universität wien

DISSERTATION

Titel der Dissertation

"Essays in Credit and Inflation Linked Derivatives"

Verfasser

Zehra Eksi

zur Erlangung des akademischen Grades Doktor der Naturwissenschaften (Dr.rer.nat.)

Wien, im September 2011

Studienkennzahl:	A 791 405
Dissertationsgebiet:	Mathematik
Betreuer:	Prof. Dr. Walter Schachermayer
Zweitbetreuer:	Prof. Dr. Damir Filipović
Gutachter:	Prof. Dr. Rama Cont
Gutachter:	Prof. Dr. Rüdiger Frey

Abstract

The objective of this study is twofold. One of them is the pricing and hedging of collateralized debt obligations (CDOs) and the other is the modeling of inflation linked derivatives. In the first part, we first review the framework introduced in Filipović et al. [2009] for the pricing and hedging of CDOs. As a first step towards the investigation of the market incompleteness, we examine the uniqueness of the martingale measure in the defaultable (T, x)-bonds market introduced in Filipović et al. [2009] and show that the equivalent local martingale measure (ELMM) is unique. Following this we specify an affine two factor stochastic drift model for the pricing and hedging of single tranche synthetic CDOs. We estimate the affine factor model on the iTraxx Europe data. The novelty of this part lies in the fact that the data covers a period, which witnessed different market conditions such as the recent credit crises. As the main tool for the estimation of the affine factor model we use quasi maximum likelihood based on a Kalman filter. Estimation results show that the two factor stochastic drift model is successful in terms of fitting the market data even for super-senior tranches. Apart from estimating the model, we analyze the real world performance of two hedging strategies, namely the variance minimizing and regression based hedging. We also run a simulation analysis where normal and extreme loss scenarios are generated via method of importance sampling. Finally we assess the hedging strategies under these more general scenarios.

The second part of this thesis deals with the pricing and hedging of inflation-indexed derivatives. Assuming that the foreign currency analogy holds we first consider a three-factor Gaussian affine model for the pricing of nominal and inflation indexed bonds. By using the theory of affine processes we get closed form bond prices. Imposing no-arbitrage assumption leads to drift restrictions that the factor process has to satisfy. In particular, one of the conditions the drift matrix of the factor process has to satisfy is the well known Fisher equation. Then, under the assumption of diagonalizable drift matrix we find conditions on the eigenvalues and the eigenvectors of the drift matrix which guarantee the hedge of an inflation indexed bond of a given maturity only by trading nominal bonds of different maturities. Combining no-arbitrage restrictions with the hedging conditions on the diagonalizable drift matrix and utilizing the market completeness criterion given in Davis and Obloj [2008] we find that cases in which it is possible to hedge inflation bonds by using nominal bonds coincide with cases where the market is spanned by the continuum of nominal bonds. That is, we show that under the assumption of diagonalizable drift matrix, hedging of inflation bonds by

using nominal bonds is possible if and only if the market is spanned by the nominal bonds.

Finally, we consider a multi-country setting where domestic and foreign nominal and real bonds are traded. We first specify the real and nominal bond prices, price index and exchange rate dynamics as Itô processes and assume that there is no- arbitrage in the market. Imposing no-arbitrage assumption immediately yields the usual definition of real exchange rate (RER) between the foreign and domestic economies. Moreover, we get drift conditions for real and nominal term structures of the domestic and foreign economies. Assuming martingale property for RER we find a relation on the real interest rate differential of the two economies. More importantly, we find that the martingale assumption on RER is equivalent to the condition that the nominal interest rate differentials between the two economies is given by the sum of appreciation rate of the exchange rate and the risk premium arising from exchange rate uncertainty. Motivated by the importance of the information on RER for central banks, we introduce a forward contract written on RER. This yields the forward real exchange rate whose value can be expressed in terms of the price of the domestic and foreign inflation indexed bonds. We further construct multi-country inflation linked derivatives such as foreign exchange inflation options and real exchange rate swaps with the idea of providing a protection for the foreign purchasing power of a domestic income. We use the change of numeraire technique to get the prices of these derivatives and under the assumption of deterministic volatility in the inflation indexed bond price dynamics, we get closed form formulae.

Zusammenfassung

Das Ziel dieser Studie ist zweifach. Einer von ihnen ist das Pricing und Hedging von Collateralized Debt Obligations (CDOs) und die andere ist die Modellierung von Inflation Linked-Derivaten. Im ersten Teil untersuchen wir die Modellierung Setup in Filipović et al. [2009] für die Pricing und Hedging von CDOs. In einem ersten Schritt auf dem Weg der Untersuchung des Marktes Unvollständigkeit, untersuchen wir die Einzigartigkeit des Martingalmass in der gegebenen (T,x)-bond Markt und zeigen, dass die entsprechenden äquivalenten lokalen Martingalmass (ELMM) einzigartig ist. Dann schlagen wir eine affine Zwei-Faktor-stochastische Drift-Modell für die Pricing und Hedging von synthetischen Single-Tranche CDOs. Wir schätzen die affine Faktor-Modell auf den iTraxx Europe-Daten. Die Neuheit dieses Teils liegt in der Tatsache, dass die Daten Periode eine Periode, die unterschiedlichen Marktbedingungen wie die Kreditkrise erlebt abdeckt. Für die Abschätzung der affine Faktor-Modell verwenden wir quasi Maximum- Likelihood- Schätzung basierend auf dem Kalman-Filter. Schätzergebnisse zeigen, dass die Zwei-Faktor- stochastische Drift-Modell erfolgreich in Bezug auf die fit der Marktdaten auch für Super- Senior- Tranchen ist. Abgesehen von der Schätzung des Modells analysieren wir die reale Welt Performance von zwei Hedge-Strategien, nämlich die Varianz minimizing und Regression based Hedging. Wir führen auch eine Simulation Analyse, wo normale und Extremschadenszenarien via Methode importance sampling generiert werden. Schliesslich bewerten wir die Hedging-Strategien im Rahmen dieser allgemeinen Szenarien.

Der zweite Teil beshäftigt sich mit die Pricing und Hedging von inflationsindexierten Derivaten. In diesem Teil, vorausgesetzt, dass die foreign currency analogy hält betrachten wir eine Drei-Faktor-Gauss-affine Modell für die Preisgestaltung der nominalen und inflationsindexierten Anleihen. Mit Hilfe der Theorie der affine Prozesse, die wir bekommen Anleihenkurse in der geschlossenen Form. No-Arbitrage-Annahme führt zu drift Bedingungen, die den Faktor Prozess hat zu befriedigen. Insbesondere ist eine der Bedingungen der bekannte Fisher-Gleichung. Dann unter der Annahme diagonalisierbar Drift-Matrix finden wir Bedingungen an die Eigenwerte und die Eigenvektoren der Drift Matrix, die die Hedging einer inflationsindexierten Anleihe nur durch den Handel mit nominalen Anleihen mit unterschiedlichen Laufzeiten zu gewährleisten. Die Kombination No-Arbitrage- Einschränkungen mit der Hedging Bedingungen auf dem diagonalisierbar Drift-Matrix und die Nutzung des Marktes Vollständigkeit Kriterium gegeben in Davis

and Obloj [2008] finden wir, dass die Hedging von inflationsindexierten Anleihen mit nominalen Anleihen ist möglich, wenn und nur wenn der Markt durch die nominale Anleihen aufgespannt wird.

Schliesslich betrachten wir eine Multi-Country Modell, wo in-und ausländischen nominalen und realen Anleihen gehandelt werden. Wir nehmen zuerst an, dass die realen und nominellen Kurs der Anleihe, Preisindex und Wechselkurs einem Itô Prozess folgen. Dann haben wir vorausgesetzten, dass der Markt Arbitrage-frei ist. No-Arbitrage-Annahme ergibt sich unmittelbar die übliche Definition des realen Wechselkurses (RW) zwischen den ausländischen und inländischen Economies. Darüber hinaus erhalten wir drift Bedingungen für reale und nominale Zinsstrukturen des in-und auslŁndischen Economies. Unter der Annahme, Martingal Eigenschaft für RW finden wir eine Beziehung auf die reale Zinsdifferenz zwischen den Economies. Motiviert durch die Bedeutung der Informationen über die RW für die Zentralbanken, führen wir einen Forward-Kontrakt auf RW geschrieben. Daraus ergibt sich die Forward realen Wechselkurs, die in Bezug auf den Preis der inländischen und ausländischen Inflation Anleihen geschrieben werden können. Wir näher vorstellen Multi-Country inflationsindexierten Derivaten, wie Foreign Exchange Inflation-Optionen und die Inflation RW-Swaps mit der Idee, einen Schutz für die ausländischen Kaufkraft der inländischen Einkünfte. Wir verwenden die change of numeraire Technik, um die Preise für diese Derivate zu erhalten und unter der Annahme von deterministischen Volatilität an den inflationsindexierten Anleihen Preisdynamik, bekommen wir die Derivate Preisen in der geschlossenen Form.

To teachers of my formal and informal education...

Acknowledgements

It is a real pleasure to thank those who have contributed to this study in one way or another.

This thesis would not have been possible without the guidance, support and patience of my principal supervisor, Prof. Dr. Damir Filipović. I appreciate all his contributions of time, ideas, and funding that he provide during my doctoral studies. I have learned a lot from him throughout the way and his logical way of thinking and explaining things clearly has been of great value for me.

I owe my sincere gratitude to Prof. Dr. Walter Schachermayer who gave me the opportunity to continue my doctoral studies in the Department of Mathematics at the University of Vienna. I feel myself very lucky and was delighted to interact with him by attending his classes on probability theory and stochastic calculus.

My sincere thanks are also due to the referees, Prof. Dr. Rama Cont and Prof Dr. Rüdiger Frey, for accepting to review my thesis.

During this work I have collaborated with colleagues for whom I have great regard. I wish to extend my warmest thanks to Irina Slinko who has shared her knowledge generously with me. I also want to thank Bezirgen Veliyev for the fruitful weekend meetings on analysis.

I am also grateful to my friends Gonca Akı, Volkan Selçuk, İrem Yıldırım, Yeliz Yolcu and Serkan Okur, Hatice Bahadır, Havva Özlem Dursun, Özge Subaşı, İren Tuna and Eberhard Mayerhofer for their friendship and encouraging me all the time.

My special gratitude is due to my mother and father in-law Süheyla and Erhan Altay for their loving support.

My greatest debts are to my family-to my grandma Emriye Eksi, to my father İsmail, my lovely mother Ayşe, my sisters Zühal and Esra, to my brother Mehmet and to my adorable nephew Levent - for their unflagging love and support throughout my life.

I owe my loving thanks to my husband Sühan. Without his love, encouragement and understanding it would have been impossible for me to finish this work.

Lastly but definitely not least, I want to thank Prof. Dr Hayri Körezlioglu for making way for me to enter the world of mathematical finance. I will never forget him.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

This thesis deals with the pricing and hedging of credit and inflation linked derivatives. **Part I** (Chapter 2) addresses the pricing and hedging of *collateralized debt obligations* (CDO). Credit derivatives market has witnessed an enormous growth between the years 1997 and 2007 increasing the necessity of models to price and hedge derivatives such as credit default swaps (CDS), index default swaps and synthetic single trance CDOs (STCDOs), to name a few. Although the financial crises of 2008 caused a decline in the global outstanding notional volume of credit derivatives, the credit derivatives market still possesses its potential benefits such as completing markets via providing the opportunity to buy or sell insurance on credit risky portfolios.

As all derivative markets, credit markets are exposed to risk. The two main sources of exposure are the *default risk* and the *market* or *spread risk*. Hence, a sound model for pricing and hedging of credit linked products is expected to incorporate these two components. However, most of the existing models, such as the Gaussian copula model, do not take the credit spread risk into account and they solely focus on the modeling of default component. On the other hand, for the consistent pricing of CDOs with different maturities and to be able to perform dynamic hedging analysis, a sensible model should also possesses a dynamic nature. That is, instead of modeling the default time distributions of constituents at a given point in time, specification of the evolution of default distributions of constituents or the portfolio loss process should be considered.

Motivated by HJM framework for default-free term structures, [Filipović et al., 2009] develop a dynamic no-arbitrage setting for the modeling of forward credit spreads. As the building block, the defaultable (T, x)-bond term structure is introduced. Then, it is shown that, as in the case of HJM default free term structure framework, any exogenous specification of the volatility and contagion parameters yields a unique consistent loss process and hence defines an arbitrage free term structure of (T, x)- bonds. Under this general setup Filipović et al. [2009] proposes doubly stochastic affine term structure models as a tractable class. The framework introduced in Filipović et al. [2009] clearly possesses the properties of being dynamic and incorporation of the credit spreads in the modeling of CDOs. Hence, we find it worthwhile to specify a factor model under this framework and test the performance of the model on the market data.

In Chapter 2 of this thesis, we first review the framework introduced in Filipović et al. [2009]. Contrary to the case in Filipović et al. [2009], where all dynamics are specified directly under an equivalent martingale measure $\mathbb{Q}$, we start building the model under the historical probability measure $\mathbb{P}$. When it comes to pricing, however, we need the risk-neutral dynamics of all modeling components. With this line of reasoning and as a first step towards the investigation of the market incompleteness, we examine the uniqueness of the martingale measure in the given market and show that the equivalent local martingale measure (ELMM) is unique for the (T, x)-bond market introduced in Filipović et al. [2009]. Following this, we specify a two factor affine stochastic drift model and give its details.

In Section 3 of Chapter 2, we estimate the affine factor model on the iTraxx Europe data. The novelty of this part lies in the fact that the data covers a period, which witnessed different market conditions such as the recent credit crises. As the main tool for the estimation of the affine factor model we use quasi maximum likelihood based on a Kalman filter. This method requires the knowledge of conditional moments of the factor process. In this context, we utilize the polynomial property of moments for an affine diffusion process and compute the first two conditional moments of the factor process explicitly. Estimation results show that the two factor stochastic drift model is successful in terms of fitting the market data even for super-senior tranches. Apart from estimating the model, we analyze the real world performance of two hedging strategies, namely the variance minimizing and regression based hedging. The detailed information on these hedging strategies is given in Section 4. We also run a simulation analysis where normal and extreme loss scenarios are generated via method of importance sampling. Finally we assess the hedging strategies under these more general scenarios.

Part II (including Chapter 3 and 4) is devoted to the pricing and hedging of inflation-linked products. Inflation indexed derivative market is important in the sense that it provides the opportunity to eliminate the inflation risk and guarantees the real interest that will be earned in a given period of time. Naturally, demand for inflation linked products is increasing as in the financial markets the amount of the inflation-linked liabilities is increasing. On the other side, monetary authorities increase their issue of inflation linked bonds to make the inflation targeting policies more reliable and to reduce the inflation premium that they have to pay when they issue nominal bonds.

Chapter 3 of the thesis focuses on the pricing and hedging of inflation indexed bond in a three factor affine framework. Under the foreign currency analogy we consider a three-factor Gaussian affine model for the pricing of nominal and inflation indexed bonds. The factor process is considered to be composed of the nominal short rate, real short rate and the logarithm of the price index process. By utilizing tools from the theory of affine processes we get closed form expressions for nominal bond price, inflation indexed bond price and the price index. Under the foreign currency analogy, imposing no-arbitrage assumption leads to drift restrictions that the factor process has to satisfy. In particular, one of the conditions the drift matrix of the factor process has to satisfy is the well known Fisher equation which states that the expected appreciation in the price index is equal to the difference between the nominal and real short rates. We also deal with the hedging question and under the assumption of diagonalizable drift matrix we find conditions on the eigenvalues and the eigenvectors of the drift matrix which guarantee the hedge of an inflation indexed bond of a given maturity only by trading nominal bonds of different maturities. The novelty if this work is due to this analysis.

Combining no-arbitrage restrictions with the hedging conditions on the diagonalizable drift matrix and utilizing the market completeness criterion given in Davis and Obloj [2008] we find that under the foreign currency analogy, cases in which it is possible to hedge inflation bonds by using nominal bonds coincide with cases where the market is spanned by the continuum of nominal bonds. Hence, as the second main contribution of this study, we show that under the above modeling setup there is no such situation that it is possible to hedge inflation bonds but hedging of other contingent claims is not granted. To sum up, our findings suggests that under the foreign currency and the assumption of diagonalizable drift matrix, hedging of inflation bonds by using nominal bonds is possible if and only if the market is spanned by the nominal bonds.

Currently we are living in a financial environment where the economies are strongly linked to each other by the exchange rates. Thus, it is natural to consider the effects of exchange rates on inflation and other macroeconomic variables. To be more clear, one can think of the situation where the appreciation of exchange rates makes imported goods more expensive in terms of the domestic currency. In such a case, an increase in the price of the imported goods might cause an overall price level increase, that is, inflation. Therefore, in inflation term structure modeling taking the exchange rates into account might be useful. With this motivation Slinko [2006] investigates the joint dynamics of the nominal exchange rate and the domestic and foreign nominal and real term structures. In Chapter 4 [1], with the same line of reasoning we propose a multi-country setting for inflation linked derivative pricing. The other source of our motivation is the fact that in a multi-country setting, presence of the inflation linked instrument might create extra information about the real exchange rate (RER) and real rate differentials between the countries.

[1] This chapter is based on a joint work with Irina Slinko

We consider a multi-country setting where domestic and foreign nominal and real bonds are traded. We first specify the real and nominal bond prices, price index and exchange rate dynamics as Itô processes and assume that there is no- arbitrage in the market. Imposing no-arbitrage assumption immediately yields the usual definition of RER between the foreign and domestic economies. Moreover we get drift conditions for real and nominal term structures of the domestic and foreign economies. Assuming martingale property for the the real exchange rate we find a relation on the real interest rate differential of the two economies. More importantly, we find that the martingale assumption on RER is equivalent to the condition that the nominal interest rate differentials between the two economies is given by the sum of appreciation rate of the exchange rate and the risk premium arising from exchange rate uncertainty.

Motivated by the importance of the information on RER for central banks, we introduce a forward contract written on RER. This yields the forward real exchange rate whose value can be expressed in terms of the price of the domestic and foreign inflation indexed bonds. We further construct multi-country inflation linked derivatives such as foreign exchange inflation options and real exchange rate swaps with the idea of providing a guarantee for the foreign purchasing power of a domestic income. We extensively use the change of numeraire technique to get the prices of these derivatives. Furthermore, we get closed form formulae under the assumption of deterministic volatility in the inflation indexed bond price dynamics.

Each chapter of this thesis is essentially self-contained with its own introduction, problem definition and conclusion and uses its own notation.

Part I

CDO Pricing and Hedging

2

A Dynamic CDO Term Structure Model

2.1 Introduction

Credit derivatives market is important in the sense that it provides opportunity to buy and sell insurance on credit risky investments. The key instruments, namely *collateralized debt obligations* (CDO), is first developed in 1987 by bankers at Drexel Burnham Lambert Inc. Within 10 years, CDOs market had become the fastest growing sector of the asset-backed synthetic securities market. CDO can be defined as a structured product which is backed by portfolio of credit risky assets. Although the structure and the underlying asset composition may vary according to the type of a CDO, the basic idea is the same. There is the originator, having the portfolio of credit risky assets such as credit card payments, mortgage payments, etc. A corporate entity, called special purpose vehicle (SPV), is constructed for the securitization of the credit risky assets. SPVs purchased the credit risky portfolio from the originator and issue various class of bonds backed by assets from the portfolio with different credit risk characteristics. These classes are called tranches. *Synthetic CDOs* are special type of CDOs where the credit risky portfolio is consist of *credit default swaps* (CDS).

CDS is a derivative instrument in which the investor (protection seller) receives the fixed periodic spread in exchange for the payment that has to be made conditional upon the occurrence of a loss due to default of the reference entity. In the same way, in an index default swap the credit risk of a equally weighted portfolio of reference entities (the index) is exchanged between the protection buyer and protections seller. In a STCDO position, the invested tranche references a specific segment of the loss distribution of the index, that is, a specific exposure to the credit risk of the underlying index is undertaken and in turn, a flow of coupon payments are received. Losses are allocated first to the equity tranche, which is the lowest tranche, and then to higher tranches as mezzanine, senior and super-senior tranches. The most liquid single-tranches are referencing the two main credit default swap indices the CDX IG index in North America and the iTraxx Europe index in Europe. CDX IG and iTraxx Europe index is a selection of 125 single name CDS. Although the attachment and detachment points are different, there are six tranches in both indices. The index market has various maturity choices such as $3, 5, 7, 10, ...$-year. Among these, the STCDOs with a maturity of 5 years are observed to be the most liquid ones.

The risk of a position taken in the credit market stems from two sources. The *default risk*, that is the risk arise from the possibility of the default of an obligor and the *market* or *spread* risk associated with the changes in the credit qualities and the interest rates. Thus, a sensible model for the pricing and hedging of credit risky securities is expected to incorporate the modeling of default and credit spread dynamics. However, in most of the portfolio credit derivatives models, the focus is solely on the modeling of the default. To give an example, consider the Gaussian copula model [Li, 2000] which has become a market practice for the pricing and hedging of portfolio credit derivatives. The idea of this model is to construct the joint distribution of defaults. Firstly, it is assumed that the default time of each constituent of the portfolio is exponentially distributed with the same parameter. Then, to put a dependency structure between the default times of different constituents, the default time of each constituent is related with a Gaussian latent factor which is decomposed as firm specific and market factors. Assuming that the market exposure of all constituents are same, default correlation between any constituents is implied to be explained by a single dependency parameter. This is restrictive and does not sound very realistic. Moreover, from its vey nature, this framework does not allow for the dynamic modeling of default time distributions and thus characterized as static.

There are number of models alternative to the Gaussian copula model for the pricing and hedging of portfolio credit derivatives. [Bielecki et al., 2010] classified the portfolio credit models under two main approaches. In the bottom-up approach the fundamental objects to be modeled are the loss processes of portfolio constituents whose sum give the total portfolio loss. While on the contrary, the top-down approach aims to model the evolution of the aggregate portfolio loss process directly. The advantage of the bottom-up approach is that it allows for the hedging of portfolio derivatives with the underlying constituents. However, there is a trade-off between the seeming realism and practical implementability of this approach. For an overview of the top-down and bottom up approaches that have been developed for pricing and hedging of portfolio credit derivatives we refer to Section 2 in [Bielecki et al., 2010] and references therein.

One may also classify the portfolio credit models as static or dynamic models. Static models, such as Gaussian copula model and some other copula based models, the particular interest is the default time distributions of constituents at a given point in time as the maturity of the credit product. The deficiency of these models is that the consistent pricing for different maturities is not possible. Moreover, these models do not allow for dynamic hedging as they do not provide a consistent basis for the assessment of the behavior of prices over time. On the other hand, the dynamic models specify the evolution of default time distributions of constituent or the total loss process depending on the top-down or bottom up framework that is followed. To our knowledge, Duffie and Garleanu [2001], in which correlated intensities are constructed for constituent names by using affine factor processes, is the first study addressing the dynamic framework for pricing of CDOs. Schönbucher [2005], Sidenius et al. [2008], Filipović et al. [2009] and Frey and Backhaus [2010] are the other examples for dynamic models for CDO pricing.

Schönbucher [2005], Sidenius et al. [2008] and Filipović et al. [2009] are very much in the same spirit that both models are inspired by the HJM (see Heath et al. [1992]) framework and model the full forward distribution of the loss process. This allows for the consistent incorporation of the dynamics of credit spreads to the modeling of multiname credit derivatives. Schönbucher [2005] introduces the forward loss distributions, and finds a Markov chain with the same marginal distribution as the loss process. The Sidenius et al. [2008] model is specified by a two-layer process. The first layer models the dynamics of portfolio loss distributions in the absence of default information. This is called the background process and calibration to the full grid of marginal loss distributions, implied by the current CDO tranche value, is performed conditional on this background process. The second layer models the loss process itself as a Markov process conditioned on the path taken by the background process.

Motivated by HJM framework for default-free term structures, [Filipović et al., 2009] also develops a dynamic no-arbitrage setting for the modeling of forward credit spreads. As the building block, the defaultable (T, x)-bond term structure is introduced and the necessary and sufficient conditions for the absence of arbitrage for this market is given. Moreover, it is shown that, as in the case of HJM default free term structure framework, any exogenous specification of the volatility and contagion parameters yields a unique consistent loss process and thus an arbitrage free term structure. This framework provides the generalization of aforementioned top-down approach and allows for feedback and contagion effects. Furthermore, under this general setup a tractable class of doubly stochastic affine term structure models is proposed. Based on [Filipović et al., 2009], Filipovic and Schmidt [2010] studies the hedging of STCDOs with the index default swap and an explicit variance minimizing hedging strategy is computed for a one-factor affine model. However, the empirical performance of the one-factor affine model and the hedging strategy is not analyzed and left as a future work. One of the main objectives of the recent study is to complete this missing part.

There are studies which focus on the empirical performance of different models for pricing and hedging of CDOs. Frey and Backhaus [2010] study the hedging of STCDOs in a dynamic setting where spread risk and default contagion are incorporated. Reckon with the incompleteness of the market arising from the presence of spread and default risk, they compute variance-minimizing strategies for consistent and dynamic hedging of STCDOs with the underlying CDSs. Moreover, they showed the impact of default contagion on sensitivity based hedge ratios via numerical comparison with the Gaussian copula model. They also showed that the variance-minimizing strategy provides a model-based endogenous interpolation between the hedging against spread risk and default risk.

In the literature, the unique study comparing various pricing models and hedging strategies for STCDOs belongs to Cont and Kan [2011]. One important result of this study suggests that the large portion of the risk in STCDOs are unhedgeable because of the market incompleteness. Another result, which is at odds with the existing literature (see, e.g., Bielecki et al. [2010]), shows that bottom-up models are not observed to perform consistently better than top-down models. Furthermore, among various hedging strategies including the variance-minimizing hedge, regression-based hedging strategy is found to be surprisingly effective. While performing the numerical analysis below, we will refer to this study again.

Inspired by the above mentioned results, we set out the specific issues to be explored in this study in terms of the following objectives:

1. To investigate the market incompleteness for the defaultable (T,x)-bond market introduced in Filipović et al. [2009];
2. To specify a tractable affine factor model under the framework of Filipović et al. [2009] which fits market data;
3. To estimate the affine factor model on the given data set including the recent financial crisis period;
4. To assess the performance of variance minimizing and regression based hedging strategy within the given data set;
5. To run a simulation analysis for the assessment of hedging strategies under more general scenarios.

Following the objectives of the study set out above, we investigate the real world performance of the one-factor affine model introduced in Filipović et al. [2009]. Experiencing the inadequacy of the one factor-model in fitting the iTraxx Europe data, we propose a two-factor affine factor model in which a catastrophic risk component is considered as a tool for explaining the dynamics of the super-senior tranches. For the estimation of the affine factor model, we use a quasi maximum likelihood approach based on the Kalman filter. The usage of the Kalman filter necessitates the knowledge of the first two moments of the factor process. In this context, we use the polynomial property of affine processes and compute the conditional means and variances of the process explicitly. We then analyze the real world performance of variance minimizing and regression based hedging strategies for the hedging of STCDOs with the underlying index default swap. Our findings suggest that two-factor affine model is successful in describing the whole data set. Furthermore, within the data period, both hedging strategies are efficient in reducing the risk on the STCDO significantly. In particular, the simulation analysis, where we use importance sampling technique to generate loss scenarios, indicates that variance minimizing hedge performs better than regression based hedge under general scenarios permitting non-zero loss trajectories.

This chapter is structured as follows. In the next section we provide the overview of the modeling framework and investigate the market completeness. Section 3 describes the cash-flow structure of STCDOs in detail. In Section 4, together with variance minimizing and regression based hedging strategies for the hedging of STCDOs, we give the hedging algorithm and two criteria for the assessment of hedging performance. Section 5 and 6 introduces the estimation and simulation methodology respectively. Section 7, which is the numerical analysis part, presents the data set and gives the results. Section 8 summarizes the results and concludes the chapter.

2.2 Modeling Framework

The framework introduced in Filipović et al. [2009] for the dynamic modeling of CDO term-structures covers a very general class of models where doubly stochastic framework, which is obtained via omitting the contagion effects, is given as a special case. Under this doubly stochastic framework, Filipović et al. [2009] proposes doubly stochastic affine term structure models as an analytically tractable class. The two-factor (stochastic drift) model that we introduce below is a particular choice among the aforementioned affine models.

In this part we first make an overview of the modeling setup given in Filipović et al. [2009]. Contrary to the case in Filipović et al. [2009], where all dynamics are specified directly under an equivalent martingale measure $\mathbb{Q}$, we start building the model under the historical probability measure $\mathbb{P}$. When it comes to pricing, however, we need the risk-neutral dynamics of all modeling components. With this line of reasoning and as a first step towards the investigation of the market incompleteness, we examine the uniqueness of the martingale measure in the given market and show that the equivalent local martingale measure (ELMM) is unique for the (T, x)-bond market introduced in Filipović et al. [2009]. We then give the HJM type no-arbitrage restrictions in terms of the $\mathbb{P}$ parameters of the model. Following this we introduce two-factor stochastic drift model and give its details. Finally discussing the implications of our model choice in terms of the yielded default intensity and loss given default distributions we conclude the section.

2.2.1 Theoretical Background

A stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions is fixed where $\mathbb{P}$ indicates the historical probability measure. We consider a CDO pool of credits with the notional normalized to 1. $x \in [0, 1]$ represents the loss fraction, that is, x represents the state where $100x\%$ of the overall nominal has defaulted.

The aggregate loss process, representing the ratio of CDO-losses realized by time t is indicated by L_t. The hypothetical (T, x)-bond paying $1_{\{L_T \leq x\}}$ at maturity T, $x \in [0, 1]$ is considered as the building-block of the CDO term-structure model. The (T, x)-bond price at time $t \leq T$ is denoted by $P(t, T, x)$ and it is assumed that bond price term structure movements is in the form

$$P(t, T, x) = 1_{\{L_t \leq x\}} e^{-\int_t^T f(t,u,x)du} \tag{2.1}$$

where $f(t, T, x)$ is the forward rate that one can contract at time t, given the condition that the aggregate loss level has not exceed the level x, on a defaultable forward investment of one euro that begins at time T and returned an instant dT later conditional on the event that the loss level L_{T+dT} is below x. This description suggests how the (T, x)-bond prices incorporates the market risk (credit spread risk) as well as the default risk and thus provides a basis for a model which considers not only the default risk but also the credit spreads for the pricing of multi-name credit derivatives.

(A1) The aggregate loss process

$$L_t = \sum_{s \leq t} \Delta L_s \tag{2.2}$$

is assumed to be an $[0, 1]$-valued, non-decreasing marked point process [1] with absolutely continuous $\mathbb{P}$-compensator $\nu^p(t, dx)dt$.

Now we assume that, for all (T, x), the $\mathbb{P}$-dynamics of the (T, x)-forward rate process

[1] See Appendix for the definition and properties of a marked point process

$f(t,T,x)$, $t \leq T$, is in the form

$$\begin{aligned} f(t,T,x) &= f(0,T,x) + \int_0^t a^p(s,T,x)ds + \int_0^t b(s,T,x)^\top \cdot dW_s^{\mathbb{P}} \\ &\quad + \int_0^t \int_{(0,1]} c(s,T,x;y)\mu(ds,dy) \end{aligned} \tag{2.3}$$

where $W^{\mathbb{P}}$ is some d-dimensional $\mathbb{P}$-Brownian motion and $\mu(dt,dx)$ denotes the integer-valued random measure associated to the jumps of L, that is

$$\mu(\omega;dt,dx) = \sum_{s>0} 1_{\{\Delta L_s(\omega)\neq 0\}}\delta_{(s,\Delta L_s(\omega))}(dt,dx)$$

where δ_a is the Dirac measure at point a.

Notice that the specification of the forward rate dynamics in (2.3), via coefficient c, allows for the contagion, or feedback effect of the loss process on the rates. Now to provide a suitable basis for further formal analysis, we make the following assumptions on the parameters of the forward rates. In the following, we denote optional and predictable σ-algebra on $\Omega \times \mathbb{R}_+$ with $\mathcal{O}$ and $\mathcal{P}$, respectively.

(A2) the initial forward curve $f(0,T,x)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}([0,1])$- measurable, and locally integrable:

$$\int_0^T |f(0,u,x)|du < \infty \quad \text{for all} \quad (T,x)$$

(A3) the drift parameter $a^p(t,T,x)$ is $\mathbb{R}$-valued $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}([0,1])$- measurable, and locally integrable:

$$\int_0^T \int_0^T |a^p(t,u,x)|dtdu < \infty \quad \text{for all} \quad (T,x),$$

(A4) the volatility parameter $b(t,T,x)$ is $\mathbb{R}^d$-valued $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}([0,1])$- measurable, and locally bounded:

$$\sup_{t \leq u \leq T} \|b(t,u,x)\| < \infty \quad \text{for all} \quad (T,x),$$

(A5) the contagion parameter $c(t,u,x;y)$ is $\mathbb{R}$-valued $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}([0,1]) \otimes \mathcal{B}([0,1])$-measurable, and locally bounded:

$$\sup_{t \leq u \leq T, y\in[0,1]} |c(t,u,x;y)| < \infty \quad \text{for all} \quad (T,x),$$

(A6) a finite time horizon $[0,T^*]$, $T^* < \infty$ is considered,

(A7) $\mathcal{F}_t$ is assumed to be the internal one, that is, $\mathcal{F}_t = \sigma\left\{W_s^{\mathbb{P}}, \mu(ds, d\xi),\ s \le t\right\}$.

Under **(A2)-(A5)**, it is guaranteed that the risk-free discount factor $e^{-\int_0^t r_s ds}$, where $r_t = f(t, t, 1)$, is well defined. Then, we denote the discounted (T, x)-bond price by

$$Z(t, T, x) = e^{-\int_0^t r_s ds} P(t, T, x)$$

Next lemma gives $\mathbb{P}$-dynamics of the $Z(t, T, x)$ implied by the relation (2.1) and the forward rate dynamics given in (2.3).

Lemma 2.1 *Assume* **(A2)-(A5)** *holds, then the* $\mathbb{P}$*-dynamics of the discounted* (T, x)*-bond price process is given by*

$$\begin{aligned}\frac{dZ(t, T, x)}{Z(t-, T, x)} &= \alpha^p(t, T, x)dt + \beta(t, T, x)^\top \cdot dW_t^{\mathbb{P}} \\ &+ \int_{(0,1]} \gamma(t, T, x, \xi)(\mu(dt, d\xi) - \nu^p(t, d\xi)dt)\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\alpha^p(t, T, x) =& - r_t - \lambda^p(t, x) + f(t, t, x) - \int_t^T a^p(t, u, x)du + \frac{1}{2}\| \int_t^T b(t, u, x)du\|^2 \\ &+ \int_{(0,1]} (e^{-\int_t^T c(t,u,x;\xi)du} - 1)1_{\{L_{t-}+y \le x\}} \nu^p(t, dy)\end{aligned} \tag{2.5}$$

$$\beta(t, T, x) = \int_t^T b(t, u, x)du \tag{2.6}$$

$$\gamma(t, T, x, \xi) = e^{-\int_t^T c(t,u,x;\xi)du} 1_{\{L_{t-}+\xi \le x\}} - 1 \tag{2.7}$$

and λ^p *is defined as*

$$\lambda^p(t, x) = \int_{(0,1]} 1_{\{L_{t-}+y>x\}} \nu^p(t, dy). \tag{2.8}$$

Proof. The proof mainly utilizes a stochastic Fubini argument and Itô's formula to get the dynamics of $e^{-\int_t^T f(t,u,x)du}$. Then, the possibility of writing

$$1_{\{L_t \le x\}} = 1 + \int_0^t \int_{(0,1]} (-1_{\{L_{s-}+y>x\}} 1_{\{L_{s-} \le x\}})\mu(ds, dy) \tag{2.9}$$

and using this in the application of the integration by parts formula to

$$P(t, T, x) = 1_{\{L_t \le x\}} e^{-\int_t^T f(t,u,x)du}$$

yields the desired results. For more details, we refer to Filipovic and Schmidt [2010], proof of Lemma 2.1. □

Remark 2.1 *Having L_t as given in* **(A1)** *implies that the default times of the (T,x)-bonds, that is*

$$\tau_x = \inf\{t | L_t > x\}$$

are totally inaccessible. Indeed the intensity of default times is given by $\lambda^p(t,x)$. This becomes more clear from the fact that $-1_{\{\tau \le t\}}$ is a càdlàg supermartingale and by Doob-Meyer decomposition, there exists an increasing predictable process A_t with $A_0 = 0$ such that

$$M_t = -1_{\{\tau \le t\}} + A_t$$

is a martingale. From the very definition of ν^p and due to (**A1**), *we have*

$$1_{\{L_t \le x\}} + \int_0^t \int_{(0,1]} 1_{\{L_{s-}+y>x\}} 1_{\{L_{s-} \le x\}} \nu^p(s,dy)ds$$

is a martingale and hence

$$A_t = \int_0^t \int_{(0,1]} 1_{\{L_{s-}+y>x\}} 1_{\{L_{s-} \le x\}} \nu^p(s,dy)ds.$$

Then, defining $\lambda^p(t,x)$ as in (2.8) *one gets the desired result.*

Conversely, the intensity processes uniquely determines the compensator $\nu^p(t,dx)$ as, (see Filipović et al. [2009], Lemma 3.1),

$$\nu^p(t,(0,x]) = \lambda^p(t,L_{t-}) - \lambda^p(t,L_{t-}+x), \quad x \in [0,1] \tag{2.10}$$

where $\lambda^p(t,x) = 0$ for $x \ge 1$. Furthermore, $\lambda^p(t,x)$ is decreasing in x for any t.

Now we define the set Θ and H of $\mathbb{R}^d$ and $\mathbb{R}$-valued processes as follows

$$\Theta := \left\{\theta \text{ predictable} : \int_0^T \|\theta\|^2 dt < \infty, \ a.s.\ \forall T > 0\right\}$$

and

$$H := \left\{h(\omega,t,x)\ \mathcal{P} \otimes \mathcal{B}([0,1])\text{-measurable} : \int_0^T \int_{(0,1]} |h(t,\xi)| \nu^p(t,d\xi)dt < \infty, \ a.s.\ \forall T\right\}$$

The following lemma proves some auxiliary facts that will be used in the proof of the main theorem on the uniqueness of the martingale measure in the above (T,x)-bond market.

Lemma 2.2 *Assume* **(A2)**-**(A6)** *holds and h be any process in H, then the following holds*

(i) $\beta(t,T,x)$ *is continuous in* $T \in [t,T^*]$,

(ii) $\int_{(0,1]} \gamma(t,T,x;\xi)h(t,\xi)\nu^p(t,d\xi)$ *is continuous in* $T \in [t,T^*]$,

(iii) $\displaystyle\int_{(0,1]} 1_{\{L_{t-}+\xi>x\}} h(t,\xi)\nu^p(t,d\xi)$ *is càdlàg in* $x \in [0,1]$.

Proof. (i) is obvious since we have $\beta(t,T,x) = \int_t^T b(t,u,x)du$. For (ii), fix $t,\ x \in [0,T] \times (0,1]$ and $\omega \in \Omega$ and define

$$F(t,x,T) := \int_{(0,1]} f(t,x,T;\xi)\nu^p(t,d\xi)$$

where

$$f(t,x,T;\xi) := (e^{-\int_t^T c(t,u,x;\xi)du} 1_{\{L_{t-}+\xi\leq x\}} - 1)h(t,\xi)$$

Clearly, f is continuous in T, that is, any sequence $T_n \in [0,T^*]$, $T_n \to T$ we have

$$\lim_{n\to\infty} f(t,x,T_n;\xi) = f(t,x,T;\xi) \quad \forall t,x,\xi.$$

Now define

$$f_n(\xi) = f(t,x,T_n;\xi) \tag{2.11}$$

from **(A5)**, we have $|c(t,u,x;\xi)|$ is bounded for $u \in [t,T^*]$ implying that for all (t,x,ξ), $|f_n(\xi)| \leq |h(t,\xi)|$ for each n. Since $h \in H$, dominated convergence theorem applies and yields the desired result as follows

$$\lim_{n\to\infty} F(t,x,T_n) = \int_{(0,1]} \lim_{n\to\infty} f(t,x,T_n;\xi) = F(t,x,T)$$

(iii) Fix $\omega \in \Omega$ and $t \in [0,T]$ and observe that

$$1_{\{L_{t-}+\xi>x\}}$$

is càdlàg in $x \in [0,1]$. Take any sequence $x_n \downarrow x$ in $[0,1]$ and define

$$f_n(\xi) = 1_{\{L_{t-}+\xi>x_n\}} h(t,\xi)$$

then

$$\lim_{n\to\infty} f_n(\xi) = 1_{\{L_{t-}+\xi>x\}} h(t,\xi)$$

Since have $|f_n(\xi)| \leq |h(t,\xi)|$ it follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \int_{(0,1]} f_n(\xi)\nu^p(t,d\xi) = \int_{(0,1]} 1_{\{L_{t-}+\xi>x\}} h(t,\xi)\nu^p(t,d\xi)$$

After showing the right continuity, now we deal with the existence of left limits. Take a sequence $x_n \uparrow x$ in $[0,1]$. We have

$$\lim_{n\to\infty} f_n(\xi) = 1_{\{L_{t-}+\xi\geq x_n\}} h(t,\xi)$$

Hence, by dominated convergence theorem we have

$$\lim_{n\to\infty}\int_{(0,1]} f_n(\xi)\nu^p(t,d\xi) = \int_{(0,1]} 1_{\{L_{t-}+\xi\geq x_n\}} h(t,\xi)\nu^p(t,d\xi)$$

□

The following definition gives the criteria for absence of arbitrage opportunities in the given (T,x)-bond market.

Definition 2.1 *Let $T^* < \infty$ and the set $\mathcal{Q}$ is defined by*

$$\mathcal{Q} := \Big\{\mathbb{Q}\sim\mathbb{P} \text{ on } \mathcal{F}_{T^*} | Z(t,T,x)\ \mathbb{Q}\text{-local martingale},\ \forall T\in[0,T^*],\ x\in[0,1]\Big\}.$$

Then, the market is called arbitrage free *if $\mathcal{Q}$ is non-empty.*

To guarantee the absence of arbitrage in the market, we now assume that $\mathcal{Q}$ is non-empty. As we mentioned before, the framework introduced in Filipović et al. [2009] is a generalization of HJM framework to defaultable term structures and naturally, assuming no arbitrage in the market yields some restrictions in the modeling components. Namely, assuming no arbitrage in the (T,x)-bond market put on one hand a relation between the short end of the defaultable forward rates and the intensity of (T,x)-bonds and on the other hand a restriction on the drift parameter of the forward rates. Next theorem gives conditions resulting from no-arbitrage assumption.

Theorem 2.1 *Assume* (**A1**)-(**A7**) *holds and the (T,x)-bond market is arbitrage-free in the sense of Definition 2.1. Then,*

$$\begin{aligned}\int_t^T a^p(t,u,x)du = \frac{1}{2}\Big\|\int_t^T b(t,u,x)du\Big\|^2 + \int_t^T b(t,u,x)^\top du\cdot\theta_t \\ + \int_{(0,1]} (e^{-\int_t^T c(t,u,x;y)du} - 1)1_{\{L_{t-}+y\leq x\}} h_t(y)\nu^p(t,dy),\end{aligned} \tag{2.12}$$

$$\lambda^q(t,x) = f(t,t,x) - r_t \tag{2.13}$$

on $L_{t-}\leq x$, $d\mathbb{P}\otimes dt$-a.s for all (T,x) where

$$\lambda^q(t,x) = \int_{(0,1]} 1_{\{L_{t-}+y>x\}} h_t(y)\nu^p(t,dy)$$

and (θ,h) are processes in $\Theta\times H$.

