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Degenerations of Lie algebras and pre-Lie algebras

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Introduction

In this thesis we are concerned with the orbit closure problem for algebras that arises in algebraic transformation group theory. The general linear group $\mathrm{GL}(V)$ over a field $\mathbb{K}$ acts on the vector space $V^* \otimes V^* \otimes V$, the space of $\mathbb{K}$-algebra structures, by the change of basis. For two $\mathbb{K}$-algebra structures λ and μ we say that μ is a degeneration of λ if μ lies in the orbit closure of λ with respect to the Zariski-topology. For this we write $\lambda \to_{\deg} \mu$. The orbit closure problem in this form is about the classification of all degenerations of a certain algebra structure of a fixed dimension. This problem also depends on a complete classification of the corresponding algebra structures. Both problems are highly complicated even in small dimensions (see for example [21]).

Thereby the choice of the field $\mathbb{K}$ is essential. If $\mathbb{K} = \mathbb{C}$ the orbit closures of the Zariski topology and the standard topology coincide. Therefore, in the past, mainly degenerations over $\mathbb{C}$ were investigated. For Lie algebras we have classifications of degenerations over $\mathbb{C}$ up to dimension four and for nilpotent ones up to dimension six ([18], [57]). Moreover in case that the field $\mathbb{K}$ is $\mathbb{C}$ the notion of a degeneration is equal to that of a contraction. Lie algebra contractions are primarily studied in physics and have a much longer history than degenerations. They are also easier to compute since contractions are always defined via the standard topology. Furthermore contractions have been studied over the fields $\mathbb{R}$ and $\mathbb{C}$ ([46]).

It is reasonable that with increasing dimension the difficulty of determining the orbit closure of an algebra increases too. To decide wether an algebra lies in the orbit closure of an other one is mainly based on methods coming from invariant theory, algebraic group theory, and algebraic geometry. Important concepts are the notions of invariance and semi-invariance. We call an $\mathbb{R}$-valued and semi-continuous function on the set of all $\mathbb{K}$-algebra structures a *semi-invariant* if it is constant on the orbit of an algebra and either increasing or decreasing in its closure. For example the dimension of the orbit space is a semi-invariant. This follows from Borel's Closed-Orbit-Lemma (Theorem 5.10), which is also the starting point for a lot of considerations with respective to the orbit closure problem for algebras. It says that in the orbit closure of a given algebra only orbits with strictly smaller dimenion can be contained. Conversely, we call a polynomial function an *invariant* of an algebra if it vanishes on the whole orbit of this algebra. Regarding the definition of the Zariski topology this polynomial function has to vanish also on the orbit closure of this algebra. An example of this invariant is commutativity. So if we have a degeneration $A \to_{\deg} B$ and A is commutative then also B is commutative.

During the work on this thesis the existence problem for degenerations, i. e., the question if for two arbitrary algebras it can always be decided if one is lying in the orbit closure of the other, was solved by Popov in greater generality (see Subsection 1.1.4). In his article [50] he presents an algorithm which solves the question of existence for degenerations by means of a finite number of effectively feasible operations. However, even in dimension two the algorithm yields a system of over hundreds of millions linear equations in hundreds of millions variables. Hence in determining the orbit closure of a certain algebra one still depends on finding new semi-invariants and invariants.

The main result in this work is the classification of all degenerations of Novikov algebras over $\mathbb{C}$ in dimension three. Such algebras form a subclass of left-symmetric algebras, so called pre-Lie algebras. Approaching this we also give the complete classification of 2-dimensional pre-Lie algebras.

In chapter one we begin with the basic definitions that are necessary to study degenerations. We touch the very interesting field of one-parameter subgroup degenerations and therein one of the oldest questions in contraction theory. That is, if every contraction (or degeneration over $\mathbb{C}$) can be realized by a generalized Inönü - Wigner contraction. We continue with some first statements about degenerations which are mostly derived from facts established in the theory of linear algebraic groups. The main result in this section is an application of Borels Closed-Orbit-Lemma which will be of great importance in all classifictions in chapter 4. It says that the dimension of the subspace of derivations of an algebra has to increase strictly in the orbit closure of this algebra. We close this introductory chapter with a disscusion of Popovs article ([50]) where he presents an algorithm that solves the the existence problem for degenerations. We show, however, that this algorithm is useless to classify degenerations. In the following two sections we give overviews about contractions and deformations. The notion contraction is rather used in physical literature where it originally first appeared. Inönü - Wigner contractions and Saletan contractions are important examples in the early development of this subject. For this reason and because they serve as a good source to get a first impression what a degeneration can look like we treat these two examples in greater detail. At the end of this chapter we give a brief discussion about deformation theory. Although deformations are a very interesting and important object to study we will keep this section short for they are not considered anywhere else in this thesis.

In chapter two we introduce pre-Lie and Novikov algebras and some of their properties. We begin to motivate the notion of a pre-Lie algebra by showing its close relationship to Lie algebras and the geometric analogon of affine structures on Lie groups. More precisely, we can identify a left-symmetric structure on a Lie algebra with a left-invariant affine structure on a Lie group. In the following we give a treatment of an important subclass of pre-Lie algebras, namely Novikov algebras. Novikov algebras were studied in [5] in form of Poisson brackets of hydrodynamic type. Refering back to this article E. Zelmanov gave some substantial results on the structure of Novikov algebras. For example, over an algebraically closed field of characteristic zero every simple Novikov algebra is a field. We continue with some crucial definitions with respect to pre-Lie algebras that turn out to be semi-invariants. We adapt the notions of upper and lower central, and derived series from Lie algebra theory (see [13]). Moreover we briefly touch some of the notions of a radical one can have for pre-Lie algebras. In the last section of this chapter we work out some technical material concerning the left-regular representation of an algebra. Specifically we present correspondences between certain restrictions on the structure constants and their effect on the left-multiplication operator of an algebra. We will need this in the next chapter as a technical tool in the proof of Theorem 3.8.

In chapter three we present the methods we will use for the classifications in chapter 4 of this work. The first section is concerned with the preservation of structural properties when we pass from the orbit of an algebra to its closure. We show that one can shift degeneration diagrams from lower dimensions up to higher ones by adding ideals. Furthermore it is proved that structures as subalgebras, ideals and factors are preserved under a degeneration. The main result in this section is the following (Theorem 3.8). If the algebra A degenerates to the algebra B then there exists an ideal $J \subset B$ such that every factor A/I by an ideal $I \subset A$ degenerates to the factor B/J. This is a very interesting result on its own, but even more it is of good use for classifying degenerations. Another crucial relation between Lie algebra and pre-Lie algebra degeneration is that every pre-Lie algebra degeneration induces a degeneration of its associated Lie algebras. In the next section we treat semi-invariants. We begin by transfering most of the known semi-invariants from the Lie algebra case and generalize them to the pre-Lie algebra case. Furthermore we show that the notions we introduced in chapter 2 indeed lead to new semi-invariants. Finally we generalize the notion of an (α, β, γ)-derivation in the way that we take certain equations in a linear operator and show that the vector space of solutions for this operator is a semi-invariant (Theorem 3.44). We close this chapter with a treatment on invariants of degenerations. The aim of this section is to generalize the well known $\mathfrak{C}_{\mathrm{p,q}}$-invariant for Lie algebras to pre-Lie algebras. We show in Lemma 3.50 that every polynomial in conjugation invariant forms defines an invariant.

Chapter 4 contains the main results of this thesis. At the beginning we present the classification of all pre-Lie algebras in dimension two. This is surprisingly complicated. For example in dimension two there are only two non-isomorphic Lie algebras. However, we have already infinitely many 2-dimensional pre-Lie algebras. Hence, the classification of degenerations of 2-dimensional pre-Lie algebras is indeed non-trivial. Out of this result we get all degenerations of 2-dimensional Novikov algebras as a corollary. In dimension three we have infinitely many Lie algebras too. The classification of 3-dimensional Novikov algebras is highly complicated. For our considerations we took a list of these algebras presented in [9] by D. Burde. To solve the orbit closure problem for 3-dimensional Novikov algebras we need to extend our tools from chapter 3 by methods that are adjusted to specific algebras. For example we construct equations in the structure constants that are zero on the whole orbit of a certain algebra. By definition of the Zariski topology this equations have to be satisfied by every algebra in the closure.

At the very end of this work we provide preliminaries from algebraic geometry, tables for all orbit closures, tables of semi-invariants, and algorithms we used for our calculations.

Contents

1 Contractions, degenerations, and deformations in algebra and physics

In this introductory chapter we present the basic definitions that are associated with the notion of a degeneration. We continue with some first examples and statements, which have its origins in the theory of linear algebraic groups. At the end of the first section we discuss a result of V. Popov. In the following two sections we give overviews about contractions and deformations.

1.1 Degenerations

We always assume that the field $\mathbb{K}$ is algebraically closed and of characteristic zero.

For preliminaries from algebraic geometry we refer to appendix A.

1.1.1 Definition of a degeneration

Let $\lambda \in \mathrm{Alg}_n(\mathbb{K})$ be an algebra law. The general linear group acts on $\mathrm{Alg}_n(\mathbb{K})$ by the change of bases:

$$(g \cdot \lambda)(x, y) = g(\lambda(g^{-1}x, g^{-1}y))$$

with $g \in \mathrm{GL}_n(\mathbb{K})$ and $x, y \in V$, the underlying vector space of an algebra. We denote by $\mathrm{O}(\lambda)$ the orbit under this action, and by $\overline{\mathrm{O}(\lambda)}$ the orbit closure with respect to the Zariski topology.

Definition 1.1. Let $\lambda, \mu \in \mathrm{Alg}_n(\mathbb{K})$ be two algebra laws. We say that λ degenerates to μ, if $\mu \in \overline{\mathrm{O}(\lambda)}$. This is denoted by $\lambda \to_{\mathrm{deg}} \mu$. If $\mu \in \mathrm{O}(\lambda)$, which means that $\lambda \cong \mu$, then the degeneration is called non-proper.

Remark 1.2. Let two n-dimensional algebras A and B be endowed with the laws λ and μ, respectively. If $\lambda \to_{\mathrm{deg}} \mu$ we frequently use the expression that the algebra A degenerates to the algebra B. In doing so we simply refer back to the fact that we have a degeneration between the corresponding multiplication structures α and μ.

Example 1.3. Every law $\lambda \in \mathrm{Alg}_n(\mathbb{C})$ degenerates to the abelian law λ_0, given by the trivial multiplication. In this sense any degeneration to an abelian algebra is called trivial.

Let $g_t = t^{-1} I_n \in \mathrm{GL}_n(\mathbb{C}(t))$.[1] We have

$$(g_t \cdot \lambda)(x, y) = t^{-1}\lambda(tx, ty) = t\lambda(x, y),$$

[1] Accordingly to subsection 1.2.2 we denote by I_n the identity matrix of an n-dimensional vector space.

and hence $\lim_{t\to 0}(g_t \cdot \lambda)(x,y) = 0$, the abelian law.

Example 1.4. Let $\mathbb{K} = \mathbb{C}$. Because of Borel's closed orbit lemma (Theorem 5.10) the orbit closures of the standard topology and those of the Zariski topology on $\mathrm{Alg}_n(\mathbb{C})$ coincide. The orbit closure of a structure $\lambda \in \mathrm{Alg}_n(\mathbb{C})$ is then formed by the convergence of $\{g_\varepsilon \cdot \lambda\}$ as $\varepsilon \to +0$, where $\{g_\varepsilon\}$ is a series of matrices in $\mathrm{GL}_n(\mathbb{C})$.

This special case of a degeneration is (particularly in the physical literature) often referred to as a contraction over $\mathbb{C}$. Because of its importance in physics and for the development of the studies of orbit closures we devote the next section to contractions.

An analogous viewpoint in the theory of obit closures is the following characterization given by Grunewald and O'Halloran [33] for Lie algebras.

Theorem 1.5. *Let λ and μ be n-dimensional Lie algebras over the field $\mathbb{K}$. The Lie algebra μ is a degeneration of λ if and only if there is a discrete valuation $\mathbb{K}$-algebra A with residue field $\mathbb{K}$ whose quotient field $\mathbb{L}$ is finitely generated over $\mathbb{K}$ of transcendence degree one, and there exists an n-dimensional Lie algebra μ_A over A such that*

$$\mu_A \otimes \mathbb{L} \cong \lambda \otimes \mathbb{L}$$

and

$$\mu_A \otimes \mathbb{K} = \mu.$$

With this characterization example 1.3 can be recovered in the following way: Let λ be an arbitrary Lie algebra law, $A = \mathbb{K}[t]_t$ be the polynomial ring localized at the prime ideal $\langle t\rangle$, and let $\mu_A = t\lambda$. Then λ is $\mathbb{K}(t)$-isomorphic to μ_A via the isomorphism $t^{-1}I_n$ and $\mu_A \otimes \mathbb{K}$ is equal to λ_0, the abelian law.

1.1.2 One parameter subgroup degeneration

The matrix g_t, which was used in example 1.3, is the special case of a so called *one-parameter subgroup degeneration.*[2]

Definition 1.6. Let $g : \mathbb{K}^* \to \mathrm{GL}_n(\mathbb{K}), t \mapsto g_t$ be a group homomorphism such that $\mu = \lim_{t\to 0} g_t \cdot \lambda$, then $\lambda \to_{\deg} \mu$ is called a one-parameter subgroup degeneration. Furthermore we call $g(\mathbb{K}^*) \subset \mathrm{GL}_n(\mathbb{K})$ a one-parameter subgroup.

One-parameter subgroups classify a special kind of degenerations. Before we can make a statement in this direction, we need the following definition.

Definition 1.7. A filtration on an algebra with underlying vector space V is a nested sequence of subspaces

$$\cdots V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset \cdots$$

such that $V_i \cdot V_j \subset V_{i+j}$. For every filtration on V there can be associated a graded algebra W, defined as follows. Let $W = \bigoplus_{l\in\mathbb{Z}} V_l/V_{l+1}$ and for $x \in V_i, y \in V_j$, define

$$\overline{x} \cdot \overline{y} = \overline{x \cdot y} \in V_{i+j}/V_{i+j+1}.$$

We note the following theorem, see [32].

[2] Abbreviated by 1-PSG degeneration.

Theorem 1.8. *If $\lambda \to_{\deg} \mu$ via a one-parameter subgroup $P(t)$, then μ is the associated graded Lie algebra given by the filtration on λ induced by $P(t)$. Conversely, if μ is the associated graded Lie algebra given by some filtration on λ, then μ is a degeneration of λ via a one-parameter subgroup.*

As a special case of a 1-PSG degeneration we have the following definition.

Definition 1.9. Let $\mu \in \mathrm{Alg}_n(\mathbb{C})$ be a degeneration over $\mathbb{C}$ of the algebra structure $\lambda \in \mathrm{Alg}_n(\mathbb{C})$. We call $\mu = \lim_{\varepsilon \to +0} g_\varepsilon \cdot \lambda$ a generalized Inönü - Wigner-contraction[3] (shortly IW-contaction) if there exist matrices $M, N \in \mathrm{GL}_n(\mathbb{C})$, which do not depend on ε, such that the matrix g_ε can be represented in the form $g_\varepsilon = M d_\varepsilon N$ where $d_\varepsilon = \mathrm{diag}(\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_n})$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. The n-tuple of exponents $(\alpha_1, \ldots, \alpha_n)$ is called the signature of the IW-contraction $\lambda \to \mu$.

All degenerations over $\mathbb{C}$ that arise in the physical literature are generalized IW-contractions. Under these circumstances it is natural to ask if there are any exceptions at all? This question has a long history ([63], [64]) and it was even believed that every degeneration over $\mathbb{C}$ can be represented by a generalized IW-contraction. However, by Burde ([14], [17]) we now know that the contrary is true. Considering characteristical nilpotent Lie algebras, which do possess only nilpotent derivations, we obtain an example of algebras that admit no poper gradings. Regarding Theorem 1.8 any 1-PSG degeneration to a characteristical nilpotent Lie algebra cannot be equivalent[4] to a generalized IW-contraction. In fact such degenerations do exist for Lie algebras with dimensions higher than 7, as was shown in [17]. Since there it was still an open question wether universality of generalized IW-contractions fails for dimensions less than 7. What we do know is that for Lie algebras of dimension up to 3 every degeneration over $\mathbb{C}$ is equivalent to a generalized IW-contraction. Nevertheless the following result was proved by D.R. Popovych and R.O. Popovych in [52]:[5]

Theorem 1.10. *There exists one and only one degeneration of complex 4-dimensional Lie algebras which is not equivalent to a generalized IW-contraction.*

Therefore we have a lowest-dimensioal example on the non-universality of generalized IW-contractions.

However, if we restrict ourselves to the subclass of 1-PSG degenerations we do have equivalence to the class of generalized IW-contractions. This was shown by D.R. Popovych and R.O. Popovych in [51]:

Theorem 1.11. *Any 1-PSG degeneration over $\mathbb{C}$ is equivalent to a generalized IW-contraction.*

Remark 1.12. In [51] this Theorem is formulated for contractions rather than for degenerations and proved for the ground fields $\mathbb{R}$ and $\mathbb{C}$.

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[3] The name generalized Inönü - Wigner-contraction was first used by Doebner and Melsheimer in [34]. There seems to be no item for this special kind of a 1-PSG degeneration in the algebraic literature, for which reason we take the existing notion of generalized IW-contractions used in physics.

[4] For the exact definition of equivalence we refer to Definition 1.23 in the next section.

[5] In this paper the term contraction is used instead of degeneration; both of which are equal because of Example 1.4.

Even more is true; we can restrict the values of the signature of a generalized IW-contraction from $\mathbb{R}$ to $\mathbb{Z}$ with out loss of generality ([51]):

Theorem 1.13. *Any generalized IW-contraction performed by a degeneration matrix $g_\varepsilon = M d_\varepsilon N$ is equivalent to a IW-contraction with integer signature (and the same associated matrices M and N).*

1.1.3 First statements about degenerations

Let G be an algebraic group acting on a variety X, then every irreducible component of X contains the orbit closures of all its elements. In the language of degenerations this means, that for an irreducible component C of $\mathrm{Alg}_n(\mathbb{K})$ with $\lambda \in C$ it follows, that $\overline{\mathrm{O}(\lambda)} \subset C$.

Definition 1.14. An algebra law λ is called rigid, if $\mathrm{O}(\lambda)$ is open in $\mathrm{Alg}_n(\mathbb{K})$.

In this case $\overline{\mathrm{O}(\lambda)}$ defines an irreducible component of $\mathrm{Alg}_n(\mathbb{K})$. Because the number of irreducible components in each dimension is finite, the number of rigid algebras in a fixed dimension is also finite.

