}$ consists of the short– and long–lived states having well defined masses $m_{S(L)}$ and decay widths $\Gamma_{(S)L}$, which are the solution of the Hamiltionian under investigation. It is the appropriate basis to discuss the kaon propagation in free space, because these states preserve their own identity in time. Due to the huge difference in the decay widths the short lived states K_S decay much faster than the long lived states K_L. Thus in order to observe if a propagating kaon is a K_S or K_L at an instant time t, one has to detect at which time it subsequently decays. Kaons which are observed to decay before $\simeq t + 4.8\,\tau_S$ have to be identified as short lived states K_S, while those surviving after this time are assumed to be long lived states K_L. Misidentifications reduce only to a few parts in 10^{-3} (see also Refs. [31, 32]). Note that the experimenter does not care about the specific decay mode, she or he records only a decay event at a certain time. This procedure was denoted as an *active* measurement of lifetime.

Neutral kaons are famous in Particle Physics as they violate the $\mathcal{CP}$ symmetry, where $\mathcal{C}$ stands for charge conjugation, i.e. interchanging a particle with an antipartice state and $\mathcal{P}$ for parity. So far no violation of the combined symmetry $\mathcal{CPT}$ has been found. Conservation of the $\mathcal{CPT}$ symmetry requires that the time reversal symmetry $\mathcal{T}$ has to be broken. The break of the $\mathcal{T}$ invariance is far from being straightforwardly to be proven experimentally, because for a decay progress $A \longrightarrow B + C$ practical considerations prevent one from creating the time reversed sequence $B + C \longrightarrow A$. The CPLEAR collaboration was able to experimentally prove the $\mathcal{T}$ violation. At the first side it might be surprising that one finds a $\mathcal{T}$ violation in a framework which is completely controlled by non-relativistic quantum mechanics. The apparent paradox is resolved by remembering that the dynamics of a quantum system is given by the equation of motions and the boundary conditions. In particular, the fact that the relative weights of the mass eigenstates are different for the states of the two strangeness states leads to the observable effects. Or differently stated the $\mathcal{T}$ violation follows from the $\mathcal{CP}$ asymmetry in the initial states. Certainly, to understand and handle these symmetry violations we have to use the framework provided by relativistic quantum field theories. The author of Ref. [33] argued that the measured $\mathcal{T}$ violation at accelerator facilities introduce destructive interference between different paths that the universe can take through time, she concludes that only two possible paths are surviving, one forward in time, the other one backward in time.

Since the neutral kaon system violates the $\mathcal{CP}$ symmetry (which will be discussed in Section IV B) the mass eigenstates are not strictly orthogonal, $\langle K_S|K_L\rangle \neq 0$. However, neglecting $\mathcal{CP}$ violation —it is of the order of 10^{-3}— the K_S's are identified by a 2π final state and K_L's by a 3π final state. One denotes this procedure as a *passive* measurement of lifetime, since the kaon decay times and decay channels used in the measurement are entirely determined by the quantum nature of kaons and cannot be in any way influenced by the experimenter.

We have introduced two conceptually different procedures −*active* and *passive*− to measure two different observables of the neutral kaon systems: strangeness or lifetime. The *active* measurement of strangeness is monitored by strangeness conservation in strong interactions while the corresponding *passive* measurement is assured by the $\Delta S = \Delta Q$ rule, i.e. the change of the strangeness number and the change of the charge in a process. *Active* and *passive* lifetime measurements are efficient thanks to the smallness of $\frac{\Gamma_L}{\Gamma_S}$ and the $\mathcal{CP}$ violation parameter, respectively. This will be deeper analyzed in terms of the Heisenberg uncertainty relation in the entropic version in Section V.

Active measurements are possible due to a huge difference in lifetime of the two mesons and, therefore, in practice are available. Thus the neutral kaon system is special concerning its natural constance of the dynamic and, therefore, we mostly stick to this system.

The set of *passive* measurements is not solely limited to the two above described basis choices, but are all possible decay modes of neutral mesons which e.g. single out different $\mathcal{CP}$ violation mechanisms. These decay modes can always be related to a certain quasispin at the moment of decay. Let us assume we find the final state f at a time t_n and we produced at time $t = 0$ a quasispin k_m, the decay rate which is the derivative of the probability is given as an integral over the amplitude squared

$$\Gamma(k_m(t_n) \longrightarrow f) = \int dph(f)|\langle f|\mathtt{T}|k_m(t)\rangle|^2 \tag{12}$$

where $\mathtt{T}$ is the transition operator and the integral is taken over the phase space. To connect the quasipin with the final state, we have to require

$$\langle k_n^\perp|k_n\rangle \stackrel{!}{=} 0 \qquad \text{and} \quad \langle f|\mathtt{T}|k_n^\perp\rangle \stackrel{!}{=} 0 \longrightarrow P_f + P_{f^\perp} = 1 \tag{13}$$

and therefore any final decay product corresponds to a certain quasispin, i.e. a certain superposition of the mass eigenstates, e.g. a two pion event corresponds to the quasispin

$$|K_{\pi^0\pi^0}\rangle \equiv |k_n\rangle \;=\; \alpha_{00}\;|K_S\rangle + \beta_{00}\;|K_L\rangle\;. \tag{14}$$

Summarizing, we have for neutral kaons different conceptual measurement procedures if we neglect $\mathcal{CP}$ violation. *Active* measurements are e.g. required when testing Bell inequalities (see Section VI) while the existence of these two procedures opens new possibilities for kaonic quantum erasure experiments which have no analog for any other two-level quantum systems [31, 32] and are in the experimental programme of the KLOE-2 collaboration [1]. If one is interested in other features of the quantum system under investigation or including $\mathcal{CP}$ violation one can consider all decay channels. For example we will calculate the Heisenberg uncertainty due to $\mathcal{CP}$ violation in the case of two pion events (see Section V). If not stated differently we neglect $\mathcal{CP}$ violation.

IV. EFFECTIVE OPERATORS - A HEISENBERG PICTURE FOR DECAYING SYSTEMS

To develop an effective formalism to derive any expectation value for the questions "*Are you in the quasispin k_n at time t_n (Yes) or not (No)*" of decaying systems

$$\begin{aligned} E(k_n,t_n) &= P(\text{Yes}:k_n,t_n) - P(\text{No}:k_n,t_n) \\ &\stackrel{P(\text{No}:k_n,t_n)+P(\text{Yes}:k_n,t_n)=1}{=} 2\,P(\text{Yes}:k_n,t_n) - 1 \end{aligned} \tag{15}$$

we have to derive the probability to find a certain quasispin k_n at time t_n for a general initial state ρ, i.e.

$$\begin{aligned} P(\text{Yes}:k_n,t_n) \;&=\; Tr(\begin{pmatrix} |k_n\rangle\langle k_n| & 0 \\ 0 & 0 \end{pmatrix}\rho(t_n)) \\ &=\; \rho_{SS}\cdot\cos^2\frac{\alpha_n}{2}e^{-\Gamma_S t_n} + \rho_{LL}\cdot\sin^2\frac{\alpha_n}{2}e^{-\Gamma_L t_n} \\ &+\; \rho_{SL}\cdot\cos\frac{\alpha_n}{2}\sin\frac{\alpha_n}{2}\;e^{i(\phi_n - t_n)}\cdot e^{-\Gamma t_n} \\ &+\; (\rho_{SL}\cdot\cos\frac{\alpha_n}{2}\sin\frac{\alpha_n}{2}\;e^{i(\phi_n - t_n)}\cdot e^{-\Gamma t_n})^*\;. \end{aligned} \tag{16}$$

where we used the following parameterizations

$$|k_n\rangle \;= \cos\tfrac{\alpha_n}{2}|K_S\rangle + \sin\tfrac{\alpha_n}{2}\cdot e^{i\phi_n}|K_L\rangle\;. \tag{17}$$

and $\rho(t)$ is derived from the master equation (5). Moreover, we used a convenient re-scaling, i.e. $\Delta m := 1$ and, consequently the decay constants are re-scaled by the same factor $\Gamma_i := \frac{\Gamma_i}{\Delta m}$.