Proof. Under the assumption of no-arbitrage, we know that there exists a measure $\mathbb{Q}\in\mathcal{Q}$ equivalent to $\mathbb{P}$. Since the filtration $\mathcal{F}_t$ is assumed to be the internal one, that is, generated by the Brownian motion $W^{\mathbb{P}}$ and the random measure μ, the result of Theorem

A.4 applies. That is, the equivalent measure $\mathbb{Q}$ is in the form $d\mathbb{Q} = M_T d\mathbb{P}$ where martingale M_t follows the dynamics

$$dM_t = M_{t-}\Big(\theta_t \cdot dW_t^{\mathbb{P}} + \int_{(0,1]} (h(t,\xi)-1)(\mu(dt,d\xi) - \nu^p(t,d\xi)dt\Big) \tag{2.14}$$

for some processes $\theta \in \Theta$ and $h \in H$. We now write the $\mathbb{Q}$-dynamics of the discounted bond price process $Z(t,T,x)$ as follows

$$\begin{aligned}\frac{dZ(t,T,x)}{Z(t-,T,x)} =& \Big(\alpha^p(t,T,x) + \beta(t,T,x)^\top \cdot \theta_t + \int_{(0,1]} \gamma(t,T,x,\xi)(h_t(\xi)-1)\nu^p(t,d\xi)\Big)\, dt \\ &+ \beta(t,T,x)^\top \cdot dW_t^{\mathbb{Q}} + \int_{(0,1]} \gamma(t,T,x,\xi)(\mu(dt,d\xi) - \nu^q(t,d\xi)dt)\end{aligned} \tag{2.15}$$

where $\nu^q(t,dx)dt = h_t(x)\nu^p(t,dx)dt$ is the $\mathbb{Q}$-compensator of μ and $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} - \theta_t dt$ is a $\mathbb{Q}$-Brownian motion.

For $\mathbb{Q}$ to be a martingale measure, the discounted price process has to be a local martingale under this measure. This implies that, the drift of equation (2.38) has to be zero, that is, for all $T \in [0,T^*],\ x \in [0,1]$

$$\alpha^p(t,T,x) + \beta(t,T,x)^\top \cdot \theta_t + \int_{(0,1]} \gamma(t,T,x,\xi)(h_t(\xi)-1)\nu^p(t,d\xi) = 0,\ d\mathbb{P}\otimes dt\text{-}a.s. \tag{2.16}$$

Now, recall that α^p satisfies (2.5). Plugging this equation into (2.16) yields

$$\begin{aligned}&-r_t - \lambda^p(t,x) + f(t,t,x) - \int_t^T a^p(t,u,x)du + \frac{1}{2}\Big\| \int_t^T b(t,u,x)du \Big\|^2 \\ &+ \int_{(0,1]} (e^{-\int_t^T c(t,u,x;y)du} - 1)1_{\{L_t + y \le x\}}\nu^p(t,dy) + \int_t^T b(t,u,x)^\top du \cdot \theta_t \\ &+ \int_{(0,1]} \gamma(t,T,x;y)(h_t(y)-1)\nu^p(t,dy) = 0, \quad d\mathbb{P}\otimes dt - a.s.\end{aligned} \tag{2.17}$$

Now denote by $\mathcal{N}_{T,x}$ the $d\mathbb{P}\otimes dt$-null set such that (2.21) holds for all $(\omega,t) \in \mathcal{N}^c_{T,x}$. Then, define $d\mathbb{P}\otimes dt$-null set

$$\mathcal{N}_x := \bigcup_{T \in Q\cap[0,T^*]} \mathcal{N}_{T,x}$$

Observe that (2.17) holds for all $(\omega,t) \in \mathcal{N}^c_x$ and for all $T \in Q \cap [0,T^*]$. Using Lemma 2.2 facts (i)-(ii) and an approximating argument we obtain that (2.17) holds for all $(\omega,t) \in \mathcal{N}^c_x$ and for all $T \in [0,T^*]$. Hence in particular for $T = t$. Now recall that $\alpha^p(t,t,x) = \beta(t,t,x) = 0$ which yields that for fix x and $T = t$, (2.17) reduces to

$$-r_t - \lambda^p(t,x) + f(t,t,x) - \int_{(0,1]} 1_{\{L_{t-}+y > x\}}(h_t(y)-1)\nu^p(t,dy) = 0 \tag{2.18}$$

for all $(\omega, t) \in \mathcal{N}_x^c$.

Now we define

$$\mathcal{N} = \bigcup_{x \in Q \cap [0,1]} \mathcal{N}_x \tag{2.19}$$

Again, an approximation argument together with fact (iii) of Lemma 2.2 implies that (2.18) holds for all $(\omega, t) \in \mathcal{N}^c$ and $x \in (L_{t-}(\omega), 1]$. Inserting the definition of $\lambda^p(t,x)$ in (2.18) and defining

$$\lambda^q(t,x) := \int_{(0,1]} 1_{\{L_{t-}+y>x\}} h_t(y) \nu^p(t, dy)$$

we get (2.13).

Finally, plugging (2.18) in to (2.17) we get the drift condition (2.12). □

Next theorem shows that for a finite time horizon $[0, T^*]$, under an assumption on the volatility parameter $\beta(t,T,x)$, the martingale measure is unique.

Theorem 2.2 *Assume that* $\mathbf{A(2)}$-$\mathbf{A(7)}$ *holds and there exists* $x_1, ..., x_d \in [0,1]$ *such that*

$$\begin{pmatrix} \beta(t, T^*, x_1)^\top \\ \cdot \\ \cdot \\ \beta(t, T^*, x_d)^\top \end{pmatrix}$$

is non-singular $d\mathbb{P} \otimes dt$ *a.e* $(\omega, t) \in \Omega \times [0, T)$. *Then, the ELMM is unique.*

Proof. We have the equivalent measure $\mathbb{Q}$ in the form $d\mathbb{Q} = M_T d\mathbb{P}$ where martingale M_t follows the dynamics

$$dM_t = M_{t-}\Big(\theta_t \cdot dW_t^{\mathbb{P}} + \int_{(0,1]} (h(t,\xi) - 1)(\mu(dt, d\xi) - \nu^p(t, d\xi)dt\Big) \tag{2.20}$$

for some processes $\theta \in \Theta$ and $h \in H$.

Now let (θ, h), $(\tilde{\theta}, \tilde{h}) \in \Theta \times H$ defining M and $\tilde{M}$, thus $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ respectively, satisfy equation (2.16). In what follows, our objective is to show

$$\theta_t(\omega) = \tilde{\theta}_t(\omega), \quad d\mathbb{P} \otimes dt - a.s$$

and

$$h(\omega, t, \xi)\nu^p(t, d\xi) = \tilde{h}(\omega, t, \xi)\nu^p(t, d\xi), \quad d\mathbb{P} \otimes dt - a.s$$

hence M and $\tilde{M}$ are indistinguishable and thus $\mathbb{Q} = \tilde{\mathbb{Q}}$. To this end, first observe that for all $T \in [0, T^*]$, $x \in (0,1]$

$$\beta(t,T,x)^\top(\theta_t - \tilde{\theta}_t) + \int_{(0,1]} \gamma(t,T,\xi)(h_t(\xi) - \tilde{h}_t(\xi))\nu^p(t, d\xi) = 0 \tag{2.21}$$

holds $d\mathbb{P} \otimes dt$-*a.s.* Now fixing $t = T$ as in the proof of Theorem 2.1 we get that

$$\int_{(0,1]} 1_{\{L_{t-}+\xi>x\}}(h_t(\xi) - \tilde{h}_t(\xi))\nu^p(t, d\xi) = 0 \tag{2.22}$$

holds for all $x \in (L_{t-}(\omega), 1]$ and $(\omega, t) \in \mathcal{N}^c$ where $\mathcal{N}$ is defined as in (2.19).

Here notice that the support of the measure $\nu^p(t, d\xi)$ is $(0, 1 - L_{t-}(\omega)]$ implying that

$$\int_{(x-L_{t-}(\omega))^+}^{1-L_{t-}(\omega)} (h(t, \xi) - \tilde{h}(t, \xi)\nu^p(t, d\xi)) = 0 \tag{2.23}$$

for all $x \in (L_{t-}(\omega), 1]$. That is, we have

$$\int_0^y (h(t, \xi) - \tilde{h}(t, \xi))\nu^p(t, d\xi) = 0, \text{ for all } y \in (0, 1 - L_{t-}(\omega)].$$

Hence, the two measures on $(0, 1]$ agree:

$$h(t, \xi)\nu^p(t, d\xi) = \tilde{h}(t, \xi)\nu^p(t, d\xi)$$

for all $(\omega, t) \in \mathcal{N}^c$. Plug in (2.16), we obtain

$$\beta(t, T, x)^\top (\theta_t - \tilde{\theta}_t) = 0 \tag{2.24}$$

for all $(\omega, t) \in \mathcal{N}_x^c$ and for all $T \in [t, T^*]$. From the non-singularity assumption of β matrix, (2.24) implies $\theta_t(\omega) = \tilde{\theta}_t(\omega)$, $d\mathbb{P} \otimes dt$-*a.s.* Hence $M = \tilde{M}$ and thus $\mathbb{Q} = \tilde{\mathbb{Q}}$ as desired. $\square$

Remark 2.2 *Note that in the current setup, where L is a marked point process with a continuous jump spectrum, the uniqueness of the martingale measure does not imply the completeness of the market in the sense that every contingent claim is replicable by a self financing hedging strategy. For default free term structure models with similar properties, the market with a unique martingale measure is characterized as approximately complete and the uniqueness of the measure implies completeness only when the mark space is finite (for a detailed discussion on this subject, we refer to (Björk et al. [1997], Sec. 4.2). In view of this information, a complete market setting can be reached, e.g., via choosing a loss process with fixed jump size, say $1/N$ where N is the number of constituent names in the index. Indeed, this is not counterintuitive for indices having equally weighted constituent names.*

Omitting the contagion effects from the dynamics of forward spreads, Filipović et al. [2009] generalizes the concept of doubly stochastic Poisson processes to marked point processes. Under this doubly stochastic framework, doubly stochastic affine term structure models are proposed as a tractable class. In the following, we first recall the definition of an affine process and then give an overview of the doubly stochastic affine term structure models.

Definition 2.2 *Fix $d \geq 1$ and a closed state space $\mathcal{X} \in \mathbb{R}^d$. A d-dimensional process X with the state space $\mathcal{X}$ is affine if the $\mathcal{F}_t$-conditional characteristic function of X_T is exponentially affine in X_t, for all $t \leq T$. That is, there exists $\mathbb{C}$ and $\mathbb{C}^d$ valued functions $\phi(t,u)$ and $\psi(t,u)$ respectively, with jointly continuous t-derivatives such that $X = X^x$ satisfies*

$$\mathbb{E}^{\mathbb{P}}[e^{u^\top \cdot X_T} | \mathcal{F}_t] = e^{\phi(T-t,u)+\psi(T-t,u)^\top \cdot X_t} \tag{2.25}$$

for all $u \in i\mathbb{R}^d$, $t \leq T$ and for all initial $x \in \mathcal{X}$.

Now, let X be some $\mathcal{X}$-valued diffusion process having the $\mathbb{Q}$-dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t) \cdot dW_t, \quad X_0 = x \tag{2.26}$$

where μ and σ are continuous functions from $\mathbb{R}^+ \times \mathcal{X}$ into $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, respectively. The idea of affine term structure models lies on the property that, the forward rates (or equivalently bond prices), are considered to be an affine function (exponentially affine function) of the state process X. That is,

$$f(t,T,x) = A'(T-t,x) + B'(T-t,,x)^\top \cdot X_t$$

for some functions $A'(T-t,,x)$ and $B'(T-t,x)$ with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. Now define

$$A(T-t,x) = \int_0^{T-t} A'(u,x)du, \quad B(T-t,x) = \int_0^{T-t} B'(u,x)du$$

In the following we recall the fundemental theorem on doubly stochastic affine term structure models given in Filipović et al. [2009].

Theorem 2.3 *Let $\tau = (T-t) \geq 0$ and assume X is an m-factor affine process given by*

$$dX_t = \left(b_0 + \sum_{i=1}^{m} X_{it} b_i \right) dt + \Sigma(X_t) dW_t^{\mathbb{Q}} \tag{2.27}$$

with an affine diffusion $\frac{1}{2}\Sigma\Sigma^\top(x) = a_0 + \sum_{i=1}^m x_i a_i$ where vectors $b_i \in \mathbb{R}^m$ and matrices $a_i \in \mathbb{R}^{m\times m}$. Then, there exists a loss process L such that

$$P(t,T,x) = 1_{\{L_t \leq x\}} e^{-A(T-t,x)-B(T-t,x)^\top X_t} \tag{2.28}$$

defines an arbitrage-free (T,x)-bond market where functions A and B satisfy the following system of Riccati equations

$$\partial_\tau A(\tau,x) = \alpha(x) + b_0^\top \cdot B(\tau,x) - B(\tau,x)^\top \cdot a_0 \cdot B(\tau,x) \tag{2.29}$$
$$A(0,x) = 0$$
$$\partial_\tau B_i(\tau,x) = \beta_i(x) + b_i^\top \cdot B(\tau,x) - B(\tau,x)^\top \cdot a_i \cdot B(\tau,x) \tag{2.30}$$
$$B(0,x) = 0$$

for some $\mathbb{R}_+$-valued bounded measurable functions $\alpha(x)$, $\beta_i(x)$ which are non-increasing and càdlàg with $\alpha(x) \geq r \geq 0$ and $\beta_i(x) = 0$ for $x \geq 1$.

Proof. The idea of the proof is to use the no arbitrage drift condition and get the Riccati equations that A and B satisfies. It is worth mentioning that the continuity of the process X is also crucial in the proof. For technical details see Section 7.1 of Filipović et al. [2009]. □

Here, it is important to emphasize that the functions α and β_i are exogenous and can be used to calibrate the model to the observable STCDO prices. This property of the model becomes even more clear with the following relations.

From the Riccati equations given in above theorem, for all $x \in [0,1]$, the short rate satisfies

$$f(t,t,x) = \alpha(x) + \beta^\top X_t.$$

Then, affine model given by (2.29)-(2.30) together with the relation (2.10) and the no-arbitrage condition (2.13) yield the following relation between $\alpha(x)$, $\beta(x)$ and the risk-neutral compensator of the loss process

$$\begin{aligned}\nu^q(t,(0,x]) &= f(t,t,L_{t-}) - f(t,t,L_{t-}+x)\\ &= \alpha(L_{t-}) - \alpha(L_{t-}+x) + (\beta(L_{t-}) - \beta(L_{t-}+x))^\top \cdot X_t \end{aligned} \tag{2.31}$$

which implies that a default event arrives with risk-neutral intensity

$$\Lambda_t = \alpha(L_{t-}) - r + \beta(L_{t-})^\top X_t \tag{2.32}$$

and an occurrence of a default causes a loss with risk-neutral cumulative distribution

$$G_L(t,x) = \frac{\nu^q(t,(0,x])}{\nu^q(t,(0,1])} \tag{2.33}$$

2.2.2 Model Specification: Two-Factor (Stochastic Drift) Model

We propose the following 2-factor affine model with the $\mathbb{P}$-dynamics

$$dY_t = \kappa_y (Z_t - Y_t)\,dt + \sigma_y \sqrt{Y_t} dW_t^y, \; Y_0 = y \in \mathbb{R}^+ \tag{2.34}$$

$$dZ_t = \kappa_z (\theta_z - Z_t)\,dt + \sigma_z \sqrt{Z_t} dW_t^z, \; Z_0 = z \in \mathbb{R}^+ \tag{2.35}$$

where $\kappa_y \geq 0$ and $\kappa_z \theta_z \geq 0$ and W^y and W^z are independent $\mathbb{P}$-Brownian motions. Here, factor Z is functioning as the stochastic long run mean reversion level of factor Y. In our empirical analysis, we also consider the nested 1-factor model where the unique factor has a constant mean reversion level.

To preserve the affine structure under a change of measure we specify the market price of risk process $\lambda_t = (\lambda_t^y, \lambda_t^z)$ in the following way

$$\lambda_t^y = \frac{\lambda_y \sqrt{Y_t}}{\sigma_y}, \quad \lambda_t^z = \frac{\lambda_z \sqrt{Z_t}}{\sigma_z} \tag{2.36}$$

Then, the $\mathbb{Q}$-dynamics of the factor process reads

$$dY_t = (\kappa_y + \lambda_y)\left(\frac{\kappa_y}{\kappa_y + \lambda_y} Z_t - Y_t\right) dt + \sigma_y \sqrt{Y_t} d\tilde{W}_t^y \tag{2.37}$$

$$dZ_t = (\kappa_z + \lambda_z)\left(\frac{\kappa_z}{\kappa_z + \lambda_z} \theta_z - Z_t\right) dt + \sigma_z \sqrt{Z_t} d\tilde{W}_t^z \tag{2.38}$$

where $\tilde{W}_t^y = W_t^y + \int_0^t \lambda_s^y ds$ and $\tilde{W}_t^z = W_t^z + \int_0^t \lambda_s^z ds$ are $\mathbb{Q}$-Brownian motions.

Given dynamics in equation(2.37)-(2.38), Theorem 2.3 immediately yields that

$$P(t, T, x) = 1_{\{L_t \le x\}} e^{-A(T-t,x) - B_y(T-t,x)Y_t - B_z(T-t,x)Z_t} \tag{2.39}$$

defines an arbitrage free (T, x)-bond market where A, B_y and B_z solves the Riccati equations

$$\partial_\tau A(\tau, x) = \alpha(x) + \kappa_z \theta_z B_z(\tau, x), \tag{2.40}$$
$$A(0, x) = 0,$$
$$\partial_\tau B_y(\tau, x) = \beta_y(x) - (\kappa_y + \lambda_y) B_y(\tau, x) - \frac{1}{2}\sigma_y^2 B_y(\tau, x)^2, \tag{2.41}$$
$$B_y(0, x) = 0,$$
$$\partial_\tau B_z(\tau, x) = \beta_z(x) + \kappa_y B_y(\tau, x) - (\kappa_z + \lambda_z) B_z(\tau, x) - \frac{1}{2}\sigma_z^2 B_z(\tau, x)^2, \tag{2.42}$$
$$B_z(0, x) = 0.$$

for some $\mathbb{R}_+$-valued functions α, $\beta_{y,z}$ which are non-increasing and càdlàg with $\alpha(x) \ge r \ge 0$ and $\beta_{y,z}(x) = 0$ for $x \ge 1$. For the above system there is no closed form solution available, however, one can solve this system numerically.

After specifying the affine factor model, the next task is to specify functions α, β_y and β_z. Here we want to point out that, particular choices for these functions imply different dynamics for the loss process via relations (2.31)-(2.33). To be able to get an exponentially decaying loss given default distribution we take

$$\alpha(x) = \gamma\left(e^{-a_0(x \wedge 1)} - e^{-a_0}\right) + r \tag{2.43}$$

$$\beta_y(x) = e^{-b_0(x \wedge 1)} - e^{-b_0} \tag{2.44}$$

where $\gamma \ge 0$, $a_0 \ge 0$, $b_0 \ge 0$. Moreover, we define

$$\beta_z(x) = c_0 1_{[0,1)}(x) \tag{2.45}$$

with $c_0 \geq 0$. Choosing function β_z as in (2.45) makes factor Z model the catastrophic level directly via giving Dirac point mass at $x = 1$ in the loss given default distribution. This is important in particular for the successful modeling of super senior tranches (see Chen et al. [2009]). We have the following Proposition giving implied loss compensator, the default intensity process and loss given default distribution under specification (2.43)-(2.45).

Proposition 2.1 *Suppose the functions α, β^y and β^z are in the form* (2.43)-(2.45). *Then, for the one and two factor models presented above the risk neutral compensator, the intensity process and loss given default distribution has the following form*

$$\nu^q\left(t,(0,x]\right) = \gamma\left(e^{-a_0(L_{t-}\wedge 1)} - e^{-a_0(L_{t-}+x\wedge 1)}\right) + \left(e^{-b_0(L_{t-}\wedge 1)} - e^{-b_0(L_{t-}+x\wedge 1)}\right)Y_t$$
$$+ c_0 1_{\{1-L_{t-}\leq x\}}Z_t$$

$$\Lambda_t = \gamma\left(e^{-a_0(L_{t-}\wedge 1)} - e^{-a_0}\right) + \left(e^{-b_0(L_{t-}\wedge 1)} - e^{-b_0}\right)Y_t + c_0 1_{[0,1)}(L_{t-})Z_t$$

$$G_L(t,x) = \frac{\gamma\left(e^{-a_0(L_{t-}\wedge 1)} - e^{-a_0(L_{t-}+x\wedge 1)}\right) + \left(e^{-b_0(L_{t-}\wedge 1)} - e^{-b_0(L_{t-}+x\wedge 1)}\right)Y_t}{\gamma\left(e^{-a_0(L_{t-}\wedge 1)} - e^{-a_0}\right) + \left(e^{-b_0(L_{t-}\wedge 1)} - e^{-b_0}\right)Y_t + c_0 1_{[0,1)}(L_{t-})Z_t}$$

$$+ \frac{c_0 1_{\{0<1-L_{t-}\leq x\}}Z_t}{\gamma\left(e^{-a_0(L_{t-}\wedge 1)} - e^{-a_0}\right) + \left(e^{-b_0(L_{t-}\wedge 1)} - e^{-b_0}\right)Y_t + c_0 1_{[0,1)}(L_{t-})Z_t} \tag{2.46}$$

Proof. Inserting function α, β_y and β_z in (2.43)-(2.45) into equations (2.31)-(2.33) yields the result. $\square$

Here notice that the loss given default distribution is not static. Its dynamics is changing with the level of the loss and the factor processes. Next corollary gives an explicit formula for the time t expected loss given default implied by the loss given default distribution given in (2.46).

Corollary 2.1 *Assume that the risk neutral loss given default distribution is as in* (2.46). *Then, at time t the $\mathbb{Q}$-expected loss given default is given by*

$$\int_{-\infty}^{\infty} xG_L(t,dx) = \frac{\frac{\gamma}{a_0}\left(e^{-a_0L_{t-}} - e^{-a_0}(a_0(1-L_{t-})+1)\right)}{\gamma\left(e^{-a_0L_{t-}} - e^{-a_0}\right) + \left(e^{-b_0L_{t-}} - e^{-b_0}\right)Y_t + c_0 1_{[0,1)}(L_{t-})Z_t}$$

$$+ \frac{\frac{1}{b_0}\left(e^{-b_0L_{t-}} - e^{-b_0}(b_0(1-L_{t-})+1)\right)Y_t + c_0(1-L_{t-})Z_t}{\gamma\left(e^{-a_0L_{t-}} - e^{-a_0}\right) + \left(e^{-b_0L_{t-}} - e^{-b_0}\right)Y_t + c_0 1_{[0,1)}(L_{t-})Z_t} \tag{2.47}$$

Proof. First, recall that L_t takes values in $[0,1]$. This immediately implies that the support of the function $G_L(t,x)$ is $[0,\infty)$ where we have $G_L(t,x)=1$ for $\{x>1-L_{t-}\}$. Then, we automatically have that the support of the function $1-G_L(t,x)$ is $[0,1-L_{t-}]$. On the other hand, it is known that for a non-negative random variable X_t we can use the following formula to compute the expectation

$$E^{\mathbb{Q}}[X_t]=\int_0^\infty \mathbb{Q}(X_t\geq x)dx=\int_0^\infty (1-\mathbb{Q}(X_t<x))dx \tag{2.48}$$

Using this and the fact that $G_L(t,x-)=G_L(t,x)\ dx-a.s.$ we get

$$\int_{-\infty}^{\infty} xG_L(t,dx)=\int_0^{1-L_{t-}} (1-G_L(t,x))dx \tag{2.49}$$

Then, from equation (2.46) we get $1-G_L(t,x)=$

$$\frac{\gamma\left(e^{-a_0((L_{t-}+x)\wedge 1)}-e^{-a_0}\right)+\left(e^{-b_0((L_{t-}+x)\wedge 1)}-e^{-b_0}\right)Y_t+c_0 1_{[0,1)}(L_{t-}+x)Z_t}{\gamma\left(e^{-a_0L_{t-}}-e^{-a_0}\right)+\left(e^{-b_0L_{t-}}-e^{-b_0}\right)Y_t+c_0 1_{[0,1)}(L_{t-})Z_t}. \tag{2.50}$$

Finally, inserting (2.50) in to (2.49) and then computing the integral finishes the proof. □

2.3 Single Tranche CDOs (STCDO)

Suppose an investor has a long position in the STCDO with attachment and detachment points x_1,x_2 and having coupon dates $0<T_1<...<T_n$. The coupon payments are determined by the pre-determined coupon rate $\kappa_0^{(x_1,x_2]}$ and the notional of the tranche, net of the losses in the tranche realized by time T_i . The attachment point indicates the point at which losses in the underlying index begin to erode the notional of the tranche and in the detachment point full tranche is written down. In case of a realization of a loss, the position holder of the respective STCDO pays the fraction of the loss which falls into the invested tranche. In turn, until the notional of the tranche gets fully written down, coupon payments on the remaining notional are received.

To formalize the cash flows of a STCDO we define,

$$H^{(x1,x2]}(x):=\int_{x_1}^{x_2} 1_{\{x\leq y\}}dy=(x_2-x)^+-(x_1-x)^+. \tag{2.51}$$

Then, the long position holder of the STCDO

- receives $\kappa_0^{(x_1,x_2]}\times H^{(x1,x2]}(L_{T_i})$ at T_i, $\{i=1,2,...,n\}$, (*coupon leg*)
- pays $-\Delta H^{(x1,x2]}(L_t)=H^{(x1,x2]}(L_{t-})-H^{(x1,x2]}(L_t)$ at any time $(T_0,T_n]$ where $\Delta L_t\neq 0$, (*protection leg*).

The value of the STCDO long position at time $t \leq T_0$ is equal to the difference between the value of the coupon leg and protection leg which, under the assumption of constant risk-free rates, can be represented as (see Lemma 4.1 in Filipović et al. [2009])

$$V_C^{(x_1,x_2]}(t) = \kappa_0^{(x_1,x_2]} \sum_{t<T_i} \int_{x_1}^{x_2} P(t,T_i,x)dx \tag{2.52}$$

and

$$V_P^{(x_1,x_2]}(t) = \int_{x_1}^{x_2} \left(1_{\{L_t \leq x\}} - P(t,T_n,x) - r \int_t^{T_n} P(t,u,x)du \right) dx \tag{2.53}$$

Now one can recover the par-coupon rate at time t, that is, the rate which makes $V_C(t) - V_P(t) = 0$ as follows

$$\kappa_t^{(x_1,x_2]} = \frac{V_P^{(x_1,x_2]}(t)}{\sum_{t<T_i} \int_{x_1}^{x_2} P(t,T_i,x)dx} \tag{2.54}$$

In practice, when κ_t is bigger than 5%, the market sets $\kappa_t = 5\%$ and quote the up-front payment $V_P(t) - V_C(t)$ instead. In the current analysis, this is valid for the equity tranche and the necessary modifications are done whenever needed.

Having κ_t, one can write the time t value, $\Gamma_t^{(x_1,x_2]}$, of the STCDO as follows

$$\Gamma_t^{(x_1,x_2]} = \left(\kappa_0^{(x_1,x_2]} - \kappa_t^{(x_1,x_2]} \right) \sum_{t<T_i} \int_{x_1}^{x_2} P(t,T_i,x)dx \tag{2.55}$$

The discounted gains process of the STCDO long position at time t, which is denoted by $G_t^{(x_1,x_2]}$, is equal to the sum of accumulated discounted cash flows $A_t^{(x_1,x_2]}$ and discounted spot value,

$$G_t^{(x_1,x_2]} = A_t^{(x_1,x_2]} + e^{-rt} \left(V_C^{(x_1,x_2]}(t) - V_P^{(x_1,x_2]}(t) \right) \tag{2.56}$$

where $A_t^{(x_1,x_2]}$ is the difference between the value of coupon and protection payments that has been realized by time t. That is,

$$A_t^{(x_1,x_2]} = \kappa_0^{(x_1,x_2]} \sum\nolimits_{T_i \leq t} e^{-rT_i} H(L_{T_i}) + \int_0^t e^{-ru} dH(L_u) \tag{2.57}$$

2.4 Hedging of STCDOs

The gains process of a STCDO long (protection seller) position is exposed to the risk arising from the credit events of any constituent. It is clear from (2.56) that, there is an exposure to the changes in the zero-coupon spreads and the loss payments. It is possible to a hedge a tranche position via taking offsetting positions in the underlying CDS, in another trance or in the index. Here, our focus is the hedging of STCDO with the index. To be able to offset negative value changes in the tranche by a dynamically rebalanced self-financing position in the index, the long position holder in STCDO would take the short (protection buyer) position in the index as long as the co-movement of the index and the tranche is assured. Certainly, the amount which is invested in to the index has to be determined via some hedging criteria. Thus, to perform the hedge, the first thing to do is finding a relevant hedging strategy.

In the current setup we are in an incomplete market setting due to the presence of infinite number of risk sources. Quadratic hedging approach, where the criterion is to minimize the hedging error in mean-square sense, is one of the alternatives for hedging in incomplete markets. By its very nature, in an incomplete market finding a self-financing strategy which at the same time allows for perfect replication is not possible. One has to either sacrifice the self-financing property of the hedging strategy to guarantee the perfect replication of a claim or vice versa. In this context, Follmer and Sondermann [1986] introduced the local risk minimization for the case where the risky asset is martingale under the physical measure. Then the idea of risk minimization is extended to a general semimartingale case by Schweizer [1988]. Under this approach, one seeks for the minimal cost strategy among the hedging strategies which minimizes the one-step ahed mean-square error between the hedging portfolio and the given contingent claim. A risk minimizing strategy is characterized by two main properties. Firstly, it possesses the mean self-financing property, meaning that the associated cost process is a martingale. Second property is that, the cost process is orthogonal to the martingale part of the underlying asset, that is the cost process represents the unhedgeble part of the claim.

An alternative method under the quadratic hedging approach is the variance minimizing hedging. This method yields a self-financing hedging strategy which approximate the contingent claim by the terminal value of the hedging portfolio. Here, the optimality criterion is taken to be the $\mathbb{Q}$-mean-square error between the value of the claim and the terminal value of the hedging portfolio. When performed under the martingale measure Q, local risk minimizing and variance minimizing approach yield the same strategies. For the review of quadratic hedging approaches see Schweizer [1999] and for different hedging approaches in incomplete markets, we refer to Chp. 10 in Cont and Tankov [2004].

Frey and Backhaus [2010] uses variance minimizing hedge for the hedging of CDO tranches with the underlying CDSs. For the hedging of STCDOs with the underlying index, variance minimizing hedge is utilized in Cont and Kan [2011] as the strategy which takes spread and default risk into account. Following the idea, Filipovic and Schmidt [2010] derives an explicit formula for the corresponding variance minimizing strategy under their modeling setup with the constant risk-free rate assumption. On the other hand, when compared to number of other hedging strategies the regression based hedging is shown to be very effective in Cont and Kan [2011]. Motivated by this, in our analysis we use variance minimizing and regression based strategies as our hedging tools. This section gives the explicit formulae of variance minimizing and regression based strategies for the two factor affine model. Furthermore, we explain the relation between the two hedging strategies. We then outline the hedging algorithm in which the objective is to cover the risky position in a STCDO via investing in the index. We also introduce criteria for the assessment of the performance of a hedging strategy.

2.4.1 Variance Minimizing Strategy

In an incomplete market setting, one can use variance-minimizing criterion in which the objective is to minimize the $\mathbb{Q}$- conditional variance of the quadratic hedging error, that is,

$$\inf_{\phi} \mathbb{E}^{\mathbb{Q}} \left[\left(G_T^{(x_1,x_2]} - c + \int_t^T \phi_s dG_s^{(x_1,x_2]} \right)^2 \Big| \mathcal{F}_t \right] \tag{2.58}$$

It is well known that, under suitable conditions on the gains process, such as the square integrable martingale property, the minimizing strategy exists and can be computed from the Galtchouk-Kunita-Watanebe decomposition. That is, along with the initial capital $c^* = G_t^{(x_1,x_2]}$, the self-financing strategy

$$\phi_t^* = -\frac{d\left\langle G_t^{(x_1,x_2]}, G_t^{(0,1]} \right\rangle^{\mathbb{Q}}}{d\left\langle G_t^{(0,1]} \right\rangle^{\mathbb{Q}}} \tag{2.59}$$

is the unique minimizer of (2.58). Here, $\langle \cdot, \cdot \rangle$ denotes the sharp bracket process and $G^{(0,1]}$ satisfies (2.56) with $x_1 = 0$ and $x_2 = 1$. Results in Filipovic and Schmidt [2010] shows that (see Sec. 5.1, Equation 20) under the assumption of deterministic risk-free interest rate, the gains process satisfies

$$dG_t^{(x_1,x_2]} = e^{-\int_0^t r_u du} \left(B_t^{(x_1,x_2]} dW_t^{\mathbb{Q}} + \int_{(0,1]} C_t^{(x_1,x_2]}(\xi)(\mu(dt, d\xi) - \nu^q(t, d\xi)dt) \right) \tag{2.60}$$

where

$$
\begin{aligned}
B_t^{(x_1,x_2]} &= \int_{(x_1,x_2]} \Big\{ s_0^{(x_1,x_2]} \sum_{t<T_i} P(t,T_i,x)\beta(t,T_i,x) \\
&+ P(t,T_n,x)\beta(t,T_n,x) + \int_t^{T_n} r_u P(t,u,x)\beta(t,u,x)du \Big\} dx
\end{aligned} \tag{2.61}
$$

$$
\begin{aligned}
C_t^{(x_1,x_2]}(\xi) &= \int_{(x_1,x_2]} \Big\{ s_0^{(x_1,x_2]} \sum_{t<T_i} P(t-,T_i,x)\gamma(t,T_i,x,\xi) \\
&+ P(t-,T_n,x)\gamma(t,T_n,x,\xi) + \int_t^{T_n} r_u P(t-,u,x)\gamma(t,u,x,\xi)du \Big\} dx
\end{aligned} \tag{2.62}
$$

where β and γ are defined as in (2.6) and (2.7) respectively.

If we further assume that the risk-free interest rate is constant and use the two-factor (stochastic drift) model specification in (2.39), $B_t^{(x_1,x_2]}$ and $C_t^{(x_1,x_2]}$ becomes

$$
\begin{aligned}
B_t^{(x_1,x_2]} &= \int_{(x_1,x_2]} \Big\{ s_0^{(x_1,x_2]} \sum_{t<T_i} P(t,T_i,x)\beta(t,T_i,x) \\
&+ P(t,T_n,x)\beta(t,T_n,x) + r\int_t^{T_n} P(t,u,x)\beta(t,u,x)du \Big\} dx
\end{aligned} \tag{2.63}
$$

$$
\begin{aligned}
C_t^{(x_1,x_2]}(\xi) &= -\int_{(x_1,x_2]} \Big\{ s_0^{(x_1,x_2]} \sum_{t<T_i} P(t-,T_i,x)1_{\{L_{t-}+\xi>x\}} \\
&+ P(t-,T_n,x)1_{\{L_{t-}+\xi>x\}} + r\int_t^{T_n} P(t-,u,x)1_{\{L_{t-}+\xi>x\}}du \Big\} dx
\end{aligned} \tag{2.64}
$$

with

$$
\beta(t,T,x) = \Big(-B_y(T-t,x)\sigma_y\sqrt{Y_t}\,,\ -B_z(T-t,x)\sigma_z\sqrt{Z_t} \Big)^\top \tag{2.65}
$$

where B_y and B_z are functions satisfying (2.41) and (2.42) respectively. Then, for the two-factor (stochastic drift model), under the assumption of constant risk-free rates, equation (2.59) together with the dynamics in (2.60) yields the variance minimizing strategy

$$
\phi_t^{VM} = -\frac{B_t^{(x_1,x_2]}B_t^{(0,1]} + \int_{(0,1]} C_t^{(x_1,x_2]}(\xi)C_t^{(0,1]}(\xi)\nu^q(t,d\xi)}{(B_t^{(0,1]})^2 + \int_{(0,1]}(C_t(0,1](\xi))^2\nu^q(t,d\xi)} \tag{2.66}
$$

where $B_t^{(x_1,x_2]}$ and $C_t^{(x_1,x_2]}$ are given by (2.63) and (2.64) respectively. Formula (2.66) reveals that, once we have the parameter estimates and the market data it is possible to get the value of the variance minimizing strategy explicitly.

2.4.2 Regression Based Strategy

Variance minimizing strategy can be criticized for suggesting an optimal hedging criterion under the risk neutral measure but not the real-world measure. Considering this drawback of the variance minimizing strategy, Cont and Kan [2011] introduces regression based hedging strategy which takes the dynamics of market observables into account. Here, the main point is that daily changes in the value of tranche is regressed to the daily value changes of the index. Following the same idea as in Section 5.7 of Cont and Kan [2011], to model the relation between the daily changes in the STCDO and index gains process we do a linear regression analysis . Formally, we assume that the daily changes in the STCDO and index gains processes follows

$$\Delta G_{t_k}^{(x_1,x_2]} = a + b\Delta G_{t_k}^{(0,1]} + \epsilon_{t_k} \tag{2.67}$$

where ϵ_{t_k} represent the independent standard Normal disturbance term. Given this statistical model, the idea is to find the daily estimates for parameters a and b. This is achieved via method of linear least squares in which we find estimates $\hat{a}$ and $\hat{b}$ which minimize the squared error of the observed data up to the current day. Then, each day we set the hedge ratio $\phi_t^{RB} = -\hat{b}$, that is, in each day the hedge ratio is computed via estimating the parameter b in above regression model by using the available data up to that day. From the very well acknowledged formula for the least squares estimate, we have

$$\phi_t^{RB} = -\frac{\sum_{t_k \le t} \left(\Delta G_{t_k}^{(0,1]} - \overline{\Delta G_t^{(0,1]}}\right)\left(\Delta G_{t_k}^{(x_1,x_2]} - \overline{\Delta G_t^{(x_1,x_2]}}\right)}{\sum_{t_k \le t} \left(\Delta G_{t_k}^{(0,1]} - \overline{\Delta G_t^{(0,1]}}\right)^2} \tag{2.68}$$

where $\overline{\Delta G_t^{(x_1,x_2]}}$ denotes the average of daily changes in the gains process by time t.

Remark 2.3 *From the theory of linear regression, we have the following well-known relation*

$$\begin{aligned} \phi_t^{RB} &= \frac{Cov\left(\Delta G_t^{(x_1,x_2]}, \Delta G_t^{(0,1]}\right)}{\sigma^2\left(\Delta G_t^{(0,1]}\right)} \\ &= \rho_t \frac{\sigma\left(\Delta G_t^{(x_1,x_2]}\right)}{\sigma\left(\Delta G_t^{(0,1]}\right)} \end{aligned} \tag{2.69}$$

where σ denotes the running standard deviation and ρ_t is the running linear correlation coefficient which are estimated by using the available data by time t. Equation (2.69) *indicates that for regression based hedge the two main determinants of the amount that* *has to invested into the index are the linear correlation and the relative values of variances of changes in tranche and index gains processes.*

Remark 2.4 *Under this setup, how well the investment in the index replicates the gains in tranche depends very much on the variance of changes in the tranche gains series and the linear correlation between the two series* $\Delta G_{t_k}^{(0,1]}$ *and* $\Delta G_{t_k}^{(x_1,x_2]}$, $t_k \leq t$. *This is because of the fact that by the regression based strategy, the remaining risk is reduced to the minimal mean square prediction error, that is*

$$\sigma^2\left(\Delta G_t^{(x_1,x_2]}\right) - \left(\phi_t^{RB}\right)^2 \sigma^2\left(\Delta G_t^{(0,1]}\right) = \sigma^2\left(\Delta G_t^{(x_1,x_2]}\right)\left(1-\rho_t^2\right) \tag{2.70}$$

where σ^2 *represents the running variance. Equation* (2.70) *suggests that, the larger the linear correlation or smaller the variance, the smaller the hedging error.*

2.4.3 Relation between Variance Minimizing and Regression Based Strategies

We first want to point out that, this part is just to give some intuition on the relation between the two hedging strategies and far from being rigorous. To be able to state the relation between the variance minimizing and regression based strategies, first recall from the previous parts that the time t value of the regression based strategy is given by

$$\phi_t^{RB} = \frac{Cov\left(\Delta G_t^{(x_1,x_2]}, \Delta G_t^{(0,1]}\right)}{\sigma^2\left(\Delta G_t^{(0,1]}\right)} \approx \frac{d\left[G^{(x_1,x_2]}, G^{(0,1]}\right]_t}{d\left[G^{(0,1]}, G^{(0,1]}\right]_t}\Delta t$$

where $[\cdot,\cdot]$ denotes the quadratic variation process. Recall that the sharp bracket process of a semimartingale, which we denote by $\langle\cdot,\cdot\rangle$ is the compensator of the quadratic variation process and for continuous semimartingales with integrable quadratic variation, the sharp bracket and quadratic variation processes are the same. Thus, we make the non-rigorous argument

$$\frac{d\left[G^{(x_1,x_2]}, G^{(0,1]}\right]_t}{d\left[G^{(0,1]}, G^{(0,1]}\right]_t}\Delta t \approx \frac{d\left\langle G^{(x_1,x_2]}, G^{(0,1]}\right\rangle_t^{\mathbb{P}}}{d\left\langle G^{(0,1]}, G^{(0,1]}\right\rangle_t^{\mathbb{P}}}\Delta t$$

This, together with (2.59) suggest that the difference between the two hedging strategies is given by

$$\phi_t^{RB} - \phi_t^{VM} \approx \frac{d\left\langle G^{(x_1,x_2]}, G^{(0,1]}\right\rangle_t^{\mathbb{P}}}{d\left\langle G^{(0,1]}, G^{(0,1]}\right\rangle_t^{\mathbb{P}}} - \frac{d\left\langle G^{(x_1,x_2]}, G^{(0,1]}\right\rangle_t^{\mathbb{Q}}}{d\left\langle G^{(0,1]}, G^{(0,1]}\right\rangle_t^{\mathbb{Q}}} \tag{2.71}$$

Now let us consider the processes X and Y with dynamics

$$dX_t = adt + \sigma dW_t^{\mathbb{Q}} + \int_{(0,1]} \gamma(\xi)(\mu(dt, d\xi) - \nu^q(t, d\xi)dt) \tag{2.72}$$

$$dY_t = bdt + \rho dW_t^{\mathbb{Q}} + \int_{(0,1]} \delta(\xi)(\mu(dt, d\xi) - \nu^q(t, d\xi)dt) \tag{2.73}$$

where all coefficients satisfy the necessary measurability and integrability conditions and ν^q is the $\mathbb{Q}$-compensator of the random measure μ. Then, we get

$$\langle X, Y\rangle_t^{\mathbb{Q}} = \int_0^t \Big(\sigma\rho + \int_{(0,1]} \gamma(\xi)\delta(\xi)\nu^q(s,\xi)\Big)ds \tag{2.74}$$

$$\langle X, Y\rangle_t^{\mathbb{P}} = \int_0^t \Big(\sigma\rho + \int_{(0,1]} \gamma(\xi)\delta(\xi)\nu^p(s,\xi)\Big)ds \tag{2.75}$$

where ν^p denotes the $\mathbb{P}$-compensator of μ. Using this result in (2.71) reveals that

$$\phi_t^{RB} - \phi_t^{VM} \approx \int_0^t \int_{(0,1]} \gamma(\xi)\delta(\xi)(\nu^p(s,\xi) - \nu^q(s,\xi))\Big)ds \tag{2.76}$$

Hence, the difference between the two strategies boils down to the difference between the $\mathbb{Q}$ and $\mathbb{P}$-compensators of μ. Here, remember that, in our two-factor affine frame work we implicitly assume that the $\mathbb{Q}$ and $\mathbb{P}$-compensator of μ are identical, that is the market price of jump risk is assumed to be zero. Thus, any difference between the two strategies is due to the approximation arguments.

2.4.4 Hedging Algorithm

Given the attachment and detachment point x_1 and x_2, we first construct a STCDO of 5-year maturity with quarterly payments. Then, the idea is to hedge this STCDO by constructing a self-financing portfolio which consists of the index and the risk-free account.

Having the parameter set and filtered factor series, we compute the par swap rate for each day of the hedging period via formula (2.54). After this preliminary work, at time t_0 we calculate the hedging strategy ϕ_{t_0} according to the hedging methodology, i.e., variance minimizing or regression based hedging, we choose. Following that, we construct the zero initial value, self financing portfolio V as follows

$$V_{t_0} = \phi_{t_0}\Gamma_{t_0}^{(0,1]} + (\Gamma_{t_0}^{(x_1,x_2]} - \phi_{t_0}\Gamma_{t_0}^{(0,1]}) \tag{2.77}$$

where the first term indicates the amount which is invested to the index and the second term with the brackets denotes the amount of borrowing from the risk-free account.