Lemma 1.15. *Let $\mathfrak{g} \in \mathrm{Lie}_n(\mathbb{K})$ be semisimple, then $\mathfrak{g}$ is rigid.*

Proof. See [47, p. 285]. A Lie algebra is semisimple if and only if its Killing form is nondegenerate. This is an open condition. □

The identification of all $\mathbb{K}$-algebra structures as a subvariety of $\mathbb{K}^{n^3}$ gives us the possibility of developing a whole bunch of methods related to algebraic geometry and the theory of algebraic groups. Borels Closed-Orbit-Lemma builds the starting point for further considerations. The most important consequence of this lemma in the context of degeneration is the following theorem. Although this is a standard result, a complete proof is hard to find in literature. So we treat the proof here in detail.

Theorem 1.16. *Let $A \to_{\deg} B$ for $A, B \in \mathrm{Alg}_n(\mathbb{K})$. Then the following two inequalities hold:*

$$\dim \mathrm{O}(A) > \dim \mathrm{O}(B)$$

$$\dim \mathrm{Der}(A) < \dim \mathrm{Der}(B)$$

Proof. Because of Borels Closed-Orbit-Lemma the first inequality follows at once. For the second one we consider the dimension formula ([41, p. 65]) for an algebraic group G acting on an affine variety W. For every point $x \in W$ we have

$$\dim G = \dim \mathrm{O}(x) + \dim G_x$$

where G_x is the isotropy group of x. In the case $G = \mathrm{GL}_n(\mathbb{K})$ and $A \in \mathrm{Alg}_n(\mathbb{K})$, the stabilizer G_A is exactly $\mathrm{Aut}(A)$. As an algebraic subgroup of $\mathrm{GL}_n(\mathbb{K})$, the tangent space (or Lie algebra) of $\mathrm{Aut}(A)$ is exactly the algebra of derivations ([37], p. 82). Using the fact that $\mathrm{GL}_n(\mathbb{K})$ contains no singular points, the dimension of any subvariety equals to that of its tangent space. Hence we find that $n^2 = \dim \mathrm{O}(A) + \dim \mathrm{Der}(A)$ and therefore $\dim \mathrm{Der}(A) < \dim \mathrm{Der}(B)$. □

Another result, that can be generalized to arbitrary algebras over an algebraically closed field, inspired by [32], is the following.

Theorem 1.17. *Let A_1 and A_0 be two algebras over the field $\mathbb{C}$, A_1 degenerating to A_0. Let C be a Zariski closed B-stable subset in $\mathrm{Alg}_n(\mathbb{C})$, where B is a Borel subgroup of a reductive algebraic group G over $\mathbb{C}$. If there exists a representative of the isomorphism class of A_1 lying in C, then there exists a representative of A_0 lying in C.*

Proof. Let λ_1 be a representative of A_1 lying in C and λ_0 some representative of A_0. With respect to the Iwasawa decomposition we can write G as KB, with K being a compact subgroup of G. Therefore, assuming $\lambda_0 \in \overline{G \cdot \lambda_1}$, we have $\lambda_0 \in \overline{K \cdot B \cdot \lambda_1}$. Because of the compactness of K it follows that $\lambda_0 \in K \cdot \overline{B \cdot \lambda_1}$. According to the hypothesis that C is a closed B-stable set, we conclude that $\lambda_0 \in K \cdot C$, for which reason we can find a $k \in K$ such that $k^{-1} \cdot \lambda_0 \in C$. □

The theorem says the following: Let there be given a certain property of an algebra A. If all algebras that share this property form a closed and B-stable set, then every degeneration of A must also have this property.

Example 1.18. Take one of the two defining laws of a Lie algebra over $\mathbb{C}$, the jacobian identity. This condition defines a Zariski closed set in $\mathrm{Alg}_n(\mathbb{C})$. It is trivially B-stable, because it is already G-stable. Therefore any degeneration of a Lie algebra is again a Lie algebra. Of course there are more subtle examples where this method comes to play, as one can see in chapter three.

1.1.4 Discussion of a result of Popov

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. Let G be a connected linear algebraic group and let M be a finite dimensional algebraic G-module. Denote by $G \cdot x$ and $G \cdot y$ the G-orbits for two points x and y in M.
During the work for this thesis an article appeared ([50]) in which V. L. Popov formulates the following question:

How can one find out whether or not the orbit $G \cdot y$ lies in the Zariski closure of the orbit $G \cdot x$ in V?

This question is referred to as the orbit closure problem in algebraic transformation group theory in its most general form. Choosing special properties for the algebraic group G and its module M results therefore in different applications of algebraic transformation group theory. As an important special case we have the following. Let V be a finite dimensional vector space over $\mathbb{K}$. Here we specify the characteristic of the ground field $\mathbb{K}$ to be zero. Let $G = GL(V)$ and $M = V^* \otimes V^* \otimes V$. The points of M are exactly all structures of $\mathbb{K}$-algebras on the vector space V. If λ and μ are points in M, Popov's question means to find out whether or not λ degenerates to μ.

In [50] a constructive method is presented that answers the orbit closure problem in general by means of a finite number of effectively feasible operations. This construction results in a finite system of linear equations (in finitely many variables), which's inconsistency is equivalent to the inclusion $G \cdot y \subset \overline{G \cdot x}$. Moreover [50] provides an algorithm that determines the orbit closure of an arbitrary linear subvariety L of M by the zero set of a finite system of polynomial functions $q_1, \dots, q_m$ on M. However, this algorithm is based on the computation of a Gröbner basis that contains the functions $q_1, \dots, q_m$ as a part of its elements. As a consequence this algorithm has lack of efficiency than the constructive method mentioned before.

Regarding the orbit closure problem in the case of degenerations one might be tempted to use these techniques for a classification as the one in chapter 4 of this thesis. However, even in dimension two the constructive method (which is the faster one) yields:[6]

$$\sum_{j=0}^{52} \frac{(j+7)!}{7!j!} = 2558620845$$

coefficients in

$$\sum_{j=0}^{312} \frac{(j+3)!}{3!j!} = 407624595.$$

variables.

In conclusion, this algorithm is not executable since it is impossible[7] to cope with that huge number of variables and coefficients that arise in the programming procedure.

1.2 Contractions

1.2.1 The general definition of a contraction

Let $\mathfrak{g}$ be an n-dimensional Lie algebra with underlying vector space V over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We denote by $[\cdot,\cdot]$ the Lie bracket of the algebra $\mathfrak{g}$ and suppose the dimension n to be finite.

Let $U\colon (0,1] \to \mathrm{GL}(V)$ be a continuous map and denote the image of an element $\varepsilon \in (0,1]$ under U by U_ε. The map U induces a parametrized family of Lie algebras $\mathfrak{g}_\varepsilon = (V, [\cdot,\cdot]_\varepsilon)$, which are isomorphic to $\mathfrak{g}$, in the following way:

$$[x,y]_\varepsilon = U_\varepsilon^{-1}([U_\varepsilon(x), U_\varepsilon(y)])$$

for all $x, y \in V$.

Definition 1.19. If for all $x, y \in V$ there exists the limit

$$\lim_{\varepsilon \to +0} [x,y]_\varepsilon = \lim_{\varepsilon \to +0} U_\varepsilon^{-1}([U_\varepsilon(x), U_\varepsilon(y)]) =: [x,y]_0$$

then $[x,y]_0$ is a well-defined Lie bracket. In this case the Lie algebra $\mathfrak{g}_0 = (V, [\cdot,\cdot]_0)$ is called a one-parametric continuous contraction (or simply contraction) of the Lie algebra $\mathfrak{g}$. The procedure that yields the algebra $\mathfrak{g}_0$ in the above explained way from the given Lie algebra $\mathfrak{g}$ is also called a contraction and abbreviated by $\mathfrak{g} \to_{\mathrm{con}} \mathfrak{g}_0$.

Remark 1.20. In contrary to the definition of a degeneration the basis change is undertaken by the right action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathfrak{g}$. This is a usual convention in physics and therefore we use it wherever the term contraction appears.

Let $(e_1, \ldots, e_n)$ be a basis of the vector space V. We write the bracket of the Lie algebra $\mathfrak{g}$ in this basis by $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$. The collection of the numbers $c_{ij}^k \in k$ is called the vector of structure constants and determines the Lie algebra $\mathfrak{g}$

[6] This and the following number was obtained by D. Burde via private communications.

[7] We mean impossible regarding the processing speed of computers at the moment.

completely. So if we fix a basis of the underlying vector space V of a Lie algebra $\mathfrak{g}$ the image of the map U lies in $\mathrm{GL}_n(\mathbb{K})$ and the limit condition of Definition 1.19 is then equivalent to requiring the existence of the limit

$$\lim_{\varepsilon\to+0}(U_\varepsilon)_{ii'}(U_\varepsilon)_{jj'}(U_\varepsilon^{-1})_{k'k}c_{ij}^k =: c_{0,ij}^k$$

for all $i', j', k' \in \{1,\dots,n\}$. Therefore the numbers $c_{0,ij}^k$ define the vector of structure constants of the Lie algebra $\mathfrak{g}_0$.

Definition 1.21. Let $[x,y]_0 := \lim_{\varepsilon\to+0} U_\varepsilon^{-1}([U_\varepsilon(x), U_\varepsilon(y)])$ be a contraction of the Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$ to the Lie algebra $\mathfrak{g}_0 = (V, [\cdot,\cdot]_0)$. We call ε the contraction parameter and the matrix-valued function U_ε the contraction matrix of the contraction $\mathfrak{g} \to_{\mathrm{con}} \mathfrak{g}_0$.

Remark 1.22. As mentioned in the previous section, the concept of a contraction can be generalized to arbitrary algebraically closed fields in terms of orbit closures in the variety of Lie algebras. A contraction defined in this way is often referred to as a degeneration.

We call a contraction $\mathfrak{g} \to_{\mathrm{con}} \mathfrak{g}_0$ trivial if $\mathfrak{g}_0$ is abelian and proper if $\mathfrak{g}_0$ is not isomorphic to $\mathfrak{g}$.

Definition 1.23. Two contractions $\mathfrak{g} \to_{\mathrm{con}} \mathfrak{g}_0$ and $\mathfrak{g}' \to_{\mathrm{con}} \mathfrak{g}_0'$ are called (weakly) equivalent if the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ are isomorphic to $\mathfrak{g}'$ and $\mathfrak{g}_0'$, respectively.

Weak equivalence doesn't take improper contractions into account. We can therefore focus on the existence problem of a contraction. A notion that emphasises the different ways a contraction can be constructed, is that of strong equivalence.

Definition 1.24. Two contractions from $\mathfrak{g}$ to $\mathfrak{g}_0$ performed by the contraction matrices U_ε and U_ε', respectively, are called strongly equivalent if there exist $\delta \in (0,1]$, mappings $G\colon (0,\delta] \to \mathrm{Aut}(\mathfrak{g})$ and $G'\colon (0,\delta] \to \mathrm{Aut}(\mathfrak{g}_0)$ and a continuous monotonic function $\varphi\colon (0,\delta] \to (0,1]$ with $\lim_{\varepsilon\to+0}\varphi(\varepsilon) = 0$, such that $U_\varepsilon' = G_\varepsilon U_{\varphi(\varepsilon)} G_\varepsilon'$, $\varepsilon \in (0,\delta]$.

In the next two subsections we give two important examples of a contraction, where the contraction matrices are of a very simple type.

1.2.2 Inönü - Wigner contractions

The study of contractions due to E. Inönü and E. P. Wigner was initiated by a problem concerning the representations of certian Lie groups. When E. Inönü determined the unitary irreducible representations of the Galilei group, it was not clear how this representations were related to physical properties as this is the case for the Poincare group ([65]). Both groups can be understood as the algebraic structures defining non-relativistic and relativistic mechanics. Inönü's and Wigner's idea was to look at the limit of the Poincare group with the velocity of light approaching infinity. This leads to a description of the Galilean group as a limiting case of the Poincare group. We will present the mathematical formalism that underlies a group contraction. To write this overview we used [31], [38], and the original papers by Inönü and Wigner [65] and [66].

Let $\mathfrak{g} = (V, [\cdot,\cdot])$ and $\mathfrak{g}_0 = (V, [\cdot,\cdot]_0)$ be two n-dimensional Lie algebras where $\mathfrak{g} \to_{\text{con}} \mathfrak{g}_0$. We consider the contraction matrix U_ε to depend linearly on the contraction parameter ε and therefore we can write

$$U_\varepsilon = U_0 + \varepsilon W$$

where U_0 and W are $n \times n$ matrices not dependend on ε. Furthermore we assume that there exist matrices $M, N \in \mathrm{GL}_n(\mathbb{K})$ such that the matrix U_ε can be transformed to the special diagonal form $MU_\varepsilon N^{-1} = \mathrm{diag}(1+\varepsilon a, \ldots, 1+\varepsilon a, \varepsilon, \ldots, \varepsilon) =: D_\varepsilon$. Without loss of generality we can set $a = 0$ and therefore, by these linear transformations, it is possible to write the matrix U_ε in the form

$$U_\varepsilon = \begin{pmatrix} I_d & 0 \\ 0 & \varepsilon I_{n-d} \end{pmatrix}$$

where I_d is the identity matrix for an d-dimensional vector space, namely:

$$I_d := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_d(\mathbb{K}).$$

In view of the last considerations we will work with the following definition:

Definition 1.25. A contraction with respect to a transformation matrix U_ε of the form

$$U_\varepsilon = \mathrm{diag}(1, \ldots, 1, \varepsilon, \ldots, \varepsilon)$$

is called an Inönü - Wigner contraction (or shortly IW-contraction).

An IW-contraction is therefore completely characterized by the dimension of the eigenspace V_1 to the eigenvalue 1 of the contraction matrix U_ε. Let $(e_1, \ldots, e_n)$ be a basis of the underlying vector space V of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_0$. We set $V = V_1 \oplus V_\varepsilon$ and $\dim V_1 =: d$. Necessarily $\dim V_\varepsilon = n - d$, which is the dimension of the eigenspace V_ε to the eigenvalue ε of the contraction matrix U_ε. We denote by c_{ij}^k and $\overline{c}_{ij}^k$ the structure constants of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively. The brackets of $\mathfrak{g}$ and $\mathfrak{g}_0$ are then related in the following way:

$$U_\varepsilon(e_{s_k}) = e_{s_k} \quad \text{for} \quad 1 \leqslant k \leqslant d$$

$$U_\varepsilon(e_{t_k}) = \varepsilon e_{t_k} \quad \text{for} \quad 1 \leqslant k \leqslant n-d.$$

We take the relations above for the transformation U_ε to calculate the new algebra structure explicitely:

$$[e_{s_i}, e_{s_j}]_0 = [e_{s_i}, e_{s_j}] = \sum_{r=1}^{d} c_{s_i s_j}^{s_r} e_{s_r} + \frac{1}{\varepsilon} \sum_{r=1}^{n-d} c_{s_i s_j}^{s_r} e_{s_r},$$

$$[e_{s_i}, e_{t_j}]_0 = \varepsilon [e_{s_i}, e_{t_j}] = \varepsilon \sum_{r=1}^{d} c_{s_i t_j}^{s_r} e_{s_r} + \sum_{r=1}^{n-d} c_{s_i t_j}^{s_r} e_{s_r},$$

$$[e_{t_i}, e_{t_j}]_0 = \varepsilon^2 [e_{t_i}, e_{t_j}] = \varepsilon^2 \sum_{r=1}^{d} c_{t_i t_j}^{s_r} e_{s_r} + \varepsilon \sum_{r=1}^{n-d} c_{t_i t_j}^{s_r} e_{s_r}.$$

Now, if we let $\varepsilon \to 0$, the only convergence problem occurs for $c_{s_i s_j}^{s_r}$. The only way for the limit $\lim_{\varepsilon\to 0} c_{ij}^k(\varepsilon) = \overline{c}_{ij}^k$ to exist, is that $c_{s_i s_j}^{s_r} = 0$ for all r with $1 \leqslant r \leqslant d$. This means that the first d basis vectors $\{e_1, \dots e_d\}$ span a subalgebra of $\mathfrak{g}$. We then say that the operation of a IW-contraction is undertaken with respect to this subalgebra. We can also see that the vector space V_1 is a subalgebra of the contracted Lie algebra $\mathfrak{g}_0$. Moreover, as $\varepsilon \to 0$ the product on V_ε vanishes completely and therefore defines an abelian ideal of $\mathfrak{g}_0$.
Finally we summarize these observations in the following Theorem by Inönü and Wigner ([66]), stated in its original wording.[8]

Theorem 1.26. *Every Lie group can be contracted with respect to any of its continuous subrgoups and only with respect to these. The contracted infinitesimal elements form an abelian invariant subgroup of the contracted group. The subgroup S with restect to which the contraction was undertaken is isomorphic with the factor group of this invariant subgroup. Conversely, the existence of an abelian invariant subgroup and the possibility to choose from each of its cosets an element so that these form a subgroup S, is a necessary condition for the possibility to obtain the group from another group by contraction.*

1.2.3 Saletan contractions

In the last subsection we observed contractions linear in one parameter, that are dependend on the choice of a basis. From a historical point of view, the next step towards degeneration theory was to get rid of that dependence. A base-free approach to contraction theory was first introduced by E. J. Saletan ([54]). In our exposition we shall follow closely his treatment.[9]

Definition 1.27. Let $\mathfrak{g} = (V, [-,-])$ and $\mathfrak{g}_0 = (V, [-,-]_0)$ be two n-dimensional Lie algebras where $\mathfrak{g} \to_{\text{con}} \mathfrak{g}_0$. Let U_ε be the matrix that performs this contraction, i. e.:

$$[x,y]_0 := \lim_{\varepsilon\to 0} U_\varepsilon^{-1}[U_\varepsilon(x), U_\varepsilon(y)].$$

If the contraction matrix U_ε depends linearily on the parameter ε we shall speak of a Saletan contraction.

For the beginning we analyze what special forms a contraction matrix $U(\varepsilon)$ can have.[10] Let U_0 be singular, so U_0 annihilates a vector subspace V_1 of V. Therefore we have a decomposition $V = V_1 \oplus \bar{V_1}$, with $U_0(V_1) = 0$. In the same way U_0 annihilates a vector subspace of $U_0(V)$ for which reason we get $V = V_2 \oplus \bar{V_2}$, with $U_0^2(V_2) = 0$. Clearly $\dim_{\mathbb{K}} U_0(V) \geqslant \dim_{\mathbb{K}} U_0^2(V)$. Repeating this argument we arrive at a subspace V_m of V for which $V = V_m \oplus \bar{V_m}$, with $U_0^m(V_m) = 0$, and $U_0^m(V) = U_0^{m+j}(V)$ for all $j \in \mathbb{N}$. This construction can be regarded as a special form of Fitting's Lemma ([53, p. 82]). Restricted to $\bar{V_m}$ the matrix U_ε is a faithful map, even as $\varepsilon \to 0$. Referring to this we shall denote the subspace $\bar{V_m}$ with the above condition by V_F. On the other hand U_0 is singular on the complement V_m which shall therefore be denoted by V_S, so we have $V = V_F \oplus V_S$.

[8] The item contraction in the following Theorem of course means Inönü - Wigner contraction.
[9] In contrary to group contractions studied by Inönü and Wigner we are here only interested in contractions of the Lie algebra.
[10] The contraction is supposed to be non-trivial.