From that we can extract a time dependent effective operator in dimensions 2×2

$$E(k_n,t_n) \;=\; Tr(O^{eff}(\alpha_n,\phi_n,t_n)\;\rho) \tag{18}$$

where ρ is any initial state which can be taken in dimensions 2×2 as at $t=0$ the decay products have not be taken into account. Herewith, we found for general decaying systems an effective operator in the Heisenberg picture which has besides the computational and interpretative advantage a conceptual one (discussed in the following Sections), i.e. it generalizes for multipartite systems simply by the usual tensor product structure

$$\begin{aligned} &E(k_{n_1},t_{n_1};k_{n_1},t_{n_1};\ldots;k_{n_k},t_{n_k}) \\ &=\; Tr(O^{eff}(\alpha_{n_1},\phi_{n_1},t_{n_1})\otimes O^{eff}(\alpha_{n_2},\phi_{n_2},t_{n_2})\otimes\cdots\otimes O^{eff}(\alpha_{n_k},\phi_{n_k},t_{n_k})\;\rho)\;. \end{aligned} \tag{19}$$

To derive these expectation values is rather cumbersome, e.g. for bipartite systems one has to derive the following four probabilities ($\mathtt{P}_i = |k_i\rangle\langle k_i|$)

$$\begin{aligned} P(Yes:k_n,t_n;Yes:k_m,t_m) &= Tr_A(\mathtt{P}_n\Lambda^{\text{single}}_{t_n}[Tr_B[\mathtt{P}_m\Lambda^{\text{bipartite}}_{t_m}[\rho]]]) \\ P(Yes:k_n,t_n;No:k_m,t_m) &= Tr_A(\mathtt{P}_n\Lambda^{\text{single}}_{t_n}[Tr_B[(\mathbb{1}-\mathtt{P}_m)\Lambda^{\text{bipartite}}_{t_m}[\rho]]]) \\ P(No:k_n,t_n;Yes:k_m,t_m) &= Tr_A((\mathbb{1}-\mathtt{P}_n)\Lambda^{\text{single}}_{t_n}[Tr_B[\mathtt{P}_m\Lambda^{\text{bipartite}}_{t_m}[\rho]]]) \\ P(No:k_n,t_n;No:k_m,t_m) &= Tr_A((\mathbb{1}-\mathtt{P}_n)\Lambda^{\text{single}}_{t_n}[Tr_B[(\mathbb{1}-\mathtt{P}_m)\Lambda^{\text{bipartite}}_{t_m}[\rho]]]) \end{aligned} \tag{20}$$

to obtain the expectation value $E(k_n,t_n;k_m,tm) = P(Yes:k_n,t_n;Yes:k_m,t_m) + P(No:k_n,t_n;No:k_m,t_m) - P(Yes:k_n,t_n;No:k_m,t_m) - P(No:k_n,t_n;Yes:k_m,t_m)$, where Λ^{single} and $\Lambda^{\text{bipartite}}$ are the Liouville operators of the two master equations (5), respectively ($t_n > t_m$).

A. What Observables are in Principle Accessible in Decaying Systems?

Explicitly the effective operator for a two state decaying system decomposed into the Pauli matrices σ is given by

$$O^{eff}(\alpha_n,\phi_n,t_n) = -n_0(\alpha_n,t_n)\mathbb{1} + \vec{n}(\alpha_n,\phi_n,t_n)\vec{\sigma} \tag{21}$$

with $\Delta\Gamma = \frac{\Gamma_L-\Gamma_S}{2}$

$$\vec{n}(\alpha_n,\phi_n,t_n) = e^{-\Gamma t_n}\begin{pmatrix}\cos(t_n+\phi_n)\sin(\alpha_n)\\ \sin(t_n+\phi_n)\sin(\alpha_n)\\ \sinh(\Delta\Gamma t_n)+\cosh(\Delta\Gamma t_n)\cos\alpha_n\end{pmatrix} \tag{22}$$

and $n_0(\alpha_n,t_n) = 1-|\vec{n}(\alpha_n,\phi_n,t_n)|$. For spin $\frac{1}{2}$ systems, the most general observable is given by $\vec{n}\vec{\sigma}$ where any normalized quantization direction ($|\vec{n}|=1$) parameterized by polar angles α_n and ϕ_n can be chosen. In case of decaying systems we can choose in principle α_n and ϕ_n but for $t_n > 0$ the "quantization direction" is no longer normalized and its loss results in an additional contribution in form of "white noise", i.e. the expectation value has a contribution independent of the initial state.

$$\begin{aligned} E(\alpha_n,\phi_n,t_n) &= Tr(O^{eff}(\alpha_n,\phi_n,t_n)\rho) \\ &= -n_0(\alpha_n,t_n) + Tr(\vec{n}(\alpha_n,\phi_n,t_n)\vec{\sigma}\rho)\;. \end{aligned} \tag{23}$$

One recognizes the involved role of the time-evolution: It is damping the "Bloch" vector $\vec{n}$ by $e^{-\Gamma t_n}$ and is responsible for the rotation or oscillation in the system, represented by the polar angle $\Phi = t_n+\phi_n$ in the x,y equatorial plane (x and y component of the "Bloch" vector $\vec{n}$ corresponding to the strangeness eigenstates). In case of $\Delta\Gamma\neq 0$ the time dependence of the z component is more involved. This complex behaviour is responsible for certain quantum features of the system which we will analyze in the following part of the paper.

Let us discuss the eigenstates of the effective operator in order to gain a more physical intuition. For that we derive its spectral decomposition

$$\begin{aligned} O_n^{eff} &\equiv O^{eff}(\alpha_n,\phi_n,t_n) \\ &= (2|\vec{n}(\alpha_n,\phi_n,t_n)|-1)\cdot|\chi(\alpha_n,\phi_n,t_n)\rangle\langle\chi(\alpha_n,\phi_n,t_n)| \\ &\quad +(-1)\cdot|\chi(\alpha_n+\pi,\phi_n+2t_n,-t_n)\rangle\langle\chi(\alpha_n+\pi,\phi_n+2t_n,-t_n)| \end{aligned} \tag{24}$$

with

$$\begin{aligned} |\chi_n\rangle &\equiv |\chi(\alpha_n,\phi_n,t_n)\rangle \\ &= \frac{1}{\sqrt{N(\alpha_n,t_n)}}\Big\{\cos\frac{\alpha_n}{2}\cdot\ e^{-\frac{\Gamma_S}{2}t_n}|K_S\rangle+\sin\frac{\alpha_n}{2}\ e^{i(t_n+\phi_n)}\cdot e^{-\frac{\Gamma_L}{2}t_n}\ |K_L\rangle\Big\} \\ &\text{with}\qquad N(\alpha_n,t_n)=|\vec{n}(\alpha_n,\phi_n,t_n)|^2\ . \end{aligned} \tag{25}$$

The first eigenvector can be interpreted as a quasispin k_n evolving in time according to the dynamics given by the non-hermitian Hamiltonian and normalized to surviving kaons, i.e. to

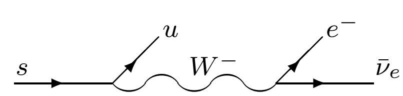
$$\begin{aligned} |\chi_n\rangle &\equiv |k_n(t_n)\rangle \\ &= \frac{1}{\sqrt{N(\alpha_n,t_n)}}\{\cos\frac{\alpha_n}{2}\ e^{i\lambda_S^* t_n}|K_S\rangle+\sin\frac{\alpha_n}{2}\ e^{i\phi_n}\cdot e^{i\lambda_L^* t_n}\ |K_L\rangle\}\ . \end{aligned} \tag{26}$$

The second eigenvector related to the time–independent eigenvalue can be interpreted besides being orthogonal to the normalized quasispin k_n as a quasispin evolving backward in time, but with no phase changes, which we discuss in the next Section IV B in more detail.

B. CP Violation in Mixing and the Effect on the Time Evolution

In 1964 Cronin and Fitch discovered in a seminal experiment that in the neutral kaon system the symmetry $\mathcal{CP}$, where $\mathcal{C}$ stands for charge conjugation, i.e. interchanging a particle state by an antiparticle state, and $\mathcal{P}$ is parity operator, is broken, for which they got the Nobel Prize in 1980. The $\mathcal{CP}$ violation (for a review see e.g. Ref. [34]) and its origin is still a hot discussed subject in particle physics. These open questions are addressed by recently approved projects as KLOE-2 and NA-62 for kaons and SuperBelle and SuperB for B–mesons.