In each point t_k, $k = 1, ..., K$, of the hedging period, having the par swap rate $\kappa_{t_k}^{(x_1,x_2]}$ we utilize formula (2.55) to get the new spot value of the STCDO and the index. In the sequel, we evaluate the profit and loss (P&L) from time t_{k-1} zero-net investment in index. This is equal to the $\phi_{t_{k-1}}$ fraction of sum of change in the nominal spot value of the index, coupon payment if due and default payment if due. That is,

$$\begin{aligned} P\&L_{t_k}^{(0,1]} = \phi_{t_{k-1}} \Big(& \Gamma_{t_k}^{(0,1]} + 1_{\{t_{k-1} \in CP\}} \kappa_{t_0}^{(0,1]} H^{(0,1]}(L_{t_{k-1}}) \\ & - (H^{(0,1]}(L_{t_k}) - H^{(0,1]}(L_{t_{k-1}})) - \Gamma_{t_{k-1}}^{(0,1]} e^{r\Delta t_k} \Big) \end{aligned} \tag{2.78}$$

where CP indicates the set of predetermined coupon payment dates and $\Delta t_k = t_k - t_{k-1}$. From equation (2.78), one can immediately get the time t_k hedging portfolio value V_{t_k} as the sum $V_{t_{k-1}} e^{r\Delta t_k} + P\&L_{t_k}^{(0,1]}$. Moreover, we get the nominal value of the gains process $e^{rt_k} G_{t_k}^{(0,1]}$ as the sum of compounded gains value from the previous date and current P&L value. In a similar way, we compute the P&L and gains process value for STCDO position. Here, we want to point out that in the $0 - 3\%$ tranche, where there is the presence of the upfront payment, the analysis is same except that the par swap rate is fixed to 5% and the nominal spot value of STCDO is taken to be equal to the negative of the upfront payment.

Both for the variance minimizing and regression based hedging, we repeat the explained procedure for each day in the sample period and report the normalized series for the hedging portfolio value, gains process of the index and sum of these two as the total portfolio P&L processes. Furthermore, we provide the series for the hedging strategy ϕ.

2.4.5 Assessment of Hedging Performance

To be able to conclude that a particular hedging strategy outperforms one another we need some criteria. Regarding this, Cont and Kan [2011] presents two different criteria one of which is the *relative hedging error* and given by the absolute value of the ratio of average daily P&L of hedge position to average daily P&L of unhedged position. In our notation, this corresponds to percentage value

$$100 \times \frac{\left(V_K + e^{rK} G_K^{(x_1,x_2]} \right)}{e^{rK} G_K^{(x_1,x_2]}} \tag{2.79}$$

The other criterion is the *reduction in volatility* and measures the reduction in the dispersion of the P&L distribution with respect to the unhedged position. Formally, one can define reduction in volatility measure as the ratio of the daily P&L volatility of hedged position to the daily P&L volatility of unhedged tranche position. According to above mentioned criteria, a hedging strategy performs better as long as the related relative hedging error and reduction in volatility values are smaller.

The first criterion that we considered for the evaluation of the performance of a hedging strategy is the reduction in volatility measure. However, instead of using the relative hedging error as the second criterion, assuming the total outstanding value equal to 100, we use the normalized total portfolio P&L series with the term

$$\frac{\left(V_{t_k} + e^{rt_k} G_{t_k}^{(x_1,x_2]}\right) \times 100}{x_2 - x_1} \tag{2.80}$$

When $G_{t_k}^{(x_1,x_2]}$ gets too small, the relative hedging error in (2.79) gets too large and may yield misleading results. The other fact about this measure is that, it strongly depends on the final date K. That is, while for some date K in the sample period the relative hedging error can get very small and implies a successful hedge, for some other day the result is observed to be the opposite. This is why instead of relative hedging error we propose the whole total portfolio P&L series as one of the assessment criterion. We conclude that a hedging strategy performs better as the P&L series stays closer to zero.

2.5 Estimation Methodology

Given the model and the data we focus on two issues. The first one is estimating model parameters and the second one is testing the performance of the model via hedging analysis. The first issue comprises fitting of the model to the available market data. Recall that in the current framework the fundamental object having been modeled is the hypothetical term-structure of (T,x)-bonds which is not a market observable data. However, given the market observable par coupon rates $\kappa_t^{(x_1,x_2]}$ for all tranches and the index, the term-structure of (T,x)-bonds can be estimated via inverting the formula (2.54). To be more precise, one can first estimate the zero-coupon discount curve

$$\tau \mapsto D(t,\tau,j) = \frac{1}{x_{j+1} - x_j} \int_{x_j}^{x_{j+1}} P(t, t+\tau, x)dx \tag{2.81}$$

for all tranches $(x_j, x_{j+1}]$. This in turn gives the implied zero-coupon spread curve

$$R(t,\tau,j) = -\frac{1}{\tau} \log D(t,\tau,j) - r \tag{2.82}$$

Finally one can get the term structure of (T,x)-bonds via interpolating (2.82) in x. In this study, to estimate the model parameters we mainly use the zero-coupon spread curve data (2.82) as the input.

In the estimation of parameters, the main difficulty stems from the unobservability of the factor process. A natural approach to overcome this problem is using filtering. In a filtering problem, the aim is to estimate a stochastic process representing the unobserved factor by using the noisy past and present observations. In a Gaussian framework, where the unobserved factor is a Gaussian process, Kalman filter yields the exact likelihood function via providing the prediction error and its variance (see Harvey [1990]). When using non-Gaussian models, however, the exact likelihood function is not available in most cases. For such cases, one can use quasi-maximum likelihood estimator (QML) approach in which the idea is to substitute the exact transition density of the non-Gaussian factor by a normal density with mean and variance being equal to the first two true moments of the factor process. This has been a popular method especially in the estimation of affine term structure models and used in series of papers. For application of the method on CIR models see Geyer and Pichler [1999], Chen and Scott [2003] and Duffee and Stanton [2004]. Both in one and two-factor models presented above, the factor process is non-Gaussian. Thus, to estimate model parameters and obtain the unobservable factor we use a QML approach based on the linear Kalman filter. Since the one factor model is nested within two-factor model, in what follows we will only give the estimation procedure for the later one.

Let $(Y_{t_k}, Z_{t_k}) \in \mathbb{R}^2$, be the value of the factor process at time t_k, $\{0 = t_0 < t_1 < ... < t_K = T\}$. In Kalman filtering, there is the **measurement (observation) equation** expressing the observed data as a linear function of the unobservable factor plus a measurement error. The discrete time evolution of the unobservable factor is, in turn, expressed by the **transition equation** as linear in $(Y_{t_{k-1}}, Z_{t_{k-1}})$. Inserting (2.39) into (2.81) reveals that $R(t,\tau,i)$ is not linear in the factor (Y_t, Z_t) as desired. Recall that we specify the function β_z as $\beta_z = c_0 1_{[0,1)}(x)$. Together with this, to be able to get a linear measurement equation, we approximate $\beta_y(x)$ appearing in (2.44) by a piecewise constant function where the values are given by averaging (2.44). That is,

$$\beta_y(x) \approx \sum_{j=1}^{6} \beta_j 1_{[x_{j-1},x_j)}(x) \tag{2.83}$$

where

$$\beta_j = \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} \beta_y(x)dx \tag{2.84}$$

This will yield the desired linear measurement equation.

Remark 2.5 *Setting $\beta_z(x) = c_0 1_{[0,1)}(x)$ and approximating $\beta_y(x)$ as in (2.83) implies that $B_z(\tau,x)$ is piecewise constant i.e., $B_z(\tau,x) = B_z(\tau,x_j)$ for $x \in [x_{j-1},x_j)$. Furthermore, recall that in the two factor model we have*

$$A(\tau,x) = \alpha(x)\tau + \kappa_z\theta_z \int_0^\tau B_z(s,x)ds$$

Thus, one can also write

$$\int_{x_{j-1}}^{x_j} e^{-A(\tau,x)}dx = e^{-\kappa_z\theta_z \int_0^\tau B_z(s,x_j)ds} \int_{x_{j-1}}^{x_j} e^{-\alpha(x)\tau}dx$$

Remark 2.6 *In (2.82), specifying α as in (2.43), taking $\beta_z = c_0 1_{[0,1)}(x)$ and approximating β_y as in (2.83) results in the cancelation of the risk-free rate r in the following way*

$$\begin{aligned}
R(t,\tau,j) &= \frac{1}{\tau}\log(x_j - x_{j-1}) - \frac{1}{\tau}\log\left(e^{-\kappa_z\theta_z \int_0^\tau B_z(s,x_j)ds} \int_{x_{j-1}}^{x_j} e^{-\gamma(e^{-a_0(x\wedge 1)}-e^{-a_0})\tau - r\tau}dx \right) \\
&\qquad + \frac{1}{\tau}B_y(\tau,x_j)Y_t + \frac{1}{\tau}B_z(\tau,x_j)Z_t - r + \epsilon(t,\tau,j) \\
&= \underbrace{\frac{1}{\tau}\log(x_j - x_{j-1}) + \frac{1}{\tau}\kappa_z\theta_z \int_0^\tau B_z(s,x_j)ds - \frac{1}{\tau}\log\left(\int_{x_{j-1}}^{x_j} e^{-\gamma(e^{-a_0(x\wedge 1)}-e^{-a_0})\tau}dx \right)}_{C_z(\tau,x_j)} \\
&\qquad + \frac{1}{\tau}B_y(\tau,x_j)Y_t + \frac{1}{\tau}B_z(\tau,x_j)Z_t + \epsilon(t,\tau,i)
\end{aligned} \tag{2.85}$$

This suggests that the risk-free rate r is not needed during the estimation .

After doing the necessary approximations, we get the linear measurement equation given by

$$R(t_k,\tau,j) = C_z(\tau,x_j) + \frac{1}{\tau}(B_y(\tau,x_j)Y_{t_k} + B_z(\tau,x_j)Z_{t_k}) + \epsilon(t_k,\tau,j) \tag{2.86}$$

where C_z is given in (2.85). Here, measurement errors $\epsilon(t_k,\tau,j)$ are assumed to be independent and $\epsilon(t_k,\tau,j) \sim N(0,h_j)$, that is the variance of the error depends on the tranche j only. This yields diagonal covariance matrix, say H, for errors which has entries h_j as the j^{th} element of the diagonal.

Let $\mathbb{P}(Y_{t_k}, Z_{t_k}|Y_{t_{k-1}}, Z_{t_{k-1}})$ denotes the transition density, which is the probability density of the factor at time t_k given its value at time t_{k-1}. As mentioned above, in the current framework the transition density is non-Gaussian. Following the QML approach, we intend to substitute the exact transition density of the factor by a Normal density, i.e.,

$$\mathbb{P}(Y_{t_k}, Z_{t_k}|Y_{t_{k-1}}, Z_{t_{k-1}}) \sim N(\mu_{t_k}, Q_{t_k})$$

where the conditional mean μ_{t_k} and the covariance matrix Q_{t_k} are distributed in such a way that the first moments of the approximate Normal and exact transition density are equal. Here, it is necessary to compute μ_{t_k} and Q_{t_k} first. In the following we use the fact that the factor process is affine and then get the desired expressions via utilizing the polynomial property of moments for affine processes.

An affine process X with state space $\mathcal{X} \in \mathbb{R}^d$ has the property that, any conditional moment of the process, when exists, is given by a polynomial function of the current state. Now in the following we will explain how the computation of moments works for the affine process $X := (Y, Z)$, for the general case we refer to Duffie et al. [2000]. Let us first denote the conditional characteristic function of X_T given X_t by

$$M_X(t, T, X_t, r) := E[e^{\langle ir, X_T \rangle} | X_t] \tag{2.87}$$

$T \geq t \geq 0$, $r \in \mathbb{R}^2$ and where $i = \sqrt{-1}$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product. From the very definition of an affine process (see Definition 2.2) we have

$$M_X(T-t, X_t, r) = e^{\phi(T-t, ir) + \langle \psi(T-t, ir), X_t \rangle} \tag{2.88}$$

where ϕ and $\psi = (\psi_y, \psi_z)$ solve the system of Riccati equations (see Theorem 10.1 in Filipović [2009])

$$\begin{aligned}
\partial_t \phi(t, ir) &= \kappa_z \theta_z \psi_z \\
\phi(0, ir) &= 0 \\
\partial_t \psi_y(t, ir) &= \frac{1}{2}\psi_y^2 - \kappa_y \psi_y \\
\partial_t \psi_z(t, ir) &= \frac{1}{2}\psi_z^2 + \kappa_y \psi_y - \kappa_z \psi_z \\
\psi(0, ir) &= ir
\end{aligned} \tag{2.89}$$

One can solve the above system explicitly and show that at point zero the functions ϕ, ψ_y and ψ_z have partial derivatives of all orders with respect to $r = (r_1, r_2)$. This implies that the conditional k^{th} cross-moments given by

$$f_k(T-t, y, z) = E[Y_T^p Z_T^q | Y_t = y, Z_t = z] = i^{-k} \left[\frac{\partial^k M_X}{\partial^p r_1 \partial^q r_2} \right]_{r=0}, \quad p, q \in \mathbb{Z}^+, \; p+q = k$$

exists for all $k \in \mathbb{Z}^+$. The exponential structure of the characteristic function and the fact

$$M_X(T-t, X_t, 0) = 1, \quad \text{for all } t \leq T, \; X_t \in \mathcal{X},$$

imply that the conditional k^{th} moment is a polynomial of order less than or equal to k of the current state (y, z).

On the other hand, being an affine diffusion, $(Y_t, Z_t)_{t \geq 0}$ possesses the Markov property (see, for instance, Chp. III in Revuz and Yor [1999] for detailed information on Markov processes). In particular $f_k(T-t,y,z)$ solves formally the Kolmogorov backward equation

$$\begin{aligned}\frac{\partial}{\partial\tau} f_k(\tau,y,z) &= \mathcal{L} f_k(\tau,y,z)\\ f_k(0,y,z) &= y^p z^q\end{aligned} \tag{2.90}$$

where $\mathcal{L}$ denotes the infinitesimal generator of the process $(Y_t, Z_t)_{t\geq 0}$ and it is given by

$$\mathcal{L} = \kappa_y(z-y)\frac{\partial}{\partial y} + \kappa_z(\theta_z - z)\frac{\partial}{\partial z} + \frac{1}{2}\sigma_y^2 y\frac{\partial^2}{\partial y^2} + \frac{1}{2}\sigma_z^2 z\frac{\partial^2}{\partial z^2} \tag{2.91}$$

In equation (2.90) we use the ansatz method via inserting the polynomial form of the moments to this equation and then matching the coefficients yields system of ordinary differential equations. Then solving these equations we get the coefficients of the polynomial moment. To prove that the result we obtain is actually the solution of (2.90), that is, the k^{th} conditional moment of the process (Y,Z), we use the fact that the function f_k satisfies a polynomial growth condition. For more formal statement we refer Appendix B.

Using the property of moments we set out above, next proposition gives explicit formulae for the conditional mean, variance and covariance of X_t.

Proposition 2.2 *Given the dynamics in* (2.34)-(2.35)*, the* $\mathbb{P}$*-conditional expectation of* Y_t *and* Z_t *is in the following form*

$$\begin{aligned}E[Y_t|Y_0=y,Z_0=z] &= \frac{\theta_z}{\kappa_z-\kappa_y}\Big(\kappa_z(1-e^{-\kappa_y t}) - \kappa_y(1-e^{-\kappa_z t})\Big) + e^{-\kappa_y t}y\\ &\quad + e^{-\kappa_z t}\frac{\kappa_y}{\kappa_z-\kappa_y}\Big(e^{t(\kappa_z-\kappa_y)}-1\Big)z\end{aligned} \tag{2.92}$$

$$E[Z_t|Y_0=y,Z_0=z] = \theta_z(1-e^{-\kappa_z t}) + e^{-\kappa_z t}z \tag{2.93}$$

Moreover, the conditional variances V_y, V_z and the conditional covariance V_{yz} given by

$$V_y(t,y,z) = \Bigg(e^{-(5\kappa_z+7\kappa_y)t}\Big(e^{(5\kappa_z+7\kappa_y)t}(\kappa_z-2\kappa_y)(\kappa_z-\kappa_y)^2(\kappa_z(\kappa_z+\kappa_y)\sigma_y^2+\kappa_y^2\sigma_z^2)\theta_z$$
$$-2e^{(5\kappa_z+6\kappa_y)t}\kappa_z(\kappa_z-2\kappa_y)(\kappa_z^2-\kappa_y^2)\sigma_y^2(\kappa_z(\theta_z-y)+\kappa_y(y-z))+e^{(3\kappa_z+7\kappa_y)t}$$
$$\times(\kappa_z-2\kappa_y)\kappa_y^3(\kappa_z+\kappa_y)\sigma_z^2(\theta_z-2z)+2e^{(4\kappa_z+7\kappa_y)t}\kappa_y^2(\kappa_y^2-\kappa_z^2)(\kappa_z(\sigma_y^2-2\sigma_z^2)$$
$$+2\kappa_y\sigma_z^2)(\theta_z-z)-4e^{(4\kappa_z+6\kappa_y)t}\kappa_z(\kappa_z-2\kappa_y)\kappa_y^2\sigma_z^2(\kappa_z\theta_z-(\kappa_z+\kappa_y)z)$$
$$+e^{5(\kappa_z+\kappa_y)t}\kappa_z(\kappa_z+\kappa_y)(\kappa_z^3\sigma_y^2(\theta_z-2y)-2\kappa_z^2\kappa_y\sigma_y^2(\theta_z-4y+z)$$
$$+2\kappa_y^3(2\sigma_y^2y-\sigma_y^2z+\sigma_z^2z)+\kappa_z\kappa_y^2(-\sigma_z^2\theta_z+\sigma_y^2(\theta_z-10y+4z)))\Big)\Bigg)$$
$$\Bigg/\Bigg(2\kappa_z(\kappa_z-2\kappa_y)(\kappa_z-\kappa_y)^2\kappa_y(\kappa_z+\kappa_y)\Bigg) \tag{2.94}$$

$$V_z(t,y,z) = \frac{\sigma_z^2e^{-2\kappa_zt}(e^{\kappa_zt}-1)((e^{\kappa_zt}-1)\theta_z+2z)}{2\kappa_z} \tag{2.95}$$

$$V_{yz}(t,y,z) = \frac{e^{-(2\kappa_z+\kappa_y)t}\sigma_z^2}{2(\kappa_z^3-\kappa_z\kappa_y^2)}\Big(e^{(2\kappa_z+\kappa_y)t}(\kappa_z-\kappa_y)\kappa_y\theta_z-e^{\kappa_yt}\kappa_y(\kappa_z+\kappa_y)(\theta-2z)$$
$$-2e^{(\kappa_z+\kappa_y)t}(\kappa_z^2-\kappa_y^2)(\theta_z-z)+2e^{\kappa_zt}\kappa_z(\kappa_z\theta_z-(\kappa_z+\kappa_y)z)\Big) \tag{2.96}$$

Proof. (i) $E[Z_t|Y_0=y,Z_0=z]=g(t,y,z)$. Function g satisfies the Kolmogorov backward equation, that is,

$$\partial_tg=\kappa_y(z-y)\partial_yg+\kappa_z(\theta_z-z)\partial_zg+\frac{1}{2}\sigma_y^2y\partial_{yy}g+\frac{1}{2}\sigma_z^2z\partial_{zz}g \tag{2.97}$$

Since $(Y_t,Z_t)_{t\geq0}$ is an affine process, we have the polynomial property of moments, that is, g is of the following form

$$g(t,y,z)=g_0(t)+g_y(t)y+g_z(t)z \tag{2.98}$$

for some functions g_0 ,g_y, g_z. Plugging (2.98) in (2.97) gives

$$\frac{d}{dt}g_0+\frac{d}{dt}g_yy+\frac{d}{dt}g_zz=\kappa_y(z-y)g_y+\kappa_z(\theta_z-z)g_z$$

Comparing the coefficients on the right and left hand side, we get the following system of equations.

$$\begin{aligned}
\frac{d}{dt}g_0 &= \kappa_z\theta_z g_z \\
g_0(0) &= 0 \\
\frac{d}{dt}g_y &= -\kappa_y g_y \\
g_y(0) &= 0 \\
\frac{d}{dt}g_z &= \kappa_y g_y - \kappa_z g_z \\
g_z(0) &= 1
\end{aligned}$$

Solving above system we get $g_z(t) = e^{-\kappa_z t}$, $g_y(t) \equiv 0$ and $g_0(t) = \theta_z(1 - e^{-\kappa_z t})$ implying that

$$E[Z_t|Y_0 = y, Z_0 = z] = \theta_z(1 - e^{-\kappa_z t}) + e^{-\kappa_z t}z \tag{2.99}$$

(ii) We set $E[Y_t|Y_0 = y, Z_0 = z] = h(t, y, z)$. Following similar arguments we have

$$\partial_t h = \kappa_y(z - y)\partial_y h + \kappa_z(\theta_z - z)\partial_z h + \frac{1}{2}\sigma_y^2 y\partial_{yy}h + \frac{1}{2}\sigma_z^2 z\partial_{zz}h \tag{2.100}$$

From the polynomial property of moments again we have

$$h(t, y, z) = h_0(t) + h_y(t)y + h_z(t)z \tag{2.101}$$

Plugging (2.101) in (2.100) gives

$$\frac{d}{dt}h_0 + \frac{d}{dt}h_y y + \frac{d}{dt}h_z z = \kappa_y(z - y)h_y + \kappa_z(\theta_z - z)h_z$$

Matching the coefficients we get

$$\begin{aligned}
\frac{d}{dt}h_0 &= \kappa_z\theta_z h_z \\
h_0(0) &= 0 \\
\frac{d}{dt}h_y &= -\kappa_y h_y \\
h_y(0) &= 1 \\
\frac{d}{dt}h_z &= \kappa_y h_y - \kappa_z h_z \\
h_z(0) &= 0
\end{aligned}$$

Solving the system, we get

$$\begin{aligned}
h_0(t) &= \frac{\theta_z}{\kappa_z - \kappa_y}(\kappa_z(1 - e^{-\kappa_y t}) - \kappa_y(1 - e^{-\kappa_z t})) \\
h_y(t) &= e^{-\kappa_y t}, \quad h_z(t) = e^{-\kappa_z t}\frac{\kappa_y}{\kappa_z - \kappa_y}(e^{t(\kappa_z - \kappa_y)} - 1)
\end{aligned}$$

implying that

$$
\begin{aligned}
E[Y_t|Y_0 = y, Z_0 = z] = &\frac{\theta_z}{\kappa_z - \kappa_y}(\kappa_z(1-e^{-\kappa_y t}) - \kappa_y(1-e^{-\kappa_z t})) + e^{-\kappa_y t}y \\
&+ e^{-\kappa_z t}\frac{\kappa_y}{\kappa_z - \kappa_y}(e^{t(\kappa_z-\kappa_y)} - 1)z
\end{aligned}
\tag{2.102}
$$

(iii) $E[Y_t Z_t|Y_0 = y, Z_0 = z] = f(t, y, z)$ and f satisfies

$$
\partial_t f = \kappa_y(z-y)\partial_y f + \kappa_z(\theta_z - z)\partial_z f + \frac{1}{2}\sigma_y^2 y\partial_{yy} f + \frac{1}{2}\sigma_z^2 z\partial_{zz} f \tag{2.103}
$$

Following exactly the same procedure as above we get

$$
f(t, y, z) = f_0(t) + f_y(t)y + f_z(t)z + f_{z^2}(t)z^2 + f_{zy}(t)zy + f_{y^2}(t)y^2 \tag{2.104}
$$

Plugging (2.104) in (2.103) gives

$$
\begin{aligned}
\frac{d}{dt}f_0 + \frac{d}{dt}f_y y + \frac{d}{dt}f_z z + \frac{d}{dt}f_{z^2}z^2 + \frac{d}{dt}f_{zy}zy + \frac{d}{dt}f_{y^2}y^2 = &\kappa_y(z-y)(f_y + 2f_{y^2}y + f_{zy}z) \\
&+ \kappa_z(\theta_z - z)(f_z + f_{zy}y + 2f_{z^2}z) \\
&+ \sigma_y^2 y f_{y^2} + \sigma_z^2 z f_{z^2}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{d}{dt}f_0 &= \kappa_z\theta_z f_z \\
\frac{d}{dt}f_y &= -\kappa_y f_y + \kappa_z\theta_z f_{zy} + \sigma_y^2 f_{y^2} \\
\frac{d}{dt}f_z &= \kappa_y f_y - \kappa_z f_z + (2\kappa_z\theta_z + \sigma_z^2)f_{z^2} \\
\frac{d}{dt}f_{z^2} &= \kappa_y f_{zy} - 2\kappa_z f_{z^2} \\
\frac{d}{dt}f_{y^2} &= -2\kappa_y f_{y^2} \\
\frac{d}{dt}f_{zy} &= 2\kappa_y f_{y^2} - (\kappa_y + \kappa_z)f_{zy}
\end{aligned}
$$

with

$$
f_0(0) = f_z(0) = f_y(0) = f_{z^2}(0) = f_{y^2}(0) = 0, \quad f_{zy}(0) = 1
$$

Solving the above system yields

$$
\begin{aligned}
f_0(t) = {} & \frac{e^{-(2\kappa_z+\kappa_y)t}\theta_z}{2(\kappa_z\kappa_y^2-\kappa_z^3)}\Big(2e^{2\kappa_z t}(\kappa_z+\kappa_y)\kappa_z^2\theta_z + e^{\kappa_y t}(\kappa_z+\kappa_y)\kappa_y(\sigma_z^2+2\kappa_z\theta_z) \\
& - 2e^{\kappa_z t}\kappa_z^2(\sigma_z^2+(\kappa_z+\kappa_y)\theta_z) - e^{(2\kappa_z+\kappa_y)t}(\kappa_z-\kappa_y)(2\kappa_z^2\theta_z+\kappa_y(\sigma_z^2+2\kappa_z\theta_z)) \\
& + 2e^{(\kappa_z+\kappa_y)t}(\kappa_z+\kappa_y)(-\kappa_y\sigma_z^2+\kappa_z^2\theta_z+\kappa_z(\sigma_z^2-2\kappa_y\theta_z))\Big),
\end{aligned}
$$

$$f_y(t) = \theta_z(e^{-\kappa_y t} - e^{-(\kappa_z+\kappa_y)t}),$$

$$
\begin{aligned}
f_z(t) = {} & \frac{e^{-(2\kappa_z+\kappa_y)t}}{\kappa_z(\kappa_z-\kappa_y)}\Big(e^{2\kappa_z t}\kappa_z\kappa_y\theta_z + e^{\kappa_y t}\kappa_y(\sigma_z^2+2\kappa_z\theta_z) - e^{\kappa_z t}\kappa_z(\sigma_z^2+(\kappa_z+\kappa_y)\theta_z) \\
& + e^{(\kappa_z+\kappa_y)t}(-\kappa_y\sigma_z^2+\kappa_z^2\theta_z+\kappa_z(\sigma_z^2-2\kappa_y\theta_z))\Big),
\end{aligned}
$$

$$f_{y^2} \equiv 0,\ \ f_{zy} = e^{-(\kappa_z+\kappa_y)t},\ \ f_{z^2}(t) = \frac{\kappa_y}{\kappa_z-\kappa_y}(e^{-(\kappa_z+\kappa_y)t} - e^{-2\kappa_z t}).$$

Inserting these expressions to (2.104) we get $E[Y_tZ_t|Y_0=y,Z_0=z]$.

(iv)$E[Z_t^2|Y_0=y,Z_0=z] = q(t,y,z)$ and q satisfies

$$\partial_t q = \kappa_y(z-y)\partial_y q + \kappa_z(\theta_z - z)\partial_z q + \frac{1}{2}\sigma_y^2 y\partial_{yy}q + \frac{1}{2}\sigma_z^2 z\partial_{zz}q \tag{2.105}$$

Also, q satisfies

$$q(t,y,z) = q_0(t) + q_y(t)y + q_z(t)z + q_{z^2}(t)z^2 + q_{zy}(t)zy + q_{y^2}(t)y^2 \tag{2.106}$$

Inserting (2.106) in (2.105) gives

$$
\begin{aligned}
\frac{d}{dt}q_0 + \frac{d}{dt}q_y y + \frac{d}{dt}q_z z + \frac{d}{dt}q_{z^2}z^2 + \frac{d}{dt}q_{zy}zy + \frac{d}{dt}q_{y^2}y^2 = {} & \kappa_y(z-y)(q_y + 2q_{y^2}y + q_{zy}z) \\
& + \kappa_z(\theta_z - z)(q_z + q_{zy}y + 2q_{z^2}z) \\
& + \sigma_y^2 y q_{y^2} + \sigma_z^2 z q_{z^2}
\end{aligned}
$$

which yields the system

$$\begin{aligned}
\frac{d}{dt}q_0 &= \kappa_z\theta_z q_z \\
\frac{d}{dt}q_y &= -\kappa_y q_y + \kappa_z\theta_z q_{zy} + \sigma_y^2 q_{y^2} \\
\frac{d}{dt}q_z &= \kappa_y q_y - \kappa_z q_z + (2\kappa_z\theta_z + \sigma_z^2) q_{z^2} \\
\frac{d}{dt}q_{z^2} &= \kappa_y q_{zy} - 2\kappa_z q_{z^2} \\
\frac{d}{dt}q_{y^2} &= -2\kappa_y q_{y^2} \\
\frac{d}{dt}q_{zy} &= 2\kappa_y q_{y^2} - (\kappa_y + \kappa_z) q_{zy}
\end{aligned}$$

with

$$q_0(0) = q_z(0) = q_y(0) = q_{y^2}(0) = q_{zy}(0) = 0, \quad q_{z^2}(0) = 1$$

We solve this system of equations and get

$$q_0(t) = \frac{e^{-2\kappa_z t}(e^{\kappa_z t} - 1)^2 \theta_z (\sigma_z^2 + 2\kappa_z\theta_z)}{2\kappa_z},$$

$$q_z(t) = \frac{e^{-2\kappa_z t}(e^{\kappa_z t} - 1)(\sigma_z^2 + 2\kappa_z\theta_z)}{\kappa_z},$$

$$q_{z^2}(t) = e^{-2\kappa_z t},$$

$$q_y(t) \equiv q_{y^2}(t) \equiv q_{zy}(t) \equiv 0.$$

Inserting above expressions to (2.106) yields the desired result.

(v) We set $E[Y_t^2 | Y_0 = y, Z_0 = z] = p(t, y, z)$. p satisfies the Kolmogorov's backward equation, that is,

$$\partial_t p = \kappa_y(z - y)\partial_y p + \kappa_z(\theta_z - z)\partial_z p + \frac{1}{2}\sigma_y^2 y \partial_{yy} p + \frac{1}{2}\sigma_z^2 z \partial_{zz} p \tag{2.107}$$

From the polynomial property of moments we have

$$p(t, y, z) = p_0(t) + p_y(t)y + p_z(t)z + p_{z^2}(t)z^2 + p_{zy}(t)zy + p_{y^2}(t)y^2 \tag{2.108}$$

Plugging (2.108) in (2.107) gives

$$\begin{aligned}
\frac{d}{dt}p_0 + \frac{d}{dt}p_y y + \frac{d}{dt}p_z z + \frac{d}{dt}p_{z^2}z^2 + \frac{d}{dt}p_{zy}zy + \frac{d}{dt}p_{y^2}y^2 &= \kappa_y(z-y)(p_y + 2p_{y^2}y + p_{zy}z) \\
&+ \kappa_z(\theta_z - z)(p_z + p_{zy}y + 2p_{z^2}z) \\
&+ \sigma_y^2 y p_{y^2} + \sigma_z^2 z p_{z^2}
\end{aligned}$$

which yields the following system of differential equations

$$\begin{aligned}
\frac{d}{dt}p_0 &= \kappa_z\theta_z p_z \\
\frac{d}{dt}p_y &= -\kappa_y p_y + \kappa_z\theta_z p_{zy} + \sigma_y^2 p_{y^2} \\
\frac{d}{dt}p_z &= \kappa_y p_y - \kappa_z p_z + (2\kappa_z\theta_z + \sigma_z^2)p_{z^2} \\
\frac{d}{dt}p_{z^2} &= \kappa_y p_{zy} - 2\kappa_z p_{z^2} \\
\frac{d}{dt}p_{y^2} &= -2\kappa_y p_{y^2} \\
\frac{d}{dt}p_{zy} &= 2\kappa_y p_{y^2} - (\kappa_y + \kappa_z)p_{zy}
\end{aligned}$$

with

$$p_0(0) = p_z(0) = p_y(0) = p_{z^2}(0) = p_{zy}(0) = 0, \quad p_{y^2}(0) = 1$$

Solving this system of linear ODEs yields

$$\begin{aligned}
p_0(t) = \frac{e^{-(3\kappa_z+\kappa_y)t}\theta_z}{2\kappa_z(\kappa_z - 2\kappa_y)(\kappa_z-\kappa_y)^2\kappa_y(\kappa_z+\kappa_y)} \Big(& e^{(\kappa_z+\kappa_y)t}(\kappa_z - 2\kappa_y)\kappa_y^3(\kappa_z+\kappa_y) \\
&\times (\sigma_z^2 + 2\kappa_z\theta_z) - 2e^{3\kappa_z t}\kappa_z^2(\kappa_z - 2\kappa_y)(\kappa_z^2 - \kappa_y^2)(\sigma_y^2 + 2\kappa_y\theta_z) - 4e^{2\kappa_z t}\kappa_z^2 \\
&\times (\kappa_z - 2\kappa_y)\kappa_y^2(\sigma_z^2 + (\kappa_z+\kappa_y)\theta_z) + e^{(3\kappa_z+\kappa_y)t}(\kappa_z - 2\kappa_y)(\kappa_z-\kappa_y)^2 \\
&\times (\kappa_z^2\sigma_y^2 + \kappa_z\kappa_y\sigma_y^2 + \kappa_y^2\sigma_z^2 + 2\kappa_z\kappa_y(\kappa_z+\kappa_y)\theta_z) + e^{(3\kappa_z-\kappa_y)t}\kappa_z^2(\kappa_z+\kappa_y) \\
&\times (\kappa_y^2(\sigma_y^2 - \sigma_z^2) + \kappa_z^2(\sigma_y^2 + 2\kappa_y\theta_z) - 2\kappa_z\kappa_y(\sigma_y^2 + 2\kappa_y\theta_z)) + 2e^{(2\kappa_z+\kappa_y)t}\kappa_y^2 \\
&\times (\kappa_y^2 - \kappa_z^2)(2\kappa_y\sigma_z^2 - 2\kappa_z^2\theta_z + \kappa_z(\sigma_y^2 - 2\sigma_z^2 + 4\kappa_y\theta_z))\Big),
\end{aligned}$$

$$p_y(t) = \frac{e^{-2\kappa_y t}((1-e^{\kappa_y t})\kappa_z(\sigma_y^2 + 2\kappa_y\theta_z) + \kappa_y((e^{\kappa_y t}-1)\sigma_y^2 + 2(e^{\kappa_y t} - e^{(\kappa_y-\kappa_z)t})\kappa_y\theta_z))}{\kappa_y(\kappa_y-\kappa_z)},$$

$$
\begin{aligned}
p_z(t) = {} & \frac{e^{-(3\kappa_z+\kappa_y)t}}{\kappa_z(\kappa_z-2\kappa_y)(\kappa_z-\kappa_y)^2}\Big(- e^{(\kappa_z+\kappa_y)t}(\kappa_z-2\kappa_y)\kappa_y^2(\sigma_z^2+2\kappa_z\theta_z) \\
& + e^{3\kappa_z t}\kappa_z(\kappa_z-2\kappa_y)(\kappa_z-\kappa_y)(\sigma_y^2+2\kappa_y\theta_z) + 2e^{2\kappa_z t}\kappa_((\kappa_z-2\kappa_y)\kappa_y \\
& \times(\sigma_z^2+(\kappa_z+\kappa_y)\theta_z) - e^{(3\kappa_z-\kappa_y)t}\kappa_z(\kappa_y^2(\sigma_y^2-\sigma_z^2)+\kappa_z^2(\sigma_y^2+2\kappa_y\theta_z) \\
& - 2\kappa_z\kappa_y(\sigma_y^2+2\kappa_y\theta_z)) - e^{(2\kappa_z+\kappa_y)t}\kappa_y(\kappa_y-\kappa_z)(2\kappa_y\sigma_z^2-2\kappa_z^2\theta_z \\
& + \kappa_z(\sigma_y^2-2\sigma_z^2+4\kappa_y\theta_z))\Big),
\end{aligned}
$$

$$
p_{zy}(t) = \frac{2\kappa_y e^{-(\kappa_z+\kappa_y)t}(e^{(\kappa_z-\kappa_y)t}-1)}{\kappa_z-\kappa_y},
$$

$$
p_{y^2}(t) = e^{-2\kappa_y t},
$$

$$
p_{z^2}(t) = \frac{\kappa_y^2 e^{-2\kappa_z t}(e^{(\kappa_z-\kappa_y)t}-1)^2}{(\kappa_z-\kappa_y)^2}.
$$

□

Corollary 2.2 *Unconditional mean, variance and covariance of Y_t and Z_t is given by the following*

$$
\begin{aligned}
\mu_y^0 &= \theta_z, && (2.109)\\
\mu_z^0 &= \theta_z, && (2.110)
\end{aligned}
$$

$$
\begin{aligned}
V_y^0 &= \frac{\sigma_y^2\theta_z}{2\kappa_y} + \frac{\kappa_y\theta_z\sigma_z^2}{2(\kappa_z+\kappa_y)\kappa_z} = \frac{\sigma_y^2\theta_z}{2\kappa_y} + V_{yz}^0, && (2.111)\\
V_z^0 &= \frac{\sigma_z^2\theta_z}{2\kappa_z}, && (2.112)\\
V_{yz}^0 &= \frac{\sigma_z^2\theta_z\kappa_y}{2\kappa_z(\kappa_z+\kappa_y)} = \frac{\kappa_y}{(\kappa_z+\kappa_y)}V_z^0. && (2.113)
\end{aligned}
$$

Proof. Letting $t \to \infty$ in conditional moments given in Proposition 2.2 yields the desired result. □

After computing the conditional moments in Proposition (2.2) we are now ready to give the **transition equation** for the two-factor model:

$$
\begin{pmatrix} Y_{t_k|t_{k-1}} \\ Z_{t_k|t_{k-1}} \end{pmatrix} = M_0(t_k) + M_1(t_k)\begin{pmatrix} Y_{t_{k-1}|t_{k-1}} \\ Z_{t_{k-1}|t_{k-1}} \end{pmatrix} + v_{t_k}, \quad v_{t_k} \sim N(0, Q(t_k)) \qquad (2.114)
$$

where

$$M_0(t_k) = \begin{pmatrix} \frac{\theta_z}{\kappa_z - \kappa_y}(\kappa_z(1 - e^{-\kappa_y \Delta t}) - \kappa_y(1 - e^{-\kappa_z \Delta t})) \\ \theta_z(1 - e^{-\kappa_z \Delta t}) \end{pmatrix} \tag{2.115}$$

$$M_1(t_k) = \begin{pmatrix} e^{-\kappa_y \Delta t} & e^{-\kappa_z \Delta t} \frac{\kappa_y}{\kappa_z - \kappa_y}(e^{\Delta t(\kappa_z - \kappa_y)} - 1) \\ 0 & e^{-\kappa_z \Delta t} \end{pmatrix} \tag{2.116}$$

$$Q(t_k) = \begin{pmatrix} V_y(\Delta t, Y_{t_{k-1}|t_{k-1}}, Z_{t_{k-1}|t_{k-1}}) & V_{yz}(\Delta t, Y_{t_{k-1}|t_{k-1}}, Z_{t_{k-1}|t_{k-1}}) \\ V_{yz}(\Delta t, Y_{t_{k-1}|t_{k-1}}, Z_{t_{k-1}|t_{k-1}}) & V_z(\Delta t, , Y_{t_{k-1}|t_{k-1}} Z_{t_{k-1}|t_{k-1}}) \end{pmatrix} \tag{2.117}$$

with $\Delta t = t_k - t_{k-1}$ and $V_y(t, y, z)$, $V_z(t, y, z)$ and $V_{yz}(t, y, z)$ are as given in (2.94), (2.95) and (2.96) respectively.

Given the parameter set $\varphi = (\kappa_z, \kappa_y, \theta_z, \lambda^z, \lambda^y, \sigma_z, \sigma_y, a_0, \gamma, b_0, c_0, H)$, Kalman filter consists of prediction and updating steps which are applied for each time step in the data sample. We give the filtering algorithm for the two-factor model where we use the unconditional moments given in Corollary (2.2) as initial values of the filter:

Initialize:

$$\begin{pmatrix} Y_{0|0} \\ Z_{0|0} \end{pmatrix} = \begin{pmatrix} \theta_z \\ \theta_z \end{pmatrix}$$

$$P_{0|0} = \begin{pmatrix} \frac{\sigma_y^2 \theta_z}{2\kappa_y} + \frac{\kappa_y \theta_z \sigma_z^2}{2(\kappa_z + \kappa_y)\kappa_z} & \frac{\sigma_z^2 \theta_z \kappa_y}{2\kappa_z(\kappa_z + \kappa_y)} \\ \frac{\sigma_z^2 \theta_z \kappa_y}{2\kappa_z(\kappa_z + \kappa_y)} & \frac{\sigma_z^2 \theta_z}{2\kappa_z} \end{pmatrix}$$

Prediction:

$$\begin{pmatrix} Y_{t_k|t_{k-1}} \\ Z_{t_k|t_{k-1}} \end{pmatrix} = M_0(t_k) + M_1(t_k) \begin{pmatrix} Y_{t_{k-1}|t_{k-1}} \\ Z_{t_{k-1}|t_{k-1}} \end{pmatrix}$$

$$P_{t_k|t_{k-1}} = M_1(t_k) P_{t_{k-1}|t_{k-1}} M_1(t_k)^\top + Q(t_k)$$

(Conditional covariance matrix of (Y_{t_k}, Z_{t_k}))

(2.118)

$$R(\tau, i)_{t_k|t_{k-1}} = C_z(\tau, x) + \frac{1}{\tau}\left(B_y(\tau, x)Y_{t_k|t_{k-1}} + B_z(\tau, x)Z_{t_k|t_{k-1}}\right)$$

$$F_{t_k}z = BP_{t_k|t_{k-1}}B^\top + H$$
$$\left(\text{VC matrix of } R_{t_k|t_{k-1}} \text{ with } B(\tau, i) = \left(\frac{B_y(\tau, i)}{\tau}, \frac{B_z(\tau, i)}{\tau}\right)\right)$$

$$e_{t_k} = R_{t_k} - R_{t_k|t_{k-1}} \qquad (n \times n \text{ prediction error vector})$$

Updating:

$$\begin{aligned} K_{t_k} &= P_{t_k|t_{k-1}}B^\top F_{t_k}^{-1} \quad \text{(Kalman gain)} \\ Y_{t_k|t_k} &= Y_{t_k|t_{k-1}} + K_{t_k}e_{t_k} \quad \text{(updated state vector)} \\ P_{t_k|t_k} &= P_{t_k|t_{k-1}} - K_{t_k}BP_{t_k|t_{k-1}} \end{aligned}$$

For a non-Gaussian factor, Kalman filter provides an approximate likelihood function which is in the following form

$$\log L(R_1, R_2, ..., R_N; \varphi) = -\frac{K}{2}\log 2\pi - \frac{1}{2}\sum_{k=1}^{K}\log|F_{t_k}| - \frac{1}{2}\sum_{k=1}^{K} e_{t_k}^\top F_{t_k}^{-1}e_{t_k} \qquad (2.119)$$

Notice that L is a function of e_t and F_t which are, in turn, depend on the parameter set φ. Thus, as the final step of the QML method, we choose φ in such a way that the likelihood function is maximized.