We now return to the contraction of a Lie algebra by a matrix U_ε depending linearily on a contraction parameter ε. Because of this linearity we are able to compute the inverse of U_0 on V_F explicitely and derive contraction criteria from that. Following Saletans approach we set without loss of generality $U_1 = I_n$ (the idetity matrix in $\mathrm{GL}_n(\mathbb{K})$) , yielding

$$U_\varepsilon = V + \varepsilon W = \varepsilon I_n + (1-\varepsilon)U_0 \quad \text{and}$$
$$U_\varepsilon^{-1} = \frac{1}{1-\varepsilon}(\delta I_n + U_0)^{-1}$$

where $\delta = \frac{\varepsilon}{1-\varepsilon}$. We notice that $\varepsilon \to 0$ implies $\delta \to 0$. With this settings the Saletan contraction takes the form

$$[x,y]_0 := \lim_{\varepsilon\to 0}(1-\varepsilon)(\delta I_n + U_0)^{-1}[(\delta I_n + U_0)(x), (\delta I_n + U_0)(y)].$$

We proceed by computing the inverses with respect to the projections to V_F and V_S. Because U_0 is faithful on V_F we have

$$(\delta I_n + U_0)^{-1} = U_0^{-1}(\delta U_0^{-1} + I_n)^{-1} \quad \text{on } V_F.$$

It follows that on V_F the matrix U_0 acts as an isomorphism:

$$\lim_{\varepsilon\to 0}(1-\varepsilon)(\delta I_n + U_0)^{-1}[(\delta I_n + U_0)(x), (\delta I_n + U_0)(y)] \to U_0^{-1}[U_0(x), U_0(y)] \quad \text{on } V_F.$$

On the subspace V_S the matrix U_0 becomes singular for which reason the inverse and hence the above expression don't exist. Instead we use a series expansion:

$$(\delta I_n + U_0)^{-1} = \delta^{-1}(I_n + \frac{U_0}{\delta})^{-1} = \frac{1}{\delta}\sum_{i=0}^{m-1}(\frac{U_0}{\delta})^i \quad \text{on } V_S,$$

where m is the lowest power of U_0 which annihilates V_S. The following straightforward calculation demonstrates the advantage of this expansion:

$$\begin{aligned}
&[x,y]_0 := \lim_{\varepsilon\to 0} U_\varepsilon^{-1}[U_\varepsilon(x), U_\varepsilon(y)] = \\
&\lim_{\varepsilon\to 0}\frac{1}{\delta}\sum_{i=0}^{m-1}(\frac{U_0}{\delta})^i(\delta^2[x,y] + \delta[U_0(x),y] + \delta[x,U_0(y)] + [U_0(x),U_0(y)]) = \\
&[U_0(x),y] + [x,U_0(y)] - U_0([x,y]) \\
&\quad + (\frac{1}{\delta} - \frac{U_0}{\delta^2} + \frac{U_0^2}{\delta^3}\dots)([U_0(x),U_0(y)] - U_0([U_0(x),y]) - U_0([x,U_0(y)]) + U_0^2([x,y])).
\end{aligned}$$

The whole computation is understood to be taken just on the subspace V_S. Moreover, from the formula of the Saletan contraction it can be seen, that we dropped the term $(1-\varepsilon)$. However, this doesn't change the result of the limit process. Back to the equation above, we remark that this limit exists if and only if the term $[U_0(x),U_0(y)] - U_0([U_0(x),y]) - U_0([x,U_0(y)]) + U_0^2([x,y])$ vanishes on V_S. We summarize our observations in the following Theorem.

Theorem 1.28. *Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be two finite dimensional Lie algebras with underlying vectorspace V. Let U_0 be a singular matrix such that for subspaces V_F and V_S of V the conditions $U_0(V_S) = 0$, $U_0(V_F) = V_F$, and $V = V_F \oplus V_S$ are satisfied. Then $\mathfrak{g}$ contractes to $\mathfrak{g}_0$ by a Saletan contraction via the matrices U_ε, $\lim_{\varepsilon\to 0} U_\varepsilon =: U_0$, if and only if*

$$[U_0(x),U_0(y)] - U_0([U_0(x),y]) - U_0([x,U_0(y)]) + U_0^2([x,y]) = 0 \quad \textit{on } V_S.$$

The next Proposition is a crucial tool for the following Theorems. The proofs for both statements can be found in [54].

Proposition 1.29. *The contraction criterion of the last Theorem can be replaced by the following, for all $p, q \geqslant 1$:*

$$[U_0^p(x), U_0^q(y)] - U_0^q([U_0^p(x), y]) - U_0^p([x, U_0^q(y)]) + U_0^{p+q}([x,y]) = 0 \quad \text{on } V_S.$$

As a consequence this equation also gives a necessary and sufficient condition for a Saletan contraction to exist.

We finally arrive at the two main Theorems, the first one is similar to that of the Inönü - Wigner contraction theory.

Theorem 1.30. *Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be two finite dimensional Lie algebras, $\mathfrak{g}_0$ a Saletan contraction of $\mathfrak{g}$ via U_ε. The following statements hold:*

1. *The subspace $U_0(V)$ of V is a subalgebra of $\mathfrak{g}$.*
2. *The subspace V_1 of V is an invariant and solvable subalgebra of $\mathfrak{g}_0$.*

Theorem 1.31. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. If $\mathfrak{g}$ contractes to some Lie algebra by a Saletan contraction via U_ε, then it also contractes to some Lie algebra via U_ε^k for all $k \in \mathbb{N}$. We abbreviate this by $\mathfrak{g}^{(k)} := U_0^k \cdot \mathfrak{g} := \lim_{\varepsilon\to 0}(U_\varepsilon^k)^{-1}[U_\varepsilon^k(x), U_\varepsilon^k(y)]$. With this definition the following statement holds:*

$$U_0^k \cdot \mathfrak{g}^{(j)} = \mathfrak{g}^{(k+j)}.$$

1.3 Deformations

In this section we shall give a short overview of the concept of an algebraic deformation, how it developed and how it is related to a degeneration. As a source for this section served the articles [43], [27], and [49].

In [40] Kodaira and Spencer introduced the concept of local and infinitesimal deformations of a complex analytic structure. They showed that infinitesimal deformations can be parametrized by related cohomology groups. Afterwards, in [42] Kuranishi established the deformation theory of compact complex structures. Based on the facts about deformation theory of analytic structures Artin and Schlessinger developed the deformation theory of algebraic manifolds ([2] and [55], 1986). Deformations of arbitrary rings and associative algebras were first studied by Gerstenhaber in [29] and [30] (1964 - 1974). Also in the work of Gerstenhaber, cohomology plays an important role. Finally, we want to remark that concerning the theory of deformations of Lie algebras Nijenhuis and Richardson have to be mentioned. In their work ([48]) they consider some general problems of this field.

A first definition introduced by Gerstenhaber and now referred to as a formal deformation, in the case of Lie algebras, is given as follows.

Definition 1.32. Let μ_0 be a Lie algebra over an arbitrary field $\mathbb{K}$. A formal deformation of μ_0 is a one-parameter family of Lie algebras μ_t in $V \otimes k[[t]]$ over the formal power series ring $k[[t]]$:

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where $\varphi_i \in \mathrm{Hom}(\Lambda^2 V, V)$.

A first and rather trivial relation between degenerations and formal deformations is the followng: Consider μ_t to be a formal deformation of μ, which is given by a convergent power series in t, then μ_0 is a degeneration of $\bigcup_{t\in k} \mathrm{O}(\mu_t)$.

Remark 1.33. Other than in degeneration theory where we presumed the underlying vector space to be finite dimensional we explicitely allow the vector space to be infinite dimensional in the case of deformations. In fact, deformations of infinite dimensional Lie algebras have been intensively studied (see [25] and [26]), in particular because of the various applications in physics.

To prove a more substantial connection between deformations and degenerations we need a more general definition. For this let A be a local finite dimensional algebra over $\mathbb{K}$ and let μ_0 be a Lie algebra with underlying vector space V over $\mathbb{K}$.[11] If μ_A is a Lie algebra over A, then any morphism $f\colon A \to B$ defines a Lie algebra $\mu_A \otimes_A B$.

Definition 1.34. Let A be a local finite dimensional algebra over $\mathbb{K}$. A deformation of a Lie algebra μ_0 is a Lie algebra μ_A over A on $V \otimes_{\mathbb{K}} A$ such that

$$\mu_A \otimes_A k = \mu_0,$$

where the tensor product is given by the residue map $A \to A/m_A = k$.

The last definition is a natural generalization of Gerstenhabers concept of a formal deformation. In fact, we can regard a formal deformation as a Lie algebra over the quotient field $k((t))$ rather than as a family of Lie algebra structures. Replacing the field $k((t))$ by a parameterring A yields the term of a deformation, sometimes also referred to as a global deformation (see [43]).

Definition 1.35. Let the two Lie algebras μ_A and μ'_A be deformations of the Lie algebra μ_0 parametrized by A. The deformations are called equivalent if there is a Lie algebra isomorphism $\mu_A \cong \mu'_A$ which induces the indentity map on μ_0. A deformation μ_A of μ_0 is called trivial if it is equivalent to $\mu_0 \otimes A$.

Using Theorem 1.5 one can prove the following statement.

Theorem 1.36. *Let μ_0 and μ_1 be two finite dimensional Lie algebras. If μ_1 degenerates to μ_0 then μ_1 is a non-trivial deformation of μ_0.*

Remark 1.37. The converse of the above theorem is not true. To see this we consider the 3-dimensional Lie algebra $\mathfrak{g}$ given by the multiplication laws $[e_1, e_2] = e_1$, $[e_1, e_3] = [e_2, e_3] = 0$ and the family of Lie algebras $\mathfrak{h}(\alpha)$ given by the brackets $[e_1, e_2] = e_1$, $[e_2, e_3] = -\alpha$, and $[e_1, e_3] = 0$. For any two distinct parameters α_1 and α_2 with $\alpha_1 \cdot \alpha_2 \neq 1$, the Lie algebras $\mathfrak{h}(\alpha_1)$ and $\mathfrak{h}(\alpha_2)$ are not isomorphic. It can be shown that the family $\mathfrak{h}(\alpha)$ is a deformation family of $\mathfrak{g}$, but $\mathfrak{g}$ is not a degeneration of any Lie algebra $\mathfrak{h}(\alpha)$ with $\alpha \neq 1$.

Although not every deformation leads to a degeneration there is an important subclass of deformations which does. These are the so called jump deformations, first introduced by Gerstenhaber in [30].

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[11] As mentioned in Remark 1.33 this vector space need not to be finite dimensional.

Definition 1.38. A formal deformation μ_t of the Lie algebra μ_0 is called a jump deformation if there exists a power series $\Phi_{s,t} \in \mathrm{GL}_n(\mathbb{K}((s,t)))$ in two variables

$$\Phi_{s,t} = I + s\varphi_1(t) + s^2\varphi_2(t) + \ldots$$

where each φ_i is a series whose coefficients are $\mathbb{K}$-linear maps $V \to V$ such that[12]

$$\mu_{(1+s)t} = \Phi_{s,t} \cdot \mu_t.$$

The condition $\mu_{(1+s)t} = \Phi_{s,t} \cdot \mu_t$ of the last definition implies that a jump deformation obtained by a convergent power series defines a degeneration.

For an overview about the relation of deformations and Hochschild cohomology we refer to [22].

[12]The action of $\mathrm{GL}_n(\mathbb{K}((s,t)))$ is supposed to be the basis change.

2 Pre-Lie and Novikov algebras

All algebras in this thesis are assumed to be finite-dimensional.

In this chapter we introduce all algebraic properties of a pre-Lie algebra that are needed in chapter 3 of this thesis. These properties will turn out to be so called semi-invariants. Moreover we explain how the notion pre-Lie algebra was motivated and how it is connected to geometric properties of Lie groups.[1] In the following section we present a subclass of pre-Lie algebras, the so called Novikov algebras, in which we are mostly interested in chapter 4 of this work.

2.1 Left-invariant affine structures on Lie groups and pre-Lie algebras

Let $\mathbb{K}$ be a field of characteristic zero. Specifically in this section we allow the ground field to be arbitrary (with respect to the algebraic closure). Later on, because of certain limit processes, we restrict ourselves to algebraically closed fields.

Definition 2.1. Let A be a finite dimensional vector space over $\mathbb{K}$ endowed with a $\mathbb{K}$-bilinear product $A \times A \to A$ that satisfies the condition

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z$$

for all $x, y, z \in A$. Then $(A, \cdot)$ is called a left-symmetric or (left-) pre-Lie algebra.[2] Defining $[x, y] := x \cdot y - y \cdot x$, the algebra A becomes a Lie algebra, denoted by $\mathfrak{g}_A$ and called the associated Lie algebra to A. Conversely we say that a Lie algebra $\mathfrak{g}$ admits a pre-Lie or left-symmetric structure if there exists a $\mathbb{K}$-bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is left-symmetric and satisfies the condition

$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y \in \mathfrak{g}$.

In view of this definition it is very natural to ask the existence question, namely if every Lie algebra admits an affine structure. Indeed, for solvable Lie groups this question was posed by Milnor in 1977 ([44]) and unsolved till the year 1992, when Yves Benoist ([7]) gave a negative answer by providing a counterexample.

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. We have the following statement, see [11].

[1] The treatment of this subsection follows [11].

[2] Originally, pre-Lie algebras were defined to be right-symmetric. In our case, however, we assume pre-Lie algebras always to be left-symmetric. That's why we omit the prefix "left "throughout the text.

Proposition 2.2. *There is a canonical one-to-one correspondence between the class of left-invariant affine structures on G and the class of affine structures on the Lie algebra $\mathfrak{g}$, up to suitable equivalence.*

Milnor asked in [44]:

"Does every solvable Lie group G admit a complete left-invariant affine structure, or equivalently, does the universal covering group $\tilde{G}$ operate simply transitively by affine transformations of $\mathbb{R}^k$?"

In [12], [15], and [16] there has been constructed a whole family of counterexamples:

Theorem 2.3. *There are filiform nilpotent Lie groups of dimension $10 \leqslant n \leqslant 13$ which do not admit any left-invariant affine structure. Any filiform nilpotent Lie group of dimension $n \leqslant 9$ admits a left-invariant affine structure.*

2.2 Novikov algebras

Novikov algebras arise in many context in mathematics and physics. Among other things they came up in [5] studying Poisson brackets of hydrodynamic type, at which one of the authors they were named after.

Definition 2.4. A pre-Lie algebra A is called a Novikov algebra, if it satisfies the following identity (right-commutativity):

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y,$$

for all $x, y, z \in A$.

Like in the case of associative, commutative and Lie algebras we can express the left symmetry and the right-commutativity in terms of the structure constants. Therefore the sets of pre-Lie and Novikov algebra laws define subvarieties of $\mathrm{Alg}_n(\mathbb{K})$, which we will denote by $\mathrm{preLie}_n(\mathbb{K})$, and $\mathrm{Nov}_n(\mathbb{K})$, respectively. The polynomials in the structure constants representing the left-symmetry take the form:

$$\sum_{l=1}^{n} (c_{ij}^l c_{lk}^m - c_{jk}^l c_{il}^m - c_{ji}^l c_{lk}^m + c_{ik}^l c_{jl}^m) = 0.$$

These polynomials together with the following polynomial identities determine the set of Novikov structures:

$$\sum_{l=1}^{n} (c_{ij}^l c_{lk}^m - c_{ik}^l c_{lj}^m) = 0.$$

Both sets of equations must hold for $1 \leqslant i, j, k, m \leqslant n$.

Consider an n-dimensional algebra A (it will be a pre-Lie algebra in most parts of this text) defined over an algebraically closed field $\mathbb{K}$ of characteristic 0. The maps $\mathrm{L}^A(x)$ and $\mathrm{R}^A(x)$ denote the left respectively right multiplication by an element $x \in A$. In sections where we consider left and right multiplications in different algebras it is important to indicate in which algebra the multiplications are taken. For this we use the upper index of $\mathrm{L}^A(x)$ and $\mathrm{R}^A(x)$.

We define the various terms of nilpotency associated to a pre-Lie algebra:

Definition 2.5. Let A be a pre-Lie algebra over an algebraically closed field of characteristic zero. We consider the subspace $R_A = \{\mathrm{R}^A(x) \mid x \in A\}$ of $\mathrm{End}_{\mathbb{K}}(A)$. The algebra A is called right-nilpotent if $R_A^n := \langle \mathrm{R}^A(x_1) \cdots \mathrm{R}^A(x_n)\rangle_{\mathbb{K}} = 0$ for some $n \geqslant 1$.

In the same way we consider the subspace $L_A = \{\mathrm{L}^A(x) \mid x \in A\}$ of $\mathrm{End}_{\mathbb{K}}(A)$. The algebra A is called left-nilpotent if $L_A^n := \langle \mathrm{L}^A(x_1) \cdots \mathrm{L}^A(x_n)\rangle_{\mathbb{K}} = 0$ for some $n \geqslant 1$.

Furthermore a pre-Lie algebra is called nilpotent if the subalgebra generated by all left- and right multiplications by all elements $x \in A$ is nilpotent. That means that there exists a natural number n such that any bracketing of n elements in A is zero.

By right-commutativity the sum of two right-nilpotent ideals I_1 and I_2 of A is again a right-nilpotent ideal of A. Therefore, in a Novikov algebra A there exists a largest right-nilpotent ideal $N(A)$ of A.

Due to the work of E. I. Zel'manov we have very substantial results about simple Novikov algebras and the decomposition of the factor algebra $A/N(A)$ into the direct sum of ideals. In what follows we give a short overview concerning the results of his note [67]. We suppose that the ground field $\mathbb{K}$ is algebraically closed and of characteristic zero.

Proposition 2.6. *Let A be a Novikov algebra then its quotient algebra $A/N(A)$ is a direct sum of fields.*

Corollary 2.7. *Let A be a Novikov algebra then A either contains a non-zero ideal with zero multiplication or is associative.*

Definition 2.8. Let A be a pre-Lie algebra. If for any two elements a and b in A we have $a \cdot b = 0$ we call the algebra A abelian. We call a pre-Lie algebra A commutative if $a \cdot b = b \cdot a$ for all $a, b \in A$.

Remark 2.9. We shall carefully distinct between the notions of an abelian pre-Lie algebra and a commutative pre-Lie algebra. The latter holds if and only if the associated Lie algebra is abelian, whereas a commutative pre-Lie algebra needs not to be abelian.

The main result of [67] follows now from Proposition 2.6 and Corollary 2.7.

Theorem 2.10. *A simple Novikov algebra over an algebraically closed field of characteristic zero is a field.*

We have a generalization of Proposition 2.6 where the field $\mathbb{K}$ is not necessarily algebraically closed.