The $\mathcal{CP}$ violation in mixing is e.g. measured by the semileptonic decay channels, i.e a strange quark s decays weakly as constituent of $\bar{K}^0$:



Due to their quark content the kaon $K^0(\bar{s}d)$ and the anti–kaon $\bar{K}^0(s\bar{d})$ have the following definite decay channels:

$$\begin{aligned} K^0(d\bar{s}) &\longrightarrow \pi^-(d\bar{u})\ l^+\ \nu_l \qquad \text{where} \qquad \bar{s}\longrightarrow \bar{u}\ l^+\ \nu_l \\ \bar{K}^0(\bar{d}s) &\longrightarrow \pi^+(\bar{d}u)\ l^-\ \bar{\nu}_l \qquad \text{where} \qquad s\longrightarrow u\ l^-\ \bar{\nu}_l\ , \end{aligned} \tag{27}$$

with l either muon or electron, $l=\mu,e$. Here the validity of the $\Delta S=\Delta Q$ rule is assumed. The Standard Model predicts negligible violations of this selection rule. When studying the leptonic charge asymmetry

$$\delta\ =\ \frac{\Gamma(K_L\to\pi^-l^+\nu_l)-\Gamma(K_L\to\pi^+l^-\bar{\nu}_l)}{\Gamma(K_L\to\pi^-l^+\nu_l)+\Gamma(K_L\to\pi^+l^-\bar{\nu}_l)}\ , \tag{28}$$

we notice that l^+ and l^- tag K^0 and $\bar{K}^0$, respectively, in the K_L state, and the leptonic asymmetry (28) is expressed by the probabilities $|p|^2$ and $|q|^2$ of finding a K^0 and a $\bar{K}^0$, respectively, in the K_L state

$$\delta\ =\ \frac{|p|^2-|q|^2}{|p|^2+|q|^2}\ , \tag{29}$$

i.e. the mass eigenstates and strangeness eigenstates are connected by

$$|K_S\rangle = \frac{1}{N}\{p|K^0\rangle - q|\bar{K}^0\rangle\}\;, \qquad |K_L\rangle = \frac{1}{N}\{p|K^0\rangle + q|\bar{K}^0\rangle\}\;. \tag{30}$$

The weights $p = 1+\varepsilon,\ q = 1-\varepsilon$ with $N^2 = |p|^2 + |q|^2$ contain the complex CP *violating parameter* ε with $|\varepsilon| \approx 10^{-3}$. CPT *invariance* is assumed ($T\ldots$ time reversal). The short–lived K–meson decays dominantly into $K_S \longrightarrow 2\pi$ with a width or lifetime $\Gamma_S^{-1} \sim \tau_S = 0.89\times 10^{-10}$ s and the long–lived K–meson decays dominantly into $K_L \longrightarrow 3\pi$ with $\Gamma_L^{-1} \sim \tau_L = 5.17\times 10^{-8}$ s. However, due to $\mathcal{CP}$ violation we observe a small amount $K_L \longrightarrow 2\pi$. Therefore, CP violation expresses that there is a difference between a world of matter and a world of antimatter.

Let us now derive the change due to $\mathcal{CP}$ violation to the effective observable. Firstly note that the length of the Bloch vector $\vec{n}$ can be rewritten by the sum of two probabilities, i.e.

$$|\vec{n}| \;=\; 1 - n_0 \;=\; |\langle k_n|K_S(t_n)\rangle|^2 + |\langle k_n|K_L(t_n)\rangle|^2\;. \tag{31}$$

The symmetry violation $\mathcal{CP}$ results in a non-orthogonality of the mass eigenstates, i.e. each amplitude leads to an interference term

$$\begin{aligned} |\langle k_n|K_S(t_n)\rangle|^2 &= e^{\Gamma_S t_n}\,|\cos\frac{\alpha_n}{2} + \delta\cdot\sin\frac{\alpha_n}{2}e^{-i\phi_n}|^2 \\ |\langle k_n|K_L(t_n)\rangle|^2 &= e^{\Gamma_L t_n}\,|\delta\cdot\cos\frac{\alpha_n}{2} + \sin\frac{\alpha_n}{2}e^{-i\phi_n}|^2 \end{aligned} \tag{32}$$

and, therefore, changes the oscillation behaviour of the system but as well the loss in the decaying system. Note that $\mathcal{CP}$ violation may as well change the state under investigation, i.e. the expectation value gets as well a "contribution" of the symmetry violation from the initial state.

The effective operator changes in detail by (we suppress the dependence on the parameters α_n, ϕ_n, t_n)

$$\begin{aligned} n_1^{\mathcal{CP}} &= n_1 - e^{-\Gamma t_n}(2\delta\cdot\cos t_n + \delta^2\cdot\sin\alpha_n\ \cos(t_n-\phi_n)) \\ n_2^{\mathcal{CP}} &= n_2 - e^{-\Gamma t_n}(2\delta\cdot\sin t_n + \delta^2\cdot\sin\alpha_n\ \sin(t_n-\phi_n)) \\ n_3^{\mathcal{CP}} &= n_3 - (\delta\cdot(e^{-\Gamma_S t_n} - e^{-\Gamma_L t_n})\sin\alpha_n\cos\phi_n \\ &\qquad + \delta^2\cdot\frac{1}{2}(e^{-\Gamma_S t_n} - e^{-\Gamma_L t_n} - (e^{-\Gamma_S t_n} + e^{-\Gamma_L t_n})\cos\alpha_n)\;. \end{aligned} \tag{33}$$

The spectral decomposition shows that the time dependent eigenvalue is changed by $\mathcal{CP}$ violation, confirming its observable character, but it has the same dependence from the Bloch vector as in case of $\mathcal{CP}$ conservation (compare with Eq. (24))

$$\begin{aligned} \lambda_1^{\mathcal{CP}} &= -1 + 2\,|\vec{n}^{\mathcal{CP}}| = 1 - 2\,n_0^{\mathcal{CP}} \\ \lambda_2^{\mathcal{CP}} &= -1\;. \end{aligned} \tag{34}$$

The two eigenvectors of the effective observable change accordingly

$$\begin{aligned} |\chi_n^{\mathcal{CP},1}\rangle &= \frac{1}{\sqrt{N(t)}}\Big\{\langle K_S|k_n\rangle\cdot e^{i\lambda_S^* t_n}\,|K_1\rangle + \langle K_L|k_n\rangle\cdot e^{i\lambda_L^* t_n}\,|K_2\rangle\Big\} \\ |\chi_n^{\mathcal{CP},2}\rangle &= \frac{1}{\sqrt{N(-t)}}\Big\{-\langle K_L|k_n\rangle^*\cdot e^{i\lambda_S t_n}\,|K_1\rangle + \langle K_S|k_n\rangle^*\cdot e^{i\lambda_L t_n}\,|K_2\rangle\Big\} \\ &\text{with} \\ N(t) &= e^{-\Gamma_S t_n}|\langle K_S|k_n\rangle|^2 + e^{-\Gamma_L t_n}|\langle K_L|k_n\rangle|^2\;. \end{aligned} \tag{35}$$

Note that if we parameterize the quasispin in the $\mathcal{CP}$ basis, $|k_n\rangle = \cos\frac{\alpha_n}{2}|K_1\rangle + \sin\frac{\alpha_n}{2}\cdot e^{i\phi_n t}\,|K_2\rangle$, we find that the weights do not add up to one generally

$$N(0) \;=\; |\langle K_S|k_n\rangle|^2 + |\langle K_L|k_n\rangle|^2 \;=\; 1 + \delta\cdot\sin\alpha_n\cos\phi_n\;. \tag{36}$$

V. THE ENTROPIC UNCERTAINTY RELATION FOR SINGLE AND BIPARTITE SYSTEMS

The entropic uncertainty relation of two non-degenerate observables is given by (introduced by D. Deutsch [35], improved in Ref. [36] and proven by Ref. [37])

$$H(O_n^{eff}) + H(O_m^{eff}) \;\geq\; -2\log_2\left(\max_{i,j}\{|\langle\chi_n^i|\chi_m^j\rangle|\}\right) \tag{37}$$

where

$$H(O_n^{eff}) = -p(n)\log_2 p(n) - (1-p(n))\log_2(1-p(n)) \tag{38}$$

is the binary entropy for a certain prepared pure state ψ and the $p(n)$'s are the probability distribution associated with the measurement of O_n^{eff} for ψ, hence $p(n) = |\langle\chi_n|\psi\rangle|^2$. This is a reformulation of the famous uncertainty principle by Robertson [38], which can be found in most textbooks on quantum theory

$$(\Delta O_n^{eff})_\psi \cdot (\Delta O_m^{eff})_\psi \;\geq\; \frac{1}{2}\,\left|\langle\psi|\left[O_n^{eff}, O_m^{eff}\right]|\psi\rangle\right|\;, \tag{39}$$

where $(\Delta A)^2_\psi = \langle A^2\rangle_\psi - \langle A\rangle^2_\psi$ are the mean square deviations. Choosing, the operators, position $\hat{x}$ and momentum $\hat{p}$, the Robertson relation turns into the famous Heisenberg relation

$$\Delta\hat{x}\cdot\Delta\hat{p} \geq \frac{1}{2}\;. \tag{40}$$

The maximal value of the right hand side of the entropic uncertainty relation is obtained for

$$|\langle\chi_n|\chi_m\rangle| \;=\; \frac{1}{\sqrt{2}}\;, \tag{41}$$

in this case the the two observables are commonly called complementary to each other (their eigenvalues have to be nondegenerate), e.g. if the operators are σ_x and σ_z. In general a non-zero value of the right hand side of Eq.(37) means that the two observables do not commute, i.e. it quantifies the complementarity of the observables. The binary entropies on the left hand side quantify the gain of information on average when we learn about the value of the random variable associated to O_n^{eff}. Alternatively, one can interpret the entropy as the uncertainty *before* we obtain the result of the random variable.