Here we want to point out that the observed data vectors may change size over the sample period. This is due to the unavailability of the data for some tranches and/or maturities . To overcome this problem, we adjust the Kalman filter algorithm in such a way that it takes the size changes in the data into account.

2.6 Simulation Methodology

In the simulation analysis our objective is to elaborate more on the performance of the model in a more general framework where scenarios with nonzero losses are permitted. In this context, we do two different simulation analysis. Recall that, in our modeling setup we deal with three stochastic processes, namely factors Y, Z and the loss process L. In the first analysis we simulate trajectories for all three processes whereas in the second one, which we call conditional simulation, we simulate the loss process L conditional on the paths Y_t and Z_t that we filtered out via estimation procedure. In what follows we only discuss the first simulation methodology since the conditional simulation is a constraint version of the first one.

We use Euler discretization to approximate the discrete time evolution of the factors Y and Z in Equation (2.34)-(2.35) on $\{0 = t_0 < t_1 < ... < t_K = T\}$. For $k = 0, 1, ..., K-1$ we have

$$Y_{t_{k+1}} = Y_{t_k} + \kappa_y(Z_{t_k} - Y_{t_k})\Delta t + \sigma_y\sqrt{Y_{t_k}}\Delta W^y_{t_{k+1}},\ Y_0 = y \in \mathbb{R}^+ \tag{2.120}$$

$$Z_{t_{k+1}} = Z_{t_k} + \kappa_z(\theta_z - Z_{t_k})\Delta t + \sigma_z\sqrt{Z_{t_k}}\Delta W^z_{t_{k+1}},\ Z_0 = z \in \mathbb{R}^+ \tag{2.121}$$

where $\Delta t = t_{k+1} - t_k$ and $\Delta W^{y,z}_{t_{k+1}} = W^{y,z}_{t_{k+1}} - W^{y,z}_{t_k}$. Here $\Delta W^{y,z}_{t_{k+1}}$s are independent of each other and distributed $N(0, \sqrt{\Delta t})$. Using this fact, to simulate trajectories of length K for each of the processes Y and Z we first generate K numbers from the standard normal distribution and then scale these numbers with $\sqrt{\Delta t}$. Inserting $\Delta W^{y,z}_{t_{k+1}}$s and the estimated parameters in (2.120) and (2.121) then yields a trajectory of length K for the processes Y and Z respectively. Euler discretization methodology that we mentioned above may give negative numbers for the values Y_{t_k} and Z_{t_k}. To avoid the negative values, whenever realized we change these negative values by 10^{-8}.

As the next step, to simulate the loss process L we use the simulated factors Y, Z, parameter estimates and another parameter Ψ which we interpret as the importance sampling parameter. The reason why we need the parameter Ψ is as follows. As it is mentioned before, there does not occur any default during the sample period we use and so the parameter set coming from the in-sample analysis is not able to generate remarkable number of jumps. Moreover, Monte Carlo simulation is known to fail in generating rare events unless the number of simulated scenarios is very large. Nevertheless, a frequently used technique in stress scenario generation is importance sampling. In this context, it is possible to manipulate the number of jumps via amplifying jump intensity Λ_t given in (2.32) with the importance sampling parameter Ψ. However, one should take care of the necessary measure change for the adjustment of probabilities assigned to each scenario. In the following we sketch the algorithm for simulating a loss trajectory of length K.

1. Initiate the jump time $\tau = 0$, number of jumps $N = 0$ and the loss process $L_{t_0} = 0$.
2. Initiate the arrival intensity $\bar{\Lambda}_{t_k} = \sum_{n=0}^{k} \Lambda_{t_n}$ at $\bar{\Lambda}_{t_0} = 0$.
3. Generate a number U from exponential distribution with parameter 1.
4. Set $t_k = \tau$.

- While $\bar{\Lambda}_{t_k} - \bar{\Lambda}_{\tau} < U$ and $k < K$ calculate $\bar{\Lambda}_{t_{k+1}}$ via

$$\bar{\Lambda}_{t_{k+1}} = \bar{\Lambda}_{t_k} + \Psi(\alpha(L_{\tau}) + \beta_y(L_{\tau})Y_{t_k} + \beta_z(L_{\tau})Z_{t_k} - r)\Delta t$$

$$\text{set} \;\; L_{t_{k+1}} = L_{t_k}, \quad k \mapsto k+1$$

- If $\bar{\Lambda}_{t_k} - \bar{\Lambda}_{\tau} \geq U$, i.e., when a jump occurs generate a number S from the standard uniform distribution. S represents the probability of having the particular jump size $\Delta L_{t_k} = L_{t_k} - L_{t_{k-1}}$. Compute jump size via

$$\Delta L_{t_k} = F^{-1}(L_{t_k}, Y_{t_k}, Z_{t_k}, S)$$

where F is the jump size distribution given by

$$F(L_{t_k}, Y_{t_k}, Z_{t_k}, x) = \frac{\alpha(L_{t_k}) + \beta_y(L_{t_k})Y_{t_k} + \beta_z(L_{t_k})Z_{t_k} - \alpha(L_{t_k}+x)}{\alpha(L_{t_k}) + \beta_y(L_{t_k})Y_{t_k} + \beta_z(L_{t_k})Z_{t_k} - r}$$

$$- \frac{\beta_y(L_{t_k}+x)Y_{t_k} + \beta_z(L_{t_k}+x)Z_{t_k}}{\alpha(L_{t_k}) + \beta_y(L_{t_k})Y_{t_k} + \beta_z(L_{t_k})Z_{t_k} - r}$$

Then update the loss path and number of jumps

$$L_{t_k} = L_{t_k} + \Delta L_{t_k}, \quad N = N+1$$

5. Set $\tau = t_k$. If $\tau > T$ stop, else return to step 3.

Using the methodology described above, we simulate 2000 scenarios. 1000 of the scenarios are normal scenarios and generated via taking importance sampling parameter $\Psi = 1$. On the other hand we take $\Psi = 100$ to simulate 1000 of stress scenarios. Since 1000 normal scenarios are equal probable, we set probability of each equal to $q(i) = 1/1000$, $i = 1, 2, ..., 1000$. For the stress scenarios we first compute the likelihood ratio $w(i)$, $i = 1, 2, ..., 1000$ via

$$w(i) = \frac{e^{(\Psi-1)\sum_{k=0}^{K-1}(\alpha(L_{t_k}) + \beta_y(L_{t_k})Y_{t_k} + \beta_z(L_{t_k})Z_{t_k} - r)\Delta t}}{\Psi^N} \tag{2.122}$$

Then, the probability of each scenario is given by

$$p(i) = \frac{w(i)}{\sum_{i=1}^{1000} w(i)} \tag{2.123}$$

Finally, to compute the probability of each scenario among 2000 scenarios we use the fact that a normal scenario and a stress scenario are equally likely to be realized. Thus, we have $\bar{q}(i) = q(i)/2$ and $\bar{p}(i) = p(i)/2$, so that

$$\sum_{i=1}^{1000} \bar{q}(i) + \bar{p}(i) = 1$$

as it should be.

2.7 Numerical Analysis

2.7.1 Data

The raw data comprises daily observations of iTraxx Europe from 30 August 2006 to 3 August 2010. The stripped data, which has been sourced from Bank Austria is the zero-coupon spreads across maturities/tranches, that is

$$R(t, \tau, j) = -\frac{1}{\tau} \log D(t, \tau, j) - r \tag{2.124}$$

where D is as given in (2.81) for four different time to maturities $\tau := T - t = 3, 5, 7, 10$ and six tranches $j = 1, ..., 6$ with standard attachment and detachment points $0\%, 3\%, 6\%, 9\%, 12\%, 22\%, 100\%$. This corresponds to 972 observation days in each of which we have a 6×4 observation matrix.

We illustrate the time series of zero-coupon spreads in Figure 2.1. Naturally, the market conditions are reflected in the data set. The index and tranche data follow relatively stable pattern from the beginning of the data period to July 2007, where the market is started to be affected from the credit crisis. In March 2008, we observe a spike in the spread data which stems from the panic due to the possibility of the collapse of the company Bear Stearns. Furthermore, a drastic upward movement is observed starting from September 2008. This time period corresponds to the breakdown of the credit market due to events such as the bankruptcy of Lehman Brothers. One other feature of the data set we use is that there does not occur any default events during the sample period.

In Figure 2.2 we provide index spreads across four maturities. Figure 2.1 and Figure 2.2 together show that the tranche data and the index data have the same up and downward trends during the considered time period. To investigate further on the co-movement of tranches and the index, in Figure 2.3 we give the series for running correlation between changes in the index and changes in the tranche data for all maturities and tranches. The striking observation in Figure 2.3a, 2.3b and 2.3c is that the correlation between the index and the $22 - 100\%$ tranche may become negative time to time. Moreover, it is understood from these figures that the correlation between the index and the $22 - 100\%$ tranche increases drastically after July 2007.

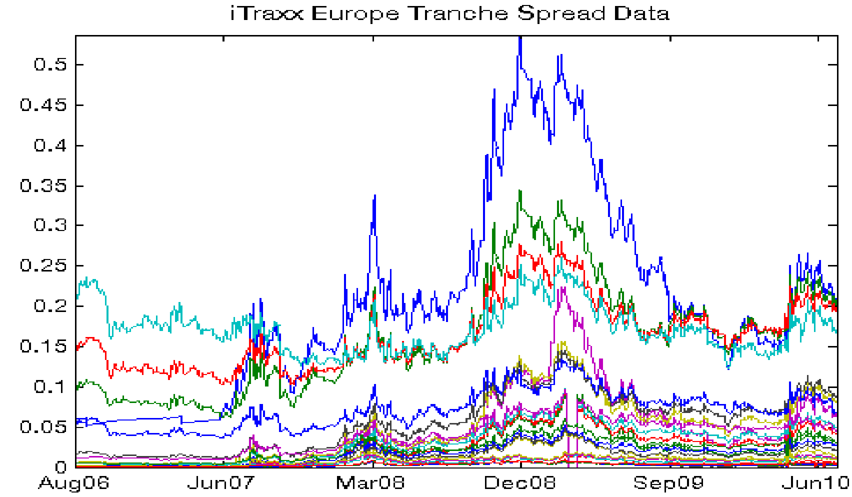


Figure 2.1: iTraxx Europe zero-coupon spread data from 30 Aug 2006 to 3 Aug 2010.

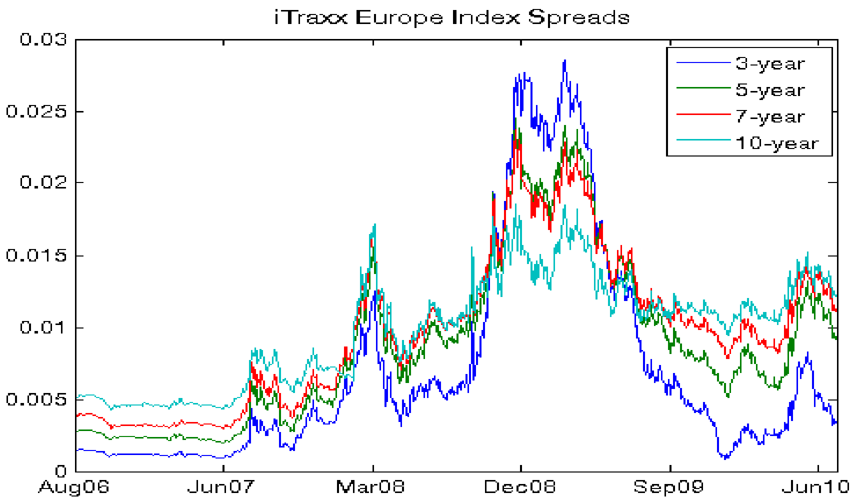
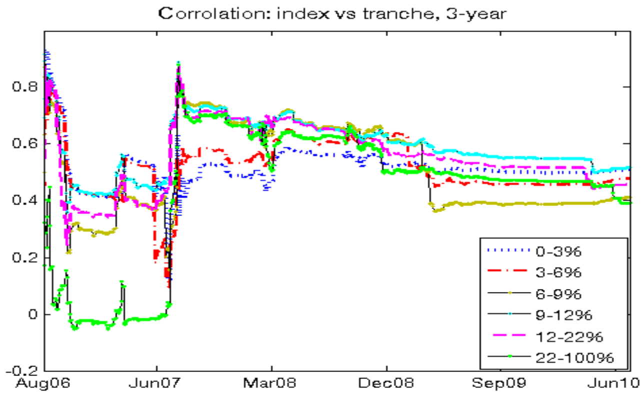
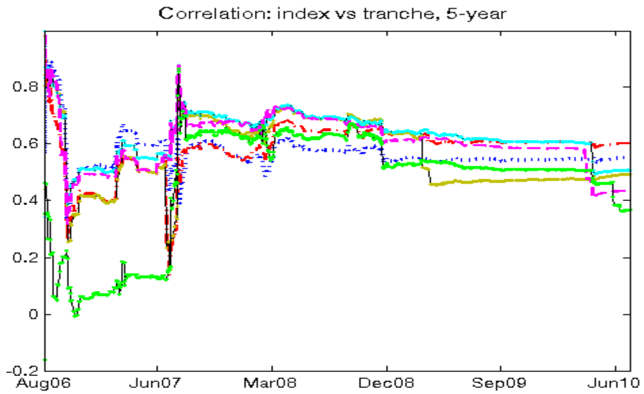


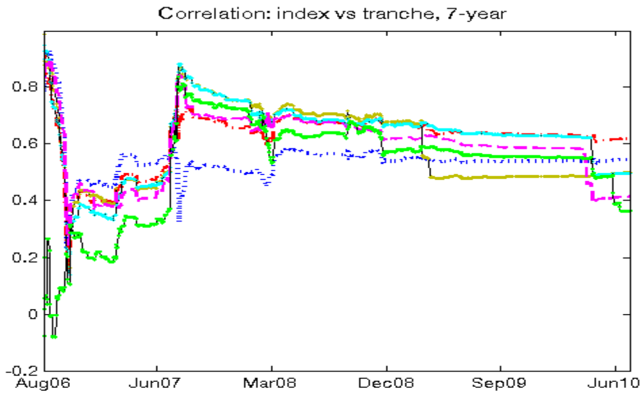
Figure 2.2: iTraxx Europe index spread data from 30 Aug 2006 to 3 Aug 2010.



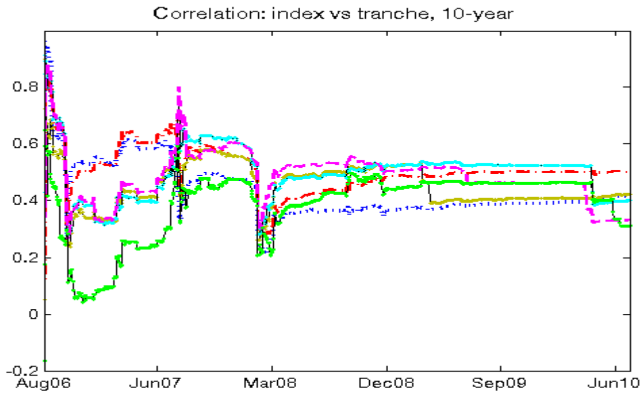
(a) 3-year



(b) 5-year



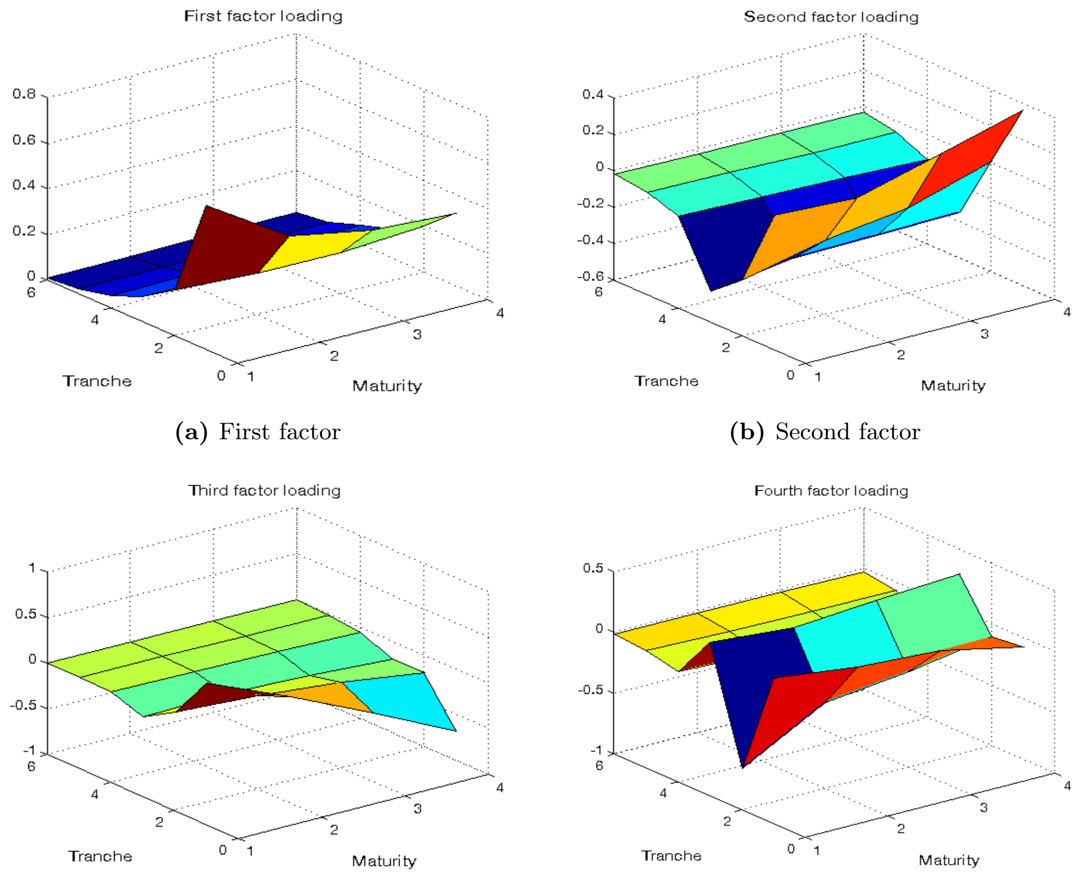
(c) 7-year



(d) 10-year

Figure 2.3: Running correlation between changes in the zero coupon tranche and index spreads from 30 Aug 2006 to 3 Aug 2010.

As we are proposing a factor model, it is indispensable to run a principal component analysis (PCA) to spread changes. PCA result suggests that first factor explains 83.36% of total variation in spreads and second to fourth factors explain 88.30%, 92.29% and 94.59% respectively.



(a) First factor **(b)** Second factor

(c) Third factor **(d)** Fourth factor

Figure 2.4: PCA factor loadings

Figure 2.4 depicts the factor loadings for four principal components. The principal component analysis suggests that one factor is not enough to explain the variation in the data. Motivated by this result, we specify a two-factor model albeit the nested one-factor version of the model is also estimated.

2.7.2 Results and Discussion

Using the estimation methodology given in previous sections, we fit the model to the data set. Then, we perform a hedging analysis where the performance of variance-minimizing and regression based hedging results are evaluated. Moreover, we make a simulation analysis in which normal and stress scenarios are generated. We use this set of scenarios to assess the two aforementioned hedging approaches. We want to point out that during whole analysis the risk-free rate is considered to be constant at $r = 0.05$ and zero recovery is assumed. In the following subsections we give the results and discussion on empirical analysis. Moreover we compare our findings with those obtained in Cont and Kan [2011].

Estimation Results

As mentioned before, the QML approach makes it possible to estimate the model parameters and filtered out the unobservable factors simultaneously. We run the estimation algorithm given in Section 2.5 for one and two factor models. Table 2.1 depicts the parameter estimates for one and two factor models.

Table 2.1: Parameter values for the sample period 30 August 2006-3 August 2010

	θ_z	κ_y	κ_z	σ_y	σ_z	λ_y	λ_z	a_0	γ	b_0	c_0
2-factor model	0.0055	0.52	0.22	0.38	0.25	-0.66	-0.30	98.78	0.32	26.40	0.09
1-factor model	0.03	-	6.96e-05	-	0.15	-	1.44e-04	3.23e-05	26.08	23.96	-

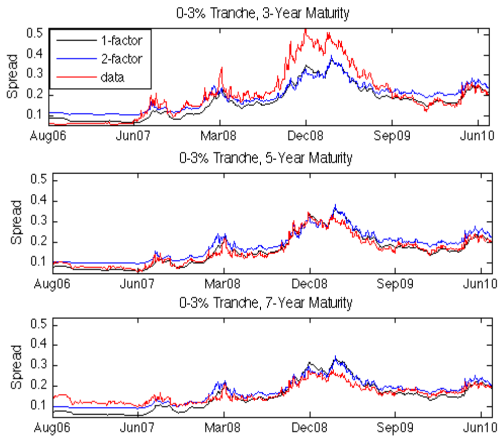
We use the parameter estimates and the filtered factors to regenerate the data. We plot actual vs estimated data in Figure 2.5 . Across all tranches/maturities the two-factor model outperforms one factor in terms of the better fit. Furthermore, it is remarkable how the one-factor model estimates are below from the actual data for $22\% - 100\%$ tranche. Here, we want to point out that, a two-factor affine factor model with the restriction of zero catastrophic component is, as the one factor-model, not able to fit to the super-senior tranche. There, the importance of the catastrophic risk component of the two-factor model comes into play. That is, under this two-factor affine framework including the catastrophic component becomes inevitable for a better fit in the super-senior tranche.

For piecewise constant $\beta_y(x)$, the changes of tranche spread equal

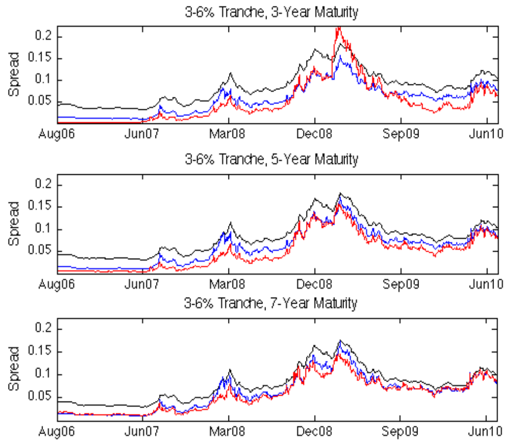
$$\Delta R(t,\tau,j) = \frac{B_y(\tau,x_j)}{\tau}\Delta Y_t + \frac{B_z(\tau,x_j)}{\tau}\Delta Z_t$$

We plot the surfaces $B_y(\tau,x)/\tau$ and $B_z(\tau,x)/\tau$ in Figure 2.6. At the first sight, it can be realized that the first factor loading in Figure 2.4, being hump shaped as function of time to maturity, looks like a combination of $B_y(\tau,x)/\tau$ and $B_z(\tau,x)/\tau$.

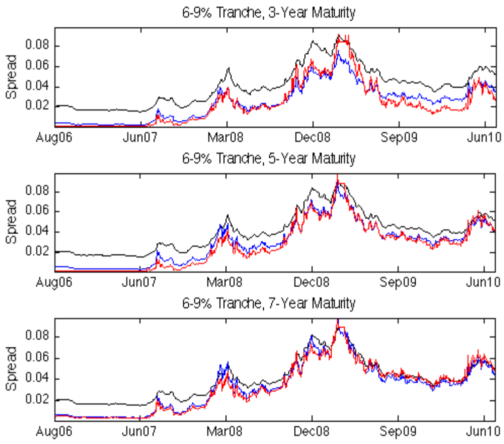
Inserting the parameter estimates and filtered factor series in formula (2.47) we get series for expected loss given default. Moreover, to investigate the effect of the catastrophic



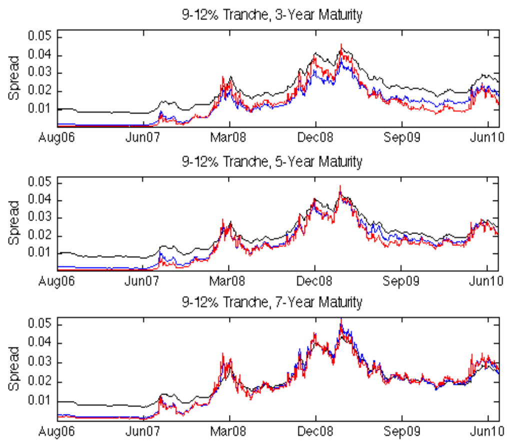
(a) $0-3\%$ tranche



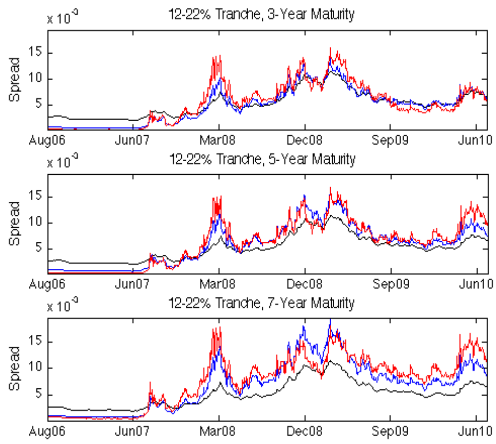
(b) $3-6\%$ tranche



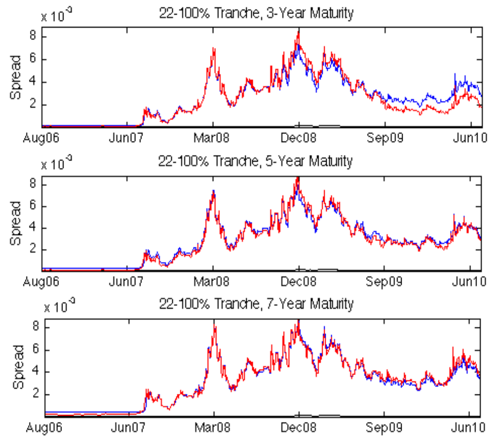
(c) $6-9\%$ tranche



(d) $9-12\%$ tranche

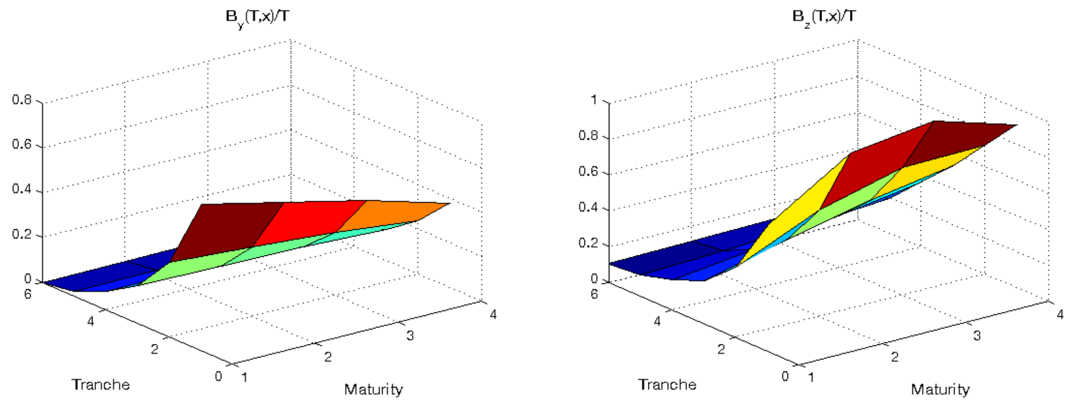


(e) $12-22\%$ tranche



(f) $22-100\%$ tranche

Figure 2.5: Actual vs Estimated Data



(a) $B_y(\tau, x)/\tau$ **(b)** $B_z(\tau, x)/\tau$

Figure 2.6: Basis surface curves $B_y(\tau, x)/\tau$ and $B_z(\tau, x)/\tau$

component, we fix the catastrophic risk parameter $c_0 = 0$ and insert the rest of the estimated parameters and filtered factors into the formula. Figure 2.7a shows how the implied expected loss given default changes in the sample period with and without catastrophic component. In the same figure Graph 2.7b to 2.7d reveals the expected loss given default series for different initial loss level (L_t) assumptions. In particular, graph 2.7d reveals the case in which the loss level is changing. In Figure 2.7 series with and without catastrophic component coincide till August 2007. This is because of the fact that, before that time, the value of the factor Z is very close to zero. The most important observation in Figure 2.7 is that, change in the initial loss level only causes parallel shifts in the series without catastrophic component. However, the shape of the series with the catastrophic component changes according to the initial loss level.

Hedging Results

For the whole sample period we perform the hedging analysis given in Section 2.4.4 for all attachment and detachment points. As it is mentioned in Remark 2.4, variance of the series $\Delta G^{(x_1,x_2]}$ and the linear correlation between changes in tranche and index gains processes are important for the performance of regression based hedging. For this reason, we provide the running variance, relative variance and correlation series in Figure 2.8. It is observed that variance series are increasing for all tranches. On the other hand, in July 2007 and March 2008 for all tranches but $12-22\%$ and $22-100\%$ there occurs two main downward shifts in the correlation series. For the $12-22\%$ the series stays relatively constant and in $22-100\%$ tranche, there is a sharp increase in June 2007.

Figure 2.9 depicts the series for the gains process and nominal spot value of the STCDO and hedging portfolio value both for regression based and variance minimizing strategy.

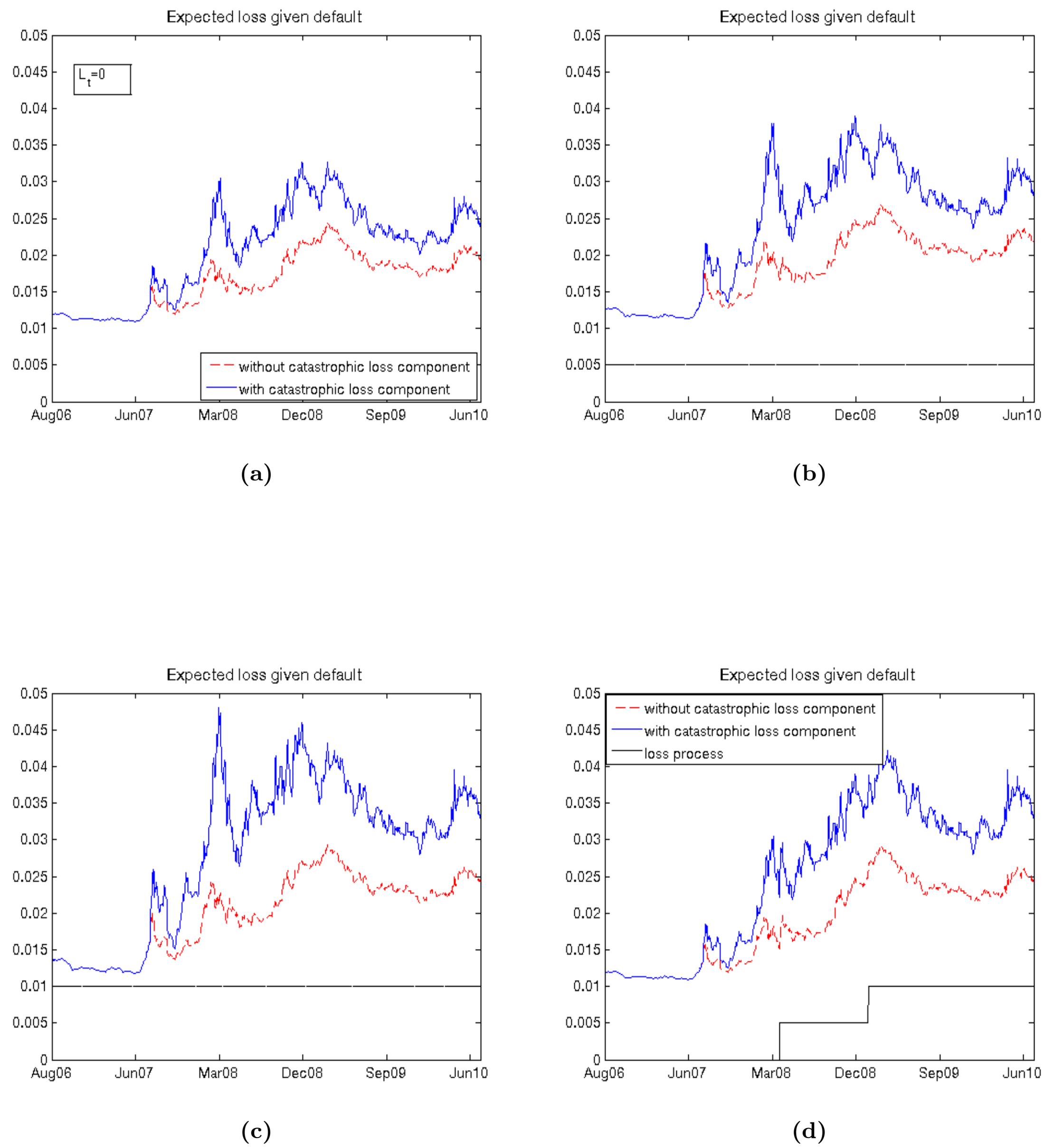
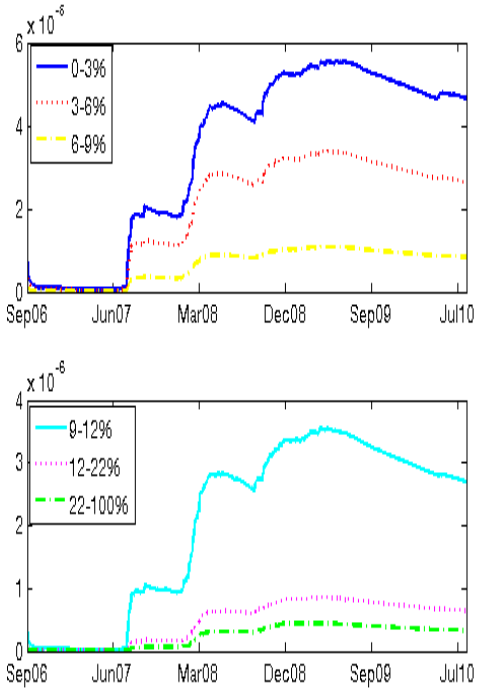
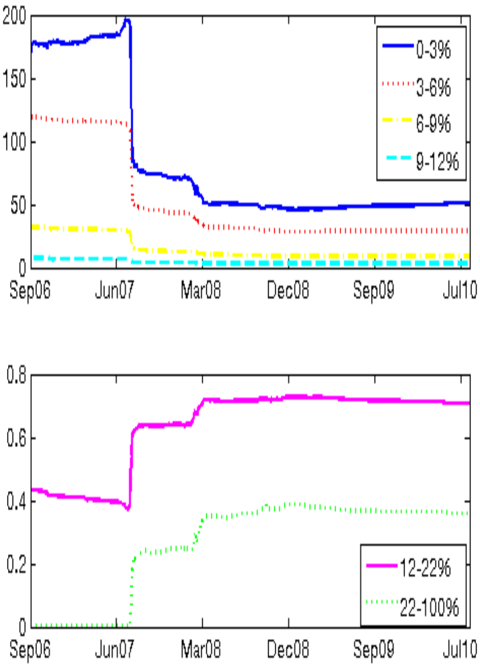


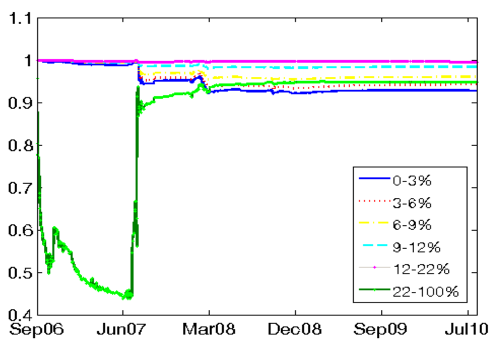
Figure 2.7: Expected loss given default with and without catastrophic component



(a) Variance of $\Delta G_t^{(x_1,x_2]}$



(b) Relative value of variances of $\Delta G_t^{(x_1,x_2]}$ and $\Delta G_t^{(0,1]}$



(c) Running linear correlation ρ_t

Figure 2.8: Running variance, relative variance and linear correlation series

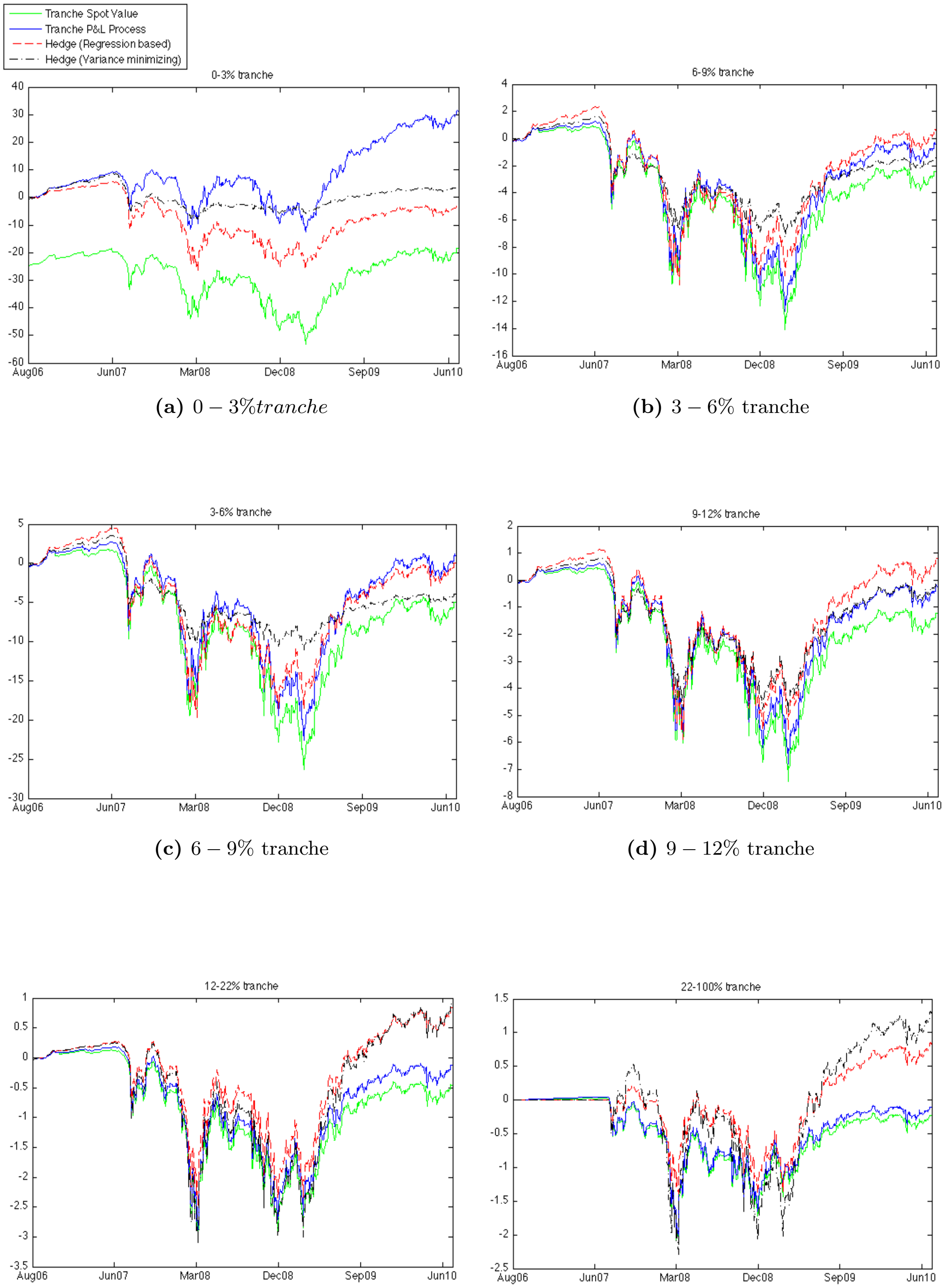
P&L processes and the series for the variance minimizing and regression based hedging strategies are plotted in Figure 2.10. According to P&L criterion, staying closer to zero during the sample period, variance minimizing strategy is observed to outperform regression based strategy for $0 - 3\%$ tranche. For other tranches the situation is more ambiguous as in some parts of the period the P&L series for the regression based strategy is closer to zero and in the rest this holds true for the variance minimizing hedge.

If we focus on the last day of the sample period, for tranches $0 - 3\%$ and $9 - 12\%$ P&L values for variance minimizing hedge are closer to zero implying a better performance whereas for $3 - 6\%$ and $12 - 22\%$ regression based strategy performs better. The better performance of the regression based strategy for the mentioned tranches can be explained via Remark 2.4 and Figure 2.8a-2.8c where very high correlation between the tranche gains process and the index is observed. Here, we want to point out that the presence of defaults in the data set may deteriorate the linear correlation structure between the index and tranches. This in turn may cause regression based strategy to perform worse. We will try to clarify this assertion in the coming section where the hedging analysis will be done under more general scenario set which permits for non-zero losses.

When we concentrate on the series for the hedging strategy ϕ in Figure 2.10, for all tranches regression based hedge is observed to be more stable during the hedging period. It is also worth mentioning that, both for variance minimizing and regression based hedge, the change in ϕ at the beginning of the crisis around July 2007 for different tranches shows different patterns. In particular, for tranches $0 - 3\%$, $3 - 6\%$, $6 - 9\%$, $9 - 12\%$, ϕ is decreasing in absolute value indicating a reduction in insurance and for tranches $12 - 22\%$ and $22 - 100\%$ there is the opposite behavior. For the regression based hedge, Remark 2.3 is helpful in understanding the different behavior of ϕ across tranches. As it is shown in Figure 2.8b for tranches $12 - 22\%$ and $22 - 100\%$, the relative variance goes almost constant till July 2007 and then exhibits a sharp increase. On the other hand, in the correlation graph 2.8c we observe that during the whole period the series for $12 - 22\%$ tranche stays constant and in July 2007 there is a large increase in $22 - 100\%$ tranche. Together with these, Remark 2.3 suggests that, the reason of the different behavior of the regression based strategy ϕ at the beginning of the crisis is the different dynamics of the relative variance and linear correlation series of tranches.

In Figure 2.11 we provide the reduction in volatility for variance minimizing and regression based strategies. It is seen that, according to the reduction in volatility criterion, regression based hedge perform better than variance minimizing hedge for all tranches.

We can summarize the results of this section as follows. Although the regression based strategy yields more favorable results under the reduction in volatility criterion, according to the P&L criterion performance of two hedging strategies depend on the tranche.