Proposition 2.11. *Let A be a Novikov algebra over a field of characteristic zero then A is decomposable into the direct sum of ideals $A = \bigoplus_i A_i$, where each summand is either right-nilpotent or $A_i/N(A_i)$ is a field.*

Whenever a power by an element occurs, the multiplication is supposed to be taken from the right. That means $x^l := (R^A(x))^{l-1}(x)$.

We start our study of right-nilpotency with the following example.

Example 2.12. Consider the two-dimensional Novikov algebra with basis (e_1, e_2) and multiplication table $\langle e_2 \cdot e_1 = -e_1\rangle$.[3] This algebra is right-nilpotent but not nilpotent.

[3] This algebra corresponds to the algebra $B_2(0)$ in the subsection 4.1.1.

Zel'manov established the following result for Novikov algebras over an algebraically closed field of characteristic zero ([67]).

Proposition 2.13. *Let A be a Novikov algebra. If A is right-nilpotent then its square A^2 is nilpotent.*

For arbitrary characteristic A. S. Dzhumadil'daev and K. M. Tulenbaev ([23]) showed the following:

Theorem 2.14. *Let A be a Novikov algebra over a field of characteristic p such that $x^n = 0$ for all $x \in A$. If $p = 0$ for $p > n$ then A^2 is nilpotent with index of nilpotency not more than n.*

But even more is true as V. T. Filippov showed in [28]:

Theorem 2.15. *Let A be a pre-Lie algebra over a field of characteristic zero, in which $(L^A(x))^l = 0$ for some $l \in \mathbb{N}$. Then A is left-nilpotent.*

Remark 2.16. We note that the converse statement of the above Theorem for right-nilpotency can easily be verified. Indeed, because the right-multiplication operators commute any power of a nilpotent operator is again nilpotent.

2.3 Ideals and series of pre-Lie and Novikov algebras

The term ideal means two-sided ideal all over this section.

2.3.1 Ideals of pre-Lie and Novikov algebras defined via the associated Lie algebra

Because to any pre-Lie algebra we can associate a Lie algebra, there are some definitions in Lie algebra theory that carry over to the pre-Lie case. The following terms for pre-Lie algebras are motivated by [13].

Definition 2.17. Let A be a pre-Lie algebra. Denote by

$$A^{(0)} := A \quad A^{(l)} := \mathfrak{g}_A^{(l)} = [A^{(l-1)}, A^{(l-1)}]$$

the terms of the derived series of A. Furthermore let

$$\gamma_1(A) := \gamma_1(\mathfrak{g}_A) = A \quad \gamma_l(A) := \gamma_l(\mathfrak{g}_A) = [A, \gamma_{l-1}(A)]$$

denote the terms of the lower central series of A and we define by

$$A_{(0)} := 0 \quad Z(\mathfrak{g}_{A/A_{(l-1)}}) = A_{(l)}/A_{(l-1)}$$

the terms of the upper central series.

Lemma 2.18. *Let A be a Novikov algebra then the subspaces $A^{(l)}$, $\gamma_l(A)$, and $A_{(l)}$ are ideals of A for all $l \in \mathbb{N}$.*

Furthermore the following is true ([13]):

Lemma 2.19. *Let A be a Novikov algebra. Then we have:*

$$\gamma_{i+1}(A) \cdot \gamma_{j+1}(A) \subset \gamma_{i+j+1}(A)$$

for all $i, j \geqslant 0$.

We proceed with a couple of structural properties of Novikov algebras which all can be found with proofs in [13]. We start with two Jacobi-like identities.

Proposition 2.20. *Let $(A, \cdot)$ be a Novikov algebra. Then we have, for all $x, y, z \in A$:*

$$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0,$$

$$x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] = 0.$$

Using these two identities the following statements can be proved.

Lemma 2.21. *Let I and J be two ideals in a Novikov algebra A. Then $I \cdot J$ and $[I, J]$ are again two-sided ideals in A.*

Lemma 2.22. *Let $(A, \cdot)$ be a Novikov algebra, then $Z(\mathfrak{g}_A) \cdot [A, A] = [A, A] \cdot Z(\mathfrak{g}_A) = 0$.*

Corollary 2.23. *Let A be a Novikov algebra, then $Z(\mathfrak{g}_A)$ is an ideal of A.*

Unfortunately, the notations $A^{(l)}$, $\gamma_l(A)$ and $A_{(l)}$ don't give any further information about a possible degeneration of some pre-Lie algebras A and B, as we will see later. In order, to get a more substantial assertion about the structure of a pre-Lie algebra, one has to define the above series using the multiplication of the pre-Lie algebra itself.

2.3.2 Ideals of pre-Lie and Novikov algebras defined via the algebra product itself

Definition 2.24. Let A be a pre-Lie algebra. We make the following definitions:

$$\delta^{(l)}(A) := \delta^{(l-1)}(A) \cdot \delta^{(l-1)}(A), \text{ with } \delta^{(0)}(A) := A,$$

$$\delta^{l}(A) := A \cdot \delta^{l-1}(A), \text{ with } \delta^{0}(A) := A,$$

$$\delta_{l}(A) := \delta_{l-1}(A) \cdot A, \text{ with } \delta_{0}(A) := A.$$

Remark 2.25. Let A be a pre-Lie algebra then the subspaces $\delta_l(A)$ define ideals of A for all $l \in \mathbb{N}$ (see [23, p. 885]).

For Novikov algebras we can establish the following results:

Lemma 2.26. *Let A be a Novikov algebra then the subspaces $\delta_l(A)$, $\delta^l(A)$, and $\delta^{(l)}(A)$ are ideals of A for all $l \in \mathbb{N}$.*

Proof. The statement of remark 2.25 is in particular true for Novikov algebras. We prove that $\delta^l(A)$ is a right ideal of A by induction over l. Clearly A^2 is an ideal of A so the case $l = 1$ is done. Now suppose that the hypothesis is true for $l - 1$. By right-commutativity we have:

$$\delta^l(A) \cdot A = (A \cdot \delta^{l-1}(A)) \cdot A = (A \cdot A) \cdot \delta^{l-1}(A) \subset A \cdot \delta^{l-1}(A) = \delta^l(A).$$

Evidently, $\delta^l(A)$ is a left ideal and therefore an ideal of A.
To show that $\delta^{(l)}(A)$ is an ideal we apply induction on l again. We know from the case $l=1$ above that $A^2=\delta^{(1)}(A)$ is an ideal of A. We now suppose that $\delta^{(l-1)}(A)$ is an ideal of A and deduce by the law of left-symmetry that $\delta^{(l)}(A)$ is a left ideal of A:

$$\begin{aligned}
&A\cdot\delta^{(l)}(A)=A\cdot(\delta^{(l-1)}(A)\cdot\delta^{(l-1)}(A))\\
&=(A\cdot\delta^{(l-1)}(A))\cdot\delta^{(l-1)}(A)+\delta^{(l-1)}(A)\cdot(A\cdot\delta^{(l-1)}(A))-(\delta^{(l-1)}(A)\cdot A)\cdot\delta^{(l-1)}(A)\\
&\subset\delta^{(l-1)}(A)\cdot\delta^{(l-1)}(A)=\delta^{(l)}(A).
\end{aligned}$$

By right-commutativity we see that $\delta^{(l)}(A)$ is also a right ideal of A:

$$\begin{aligned}
\delta^{(l)}(A)\cdot A&=(\delta^{(l-1)}(A)\cdot\delta^{(l-1)}(A))\cdot A=\\
&=(\delta^{(l-1)}(A)\cdot A)\cdot\delta^{(l-1)}(A)\subset\delta^{(l-1)}(A)\cdot\delta^{(l-1)}(A)=\delta^{(l)}(A).
\end{aligned}$$

Hence, $\delta^{(l)}(A)$ is an ideal of A. □

Definition 2.27. We define the center of a pre-Lie algebra as

$$Z(A):=\{x\in A\mid x\cdot y=y\cdot x=0\,\forall y\in A\}.$$

Because of the last Lemma we are now able to give the following definition.

Definition 2.28. Let A be a Novikov algebra. We call the chain of ideals

$$\delta^{(0)}(A)\supset\delta^{(1)}(A)\supset\cdots\supset\delta^{(l)}(A)\supset\cdots$$

the derived series of A. Similar to the Lie case we call the following two chains of ideals

$$\delta^{0}(A)\supset\delta^{1}(A)\supset\cdots\supset\delta^{l}(A)\supset\cdots$$

$$\delta_{0}(A)\supset\delta_{1}(A)\supset\cdots\supset\delta_{l}(A)\supset\cdots$$

the lower left- and right-central series, respectively. Finally we define $\delta_{(l)}(A)$ implicitly by $Z(A/\delta_{(l-1)}(A))=\delta_{(l)}(A)/\delta_{(l-1)}(A)$, with $\delta_{(0)}(A):=0$ and denote by

$$\delta_{(0)}(A)\subset\delta_{(1)}(A)\subset\cdots\subset\delta_{(l)}(A)\subset\cdots$$

the upper central series of A.

Clearly $\delta_{(1)}(A)=Z(A)$, which is an abelian ideal and contained in the center of the associated Lie algebra.

Remark 2.29. Let A be a finite dimensional pre-Lie algebra. We call A solvable, if $\delta^{(l)}(A)=0$ for some $l\in\mathbb{N}$. We see that A is left- resp. right-nilpotent, if $\delta_l(A)=0$ resp. $\delta^l(A)=0$ with $l\in\mathbb{N}$.

2.4 The radical of a pre-Lie algebra

Different from Lie algebra theory there is possibly more than one way of defining the radical of a pre-Lie algebra. This observation is based on the fact that for pre-Lie algebras left and right ideals need not to coincide. Usually the radical should be a 2-sided ideal in the algebra. Therefore we start with a definition, that is motivated by Lie algebra theory.

Definition 2.30. Let A be a finite dimensional pre-Lie algebra. We denote by $\operatorname{sol}(A)$ the maximal solvable ideal in A and call it the solvable radical. Furthermore let $\operatorname{nil}(A)$ be the maximal left-nilpotent ideal in A. We call it the left-nilpotent radical of A.

Remark 2.31. The ideals $\operatorname{sol}(A)$ and $\operatorname{nil}(A)$ are unique, since the sum of two solvable (left-nilpotent) ideals in A are again solvable (left-nilpotent). We have $\operatorname{nil}(A) \subset \operatorname{sol}(A)$. For a proof of these statements we refer to [11] and the citations given therein.

In addition let us consider the symmetric bilinear form s on A definded by

$$s(x,y) = \operatorname{tr} \mathrm{R}^A(x)\mathrm{R}^A(y).$$

Another way of associating a radical to the pre-Lie algebra A is by the kernel of the form s:

$$A^{\perp} := \ker s = \{a \in A \mid s(a,b) = 0 \ \forall b \in A\}.$$

Finally we give a somewhat different but perhaps more comprehensive definition of a radical. Before we can do so, we need the concept of completeness.

Definition 2.32. A pre-Lie algebra A is called complete if for every $x \in A$, the linear transformation $\mathrm{Id}_A + \mathrm{R}^A(x) : A \to A$ is bijective.

Due to the work of D. Segal ([60]) we have the following result.

Theorem 2.33. *Let A be a finite-dimensional pre-Lie algebra over a field $\mathbb{K}$ of characteristic zero. Then the following conditions are equivalent:*

1. *A is complete, i. e. $\mathrm{Id}_A + \mathrm{R}^A(x)$ is bijective for all $x \in A$.*
2. *The linear transformation $\mathrm{R}^A(x)$ has no eigenvalue in $\mathbb{K} \setminus \{0\}$ for all $x \in A$.*
3. *A is right nil, meaning $\mathrm{R}^A(x)$ is nilpotent for all $x \in A$.*
4. *$\operatorname{tr}(\mathrm{R}^A(x)) = 0$ for all $x \in A$.*

We remark, that point four of the last Theorem is in practice probably the easiest way of deciding whether an algebra is complete or not. Now we are able to give a definition of a radical, which is due to Koszul, see [35].

Definition 2.34. Let A be a pre-Lie algebra and $T(A) = \{x \in A \mid \operatorname{tr} \mathrm{R}^A(x) = 0\}$. The largest left ideal of A contained in $T(A)$ is called the radical of A and is denoted by $\operatorname{rad}(A)$.

It was remarked by Helmstetter ([35]) that $\mathrm{rad}(A)$ need not to be a 2-sided ideal in general. He constructed a pre-Lie algebra $B = \mathrm{End}(A) \oplus A$ with an arbitrary pre-Lie algebra A by defining a product

$$(f, a).(g, b) = (fg + [L(a), g], a \cdot b + f(b) + g(a))$$

for $a, b \in A$ and $f, g \in \mathrm{End}(A)$. If A is not complete then $\mathrm{rad}(B) = 0$. If A is complete and not abelian then $\mathrm{rad}(B)$ is not a 2-sided ideal in A.

The reason why we introduced all these different definitions is because they all define semi-invariants with respect to degeneration. Moreover we have the following relation ([19]).

Theorem 2.35. *Let A be a finite-dimensional pre-Lie algebra over $\mathbb{C}$. Then we have*

$$\mathrm{nil}(A) \subset \mathrm{rad}(A) \subset A^{\perp} \subset T(A).$$

2.5 The left-multiplication operator $\mathrm{L}^A(x)$

The map $\mathrm{L}^A(x) : A \to A$ is linear and can therefore be written as a matrix by choosing a fixed basis. The map $L : A \to \mathrm{End}_{\mathbb{K}}(A)$ that associates to each element $x \in A$ its left multiplication is an algebra representation of A. If not mentioned otherwise we will always regard $\mathrm{L}^A(x)$ as a matrix in $M_n(\mathbb{K})$. Furthermore this representation, and hence the matrix $\mathrm{L}^A(x)$ determines the algebra structure of A completely. Indeed, let $(e_1, \ldots, e_n)$ be a basis of the algebra A. An element $x \in A$ can therefore be written by $x = \sum_k x_k e_k$. We want to see how the structure constants refer to the matrix $\mathrm{L}^A(x)$ and therefore compute $\mathrm{L}^A(e_i)$:

$$\mathrm{L}^A(e_i) = \begin{pmatrix} c_{i1}^1 & \cdots & c_{in}^1 \\ \vdots & \ddots & \vdots \\ c_{i1}^n & \cdots & c_{in}^n \end{pmatrix}.$$

We have $\mathrm{L}^A(x) = \sum_{i=1}^n x_i \mathrm{L}^A(e_i)$ and so $\mathrm{L}^A(x)$ takes the form:

$$\mathrm{L}^A(x) = \begin{pmatrix} \sum_{i=1}^n x_i c_{i1}^1 & \cdots & \sum_{i=1}^n x_i c_{in}^1 \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n x_i c_{i1}^n & \cdots & \sum_{i=1}^n x_i c_{in}^n \end{pmatrix}.$$

In what follows we identify some useful structural properties of the algebra A within the matrix $\mathrm{L}^A(x)$. The reason for this is, that in most cases it is easier to work with $\mathrm{L}^A(x)$ rather than with methods using structure constants.

Definition 2.36. Denote by $M_{s,t}(\mathbb{K})$ the ring of matrices with s rows and t columns. Let the projection maps $p_r, q_r : M_{s,t}(\mathbb{K}) \to M_{s-r,t}(\mathbb{K})$ be defined by

$$p_r(m_{ij})_{1 \leqslant i \leqslant s} = (m_{ij})_{1 \leqslant i \leqslant s-r}$$

$$q_r(m_{ij})_{1 \leqslant i \leqslant s} = (m_{ij})_{r+1 \leqslant i \leqslant s}$$

with $1 \leqslant j \leqslant t$. We can also say that the map q_r cuts off the first r rows from a matrix $M = (m_{ij})$, while p_r cuts off the last r rows from a matrix $M = (m_{ij})$.

Now, consider a d-dimensional vector subspace W of the algebra A generated by the first d basis vectors. The subspace $\overline{W}$ formed by the last $n-d$ basis vectors completes W to yield $A = W \oplus \overline{W}$ as a vector space. An element in A shall now be given by $x \oplus \overline{x}$, where $x \in W$ and $\overline{x} \in \overline{W}$. Regarding this decomposition we can write $\mathrm{L}^A(x)$ in terms of block matrices. We make the following definitions:

$$\begin{aligned} R &:= p_{n-d}(\mathrm{L}^A(x \oplus \overline{x})|_W) \in M_{d,d}(\mathbb{K}) \\ S &:= p_{n-d}(\mathrm{L}^A(x \oplus \overline{x})|_{\overline{W}}) \in M_{d,n-d}(\mathbb{K}) \\ T &:= q_d(\mathrm{L}^A(x \oplus \overline{x})|_W) \in M_{n-d,d}(\mathbb{K}) \\ U &:= q_d(\mathrm{L}^A(x \oplus \overline{x})|_{\overline{W}}) \in M_{n-d,n-d}(\mathbb{K}) \end{aligned} \tag{2.1}$$

Proposition 2.37. *With the considerations made above the left multiplication of an element $x \oplus \overline{x}$ in A takes the form:*

$$\mathrm{L}^A(x \oplus \overline{x}) = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$$

Proof. The proof will be examined for R, the other cases are similar. The block matrix of $\mathrm{L}^A(x \oplus \overline{x})$ formed by the first d rows and columns is defined by the value of the basis vectors e_i for $1 \leqslant i \leqslant d$. Applying $\mathrm{L}^A(x \oplus \overline{x})$ to e_i gives the i-th column of $\mathrm{L}^A(x \oplus \overline{x})$. The projection map p_{n-d} takes the first d components of this vector. □

We are now able to proceed with the following Lemma.