The reformulation of the Heisenberg relation (37) has —besides its different information-theoretic interpretation and its stronger bound [39]— the advantage that the right hand side of the inequality is independent of the prepared state and only depends on the eigenvectors of the observables, hence puts a stronger limit on the extent to which the two observables can be simultaneously peaked.

Remarkably, the right hand side of the entropic uncertainty relation also does not depend on the eigenvalues (except to test the non-degeneracy), this means that if the state is prepared in an eigenstate say of O_n^{eff} then the two eigenvalues of O_m^{eff} are equally probable as measured values, i.e. the exact knowledge of the measured value of one observable implies maximal uncertainty of the measured value of the other, independent of the eigenvalues.

A. An Information Theoretic View on Measurements at Different Times at Accelerator Facilities

Particle detectors at accelerator facilities detect or reconstruct different decay products at different distances from the creation point, usually by a *passive* measurement procedure, more rarely by an *active* measurement procedure. Let us here discuss what is learnt by finding a certain quasispin $|k_n\rangle$ at a certain time t_n or not which can correspond to a certain decay channel, compared to the situation to find a k_m at the creation point $t_m = 0$ or not. Certainly, this result also quantifies our uncertainty *before* we learn the result (Yes, No) at t_n and (Yes, No) at t_m. In particular, if we compare observables of same quasispin at different time, we obtain the uncertainty due to the time evolution.

Differently stated, we can view it in the following way [40], two experimenters, Alice and Bob, choose two different measurements corresponding to the observables O_n^{eff}, O_m^{eff}. Alice prepares a certain state ψ and sends it to Bob. Bob carries out one of the two measurements O_n^{eff}, O_m^{eff} and announces his choice n or m to Alice. She wants to minimize her uncertainty about Bob's measurement result. Alice's result is bounded by the equation (37).

In case of unstable systems the right hand side of the entropic uncertainty relation (37), for which we have to find the maximum, is given by

$$\max\left\{\langle\chi_m^1|\chi_n^1\rangle, \langle\chi_m^1|\chi_n^2\rangle, \langle\chi_m^2|\chi_n^1\rangle, \langle\chi_m^2|\chi_n^2\rangle\right\} \tag{42}$$

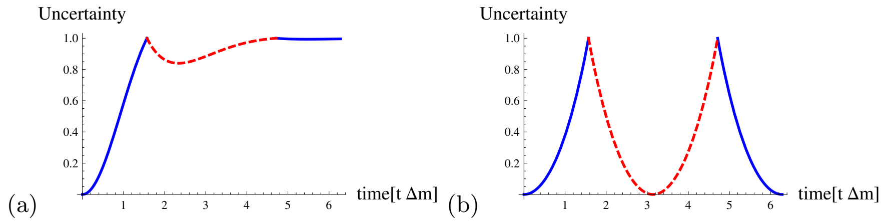


FIG. 1: Here the lower bound of the entropic quantum uncertainty inequality (37) is plotted in case of a strangeness event at $t = 0$ compared to a strangeness event at a later time, i.e. for the observables $A = O_{eff}(\frac{\pi}{2}, \phi_n, 0)$ and $B = O_{eff}(\frac{\pi}{2}, \phi_m, t)$ with $\phi_n = \phi_m = 0, \pi$ for (a) Γ_S and (b) $\Gamma \approx \Gamma_L$ including $\Gamma_S = \Gamma_L = 0$. The solid blue line shows when the eigenvectors both propagating forward in time or both propagating backward in time overlap maximally, whereas the red dashed line shows the case when forward and backward propagating quasipins overlap. Figure (b) shows the case of a slow decaying system or all other meson systems, i.e. B_d, B_s, except maybe the D meson system for which not much precise data is available. If the decay constants are considerably different there is always missing information in the system.

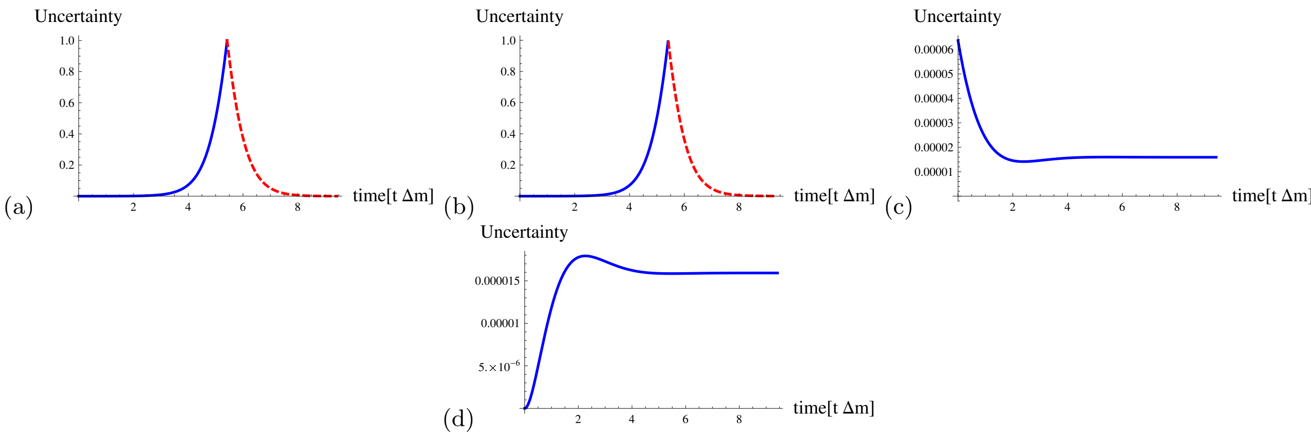


FIG. 2: This graphs depict the lower bound of the entropic quantum uncertainty inequality (37) by comparing measurements at time $t = 0$ to measurements at later times t for (a) short lived state ($t = 0$) versus short lived state at t, (b) long lived state ($t = 0$) versus short lived state at t, (c) short lived state ($t = 0$) versus long lived state at t and (d) long lived state ($t = 0$) versus long lived state at t. This shows the uncertainty introduced by breaking the $\mathcal{CP}$ symmetry in the time evolution. If Alice and Bob agree to ask about a short lived state at the complementary time $t = 5.4[t\Delta m] \equiv 11.4\tau_S$ the uncertainty becomes the maximal possible value. In case Alice and Bob agree to ask for any time $t > 0$ for a long lived state, the uncertainty is nonzero.

with $|\chi_n^1\rangle = |\chi(\alpha_n, \phi_n, t_n)\rangle$ and $|\chi_n^2\rangle = |\chi(\alpha_n + \pi, \phi_n + 2t_n, -t_n)\rangle$ being the eigenvectors of the effective operators or the quasispin propagating forward or backward in time, respectively. Any product derives to

$$\langle\chi(\alpha_n, \phi_n, t_n)|\chi(\alpha_m, \phi_m, t_m)\rangle = \frac{\cos\frac{\alpha_n}{2}\cos\frac{\alpha_m}{2} + \sin\frac{\alpha_n}{2}\sin\frac{\alpha_m}{2}\; e^{i(t_m - t_n + \phi_m - \phi_n)} \cdot e^{-\Delta\Gamma(t_n + t_m)}}{\frac{1}{2}\sqrt{1 + e^{-2\Delta\Gamma t_n} + \cos\alpha_n(1 - e^{-2\Delta\Gamma t_n})}\sqrt{1 + e^{2\Delta\Gamma t_m} - \cos\alpha_m(1 - e^{2\Delta\Gamma t_m})}}\,. \tag{43}$$

In Fig. 1 we plotted the complementarity for the observable asking the question "*Is the neutral kaon system in the state* $|K^0\rangle$ *or not at time* $t = 0$" compared to the question "*Is the neutral kaon system in the state* $|K^0\rangle$ *or not at time* t", i.e. comparing the complementary introduced by the time evolution in the case of strangeness measurements. Here Fig. 1 (a) refers to the neutral kaon case and (b) to a slowly decaying system ($\Gamma_S \to 100\Gamma_S$) or any of the other meson systems $\Delta\Gamma = 0$. One notices that for times being odd multiples of $\frac{\pi}{2}$ the complementary of the two observables becomes minimal, while for even multiples it maximizes.