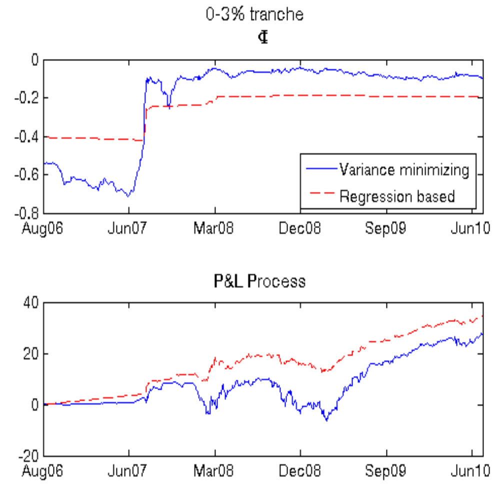


(a) $0-3\% tranche$ **(b)** $3-6\%$ tranche

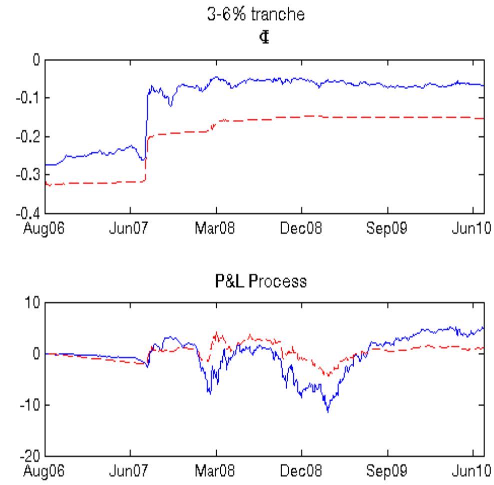
(c) $6-9\%$ tranche **(d)** $9-12\%$ tranche

(e) $12-22\%$ **(f)** $22-100\%$ tranche

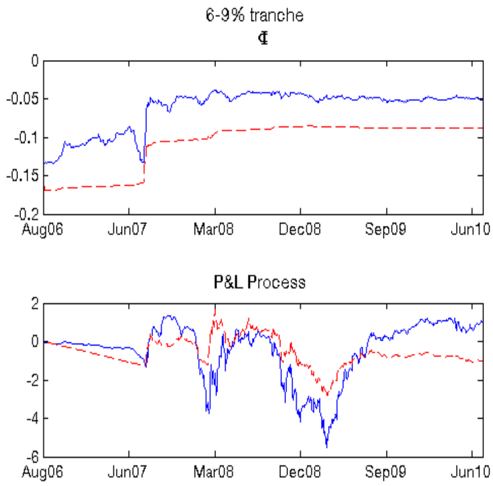
Figure 2.9: Hedging results for the sample period: gains process, nominal spot value and hedging portfolio value



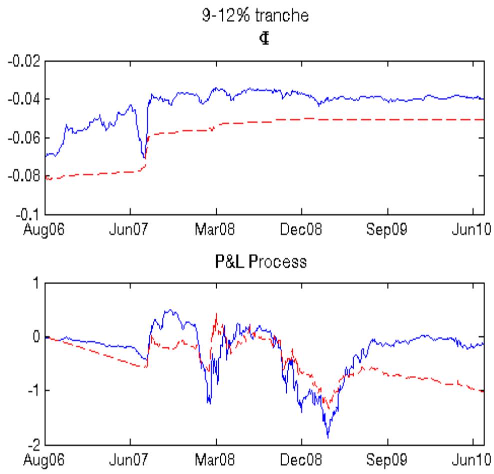
(a) $0-3\%$ tranche



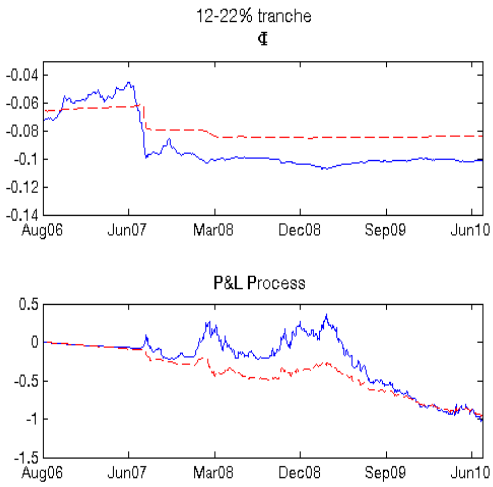
(b) $3-6\%$ tranche



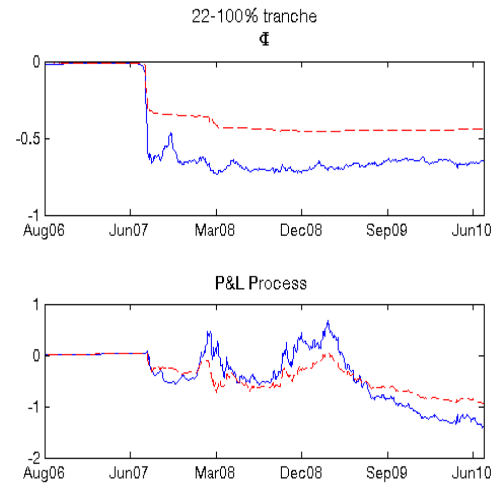
(c) $6-9\%$ tranche



(d) $9-12\%$ tranche



(e) $12-22\%$ tranche



(f) $22-100\%$ tranche

Figure 2.10: Hedging results for the sample period: the hedging strategy ϕ and total portfolio P&L process

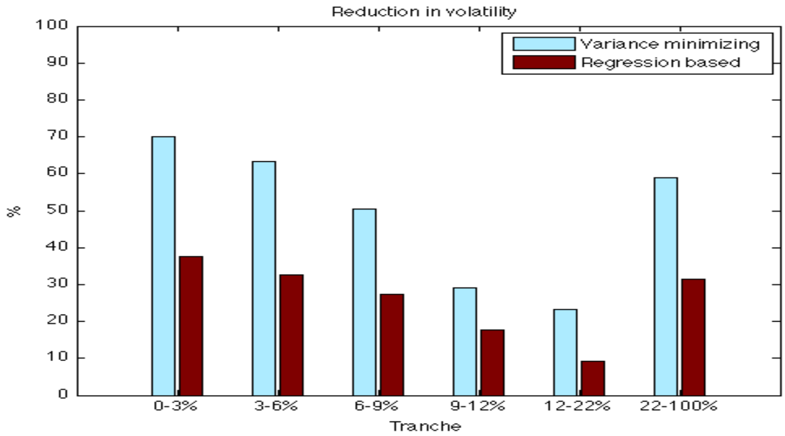


Figure 2.11: Reduction in volatility for variance minimizing and regression based hedge

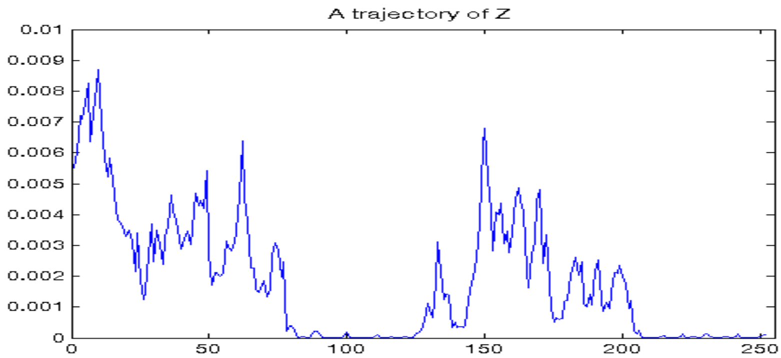
In the next section we will do a simulation analysis to compare the performance of two hedging strategies under more general scenarios.

Simulation Results

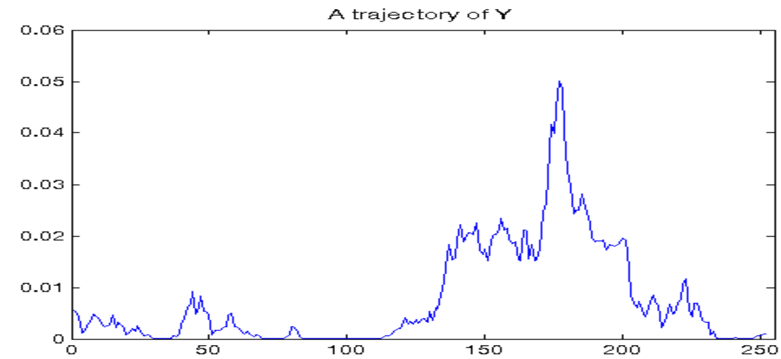
Using the method given in Section 2.6 and the estimated parameters of the two-factor model we simulate 2000 trajectories for the time horizon of 252 days. 1000 of trajectories correspond to normal scenarios and the rest represent stress scenarios. We then investigate the performance of variance minimizing and the regression based hedging strategies on simulated scenarios. Moreover, we perform a conditional simulation analysis in which trajectories coming from the estimation are fixed and conditional on these trajectories 2000 loss scenarios are generated. We first give the results on general simulation analysis, then the conditional simulation results follow.

In Figure 2.12 we provide sample trajectories for generated factors Y, Z and the loss process L. In the given particular path, the loss process is observed to have two jumps and reaches almost the value 0.03 indicating that the largest part of the $0-3\%$ tranche is eroded. Effect of jumps in the loss trajectory can be seen in Figure 2.16a where the tranche spot value becomes almost zero due to the fact that the remaining notional for the $0-3\%$ tranche becomes very close to zero.

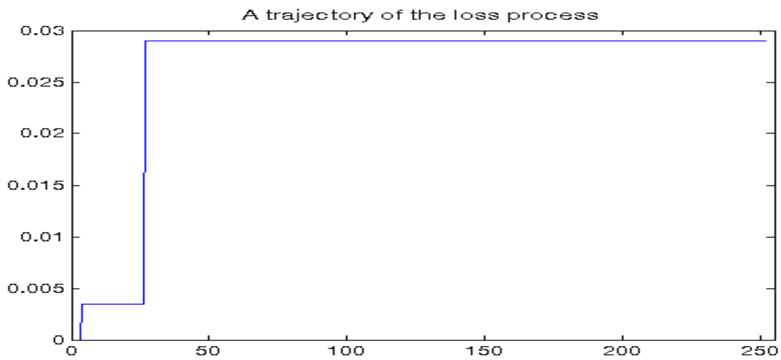
Figure 2.14 depicts P&L series for all tranches. This figure indicates that the performance of the variance minimizing and regression based hedge similar during the 252 day period and when concentrated on the last day of the sample, except the $3-6\%$ tranche, variance minimizing hedge is observed to outperform the regression based strategy as the total P&L value is closer to zero.



(a) Simulated factor Z

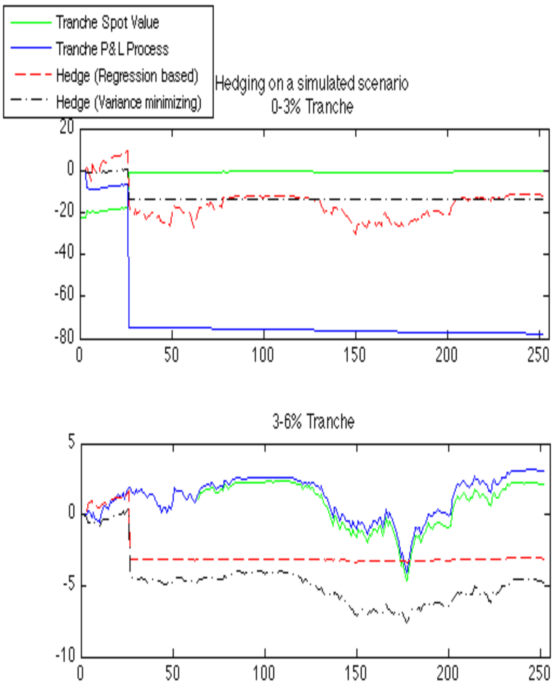


(b) Simulated factor Y

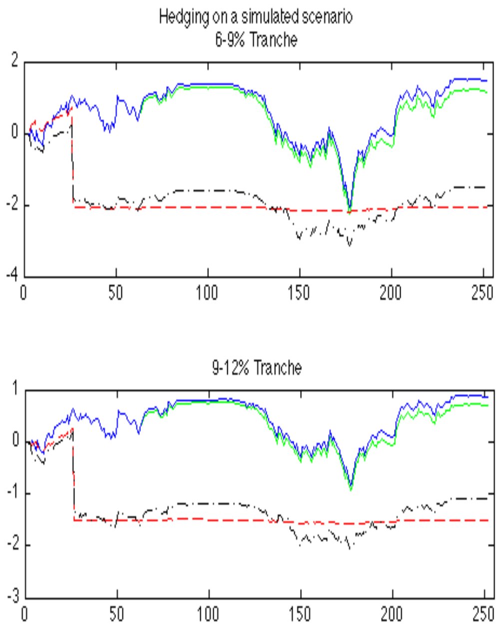


(c) Simulated loss path

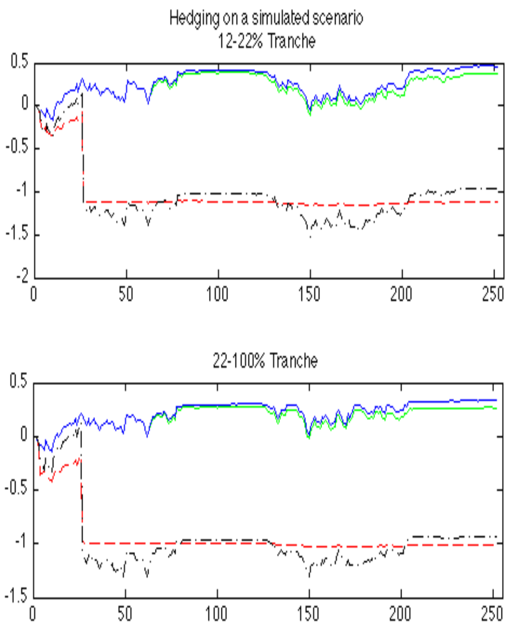
Figure 2.12: Hedging on a simulated scenario: sample trajectories



(a) $0-3\%$, $3-6\%$ tranche

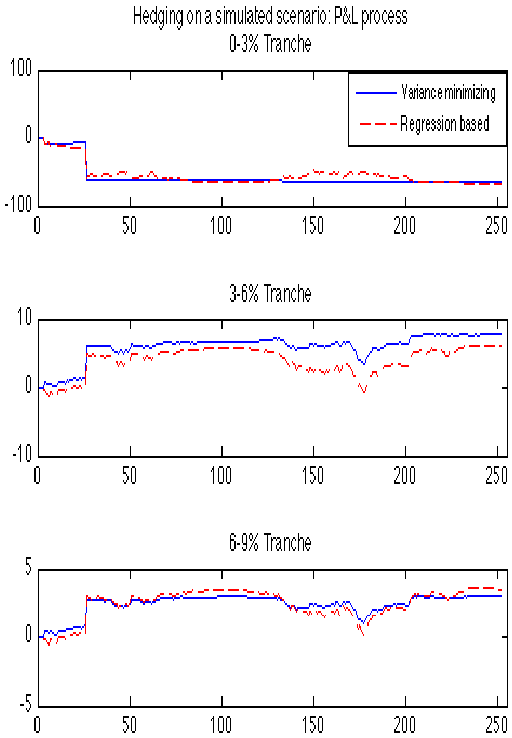


(b) $6-9\%$, $9-12\%$ tranche

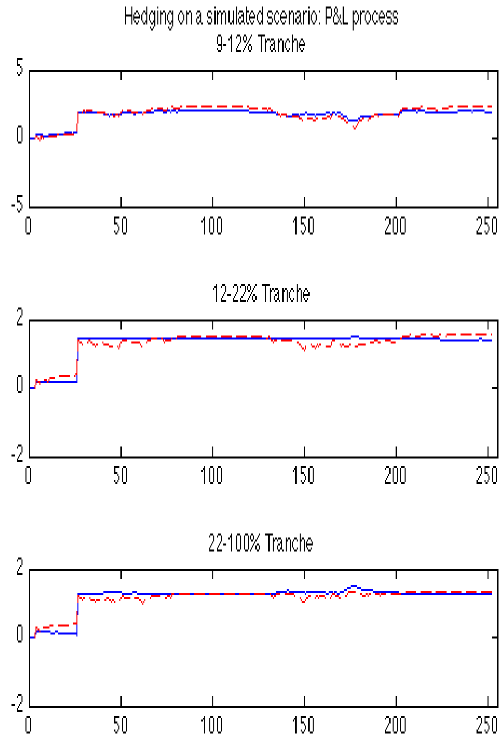


(c) $12-22\%$, $22-100\%$ tranche

Figure 2.13: Hedging on a simulated scenario: gains process, nominal spot value and hedging portfolio value



(a) $0 - 3\%$, $3 - 6\%$,$6 - 9\%$ tranche



(b) $9 - 12\%$, $12 - 22\%$, $22 - 100\%$ tranche

Figure 2.14: Hedging on a simulated scenario: total portfolio P&L value

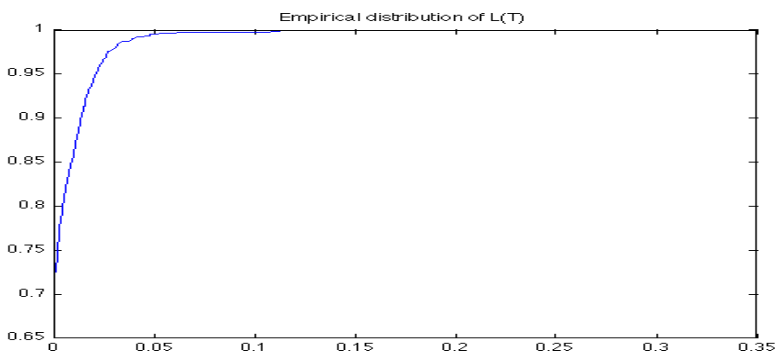


Figure 2.15: Empirical distribution of L(T), 2000 scenarios with $\psi = 1$ and $\psi = 100$

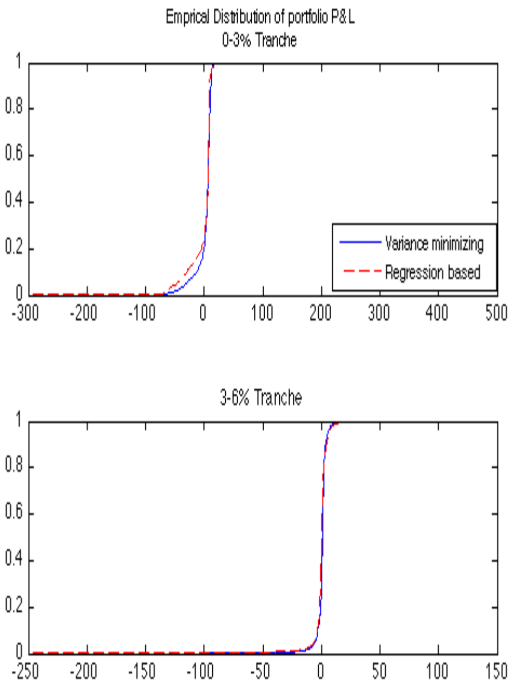
We take the set of all simulated scenarios and focus on the last date T of the simulation period. The empirical cumulative distribution function for the loss process at time T is given in Figure 2.15 showing that the simulation procedure is successful in the sense that it produces loss scenarios ranging between 0 and 0.1.

To show the effect of the hedge, we plot the date T cumulative distribution of the total hedging portfolio P&L distribution for variance minimizing and regression based strategies. Results are depicted in Figure 2.16. Figure 2.16 implies that in most of simulated trajectories both variance minimizing and regression based hedging strategies yield total portfolio P&L values which are close to zero. In other words, both strategies are successful in average. However, for all tranches, regression based strategy, which is indicated by the dashed line, is observed to produce more extreme losses, that is the density for the time T P&L values coming from the regression based strategy has longer left tail. According to this observation, the variance minimizing strategy is more successful than the regression based hedge for scenarios which permit non-zero losses.

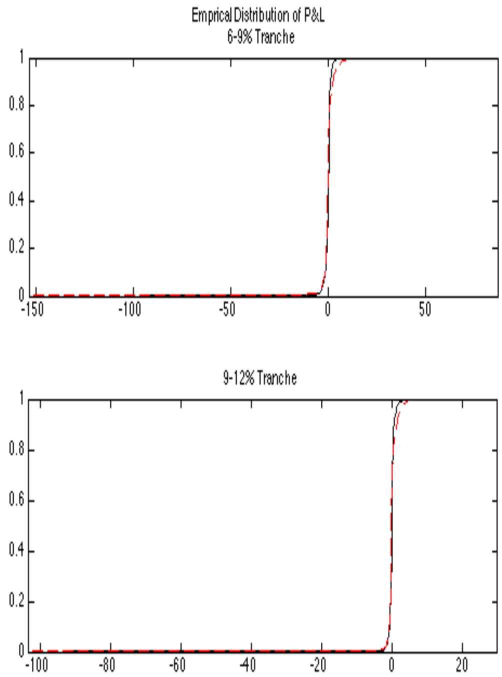
To investigate further on the relative performances of hedging strategies, sitting at time 0, we estimate the riskiness of hedging portfolios for each day of the sample period via computing value at risk (VAR) and expected shortfall at the confidence levels 99% and 99.9% respectively. Results are given in in Figure 2.17. For all tranches, VAR and expected shortfall series for regression based strategy is observed to lie above the respective VAR and expected shortfall series of the variance minimizing hedge. In other words, when compared to variance minimizing strategy, the regression based strategy yields a riskier hedging portfolio.

As the second criterion, for each scenario we compute reduction in volatility for all tranches. We then compute the descriptive statistics by taking the Radon-Nikodym densities into account. Results for the regression based and variance minimizing hedge is given in Table 2.2 and Table 2.3 respectively. According to these tables, for all tranches mean values of reduction in volatility for regression based hedge are higher than the mean values for variance minimizing hedge. This suggests that, with respect to the reduction in volatility criterion, for all tranches variance minimizing hedge performs better in average. On the other hand, regression based hedge yields more right-skewed distribution for reduction in volatility implying that most of the values lie to the left of the mean. Moreover, for all tranches regression based hedge yields more dispersed reduction in volatility values as it is suggested by the coefficient of variation (CV) value.

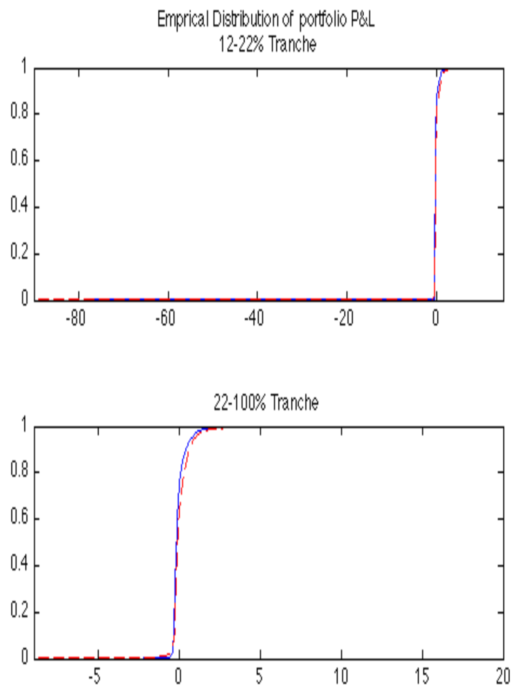
Our next goal is to get the density function of reduction in volatility for regression based and variance minimizing strategies. To achieve this, one should first compute the related frequencies of the possible values for reduction in volatility. The important point in this step is to adjust the frequencies coming from the stress scenarios, that is,



(a) $0-3\%$, $3-6\%$ tranche

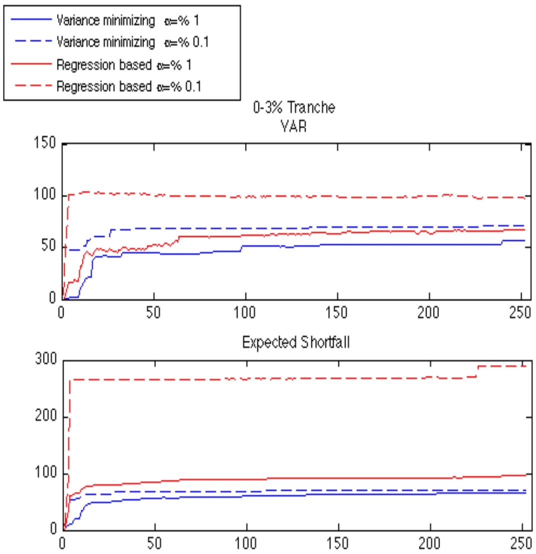


(b) $6-9\%$, $9-12\%$ tranche

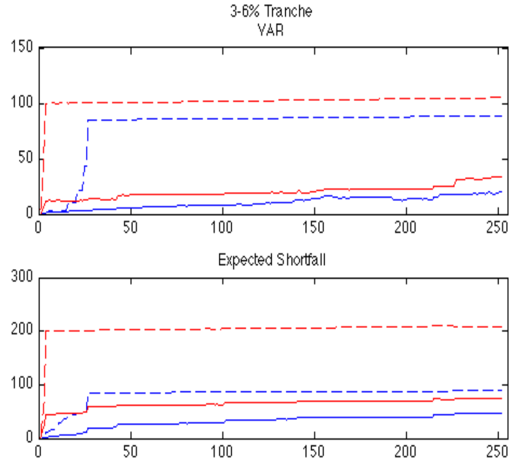


(c) $12-22\%$, $22-100\%$ tranche

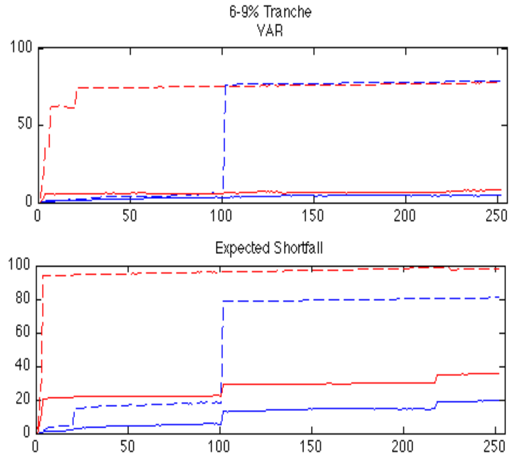
Figure 2.16: Empirical distribution of P&L at T for variance minimizing and regression based strategies



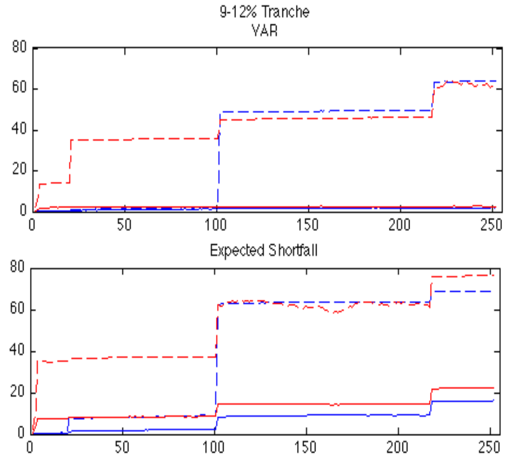
(a) $0-3\%$ tranche



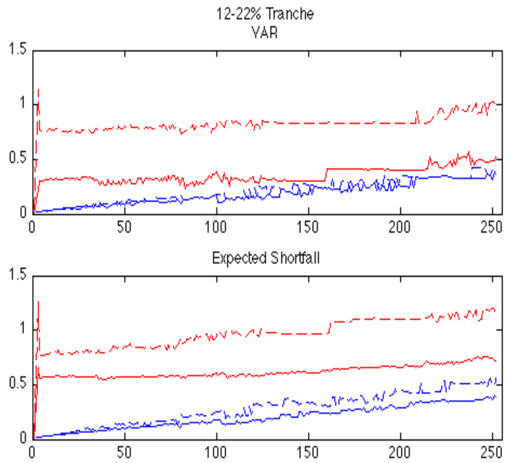
(b) $3-6\%$ tranche



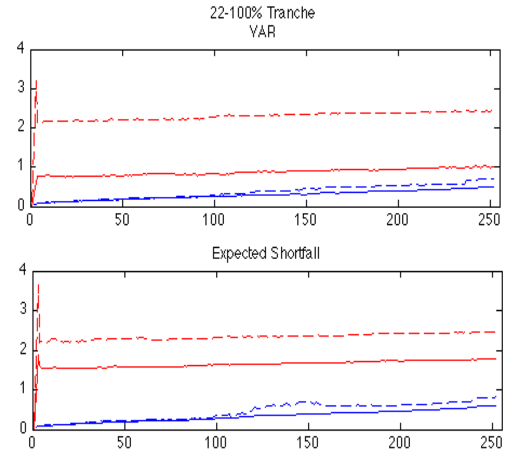
(c) $6-9\%$ tranche



(d) $9-12\%$ tranche



(e) $12-22\%$ tranche



(f) $22-100\%$ tranche

Figure 2.17: Time Series for VAR and Expected Shortfall values at 1% and 0.1% levels

Table 2.2: Descriptive Statistics: Reduction in volatility for regression based hedge

	Mean	Median	Std	CV	Max	Min
0-3%	69.17	54.73	73.92	1.06	1257.5	9.86
3-6%	74.09	47.97	140.56	1.89	3531.4	6.96
6-9%	72.20	40.57	130.60	1.80	3938.8	3.86
9-12%	57.62	27.13	103.57	1.79	2766.7	5.21
12-22%	44.08	13.03	95.89	2.17	2601.9	1.61
22-100%	72.10	42.09	112.75	1.56	3761.5	5.43

Table 2.3: Descriptive Statistics: Reduction in volatility for variance minimizing hedge

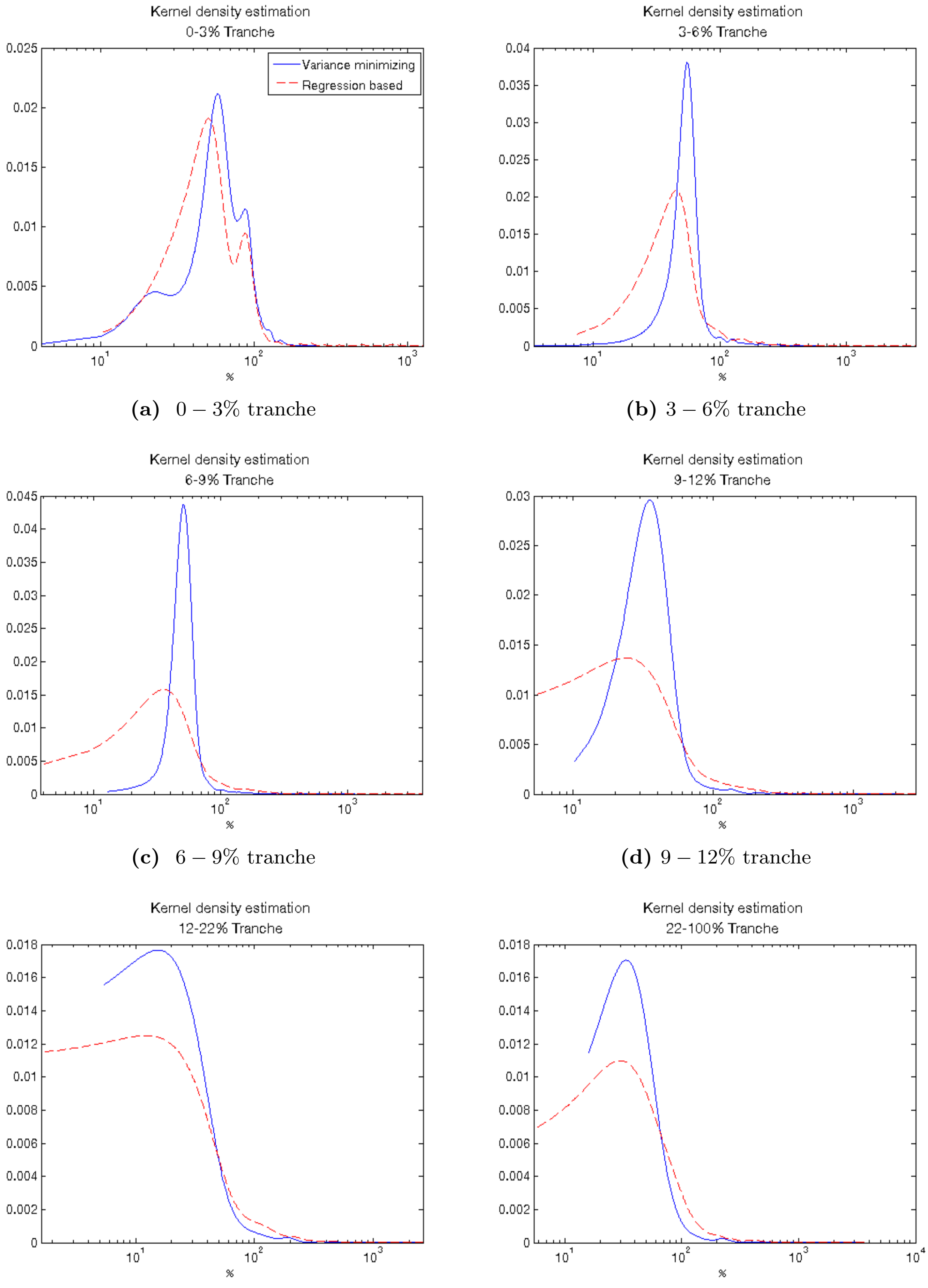
	Mean	Median	Std	CV	Max	Min
0-3%	66.3	62.84	24.62	0.37	258.1	4.12
3-6%	64.95	55.72	43.56	0.67	601.6	3.36
6-9%	56.33	51.65	28.16	0.50	1131.8	12.95
9-12%	43.28	36.01	34.33	0.79	2223.6	10.26
12-22%	30.36	15.77	58.33	1.92	2361.4	5.45
22-100%	50.35	34.40	62.19	1.23	3261.6	15.98

scenarios which are generated by taking the importance sampling parameter $\psi = 100$. However, this is easy as we have the related Radon-Nikodym densities. After we get the adjusted frequencies, Kernel smoothing technique is used to get the density function of reduction in volatility for regression based and variance minimizing strategies. For each tranche, we plot the density of reduction in volatility in Figure 2.18 where for illustration purposes logarithmic scale is used in the horizontal axis.

Conditional simulation analysis: We shall now present the results of conditional simulation analysis. To begin with, in Figure 2.19 we provide the factor series Y_t and Z_t that we filtered out from the data.

We fix the filtered factor series in Figure 2.19 and conditional on these trajectories we simulated 2000 trajectories for the loss process again with importance sampling parameter $\psi = 1$ and $\psi = 100$. Conditional distribution of the simulated loss process at time T is depicted in Figure 2.20. One striking result is that, when compared with the loss distribution function given in Figure 2.15, conditional loss distribution in Figure 2.20 gives higher probability to losses greater than 0.1. Moreover, simulation results suggest that for $\psi = 1$, in 815 of 1000 simulated loss trajectories, there occurred a jump, that is a default. This may imply that we were lucky in the crisis times that the market did not experienced any defaults.

Our main objective for the reminder of this section is to compare our main findings with Cont and Kan [2011].



(a) 0 – 3% tranche

(b) 3 – 6% tranche

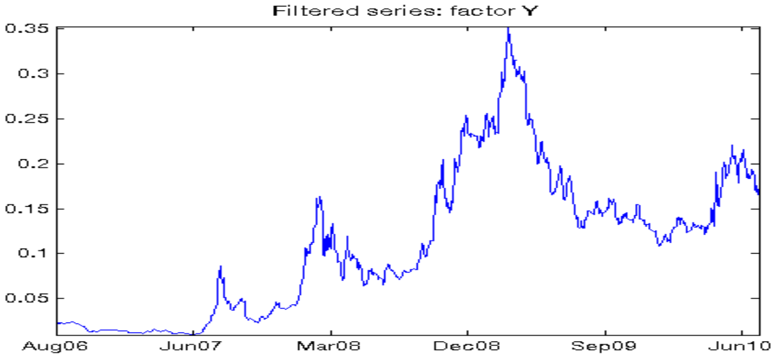
(c) 6 – 9% tranche

(d) 9 – 12% tranche

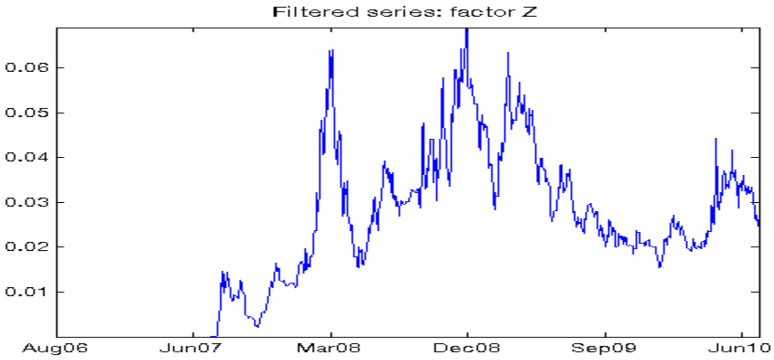
(e) 12 – 22% tranche

(f) 22 – 100% tranche

Figure 2.18: Kernel density estimate of reduction in volatility in logarithmic scale



(a) Filtered series: factor Y



(b) Filtered series: factor Z

Figure 2.19: Filtered series for the factors Y and Z from 30 Aug 2006 to 3 Aug 2010.

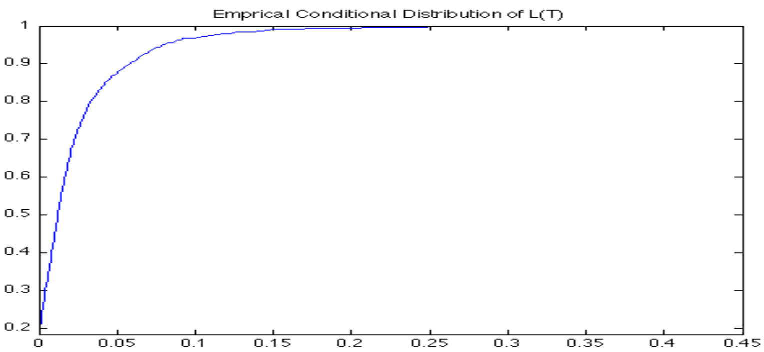
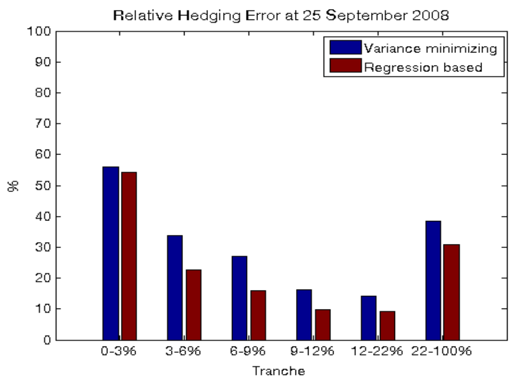


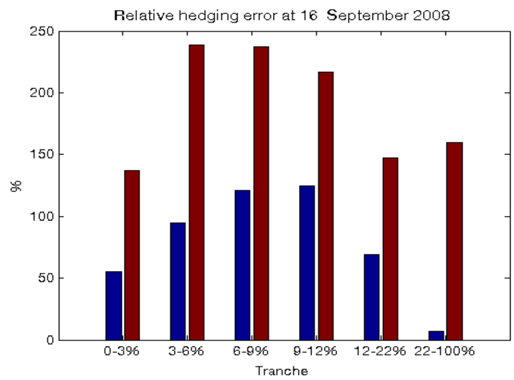
Figure 2.20: Empirical conditional distribution function of $L(T)$

2.7.3 Comparison with Cont and Kan [2011]

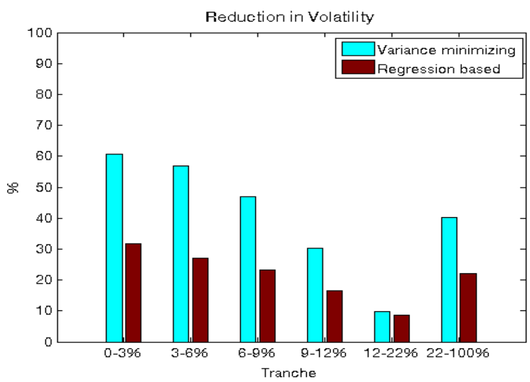
Although the data set and the sample period of the current study and Cont and Kan [2011] differs, it makes sense to provide a comparison between our findings and those obtained in Cont and Kan [2011]. To provide a more appropriate basis for a comparison, we first redo the in sample hedging analysis for the period covered in Cont and Kan [2011], that is, the period 25 March-25 September 2008. According to the results of this analysis, the regression based hedge is observed to be more efficient in terms of the reduction in volatility criterion. This match up with the findings of Cont and Kan [2011]. Moreover, we compare the variance minimizing and regression based strategies under the criterion of relative hedging error (see Equation (2.79)). In particular, we compute the relative hedging error values at dates 16 September and 25 September 2008. The results are depicted in Figure 2.21 where bar graphs 2.21a and 2.21b show that the relative hedging error criterion depends very much on the final date of the hedging period. When we compare these graphs with the ones obtained in Cont and Kan [2011], the results are very much in the same direction.



(a) Relative hedging error at 25 September 2008



(b) Relative hedging error at 16 September 2008



(c) Reduction in volatility: 25 March 2008-25 September 2008

Figure 2.21: Relative hedging error and reduction in volatility

Thus, the relative success of the regression based hedging in the given data period is the common finding of the two study. However, our findings show that under more general scenarios regression based hedge strategy can not outperform the variance minimizing hedge. Moreover, according to the VAR and expected shortfall results, regression based strategy is observed to engender riskier hedging portfolios.

2.8 Conclusion

In this part of this thesis, following the framework given in Filipović et al. [2009] we propose a two-factor affine factor model in which a catastrophic risk component is considered as a tool for explaining the dynamics of the super-senior tranches. We then investigate the uniqueness of the martingale measure and the market incompleteness for the setup given in Filipović et al. [2009]. Moreover, we analyze the real world performance of variance minimizing and regression based hedging strategies for the hedging of STCDOs with the underlying index default swap. We conclude our analysis with a simulation analysis, in which the objective is to test the performance of hedging strategies under more general loss scenarios.

The results of this part can be summarized as follows. We showed that the two-factor model yields satisfactory results in terms of the successful fit to iTraxx Europe data. Uniqueness of the martingale measure for the current modeling setup is proved. However, due to the presence of infinite number of possible jump sizes for the loss process, the market is characterized as incomplete.

Our findings also suggest that within the data period, both hedging strategies are efficient in reducing the risk on the STCDO significantly. However, according to the reduction in volatility criterion, the regression based strategy is observed to be more successful than the variance minimizing hedge. This result agrees with the findings of Cont and Kan [2011]. On the other hand, the simulation analysis, in which we use importance sampling technique to generate loss scenarios, indicates that variance minimizing hedge performs better than regression based hedge under more general scenarios permitting non-zero loss trajectories.

Part II

Pricing and Hedging of Inflation Indexed Bonds

3

Pricing and Hedging in an Affine Framework

3.1 Introduction

Inflation indexed derivatives are becoming popular as the amount of inflation linked liabilities is increasing in financial markets. One can consider, for example, the insurance sector where inflation linked products are purchased to cover inflation risk associated with the pension payments. Moreover, in the developing economies, where a sustained high-medium level of inflation is observed, investors prefer inflation linked products as long-term investments with the idea of preserving the purchasing power of their nominal income. Additionally, the current financial crisis and the rising commodity prices caused an increase in inflation expectations resulting in an increase in the demand for inflation linked products. On the supply side, monetary authorities increase their issue of inflation linked bonds to make the inflation targeting policies more reliable and to reduce the inflation premium paid in the issue of nominal bonds.

Inflation is defined as the percentage change in the value of a fixed basket of goods and services. A *zero-coupon inflation indexed bond*, also called *treasury inflation protected security* (TIPS), is a debt security which pays not the issue date face value but the inflation-adjusted value when the maturity date comes. That is, the amount received by the investor has the same purchasing power with the issue date purchasing power of the face value. In this way, indexed bonds provide a protection against inflation. In case of a coupon paying indexed bond, the pre-determined coupon rate is paid over the inflation adjusted notional of the bond. In other words, unlike regular bond, an inflation indexed bond when it is hold until the maturity, guarantees the real interest regardless of the future realized inflation. Naturally, in this case the nominal interest is not known a priori and varies with the realized inflation. Inflation indexed bonds are not the only inflation sensitive instruments. Options on inflation and inflation swaps are the other liquid inflation linked products.

The *foreign currency analogy* is the most widely used approach for the modeling of inflation derivatives. The rationale of this approach is that specifying the real and nominal interest rate term structures and considering the nominal and real part of the economy as the domestic and foreign economy make it possible to treat the price index process as an exchange rate between the real and nominal economies. To be more precise, let us consider a real bond which is defined as the instrument paying one unit of price index basket in real terms at the maturity. Let us now consider that we are investors in the nominal economy, that is, the economy in which the value of all assets are described by the nominal units, i.e., money terms. Foreign currency analogy suggests that when converted to the nominal units via price index, real bond is a traded instrument in the nominal economy.

To our knowledge, the first pricing model for inflation linked products is proposed by Hughston [1998] where a foreign exchange analogy for inflation derivatives pricing is used. To price TIPS and related derivatives, Jarrow and Yildirim [2003] modeled the evolution of the nominal and real term structures and the consumer price index by using an HJM foreign currency analogy. More specifically, they build a arbitrage free term structure model and derive the no arbitrage drift conditions under the assumption that the real bank account is a tradeable asset in the nominal economy. An alternative approach is proposed independently by Belgrade et al. [2004] and Mercurio [2005] in which the lognormal Libor market model is adapted to inflation modeling. In this setting, the underlying variables are the forward price indices and via modeling the price indices they propose pricing models for inflation swaps. As an extension of the HJM approach proposed by Jarrow and Yildirim [2003], Hinnerich [2008] proposes a pricing model for inflation swaps in which the forward interest rates and the consumer price index are allowed to be driven by a standard multidimensional Wiener process and a general marked point process. Considering a three-factor Gaussian framework, Kjaergaard [2007] models inflation dynamics and gives close form expressions for the index, discount factors and year-on-year inflation swaps.