Lemma 2.38. *Let A be an algebra of dimension n. Let $(e_1, \dots, e_n)$ be a basis of A and $W := \langle e_1, \dots, e_d \rangle_{\mathbb{K}}$. Consider the decomposition $\mathrm{L}^A(x \oplus \overline{x}) = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right)$ from Proposition 2.37. The following statements are equivalent:*

1. *The subspace W generates a subalgebra of A.*
2. *The structure constants c_{ij}^k are zero for $1 \leqslant i, j \leqslant d < k \leqslant n$.*
3. *The block T is independent of x.*

Proof. (2) $\Leftrightarrow$ (3): Regarding the basis $(e_1, \dots, e_n)$ we can write an element of A by $x \oplus \overline{x} = \sum_{i=1}^d x_i e_i \oplus \sum_{i=d+1}^n \overline{x}_i e_i$. The direct sum used for elements shall indicate that x and $\overline{x}$ refer to two different subspaces, namely W and $\overline{W}$. With this notation we evaluate $\mathrm{L}^A(x \oplus \overline{x})$ at e_j:

$$\begin{aligned} \mathrm{L}^A(x \oplus \overline{x})(e_l) &= \sum_{i=1}^d x_i e_i \cdot e_l \oplus \sum_{i=d+1}^n \overline{x}_i e_i \cdot e_l \\ &= \begin{cases} \sum_{i,k=1}^d x_i c_{il}^k e_k \oplus \sum_{i=d+1}^n \sum_{k=1}^n \overline{x}_i c_{il}^k e_k & \text{for } 1 \leqslant l \leqslant d, \\ \sum_{i=1}^d \sum_{k=1}^n x_i c_{il}^k e_k \oplus \sum_{i=d+1}^n \sum_{k=1}^n \overline{x}_i c_{il}^k e_k & \text{for } d+1 \leqslant l \leqslant n. \end{cases} \end{aligned} \tag{2.2}$$

From this we can immediately deduce that $c_{ij}^k = 0$ for $1 \leqslant i, j \leqslant d < k \leqslant n$ if and only if $q_d(\mathrm{L}^A(x \oplus \overline{x})|_W)$ is independent of x.
(1) $\Leftrightarrow$ (2): A subspace W is a subalgebra if and only if for any two basis vectors $e_i, e_j \in W$ the product $e_i \cdot e_j$ lies again in W. By definition of the structure constants this is the case if and only if $c_{ij}^k = 0$ for $1 \leqslant i, j \leqslant d < k \leqslant n$. □

Lemma 2.39. *Let A be an algebra of dimension n. Let $(e_1, \dots, e_n)$ be a basis of A and $I := \langle e_1, \dots, e_d \rangle_{\mathbb{K}}$. Consider the decomposition $\mathrm{L}^A(x \oplus \overline{x}) = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right)$ from Proposition 2.37. The following statements are equivalent:*

1. *The subspace I generates an ideal of A.*
2. *The structure constants c_{ij}^k are zero for $1 \leqslant i \leqslant d < k \leqslant n$ and $1 \leqslant j \leqslant d < k \leqslant n$.*
3. *The block U is independent of x and $T = 0$.*

Proof. (1) $\Leftrightarrow$ (2): In addition to a subalgebra a subspace I of A is a two-sided ideal if and only if for any $e_i \in I$ and $e_j \in A$ the products $e_i \cdot e_j$ and $e_j \cdot e_i$ lie again in I. In terms of structure constants this condition can be written by $c_{ij}^k = 0$ for $1 \leqslant i \leqslant d < k \leqslant n$ and $1 \leqslant j \leqslant d < k \leqslant n$.
(2) $\Leftrightarrow$ (3): In view of equation (2.2) we have

$$\mathrm{L}^A(x \oplus \overline{x})(e_l) = \begin{cases} \sum_{i,k=1}^{d} x_i c_{il}^k e_k \oplus \sum_{i=d+1}^{n} \sum_{k=1}^{d} \overline{x}_i c_{il}^k e_k & \text{for } 1 \leqslant l \leqslant d, \\ \sum_{i=1}^{d} \sum_{k=1}^{n} x_i c_{il}^k e_k \oplus \sum_{i=d+1}^{n} \sum_{k=1}^{d} \overline{x}_i c_{il}^k e_k & \text{for } d+1 \leqslant l \leqslant n. \end{cases} \tag{2.3}$$

We see that in the above equation for $l \leqslant d$ no basis vector e_k with k higher than d appears. This is equivalent to I is a left ideal or $T = 0$. Conversely, if $T = q_d(\mathrm{L}^A(x \oplus \overline{x})|_W) = 0$ then c_{il}^k has to be zero for $l \leqslant d$ and $d + 1 \leqslant k$. Similar, like in the case for a subalgebra I is a right ideal if and only if U is independent of x. □

As a direct consequence of the proofs of Lemma 2.38 and Lemma 2.39 we note:

Corollary 2.40. *Let A be an algebra of dimension n. Let $(e_1, \dots, e_n)$ be a basis of A and $I := \langle e_1, \dots, e_d \rangle_{\mathbb{K}}$. Consider the decomposition $\mathrm{L}^A(x \oplus \overline{x}) = \left(\begin{smallmatrix} R & S \\ T & U \end{smallmatrix}\right)$ from Proposition 2.37. We have the following statements:*

1. *The subspace I generates a left ideal of A if and only if $T = 0$.*
2. *The subspace I generates a right ideal of A if and only if U is independent of x.*

3 Semi-invariants of degenerations

We refer for this chapter to the various definitions made in chapter 2. The ground field $\mathbb{K}$ is always supposed to be $\mathbb{C}$, except otherwise noted.

In this chapter we present methods to gain information about whether a given pre-Lie algebra degenerates to another one or not. Like this was done in the Lie algebra case (see for example [46], [18] and [32]), one tries to associate certain quantities to an algebra, that transfer algebraic properties. If these carry over within the degeneration process, we can restrict the possible degenerations by filtering out those structures which share the same algebraic properties. Such quantities will be called invariants. Unfortunately, most of the quantities, for which there can be made an assertion under degeneration, we find that they perform inequality rather than equality. Nevertheless they play an important role in the theory of degenerations and are called semi-invariants. The aim of this chapter is to generalize the existing invariants and semi-invariant for Lie algebras to pre-Lie algebras if this is possible. Furthermore we add new results and demonstrate the close relationship between Lie algebra and pre-Lie algebra degeneration.

3.1 Getting new degenerations from old one

Definition 3.1. Let $\{A_i\}$, $i \in I$ be a family of pre-Lie algebras. We take the direct sum of vector spaces:

$$A = \bigoplus_{i \in I} A_i.$$

Let $x, y \in A$ where $x = (x_i)$ and $y = (y_i)$. We define $x \cdot y := (x_i \cdot y_i)$. In this way A becomes a pre-Lie algebra, called the direct sum of the pre-Lie algebras A_i.

Lemma 3.2. *Let A and B be two n-dimensional pre-Lie algebras with $A \to_{\deg} B$. Furthermore let C and D be two d-dimensional pre-Lie algebras with $C \to_{\deg} D$, then*

$$A \oplus C \to_{\deg} B \oplus D.$$

Proof. By definition of the direct sum the left multiplication by an element $x \oplus \overline{x}$ in $A \oplus C$ can be written as follows:

$$\mathrm{L}^{A\oplus C}(x \oplus \overline{x})(y \oplus \overline{y}) = (x \oplus \overline{x}) \cdot (y \oplus \overline{y}) = x \cdot y \oplus \overline{x} \cdot \overline{y} = \mathrm{L}^A(x)(y) \oplus \mathrm{L}^C(\overline{x})(\overline{y}).$$

If we write this left multiplication in form of a matrix, we get

$$\mathrm{L}^{A\oplus C}(x \oplus \overline{x}) = \mathrm{L}^A(x) \oplus \mathrm{L}^C(\overline{x}) = \begin{pmatrix} \mathrm{L}^A(x) & 0 \\ 0 & \mathrm{L}^C(\overline{x}) \end{pmatrix}$$

Now, by assumption there exist matrices g_ε and h_ε with $\mathrm{L}^B(x) = \lim_{\varepsilon\to 0} g_\varepsilon \cdot \mathrm{L}^A(x)$ and $\mathrm{L}^D(\overline{x}) = \lim_{\varepsilon\to 0} h_\varepsilon \cdot \mathrm{L}^C(\overline{x})$. We define $G_\varepsilon := \begin{pmatrix} g_\varepsilon & 0 \\ 0 & h_\varepsilon \end{pmatrix}$. Then it follows that the inverse of the matrix G_ε is formed by the inverse of the matrices g_ε and h_ε. Therefore we have[1]

$$\begin{aligned}\lim_{\varepsilon\to 0} G_\varepsilon \cdot (\mathrm{L}^A(x) \oplus \mathrm{L}^C(\overline{x})) &= \lim_{\varepsilon\to 0} \begin{pmatrix} g_\varepsilon & 0 \\ 0 & h_\varepsilon \end{pmatrix} \begin{pmatrix} \mathrm{L}^A(g_\varepsilon^{-1}(x)) & 0 \\ 0 & \mathrm{L}^C(h_\varepsilon^{-1}(\overline{x})) \end{pmatrix} \begin{pmatrix} g_\varepsilon^{-1} & 0 \\ 0 & h_\varepsilon^{-1} \end{pmatrix} \\ &= \lim_{\varepsilon\to 0} \begin{pmatrix} g_\varepsilon \mathrm{L}^A(g_\varepsilon^{-1}(x)) g_\varepsilon^{-1} & 0 \\ 0 & h_\varepsilon \mathrm{L}^C(h_\varepsilon^{-1}(\overline{x})) h_\varepsilon^{-1} \end{pmatrix} \\ &= \lim_{\varepsilon\to 0} \begin{pmatrix} g_\varepsilon \cdot \mathrm{L}^A(x) & 0 \\ 0 & h_\varepsilon \cdot \mathrm{L}^C(\overline{x}) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{L}^B(x) & 0 \\ 0 & \mathrm{L}^D(\overline{x}) \end{pmatrix} = \mathrm{L}^B(x) \oplus \mathrm{L}^D(\overline{x})\end{aligned}$$

In conclusion we have proved that $A \oplus C \to_{\deg} B \oplus D$ with degeneration matrix G_ε. □

Corollary 3.3. *Let A and B be two n-dimensional pre-Lie algebras with $A \to_{\deg} B$. For an arbitrary d-dimensional pre-Lie algebra C the direct sum $A \oplus C$ degenerates to $B \oplus C$. In particular we have $A \oplus \mathbb{C}^d \to_{\deg} B \oplus \mathbb{C}^d$ where $A \oplus \mathbb{C}^d$ is the $(n+d)$-dimensional pre-Lie algebra formed by adding a d-dimensional abelian component.*

Remark 3.4. This corollary enables us to shift a degeneration diagram from an arbitrary dimension to any higher dimension. We will explicitely use this technique in chapter 4 of this work by classifying all degenerations of 3-dimensional Novikov algebras with associated abelian Lie algebra.

The following Lemma was established for complex Leibniz algebras in [1].

Lemma 3.5. *Let A be a pre-Lie algebra of dimension n that contains an element, which does not generate a one-dimensional subalgebra of A. Then $A \to_{\deg} B \oplus \mathbb{C}^{n-2}$, where B is a two-dimensional non-abelian nilpotent pre-Lie algebra.*

Proof. Let $x \in A$ be nonzero. Because x doesn't generate a one-dimensional subalgebra of A, we have $x \cdot x = y$ with x and y linearily independent. Therefore x and y can be included in a basis of A: $e_1 = x$, $e_2 = y$, $e_3 \dots e_n$. By setting $g_t(e_1) = t^{-1}e_1$ and $g_t(e_i) = t^{-2}e_i$ $(2 \leqslant i \leqslant n)$ for $g_t \in \mathrm{GL}_n(\mathbb{C}(t))$, we obtain a degeneration $A \to_{\deg} B \oplus \mathbb{C}^{n-2}$, where by subsection 4.1.1[2], B is a non-abelian, nilpotent pre-Lie algebra that is unique up to isomorphism. □

3.1.1 Degenerations of quotients

Proposition 3.6. *Let A and B be two n-dimensional algebras over the field $\mathbb{C}$. If $A \to_{\deg} B$ and I is a left respectively right ideal of A then there exists a left respectively right ideal J of B, which is as a subalgebra a degeneration of I. As a consequence any two-sided ideal in A degenerates to a two-sided ideal in B. In particular every subalgebra of A degenerates to a subalgebra of B.*

Proof. We perform a change of bases so that the ideal I is generated by the first $d := \dim_{\mathbb{C}} I$ basis vectors. We denote this new algebra by $\widetilde{A}$ and again $I := \langle e_1, \dots, e_d \rangle \subset \widetilde{A}$. By Theorem 1.17 every degeneration can be accomplished by

[1] We remark that a central dot behind a matrix denotes the action of basis change by this matrix. If there is no dot we have the ordinary matrix multiplication.

[2] The algebra B is isomorphic to the algebra U_5 in the notation of subsection 4.1.1.

the orbit closure with an upper triangular matrix B_ε followed by an appropriate isomorphism. Because every upper triangular matrix stabalizes a subspace of the form $\langle e_1, \dots, e_d \rangle$ for $1 \leqslant d \leqslant n$, we have:

$$I \cdot_\varepsilon \widetilde{A} := B_\varepsilon(B_\varepsilon^{-1}(I) \cdot B_\varepsilon^{-1}(\widetilde{A})) \subset B_\varepsilon(I \cdot \widetilde{A}) \subset B_\varepsilon(I) \subset I \text{ and}$$
$$\widetilde{A} \cdot_\varepsilon I := B_\varepsilon(B_\varepsilon^{-1}(\widetilde{A}) \cdot B_\varepsilon^{-1}(I)) \subset B_\varepsilon(\widetilde{A} \cdot I) \subset B_\varepsilon(I) \subset I,$$

for a left respectively right ideal I of $\widetilde{A}$. What we get is a sequence of isomorphic ideals, formed by the first $\dim I$ basis vectors, that converges as the parameter ε approaches zero:

$$I \cdot_0 \widetilde{A} := \lim_{\varepsilon \to 0} I \cdot_\varepsilon \widetilde{A} \subset I \text{ and}$$
$$\widetilde{A} \cdot_0 I := \lim_{\varepsilon \to 0} \widetilde{A} \cdot_\varepsilon I \subset I.$$

The same argument can be applied for the degeneration of subalgebras. □

Remark 3.7. Every ideal is an algebra and therefore we can find for $d := \dim I$ a matrix $G_\varepsilon \in \mathrm{GL}_d(\mathbb{C})$ that degenerates an ideal of A to one in B. In fact, we will show by the next Theorem that this degeneration of ideals is so to say "embedded" in the corresponding degeneration of algebras.

The following theorem has already been formulated and proved for Lie algebras by Roman Popovych. A draft of the paper in which this theorem appears reached the author via private communications.

Theorem 3.8. *Let A and B be n-dimesional algebras defined over the field $\mathbb{C}$ and suppose that $A \to_{\deg} B$. Let*

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_m \subset A$$

be a nested sequence of left respectively right ideals of A then there exists a nested sequence

$$0 \subset J_0 \subset J_1 \subset \cdots \subset J_m \subset B$$

of right respectively left ideals of B such that for $1 \leqslant i \leqslant m$ there exists a degeneration $I_i \to_{\deg} J_i$. If moreover all ideals are supposed to be two-sided then there exists for all i and j with $1 \leqslant i \leqslant j \leqslant m$ a degeneration $I_j/I_i \to_{\deg} J_j/J_i$.

Proof. The first statement concerning the degeneration of a nested sequence of right or left ideals follows from Proposition 3.6 via induction over $m \in \mathbb{N}$.

The second statement concering the degeneration of factors is a little bit more complicated. Nevertheless we can again use induction for which it suffices to prove:

$$A/I \to_{\deg} B/J,$$

if $A \to_{\deg} B$ and I is an arbitrary two-sided ideal of A. The idea of our proof is the following. From Proposition 3.6 we know that there exists an ideal J in B such that $I \to_{\deg} J$. Therefore the factor B/J also exists. Moreover for any to A isomorphic algebra $G_\varepsilon \cdot A$, G_ε being the degeneration matrix, we have ideals I_ε which are isomorphic to I and hence $A/I \cong G_\varepsilon \cdot (A/I_\varepsilon)$. The result is a sequence of factors isomorphic to A/I that converges to B/J. What we have to do now is to find a degeneration matrix that corresponds to this sequence.

So fix a basis $(e_1,\dots,e_n)$ and let (c_{ij}^k) be a point in $\mathrm{Alg}_n(\mathbb{C})$. Consider the projection map $\pi : \mathrm{Alg}_n(\mathbb{C}) \to \mathrm{Alg}_{n-d}(\mathbb{C})$ defined by $\pi(c_{ij}^k) = (c_{ij}^k)_{d+1\leqslant i,j,k\leqslant n}$. This map describes formally the process of factoring out the first d basis vectors of an arbitrary $\mathbb{C}$-algebra and is a morphism of affine algebraic varieties. Of course, the resulting vector of structure constants $(c_{ij}^k)_{d+1\leqslant i,j,k\leqslant n}$ defines a $\mathbb{C}$-algebra if and only if the first d basis vectors form an ideal of the algebra. That's why we assume, without loss of generality, that the ideal I is generated by $(e_1,\dots,e_d)$. Let $\overline{I} := \langle e_{d+1},\dots,e_n\rangle_{\mathbb{C}}$, which need not to be an ideal of A. As a vector space we now can decompose A by $A = I \oplus \overline{I}$. In view of Lemma 2.39 we can describe the fact, that the first d basis vectors generate an ideal, in terms of the left multiplication by an element $x \oplus \overline{x} \in A$, $x \in I$ and $\overline{x} \in \overline{I}$, by

$$\mathrm{L}^A(x \oplus \overline{x}) = \begin{pmatrix} R & S \\ 0 & U \end{pmatrix}$$

where $R := p_{n-d}(\mathrm{L}^A(x \oplus \overline{x})|_I) \in M_{d,d}(\mathbb{C})$, $S := p_{n-d}(\mathrm{L}^A(x \oplus \overline{x})|_{\overline{I}}) \in M_{d,n-d}(\mathbb{C})$, and $U := q_d(\mathrm{L}^A(x \oplus \overline{x})|_{\overline{I}}) \in M_{n-d,n-d}(\mathbb{C})$.[3] Notice that the $(n-d)\times(n-d)$-block matrix U does not depend on x.

Now, we rewrite the projection map π in terms of $\mathrm{L}^A(z)$ for $z \in A$. Therefore we define the projections $\psi : A \to \overline{I}$ by $\psi(z) = \sum_{k=d+1}^n z_k e_k$ and $\varphi : M_{n,n}(\mathbb{C}) \to M_{n-d,n-d}(\mathbb{C})$ by $\varphi : (a_{ij})_{1\leqslant i,j\leqslant n} \mapsto (a_{ij})_{d+1\leqslant i,j\leqslant n}$. Hence, $\pi(\mathrm{L}^A(x\oplus\overline{x})) := \varphi(\mathrm{L}^A(\psi(x\oplus\overline{x})))$ corresponds to the projection map π on the space of structure constants.

To prove the theorem we have to show that there exists a family of matrices $\{g_\varepsilon\}$ such that

$$\pi(\mathrm{L}^B(x)) = \lim_{\varepsilon\to 0} g_\varepsilon \cdot \pi(\mathrm{L}^A(x)).$$

By Theorem 1.17 we can assume that the degeneration matrix for $A \to_{\deg} B$ has the form $G_\varepsilon = \left(\begin{smallmatrix} C & D \\ 0 & E \end{smallmatrix}\right)$, with $C \in \mathrm{GL}_d(\mathbb{C})$. Note that the blocks C, D, and E also depend on the degeneration parameter ε. Setting $\left(\begin{smallmatrix} y \\ \overline{y} \end{smallmatrix}\right) := \left(\begin{smallmatrix} C & D \\ 0 & E \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} x \\ \overline{x} \end{smallmatrix}\right)$, and

$$\overline{D} := -C^{-1}DE^{-1}$$

we calculate the basis change of $\mathrm{L}^A(x\oplus\overline{x})$ by G_ε:

$$\begin{aligned} G_\varepsilon \cdot \mathrm{L}^A(x\oplus\overline{x}) &= G_\varepsilon \mathrm{L}^A(y\oplus\overline{y})G_\varepsilon^{-1} \\ &= \left(\begin{smallmatrix} C\overline{R}C^{-1} & C(\overline{RD}+\overline{S}E^{-1})+D\overline{U}E^{-1}) \\ 0 & E\overline{U}E^{-1} \end{smallmatrix}\right). \end{aligned}$$

where $\overline{R} := p_{n-d}(\mathrm{L}^A(y\oplus\overline{y})|_I)$, $\overline{S} := p_{n-d}(\mathrm{L}^A(y\oplus\overline{y})|_{\overline{I}})$, and $\overline{U} := q_d(\mathrm{L}^A(y\oplus\overline{y})|_{\overline{I}})$. As long as $\varepsilon > 0$ the action of G_ε on $\mathrm{Alg}_n(\mathbb{C})$ is an isomorphism. Therefore, as can be seen by the above equation, the matrix G_ε leaves the subspace I invariant. Furthermore the block $E\overline{U}E^{-1}$ does not depend on y and hence not on x. Consulting Theorem 1.17 again, we see that the subspace I is still an ideal of $G_\varepsilon \cdot A$.