Asking about the mass-eigenstates we find no complementary of the observables for any time, this certainly only changes if we include CP violation. The uncertainty, i.e. the overlap of the measurement of a short lived state at a later time point to that at time zero, is moderated by δ, i.e. for small times the maximum is obtained by the overlap

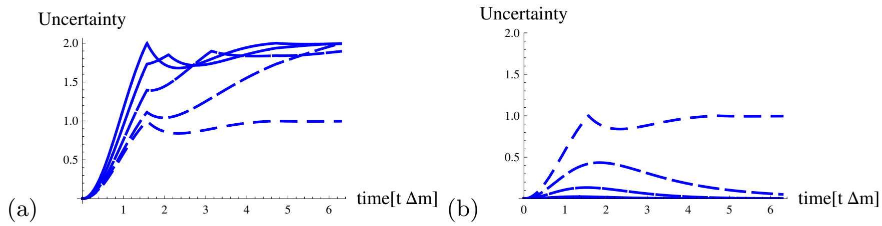


FIG. 3: The right hand side of the entropic quantum uncertainty inequality (37) for the two observables (a) $O_{eff}(\frac{\pi}{2},0,0)\otimes O_{eff}(\frac{\pi}{2},0,t)$ versus $O_{eff}(\frac{\pi}{2},0,t_1)\otimes O_{eff}(\frac{\pi}{2},0,0)$ and (b) $O_{eff}(\frac{\pi}{2},0,0)\otimes O_{eff}(\frac{\pi}{2},0,t)$ versus $O_{eff}(\frac{\pi}{2},0,0)\otimes O_{eff}(\frac{\pi}{2},0,t_1)$ for $t_1=0,t/4,\dots,t$ is plotted, where $t_1=0$ is the dashed line. One recognizes that one can increase or decrease the maximal uncertainty if the role of the first and second observable in the tensor product, i.e. Alice and Bob's role, is changed.

of the first two eigenvalues Eq.(35)

$$\left|\langle\chi^{\mathcal{CP},1}(K_S,t_n)|\chi^{\mathcal{CP},1}(K_S,t_m=0)\rangle\right| \;=\; \frac{\left|e^{-\frac{\Gamma_S}{2}t_n}+\delta^2 e^{-it_n}\cdot e^{-\frac{\Gamma_L}{2}t_n}\right|}{\sqrt{(1+\delta^2)(e^{-\Gamma_S t_n}+\delta^2 e^{-\Gamma_L t_n})}} \tag{44}$$

and the maximum uncertainty $-2\log_2\max\{|\langle\chi_n^i|\chi_m^j\rangle|\}$ is reached for the overlap $\frac{1}{\sqrt{2}}$ for $t_n=11.4\tau_S$ and choosing the $\mathcal{CP}$ violation parameter $\delta=3.322\cdot 10^{-3}$ (world average [41]). This is just the case when the overlaps of all possibilities are equal, i.e. the two bases are mutually unbiased bases ($MUBS$). The same complementary time $t_n=11.4\tau_S$ is obtained when we compare the measurement of the long lived state at time $t_m=0$ and a measurement of the short lived state. For the two other options the maximal uncertainty can never be reached. Initially, the uncertainty is zero in case of measuring long lived states and then oscillates due to δ and reaches after $t_n=11.4\tau_S$ a constant value close to zero. This is summarized in Fig. 2.

Certainly, at this time the probability to find a short lived state is for all practical purposes zero. Remember that we have chosen for active measurements of lifetime a time of $4.8\tau_S$, which is the time when the probability of not finding a short lived state when it was produced as a short lived state equals the probability to find a long lived state when it was produced as a long lived state, i.e. $1-e^{-\Gamma_S t}\stackrel{!}{=}e^{-\Gamma_L t}$. This time is by more than a factor 2 different to the complementary time which strongly depends on the amount of $\mathcal{CP}$ violation. We can revert the issue and ask how big δ needs to be in order that the two times would be equal: it would need to be 25 times the value of δ. Therefore, *active* and *passive* measurements of lifetime are efficient.

B. The Uncertainty of Measurement Settings for Bipartite Kaons

The effective operator formalism guarantees that the tensor product structure is conserved, i.e. the most general expectation value of a bipartite system is given by

$$E(k_n,k_m)=Tr(O_n^{eff}\otimes O_m^{eff}\rho) \tag{45}$$

for any initial bipartite state ρ. In this case one studies e.g. symmetry violations or Bell inequalities where one compares measurements of different quasispins at different times. In this section we want to investigate the uncertainty of such different measurement settings and herewith obtain a different view and intuition on how certain properties of quantum states are revealed, in particular we will then proceed to analyze the maximum violation of a Bell inequality.

To compute the right hand side of the entropic uncertainty relation we have to find the maximum of all eigenvectors of the operator $O_n\otimes O_m$ which is straightforward as it is simply the product of the eigenvectors of the single operators $O_{n/m}$ of Alice and Bob, respectively

$$\max\left\{\langle\chi_m^i|\chi_n^j\rangle\cdot\langle\chi_m^k|\chi_n^l\rangle\right\}\quad\text{with } i,j,k,l=1,2\;. \tag{46}$$

In Fig. 3 we show how the uncertainty is changed for different observables in the bipartite kaons system, which gives an intuition when a certain Bell operator may yield a violation (see next Section VI).

VI. THE BELL-CHSH INEQUALITY

In accelerator experiments one can produce a spin singlet state, e.g. by the decay of a Φ meson at the DAPHNE machine. One has the same scenario as Einstein, Podolsky and Rosen considered in 1935 which we write down for different quantum systems (spin–$\frac{1}{2}$, ground/excited state, polarisation, K–meson, B–mesons, molecules arriving early/late [42] or single neutrons in an interferometer) to show its similarity:

$$\begin{aligned} |\psi^-\rangle &= \frac{1}{\sqrt{2}}\big\{|\Uparrow\rangle_l \otimes |\Downarrow\rangle_r - |\Downarrow\rangle_l \otimes |\Uparrow\rangle_r\big\} \\ &= \frac{1}{\sqrt{2}}\big\{|0\rangle_l \otimes |1\rangle_r - |1\rangle_l \otimes |0\rangle_r\big\} \\ &= \frac{1}{\sqrt{2}}\big\{|H\rangle_l \otimes |V\rangle_r - |V\rangle_l \otimes |H\rangle_r\big\} \\ &= \frac{1}{\sqrt{2}}\big\{|K^0\rangle_l \otimes |\bar{K}^0\rangle_r - |\bar{K}^0\rangle_l \otimes |K^0\rangle_r\big\} \\ &= \frac{1}{\sqrt{2}}\big\{|B^0\rangle_l \otimes |\bar{B}^0\rangle_r - |\bar{B}^0\rangle_l \otimes |B^0\rangle_r\big\} \\ &= \frac{1}{\sqrt{2}}\big\{|late\rangle_l \otimes |early\rangle_r - |early\rangle_l \otimes |late\rangle_r\big\} \\ &= \frac{1}{\sqrt{2}}\big\{|I\rangle_l \otimes |\Uparrow\rangle_r - |II\rangle_l \otimes |\Downarrow\rangle_r\big\} \\ &= \dots \,. \end{aligned} \tag{47}$$

Analog to entangled photon systems for these systems Bell inequalities can be derived, i.e. the most general Bell inequality of the CHSH–type is given by (see Ref. [43])

$$\begin{aligned} & S_{k_n,k_m,k_{n'},k_{m'}}(t_1,t_2,t_3,t_4) = \\ & \quad \big|E_{k_n,k_m}(t_1,t_2) - E_{k_n,k_{m'}}(t_1,t_3)\big| \\ & \qquad + |E_{k_{n'},k_m}(t_4,t_2) + E_{k_{n'},k_{m'}}(t_4,t_3)| \leq 2 \,. \end{aligned} \tag{48}$$

Here Alice can choose on the kaon propagating to her left hand side to raise the question if the neutral kaon is in the quasispin $|k_n\rangle = \cos\frac{\alpha_n}{2}|K^0\rangle + \sin\frac{\alpha_n}{2}e^{i\phi_n}|\bar{K}^0\rangle$ or not, and how long the kaon propagates, the time t_n. The same options are given to Bob for the kaon propagating to the right hand side. As in the usual photon setup, Alice and Bob can choose among two settings.

Differently to commonly investigated systems one has more options. One can vary in the quasispin space or vary the detection times or both.