In this chapter, under the foreign currency analogy we consider a three-factor Gaussian affine model for the pricing of nominal and inflation indexed bonds. The factor process is considered to be composed of the nominal short rate, real short rate and the logarithm of the price index process. By utilizing tools from the theory of affine processes we get closed form expressions for nominal bond price, inflation bond price and the price index. Imposing no-arbitrage assumption under the foreign currency analogy leads to drift restrictions that the factor process has to satisfy. In particular, one of the conditions that the drift matrix of the factor process has to satisfy is the well known Fisher equation which states that the expected appreciation in the price index is equal to the difference between the nominal and real short rates.

When we consider the continuum of nominal bonds, indexed bonds and the real bank account as the tradable instruments of the nominal economy, we are in a complete market setting in the sense that every contingent claim is attainable. In such a case there is no special hedging problem and complete hedge is provided where the hedging strategy follows from delta hedging arguments. In the current study we investigate whether it is possible to hedge an inflation bond of a given maturity by using nominal bonds with different maturities only. Clearly, this question is related with the question of whether the continuum of nominal bonds span the above mentioned affine factor market. That is, when the set of hedging instruments is restricted to the continuum of nominal bonds, does the market still possesses the completeness property. Naturally, the positive answer is not for free in the sense that it necessitates some restrictions on the parameters of the factor process. In this regard, the completeness problem for the market which is spanned by a d-dimensional diffusion factor process and there are d tradable assets is solved by Davis and Obloj [2008] and a criterion, depending on the partial derivatives of the d- price processes with respect to d-factors, is given. However, as we pointed out above, the main question is not on the market completeness, we ask for something less. Our main goal is to find conditions on the parameters of the factor process which make it possible to hedge inflation bonds.

As the second task, this chapter deals with the hedging question we asked above and under the assumption of diagonalizable drift matrix for the factor process, it introduces conditions on the eigenvalues and the eigenvectors of the drift matrix which guarantee the hedge of an inflation indexed bond of a given maturity only by trading nominal bonds of different maturities. The novelty of this work is due to this hedging analysis. Combining no-arbitrage restrictions, that is, the conditions on the drift matrix, with the conditions regarding the hedge and utilizing the criterion given in Davis and Obloj [2008] to investigate the market completeness we find that under the foreign currency analogy and the assumption of diagonalizable drift matrix, cases in which it is possible to hedge inflation bonds by using nominal bonds coincide with cases where the market is spanned by the continuum of nominal bonds. Hence, as the second main contribution of this study, we show that in the current modeling setup there is no such situation in which it is possible to hedge inflation bonds but hedging of other contingent claims is not granted. To sum up, our findings suggests that under the foreign currency analogy and diagonalizable drift matrix assumption hedging of inflation bonds by using nominal bonds is possible if and only if the market is spanned by the nominal bonds.

This chapter is structured as follows. Section 2 describes the underlying modeling framework. In Section 3 we provide pricing equations for nominal and inflation bonds. Section 4 introduces the results on hedging of inflation bonds and Section 5 concludes.

3.2 Model Specification

In the current chapter, we will use the notation $X^\top$ to denote the transpose of the matrix or the vector x. For a matrix X, $X^{(i,j)}$ will represent the $(i,j)^{th}$ entry of the matrix and $X^{(i)}$ will address the i^{th} column. For a vector, $x^{(i)}$ will indicate the i^{th} entry. We will occasionally denote a matrix with columns x_i, $i = 1, \cdots, n$ with $(x_1|x_2|\cdot\cdot|x_n)$. The cross product of two vectors $x,\ y$ is indicated by $x \times y$. The basis vector having 1 in the n^{th} entry and other entries zero is represented by e_n.

We consider a finite time horizon $[0, T^*]$ and a frictionless market, meaning that there are no transaction costs or taxes and short selling is allowed. The uncertainty in the market is represented by the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ where $\mathbb{P}$ is the statistical probability measure. Let $W^{\mathbb{P}}$ be a 3-dimensional standard Brownian motion defined on this probability space. The filtration $(\mathcal{F}_t)$ is assumed to be the natural filtration generated by $W^{\mathbb{P}}$ and to satisfy the usual conditions.

We assume that in the given market the continuum of zero-coupon conventional (nominal) and inflation-indexed bonds are traded. The zero-coupon conventional bond is defined as the instrument which pays one unit of cash at the maturity date T and time $t \le T$ of this bond is denoted by $P(t,T)$. Whereas, at the maturity date T a zero-coupon inflation-indexed bond pays the nominal value of one unit of price index at time T. That is, it pays the nominal amount which has the same purchasing power with the purchasing power of one unit of cash at the issue date. We indicate the time $t \le T$ price of the zero-coupon inflation-indexed bond by $\widetilde{\Pi}(t,T)$.

We assume that prices of conventional and inflation-indexed bonds are driven by nominal interest rate r, real interest rate ρ and the price index I. These three factors are represented by the following vector process X

$$X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ X^{(3)} \end{pmatrix} = \begin{pmatrix} r \\ \rho \\ log(I) \end{pmatrix} \tag{3.1}$$

Under the physical measure $\mathbb{P}$ the following Gaussian dynamics is assumed for the factor process X

$$dX_t = (B^{\mathbb{P}} + \beta^{\mathbb{P}} \cdot X_t)dt + \Sigma \cdot dW_t^{\mathbb{P}}, \quad X_0 = x \in \mathbb{R}_+ \times \mathbb{R}^2, \tag{3.2}$$

where the vector $B^p \in \mathbb{R}^3$ and the matrices β^p and $\Sigma \in \mathbb{R}^{3\times 3}$.

The theory of affine processes suggests that, given the dynamics in (3.2), X follows an affine process (see Definition 2.2 and for details see e.g, Filipović [2009], Chp. 10). Affine processes are widely used in finance due to their analytical tractability. Furthermore, in most cases, affine factor models yield closed form pricing formulae while for others this class of processes make it possible to compute prices numerically via their defining property given in (2.25). In this study, we will also utilize the affine property of the factor processes to get closed form formulae for conventional and inflation indexed bond prices.

The dynamics given in (3.2) are Gaussian implying that the value of the factor may become negative with positive probability. This is not unrealistic for the real short rate and the inflation process as they are observed to be negative time to time. However, this particular choice can be criticized for the nominal short rates. Despite this drawback Gaussian term structure models still receives significant interest due to their computational tractability. At this point, we also want to point out that, in addition to the simplicity it provides for the computation of explicit pricing formulae, this particular choice for the dynamics of the factor process is to provide an extensive hedging analysis that will be given in a later section. Moreover, contrary to the most of the existing models in the literature, the specification of the factor process as in (3.2) allows for the long run as well as instantaneous relations between the real rate, nominal rate and inflation.

Given the dynamics of nominal and real short rate processes r and ρ in (3.2), we can immediately define the nominal and real savings account processes S^r and S^ρ as follows

$$\begin{aligned} S_t^r &= e^{\int_0^t r_s ds}, && (3.3)\\ S_t^\rho &= e^{\int_0^t \rho_s ds}, \quad \forall t \in [0, T^*]. && (3.4) \end{aligned}$$

Now we are ready to define a no-arbitrage criterion for the given bond market as follows.

Definition 3.1 *The bond market is called* arbitrage-free *if*

1. *$P(T,T) = 1$ and $\widetilde{\Pi}(T,T) = I(T)$ for every $T \in [0, T^*]$,*
2. *there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbb{P}$ such that the discounted prices $\frac{P(t,T)}{S_t^r}$, $\frac{\widetilde{\Pi}(t,T)}{S_t^r}$ and $\frac{I(t)S_t^\rho}{S_t^r}$ are $\mathbb{Q}$-martingales.*

In a market which posses the the arbitrage-free property in the sense of Definition 3.1 the conventional and inflation-indexed bond prices can be represented as

$$\begin{aligned} P(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] && (3.5)\\ \widetilde{\Pi}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} I_T \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{X_T^{(3)}} \middle| \mathcal{F}_t \right] && (3.6) \end{aligned}$$

Note that here we implicitly assume that the real bank account $I(t)S_t^\rho$ is tradable. This is also one of the assumptions in Jarrow and Yildirim [2003]. Making this assumption is equivalent to say that the foreign exchange analogy holds under which a drift condition for the inflation process can be deduced. However, the results in Hinnerich [2008] shows that in the HJM framework, the foreign exchange analogy is indeed hold (see Lemma 2.1 in Hinnerich [2008]) irrespective of the assumption on the tradebility of $I(t)S_t^{\rho}$. Unfortunately, as we are following a short-rate modeling framework, the result given in Hinnerich [2008] does not apply to our case. This is because, in the current framework we are not able to define real bond prices in terms of the forward rates. Thus, to be able to get drift conditions on the inflation process, which we will state later on this section, we choose to make this assumption. Here, we wish to remind the worth of having these conditions. The importance is that, without these conditions one can not guarantee the consistency between the real and nominal economies.

In (3.2), the dynamics of the process X is specified under the physical measure $\mathbb{P}$. On the other hand, for pricing purposes we have to use the Q-dynamics of the factor process. With this purpose in mind, in the following part we deal with the equivalent change of measure.

Let λ be an adapted vector process in $\mathbb{R}^d$ such that the stochastic exponent

$$\mathcal{E}_t\left(\int_0^{\cdot} \lambda_s \cdot dW_s^{\mathbb{P}}\right) = e^{\int_0^t \lambda_u \cdot dW_u - \frac{1}{2}\int_0^t \|\lambda_u\|^2 du} \tag{3.7}$$

is a true $\mathbb{P}$-martingale. Then, Girsanov's theorem states that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_{T^*}\left(\int_0^{\cdot} \lambda_s \cdot dW_s^{\mathbb{P}}\right)$$

defines a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and

$$W_t = W_t^{\mathbb{P}} - \int_0^t \lambda_s ds$$

is a d-dimensional $\mathbb{Q}$-Brownian motion. It is a well known feature of the short- rate models that to be able to use arbitrage pricing one has to specify the process λ exogenously and then each specification of λ satisfying the conditions of Girsanov's theorem yields different measures $\mathbb{Q}$. As we stated above, to price inflation indexed and conventional bonds we want to make use of the affine processes theory. In this context, it is crucial to preserve the affine property of the factor process X under a measure transformation. To this end, we follow the results given in Cheridito et al. [2007] and specify λ as an affine function of the state vector X_t.

$$\lambda_t = \lambda_1 + \lambda_2 X_t \tag{3.8}$$

where λ_1 is a 3-vector of constants and λ_2 is a 3×3 matrix. At this point one needs to check whether this specification of λ guarantees that (3.7) is a $\mathbb{P}$ -martingale. The results, more specifically Theorem 2.4 of Cheridito et al. [2005] implies that this condition is satisfied and thus (3.7) is a true $\mathbb{P}$-martingale. This shows the existence of the equivalent measure under which the process X possesses the affine property. In particular, specifying λ as in (3.8) yields the following $\mathbb{Q}$-dynamics for the process X

$$dX_t = (B + \beta \cdot X_t)dt + \sigma \cdot dW_t \tag{3.9}$$

with $B = B^{\mathbb{P}} + \Sigma \cdot \lambda_1$ and $\beta = \beta^{\mathbb{P}} + \Sigma \cdot \lambda_2$.

To guarantee the absence of arbitrage in the market, from now on we will take the equivalent measure $\mathbb{Q}$ specified by λ given in (3.8) as the martingale measure. That is, we assume that the discounted prices are martingales under this specific measure. Indeed, this assumption yields some restrictions on the drift of the factor process X that we will state later on.

3.3 No-arbitrage Pricing

In this part our objective is to find the price of conventional and inflation indexed bonds under the aforementioned modeling framework. Having an arbitrage-free market, we achieve this goal via using the formula (3.5) and (3.6). The following theorem is the first step towards the computation of bond prices and it mainly utilizes the affine property of the factor X under the martingale measure $\mathbb{Q}$.

Theorem 3.1 *The discounted transform $\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s ds} e^{u^\top \cdot X_T} | \mathcal{F}_t]$ of the process X having dynamics* (3.9) *is in the following form*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{u^\top \cdot X_T} \middle| \mathcal{F}_t \right] = e^{\Phi(T-t,u) + \Psi(T-t,u)^\top \cdot X_t} \tag{3.10}$$

for $u \in i\mathbb{R}^3$ and $t \leq T$ and where Φ and $\Psi = (\Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)})^\top$ are given by

$$\Psi(t,u) = e^{\beta^\top t} \cdot u - \int_0^t e^{\beta^\top (t-s)} \cdot (1,0,0)^\top ds \tag{3.11}$$

and

$$\Phi(t,u) = \int_0^t \left(\ \tfrac{1}{2} \Psi(r,u)^\top \cdot \Sigma \cdot \Sigma^\top \cdot \Psi(r,u) + B^\top \cdot \Psi(r,u) \ \right) dr. \tag{3.12}$$

Proof. It follows from Theorem 10.4 of Filipović [2009] that the discounted transform satisfies (3.10) where Φ and Ψ solve the following system of differential equations

$$\begin{aligned}\partial_t\Phi(t,u) &= \frac{1}{2}\Psi(t,u)^\top\cdot\Sigma\cdot\Sigma^\top\cdot\Psi(t,u)+B^\top\cdot\Psi(t,u), \\ \Phi(0,u) &= 0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}\partial_t\Psi(t,u) &= \beta^\top\cdot\Psi(t,u)-(1,0,0)^\top, \\ \Psi(0,u) &= u.\end{aligned}\tag{3.14}$$

for $u\in iR^3$ and $t\leq T$.

We first solve the linear differential equation (3.14) as follows

$$\begin{aligned}e^{-\beta^\top t}\cdot\partial_t\Psi(t,u) &= e^{-\beta^\top t}\cdot\beta^\top\cdot\Psi(t,u)-e^{-\beta^\top t}\cdot(1,0,0)^\top \\ \partial_t(e^{-\beta^\top t}\cdot\Psi(t,u)) &= -\,e^{-\beta^\top t}\cdot(1,0,0)^\top\end{aligned}\tag{3.15}$$

Here, notice that all exponentials appearing in (3.15) are matrix exponentials, i.e., for the square matrix β we have

$$e^{\beta t}=\sum_{k=0}^{\infty}\frac{\beta^k}{k!}t^k.$$

Also recall that exponential of a matrix is always an invertible matrix, thus using $(e^{-\beta^\top t})^{-1}$ we get

$$\Psi(t,u)=(e^{-\beta^\top t})^{-1}\cdot u-(e^{-\beta^\top t})^{-1}\int_0^t e^{-\beta^\top s}\cdot(1,0,0)^\top ds.\tag{3.16}$$

Using the fact that $(e^{-\beta^\top t})^{-1}=e^{\beta^\top t}$ one can write (3.16) as follows

$$\Psi(t,u)=e^{\beta^\top t}\cdot u-\int_0^t e^{\beta^\top(t-s)}\cdot(1,0,0)^\top ds.\tag{3.17}$$

Then, as the last step one can find Φ via simple integration

$$\Phi(t,u)=\int_0^t\left(\ \tfrac{1}{2}\Psi(r,u)^\top\cdot\Sigma\cdot\Sigma^\top\cdot\Psi(r,u)+B^\top\cdot\Psi(r,u)\ \right)dr.\tag{3.18}$$

□

After computing the discounted transform for the factor process X, we can easily compute the conventional and inflation indexed bond prices given by (3.5) and (3.6) respectively. The following corollary of the Theorem 3.1 gives the desired result.

Corollary 3.1 *Let X has the dynamics given in* (3.2)*, then the time $t\leq T$ prices $P(t,T)$ and $\widetilde{\Pi}(t,T)$ are given by*

$$P(t,T) = e^{\phi_1(T-t)+\psi_1(T-t)^\top\cdot X_t}\tag{3.19}$$
$$\widetilde{\Pi}(t,T) = e^{\phi_2(T-t)+\psi_2(T-t)^\top\cdot X_t}\tag{3.20}$$

where we have

$$\psi_1(t) = -\int_0^t e^{\beta^\top(t-s)} \cdot (1,0,0)^\top ds, \tag{3.21}$$

$$\phi_1(t) = \int_0^t \left(\tfrac{1}{2}\psi_1(r)^\top \cdot \Sigma \cdot \Sigma^\top \cdot \psi_1(r) + B^\top \cdot \psi_1(r) \right) dr \tag{3.22}$$

and

$$\psi_2(t) = e^{\beta^\top t} \cdot (0,0,1)^\top - \int_0^t e^{\beta^\top(t-s)} \cdot (1,0,0)^\top ds, \tag{3.23}$$

$$\phi_2(t) = \int_0^t \left(\tfrac{1}{2}\psi_2(r)^\top \cdot \Sigma \cdot \Sigma^\top \cdot \psi_2(r) + B^\top \cdot \psi_2(r) \right) dr. \tag{3.24}$$

Proof. To find the bond price $P(t,T)$, we set $u = (0,0,0)$ in (3.10), (3.11) and (3.12) respectively. This yields

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\middle|\mathcal{F}_t\right] = e^{\phi_1(T-t)+\psi_1(T-t)^\top \cdot X(t)}$$

where

$$\begin{aligned}
\psi_1(t) &:= \Psi(t,(0,0,0)^\top) = -\int_0^t e^{\beta^\top(t-s)} \cdot (1,0,0)^\top ds, \\
\phi_1(t) &:= \Phi(t,(0,0,0)^\top) = \int_0^t \left(\tfrac{1}{2}\psi_1(r)^\top \cdot \Sigma \cdot \Sigma^\top \cdot \psi_1(r) + B^\top \cdot \psi_1(r) \right) dr.
\end{aligned}$$

Following a similar strategy, for the inflation indexed bond price we take $u = (0,0,1)^\top$ in (3.10) and get

$$\widetilde{\Pi}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{X_T^{(3)}}\middle|\mathcal{F}_t\right] = e^{\phi_2(T-t)+\psi_2(T-t)^\top \cdot X_t}$$

with

$$\begin{aligned}
\psi_2(t) &:= \Psi(t,(0,0,1)^\top) = e^{\beta^\top t} \cdot (0,0,1)^\top - \int_0^t e^{\beta^\top(t-s)} \cdot (1,0,0)^\top ds, \\
\phi_2(t) &:= \Phi(t,(0,0,1)^\top) = \int_0^t \left(\tfrac{1}{2}\psi_2(r)^\top \cdot \Sigma\Sigma^\top \cdot \psi_2(r) + B^\top \cdot \psi_2(r) \right) dr.
\end{aligned}$$

This finishes the proof. □

Remark 3.1 *To determine the value of the bond prices for a given set of parameters one needs to compute the value of $e^{\beta^\top}$. Fortunately, there are methods based on some approximation techniques to compute the matrix exponent numerically. In the particular case, where $\beta^\top$ is diagonalizable, computation of the matrix exponential even reduces to an easier task.*

As we mentioned above, absence of arbitrage in the market implies some restrictions on the $\mathbb{Q}$-drift of the factor process X. The following lemma states and proves drift restrictions that arise from no-arbitrage condition.

Lemma 3.1 *Let the foreign currency analogy hold and assume there is no arbitrage in the market. Then the matrix β and the vector B are in the following form*

$$\beta = \begin{pmatrix} \beta^{(1,1)} & \beta^{(1,2)} & \beta^{(1,3)} \\ \beta^{(2,1)} & \beta^{(2,2)} & \beta^{(2,3)} \\ 1 & -1 & 0 \end{pmatrix} \tag{3.25}$$

and

$$B = \left(B^{(1)}, B^{(2)}, -\frac{1}{2} \left\| \Sigma^{\top(3)} \right\|^2 \right)^{\top} \tag{3.26}$$

where $B^{(i)}$ and $\beta^{(i,j)}$, for $i = 1, 2$ and $j = 1, 2, 3$, are constants in $\mathbb{R}$.

Proof. The condition 2 of Definition 3.1 states that in an arbitrage-free market

$$\frac{I(t) S_t^{\rho}}{S_t^r}$$

is a martingale under the risk neutral probability measure $\mathbb{Q}$. Imposing this martingale property to the price process will yield the desired result.

To begin with, we derive the $\mathbb{Q}$ dynamics of the index process I by the Itô's formula as

$$\frac{dI(t)}{I(t)} = \left(B^{(3)} + \beta^{\top(3)} \cdot X_t + \frac{1}{2} \left\| \Sigma^{\top(3)} \right\|^2 \right) dt + \Sigma^{\top(3)} \cdot dW(t) \tag{3.27}$$

Following that, the risk neutral dynamics of the process $Y_t = \frac{I(t) S_t^{\rho}}{S_t^r}$ reads

$$\frac{dY_t}{Y_t} = \left(B^{(3)} + \beta^{\top(3)} \cdot X_t + \frac{1}{2} \left\| \Sigma^{\top(3)} \right\|^2 + \rho_t - r_t \right) dt + \Sigma^{\top(3)} \cdot dW(t) \tag{3.28}$$

Imposing the martingale property, that is, making the drift of (3.28) equal to zero $dt \otimes d\mathbb{Q} - a.s$, we get

$$B^{(3)} + \beta^{\top(3)} \cdot X_t + \frac{1}{2} \|\Sigma^{\top(3)}\|^2 = r_t - \rho_t, \tag{3.29}$$

which can be rewritten as

$$B^{(3)} + \beta^{\top(3)} \cdot X_t + \frac{1}{2} \|\Sigma^{\top(3)}\|^2 = (1, -1, 0) \cdot X_t \tag{3.30}$$

implying that we have

$$\beta^{\top(3)} = (1, -1, 0) \tag{3.31}$$

$$B^{(3)} = -\frac{1}{2} \left\| \Sigma^{\top(3)} \right\|^2 \tag{3.32}$$

and this finishes the proof. □

Remark 3.2 *Equation* (3.29) *is the well known* Fisher *equation which states that the risk neutral appreciation rate of the price index is equal to the difference between nominal and real short rate.*

In the next section, we will continue with the hedging problem of inflation indexed bonds.

3.4 Hedging Problem

When we take the continuum of nominal bonds, indexed bonds and the real bank account as the tradable instruments in the above introduced market, we are in a complete market setting in the sense that every contingent claim depending on the factor process is replicable by a self financing portfolio consisting of the tradable assets. In such a case, hedging of an inflation bond is a trivial task as any indexed bond of a given maturity can be replicated by a portfolio consisting of nominal bonds, indexed bonds and the real bank account where the hedging strategies are provided by the deltas. However, when we restrict the set of hedging instruments to the continuum of nominal bonds, the hedging problem becomes non-trivial and becomes equivalent to the question of whether the continuum of nominal bonds spans the market that we described above. In the following, we will in particular narrow this problem down to the hedging of inflation indexed bonds and ask whether inflation bonds can be replicated by nominal bonds or not. More explicitly, the main question that we will try to answer is the following

Problem 1 *Is it possible to replicate an inflation indexed bond of fix maturity by a portfolio formed by dynamically trading in the risk free account and nominal bonds with different maturities?*

Clearly, this question is less restricted than the market completeness question and we expect that the restrictions guaranteeing the hedge of inflation bonds should be less than the ones needed for the market completeness. If we go back to the market completeness question, for our current setup the market completeness question can be investigated via results given in Davis and Obloj [2008]. This study introduces a completeness criterion for the market which is spanned by a d-dimensional diffusion factor process and there are d tradable assets. If in our setup, we consider the bonds with $d = 3$ different maturities as the tradable assets in the market and imagine the inflation indexed bond as a contingent claim depending on the inflation, we fall into the frame work given in Davis and Obloj [2008]. Thus, we can use the results therein to investigate the market completeness for our setting. Then, having the market completeness, that is, the spanning property of nominal bonds, means the possibility of using nominal bonds in the hedging of other derivative instruments, such as inflation swaps. In the following, in addition to provide an answer to Problem 1, employing the results given in Davis and Obloj [2008] we will also investigate the market completeness.

To answer the question 1, we construct a self financing portfolio which consists of savings account and n nominal bonds with different maturities. The $\mathbb{Q}$-dynamics of the hedging portfolio V reads

$$\begin{aligned} dV_t &= \sum_{i=1}^{n} c_t^i dP(t,T_i) + r_t \underbrace{\left(V_t - \sum_{i=1}^{n} c_t^i dP(t,T_i) \right)} dt \\ &= \sum_{i=1}^{n} c_t^i P(t,T_i) \left(r_t dt + \psi_1(T_i - t)^\top \cdot \Sigma \cdot dW_t \right) + r_t \left(V_t - \sum_{i=1}^{n} c_t^i P(t,T_i) \right) dt \quad (3.33) \end{aligned}$$

with $T \leq T_1 < T_2 < ... < T_n < T^*$ and where $c_t^i \in \mathbb{R}$, $(i = 1, ..., n)$, indicates the number of the nominal bond with maturity T_i we hold at time t and the term with the brace denotes the amount deposited on the savings account or borrowed from the savings account. Here notice that we impose the condition that $T \leq T_i$, that is maturities of the nominal bonds that we are using for hedge must be larger than the maturity of the inflation bond. Also notice that it is sufficient to use $n = 3$ different nominal bonds since we have $\widetilde{W}$ a 3-dimensional Brownian motion, that is 3 independent risk sources in the market. Now, we apply the Itô formula and write the discounted portfolio dynamics as

$$d\left(\frac{V_t}{S_t}\right) = \sum_{i=1}^{3} c_t^i \frac{P(t,T)}{S_t} \psi_1(T_i - t)^\top \cdot \Sigma \cdot dW_t \quad (3.34)$$

On the other hand the discounted inflation indexed bond price process satisfies

$$d\left(\frac{\widetilde{\Pi}(t,T)}{S_t}\right) = \frac{\widetilde{\Pi}(t,T)}{S_t} \psi_2(T - t)^\top \cdot \Sigma \cdot dW_t \quad (3.35)$$

In order to have $V_t = \widetilde{\Pi}(t,T)$ for all $t \leq T$, the vector $\left(c_t^1, c_t^2, c_t^3\right)$ should be chosen in such a way that the quadratic variation terms that appear in dynamics (3.34) and (3.35) become equal at all times t. Existence of such a choice guarantees the hedge of T-inflation bond by a portfolio consisting of nominal bonds.

Assumption 1 *Assume Σ given in the dynamics* (3.2) *is non-degenerate.*

Under Assumption 1, finding an hedging portfolio which replicates $\widetilde{\Pi}(t,T)$ is equivalent to

$$\frac{\widetilde{\Pi}(t,T)}{S_t} \psi_2(T - t)^\top = \sum_{i=1}^{3} c_t^i \frac{P(t,T_i)}{S_t} \psi_1(T_i - t)^\top \quad (3.36)$$

defining

$$\hat{c}_t^i = c_t^i \frac{P(t,T_i)}{\widetilde{\Pi}(t,T)} \tag{3.37}$$

we get

$$\psi_2(T-t) = \sum_{i=1}^{3} \hat{c}_t^i \psi_1(T_i - t)^\top \tag{3.38}$$

Equation (3.38) reveals that, our problem reduces to determine whether for each $t \geq 0$ the vector $\psi_2(T-t)$, for fixed $T \leq T^*$, can be represented as a linear combination of $\psi_1(T_i - t)$, $i = 1, 2, 3$, for some $T \leq T_1 < T_2 < T_3 \leq T^*$. That is, whether for each t the vector $\psi_2(T-t)$ lies in the subset consisting of all possible linear combinations of the vectors $\psi_1(T_i - t)$, $i = 1, 2, 3$. Here, notice that this subset is by definition the span of vectors $\psi_1(T_i - t)$, $i = 1, 2, 3$ and thus a subspace of $\mathbb{R}^3$. Therefore, solving problem 1 becomes equivalent to find the answer of the following question.

Problem 2 *Fix $T \leq T^*$, is the vector $\psi_2(T-t)$ in span$\{\psi_1(T_i - t)|i = 1, ..., 3\}$, $\forall t \leq T$ and for some $T_i, i = 1, 2, 3$, $T \leq T_1 < T_2 < T_3 \leq T^*$?*

Here, we want to emphasize that $T_1 < T_2 < T_3$ are fixed throughout. That is, when at $t = 0$ we start with an hedging portfolio consisting of nominal bonds with maturities T_1, T_2, T_3 it is allowed to trade only that specific bonds at the following time points. Instead, one can also insists on the convention that the time to maturity of the nominal bonds are fixed. However, this convention will lead us to another problem where a different kind of analysis is needed. Here, we first fix t. In particular, without loss of generality we can fix $t = 0$. For fixed T_i, we use equations (3.21) and (3.23) and write the 3-vectors

$$\psi_1(T_i) = -\int_0^{T_i} e^{\beta^\top (T_i - s)}(1,0,0)^\top ds \tag{3.39}$$

$$\psi_2(T) = e^{\beta^\top T}(0,0,1)^\top - \int_0^{T} e^{\beta^\top (T - s)}(1,0,0)^\top ds \tag{3.40}$$

Then, we consider (3.39) as the value of the solution of the following system evaluated at T_i.

$$\begin{aligned} \psi'(z) &= \beta^\top \psi(z) - (1,0,0)^\top \\ \psi(0) &= 0 \end{aligned} \tag{3.41}$$

for $z \geq 0$.

For the reminder of this section, one of the main objectives is to characterize the span generated by the vectors in the range of the function ψ. The next lemma is the first step towards this goal and will significantly ease the later development of our analysis.

Lemma 3.2 *The span of the function ψ, defined as $\mathcal{M} := span\{\psi(z)|z \geq 0\}$, satisfies*

$$\mathcal{M} = span\left\{e^{\beta^\top r}(1,0,0)^\top |r \geq 0\right\} \tag{3.42}$$

Proof. Let us first define

$$\mathcal{N} = span\{e^{\beta^\top r}(1,0,0)^\top |r \geq 0\}$$

We have $\psi : \mathbb{R}_+ \to \mathcal{R}(\psi) \subseteq \mathbb{R}^3$ given by

$$\psi(z) = -\int_0^z e^{\beta^\top (z-s)} \cdot (1,0,0)^\top ds \tag{3.43}$$

$\mathcal{M} \subset \mathcal{N}$ follows from the fact that an integral is a limit of Riemann sums and being a subspace of $\mathbb{R}^3$, $\mathcal{N}$ is closed. To be more precise, let us define $\phi : \mathbb{R} \to \mathcal{R}(\phi) \subseteq \mathbb{R}^3$ with

$$\phi(r) \quad = \quad e^{\beta^\top r} \cdot (1,0,0)^\top, \quad r \geq 0 \tag{3.44}$$

So we have

$$\psi(z) = -\int_0^z \phi(z-s)ds \tag{3.45}$$

It is clear that for all $s \in [0,z]$, the integrand in (3.45) takes values in $\mathcal{R}(\phi)$. Thus, the nth Riemann sum is in $\mathcal{N}$. Using the fact that $\mathcal{N}$ is a closed subspace in $\mathbb{R}^3$, we conclude that the integral in (3.45), being the limit of Riemann sums, is in $\mathcal{N}$.

For the reverse inclusion, again recall the closedness property of $\mathcal{M}$ and deduce that $\psi'(z)$ given by (3.41) is in $\mathcal{M}$, $\forall z \in \mathbb{R}_+$. In particular, $\psi'(0) = (1,0,0)^\top \in \mathcal{M}$ is satisfied. From (3.41) and the linearity property of $\mathcal{M}$, this immediately implies

$$\beta^\top \cdot \psi(z) \in \mathcal{M} \quad \forall z \geq 0 \tag{3.46}$$

and repeating the above argument for the second derivative of ψ we get

$$\beta^\top \cdot \psi'(z) \in \mathcal{M} \quad \forall z \geq 0 \tag{3.47}$$

Then, taking $z = 0$ implies that $\beta^\top \cdot (1,0,0)^\top \in \mathcal{M}$ and thus

$$(\beta^\top)^2 \cdot \psi(z) \in \mathcal{M}, \ \forall z \geq 0 \tag{3.48}$$

Applying this procedure for higher order derivatives, we get the following

$$(\beta^\top)^k \cdot (1,0,0)^\top \in \mathcal{M}, \ \forall k \geq 0 \tag{3.49}$$

Now, using (3.49) and recalling the definition of a matrix exponential, we get

$$e^{\beta^\top r}(1,0,0)^\top = \sum_{k\geq 0} r^k \frac{(\beta^\top)^k}{k!} \cdot (1,0,0)^\top \in \mathcal{M}, \quad \forall r \geq 0. \tag{3.50}$$

Showing that $\mathcal{N} \subset \mathcal{M}$, this finishes the proof. $\square$

Above lemma clearly shows that $\mathcal{M}$ is a $\beta^\top$ invariant subspace in the following sense.

Definition 3.2 *Suppose $A \in \mathbb{R}^{n\times n}$ and $\mathcal{Z}$ is a subspace of $\mathbb{R}^n$. $\mathcal{Z}$ is called A-*invariant *if $A\mathcal{Z} \subseteq \mathcal{Z}$, that is, we have $z \in \mathcal{Z}$ implies $A \cdot z \in \mathcal{Z}$.*

Clearly for $A \in \mathbb{R}^{n\times n}$, $\{0\}$ and $\mathbb{R}^n$ are A-invariant sets. Moreover, the subspace generated by the real eigenvector of a matrix forms a one-dimensional invariant subspace. Furthermore, the plane generated by the eigenvectors which correspond to the complex-conjugate pair of eigenvalues of a matrix is a two-dimensional invariant subspace.

Using the invariance property of $\mathcal{M}$, next corollary extends Lemma 3.2.

Corollary 3.2 *Let $\mathcal{M}$ is defined as in* (3.42)*. Then, $\mathcal{M}$ satisfies*

$$\mathcal{M} = span\left\{e^{\beta^\top r}(1,0,0)^\top | r \geq T\right\} \tag{3.51}$$

Proof. First recall the invariance property of $\mathcal{M}$. That is,

$$e^{\beta^\top T}\mathcal{M} \subseteq \mathcal{M} \tag{3.52}$$

On the other hand, it is acknowledged that an exponential of a matrix is always non-singular. This implies

$$\dim\left(e^{\beta^\top T}\mathcal{M}\right) = \dim\left(\mathcal{M}\right) \tag{3.53}$$

This, together with (3.52) suggests

$$e^{\beta^\top T}\mathcal{M} = \mathcal{M} \tag{3.54}$$

Inserting the definition of $\mathcal{M}$ to the left hand side of (3.54) we get

$$\begin{aligned}\mathcal{M} &= e^{\beta^\top T} span\left\{e^{\beta^\top r}(1,0,0)^\top | r \geq 0\right\} \\ &= span\left\{e^{\beta^\top (T+r)}(1,0,0)^\top | r \geq 0\right\}\end{aligned}$$

Showing that

$$\mathcal{M} = span\left\{e^{\beta^\top r}(1,0,0)^\top | r \geq T\right\} \tag{3.55}$$

this finishes the proof. □

At this point, we want to recall that our objective is to investigate whether

$$\psi_2(T) = e^{\beta^\top T}(0,0,1)^\top - \int_0^T e^{\beta^\top (T-s)}(1,0,0)^\top ds \in \mathcal{M} \tag{3.56}$$

holds. It is clear from the definition of $\mathcal{M}$ that, being equal to $\psi(T)$ the second term of the above expression lies in $\mathcal{M}$. If we can show that the first term is in $\mathcal{M}$, linearity property of $\mathcal{M}$ immediately implies that (3.56) holds. This is what we will try to answer in the following and the following corollary of Lemma 3.2 will convert this problem into an equivalent one.

Corollary 3.3 *Let $\mathcal{M}$ defined as in* (3.42)*, then the following are equivalent:*

(i) $e^{\beta^\top T} \cdot (0,0,1)^\top \in \mathcal{M}$*, for all* $T \leq T^*$.

(ii) $(0,0,1)^\top \in \mathcal{M}$.

Proof. Assume (i) holds. Then, taking $T = 0$ immediately implies $(0,0,1)^\top \in \mathcal{M}$. Now let (ii) holds true. Then from the invariance property of $\mathcal{M}$

$$\begin{aligned} \beta^\top \cdot (1,0,0)^\top &\in \mathcal{M} \\ \left(\beta^\top\right)^2 \cdot (1,0,0)^\top &\in \mathcal{M} \\ &\cdot \\ &\cdot \\ \left(\beta^\top\right)^k \cdot (1,0,0)^\top &\in \mathcal{M},\ \forall k \end{aligned} \tag{3.57}$$

holds. Hence, $e^{\beta^\top T} \cdot (0,0,1)^\top \in \mathcal{M}$ is satisfied. □

We now investigate the structure of the subspace $\mathcal{M}$. Here, it is clear from Lemma 3.2 that the subspace $\mathcal{M}$ is generated by the range of the solution of the following system

$$\phi'(r) = \beta^\top \cdot \phi(r), \qquad r \geq 0 \tag{3.58}$$

$$\phi(0) = (1,0,0)^\top \tag{3.59}$$

and the dynamics of this system depends on the properties of the matrix $\beta^\top$. Also, we wish to point out that $\mathcal{Z} \in \mathbb{R}^3$ is $\beta^\top$ invariant if and only if $\phi(0) \in \mathcal{Z}$ implies $\phi(r) \in \mathcal{Z}$ for all $r \geq 0$. This information, together with the initial condition of the system (3.58) immediately implies that, if $(1,0,0)^\top$ is an eigenvector of the matrix $\beta^\top$, the system stays in the invariant line $\{(c,0,0)^\top | c \in \mathbb{R}\}$. This means that on this line the dynamics of the system is simply the multiplication by the corresponding eigenvalue κ in every time step. In such a case, the system fails to generate any vector of the form $(0,0,x)^\top, x \in \mathbb{R}$ which, in light of Corollary 3.3 implies that the hedge is not possible. Here, remember that the dimension of $\mathcal{M}$ can be at most 3. Next proposition elaborates more on the structure of $\mathcal{M}$ and states the suitable dimension that this subspace should posses for hedging.

Proposition 3.1 *Let foreign currency analogy hold and assume there is no arbitrage in the market. Let $\mathcal{M}$ is defined as in* (3.42)*. If* $\dim(\mathcal{M}) \leq 2$*, then* $(0,0,1)^\top$ *can not lie in* $\mathcal{M}$.

Proof. Let $\dim(\mathcal{M}) = 2$ and assume $(0,0,1)^\top \in \mathcal{M}$. Then, for all $r_1, r_2 \geq 0$ the determinant

$$\left|\Big((0,0,1)^\top|\phi(r_1)|\phi(r_2)\Big)\right| = 0 \tag{3.60}$$

In particular, for $r_1 = 0$

$$\left|\Big((0,0,1)^\top|e^{\beta^\top r_2}\cdot(1,0,0)^\top|(1,0,0)^\top\Big)\right| = 0 \tag{3.61}$$

holds. This suggests,

$$e^{\beta^\top r_2}\cdot(1,0,0)^\top = (x,0,y)^\top \tag{3.62}$$

with some $x,\ y\ \in \mathbb{R}$ and hence

$$(0,1,0)^\top \cdot e^{\beta^\top r_2}\cdot(1,0,0)^\top = 0. \tag{3.63}$$

Thus, we get

$$\mathcal{M} = span\{(1,0,0),(0,0,1)\}. \tag{3.64}$$

Now recall from the proof of the Corollary 3.2 that for $r \geq 0$

$$e^{\beta^\top r}\mathcal{M} = \mathcal{M}$$

implying that

$$\beta^\top\mathcal{M} = \mathcal{M} \tag{3.65}$$

Then, it follows from (3.64) and (3.65) that the matrix $\beta^\top$ is in the following form

$$\beta^\top = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & 0 \\ x_5 & x_6 & x_7 \end{pmatrix} \tag{3.66}$$

for some $x_i \in \mathbb{R}$, $i = 1, ..., 7$. However, this contradicts with the no arbitrage drift restriction given in (3.25) since the last column in (3.66) can never become $(1,-1,0)^\top$.

For $\dim(\mathcal{M}) = 1$ the result immediately follows since starting from the point $(1,0,0)^\top$, $\mathcal{M} = span\{(1,0,0)^\top\}$ and thus $(0,0,1)^\top \notin \mathcal{M}$. □

Ruling out the case $\dim(\mathcal{M}) \leq 2$ in the investigation of hedging possibilities we left with $\mathcal{M} = \mathbb{R}^3$. Our next objective is to find necessary and sufficient conditions on $\beta^\top$ such that $\mathcal{M} = \mathbb{R}^3$ is satisfied. To be able to do this systematically, we find it beneficial to use the Jordan normal form of the matrix $\beta^\top$

Let the matrix $\beta^\top$ has the following Jordan canonical form

$$J = Q^{-1}\beta^\top Q \tag{3.67}$$

with some invertible matrix Q.

Next proposition recalls a result from linear algebra on the connection between invariant subspaces of similarity transformations. This result is needed in the proof of the next theorem

Proposition 3.2 *Let transformation A and J be similar, with the transformation $J = Q^{-1}AQ$. Then a subspace $\mathcal{Z} \in \mathbb{R}^n$ is J invariant if and only if the space*

$$Q\mathcal{Z} = \{Qz | z \in \mathcal{Z}\}$$

is A invariant.

Proof. See Gohberg et al. [2006], Proposition 1.4.2. □

Naturally, the form of J depends very much on the structure of the matrix $\beta^\top$. In the current situation, where $\beta^\top \in \mathbb{R}^{3\times 3}$, there are three main possibilities we summarize below.

1. $\beta^\top$ diagonalizable.
2. $\beta^\top$ has one real and and a pair of complex-conjugate eigenvalues.
3. $\beta^\top$ has real eigenvalues with multiplicity greater than one and the number of independent eigenvectors corresponding to all eigenvalues is less than three.

In the following, we will focus on the case where $\beta^\top$ is diagonalizable. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable if and only if it has n linearly independent eigenvectors. In such a case, the Jordan form of the matrix has a diagonal form having eigenvalues in the diagonal entries. However, not all matrices can be diagonalized. This may be the case when the eigenvalues, given by the roots of the characteristic polynomial are not distinct or not real. That is, when we are not able to find a complete basis of eigenvectors relative to which the matrix A has a diagonal form. However, we want to point out that defective matrices, matrices which are not diagonalizable, are rare in the sense that in the space of real $n \times n$ matrices, defective matrices has Lebesgue measure zero. In other words, if we pick a random square matrix from $\mathbb{R}^{n\times n}$, it will almost surely be diagonalizable. This suggests that, in many of the real world applications, we will face with a matrix $\beta^\top$ which is diagonalizable.

Next theorem gives necessary and sufficient conditions for $\mathcal{M} = \mathbb{R}^3$ in the case where the matrix $\beta^\top$ is diagonalizable and thus J is diagonal given by

$$J = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} \tag{3.68}$$

where κ_1, κ_2, $\kappa_3 \in \mathbb{R}$ are eigenvalues of $\beta^\top$. Here note that, κ_i's, $i = 1, 2, 3$, do not have to be distinct. As long as the analytic and geometric multiplicities of each eigenvalue are same, the $\beta^\top$ matrix has a Jordan form as given in (3.68).

Theorem 3.2 *Let $\mathcal{M}$ is defined as in* (3.42) *and assume $\beta^\top$ be diagonalizable. Let κ_i denotes the i^{th} eigenvalue and v_i, $i = 1, 2, 3$, is the corresponding eigenvector of $\beta^\top$. Then, $\mathcal{M} = \mathbb{R}^3$ if and only if the determinant*

$$\left|(v_i|v_j|e_1^\top)\right| \neq 0$$

and $\kappa_i \neq \kappa_j$ $i, j = 1, 2, 3,$ $i \neq j$.