Now, as $A \to_{\deg} B$, the limit of $G_\varepsilon \cdot \mathrm{L}^A(x\oplus\overline{x})$ exists and so do the limits:

$$\begin{aligned} &\lim_{\varepsilon\to 0} C\overline{R}C^{-1}, \\ &\lim_{\varepsilon\to 0} C(\overline{RD}+\overline{S}E^{-1})+D\overline{U}E^{-1}), \\ &\lim_{\varepsilon\to 0} E\overline{U}E^{-1}. \end{aligned}$$

[3] For the exact definitions of p_{n-d} and q_d we refer to section 2.5.

As a consequence

$$\lim_{\varepsilon\to 0} E\overline{U}E^{-1}$$

does also not depend on x, which is beside $q_d(\mathrm{L}^B(x\oplus\overline{x})|_J)=0$ a sufficient condition for the subspace $J:=\langle e_1,\dots,e_d\rangle_{\mathbb{C}}$ of B to generate an ideal of B. The structure of J is given by $\lim_{\varepsilon\to 0} C\overline{R}C^{-1}$. We can therefore consider $\pi(\mathrm{L}^B(x\oplus\overline{x}))$. Moreover the limit of $E\overline{U}E^{-1}$ is independent of the other blocks in the matrix representation of $\mathrm{L}^B(x\oplus\overline{x})$, which means that the projection π and lim commute. The calculation of $\pi(\mathrm{L}^B(x\oplus\overline{x}))$ is done as follows:

$$\begin{aligned}\pi(\mathrm{L}^B(x\oplus\overline{x})) &= \lim_{\epsilon\to 0}\pi(G_\epsilon\cdot\mathrm{L}^A(x\oplus\overline{x}))\\ &= \lim_{\varepsilon\to 0}\varphi(G_\epsilon\cdot\mathrm{L}^A(\overline{x}))\\ &= \lim_{\varepsilon\to 0} E\cdot q_d(\mathrm{L}^A(\overline{x})|_{\overline{I}})\\ &= \lim_{\varepsilon\to 0} E\cdot\pi(\mathrm{L}^A(x\oplus\overline{x})).\end{aligned}$$

We see that if we choose $g_\varepsilon=E$ the projection of $\mathrm{L}^A(x\oplus\overline{x})$ degenerates to the projection of $\mathrm{L}^B(x\oplus\overline{x})$. □

This Theorem has some immediate consequences for the orbit closure problem of degenerations. It relates a degeneration in a certain dimension with degenerations in lower dimension. The following Corollary demonstrates how the Theorem can be used to classify degenerations.

Corollary 3.9. *Let $A,B\in\mathrm{Alg}_n(\mathbb{C})$ and I be a d-dimensional ideal in A. If A/I doesn't degenerate to any $(n-d)$-dimensional $\mathbb{C}$-algebra or I doesn't degenerate to any d-dimensional ideal of B, then A cannot degenerate to B.*

As another consequence we get a statement that was already proved by C. Seeley for Lie algebras.

Corollary 3.10. *Let $A,B\in\mathrm{Alg}_n(\mathbb{C})$ and $A\to_{\deg} B$ then*

$$A/Z(A)\to_{\deg} B/Z(B)\oplus\mathbb{C}^d$$

where $d:=\dim Z(B)-\dim Z(A)$.

Proof. Without loss of generality we can assume that the centers of A and B are formed by the very first basis vectors. Hence, with degeneration matrix g_ε we have $J:=\lim_{\varepsilon\to 0} g_\varepsilon\cdot Z(A)\subset Z(B)$ and

$$A/Z(A)\to_{\deg} B/J.$$

The center of B/J is of dimension $\dim Z(B)-\dim Z(A)=:d$ and clearly degenerates to $\mathbb{C}^d$. □

3.1.2 Degenerations related to associated structures

In the definition of section 2.1 we associated to a given pre-Lie algebra A a Lie algebra $\mathfrak{g}_A$. The degeneration of these two structures are related in the following way.

Lemma 3.11. *If $A \to_{\deg} B$, then $\mathfrak{g}_A \to_{\deg} \mathfrak{g}_B$, for the associated Lie algebras.*

Proof. Let $(e_1, \dots, e_n)$ be a basis of the underlying vectorspace. Denote the product in A by $e_i \cdot e_j$ and that in B by $e_i * e_j$. The Lie products are then given by

$$[e_i, e_j]_A = e_i \cdot e_j - e_j \cdot e_i \quad \text{and} \quad [e_i, e_j]_B = e_i * e_j - e_j * e_i.$$

We have

$$\begin{aligned}
\lim_{\varepsilon \to 0} g_\varepsilon([g_\varepsilon^{-1}(e_i), g_\varepsilon^{-1}(e_j)]_A) &= \lim_{\varepsilon \to 0} g_\varepsilon(g_\varepsilon^{-1}(e_i) \cdot g_\varepsilon^{-1}(e_j) - g_\varepsilon^{-1}(e_j) \cdot g_\varepsilon^{-1}(e_i)) \\
&= \lim_{\varepsilon \to 0} g_\varepsilon(g_\varepsilon^{-1}(e_i) \cdot g_\varepsilon^{-1}(e_j) - \lim_{\varepsilon \to 0} g_\varepsilon^{-1}(e_j) \cdot g_\varepsilon^{-1}(e_i) \\
&= e_i * e_j - e_j * e_i \\
&= [e_i, e_j]_B
\end{aligned}$$

□

Remark 3.12. This lemma will give us a useful tool to work with in chapter 4, where we classify all possible degenerations of pre-Lie algebras with dimension two and of Novikov algebras with dimension three. Considering the Hasse diagram of orbit closures in a fixed dimension we see that pre-Lie algebra degenerations form a refinement of Lie algebra degenerations. These diagrams for Lie algebras up to dimension four have been studied well (for example [18], [46]).

We give another example which is in some sense dual to that of an associated Lie algebra.

Definition 3.13. Let A be a pre-Lie algebra. We define an algebra structure associated to A by $\widetilde{\mathrm{L}}^A(x) := \mathrm{L}^A(x) + \mathrm{R}^A(x)$ and denote it by $\mathfrak{j}_A$.

Lemma 3.14. *Let A and B be two pre-Lie algebras with $A \to_{\deg} B$. Then $\mathfrak{j}_A \to_{\deg} \mathfrak{j}_B$.*

Proof. The proof is similar to that of Lemma 3.11. □

Remark 3.15. If A happens to be an associative algebra the associated algebra $\mathfrak{j}_A$ is a Jordan algebra. Other than in the case of special Jordan algebras we drop the factor $1/2$, for it makes computations easier here.

The argument used in Lemma 3.11 can be adapted for arbitrary associated structures.

In general for those structures we don't have a full degeneration diagram like in the Lie algebra case, but for practice this is not necessary.

3.2 Semi-invariants under degeneration

Like in the sections before we want to state our theorems for the most general class of algebras, namely $\mathrm{Alg}_n(\mathbb{C})$. Only in the cases where a special definition of pre-Lie algebras is used, we restrict ourselves to that kind of algebras.

Definition 3.16. Let $f\colon \mathrm{Alg}_n(\mathbb{C}) \to \mathbb{R}$ be a function, that is semi-continuous. We call this function a semi-invariant, if f is either increasing or decreasing with respect to a degeneration. In other words, for all $A, B \in \mathrm{Alg}_n(\mathbb{C})$ with $A \to_{\deg} B$ we have either $f(A) \leqslant f(B)$ or $f(A) \geqslant f(B)$.

Definition 3.17. For an arbitrary algebra A let $\mathrm{ab}(A)$ be the maximal dimension of an abelian subalgebra of A, where a subalgebra $W \subset A$ is called abelian, if $x \cdot y = 0$ for all $x, y \in W$. Furthermore we denote by $\mathrm{ab}_I^r(A)$ the maximal dimension of an abelian right ideal of A and by $\mathrm{ab}_I^l(A)$ the maximal dimension of an abelian left ideal of A. The maximal dimension of a two-sided abelian ideal is denoted by $\mathrm{ab}_I(A)$.

To prove the next Lemma and the following Corollaries we use a technique of Grunewald and O'Halloran [32] developed to classify degenerations of nilpotent Lie algebras of dimension 5. We show that this method can be generalized for arbitrary algebras defined over $\mathbb{C}$.

Lemma 3.18. *Let $A, B \in \mathrm{Alg}_n(\mathbb{C})$. If $A \to_{\deg} B$, then $\mathrm{ab}(A) \leqslant \mathrm{ab}(B)$.*

Proof. Construct a subset Δ_z of $\mathrm{Alg}_n(\mathbb{C})$ in the following way:

$$\Delta_z = \{A = (c_{ij}^k)_{ij,k} \mid c_{ij}^k = 0 \text{ if } n - z + 1 \leqslant i, j \leqslant n\} \tag{3.1}$$

This set collects all $\mathbb{C}$-algebras that contain an abelian subalgebra formed by the last z basis vectors. Because every $\mathbb{C}$-algebra A with $\mathrm{ab}(A) \geqslant z$ has a representative in Δ_z, this set describes formally all $\mathbb{C}$-algebras A with $\mathrm{ab}(A) \geqslant z$. The set Δ_z is clearly Zariski closed and $\mathrm{B}_n(\mathbb{C})$-stable because of [32] (1.5). If $W \subset A$ is an abelian subalgebra of dimension z, we conclude by Theorem 1.17 that the algebra B lies in $\mathrm{GL}_n \cdot \Delta_z$. □

Corollary 3.19. *Let $A, B \in \mathrm{Alg}_n(\mathbb{C})$ and $A \to_{\deg} B$, then $\mathrm{ab}_I^r(A) \leqslant \mathrm{ab}_I^r(B)$ and $\mathrm{ab}_I^l(A) \leqslant \mathrm{ab}_I^l(B)$.*

Proof. We have to modify the set Δ_z in the following way. Let

$$\Delta_R(z) = \{A = (c_{ij}^k)_{ij,k} \mid c_{ijk} = 0 \text{ if } 1 \leqslant k \leqslant n - z + 1 \leqslant i \leqslant n\},$$
$$\Delta_L(z) = \{A = (c_{ij}^k)_{ij,k} \mid c_{ijk} = 0 \text{ if } 1 \leqslant k \leqslant n - z + 1 \leqslant j \leqslant n\}.$$

The set $\Delta_R(z)$ collects all $\mathbb{C}$-algebras, that contain a right ideal formed by the last z basis vectors. The set $\Delta_L(z)$ does the same for left ideals. Let Δ_z be as in the lemma before. Because $\Delta_R(z)$ and $\Delta_L(z)$ are closed and B-stable, the sets $\Delta_R(z) \cap \Delta_z$ and $\Delta_L(z) \cap \Delta_z$ are closed and B-stable and therefore Theorem 1.17 applies again. □

Corollary 3.20. *Let $A, B \in \mathrm{Alg}_n(\mathbb{C})$ and $A \to_{\deg} B$, then $\mathrm{ab}_I(A) \leqslant \mathrm{ab}_I(B)$.*

Proof. We have $\mathrm{ab}_I(A) = \mathrm{ab}_I^r(A) \cap \mathrm{ab}_I^l(A)$. As the intersection of two B-stable sets, a two-sided ideal is again B-stable and it is closed by definition of the Zariski topology. Hence, by Theorem 1.17 we are done. □

As an immediate consequence we get:

Corollary 3.21. *Let $A, B \in \mathrm{preLie}_n(\mathbb{C})$ and $A \to_{\deg} B$. According to definition 2.17 we have $\dim A^{(l)} \geqslant \dim B^{(l)}$ and $\dim \gamma_l(A) \geqslant \dim \gamma_l(B)$.*

Proof. For a proof see [46]. □

Because of Lemma 3.11 the last Corollary and furthermore all invariants and semi-invariants of Lie algebras associated to a pre-Lie algebra give no additional information about a possible degeneration of two pre-Lie algebras.

Fortunately, we are able to show that out of series of pre-Lie algebras, as defined in section 2.3, there also emerge new semi-invariants. To prove this we use a technique introduced by Roman Popovych in [46] for Lie algebras.

Proposition 3.22. *Let $\mathbb{C}$ be the field of real or complex numbers and $M_p \in M_{m,n}(\mathbb{C})$, $p \in \mathbb{N}$. Let (M_p) be a sequence of matrices (parametrized by p) for which there exists a componentwise limit $\lim_{p\to\infty} M_p =: M_0$. If $\operatorname{rank} M_p = r$ for all $p \in \mathbb{N}$, then $\operatorname{rank} M_0 \leqslant r$.*

Proof. We identify the space of all $m \times n$ matrices with $\mathbb{C}^{mn}$. The subset of all matrices $M \in M_{m,n}$ with $\operatorname{rank} M \leqslant r$ is an algebraic subset of $\mathbb{C}^{mn}$. □

Lemma 3.23. *Let A and B be two n-dimensional pre-Lie algebras with $A \to_{\deg} B$. Then $\dim A \cdot A \geqslant \dim B \cdot B$.*

Proof. First, if $A^2 := A \cdot A = A$, then of course $\dim B^2 \leqslant n = \dim A = \dim A^2$, causing no contradiction with the hypothesis. So let us suppose that $\dim A^2 < n$. The key argument lies in the realization of $\dim A^2$ in terms of the structure constants. For an exact language we write the structure constants by the algebra product, using the dual vector space V^*. So let $(e_1, \dots, e_n)$ be a basis of the underlying vector space V, then $(e^1, \dots, e^n)$ is a basis of V^*, where $\langle e^i, e_j\rangle = \delta^i_j$.[4] Under this notation we can write $c^k_{ij} = \langle e^k, e_i \cdot e_j\rangle$. By assumption, $\dim A^2 < n$, and therefore we can fix at least one k, for which $c^k_{ij} = 0$ for $1 \leqslant i, j \leqslant n$. Building up a matrix $C := (c^k_{ij})_{k,(i,j)}$, consisting of all structure constants, where the index k runs the row range and the index pair (i, j) runs the column range, we see that the rank of this matrix corresponds to $\dim A^2$. Moreover, C defines the algebra completely. It takes the form:

$$\begin{pmatrix} c^1_{11} & \cdots & c^1_{1n} & c^1_{21} & \cdots & c^1_{2n} & \cdots & c^1_{n1} & \cdots & c^1_{nn} \\ c^2_{11} & \cdots & c^2_{1n} & c^2_{21} & \cdots & c^2_{2n} & \cdots & c^2_{n1} & \cdots & c^2_{nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c^n_{11} & \cdots & c^n_{1n} & c^n_{21} & \cdots & c^n_{2n} & \cdots & c^n_{n1} & \cdots & c^n_{nn} \end{pmatrix}$$

Let λ be a structure representing A. Assuming the existence of a degeneration, we can find a sequence of algebra structures $g_p \cdot \lambda =: \lambda_p$ all isomorphic to λ and $\lim_{p\to\infty} g_p \cdot \lambda = \lambda_0$, where λ_0 represents B. Letting $c^k_{p,ij} = \langle e^k, \lambda_p(e_i, e_j)\rangle$ denote the structure constants of λ_p, we define C_p in the same way like C, but formed by $c^k_{p,ij}$ instead of $c^k_{i,j}$. Then, the degeneration $A \to_{\deg} B$ can be reformulated by the componentwise convergence of the matrices C_p: $\lim_{p\to\infty} C_p =: C_0$. In terms of the structure constants we have $c^k_{p,ij} = \langle e^k, \lambda_p(e_i, e_j)\rangle \to c^k_{0,ij} = \langle e^k, \lambda_0(e_i, e_j)\rangle$ as $p \to \infty$.

Clearly, $\operatorname{rank} C_p = \operatorname{rank} C =: r$, and therefore, by Proposition 3.22 $\operatorname{rank} C_0 \leqslant r$, i.e. $\dim B^2 \leqslant \dim A^2$. □

Corollary 3.24. *Let A and B be two pre-Lie algebras with $A \to_{\deg} B$, then*

$$\dim \delta^{(l)}(A) \geqslant \dim \delta^{(l)}(B).$$

Proof. We can prove this corollary in a similar way as in Lemma 3.23. We can find a matrices C_l for which $\operatorname{rank} C_l = \dim \delta^{(l)}(A)$. □

Corollary 3.25. *Let A and B be two pre-Lie algebras with $A \to_{\deg} B$. If A is solvable, then so is B.*

[4]The symbol δ^i_j denotes the Kronecker delta.