With our effective framework we can rewrite the Bell-CHSH-inequality in a witness type, i.e. with the Bell operator

$$\mathbf{Bell}^{eff} = O_n^{eff} \otimes (O_m^{eff} - O_{m'}^{eff}) + O_{n'}^{eff} \otimes (O_m^{eff} + O_{m'}^{eff}) \tag{49}$$

any local realistic hidden parameter theory has to satisfy

$$|Tr(\mathbf{Bell}^{eff}\rho)| \leq 2 \,. \tag{50}$$

This operator form of the generalized Bell-CHSH inequality [28] gives us the opportunity to find for a given choice of Bell settings without optimization over all possible initial states whether the Bell inequality can be violated. In particular, the eigenvalues of the Bell operator give us the upper and lower bound that can be reached for the optimal initial state, i.e. the one which maximizes or minimizes the Bell inequality. Determining whether a Bell inequality is preserved or violated for a given state ρ is in general a high-dimensional nonlinear constrained optimization problem. In Ref. [44] a numerical method was shown by introducing a proper parameterization [45] for unitary matrices to derive bounds on Bell inequalities for any qudit system (d–level system). This certainly is a benefit of our introduced effective formalism as optimization in this case is not needed. In any numerical optimization there is no guarantee that the global extremum was reached. In some exemplary cases we checked for the agreement and in many case the optimization failed.

We present first a generalized Bell inequality which has been discussed in literature [28, 29, 43, 46] and shows a relation between $\mathcal{CP}$ violation and the nonlocality detected by the above Bell inequality. Then we proceed to a Bell setting that can be realized in a direct experiment.

A. A Bell Inequality Sensitive to $\mathcal{CP}$ Violation

Let us choose all times equal zero and choose the quasispin states $k_n = K_S, k_m = \bar{K}^0, k_{n'} = k_{m'} = K_1^0$ where K_1^0 is the $\mathcal{CP}$ plus eigenstate.

In Ref. [43] the authors showed that after optimization the CHSH–Bell inequality can be turned for an initial spin singlet state into

$$\delta \leq 0 \tag{51}$$

where δ is the $\mathcal{CP}$ violating parameter in mixing, Eq.(28). Experimentally, δ corresponds to the leptonic asymmetry of kaon decays which is measured to be $\delta = (3.322 \pm 0.055) \cdot 10^{-3}$. This value is in clear contradiction to the value required by the CHSH-Bell inequality above, i.e. by the premises of local realistic theories! The result can be also made stronger by changing the Bell setting by $K_S \longrightarrow K_L$, then one obtains $\delta \geq 0$, thus both CHSH-Bell inequalities require

$$\delta = 0 \; , \tag{52}$$

i.e. any local realistic hidden variable theory is in contradiction to $\mathcal{CP}$ violation, a difference of a world of particles and antiparticles. In this sense the violation of a symmetry in high energy physics is connected to the violation of a Bell inequality, i.e. to nonlocality. This is clearly not available for photons, they do not violate the $\mathcal{CP}$ symmetry.

We also want to remark that the considered Bell inequality, since it is chosen at time $t = 0$ is connected to a test of contextuality rather than nonlocality. *Noncontextuality*, the independence of the value of an observable on the experimental context due to its predetermination —a main hypothesis in hidden variable theories— is definitely ruled out! So the contextual quantum feature is demonstrated for entangled kaonic qubits.

Although the Bell inequality (51) is as loophole free as possible, the probabilities or expectation values involved are not directly measurable, because experimentally there is no way to distinguish the short–lived state K_S from the $\mathcal{CP}$ plus state K_1^0 directly.

B. A Bell Inequality Sensitive to Strangeness

Let us now proceed to another choice for the Bell inequality (48), i.e. all quasispins equal $\bar{K}^0$, but we are going to vary all four times

$$\begin{aligned} &S_{\bar{K}^0,\bar{K}^0,\bar{K}^0,\bar{K}^0}(t_1,t_2,t_3,t_4) = \\ &\quad |E(\bar{K}^0,t_1;\bar{K}^0,t_2) - E(\bar{K}^0,t_1;\bar{K}^0,t_3)| + |E(\bar{K}^0,t_4;\bar{K}^0,t_2) + E(\bar{K}^0,t_4;\bar{K}^0,t_3)| \\ &\qquad \leq 2 \; . \end{aligned} \tag{53}$$

This has the advantage that it can in principle be tested in experiments. Alice and Bob insert at a certain distance from the source (corresponding to the detection times) a piece of matter forcing the incoming neutral kaon to react. Because the strong interaction is strangeness conserving one knows from the reaction products if it is an antikaon or not. Note that different to photons a *NO* event does not mean that the incoming kaon is a K^0 but also includes that it could have decayed before. In principle the strangeness content can also be obtained via decay modes, but Alice and Bob have no way to force their kaon to decay at a certain time, the decay mechanism is a spontaneous event. However, a necessary condition to refute local realistic theories are *active* measurements, i.e. exerting the free will of the experimenter (for more details consult [27]).

In Refs. [27, 43] the authors studied the problem for an initial maximally entangled Bell state, i.e., $\psi^- \simeq K^0\bar{K}^0 - \bar{K}^0K^0$, and found that a value greater than 2 cannot be reached, i.e. one cannot refute any local realistic theory. The reason is that the particle–antiparticle oscillation is too slow compared to the decay or vice versa, i.e., the ratio of oscillation to decay $x = \frac{\Delta m}{\Gamma}$ is about 1 for kaons and not 2 necessary for a violation. A different view is that the decay property acts as a kind of "decoherence" as we introduced in Section II. From decoherence studies we know that some states are more "robust" against a certain kind of decoherence than others, this leads to the question if another maximally entangled Bell state or maybe a different initial state would lead to a violation which is indeed the case.

In Ref. [29] it was shown that such states exists. This shifts the problem to finding methods to produce these initial states leading to a violation of the generalized CHSH–Bell inequality. This is still an open problem. In Ref. [29] also the interplay between entanglement and entropy was studied and as also shown by the authors of Ref. [30], who studied the dynamics of two qubits interacting with a common zero-temperature non-Markovian reservoi, the picture that entanglement loss due to environmental decoherence is accompanied by loss of the purity of the state of the system does not apply to these systems.

Given our effective operator formalism we can answer the question how much nonlocality is there for the given Bell setting if we vary the times. In Fig. 4 we plotted the eigenvalues of the Bell operator for different choices corresponding to the maximal/minmal value of the Bell inequality as well as the uncertainty. We find only a small amount of violation (about 2.1) but huge time regions of possible violations. Moreover, we notice an asymmetric behaviour of the minimal and maximal eigenvalues of the Bell operator which is due to the two different decay constants, as also plotted in Fig. 5 for a slow decaying system and for the B–meson system.

In Ref. [47] the authors showed that the maximal violation of the CHSH–Bell inequality is reached when the two operators in the sum of the Bell operator, Eq. (49), commute. This fact the authors used to construct other relevant Bell inequalities for two–qubit systems. For unstable systems we do not see a one–to–one correspondence between the uncertainty of the summands in the Bell operator and the amount of violation, moreover, $O_m^{eff} \pm O_{m'}^{eff}$ does not necessary describe an observable obtainable in a single measurement.

VII. SUMMARY AND CONCLUSIONS

We studied the phenomenology of decaying two–state systems and discussed quantum features from an information theoretic view. For that we developed an effective formalism which allows to handle unstable two-state systems with the usual well developed formalism in Quantum Information Theory. We applied it to the neutral kaon system including the $\mathcal{CP}$ violation, the observed imbalance between matter and antimatter in our universe.

We presented the effective operator in decomposition of the Pauli-matrices and the unity, which shows the complicated change of the Bloch vector in time. The spectral decomposition shows that only one eigenvalue depends on measurement settings and that the corresponding eigenvectors can be interpreted as quasispins evolving in (forward and backward) time normalized to surviving pairs. The second eigenvalue is always -1, i.e. it does not depend on the chosen measurement settings. This expresses the fact that we are only interested in quantum features intrinsic to neutral kaons and not about the properties of the different decay channels.

The lower bound on the binary entropies of two chosen observables is given by maximal overlap of the eigenvectors of both observables and encodes the limitations on the available information obtainable by the chosen observables. To obtain this Heisenberg uncertainty in time for meson-antimeson systems we compared measurement settings at time $t = 0$ to the same measurement settings at a later time t. We find for flavor measurements that the uncertainty becomes maximal for times which are odd multiples of $\frac{\pi}{2}$, while for times which are multiples of π only in the case both decay rates are equal the uncertainty becomes zero again as it is the case for non–decaying systems. For considerably different decay constants as in the neutral kaon system the uncertainty never vanishes for any later time measurement, i.e. introducing an persisting lack of information; this is depicted in Fig. 1.