Proof. if part:

To start with, we take basis vectors

$$e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)$$

and any index $1 \leq i_1 < ... < i_k \leq 3$. Now we choose a vector $z \in \mathbb{R}^3$ which can be written in the following form

$$z = \sum_{j=1}^{k} p_j e_{i_j} \tag{3.69}$$

with some $p_j \in \mathbb{R}$. Clearly, we have $z \in span\{e_{i_1}, ..e_{i_k}\}$. If we apply matrix J to the vector z, we get

$$J \cdot z = \sum_{j=1}^{k} p_j \kappa_{i_j} e_{i_j} \tag{3.70}$$

which also lies in $span\{e_{i_1}, ..e_{i_k}\}$. This shows that

$$\begin{aligned} \mathcal{Z}_i &:= span\{e_i\},\ i = 1, 2, 3, \\ \mathcal{Z}_{i,j} &:= span\{e_i, e_j\},\ i, j = 1, 2, 3; i \neq j, \\ \mathcal{Z} &:= span\{e_1, e_2, e_3\} = \mathbb{R}^3 \end{aligned}$$

are all J invariant subspaces. Then, Proposition 3.2 implies that $Q\mathcal{Z}_i$, $Q\mathcal{Z}_{i,j}$ and $Q\mathcal{Z}$ are all $\beta^\top$ invariant subspaces. For the current case, where we have three independent eigenvectors v_1, v_2, v_3 for matrix $\beta^\top$, Q is nothing but a matrix having these eigenvectors as the columns. Thus, we get

$$Q\mathcal{Z}_i = span\{v_i\}, \quad Q\mathcal{Z}_{i,j} = span\{v_i, v_j\}$$

and

$$Q\mathcal{Z} = span\{v_1, v_2, v_3\} = \mathbb{R}^3.$$

Here, $span\{v_1, v_2, v_3\} = \mathbb{R}^3$ follows from the fact that v_i's are independent 3-vectors and thus form a basis for $\mathbb{R}^3$.

Now assume that matrices $(v_i|v_j|e_1^\top)$, $i,j=\{1,...,3\}$, $i\neq j$, have nonzero determinant. This suggests that $(1,0,0)^\top$ does not lie in one of the one or two dimensional $\beta^\top$ invariant subspaces.

This assumption also yields via Proposition 3.2 that

$$Q^{-1}\cdot(1,0,0)^\top\notin span\{e_i,e_j\}, i=1,2,3 \tag{3.71}$$

and thus $Q^{-1}\cdot(1,0,0)^\top$ is in the form

$$q=(q^{(1)},q^{(2)},q^{(3)})^\top$$

with some non-zero $q^{(1)},\ q^{(2)},\ q^{(3)}$.

On the other hand, for the current case, the solution of the system (3.58) can be written as

$$\phi(r)=Q\cdot e^{Jr}\cdot Q^{-1}\cdot(1,0,0)^\top \tag{3.72}$$

and since the matrix exponential of a diagonal matrix can be performed by exponentiating each of the diagonal elements and Q is nothing but a matrix having eigenvectors of $\beta^\top$ as columns, we have

$$\phi(r)=q^{(1)}e^{\kappa_1 r}v_1+q^{(2)}e^{\kappa_2 r}v_2+q^{(3)}e^{\kappa_3 r}v_3,\quad r\in\mathbb{R} \tag{3.73}$$

Now we shall show that for any time points $r_1<r_2<r_3$, the realizations $\phi(r_1),\ \phi(r_2),\ \phi(r_3)$ are independent vectors and thus form a basis for $\mathbb{R}^3$, implying that the subspace $\mathcal{N}=\mathbb{R}^3$ and hence contains the vector $(0,0,1)^\top$. To this end, we form a matrix $D(t)$, having $\phi(r_1-t),\ \phi(r_2-t),\ \phi(r_3-t)$ in the columns and for $t=0$ we write it in the following form

$$D(0)=(v_1|v_2|v_3)\begin{pmatrix} q^{(1)} & 0 & 0\\ 0 & q^{(2)} & 0\\ 0 & 0 & q^{(3)}\end{pmatrix}\begin{pmatrix} e^{\kappa_1 r_1} & e^{\kappa_1 r_2} & e^{\kappa_1 r_3}\\ e^{\kappa_2 r_1} & e^{\kappa_2 r_2} & e^{\kappa_2 r_3}\\ e^{\kappa_3 r_1} & e^{\kappa_3 r_2} & e^{\kappa_3 r_3}\end{pmatrix} \tag{3.74}$$

Determinants of the first two matrices above are non-zero since $q^{(1)},\ q^{(2)},\ q^{(3)}$, are non-zero and $v_i,\ i=1,2,3$, are independent. Also by inspection the determinant of the last matrix is non-zero provided that $\kappa_i,\ i=1,2,3$ are distinct. This implies that $\mathcal{M}=\mathcal{N}=\mathbb{R}^3$.

only if part:

Now assume $\mathcal{M}=\mathbb{R}^3$. This necessarily implies there exists $0<r_1<r_2$ such that the determinant

$$\left|\Big((1,0,0)^\top|\phi(r_1)|\phi(r_2)\Big)\right|\neq 0 \tag{3.75}$$

That is,

$$\left|\Big((1,0,0)^\top|\ Qe^{Jr_1}Q^{-1}\cdot(1,0,0)^\top|\ Qe^{Jr_2}Q^{-1}\cdot(1,0,0)^\top\Big)\right|\neq 0 \tag{3.76}$$

(3.76) can be written as

$$|(v_1|v_2|v_3)| \left| \begin{pmatrix} q^{(1)} & 0 & 0 \\ 0 & q^{(2)} & 0 \\ 0 & 0 & q^{(3)} \end{pmatrix} \right| \left| \begin{pmatrix} 1 & e^{\kappa_1 r_1} & e^{\kappa_1 r_2} \\ 1 & e^{\kappa_2 r_1} & e^{\kappa_2 r_2} \\ 1 & e^{\kappa_3 r_1} & e^{\kappa_3 r_2} \end{pmatrix} \right| \neq 0 \tag{3.77}$$

The first matrix with the eigenvectors in the columns has a non-zero determinant for $\beta^\top$ diagonalizable. Thus (3.77) implies

$$q^{(1)},\ q^{(2)},\ q^{(3)} \neq 0$$

Here recall that we have $q = (q^{(1)}, q^{(2)}, q^{(3)})^\top = Q^{-1} \cdot (1,0,0)^\top$. Now, using a formula for the matrix inverse we will write the vector q in terms of the eigenvectors v_i, $i = 1,2,3$. We have

$$Q^{-1} = \frac{1}{|Q|} \begin{pmatrix} v_2 \times v_3 \\ v_3 \times v_1 \\ v_1 \times v_2 \end{pmatrix} \tag{3.78}$$

where the cross product $(v_i \times v_j)$ is a row vector and given by

$$v_i \times v_j = \left| \begin{pmatrix} v_i^{(2)} & v_i^{(3)} \\ v_j^{(2)} & v_j^{(3)} \end{pmatrix} \right| e_1 - \left| \begin{pmatrix} v_i^{(1)} & v_i^{(3)} \\ v_j^{(1)} & v_j^{(3)} \end{pmatrix} \right| e_2 + \left| \begin{pmatrix} v_i^{(1)} & v_i^{(2)} \\ v_j^{(1)} & v_j^{(2)} \end{pmatrix} \right| e_3 \tag{3.79}$$

Using the formula (3.79) in (3.78) we get

$$q^{(1)} = \frac{v_2^{(2)} \cdot v_3^{(3)} - v_2^{(3)} \cdot v_3^{(2)}}{|Q|} \neq 0 \tag{3.80}$$

$$q^{(2)} = \frac{v_3^{(2)} \cdot v_1^{(3)} - v_3^{(3)} \cdot v_1^{(2)}}{|Q|} \neq 0 \tag{3.81}$$

$$q^{(3)} = \frac{v_1^{(2)} \cdot v_2^{(3)} - v_1^{(3)} \cdot v_2^{(2)}}{|Q|} \neq 0 \tag{3.82}$$

(3.80)-(3.82) necessitates

$$\frac{v_i^{(3)}}{v_i^{(2)}} \neq \frac{v_j^{(3)}}{v_j^{(2)}}, \quad i,j = 1,2,3,\ i \neq j \tag{3.83}$$

We have the determinant given by

$$\left| (v_i | v_j | (1,0,0)^\top) \right| = v_i \cdot (v_j \times (1,0,0)^\top)$$

which is equal to

$$= v_i \cdot \left(\left| \begin{pmatrix} v_j^{(2)} & v_j^{(3)} \\ 0 & 0 \end{pmatrix} \right| e_1 - \left| \begin{pmatrix} v_j^{(1)} & v_j^{(3)} \\ 1 & 0 \end{pmatrix} \right| e_2 + \left| \begin{pmatrix} v_j^{(1)} & v_j^{(2)} \\ 1 & 0 \end{pmatrix} \right| e_3 \right)$$

$$= v_i^{(2)} v_j^{(3)} - v_i^{(3)} v_j^{(2)} \tag{3.84}$$

(3.83) implies that (3.84) is non zero and thus we show that

$$\left| (v_i | v_j | (1,0,0)^\top) \right| \neq 0, \ i,j = 1,2,3, \ i \neq j$$

must hold.

Now we go back to the last matrix appearing in (3.77). We have

$$\left| \begin{pmatrix} 1 & e^{\kappa_1 r_1} & e^{\kappa_1 r_2} \\ 1 & e^{\kappa_2 r_1} & e^{\kappa_2 r_2} \\ 1 & e^{\kappa_3 r_1} & e^{\kappa_3 r_2} \end{pmatrix} \right| \neq 0$$

which is equivalent to

$$e^{\kappa_1 r_2} \left(e^{\kappa_3 r_1} - e^{\kappa_2 r_1} \right) - e^{\kappa_1 r_1} \left(e^{\kappa_3 r_2 - e^{\kappa_2 r_2}} \right) + e^{\kappa_2 r_1 + \kappa_3 r_2} - e^{\kappa_3 r 1 + \kappa_2 r_2} \neq 0 \tag{3.85}$$

This necessarily implies

$$\kappa_i \neq \kappa_j, \ i,j = 1,2,3, \ i \neq j$$

and this finishes the proof. □

Previous theorem gives conditions on the eigenvalues and eigenvectors of $\beta^\top$ such that $\mathcal{M} = \mathbb{R}^3$ is satisfied. Now we want to check whether these conditions contradict with the specific form of matrix β given in (3.25). For a diagonalizable $\beta^\top$ next theorem states the conditions on the eigenvalues and eigenvectors of $\beta^\top$ such that the matrix β is in the form (3.25).

Theorem 3.3 *Let $\beta^\top$ be diagonalizable. Then, the matrix β has the form as given in* (3.25) *only if the determinant* $\left| (v_i | v_j | e_3^\top) \right| \neq 0$

Proof. Assume β is in the from (3.25). We have

$$\beta^\top = (\cdot | \cdot | (1,-1,0)^\top) = QJQ^{-1}$$

Multiplying both sides with Q^{-1} we get

$$Q^{-1} \beta^\top = JQ^{-1} \tag{3.86}$$

Using formula (3.78) in (3.86) we get

$$\begin{pmatrix} v_2 \times v_3 \\ v_3 \times v_1 \\ v_1 \times v_2 \end{pmatrix} \begin{pmatrix} & & & & 1 \\ \cdot & \Big| & \cdot & \Big| & -1 \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} \underbrace{\begin{pmatrix} v_2 \times v_3 \\ v_3 \times v_1 \\ v_1 \times v_2 \end{pmatrix}}_{V} \tag{3.87}$$

This suggests

$$\kappa_1 V^{(1,3)} = V^{(1,1)} - V^{(1,2)} \tag{3.88}$$
$$\kappa_2 V^{(2,3)} = V^{(2,1)} - V^{(2,2)} \tag{3.89}$$
$$\kappa_3 V^{(3,3)} = V^{(3,1)} - V^{(3,2)} \tag{3.90}$$

We utilize the cross product formula in (3.79) to calculate $V^{(i,j)}$,$i, j = 1, 2, 3$. Inserting this in (3.88)-(3.90) we get

$$\kappa_1 \left(v_2^{(1)} v_3^{(2)} - v_2^{(2)} v_3^{(1)} \right) = v_3^{(3)} \left(v_2^{(2)} + v_2^{(1)} \right) - v_2^{(3)} \left(v_3^{(2)} + v_3^{(1)} \right) \tag{3.91}$$
$$\kappa_2 \left(v_3^{(1)} v_1^{(2)} - v_3^{(2)} v_1^{(1)} \right) = v_1^{(3)} \left(v_3^{(2)} + v_3^{(1)} \right) - v_3^{(3)} \left(v_1^{(2)} + v_1^{(1)} \right) \tag{3.92}$$
$$\kappa_3 \left(v_1^{(1)} v_2^{(2)} - v_1^{(2)} v_2^{(1)} \right) = v_2^{(3)} \left(v_1^{(2)} + v_1^{(1)} \right) - v_1^{(3)} \left(v_2^{(2)} + v_2^{(1)} \right) \tag{3.93}$$

Here notice that under the assumption that $\beta^\top$ is diagonalizable

$$\begin{aligned} \left(v_2^{(1)} v_3^{(2)} - v_2^{(2)} v_3^{(1)} \right) &\neq 0 \\ \left(v_3^{(1)} v_1^{(2)} - v_3^{(2)} v_1^{(1)} \right) &\neq 0 \\ \left(v_1^{(1)} v_2^{(2)} - v_1^{(2)} v_2^{(1)} \right) &\neq 0 \end{aligned} \tag{3.94}$$

has to be satisfied. Otherwise, eigenvectors v_i, $i = 1, 2, 3$, becomes pairwise linearly dependent and cannot form a basis in $\mathbb{R}^3$. Hence, (3.91)-(3.93) are equivalent to the following equations giving the relation between eigenvalues and eigenvectors.

$$\kappa_1 = \frac{v_3^{(3)} \left(v_2^{(2)} + v_2^{(1)} \right) - v_2^{(3)} \left(v_3^{(2)} + v_3^{(1)} \right)}{v_2^{(1)} v_3^{(2)} - v_2^{(2)} v_3^{(1)}} \tag{3.95}$$

$$\kappa_2 = \frac{v_1^{(3)} \left(v_3^{(2)} + v_3^{(1)} \right) - v_3^{(3)} \left(v_1^{(2)} + v_1^{(1)} \right)}{v_3^{(1)} v_1^{(2)} - v_3^{(2)} v_1^{(1)}} \tag{3.96}$$

$$\kappa_3 = \frac{v_2^{(3)} \left(v_1^{(2)} + v_1^{(1)} \right) - v_1^{(3)} \left(v_2^{(2)} + v_2^{(1)} \right)}{v_1^{(1)} v_2^{(2)} - v_1^{(2)} v_2^{(1)}} \tag{3.97}$$

Moreover (3.94) is equivalent to

$$\frac{v_i^{(1)}}{v_i^{(2)}} \neq \frac{v_j^{(1)}}{v_j^{(2)}}, \quad i,j=1,2,3 \tag{3.98}$$

Since we have

$$\begin{aligned}
&\left|(v_i|v_j|e_3^\top)\right| = \\
&= v_i \cdot \left(\left| \begin{pmatrix} v_j^{(2)} & v_j^{(3)} \\ 0 & 1 \end{pmatrix} \right| e_1 - \left| \begin{pmatrix} v_j^{(1)} & v_j^{(3)} \\ 0 & 1 \end{pmatrix} \right| e_2 + \left| \begin{pmatrix} v_j^{(1)} & v_j^{(2)} \\ 0 & 0 \end{pmatrix} \right| e_3 \right) \\
&= v_i^{(1)} v_j^{(2)} - v_i^{(2)} v_j^{(1)}
\end{aligned}$$

(3.98) yields the desired result. □

Remark 3.3 *Theorem 3.2 and Theorem 3.3 together show that when $\beta^\top$ is assumed to be diagonalizable the necessary conditions on β for the form* (3.25) *and $\mathcal{M} = \mathbb{R}^3$ simultaneously are as follows*

1. $\left|(v_i|v_j|e_1^\top)\right| \neq 0$
2. $\left|(v_i|v_j|e_3^\top)\right| \neq 0$
3. $\kappa_i \neq \kappa_j$, $i,j = 1,2,3$.

In Theorem 3.2, we give the conditions which guarantees $\mathcal{M} = \mathbb{R}^3$ and hence $\psi_2(T - t_0)$ lies in $\mathcal{M}$ for fixed $t_0 \in [0,T]$. Now in the next theorem we will first show that when $\mathcal{M} = \mathbb{R}^3$, $\psi_2(T-t) \in \mathcal{M}$ holds for almost all $t \leq T$. Then, the hedging result will follow.

Theorem 3.4 *Assume foreign exchange analogy holds and there is no arbitrage in the market. Assume further that the drift matrix β in* (3.9) *is diagonalizable. Then, it is possible to hedge an inflation index bond with maturity T by a portfolio formed by dynamically trading in the risk free account and nominal bonds with maturities T_i, $i = 1,2,3$, $T \leq T_1 < T_2 < T_3 < T^*$, if and only if*

$$\begin{gathered}
\left|(v_i|v_j|e_1^\top)\right| \neq 0 \\
\kappa_i \neq \kappa_j, i,j = 1,2,3
\end{gathered} \tag{3.99}$$

are satisfied.

Proof. Since the conditions of Theorem3.2 is satisfied $\mathcal{M} = \mathbb{R}^3$. Then, due to Lemma 3.2 there exists $T \leq T_1 < T_2 < T_3$ such that

$$\mathcal{M} = span\left\{\psi(T_1),\ \psi(T_2),\ \psi(T_3)\right\} \tag{3.100}$$

holds. Now our aim is to show

$$\mathcal{M} = span\,\{\psi(T_1 - t), \psi(T_2 - t), \psi(T_3 - t)\} \tag{3.101}$$

is satisfied for all $t \in [0, T]\backslash\mathcal{I}$ where $\mathcal{I}$ is a set which has no accumulation point.

Assume (3.101) does not hold, that is,

$$span\,\{\psi(T_1 - t), \psi(T_2 - t), \psi(T_3 - t)\} \neq \mathcal{M}$$

for all t in some open interval $\mathcal{J} \subset [0, T]$. Then, the determinant function

$$\mathbb{D}(t) = |(\psi(T_1 - t)|\psi(T_2 - t)|\psi(T_3 - t))| = 0$$

foe all $t \in \mathcal{J}$. But since $\psi(z), z \geq 0$ is a real analytic function and so does the sum and product of it, $\mathbb{D}$ is also analytic on $[0, T]$. This implies that $\mathbb{D}(t) = 0$ for all $t \in [0, T]$ (See Krantz and Parks [2002], Corollary 1.2.6). However, this contradicts with (3.100). Showing that (3.101) holds for almost all t and for some T_i $i = 1, 2, 3$ such that $T \leq T_1 < T_2 < T_3 < T^*$ this provides a positive answer for 2. Hence, hedging of an inflation indexed bond with portfolio of nominal bonds is granted. □

Remark 3.4 *In Davis and Obloj [2008], for a market which is spanned by a d- dimensional diffusion factor process and there are d tradable assets, matrix of partial derivatives of pricing functions with respect to the factors is denoted by*

$$G(t, x) = \left(\frac{\partial p_i(t, x)}{\partial x_j}\right)_{1 \leq i,j \leq d}$$

where x_i, $i = 1, ..., d$ indicate factors and p_i, $i = 1, ..., d$ denote the pricing functionals of the traded assets. Then, for this market the completeness criterion is given by

$$G(t_0, x_0) \neq 0$$

for some point (t_0, x_0), provided that p_i, $i = 1, .., d$ are real analytic functions (see Corollary 4.2 of Davis and Obloj [2008]). Since our market setup satisfies the assumptions of Corollary 4.2 of Davis and Obloj [2008], we can use this criterion to check whether we fall into a complete market setting when the conditions of Theorem 3.4 are fulfilled. Indeed, the answer is positive, that is, provided that the conditions in Theorem 3.4 are satisfied, the market is complete in the sense that any contingent claim depending on the factor X can be replicated by a self-financing portfolio of nominal bonds. This is because, for the current setup the matrix G looks like

$$(\psi_1(T_1 - t)\Sigma|\psi_1(T_2 - t)\Sigma|\psi_1(T_3 - t)\Sigma)$$

and we have already shown that this has zero determinant for some T_1, T_2, T_3 and almost all $t \in (0, T)$.

As highlighted in the above remark, when the conditions in Theorem 3.4 are fulfilled, the current market is spanned by the nominal bonds and hence any contingent claim, depending on the factor dynamics can be replicated by a portfolio of nominal bonds with different maturities.

Finding the hedge ratios When hedge is possible, to replicate an inflation indexed bond of maturity T, we first choose three nominal bonds with maturities T_i, $i = 1, 2, 3$, $T \leq T_1 < T_2 < T_3$ and then at each time t, given the observations of nominal and real bond prices we find $\hat{c}_t^i$, $i = 1, 2, 3$ via solving the system of linear equations

$$(\psi_1(T_1 - t)|\psi_1(T_2 - t), \psi_1(T - 3)) \cdot (\hat{c}_t^1, \hat{c}_t^2, \hat{c}_t^3)^\top = \psi_2(T - t) \tag{3.102}$$

More explicitly, we get

$$(\hat{c}_t^1, \hat{c}_t^2, \hat{c}_t^3)^\top = (\psi_1(T_1 - t)|\psi_1(T_2 - t), \psi_1(T - 3))^{-1} \cdot \psi_2(T - t). \tag{3.103}$$

Then, from equation (3.37) one can get

$$c_t^i = \hat{c}_t^i \frac{\widetilde{\Pi}(t, T)}{P(t, T_i)}$$

for $i = 1, 2, 3$.

3.5 Summary and Outlook

In this chapter we propose a three factor affine Gaussian model for the pricing and hedging of inflation indexed bonds. Using the tools from affine processes theory we compute formulae for nominal and inflation indexed bond prices explicitly. We then proceed with an hedging analysis where the objective is to hedge an inflation indexed bond of given maturity with the portfolio of nominal bonds. We are able to give criterion for hedge under the convention that the drift matrix of the factor process is diagonalizable. Moreover, we show that in the current market setup the criterion for hedging of inflation indexed bonds coincides with the market completeness criterion, that is criterion for the spanning property of the continuum of nominal bonds.

Although it is not very likely, in application to real data one may encounter with cases where the drift matrix of the factor process is not diagonalizable. Our future plan is to complete the hedging analysis via providing an answer for such cases. Moreover, we find it interesting to investigate more general cases such as the one where the factor process has non-Gaussian dynamics. Also, testing our model on the real market data is left as a future work.

4

Pricing Model for a Multi-Country Setting[1]

4.1 Introduction

Currently we are living in a financial environment where the economies are strongly linked to each other by the exchange rates. Thus, it is natural to consider the effects of exchange rates on inflation and other macroeconomic variables. To be more precise, one can think of the situation where the appreciation of exchange rates makes imported goods more expensive in terms of the domestic currency. In such a case increase in the price of the imported goods might cause an overall price level increase, that is, inflation. Therefore, in inflation term structure modeling taking the exchange rates into account might be useful.

With the motivation we set out above Slinko [2006] investigates the joint dynamics of the nominal exchange rate, the domestic and foreign term structures and the real exchange rate. In this study, with the same line of reasoning we propose a multi-country setting for inflation linked derivative pricing. The other source of our motivation is the fact that in a multi-country setting, presence of the inflation linked instrument might create extra information about the real exchange rate (RER) and real rate differentials between the countries. The information on RER is important especially for the central banks.The importance of the real exchange rate for a central bank is due to the fact that any changes in the RER is considered as a signal on the future inflation. Thus, having the correct information provides the possibility of taking the sensible monetary action for the price stability (see Kipici and Kesriyeli [1997] for more details).

We summarize the main objectives of this part as follows:

[1]This chapter is based on a joint work with Irina Slinko

1. To find no-arbitrage conditions for a multi-country setting where the continuum of nominal and real bonds are traded;
2. To investigate the implications of the existence of domestic and foreign inflation bonds on real exchange rates;
3. To show the effects of the assumptions about the real exchange rates on domestic and foreign, nominal and real term structures;
4. To introduce and price multi-country inflation linked derivatives such as foreign exchange inflation options and real exchange rate swaps.

Motivated by the goals we set out above, in this part we first consider a multi-country setting where domestic and foreign nominal and real bonds are traded. The price processes of nominal and real bonds of domestic and foreign economies, the price index processes and the exchange rate is assumed to follow Itô process. We then impose no-arbitrage condition to the two country model and this immediately yields drift conditions for real and nominal term structures of the domestic and foreign economies. Moreover, under the no-arbitrage assumption presence of real bonds in the domestic and real economies automatically yields the usual definition of real exchange rate (RER). At this point, we recall the debate on the behavior of real exchange rates. Our results suggest that, theoretically there is not any strict evidence showing whether the RER follows a martingale process or it is mean reverting. However, one can assume one of these conventions and search for the implications of it. To this end, we assume martingale property for RER and find a relation between the real interest rates of the two economies. We further introduce a forward contract written on RER into our model. This yields the forward real exchange rate whose value can be expressed in terms of the price of the domestic and foreign inflation indexed bonds. As mentioned above, this might be considered as a valuable information for the policy makers, in particular for the central banks. We further construct multi-country inflation linked derivatives such as foreign exchange inflation options and real exchange rate swaps. We extensively use the change of numeraire technique to get prices of these derivatives. In particular, we get closed form formulae under the assumption of deterministic volatility in the inflation indexed bond price dynamics.

This part of the thesis is organized as follows. In section 2 we give the notation and dynamics that we used in our model. Section 3 introduces the no-arbitrage and gives the immediate implications of this assumption. Section 4 reveals all results on nominal, real and inflation term structures of the domestic and foreign economies. By giving basics on real exchange rates, Section 5 considers the case where RER is assumed to follow a martingale process and mentions to the forward real exchange rates. Section 6 constructs and and deals with the pricing of inflation derivatives for the two-country setting. Section 7 concludes the paper.

4.2 Notation and Dynamics

For the domestic and foreign economies we consider

- **Nominal bond**: as a debt security which pays the face value of 1 in currency units at the maturity date T.
- **Price index**: as the index number measuring the average price of a specified basket of goods and services.
- **Real bond**: as the debt security that gives its holder one price index unit at maturity T. The price of the real bond is expressed in terms of the respective price index.
- **Inflation indexed bond**: as the bond which pays out the nominal value of the one unit of the price index basket in term of the respective currency. In particular, using the risk-neutral valuation, the price of the inflation indexed bond can be found as the multiple of the time t price of a real bond price and the value of the price index.

As a link between the two economies, nominal and real exchange rates are introduced with the following definitions:

- **Nominal exchange rate**: is defined as the market price of a foreign currency which is expressed in terms of the domestic currency.
- **Real exchange rate**: is defined as the rate at which the domestic and foreign basket of goods and services can be exchanged. That is, it is the domestic basket value of a foreign basket of goods and services.

An underlying probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is given. W is a $d-$dimensional standard Brownian motion living in this probability space and $\mathcal{F}_t = \sigma\{W_s, s \leq t\}$ is assumed. Under the physical measure $\mathbb{P}$ we assume the following dynamics given for the nominal bond price, price index, real bond price, nominal spot exchange rate, real spot exchange rate, nominal savings account and real savings account for domestic and foreign economies where $x \in \{d, f\}$ and 'd' and 'f' indicates domestic and foreign economies respectively.

- Nominal bond price process:

$$\frac{dP^x(t,T)}{P^x(t,T)} = a^x(t,T)dt + b^x(t,T) \cdot dW(t) \tag{4.1}$$

- Real bond price process:

$$\frac{d\Pi^x(t,T)}{\Pi^x(t,T)} = \alpha^x(t,T)dt + \beta^x(t,T) \cdot dW(t) \tag{4.2}$$

- Price index value process:

$$\frac{dI^x(t)}{I^x(t)} = p^x(t)dt + q^x(t) \cdot dW(t) \tag{4.3}$$

- Nominal exchange rate value process:

$$\frac{de(t)}{e(t)} = m(t)dt + n(t) \cdot dW(t) \tag{4.4}$$

- Real exchange rate value process:

$$\frac{d\epsilon(t)}{\epsilon(t)} = \mu(t)dt + \nu(t) \cdot dW(t) \tag{4.5}$$

- Nominal savings account process:

$$dS^x(t) = r^x(t)S^x(t)dt \tag{4.6}$$

- Real savings account process:

$$d\Sigma^x(t) = \rho^x(t)\Sigma^x(t)dt \tag{4.7}$$

where it is assumed that volatility coefficients are $d-$row vectors and all coefficients satisfy the necessary measurability and integrability conditions.

4.3 No Arbitrage

We assume that there is no arbitrage between any of the economies considered. First implication of this assumption is that the real exchange rate satisfies the following:

$$\epsilon(t) = \frac{I^f(t)e(t)}{I^d(t)} \tag{4.8}$$

No-arbitrage assumption necessitates equation (4.8) holds true because of the fact that we have a setting analogues to the four country model in which real exchange rate connects domestic and foreign real economies. To be more clear let us assume we have one unit of foreign real bond and we want to calculate the nominal value of this bond in terms of domestic currency. To achieve this we can follow two distinct routes:

1.) An investor has a unit of foreign real bond whose time t value is $\Pi^f(t,T)$ in terms of foreign price index basket. Nominal value of this bond is $\Pi^f(t,T)I^f(t)$ in units of foreign currency and by definition this is equal to the value of the foreign

inflation indexed bond. Converting this amount with the nominal exchange rate yields $\Pi^f(t,T)I^f(t)e(t)$ which is a value of a traded asset in terms of domestic currency.

2.) Another investor having a unit of foreign real bond chooses to convert $\Pi^f(t,T)$ to an asset in domestic real terms. This is achieved by using the real exchange rate. Corresponding value is $\Pi^f(t,T)\epsilon(t)$ in units of the domestic price index basket. Finally domestic nominal value of the bond is obtained by considering the domestic price index value as $\Pi^f(t,T)\epsilon(t)I^d(t)$.

It is clear that for law of one price to hold

$$\Pi^f(t,T)I^f(t)e(t) = \Pi^f(t,T)\epsilon(t)I^d(t) \tag{4.9}$$

should be satisfied. This implies that equation (4.8) holds showing that the real exchange rate is the nominal rate which is adjusted for the inflation. Indeed this is the way how real exchange rates are defined in the macroeconomics literature and we will manipulate on this result in the next sections.
For given price processes $I^f(t)$, $I^d(t)$ and $e(t)$ application of the Itô formula to equation (4.8) yields the following dynamics for the real exchange rate.

$$\begin{aligned}\frac{d\epsilon}{\epsilon} &= \Big(m(t) + p^f(t) - p^d(t) + n(t)\cdot q^f(t)^\top + (q^d(t) - q^f(t) - n(t))\cdot q^d(t)^\top\Big)\,dt \\ &\quad + \Big(n(t) + q^f(t) - q^d(t)\Big)\cdot dW(t)\end{aligned} \tag{4.10}$$

By the fundamental theorem of asset pricing, assuming no-arbitrage in the model is equivalent to the existence of the risk neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the elements of the following table, which consists of the traded assets in the domestic economy, are $\mathbb{Q}$- martingales.

	Domestic	**Foreign**
nominal bond	$\frac{P^d(t,T)}{S^d(t)}$	$\frac{P^f(t,T)e(t)}{S^d(t)}$
real bond	$\frac{\Pi^d(t,T)I^d(t)}{S^d(t)}$	$\frac{\Pi^f(t,T)I^f(t)e(t)}{S^d(t)}$
nominal savings	$\frac{S^d(t)}{S^d(t)} = 1$	$\frac{S^f(t)e(t)}{S^d(t)}$
real savings	$\frac{\Sigma^d(t)I^d(t)}{S^d(t)}$	$\frac{\Sigma^f(t)I^f(t)e(t)}{S^d(t)}$

4.4 Main Results

By Girsanov‘s theorem, given that $W(t)$ is a $\mathbb{P}$-Brownian motion and $\mathbb{Q}$ is equivalent to $\mathbb{P}$, there exists a predictable, square integrable process $\lambda(t) \in \mathbb{R}^d$ such that

$$d\widetilde{W}(t) = dW(t) - \lambda(t)^\top dt \tag{4.11}$$

is a $\mathbb{Q}$-Brownian motion. Now we are going to examine some fundamental results that follow from the no arbitrage assumption.

1.)
$$\nu(t) = q^f(t) - q^d(t) + n(t) \tag{4.12}$$

   This is a condition which shows the relation between the volatility parameters of the real exchange rate, nominal exchange rate and foreign and domestic price index level. It directly follows from equations (4.5) and (4.10).

2.)
$$r^d(t) = m(t) + r^f(t) + n(t) \cdot \lambda(t)^\top \tag{4.13}$$

   Indeed this relation looks like the well known interest rate parity condition with an additional risk premium term arising from the exchange rate risk. That additional term is important in terms of explaining the puzzles related with the interest rates and nominal exchange rate. To be more precise there are situations, like the ones we witnessed on July 2007 due to the liquidity crunch, in which domestic rate is increasing with the decreasing exchange rates and foreign rate and the additional risk premium term in (4.13) might be useful for explaining such situations.

3.)
$$r^d(t) = p^d(t) + \rho^d(t) + q^d(t) \cdot \lambda(t)^\top \tag{4.14}$$

   This is the very well known Fisher equation with an additional term of the inflation risk premium arising due to the fact that the investors might fail in their inflation expectations, $p^d(t)$, and so ask for a compensation of the amount $q^d(t)\lambda(t)$.

4.)
$$r^f(t) = p^f(t) + \rho^f(t) + n(t) \cdot q^f(t)^\top + q^f(t) \cdot \lambda(t)^\top \tag{4.15}$$

   This is the Fisher equation for the foreign economy.

5.)
$$\rho^d(t) = \rho^f(t) + \mu(t) + \nu(t) \cdot q^d(t)^\top + \nu(t) \cdot \lambda(t)^\top \tag{4.16}$$

   Equation (4.16) is the real interest rate parity condition.

6.)
$$a^d(t,T) - r^d(t) = -b^d(t,T) \cdot \lambda(t)^\top \tag{4.17}$$

7.)
$$a^f(t,T) - r^f(t) = -b^f(t,T) \cdot n(t)^\top - b^f(t,T) \cdot \lambda(t)^\top \tag{4.18}$$

8.)
$$\alpha^d(t,T) - \rho^d(t) = -\beta^d(t,T) \cdot q^d(t)^\top - \beta^d(t,T) \cdot \lambda(t)^\top \tag{4.19}$$

9.)
$$\alpha^f(t,T) - \rho^f(t) = -\beta^f(t,T) \cdot (q^f(t) + n(t))^\top - \beta^f(t,T) \cdot \lambda(t)^{top} \tag{4.20}$$

Equations (4.17)-(4.20) are the drift conditions for the real and nominal term structures of the foreign and the domestic economies.

Following above discussions one of the interesting results we find is that in a multi-country setting with the no-arbitrage assumption, relation (4.8) naturally arises as we introduce the real bonds as the tradable instruments in domestic and foreign economies. As it is noted earlier, this identification of the real exchange rates is very common in macroeconomics literature. Moreover, there is a debate on the behavior of the real exchange rates. One side argues that RER should follow a mean reverting process under the objective measure whereas there are empirical studies which can not reject the hypothesis that RER follows a martingale process. In this study we will investigate implications of assuming martingale property albeit mean reversion hypothesis for real exchange rates is also a good candidate to exploit.

4.5 Results on Real Exchange Rates

Before the World War I there was the gold standard enabling currencies to be converted into gold from a fix rate. As a result there was implied exchange rates between the currencies. However, during the war this system collapsed and there was a need for another system which makes it possible to set exchange rates between the currencies. At that point Cassel [1921] proposed the use of purchasing power parity (PPP) to determine exchange rates. PPP states that when represented in a common currency, price levels of the countries should be equal. This is mainly based on the law of one price under the assumptions that the price index baskets of the countries consist of the same goods, all final goods and factors of production are tradable between the countries and etc. (see Rogoff [1996] for details about PPP). In more formal terms PPP means that the value of the domestic price index is equal to the product of the foreign price index value and the nominal exchange rate. In the previous sections it is shown that real exchange rates are the nominal rates which are adjusted for the relative national price levels. Because of this, a non-constant real exchange rate indicates that there are deviations from PPP.

It is observed that RERs deviate from the value assigned by the purchasing power parity (PPP) in the short run (see Rogoff [1996]). The line of reasoning behind this result is given by the fact that the composition of the price index baskets of two countries are rarely the same. Indeed this is not the unique reason that prevents real exchange rates to follow the PPP value. If this was the case, being identical all around the world, Big-Mac prices would be able to reflect the true market rates[3]. In Rogoff [1996] and Taylor and Taylor [2004] one of the main reasons for the possibility for prices of identical goods to differ across the countries is given by that some of the factors of production, such as labor and land, can not be traded across the countries. Besides, tax differentials across the countries, tariffs or other trade restrictions are listed as other reasons. In a nut shell, the reason why the law of one price might fail to apply in such an environment is that, economic agents are not able to trade freely over the price index baskets. However, the existence of the inflation indexed bonds in the foreign and domestic economies and possibility of trading them is equivalent to the trading of the price index baskets. Thus, it is reasonable to speculate on the movement of RER via considering the dynamics of the inflation indexed bonds of the domestic and foreign economies.

4.5.1 Martingale Property for Real Exchange Rates

In the literature there are two distinct idea about the dynamics of real exchange rates. On one hand RER is argued to follow a mean reverting process under the physical measure. On the other hand it is driven by a martingale process. Most of the empirical studies failed to reject the hypothesis that the real exchange rate follows a martingale process (see Rogoff [1996], Adler and Lehmann [1983]). If we assume the martingale property of RER this implies via (4.10) the following

$$\mu(t) = m(t) + p^f(t) - p^d(t) + n(t) \cdot q^f(t)^\top - \nu(t) \cdot q^d(t)^\top = 0 \tag{4.21}$$

To see the further implication of such an assumption we use our previous findings. From (4.16) we get that the martingale property for RER has the following implication on the real rate differential between the two economies

$$\rho^d(t) - \rho^f(t) = \nu(t) \cdot q^d(t)^\top + \nu(t) \cdot \lambda(t)^\top. \tag{4.22}$$

[3]Big Mac Index: is based on the theory of purchasing-power parity, i.e., the notion that a dollar should buy the same amount in all countries. Thus in the long run, the exchange rate between two countries should move towards the rate that equalizes the prices of an identical basket of goods and services in each country. McDonald's Big Mac is treated as a basket, which is produced in about 120 countries. The Big Mac PPP is the exchange rate that would mean hamburgers cost the same in America as abroad. Comparing actual exchange rates with PPPs indicates whether a currency is under- or overvalued. From (http://www.economist.com/markets/bigmac/about.cfm).

From (4.21) we get

$$p^d(t) - p^f(t) = m(t) + n(t) \cdot q^f(t)^\top - \nu(t) \cdot q^d(t)^\top \tag{4.23}$$

Summing (4.22) and (4.23) and inserting the expressions for $p^d(t)$ and $p^f(t)$ implied by (4.14) and (4.15) respectively, we have

$$r^d(t) - r^f(t) - (q^d(t) - q^f(t)) \cdot \lambda(t)^\top = m(t) + \nu(t) \cdot \lambda(t)^\top \tag{4.24}$$

Using (4.12) we deduce that

$$r^d(t) - r^f(t) = m(t) + n(t) \cdot \lambda(t)^\top \tag{4.25}$$

Equation (4.25) tells that, martingale assumption on RER is equivalent to argue that the nominal rate differentials between the two economies is given by the sum of appreciation rate of the exchange rate and the risk premium arising from exchange rate uncertainty.

4.5.2 Forward Real Exchange Rates

In this part we will derive a formula for the forward real exchange rate implied by a trade in the price index baskets of the domestic and foreign countries. Assume we have two time instants with $t < T$ where t is the current time and T is the expiration time. At time t set a forward contract between the two parties which makes it possible to exchange one unit of the foreign price index basket for the fraction A of the domestic basket at time T. As an example 1 Euro zone price index basket for a 1.5 US basket, et cetera. Indeed this is a forward contract on real exchange rate. Under the domestic risk neutral measure $\mathbb{Q}$ such a contract has the following value at time t:

$$F(t,T) = \mathbb{E}^Q[e^{-\int_t^T r^d(s)ds}(I^f(T)e(T) - A(t,T)I^d(T)|F_t] \tag{4.26}$$

which is equal to

$$= \mathbb{E}^Q[e^{-\int_t^T r^d(s)ds} I^f(T)e(T)|F_t] - \mathbb{E}^Q[e^{-\int_t^T r^d(s)ds} A(t,T)I^d(T))|F_t]$$

Let us define the likelihood ratio L_t with the following

$$L_T = \frac{e(T)}{e(t)} e^{-\int_t^T (r^d(s) - r^f(s))ds}$$

One can show that L_T is a martingale with an expected value equal to 1. Now let us define the foreign martingale measure with the following

$$\frac{dQ^f}{dQ} = L_T \tag{4.27}$$

In the first term of the expression (4.26) if we change the measure from Q to Q^f we have the following

$$\mathbb{E}^Q[e^{-\int_t^T r^d(s)ds} I^f(T)e(T)|F_t] = e(t)\widetilde{\Pi}^f(t,T)$$

Here after changing the measure and by using the definition of inflation indexed bond in the first and second terms we get the following formula for the value of the forward contract:

$$F(t,T) = e(t)\widetilde{\Pi^f}(t,T) - A(t,T)\widetilde{\Pi^d}(t,T)$$

We know from the general theory that,for the initial date t, F(t,T)=0 must be satisfied. This yields the following expression for the rate A(t,T):

$$A(t,T) = \frac{e(t)\widetilde{\Pi^f}(t,T)}{\widetilde{\Pi^d}(t,T)} \tag{4.28}$$

Equation (4.28) tells us that one can find the forward real exchange rate by observing the spot nominal exchange rate and the price of foreign and domestic inflation indexed bonds. This might be a valuable information for policy makers and in particular for the central banks due to reasons we mentioned before and to the fact that RER is accounted as one of the indicators which shows the comparative advantage of a country.

4.6 Inflation Linked Foreign Exchange Derivatives or Real Exchange Derivatives

Inflation derivatives market is enlarging with the growing amount of the inflation features, inflation swaps and inflation options traded. One of the reasons why investors demand such products is that these contracts guarantee a bound for the purchasing power of the nominal income, in other words they ensure the lowest bound for the number of the domestic baskets that a unit of domestic currency can buy. However, to our knowledge there are not any contracts which covers the joint risk based on the unfavorable movements of the foreign inflation and exchange rates. Such a contract would guarantee the foreign purchasing power of a domestic income. To achieve this, our aim is to construct derivative instruments enabling the simultaneous hedging of foreign inflation and exchange rates with guaranteeing an amount of foreign real income to the domestic investor.