Proof. If A is solvable, there exists a number $m \in \mathbb{N}$ such that $\delta^{(m)}(A) = 0$. By the last corollary we get $\dim \delta^{(m)}(B) \leqslant \dim \delta^{(m)}(A) = 0$, so B is solvable. □

Lemma 3.26. *Let A and B be two pre-Lie algebras with $A \to_{\deg} B$, then*

$$\dim \delta^l(A) \geqslant \dim \delta^l(B) \quad \textit{and} \quad \dim \delta_l(A) \geqslant \dim \delta_l(B).$$

Proof. We want to use the technique that was arranged in the proof of Lemma 3.23. For our purposes here, we construct matrices $C^{l-1}(A)$, for which

$$\operatorname{rank} C^{l-1}(A) = \dim \delta^l(A).$$

We can then use the argument of the convergence of minors, established in Proposition 3.22, again. The next few arguments are very technical, so we better examine them step by step. First we treat the case $\delta^2(A)$. For this we calculate

$$e_l \cdot (e_i \cdot e_j) = e_l \sum_{k=1}^{n} c_{ij}^k e_k = \sum_{k,m=1}^{n} c_{ij}^k c_{lk}^m e_m$$

and therefore find that the rank of the matrix $(\sum_{k=1}^n c_{ij}^k c_{lk}^m)_{m,(i,j,l)}$ is equal to $\dim \delta^2(A)$. Similar for $\delta^3(A)$ we compute the matrix $(\sum_{k,m=1}^n c_{ij}^k c_{lk}^m c_{sm}^p)_{p,(i,j,l,s)}$, which rank is equal to $\dim \delta^3(A)$. We can now see how those matrices can be built up in the general case. Let l be arbitrary and

$$C^{l-1}(A) := \Big(\sum_{j_1,\dots,j_{l-2}=1}^{n} c_{i_1 i_2}^{j_1} c_{i_3 j_1}^{j_2} \cdots c_{i_l j_{l-2}}^{j_{l-1}} \Big)_{j_{l-1},(i_1,\dots,i_l)},$$

then $\dim \delta^l(A) = \operatorname{rank} C^{l-1}(A)$. The proof for $\dim \delta_l(A)$ is very similar. □

Corollary 3.27. *Let A and B be two pre-Lie algebras with $A \to_{\deg} B$. If A is right-nilpotent, then also is B right-nilpotent. The same condition holds for left-nilpotency.*

Proof. As in the case of solvability, if A is right-nilpotent we can find a number $m \in \mathbb{N}$ such that $\delta^m(A) = 0$. Because of the last lemma we have

$$\dim \delta^m(B) \leqslant \dim \delta^m(A) = 0$$

and therefore B is right-nilpotent. The same argument holds for left-nilpotency. □

Definition 3.28. We call the following sets the left resp. right annihilator of a pre-Lie algebra A:

$$\operatorname{Ann}_L(M) := \{x \in A \mid y \cdot x = 0 \quad \forall y \in M\}$$

$$\operatorname{Ann}_R(M) := \{x \in A \mid x \cdot y = 0 \quad \forall y \in M\}$$

Lemma 3.29. *Let $A, B \in \operatorname{preLie}_n(\mathbb{C})$ and $A \to_{\deg} B$. Then*

$$\dim \operatorname{Ann}_L(A) \leqslant \dim \operatorname{Ann}_L(B) \quad \textit{and} \quad \dim \operatorname{Ann}_R(A) \leqslant \dim \operatorname{Ann}_R(B).$$

Proof. Let $x \in \mathrm{Ann}_L(A)$ and $x = \sum_{j=1}^n x_j e_j$. It follows that

$$e_i \cdot x = \sum_{k=1}^{n} (\sum_{j=1}^{n} x_j c_{ij}^k) e_k = 0$$

and therefore $\sum_{j=1}^n x_j c_{ij}^k = 0$ for $1 \leqslant i, k \leqslant n$. We can express this fact by a matrix equation:

$$\begin{pmatrix} c_{11}^1 & c_{12}^1 & \cdots & c_{1n}^1 \\ c_{21}^1 & c_{22}^1 & \cdots & c_{2n}^1 \\ \vdots & \ddots & \cdots & \vdots \\ c_{n1}^n & c_{n2}^n & \cdots & c_{nn}^n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$$

where the index j runs the column range and the index pair (i, k) in the structure constant runs the row range. We can now say that an element x is contained in $\mathrm{Ann}_L(A)$, exactly when it is in the kernel of the above matrix C. Using

$$\dim \mathrm{Ann}_L(A) = \dim \ker C = n^2 - \operatorname{rank} C$$

and Proposition 3.22 we find that $\dim \mathrm{Ann}_L(A) \leqslant \dim \mathrm{Ann}_L(B)$.

For the second identity we compute $x \cdot e_i = \sum_{k=1}^n (\sum_{j=1}^n x_j c_{ji}^k) e_k = 0$ and find $\sum_{j=1}^n x_j c_{ji}^k = 0$ for $1 \leqslant i, k \leqslant n$. Therefore we just have to change the indices i and j in the equation for $\mathrm{Ann}_L(A)$ and conclude that $\mathrm{Ann}_R(A)$ also defines the kernel of a matrix consisting only of structure constants. □

Lemma 3.30. *Let* $A, B \in \mathrm{preLie}_n(\mathbb{C})$ *and* $A \to_{\deg} B$. *Then* $\dim Z(A) \leqslant \dim Z(B)$.

Proof. Refering to the proof of Lemma 3.18 we immediately see that the set of structures with a center formed by the last z basis vectors, is a subset of Δ_z. It is therefore B-stable and of course closed. □

Corollary 3.31. *Let* $A, B \in \mathrm{preLie}_n(\mathbb{C})$ *and* $A \to_{\deg} B$. *Then* $\dim \delta_{(l)}(A) \leqslant \dim \delta_{(l)}(B)$.

Proof. This proof is similar to that of Lemma 3.23 as can be seen in [46]. □

We recall the various definitions made about the radical of a pre-Lie algebra that resulted in Theorem 2.35.

Theorem 3.32. *Let* $A, B \in \mathrm{preLie}_n(\mathbb{C})$ *and* $A \to_{\deg} B$. *Then the following relations hold:*

1. $\dim_{\mathbb{C}} T(A) \leqslant \dim_{\mathbb{C}} T(B)$.
2. $\dim_{\mathbb{C}} A^{\perp} \leqslant \dim_{\mathbb{C}} B^{\perp}$.
3. $\dim_{\mathbb{C}} \mathrm{rad}(A) \leqslant \dim_{\mathbb{C}} \mathrm{rad}(B)$.
4. $\dim_{\mathbb{C}} \mathrm{nil}(A) \leqslant \dim_{\mathbb{C}} \mathrm{nil}(B)$.

Proof. (1) According to our previous notation let

$$\begin{aligned} \Delta_z :&= \{A \in \mathrm{preLie}_n(\mathbb{C}) \mid \mathrm{tr}(\mathrm{R}^A(e_i)) = 0 \text{ if } 1 \leqslant i \leqslant z\} \\ &= \{A \in \mathrm{preLie}_n(\mathbb{C}) \mid \dim T(A) \geqslant z\}. \end{aligned}$$

Then Δ_z is $\mathrm{B}_n(\mathbb{C})$-stable for some Borel subgroup $\mathrm{B}_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$. Indeed, let $A \in \Delta_z$ and $b \in \mathrm{B}_n(\mathbb{C})$, so $\mathrm{tr}(b \cdot \mathrm{R}^A(e_i)) = \mathrm{tr}(b\mathrm{R}^A(b^{-1}(e_i))b^{-1}) = \mathrm{tr}(\mathrm{R}^A(b^{-1}(e_i)))$. If $1 \leqslant i \leqslant z$, which means that $e_i \in T(A)$, then clearly $b^{-1}(e_i) \in T(A)$. We conclude that $\mathrm{tr}(b \cdot \mathrm{R}^A(e_i)) = 0$. Of course, Δ_z is Zariski closed as its elements are roots of the polynomials $\mathrm{tr}(\mathrm{R}^A(e_i)) = 0$. Theorem 1.17 applies and we are done.
(2) The dimension of the kernel of every bilinear form on A defines a semi-invariant.
(3) We can easily conclude this from point one. We have already seen that a maximal ideal of a fixed dimension defines for itself a B-stable and Zariski closed set Δ_I. As an intersection of Δ_z with Δ_I, the ideal $\mathrm{rad}(A)$ also defines such a set. Theorem 1.17. applies again.
(4) The same argument as in (3) applies for left-nilpotency. □

3.2.1 Semi-invariants given by dimensions of certain vector spaces

In this subsection we show that there are innumerable many possibilities for finding semi-invariants by defining certain equations in linear operators. To motivate the general procedure we start with two examples.

Definition 3.33. Let A be a pre-Lie algebra. Let $\alpha, \beta, \gamma \in \mathbb{C}$ and define $\mathrm{Der}_{(\alpha,\beta,\gamma)}(A)$ to be the space of all $D \in \mathrm{End}(A)$ satisfying

$$\alpha D(x \cdot y) = \beta D(x) \cdot y + \gamma x \cdot D(y)$$

for all $x, y \in A$. We call the elements $D \in \mathrm{Der}_{(\alpha,\beta,\gamma)}(A)$ (α, β, γ)-derivations.

The proof of the following Lemma can also be found in [6]. We bring it here once more because we adapt the key argument for our main theorem in this subsection.

Lemma 3.34. *Let $A, B \in \mathrm{preLie}_n(\mathbb{C})$. If $A \to_{\deg} B$, then*

$$\dim \mathrm{Der}_{(\alpha,\beta,\gamma)}(A) \leqslant \dim \mathrm{Der}_{(\alpha,\beta,\gamma)}(B)$$

for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Let $\lambda, \mu \in \mathrm{preLie}_n(\mathbb{C})$ represent A and B. Fix a basis $(e_1, \ldots, e_n)$ of the underlying vector space. Then

$$\lim_{\varepsilon \to 0} (g_\varepsilon \circ \lambda)(e_i, e_j) = \mu(e_i, e_j)$$

for operators $g_\varepsilon \in \mathrm{GL}_n(\mathbb{C})$. For $D \in \mathrm{Der}_{(\alpha,\beta,\gamma)}(A)$ we write $D = (d_{i,j})_{1 \leqslant i,j \leqslant n}$, and $D(e_i) = \sum_{l=1}^n d_{li} e_l$. We have $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ in A, with the structure constants c_{ij}^k. Since D is an (α, β, γ)-derivation we have

$$\sum_{l=1}^n (\alpha c_{ij}^l d_{kl} - \beta c_{lj}^k d_{li} - \gamma c_{il}^k d_{lj}) = 0$$

for all i, j, k. We can rewrite these n^3 equations as a matrix equation $Md = 0$ where d is the vector formed by the columns of the matrix $D = (d_{ij})$, and M is a $n^3 \times n^2$ matrix depending on c_{ij}^k and α, β, γ. Thus we have $\ker(M) = \mathrm{Der}_{(\alpha,\beta,\gamma)}(A)$. If A degenerates to B via g_ε we obtain a sequence of matrices M_ε with $\lim_{\varepsilon\to 0} M_\varepsilon = M_0$ by componentwise convergence of the structure constants, where $\ker(M_0) = \mathrm{Der}_{(\alpha,\beta,\gamma)}(B)$. Let m be the rank of the matrix M. Then every submatrix of size $(m+1) \times (m+1)$ has zero determinant. Because of the convergence the same is true for M_0. It follows that $\mathrm{rank}(M) \geqslant \mathrm{rank}(M_0)$, or $\dim\ker(M) \leqslant \dim\ker(M_0)$. □

This semi-invariant is also an important tool for determining the degenerations of an algebra A. For Lie algebras it is not necessary to compute $\dim \mathrm{Der}_{(\alpha,\beta,\gamma)}(A)$ for all values of $\alpha, \beta, \gamma \in \mathbb{C}$, as was proved by Novotny and Hrivnak in [36].

Theorem 3.35. *Let $\mathfrak{g}$ be a Lie algebra. For any $\alpha, \beta, \gamma \in \mathbb{C}$ there exists $\delta \in \mathbb{C}$ such that the subspace $\mathrm{Der}_{(\alpha,\beta,\gamma)}(\mathfrak{g})$ is equal to some of the four spaces:*

1. $\mathrm{Der}_{(\delta,0,0)}(\mathfrak{g})$
2. $\mathrm{Der}_{(\delta,1,-1)}(\mathfrak{g})$
3. $\mathrm{Der}_{(\delta,1,0)}(\mathfrak{g})$
4. $\mathrm{Der}_{(\delta,1,1)}(\mathfrak{g})$.

Another important example motivated from Lie algebra theory is the following.

Definition 3.36. Let A be a pre-Lie algebra. A linear operator $P \in \mathrm{End}_{\mathbb{C}}(A)$ is called a pre-derivation if there holds the following equation for all $x, y, z \in A$:

$$P(x \cdot (y \cdot z)) = (P(x) \cdot (y \cdot z)) + (x \cdot (P(y) \cdot z)) + (x \cdot (y \cdot P(z))).$$

The vector space of all pre-derivations of an algebra A is denoted by $\mathrm{Pder}(A)$.

Lemma 3.37. *If $A, B \in \mathrm{Alg}_n(\mathbb{C})$ and $A \to_{\deg} B$, then* $\dim \mathrm{Pder}(A) \leqslant \dim \mathrm{Pder}(B)$.

Proof. The proof follows as a special case from the next theorem. □

One thing that both examples have in common is the following. We are given a product of a fixed number of elements in an algebra A with a certain kind of bracketing. Moreover we have a linear operator, let's say $T \in \mathrm{End}(A)$, that permutes its position within this product. We can now define a sum where each position of T in the product corresponds to summand with its own coefficient in $\mathbb{C}$. Setting this sum zero gives an equation in the operator T. We are now going to formulate this procedure in an exact way.

At first, we note that every operator $T \in \mathrm{End}(A)$ may appear only once in each product. Let $p(x_1, \ldots, x_m)$ denote an m-fold product in an arbitrary algebra A given by the elements $x_1, \ldots, x_m \in A$. For such a product there are exactly $b(m) := \frac{1}{m}\binom{2m-2}{m-1}$ possiblities for bracketing types[5]. In everyone of those bracketings an operator can take $2m-1$ positions, that is in front of every bracket and in front of every factor. In terms of the left multiplication we can write $p(x_1, \ldots, x_m)$ as a composition $C_{L_1,\ldots,L_{m-1}}(x_1, \ldots, x_m)$ of $m-1$ left multiplication operators.

[5] Compare with [23].

Example 3.38. One can find this example also in [23]. For $m = 4$ we have 5 bracketing types:[6]

$$
\begin{aligned}
x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)) &= L_{x_1} \circ L_{x_2} \circ L_{x_3}(x_4), \\
((x_1 \cdot x_2) \cdot x_3) \cdot x_4 &= L_{L_{L_{x_1}(x_2)}(x_3)}(x_4), \\
(x_1 \cdot x_2) \cdot (x_3 \cdot x_4) &= L_{L_{x_1}(x_2)}(L_{x_3}(x_4)), \\
(x_1 \cdot (x_2 \cdot x_3)) \cdot x_4 &= L_{L_{x_1}(L_{x_2}(x_3))}(x_4), \\
x_1 \cdot ((x_2 \cdot x_3) \cdot x_4) &= L_{x_1}(L_{L_{x_2}(x_3)}(x_4)).
\end{aligned}
$$

The following definition shall help us to indicate the exact position of the operator $T \in \mathrm{End}(A)$ in the product.

Definition 3.39. Let $A \in \mathrm{Alg}_n(\mathbb{C})$ and $C_{L_1,\dots,L_{n-1}}(x_1,\dots,x_m)$ denote an m-fold product in A with arbitrary bracketing. For $T \in \mathrm{End}(A)$ we define:

$$
\begin{aligned}
\pi_i(T)(C_{L_1,\dots,L_{m-1}}(x_1,\dots,x_m)) &:= C_{L_1,\dots,L_{m-1}}(x_1,\dots,T(x_i),\dots,x_m), \\
\rho_i(T)(C_{L_1,\dots,L_{m-1}}(x_1,\dots,x_m)) &:= C_{L_1,\dots,T\circ L_i,\dots,L_{m-1}}(x_1,\dots,x_m).
\end{aligned}
$$

Definition 3.40. Let $A \in \mathrm{Alg}_n(\mathbb{C})$ and $\alpha_i, \beta_i \in \mathbb{C}$ for all i. For a linear map $T \in \mathrm{End}(A)$ and an m-fold product $C_{L_1,\dots,L_{m-1}}(x_1,\dots,x_m)$ we define the following function:

$$
f_C(T) := \sum_{i=1}^{m} \alpha_i \pi_i(T)(C_{L_1,\dots,L_{m-1}}(x_1,\dots,x_m)) + \sum_{i=1}^{m-1} \beta_i \rho_i(T)(C_{L_1,\dots,L_{m-1}}(x_1,\dots,x_m)).
$$

Furthermore we define the following set:

$$
V_{(\overline{\alpha};\overline{\beta})}(A) := \{T \in \mathrm{End}(A) \mid \sum_{i=1}^{b(m)} f^{\beta_1,\dots,\beta_{m-1}}_{C_i,\alpha_1,\dots,\alpha_m}(T) = 0, \text{ for all } x_1,\dots,x_m \in A\}
$$

The products C_i run over all bracketing types, of which we got $b(m) = \frac{1}{m}\binom{2m-2}{m-1}$ many. Also note, that the $\alpha's$ and $\beta's$ correspond to the products C_i. For every C_i we therefore have other $\alpha's$ and $\beta's$. The collection of all $\alpha's$ and $\beta's$, which are associated to the various products, will be denoted by the index $(\overline{\alpha};\overline{\beta})$ in $V_{(\overline{\alpha};\overline{\beta})}(A)$.

Clearly, the set $V_{(\overline{\alpha};\overline{\beta})}(A)$ is a vector subspace of $\mathrm{End}(A)$ and therefore of dimension $\leqslant n^2$.

Remark 3.41. The above definitions look a little bit complicated, but their motivation is very simple. As we contemplated in the introduction of this subsection, every equation linear in an operator $T \in \mathrm{End}(A)$ might define a new semi-invariant.[7] To write down all the different terms that arise in an equation an operator can appear in, we have to determine the position of the operator. In fact this is what we do in the above definitions.

[6] The last bracket, surrounding the whole term, is ommited.
[7] That this is actually true follows from the next Theorem.

To derive our previous special cases $\mathrm{Der}_{(\alpha,\beta,\gamma)}(A)$ and $\mathrm{Pder}(A)$ from this generalization we proceed with the following two examples.

Example 3.42. Let $m = 2$ then there exists only one kind of product, that is

$$C_L(x, y) = L_x(y) = x \cdot y.$$

The function $f_C(T)$ then takes the form:

$$f_C(T) = \alpha_1 T(x) \cdot y + \alpha_2 x \cdot T(y) + \beta_1 T(x \cdot y).$$

If we set $\alpha_1 = \beta$, $\alpha_2 = \gamma$, and $\beta_1 = \alpha$, we see that for $(\overline{\alpha}; \overline{\beta}) = (\alpha_1, \alpha_2; -\beta_1)$

$$V_{(\overline{\alpha};\overline{\beta})}(A) = \mathrm{Der}_{(\alpha,\beta,\gamma)}(A).$$

Example 3.43. Let $m = 3$ and $C_{L_1,L_2}(x, y, z) = L_x \circ L_y(z) = (x \cdot (y \cdot z))$. If we set $\alpha_i = -\beta_1 = 1$ for all i and $\beta_2 = 0$ then we have:

$$f_C(T) = (T(x) \cdot (y \cdot z)) + (x \cdot (T(y) \cdot z)) + (x \cdot (y \cdot T(z))) - T(x \cdot (y \cdot z)).$$

Setting all indices corresponding to the other products $C_{L_1,L_2}(x, y, z)$ equal to zero guarantees that no other product than $C_{L_1,L_2}(x, y, z)$ appears in the defining functions of the set $V_{(\overline{\alpha};\overline{\beta})}(A)$. For $(\overline{\alpha}; \overline{\beta}) = (1, 1, 1; -1, 0)$ we then have:

$$V_{(\overline{\alpha};\overline{\beta})}(A) = \mathrm{Pder}(A).$$

With the above definitions we can now formulate the main Theorem of this subsection.