Due to imbalance of matter and antimatter we derived a maximal uncertainty for short lived measurements at a "complementary" time depending on the precise values of the $\mathcal{CP}$ violating parameter δ. This "complementary" time is more than twice the time of the time duration for which the probability to misidentify a long lived state as a short lived state or vice versa is equal. In case of long lived measurements the lower bound on the uncertainty relation is constant (about the amount of the $\mathcal{CP}$ violating parameter). This is illustrated in Fig. 2 and shows the effect of indirect $\mathcal{CP}$ violation on the states persisting their nature in the time evolution.

Then we proceed to entangled bipartite systems. The effective observables simply generalize for multipartite systems by the usual tensor product structure which is a clear advantage to the open quantum approach. The uncertainty for bipartite systems is straightforwardly obtained as it is the maximum of the product of the scalar products of the eigenvectors of the single effective operator.

Due to the developed effective formalism Bell inequalities, i.e. inequalities deciding whether a local realistic view for kaons is valid, can be formulated in a mathematically more simple form, i.e. as a witness operator. Herewith, we do not need to optimize over the state space parameters and the four different measurement settings, but can simply compute the eigenvalues of the Bell operator to obtain the maximal possible value given by the quantum theory. In case of strangeness measurements we find that the violation is not big, but can be obtained for long time regions. Indeed, also for times when the short lived component has already died out for all practical purposes, i.e. no oscillation can be seen, but since the probability is still nonzero, non-negligible contributions in the Bell operator exist.

We believe with this information theoretic view on unstable two–state systems and, in particular, on the meson-antimeson systems in high energy physics we could enlighten the quantum features in these massive systems and, in particular, the threefold role of time, being responsible for strangeness oscillations, oscillation due to $\mathcal{CP}$ violation and characterizing the decay property.

Acknowledgements: Marcus Huber and Christoph Spengler gratefully acknowledge the Austrian Fund project FWF-

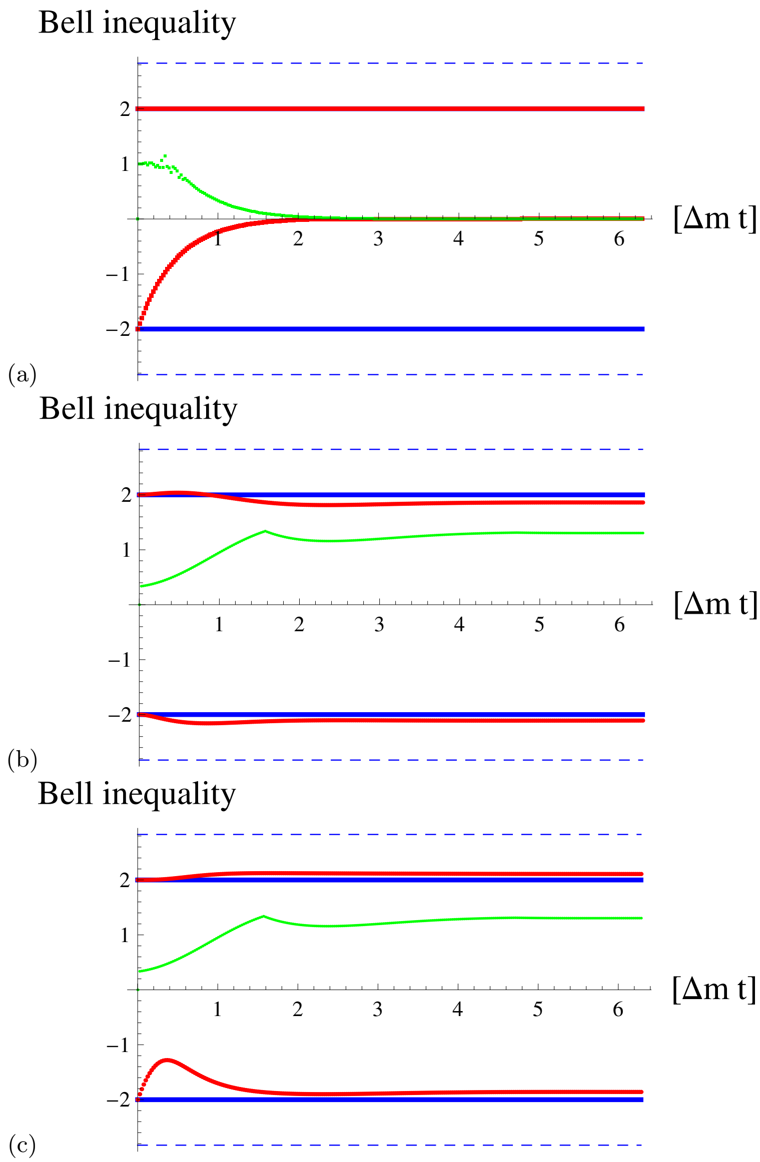


FIG. 4: (Color online) The maximal violations of the Bell inequality, i.e. the maximal and minimal eigenvalues of the operator (49) for strangeness questions for time choices (a) $\{t_n = t_m = t_{n'} = t_{m'} = t\}$, (b) $\{t_n = 0, t_m = t, t_{n'} = t, t_{m'} = 0\}$ and (c) $\{t_n = t, t_m = 0, t_{n'} = 0, t_{m'} = t\}$ are plotted (red big dots). Green dots (smaller dots) represent a lower bound on the entropic uncertainty relation (37) between the two summands of the Bell operator which is zero for $t = 0$ and then immediately jumps to a certain value and is equal for the time settings (b) and (c). The dashed blue lines are the upper bounds on the CHSH-Bell inequality, i.e. $\pm 2\sqrt{2}$, and the solid blue line represent the bound ± 2 given by local realistic theories. One notices that even for long times a violation can be found, though the short lived component can no longer directly be measured.

P21947N16. Andreas Gabriel is supported by the University of Vienna's research grant. Beatrix C. Hiesmayr acknowledges the EU project QESSENCE and wants to thank in particular Reinhold A. Bertlmann for introducing her to the world of neutral kaons and wants to devote this paper to him.

---

[1] G. Amelino-Camelia et.al., Physics with the KLOE-2 experiment at the upgraded DAPHNE, European Physical Journal C 68, Number 3-4, 619 (2010)
[2] N.E. Mavromatos, Decoherence and CPT Violation in a Singy Model of Space-Time Foam, arXive:0906.2712
[3] N.E. Mavromatos and S. Sarkar, Liouville Decoherence in a Model of Flavour Oscillations in the Presence of Dark Energy, Phys. Rev. D 72, 065016 (2005)

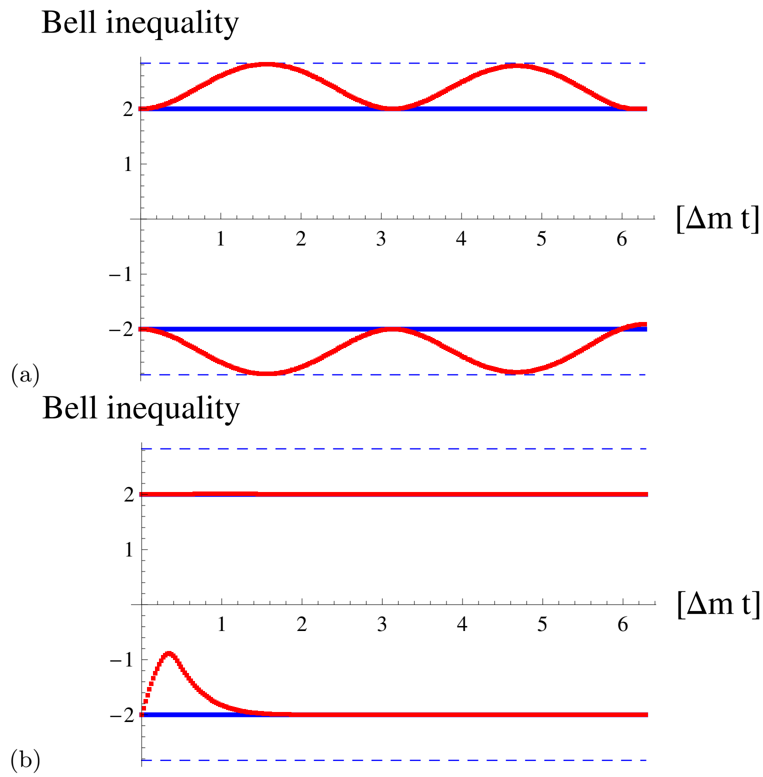


FIG. 5: (Color online) The maximal violations of the Bell inequality, i.e. the maximal and minimal eigenvalues of the operator (49) for flavor questions for time choices $\{t_n = 0, t_m = t, t_{n'} = t, t_{m'} = 0\}$ or $\{t_n = t, t_m = 0, t_{n'} = 0, t_{m'} = t\}$ for (a) $\Gamma_S = \Gamma_L$ and (b) $\Gamma_1 = \Gamma_2 = 1/0.776$ are plotted (red big dots). Here (a) shows the violation if both mass eigenstates wer e long lived in the neutral kaon system and (b) the values for the B–meson system.