4.6.1 Forward Contracts on Inflation Indexed Bonds

We consider the domestic and foreign inflation bonds maturing at time S. Let X be the value of the contract which makes it possible to exchange domestic and foreign inflation indexed bonds at rate A, with an expiration date T satisfying $T \leq S$. Here, the rate $A(t,T,S)$ indicates the fraction of domestic inflation bonds that will be exchanged for foreign inflation bonds, that is, A can be considered as the domestic relative forward price of two countries inflation bonds. We first find the value of A which makes the contract fair, i.e, value which makes the initial value of the contract equal zero.

$$X(t,T) = E^{Q}[e^{-\int_t^T r_u du}(e(T)\widetilde{\Pi}^f(T,S) - A^*(t,T,S)\widetilde{\Pi}^d(T,S))|F_t] = 0 \tag{4.29}$$

$$E^{Q}[\underbrace{e^{-\int_t^T r_u du}e(T)\widetilde{\Pi}^f(T,S)}_{I}|F_t] = E^{Q}[\underbrace{e^{-\int_t^T r_u du}A^*(t,T,S)\widetilde{\Pi}^d(T,S)}_{II}|F_t] \tag{4.30}$$

In equation (4.30) terms I and II are the discounted value of two domestically tradeable assets. Thus they are martingale under the domestic nominal risk neutral measure Q. As a result, equation (4.30) can be written as follows

$$e(t)\widetilde{\Pi}^f(t,S) = A^*(t,T,S)\widetilde{\Pi}^d(t,S)$$

Implying that we have

$$A^*(t,T,S) = \frac{e(t)\widetilde{\Pi}^f(t,S)}{\widetilde{\Pi}^d(t,S)} \tag{4.31}$$

Here notice that when S goes to T we have the forward real exchange rate and when S=T=t we have the equation for spot real exchange rates (see previous section). Now let us take the ratio $\frac{e(t)\widetilde{\Pi}^f(t,S)}{\widetilde{\Pi}^d(t,S)}$. When we take the $\widetilde{\Pi}^d(t,S)$ bond as a numeraire we have the following expectation under the S-forward inflation measure $Q^{I_S^d}$:

$$\widetilde{\Pi}^d(t,S)E^{Q^{I_S^d}}[\frac{e(T)\widetilde{\Pi}^f(T,S)}{\widetilde{\Pi}^d(T,S)}|F_t] \tag{4.32}$$

which is equal to

$$= B(t)E^{Q}[\frac{e(T)\widetilde{\Pi}^f(T,S)}{B(T)}|F_t] = e(t)\widetilde{\Pi}^f(t,S) \tag{4.33}$$

since in (4.33) the term inside the expectation is a martingale. Therefore we have

$$E^{Q^{I_S^d}}[\frac{e(T)\widetilde{\Pi}^f(T,S)}{\widetilde{\Pi}^d(T,S)}|F_t] = \frac{e(t)\widetilde{\Pi}^f(t,S)}{\widetilde{\Pi}^d(t,S)} \tag{4.34}$$

That is $\frac{e(T)\widetilde{\Pi}^f(T,S)}{\widetilde{\Pi}^d(T,S)}$ is a $Q^{I_S^d}$ martingale. It is known that forward rates are the unbiased predictors of the future spot rates under the respective forward measure. Thus if we use the same analogy and the fact that $\widetilde{\Pi}^f(S,S) = I(S)$ we have the following:

$$\frac{e(t)\widetilde{\Pi}^f(t,S)}{\widetilde{\Pi}^d(t,S)} = E^{Q^{I_S^d}}[\frac{e(S)\widetilde{\Pi}^f(S,S)}{\widetilde{\Pi}^d(S,S)}|F_t] = E^{Q^{I_S^d}}[\varepsilon(S)] \tag{4.35}$$

We will use this result in the next sections.

4.6.2 Inflation Linked Foreign Exchange Option

We consider a contract where domestic and foreign parties aim to fix purchasing power of their income in terms of the cross economy's basket. Namely, we consider the call option which gives the holder the right but not the obligation of changing the S-inflation linked bond of foreign country for the bond of domestic country at expiration time T. At time t, the payoff of the option is as follows:

$$X = Max[e(T)\widetilde{\Pi}^f(T,S) - \widetilde{\Pi}^d(T,S), 0] \tag{4.36}$$

As it is very well known that the fair price at time t of a T-contingent claim X is given by the formula

$$\Theta(t,X) = E^Q[e^{-\int_t^T r(s)ds} X|F_t]. \tag{4.37}$$

where Q represents the domestic nominal risk neutral measure. Whence, time t value of the option equals

$$E^Q[e^{-\int_t^T r(s)ds}(e(T)\widetilde{\Pi}^f(T,S) - \widetilde{\Pi}^d(T,S))\mathfrak{I}_{\{e(T)\widetilde{\Pi}^f(T,S)\geq\widetilde{\Pi}^d(T,S)\}}|F_t] \tag{4.38}$$

By using the linearity of the expectation operator we can rewrite the equation (4.38) as

$$= \underbrace{E^Q[e^{-\int_t^T r(s)ds} e(T)\widetilde{\Pi}^f(T,S)\mathfrak{I}_{\{e(T)\widetilde{\Pi}^f(T,S)\geq\widetilde{\Pi}^d(T,S)\}}|F_t]}_{I} \tag{4.39}$$

$$\underbrace{E^Q[e^{-\int_t^T r(s)ds}\widetilde{\Pi}^d(T,S)\mathfrak{I}_{\{e(T)\widetilde{\Pi}^f(T,S)\geq\widetilde{\Pi}^d(T,S)\}}|F_t]}_{II}$$

In what follows we will use the standard change of numeraire technic to compute the value of the equation (4.39). In equation (4.39), part I the domestic value of the foreign bond is a domestically traded asset with positive price process. Thus we can use it as a numeraire and get the following

$$I = e(t)\widetilde{\Pi}^f(t,S)E^{QI_S^f}[\mathfrak{I}_{\{e(T)\widetilde{\Pi}^f(T,S)\geq\widetilde{\Pi}^d(T,S)\}}|F_t] \tag{4.40}$$

where $Q^{I_S^f}$ is the S-foreign inflation measure. It is obvious that (4.40) is equal to

$$I = e(t)\widetilde{\Pi}^f(t,S)Q^{I_S^f}\left(\frac{\widetilde{\Pi}^d(T,S)}{e(T)\widetilde{\Pi}^f(T,S)} \leq 1\right) \tag{4.41}$$

To compute the expectation in II we follow a similar way and take the domestic inflation bond as a numeraire. This yields the following

$$II = \widetilde{\Pi}^d(t,S)E^{QI_S^d}[\mathfrak{I}_{\{e(T)\widetilde{\Pi}^f(T,S)\geq\widetilde{\Pi}^d(T,S)\}}|F_t] \tag{4.42}$$

which becomes

$$II = \widetilde{\Pi}^d(t,S) Q^{I_S^d}(e(T)\frac{\widetilde{\Pi}^f(T,S)}{\widetilde{\Pi}^d(T,S)} \geq 1) \tag{4.43}$$

where $Q^{I_S^d}$ is the S-domestic inflation measure. It is clear from the equations (4.41) and (4.43) that if we can compute the probabilities in these equations, the closed form formula for the call option price directly follows. We start with introducing the process $Z(t) = \frac{e(T)\widetilde{\Pi}^f(T,S)}{\widetilde{\Pi}^d(T,S)}$. Obviously Z is a price process which is normalized by the S-domestic inflation indexed bond. Thus, it is a martingale with respect to the domestic inflation measure $Q^{I_S^d}$. That is, Z has the following dynamics

$$dZ(t) = Z(t)\sigma_Z dW^{I_S^d}(t) \tag{4.44}$$

where $W^{I_S^d}(t)$ is a multi-dimensional $Q^{I_S^d}$ Wiener Process. Indeed (4.44) is a Log-normal process and the solution is given by

$$Z(T) = Z(t)\exp\{-\frac{1}{2}\int_t^T ||\sigma_Z(u)||^2 du + \int_t^T \sigma_Z(u) dW^{I_S^d}(u)\} \tag{4.45}$$

Therefore, our problem reduces to find the σ_Z. Indeed if σ_Z is non-stochastic we can guarantee an explicit solution for our option pricing problem. Namely, if we have σ_Z deterministic then the stochastic integral in (4.45) has a normal distribution with mean zero and variance $\Sigma^2(T) = \int_t^T ||\sigma_Z(u)||^2 du$. For the sake of completeness of the explicit pricing formula from now on we will assume that σ_Z is non-stochastic. Now the aim is to find the explicit form of σ_Z where under the domestic martingale measure the domestic and foreign inflation bond price processes and the nominal exchange rate process is in the form (4.2) and (4.4) respectively. To achieve this we apply the integration by parts formula to Z, get the Q dynamics and derive the following formula for σ_Z

$$\sigma_Z(t) = n(t) + q^f(t) - q^d(t) + \beta^f(t) - \beta^d(t) \tag{4.46}$$

Utilizing the no arbitrage conditions that we found above, we can write (4.46) as follows

$$\nu(t) - (\beta^d(t) - \beta^f(t)) \tag{4.47}$$

which makes it possible to compute the probability in the second part of the equation (4.39). To derive the probability in the part I of equation (4.39), we introduce the following process

$$Y(t) = \frac{1}{Z(t)} = \frac{\widetilde{\Pi}^d(T,S)}{e(T)\widetilde{\Pi}^f(T,S)} \tag{4.48}$$

Y is a martingale under the numeraire $e(T)\widetilde{\Pi}^f(T,S)$ and it has the following dynamics

$$dY(t) = Y(t)\sigma_Y dW^{I_S^f}(t) \tag{4.49}$$

where $W^{I_S^f}(t)$ is a multi-dimensional $Q^{I_S^f}$ Wiener Process. One can follow the similar way as above to clarify that σ_Y satisfies

$$\sigma_Y = -\sigma_Z \tag{4.50}$$

Finally, via following the general option pricing formula given in Björk [2009] we get the final valuation formula for an inflation-linked foreign exchange option.

$$\Theta(t, I^f, I^d, e) = e(t)\widetilde{\Pi}^f(t,S)N(d_1) - \widetilde{\Pi}^d(t,S)N(d_2) \tag{4.51}$$

Here N represents the cumulative Standard Normal Distribution function and d_1 and d_2 are given by the following

$$d_2 = \frac{\ln(\frac{e(t)\widetilde{\Pi}^f(t,S)}{\widetilde{\Pi}^d(t,S)}) - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}} \tag{4.52}$$

$$d_1 = d_2 + \sqrt{\Sigma^2(T)} \tag{4.53}$$

where

$$\Sigma^2(T) = \int_0^T ||\sigma_Z(t)||^2 dt$$

4.6.3 Zero Coupon Relative Inflation Swaps

A swap is a contract to exchange the cash flows between the two parties. Traditionally investors enter into a swap contract to exchange securities to change the maturity, currency or type of the rate from fix to floating or nominal to real. In this study we consider a zero coupon swap with two parties A and B. Given a set of dates $T_1, T_2, ..., T_M$ party A pays party B the change in the relative inflation index levels of two countries over a predefined period T_0, while part B pays the pre-specified fix rate K. Let us assume we have $T_M = M$ years. In zero coupon relative inflation swap (ZCRIS) Party B pays

$$N[(1+K)^M - 1] \tag{4.54}$$

and get

$$N[\frac{\frac{e(T_M)I^f(T_M)}{I^d(T_M)}}{\frac{e(T_0)I^f(T_0)}{I^d(T_0)}}] \tag{4.55}$$

Thus, from very fundamental no arbitrage arguments the value at time t ,$0 \leq t < T_M$, of the inflation linked part of the swap is

$$ZCRIS(t, T_M, N, I^f, I^d, e) = NE^Q[e^{-\int_t^{T_M} r(u)d(u)}[\frac{\frac{e(T_M)I^f(T_M)}{I^d(T_M)}}{\frac{e(T_0)I^f(T_0)}{I^d(T_0)}} - 1]|F_t] \tag{4.56}$$

To evaluate (4.56) we use very simple replicating strategies. Namely, time t value of 1 unit which will be paid in time T_M is obviously equals to the nominal domestic zero bond price at t, i.e., $P^d(t, T_M)$. Similarly time t value of $\frac{e(T_M)I^f(T_M)}{I^d(T_M)}$ is equal to $\frac{e(t)\widetilde{\Pi}^f(t,T_M)}{\widetilde{\Pi}^d(t,T_M)}$. This is because of the fact that buying and holding a T_M inflation bond up to time T_M yields $I(T_M)$. Hence, we have

$$ZCRIS(t, T_M, N, I^f, I^d, e) = N[\frac{\frac{e(t)\widetilde{\Pi}^f(t,T_M)}{\widetilde{\Pi}^d(t,T_M)}}{\frac{e(T_0)I^f(T_0)}{I^d(T_0)}} - P^d(t, T_M)] \tag{4.57}$$

We hereby use the definition of inflation bonds and equation (4.57), to calculate the $t = T_0$ value of the inflation linked payment. The result is as follows

$$ZCRIS(T_0, T_M, N, I^f, I^d, e) = \frac{\Pi^f(T_0, T_M)}{\Pi^d(T_0, T_M)} - P^d(T_0, T_M) \tag{4.58}$$

It is known that, no arbitrage necessitate that the time T_0 value of the swap should be equal to 0. That is,

$$N[\frac{\Pi^f(T_0, T_M)}{\Pi^d(T_0, T_M)} - P^d(T_0, T_M)] = NP^d(T_0, T_M)[(1 + K)^M - 1] \tag{4.59}$$

should be satisfied. (4.59) can be written as

$$\frac{\Pi^f(T_0, T_M)}{\Pi^d(T_0, T_M)} = P^d(T_0, T_M)(1 + K)^M \tag{4.60}$$

Thus, if the market quotes the value for K , using the quoted nominal bond prices as well, we can find the ratio of the real rates of the two countries. Indeed, one can also check the results of the martingale property of real exchange rates on the rate K.

4.6.4 Year on Year Relative Inflation Swap

Given the M periods with $T_1, T_2, ...T_M$ cash flow structure of the YYRIS is that at the time point T_i party A pays

$$N\phi_i[\frac{\frac{e(T_i)I^f(T_i)}{I^d(T_i)}}{\frac{e(T_{i-1})I^f(T_{i-1})}{I^d(T_{i-1})}} - 1] \tag{4.61}$$

whereas party B pays the fix amount

$$N\phi_i K \tag{4.62}$$

where ϕ_i is the year fraction of the length of the interval $[T_{i-1}, T_i]$. For $t < T_i$ we have the following value for the floating leg of the YYRIS

$$YYRIS(t, T_M, N, I^f, I^d, e) = N\phi_i E^Q[e^{-\int_t^{T_i} r(u)du}[\frac{\frac{e(T_i)I^f(T_i)}{I^d(T_i)}}{\frac{e(T_{i-1})I^f(T_{i-1})}{I^d(T_{i-1})}} - 1]|F_t] \tag{4.63}$$

Equation (4.63) can be written as

$$N\phi_i E^Q[e^{-\int_t^{T_{i-1}} r(u)du} \underbrace{E^Q[e^{-\int_{T_{i-1}}^{T_i} r(u)du} \frac{\frac{e(T_i)I^f(T_i)}{I^d(T_i)}}{\frac{e(T_{i-1})I^f(T_{i-1})}{I^d(T_{i-1})}}|F_{T_{i-1}}}|F_t] \tag{4.64}$$

In equation (4.64) the term with the underbrace is nothing but the time T_{i-1} value of a ZCRIS which will mature at time T_i. Thereby (4.64) becomes

$$= N\phi_i E^Q[e^{-\int_t^{T_{i-1}} r(u)du}[\frac{\Pi^f(T_{i-1}, T_i)}{\Pi^d(T_{i-1}, T_i)} - P^d(T_{i-1}, T_i)]|F_t] \tag{4.65}$$

which is equal to

$$= N\phi_i E^Q[e^{-\int_t^{T_{i-1}} r(u)du}\frac{\Pi^f(T_{i-1}, T_i)}{\Pi^d(T_{i-1}, T_i)}|F_t] - N\phi_i P^d(t, T_i) \tag{4.66}$$

Here the value of the expectation depends on the model specification of the real bonds of domestic and foreign economies.

4.6.5 Inflation Linked Foreign Exchange Swap

Assume now again that we have M periods with $T_1, T_2, ...T_M$. ILFES is set in such a way that at each T_i, for $i = 1...M$ the following transaction occurs between the two parties. Party A pays the floating amount $e(T_i)I^f(T(i))$ and party B pays the floating amount $KI^d(T_i)$ where K is the pre-specified swap rate. Time 0 value of the associated swap is equal to

$$\begin{aligned} E^Q[e^{-\int_0^{T_1} r(u)du}(e(T_1)I^f(T_1) - KI^d(T_1))] + E^Q[e^{-\int_0^{T_2} r(u)du}(e(T_2)I^f(T_2) - KI^d(T_2))] + \cdots \\ + E^Q[e^{-\int_0^{T_M} r(u)du}(e(T_M)I^f(T_M) - KI^d(T_M))] \end{aligned} \tag{4.67}$$

Following the similar arguments as in section 6.1 we can write equation (4.67) as follows

$$\begin{aligned} e(0)\widetilde{\Pi}^f(0, T_1) - K\widetilde{\Pi}^d(0, T_1) + e(0)\widetilde{\Pi}^f(0, T_2) - K\widetilde{\Pi}^d(0, T_2) + \cdots \\ + e(0)\widetilde{\Pi}^f(0, T_M) - K\widetilde{\Pi}^d(0, T_M) \end{aligned} \tag{4.68}$$

Setting the time 0 value equal to zero implies the following value for K

$$K = \frac{e(0) \sum_{i=1}^{M} \widetilde{\Pi}^f(0, T_i)}{\sum_{i=1}^{M} \widetilde{\Pi}^d(0, T_i)} \tag{4.69}$$

Using the definition of inflation linked bonds we can rewrite K as follows

$$K = \frac{e(0) I^f(0) \sum_{i=1}^{M} \Pi^f(0, T_i)}{I^d(0) \sum_{i=1}^{M} \Pi^d(0, T_i)} = \frac{\varepsilon(0) \sum_{i=1}^{M} \Pi^f(0, T_i)}{\sum_{i=1}^{M} \Pi^d(0, T_i)} \tag{4.70}$$

4.7 Summary and Outlook

This part proposes a multi-country modeling framework for the pricing of inflation indexed products. Considering a multi-country setting where the continuum of domestic and foreign nominal and real bonds are traded we first specify the price processes of nominal and real bonds of domestic and foreign economies, the price index processes and the exchange rate as Itô processes. We then impose no-arbitrage condition to the two country model and this immediately yields drift conditions for real and nominal term structures of the domestic and foreign economies. Under the no-arbitrage assumption presence of real bonds in the domestic and real economies yields the usual definition of real exchange rate (RER). There, recalling the debate on the behavior of real exchange rates we make the assumption that the RER follows a martingale process under the statistical measure. This yields condition on the real interest rate differentials of the domestic and nominal economies. Furthermore, we showed that the martingale assumption on RER is equivalent to the condition that the nominal interest rate differentials between the two economies is given by the sum of appreciation rate of the exchange rate and the risk premium arising from exchange rate uncertainty.

Motivated by the importance of the information on RER for central banks, we introduce a forward contract written on RER. This yields the forward real exchange rate which can be written in terms of the price of the domestic and foreign inflation indexed bonds.

We further construct multi-country inflation linked derivatives such as foreign exchange inflation options and real exchange rate swaps with the idea of providing a guarantee for the foreign purchasing power of a domestic income. We extensively use the change of numeraire technique to get prices of these derivatives. In particular, we get closed form formulae under the assumption of deterministic volatility in the inflation indexed bond price dynamics.

Application of the model to the real data and comparison with a nested one-country model is left as a future study. Moreover, we are considering to investigate further on the RER implications of the two-country frame work for the pricing of inflation-linked products.

Appendix A

Review for Marked Point Processes

This appendix gives a short overview of marked point processes via recalling definitions and stating some important theorems without giving proofs. For the sake of completeness we will first provide a background on point processes and then the theory of marked point processes will follow. The main references that we used are Brémaud [1981], Björk et al. [1997], Jacod and Shiryaev [1987], Jeanblanc et al. [2009] and Runggaldier [2003].

A.1 Univariate and Multivariate Point Processes

We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A univariate point process describes events that occur randomly over time. It is possible to view a point process in three different ways; as a sequence of nonnegative random variables, as an integer-valued random measure or via its associated counting process. Following the first view, we may consider

$$0 = T_0 < T_1 < T_2 < \cdots$$

where T_i generally indicates the time of the i^{th} occurrence of an event. The point process is called nonexplosive if and only if

$$T_\infty = \lim \uparrow T_n = +\infty$$

implying that finite number of events occur in a finite time interval. Now recall that a counting process N_t is a non-negative increasing process with $N_0 = 0$ which increases by one at isolated times and stays constant between the times of increase. Under the assumption of non-explosion for each T_n, the corresponding random variable N_t is defined as

$$N_t = n \quad \text{if} \quad t \in [T_n, T_{n+1}), \ n \geq 0$$

or equivalently

$$N_t = \sum_{n \geq 1} 1_{\{T_n \leq t\}}$$

Clearly, the process T_n and N_t carries the same information.

Let $\mathbb{G}$ be a given filtration. A counting process is $\mathbb{G}$-adapted if and only if the random variables $(T_n,\ n \geq 1)$ are $\mathbb{G}$-stopping times. In such a case, $\{N_t \leq n\} = \{T_{n+1} > t\}$ belongs to $\mathbb{G}_t$

The stochastic integral, $\int_0^t C_s dN_s$ is defined pathwise as a Stieltjes integral for every bounded measurable process C by

$$(C \bullet N)_t := \int_{(0,t]} C_s dN_s := \sum_{n=1}^{\infty} C_{T_n} 1_{\{T_n \leq t\}}$$

One can also associate a random measure to the counting process N_t. For any Borel set $A \in \mathbb{R}^+$, for any ω,

$$\mu(\omega, A) = \#\{n \geq 1 : T_n(\omega) \in A\}.$$

For any ω, the map $A \to \mu(\omega, A)$ is a positive measure on $\mathbb{R}^+$. Moreover, one can write

$$\mu(\omega, dt) = \sum_n \delta_{T_n(\omega)}(dt)$$

where δ_a indicates the Dirac measure at point a. The random variable N_t can be written as

$$N_t(\omega) = \mu(\omega, (0,t]) = \int_{(0,t]} \mu(\omega, ds)$$

and the stochastic integral as $\int_{(0,t]} C_s dN_s = \int_{(0,t]} C_s \mu(ds)$.

Now let T_n be a univariate point process and $Y_n,\ n \geq 1$ a sequence of random variables with values in $\{1, 2, \cdots, K\}$, all defined on the same probability space. For each $k = 1, 2, \cdots, K$ we associate the counting process

$$N_t(k) := 1_{\{T_n \leq t\}} 1_{\{Y_n = k\}}, \quad n \geq 1$$

Each $N_t(k)$ is a univariate point process and theses processes have no common jumps. The double sequence (T_n, Y_n) or equivalently the vector process $N_t = (N_t(1), \cdots, N_t(K))$ called a $m$ultivariate marked point process. The first representation reveals that, one may interpret T_n as the n^{th} occurrence of some event and Y_n as an attribute or mark of this event. We are now ready to speak of the marked point processes.

A.2 Marked Point Processes

Let (Y_n) be a sequence of random variables taking values in the measurable space $(E, \mathcal{E})$, and T_n (or N_t) a point process with $T_\infty = +\infty$. The double sequence $(T_n, Z_n,\ n \geq 1)$ is called an E-marked point process . The measurable space $(E, \mathcal{E})$ is called the *m*arked space. Sometimes, marked point processes are called *s*pace-time point processes. Marked point processes can be considered as the generalization of compound Poisson processes where the jump sizes are no longer i.i.d random variables and the time intervals between two consecutive jumps are no longer independent.

It is possible to associate to each $A \in \mathcal{E}$ the counting process $N_t(A)$ defined by

$$N_t(A) = \sum_{n \geq 1} 1_{\{Y_n \in A\}} 1_{\{T_n \leq t\}}$$

and in particular, $N_t(E) = N_t$.

The natural filtration associated with the process N_t is defined by

$$\mathcal{F}_t^\mu := \sigma\Big\{N_s(A); s \leq t,\ A \in \mathcal{E}\Big\}$$

Note that each T_n is an $\mathcal{F}_t^\mu$-stopping time.

Now recall that a random measure on $\mathbb{R}^+ \times E$ is a family of measures

$$\Big(\mu(\omega; dt, dy) : \omega \in \Omega\Big)$$

defined on $\mathbb{R}^+ \times E$ such that, for $[0,t] \times A \in \mathcal{B} \otimes \mathcal{E}$, the map $\omega \to \mu(\omega; [0,t], A)$ is $\mathcal{F}$-measurable and $\mu(\omega; \{0\} \times E) = 0$. One can associate with N_t a random measure μ by

$$\mu(\omega; (0,t], A) = N_t(\omega, A), \quad t \geq 0,\ A \in \mathcal{E}$$

For a given $\omega \in \Omega$, $\mu(w; dt, dy)$ is σ-finite if and only if the realization $T_n(\omega)$ is non-explosive.

We say that a map $H : \mathbb{R}^+ \times \Omega \times E \ \to \mathbb{R}$ is predictable E-marked process if it is $\mathcal{P} \otimes \mathcal{E}$ measurable where $\mathcal{P}$ is the predictable σ-field on $(\mathbb{R}^+ \times \Omega)$. The random measure $\mu(\omega; ds, dy)$ acts on the set of predictable E-marked processes H as

$$(H \bullet \mu)_t = \int_{(0,t]} \int_E H(s,y)\mu(ds, dy) = \sum_n H(T_n, Y_n) 1_{\{T_n \leq t\}} = \sum_{n=1}^{N_t} H(T_n, Y_n)$$

where ω is suppressed in the notation .

Now we recall the definition of the compensator of the random measure μ. The *c*ompensator of μ is the unique random measure ν such that, for every predictable process H,

(i) the process $H \bullet \nu$ is predictable

(ii) if, moreover, the process $|H| \bullet \nu$ is increasing and locally integrable, the process $(H \bullet \mu - H \bullet \nu)$ is a local martingale.

The existence of compensator is proved e.g. in Jacod and Shiryaev [1987] (see page 66, Thm 1.8). In the following we will assume that for every $A \in \mathcal{E}$, the process $N_t(A)$ admits the $\mathcal{F}$-predictable intensity $\lambda_t(A)$, that is, there exists a predictable process (λ_t) such that

$$N_t(A) - \int_0^t \lambda_s(A) ds$$

is a martingale. In such a case, we clearly have $\nu(\omega; dt, dy) = \lambda_t(\omega, dy)dt$.

Theorem A.1 *Let $\mu(dt, dy)$ be a E-marked point process with the $(P, \mathcal{F})$-intensity kernel $\lambda_t(dy)$. Then for each non-negative $\mathcal{F}_t$-predictable E-marked process H*

$$E\Big[\int_0^\infty \int_E H(s, y)\mu(ds, dy)\Big] = E\Big[\int_0^\infty \int_E H(s, y)\lambda_s(dy)ds\Big]$$

Proof. See Brémaud [1981], page 235, T3. □

Now define $\tilde{\mu}(ds, dy) = \mu(ds, dy) - \lambda_s(dy)ds$. It follows from the above theorem that

$$\int_{(0,t]} \int_E H(s, y)\tilde{\mu}(ds, dy)$$

is a $(P, \mathcal{F}_t)$-local martingale provided that H is a predictable process satisfying

$$\int_{(0,t]} \int_E |H(s, y)|\lambda_s(dy)ds \ < \infty \quad P - a.s$$

For Brownian martingales there is the well-known result stating that every square integrable martingale with respect to the filtration generated by a Brownian motion is, up to an additive constant, a stochastic integral of the Itô type. Next theorem states a representation result for a general case.

Theorem A.2 *A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ given. Let W be a Brownian motion and $\mu(ds, dy)$ a marked point process and*

$$\mathcal{F}_t := \sigma\Big\{W_s, \mathcal{F}^\mu_s,\ s \leq t\Big\}$$

completed. Then, any $(\mathbb{P}, \mathcal{F}_t)$-local martingale M_t has the representation

$$M_t = M_0 + \int_0^t \phi_s dW_s + \int_0^t \int_E H(s, y)\tilde{\mu}(ds, dy)$$

where ϕ is a predictable square integrable process and H is predictable process such that

$$\int_{(0,t]} \int_E |H(s,y)| \lambda_s(dy) ds \ < \infty.$$

Proof. See e.g. Jacod and Shiryaev [1987], Part III. □

Now we consider process of the general type

$$X_t = X_0 + \int_{[0,t]} \alpha_s ds + \int_{[0,t]} \beta_s dWs + \int_{(0,t]} \int_E \gamma(s,y) \mu(ds,dy)$$

where all coefficients assumed to satisfy the integrability conditions and β and γ is predictable. Next theorem gives the general Itô formula for processes of this type.

Theorem A.3 *Assume that X has the dynamics of the form*

$$dX_t = \alpha_t dt + \beta_t dW_t + \int_E \gamma(t,y) \mu(dt,dy)$$

where β and γ are predictable. Let $F(t,X)$ be a $\mathcal{C}^{1,2}$ function. Then the following Itô formula holds

$$\begin{aligned} dF(t,X_t) &= F_t(\cdot)dt + F_X(\cdot)\alpha_t dt + \frac{1}{2} F_{XX}(\cdot)\beta_t^2 dt + F_X(\cdot)\beta_t dW_t \\ &+ \Big(F(t, X_{t-} + \gamma(t,Y_t)) - F(t,X_{t-}) \Big) dN_t \end{aligned}$$

where $N_t = N_t(E) = \mu((0,t],E)$, $(\cdot)$ denotes (t,X_t) and the subscribts in F indicates partial derivatives.

Proof. See Runggaldier [2003], Section 2.4. □

As a specific case, now we take an equation of the form

$$dX_t = X_{t-}(\alpha_t dt + \beta_t dW_t + \int_E \gamma(t,y) \mu(dt,dy)) \tag{A.1}$$

where $\gamma(t,y) > -1$. Application of the Itô formula to $F(t,X) = \log(X)$ yields that

$$dF = \alpha_t dt - \frac{1}{2}\beta_t^2 dt + \beta dW_t + \log(1 + \gamma(t,Y_t)) dN_s$$

which implies that the solution of (A.1) is in the form

$$X_t = X_0 \exp\Big\{ \int_0^t (\alpha_s - \frac{1}{2}\beta_s^2) ds \ + \int_0^t \beta_s dW_s + \ \int_0^t \log(1 + \gamma(s,Y_s)) dN_s \Big\}$$

Next theorem is a Girsanov type measure transformation for the finite time horizon $[0,T^*]$.

Theorem A.4 *A filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is given. On the finite time interval $[0, T^*]$, let $\mu(dt, dy)$ be an E-marked point process with $(\mathbb{P}, \mathcal{F}_t)$-compensator $\nu(dt, dy)$ and W_t a Brownian motion. Let θ_t be a predictable square integrable process and $h(t, y)$ be predictable E-marked process satisfying*

$$\int_0^T \int_E |h(s,y)|\nu(ds,dy) \ < \infty, \quad \forall T \in [0, T^*].$$

Define $M_t = M_t^c.M_t^d$ where M_t^c and M_t^d satisfies

$$\begin{aligned} dM_t^c &= M_t^c \theta_d dW_t \\ dM_t^d &= \int_E (h(t,x) - 1) M_{t-}^d (\mu(dt,dx) - \nu(dt,dx)) \end{aligned}$$

Suppose M is a true martingale. Then there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}$, equivalent to $\mathbb{P}$ such that

$$\begin{aligned} dW_t^Q &= dW_t - \theta_t dt \ \textit{is} \ \ a \ \ \mathbb{Q} \ \textit{Brownian} \ \ \textit{motion}, \\ \nu^q(dt,dy) &= h(t,y)\nu(dt,dy) \ \textit{is} \ \mathbb{Q} \ \textit{compensator} \ \ \textit{of} \ \ \mu(dt,dy). \end{aligned}$$

Conversely, if $\mathcal{F}_t := \sigma\Big\{W_s, \mu(ds,dy), \ s \le t\Big\}$ completed, then every probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, has the above structure.

Proof. See Runggaldier [2003], Thm. 2.5. □

Appendix B

In the following we will make the complimentary proof that is needed for the polynomial property of affine processes.

First recall that we have the process X satisfying the stochastic differential equation

$$dX_t := \begin{pmatrix} dY_t \\ dZ_t \end{pmatrix} = \left(\begin{bmatrix} 0 \\ \kappa_z \theta_z \end{bmatrix} + \begin{bmatrix} -\kappa_y & \kappa_y \\ 0 & -\kappa_z \end{bmatrix} \begin{bmatrix} Y_t \\ Z_t \end{bmatrix} \right) dt + \begin{bmatrix} \sigma_y \sqrt{Y_t} & 0 \\ 0 & \sigma_z \sqrt{Z_t} \end{bmatrix} \begin{bmatrix} dW_t^y \\ dW_t^z \end{bmatrix} \tag{B.1}$$

Here, notice that the drift and diffusion parts of the above equation satisfies the linear growth condition. That is, for any $(y, z) \in \mathcal{X}$ for the diffusion matrix, i,e., we write

$$\left\| \begin{bmatrix} \sigma_y \sqrt{Y_t} & 0 \\ 0 & \sigma_z \sqrt{Z_t} \end{bmatrix} \right\|^2 \leq K(1 + \|(y, z)\|^2) \tag{B.2}$$

Lemma B.1 *Suppose u_0 is a C^2-function on $\mathcal{X}$, and u is a $C^{1,2}$ -function on $R^+ \times \mathcal{X}$ whose spatial derivatives satisfies the polynomial growth condition*

$$\left\| \left(\frac{\partial u}{\partial y} , \frac{\partial u}{\partial z} \right) \right\| \leq K(1 + \|(y, z)\|^p), \; t \leq T, \quad (y, z) \in \mathcal{X} \tag{B.3}$$

for some constant $K = K(T) \leq \infty$ and some $p \geq 1$, for all $T < \infty$.

If $u(t, y, z)$ satisfies the Kolmogorov backward equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa_y (z - y) \frac{\partial u}{\partial y} + \kappa_z (\theta_z - z) \frac{\partial u}{\partial z} + \frac{1}{2} \sigma_y^2 y \frac{\partial^2 u}{\partial y^2} + \frac{1}{2} \sigma_z^2 z \frac{\partial^2 u}{\partial z^2} \\ u(0, y, z) &= u_0(y, z) \end{aligned} \tag{B.4}$$

for all $t \geq 0$ and $(x, y) \in \mathcal{X}$, then for all $t \leq T < \infty$

$$u(T - t, Y_t, Z_t) = E[u_0(Y_T, Z_T) | Y_t, Z_t] \tag{B.5}$$

Proof. Since u is assumed to be $C^{1,2}$, in view of Itô formula we get

$$\begin{aligned} du(T-t,Y_t,Z_t) = \Bigg(& -\frac{\partial u(T-t,Y_t,Z_t)}{\partial t} + \frac{\partial u(T-t,Y_t,Z_t)}{\partial y}\kappa_y(Z_t - Y_t) \\ & + \frac{\partial u(T-t,Y_t,Z_t)}{\partial z}\kappa_z(\theta_z - Z_t) + \frac{1}{2}\sigma_y^2 Y_t \frac{\partial^2 u(T-t,Y_t,Z_t)}{\partial y^2} \\ & + \frac{1}{2}\sigma_z^2 Z_t \frac{\partial^2 u(T-t,Y_t,Z_t)}{\partial z^2}\Bigg) dt \\ & + \frac{\partial u(T-t,Y_t,Z_t)}{\partial y}\sigma_y\sqrt{Y_t}dW_t^y + \frac{\partial u(T-t,Y_t,Z_t)}{\partial z}\sigma_z\sqrt{Z_t}dW_t^z \end{aligned} \tag{B.6}$$

Now suppose u satisfies (B.4). Then, the drift term in (B.6) immediately vanishes implying that $u(T-t,Y_t,Z_t)$ is a local martingale with $u(0,Y_T,Z_T) = u_0(Y_T,Z_T)$. We now write

$$du(T-t,Y_t,Z_t) = \frac{\partial u(T-t,Y_t,Z_t)}{\partial y}\sigma_y\sqrt{Y_t}dW_t^y + \frac{\partial u(T-t,Y_t,Z_t)}{\partial z}\sigma_z\sqrt{Z_t}dW_t^z \tag{B.7}$$

In what follows our main objective is to show that under the assumptions of the lemma, $u(T-t,Y_t,Z_t)$ is indeed a true martingale. We have

$$\begin{aligned} & E\left[\int_0^T \left\| \left(\frac{\partial u(T-s,Y_s,Z_s)}{\partial y}, \ \frac{\partial u(T-s,Y_s,Z_s)}{\partial z} \right) \begin{bmatrix} \sigma_y\sqrt{Y_t} & 0 \\ 0 & \sigma_z\sqrt{Z_t} \end{bmatrix} \right\|^2 ds \right] \\ \leq & E\left[\int_0^T \left\| \left(\frac{\partial u(T-s,Y_s,Z_s)}{\partial y}, \ \frac{\partial u(T-s,Y_s,Z_s)}{\partial z} \right) \right\|^2 \left\| \begin{bmatrix} \sigma_y^2 Y_t & 0 \\ 0 & \sigma_z^2 Z_t \end{bmatrix} \right\| ds \right] \\ \leq & K\left(1 + E\left[\sup_{s\leq T} \|(Y_s,Z_s)\|^{2p}\right]\right) \end{aligned} \tag{B.8}$$

where the last inequality follows from the assumption (B.3) and due to the fact that the diffusion parameter of the process X satisfies the linear growth condition. One can show that (see Karatzas and Shreve [1991], Problem 5.3.15) the expectation in (B.8) is finite and this yields the desired result. □

References

M. Adler and B. Lehmann. Deviations from purchasing power parity in the long run. *The Journal of Finance*, 38(5):1471–1487, 1983. 110

N. Belgrade, E. Benhamou, and E. Koehler. A market model for inflation. *CDC IXIS*, 2004. 78

T.R. Bielecki, S. Crépey, and M. Jeanblanc. Up and down credit risk. *Quantitative Finance*, 10(10):1137–1151, 2010. ISSN 1469-7688. 8, 10

T. Björk. *Arbitrage theory in continuous time*. Oxford University Press, USA, 2009. 116

T. Björk, Y. Kabanov, and W. Runggaldier. Bond market structure in the presence of marked point processes. *Mathematical Finance*, 7(2):211–239, 1997. ISSN 1467-9965. 20, 121

P. Brémaud. *Point processes and queues, martingale dynamics*. Springer, 1981. 121, 124

G. Cassel. The world's money problems. *EP Dutton and Co., New York*, 1921. 109

L. Chen, P. Collin-Dufresne, and R.S. Goldstein. On the relation between the credit spread puzzle and the equity premium puzzle. *Review of Financial Studies*, 22(9): 3367, 2009. ISSN 0893-9454. 24

R.R. Chen and L. Scott. Multi-factor Cox-Ingersoll-Ross models of the term structure: Estimates and tests from a Kalman filter model. *The Journal of Real Estate Finance and Economics*, 27(2):143–172, 2003. ISSN 0895-5638. 35

P. Cheridito, D. Filipović, and M. Yor. Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability*, 15(3):1713–1732, 2005. ISSN 1050-5164. 82

P. Cheridito, D. Filipović, and R.L. Kimmel. Market price of risk specifications for affine models: Theory and evidence. *Journal of Financial Economics*, 83(1):123–170, 2007. ISSN 0304-405X. 82

R. Cont and Y.H. Kan. Dynamic hedging of portfolio credit derivatives. *SIAM Journal on Financial Mathematics*, 2(1):112–140, 2011. iii, 10, 27, 28, 30, 33, 54, 69, 72, 73

R. Cont and P. Tankov. *Financial modelling with jump processes*, volume 2. Chapman & Hall, 2004. 27

M. Davis and J. Obloj. Market completion using options. *Advances in mathematics of finance*, 83:49, 2008. iii, v, 3, 79, 87, 88, 101

G. Duffee and R. Stanton. Estimation of dynamic term structure models. *Manuscript, Haas School of Business*, 2004. 35

D. Duffie and N. Garleanu. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, pages 41–59, 2001. 9

D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000. 37

D. Filipović. *Term-structure models: a graduate course*. Springer Verlag, 2009. ISBN 3540097260. 37, 80, 83

D. Filipovic and T. Schmidt. Pricing and Hedging of CDOs: A Top Down Approach. *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, page 231, 2010. 9, 14, 27, 28

D. Filipović, L. Overbeck, and T. Schmidt. Dynamic CDO term structure modeling. *Mathematical Finance*, 2009. ISSN 1467-9965. iii, v, 1, 2, 9, 10, 11, 12, 15, 17, 20, 21, 22, 26, 73

H. Follmer and D. Sondermann. Hedging of non-redundant contingent claims. *Contributions to mathematical economics in honor of Gérard Debreu*, page 205, 1986. 27

R. Frey and J. Backhaus. Dynamic hedging of synthetic CDO tranches with spread risk and default contagion. *Journal of Economic Dynamics and Control*, 34(4):710–724, 2010. ISSN 0165-1889. 9, 10, 27

A.L.J. Geyer and S. Pichler. A state-space approach to estimate and test multifactor Cox-Ingersoll-Ross models of the term structure. *Journal of Financial Research*, 22: 107–130, 1999. ISSN 0270-2592. 35

I. Gohberg, P. Lancaster, and L. Rodman. *Invariant subspaces of matrices with applications*. Society for Industrial Mathematics, 2006. 94

A.C. Harvey. *Forecasting, structural time series models and the Kalman filter*. Cambridge university press, 1990. ISBN 0521405734. 35

D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60(1):77–105, 1992. 9

M. Hinnerich. Inflation-indexed swaps and swaptions. *Journal of Banking & Finance*, 32(11):2293–2306, 2008. 78, 81, 82

LP Hughston. Inflation derivatives. *Working Paper, Merrill Lynch and King's College*, 1998. 78

J. Jacod and A.N. Shiryaev. *Limit theorems for stochastic processes*, volume 2003. Springer, 1987. 121, 124, 125

R. Jarrow and Y. Yildirim. Pricing treasury inflation protected securities and related derivatives using an HJM model. *Journal of Financial and Quantitative Analysis*, 38 (02):337–358, 2003. ISSN 1756-6916. 78, 81

M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical methods for financial markets.* Springer Verlag, 2009. 121

I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus.* Springer, 1991. 128

A.N. Kipici and M. Kesriyeli. The real exchange rate definitions and calculation. *Central Bank of the Republic of Turkey Working Paper*, 1997. 103

L. Kjaergaard. Modelling inflation. *RISK-LONDON-RISK MAGAZINE LIMITED-*, 20 (6):89, 2007. ISSN 0952-8776. 78

S.G. Krantz and H.R. Parks. *A primer of real analytic functions.* Birkhauser, 2002. 101

D.X. Li. On default correlation: a copula function approach. *Journal of Fixed Income*, 9(4):43–54, 2000. ISSN 1059-8596. 8

F. Mercurio. Pricing inflation-indexed derivatives. *Quantitative Finance*, 5(3):289–302, 2005. ISSN 1469-7688. 78

D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer Verlag, 1999. 37

K. Rogoff. The purchasing power parity puzzle. *Journal of Economic literature*, 34(2): 647–668, 1996. 109, 110

W.J. Runggaldier. Jump diffusion models. *Handbook of heavy tailed distributions in finance*, 1:170–209, 2003. 121, 125, 126

P. Schönbucher. Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives. *Department of Mathematics, ETH Zurich*, 2005. 9

M. Schweizer. Hedging of options in a general semimartingale model. *Diss. ETHZ*, 1988. 27

M. Schweizer. A guided tour through quadratic hedging approaches. 1999. 27

J. Sidenius, V. Piterbarg, and L. Andersen. A new framework for dynamic credit portfolio loss modelling. *International Journal of Theoretical and Applied Finance*, 11(2):163–197, 2008. 9

I. Slinko. *Essays in option pricing and interest rate models.* Economic Research Institute, Stockholm School of Economics [Ekonomiska forskningsinstitutet vid Handelshögskolan i Stockholm](EFI), 2006. 3, 103

A.M. Taylor and M.P. Taylor. The purchasing power parity debate. *The Journal of Economic Perspectives*, 18(4):135–158, 2004. 110

Curriculum Vitae

Zehra Eksi-Altay

Personal Data

Address	Strozzigasse, 13-3 A–1080 Wien
Email	zehra.eksi@gmail.com
Date of birth	September 12, 1980
Place of birth	Izmit (Turkey)
Citizenship	Turkey

Education

Oct 2007 –	**Ph.D. candidate in Mathematical Finance** University of Vienna, Austria, "Essays in Credit and Inflation Linked Derivatives " Advisor: Prof. Damir Filipović
Sept 2005 – Oct 2007	Ph.D. studies in Mathematical Finance Inst. of Applied Mathematics, Middle East Technical University, Ankara, Turkey
Sept 2003 – August 2005	**M.Sc. in Financial Mathematics** Inst. of Applied Mathematics, Middle East Technical University, Ankara, Turkey "Comparative Study of Risk Measures" Advisor: Prof. Hayri Körezlioglu
Sept 1999 – June 2003	**B.Sc. in Economics** Faculty of Economics and Administrative Sciences, Middle East Technical University, Ankara, Turkey

Employment

Oct 2007 –	Junior Scientist Vienna Institute of Finance, Vienna, Austria
Sept 2005 – Oct 2007	Research Assistant, Teaching Assistant Inst. of Applied Mathematics, Middle East Technical University, Ankara, Turkey
July 2004 – Sept 2004	Trainee Research Department, Nurol Securities, İstanbul, Turkey
Feb 2003	Trainee İstanbul Stock Exchange, İstanbul, Turkey