[4] M. Blasone, A. Capolupo, G. Vitiello, Quantum Field Theory of three flavor neutrino mixing and oscillations with CP violation, Phys.Rev. D 66, 025033 (2002)

[5] M. Blasone, F. Dell'Anno, S. De Siena, M. Di Mauro, F. Illuminati, Multipartite entangled states in particle mixing, Phys. Rev. D 77, 096002 (2008)

[6] A. Capolupo, S. Capozziello, G. Vitiello, Dark energy and particle mixing, Phys.Lett. A 373, 601 (2009)

[7] R.A. Bertlmann, K. Durstberger and B.C. Hiesmayr, Decoherence of entangled kaons and its connection to entanglement measures, Phys. Rev. A 68, 012111 (2003)

[8] M. Genovese, On the distnaces between entangled pseudoscalar mesons states, Eur. Phys. C 55, 683 (2008)

[9] R.A. Bertlmann, W. Grimus and B.C. Hiesmayr, Quantum mechanics, Furry's hypothesis and a measure of decoherence, Phys. Rev. D 60, 114032 (1999)

[10] M. Genovese, About entanglement properties of kaons and tests of hidden variables models, Phys. Rev. A 69, 022103 (2004)

[11] P. Caban, J. Rembielinski, K.A. Smolinski, Z. Walczak and M. Wlodarczyk, An open quantum system approach to EPR correlations in K0-K0 systems, Phys. Lett. A 357, 6 (2006)

[12] B.C. Hiesmayr and M. Huber, Bohr's complementarity relation and the violation of the CP symmetry in high energy physics, Physics Letters A 372, 3608 (2008)

[13] A. Bramon, R. Escribano and G. Garbarino, Bell's inequality tests with meson-antimeson pairs, Found. Phys. 26, 563 (2006)

[14] A. Bramon, R. Escribano and G. Garbarino, Bell's inequality tests: from photons to B-mesons, J. Mod. Opt. 52, 1681 (2005)

[15] M.M. Courbage, T.T. Durt and S.M. Saberi Fathi, A new formalism for the estimation of the CP-violation parameters, arXiv:0907.2514

[16] J. Bernabeu, J. Ellis, N.E. Mavormatos, D.V. Nanopoulos and J. Pavassiliou, CPT and Quantum Mechanics Tests with Kaons, DAFNE Physics Handbook, arXive:hep-ph/0607322

[17] J. Li and C.F. Qiao, New Possibilities for Testing Local Realism in High Energy Physics, Phys. Lett. A 373, 4311 (2009)

[18] A. Bramon, G. Garbarino and B.C. Hiesmayr, Quantitative complementarity in two-path interferometry, Phys. Rev. A 69, 022112 (2004)

[19] B.D. Yabsley, Quantum entanglement at the psi(3770) and Upsilon (4S), Flavor Physics & CP Violation Conference, Taipei (2008)

[20] I.I. Bigi, Flavour Dynamics & CP Violation in the Standard Model: A Crucial Past – and an Essential Future, arXiv:hep-ph/0701273
[21] A. Di Domenico and KLOE coll., CPT symmetry and quantum mechanics tests in the neutral koan system at KLOE, Found. Phys. 40, 852 (2010)
[22] A. Capolupo, C.-R. Ji, Y. Mishchenko and G. Vitiello, Phenomenology of flavor oscillations with non-perturbative effects from quantum field theory,Physics Letters B 594, 135 (2004)
[23] M. Beuthe, Oscillations of neutrinos and mesons in quantum field theory, Physics Reports 375, 105 (2003)
[24] R.A. Bertlmann, W. Grimus and B.C. Hiesmayr, An open–quantum–system formulation of particle decay, Phys. Rev. A 73, 054101 (2006)
[25] G. Lindblad, Comm. Math. Phys. **48**, 119 (1976)
[26] V. Gorini, A. Kossakowski, and E.C.G. Sudarshan, J. Math. Phys. **17**, 821(1976)
[27] R.A. Bertlmann, A. Bramon, G. Garbarino and B.C. Hiesmayr, A Bell inequality in high energy physics really experimentally violated?, Phys. Lett. A 332, 355 (2004)
[28] R.A. Bertlmann and B.C. Hiesmayr, Bell inequalities for entangled kaons and their unitary time evolution, Phys. Rev. A 63, 062112 (2001)
[29] B.C. Hiesmayr, Nonlocality and entanglement in a strange system, European Physical Journal C, Vol. 50, 73 (2007)
[30] L. Mazzola, S. Maniscalco, J. Piilo and K.A, Suominen, Interplay between entanglement and entropy in two-qubit systems, J. of Phys. B **43**, 085505 (2010).
[31] A. Bramon, G. Garbarino and B.C. Hiesmayr, Quantum marking and quantum erasure for neutral kaons, Phys. Rev. Lett. 92, 020405 (2004)
[32] A. Bramon, G. Garbarino and B.C. Hiesmayr, Active and passive quantum eraser for neutral kaons, Phys. Rev. A 69, 062111 (2004)
[33] J.A. Vaccaro, Large scale physical effects of T violation in mesons, arXiv:0911.4528
[34] I.I. Bigi, Matter–antimatter oscillations and CP violation as manifested through quantum mysteries, Rep. Prog. Phys. 79, 1869 (2007)
[35] D. Deutsch, Uncertainty in quantum measurements, Phys. Rev. Lett. 50, 631 (1983)
[36] K. Kraus, Complementary observables and uncertainty relations, Phys. Rev. D 35, 3070 (1987)
[37] H. Maassen and J.B.M. Uffink, Generalized Entropy Uncertainty Relation, Phys. Rev. Lett. 60, 1103 (1988)
[38] H.P. Robertson, The Uncertainty Principle, Phys. Rev. 34, 163 (1929)
[39] I. Bialynici-Birula and L. Rudnicki, Entropic Relation in Quantum Physics, arXiv:1001.4668
[40] M. Berta, M. Christandl, R. Cobeck, J.M. Renes and R. Renner, The uncertainty principle in the presence of quantum memory, Nature Physics 6, 659 (2010)
[41] K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010)
[42] C. Gneiting and K. Hornberger, Non-classical correlations from dissociation time entanglement, Appl. Phys. B 95, 237 (2009)
[43] R.A. Bertlmann, W. Grimus and B.C. Hiesmayr, Bell inequality and CP violation in the neutral kaon system, Phys. Lett. A 289, 21 (2001)
[44] Ch. Spengler, M. Huber and B.C. Hiesmayr, A geometric comparison of entanglement and quantum nonlocality in discrete system, J. Phys. A: Math. Theor. 44, 065304 (2011)
[45] Ch. Spengler, M. Huber and B.C. Hiesmayr, A composite parameterization of unitary groups, density matrices and subspaces, J. Phys. A: Math. Theor. 43, 385306 (2010)
[46] B.C. Hiesmayr, A generalized Bell inequality and decoherence for the K0 anti-K0 system, Found. of Phys. Lett. 14, 231 (2001)
[47] T. Isobe, S. Tanimura, A method for systematic construction of Bell-like inequalities and a proposal of a new type of test, Prog. Theor. Phys. 124 , 191 (2010)

Lebenslauf

Sasa Radic
Fleischmanngasse 3, 44
A-1040 Wien
Mobil: 0699/ 17 202 205
Mail: sasa_radic@gmx.net

Persönliches

Geboren am 16. Juni 1986
in Sarajevo / Bosnien & Herzegowina
Ledig

Berufliche Erfahrung (Studiumsbegleitend)

01/2007 – 06/2011	**Intercontainer Austria GesmbH, Rail Cargo Ausria Group** **Disposition/Prozessmanagement**

Studium und Ausbildung

10/2005 –	**Universität Wien, Wien** **Fakultät für Physik** **Magisterstudium**
09/2000 – 05/2005	**AHS Amerlinggymnasium (1060 Wien)**
09/1997 – 07/2000	**AHS Okengymnasium (77652 Offenburg, Deutschland)**
08/1993 – 07/1997	**Grundschule (77767 Appenweier, Deutschland)**

Publikationen

Heisenberg's Uncertainty Relation and Bell Inequalities in High Energy Physics
A. Di Domenico, A. Gabriel, B.C. Hiesmayr, F. Hipp, M. Huber, G. Krizek, K. Mühlbacher, S. Radic, Ch. Spengler, L. Theussl
arXiv:1101.4517

A Computable Criterion for Partial Entanglement in Continuous Variable Quantum Systems
Andreas Gabriel, Marcus Huber, Sasa Radic, Beatrix C. Hiesmayr
arXiv:1101.0972