Theorem 3.44. *Let $A, B \in \mathrm{Alg}_n(\mathbb{C})$. Suppose that $A \to_{\deg} B$ then*

$$\dim V_{(\overline{\alpha};\overline{\beta})}(A) \leqslant \dim V_{(\overline{\alpha};\overline{\beta})}(B).$$

Proof. The proof of the statement follows closely the argument brought in Lemma 3.34. Therefore we take the notation over from there. We only have to show that there exists a matrix M such that $\ker M = \dim V_{(\overline{\alpha};\overline{\beta})}(A)$ and $M(d) = 0$. For this we compute the functions $f_C(T)$ on the basis $(e_1, \dots, e_n)$ and find that the resulting equations are linear in the elements d_{ij}. Hence we can isolate those $d'_{ij}s$ and put the remaining factor together to form an element in the matrix M_C. Summing over all possible compositions C we get the desired matrix M. □

Remark 3.45. We can generalize this Theorem even more. Take for example more than one equation and the resulting set of zeros still defines a vector space, which dimension is a semi-invariant. In the same way we can choose more than one operator T, having a similar effect.

3.3 Invariants under degeneration

As we have seen, most of our semi-invariants are not of a strict kind, by means of the relations $\leqslant$ and $\geqslant$. Therefore, in most of our cases such arguments won't give as much restrictions for degenerations as desired. One of the potentially most powerful concepts of showing that a certain degeneration is impossible, is that of

an invariant. Here we want to reserve this notion for a polynomial in terms of the structure constants which is zero on the whole orbit of an algebra. If this is the case for an algebra A, the same polynomial has to be zero on the orbit closure $\overline{\mathrm{O}(A)}$. So, if for any other algebra B this polynomial does not vanish, B cannot lie in the orbit closure of A.
For example, commutativity is such an invariant. If $\mathrm{L}^A(x)$ respectively $\mathrm{R}^A(x)$ denotes the left respectively right multiplication operator in $\mathrm{End}(A)$, then commutativity of A means that the polynomial $T(x) = \mathrm{L}^A(x) - \mathrm{R}^A(x)$ satisfies $T(x) = 0$ for all $x \in A$. This is clearly a polynomial invariant on the orbit of A. Hence if A is commutative and B is not, a degeneration $A \to_{\deg} B$ is impossible. Another operator identity is $T(x,y) = [\mathrm{L}^A(x), \mathrm{R}^A(y)] = \mathrm{L}^A(x)\mathrm{R}^A(y) - \mathrm{L}^A(y)\mathrm{R}^A(x) = 0$, which means that the algebra A is associative. The formal details are recorded in the following lemma, which also can be found in [6].

Lemma 3.46. *Let A and B be two K-algebras of dimension n, where K is a field of characteristic zero. Let $T(x_1,\dots,x_l)$ be a polynomial in $\mathrm{L}^A(x_1),\dots,\mathrm{L}^A(x_l)$ and $\mathrm{R}^A(x_1),\dots,\mathrm{R}^A(x_l)$, the left and right multiplications by the elements $x_1,\dots x_n$. Suppose that $T(x_1,\dots,x_l) = 0$ in A, but not in B, then $B \notin \overline{\mathrm{O}(A)}$.*

Proof. Let $\varphi\colon A \to A'$ be an isomorphism of K-algebras. Then $\varphi(x\cdot y) = \varphi(x).\varphi(y)$ implies

$$\mathrm{L}^A(x) = \varphi \circ \mathrm{L}^{A'}(\varphi^{-1}(x)) \circ \varphi^{-1}, \quad \mathrm{R}^A(x) = \varphi \circ \mathrm{R}^{A'}(\varphi^{-1}(x)) \circ \varphi^{-1}.$$

If a polynomial T in the left- and right multiplications of A vanishes, then the same is true for the left- and right multiplications of all K-algebra structures in the GL_n-orbit representing A, since a base change just induces a conjugation of the operator polynomial. It follows that the operator identity holds also for all K-algebra structures in the orbit closure. This completes the proof. □

This lemma gives us already a huge number of possibilities how to construct invariants, because we have no restriction on the polynomial $T(x)$. We will see in the next chapter, that considering $T(x)$ as a polynomial in linear combinations of all quadratic monomials in $\mathrm{L}^A(x), \mathrm{L}^A(y), \mathrm{R}^A(x)$, and $\mathrm{R}^A(y)$ solves the case of all orbit closures of 2-dimensional pre-Lie algebras. Unfortunately, not all polynomials in the structure constants can be expressed in form of operator identities, that only involve left and right multiplication by an element of A. For example we have (see [47]):

Proposition 3.47. *Let A be an algebra over $\mathbb{C}$. Let*

$$T_{p,q}(x,y) := \mathrm{tr}(\mathrm{L}^A(x)^p) \cdot \mathrm{tr}(\mathrm{L}^A(y)^q) - c \cdot \mathrm{tr}(\mathrm{L}^A(x)^p \circ \mathrm{L}^A(y)^q)$$

with p and q in $\mathbb{N}$. If $T_{p,q}(x,y) = 0$ for all $x, y \in A$ then $T_{p,q}(x,y) = 0$ for all $x, y \in B$ with $B \in \overline{\mathrm{O}(A)}$.[8]

Proof. Follows from the next Lemma. □

[8]By definition $T_{p,q}(x,y)$ involves the operators $\mathrm{L}^A(x)$ and $\mathrm{L}^A(y)$ and is therefore dependend on the choice of a basis. If we say that the expression $T_{p,q}(x,y)$ equals zero for an algebra $B \in \overline{\mathrm{O}(A)}$ we mean that $T_{p,q}(x,y)$ stays zero if we replace $\mathrm{L}^A(x)$ by $\mathrm{L}^B(x)$.

Definition 3.48. Consider $A \in \mathrm{Alg}_n(\mathbb{C})$. If the equations

$$T_{p,q}(x,y) = \mathrm{tr}(\mathrm{L}^A(x)^p) \cdot \mathrm{tr}(\mathrm{L}^A(y)^q) - c \cdot \mathrm{tr}(\mathrm{L}^A(x)^p \circ \mathrm{L}^A(y)^q) = 0$$

have a unique solution in c for all $x, y \in A$, then the number c is called the $\mathfrak{C}_{\mathrm{p,q}}$-invariant of A. In this case we write $c =: \mathfrak{C}_{\mathrm{p,q}}(A)$.

Remark 3.49. As already indicated in the definition above the equations $T_{p,q}(x,y) = 0$ need not to have a unique solution in c. Two different cases can occur. First, the expression $T_{p,q}(x,y)$ is dependend on x and y. Second, the trace of $\mathrm{L}^A(x)^p \circ \mathrm{L}^A(y)^q$ is zero for all $x, y \in A$.

The important property for the sucessful completion of the last proof lies in the fact that the linear form $\mathrm{tr}\colon \mathrm{End}(V) \to \mathbb{C}$ is conjugation invariant. Similarly to the Killing form in the case of Lie algebras, we can regard the trace as a bilinear form by $\psi(A,B) = \mathrm{tr}(A \cdot B)$. The matrices A and B would usually be the left and right multiplication by an element of some algebra. We know that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, which is a sufficient condition for conjugation invariance. So, trying to generalize the trace in this way, we can take an arbitrary linear form φ with

$$\varphi(AB) = \varphi(BA), \tag{3.2}$$

which is equal to $\varphi([A,B]) = 0$. This means, that searching for linear forms satisfying the identity above is equal to looking for elements in $(\mathfrak{gl}_n/[\mathfrak{gl}_n, \mathfrak{gl}_n])^*$, hence one-dimensional characters of $\mathfrak{gl}_n$. Because $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$ and $\mathfrak{gl}_n/\mathfrak{sl}_n = \{k \cdot Id_n \in \mathrm{End}(V) \mid k \in \mathbb{C}^*\}$ we have $\varphi\colon \mathbb{C}^* \to \mathbb{C}^*$ with $\varphi(c) = \alpha c$ for $\alpha \in \mathbb{C}^*$. Computing the trace under the projection map $\pi\colon \mathfrak{gl}_n \to \mathfrak{gl}_n/\mathfrak{sl}_n$ we find that $\mathrm{tr}(c) = n \cdot c$ and therefore $\varphi(c) = \alpha \cdot \mathrm{tr}(c)$ by rescaling α in a proper way. In conclusion, searching for new degeneration invariants by looking for conjugation invariant linear forms on $\mathrm{End}(V)$ fails, because modulo scalars the trace is the only form of this kind.

Lemma 3.50. *Let A be an n-dimensional algebra over $\mathbb{C}$ with structure λ and denote by $\mathrm{L}^A(x)$ resp. $\mathrm{R}^A(x)$ the left resp. right multiplication with the element $x \in A$. Let $\varphi_1, \ldots, \varphi_r$ be conjugation invariant forms on $\mathrm{End}(A)$. Take polynomials $h_1, \ldots, h_r \in k[X_1, Y_1, \ldots, X_s, Y_s]$ and $f \in k[Z_1, \ldots, Z_r]$. Finally we define*

$$\begin{aligned} T^{(\lambda)}_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(x_1, \ldots x_s) = f\Big(&\varphi_1\big(h_1(\mathrm{L}^A(x_1), \mathrm{R}^A(x_1), \ldots, \mathrm{L}^A(x_s), \mathrm{R}^A(x_s))\big), \ldots, \\ &\varphi_r\big(h_r(\mathrm{L}^A(x_1), \mathrm{R}^A(x_1), \ldots, \mathrm{L}^A(x_s), \mathrm{R}^A(x_s))\big)\Big) \end{aligned} \tag{3.3}$$

with r and s in $\mathbb{N}$. If $T^{(\lambda)}_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(x,y) = 0$ for all $x, y \in A$ then also on $\overline{\mathrm{O}(\lambda)}$.

Proof. For this proof we abbreviate

$$h(\mathrm{L}^A(x_1), \mathrm{R}^A(x_1), \ldots, \mathrm{L}^A(x_s), \mathrm{R}^A(x_s)) \text{ by } h(\mathrm{L}^A(X), \mathrm{R}^A(X))$$

without loss of exactness. By hypothesis we have $T_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(x_1, \ldots x_s) = 0$ for all $x, y \in A$. We note that $\mathrm{L}^{A'}(x) = g \circ \mathrm{L}^A(g^{-1}(x)) \circ g^{-1}$ for an arbitrary algebra A' (with structure μ) isomorphic to A via $g \in \mathrm{GL}_n(\mathbb{C})$. In the same way $\mathrm{R}^{A'}(x) = g \circ \mathrm{R}^A(g^{-1}(x)) \circ g^{-1}$. For a polynomial h in $k[X_1, Y_1, \ldots, X_s, Y_s]$ we have

$h(\mathrm{L}^{A'}(X), \mathrm{R}^{A'}(X)) = g \cdot h(\mathrm{L}^{A}(g^{-1}(X)), \mathrm{R}^{A}(g^{-1}(X))) \cdot g^{-1}$. Thus, for a conjugation invariant form φ we conclude that

$$\varphi(h(\mathrm{L}^{A'}(X), \mathrm{R}^{A'}(X))) = \varphi(h(\mathrm{L}^{A}(g^{-1}(X)), \mathrm{R}^{A}(g^{-1}(X))).$$

Now, we put those things together, to compute the identity $T^{(\mu)}_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(x_1,\ldots x_s)$:

$$\begin{aligned}
&T^{(\mu)}_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(x_1,\ldots x_s) = \\
&f(\varphi_1(h_1(\mathrm{L}^{A}(X), \mathrm{R}^{A}(X))), \ldots, \varphi_r(h_r(\mathrm{L}^{A}(X), \mathrm{R}^{A}(X)))) = \\
&f(\varphi_1(h_1(\mathrm{L}^{A'}(g^{-1}(X)), \mathrm{R}^{A'}(g^{-1}(X)))), \ldots, \varphi_r(h_r(\mathrm{L}^{A'}(g^{-1}(X)), \mathrm{R}^{A'}(g^{-1}(X))))) = \\
&T^{(\lambda)}_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(g^{-1}(x_1),\ldots g^{-1}(x_s)) = 0
\end{aligned}$$

Now where we have a Zariski equation on the whole orbit of A, the identity

$$T^{(\lambda)}_{f,h_1,\ldots,h_r,\varphi_1,\ldots,\varphi_r}(x_1,\ldots x_s)$$

even vanishes on the orbit closure of A by definition of the Zariski topology. □

Example 3.51. Take $r = s = 1$ and $\varphi = \mathrm{tr}$. If we choose $h(X,Y) = Y$ and $f(Z) = Z$ then the expression $T^{(\lambda)}_{f,h,\varphi}(x)$ takes the from:

$$T^{(\lambda)}_{f,h,\varphi}(x) = \mathrm{tr}(\mathrm{R}^{A}(x)).$$

Recalling Theorem 2.33 we see that if $T^{(\lambda)}_{f,h,\varphi}(x) = 0$ then λ is complete. By the last Lemma every $\mu \in \overline{\mathrm{O}(\lambda)}$ is also complete.

Because of its importance for the classification of degenerations we note the conclusion of the last example seperately.

Corollary 3.52. *Let $A, B \in \mathrm{Alg}_n(\mathbb{C})$ and $A \to_{\deg} B$. If A is complete then also B is complete.*

Example 3.53. We show that Lemma 3.50 is a generalization of Proposition 3.47. For this set $r = 3$, $s = 2$ and $\varphi_i = \varphi = \mathrm{tr}$ for $i = 1,2,3$. We make the following definitions:

$$\begin{aligned}
h_1(X_1, Y_1, X_2, Y_2) &= X_1^p, \\
h_2(X_1, Y_1, X_2, Y_2) &= X_2^q, \\
h_3(X_1, Y_1, X_2, Y_2) &= X_1^p X_2^q, \\
f(Z_1, Z_2, Z_3) &= Z_1 \cdot Z_2 - cZ_3.
\end{aligned}$$

With these definitions we see that:

$$T^{(\lambda)}_{f,h_1,h_2,h_3,\varphi}(x,y) = T_{p,q}(x,y).$$

Example 3.54. We will use the following invariant several times troughout the classification in chapter 4. We set $r = 3$, $s = 2$ and $\varphi_i = \varphi = \mathrm{tr}$ for $i = 1,2,3$. We define the polynomials h and f by

$$\begin{aligned}
h_1(X_1, Y_1, X_2, Y_2) &= Y_1^p, \\
h_2(X_1, Y_1, X_2, Y_2) &= Y_2^q, \\
h_3(X_1, Y_1, X_2, Y_2) &= Y_1^p Y_2^q, \\
f(Z_1, Z_2, Z_3) &= Z_1 \cdot Z_2 - rZ_3.
\end{aligned}$$

In conclusion we get

$$T^{(\lambda)}_{f,h,\varphi}(x,y) = \operatorname{tr}(\mathrm{R}^A(x)^p) \cdot \operatorname{tr}(\mathrm{R}^A(y)^q) - r \cdot \operatorname{tr}(\mathrm{R}^A(x)^p \circ \mathrm{R}^A(y)^q).$$

Similar to the case of Proposition 3.47 we can associate the complex number r to an algebra A whenever it is defined, i. e. the equations $T^{(\lambda)}_{f,h,\varphi}(x,y) = 0$ have a unique solution in r for all $x, y \in A$. If this is the case we call the number $r \in \mathbb{C}$ the $\mathfrak{R}_{\mathrm{p,q}}$-invariant.

Remark 3.55. Let A be a pre-Lie algebra of arbitrary dimension for which the associated Lie algebra $\mathfrak{g}_A$ is abelian. Then $\mathfrak{C}_{\mathrm{p,q}}(A) = \mathfrak{R}_{\mathrm{p,q}}(A)$ for all $p, q \in \mathbb{N}$.

4 Orbit closures of Novikov algebras in dimension three

In this chapter we apply the methods we developed in the previous part to the case of pre-Lie algebras of dimension ≤ 2 and Novikov algebras of dimension three. We start with the classification of all pre-Lie algebras of dimension ≤ 2 and their degenerations. This study is heavily based on the article [6], which gives a concise treatment of what will be presented here in more detail. For the case of three dimensional Novikov algebras the list presented in Burde: "The variety of complex Novikov algebras" ([9]) seems to be best to work with and will be sketched at the beginning of that part of the chapter. In all what follows, we take over the notation for Novikov algebras in any dimension from this article. We mention, that the classification of three dimensional Novikov algebras one can also find in the literature ([4]). Both lists are equivalent.

4.1 Degenerations of pre-Lie algebras of dimension less than two

Following this classification of pre-Lie algebras gives the guideline how to handle the classification in case of degenerations more easily. Because of Lemma 3.11 a degeneration of pre-Lie algebras is only possible, if there exists a degeneration of the associated Lie algebras. Therefore it is best starting to find all degenerations of pre-Lie algebras with the same associated Lie algebra. After that, we proceed with classifying degenerations between classes of different associated Lie algebras.

As an introducing example we examine the case of 1-dimensional pre-Lie algebras, all of which have the abelian complex Lie algebra $\mathbb{C}$ associated. Moreover, we remark that in this case we even have just two algebra laws at all and so $\mathrm{preLie}_1(\mathbb{C})$ equals $\mathrm{Alg}_1(\mathbb{C})$. Let e be a basis vector of $\mathbb{C}$. There are two non-isomorphic pre-Lie algebras, denoted by P_1 for the abelian pre-Lie algebra and P_2 for the algebra with product $e \cdot e = e$. Both of them are clearly Novikov too. Because every non-trivial algebra degenerates to the abelian one (example 1.3), we immediately conclude for $\mathrm{preLie}_1(\mathbb{C})$ in form of the Hasse diagram:

$$\begin{array}{c} P_2 \\ \downarrow \\ P_1 \end{array}$$

4.1.1 Classification of pre-Lie algebras of dimension 2

In dimension two there are two non-isomorphic Lie algebras, $\mathfrak{g}_1 = \mathbb{C}^2$ and $\mathfrak{g}_2 = \mathfrak{r}_2(\mathbb{C})$. For $\mathfrak{g}_2$ we can choose $[e_1, e_2] = e_1$ as a representative law. The classification of 2-dimensional complex pre-Lie algebras is well known and can be found in [10]:

U	Products	$\mathfrak{g}_U$	$\dim \mathrm{Der}(U)$
U_1	-	$\mathbb{C}^2$	4
U_2	$e_1 \cdot e_1 = e_1$	$\mathbb{C}^2$	1
U_3	$e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2$	$\mathbb{C}^2$	0
U_4	$e_1 \cdot e_2 = e_1, e_2 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_2$	$\mathbb{C}^2$	1
U_5	$e_2 \cdot e_2 = e_1$	$\mathbb{C}^2$	2
$W_1(\alpha)$	$e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = \alpha e_2$	$\mathfrak{r}_2(\mathbb{C})$	1 if $\alpha \neq -1$, 2 if $\alpha = -1$
$W_2(\beta)$ $\beta \neq 0$	$e_1 \cdot e_2 = \beta e_1, e_2 \cdot e_1 = (\beta - 1)e_1,$ $e_2 \cdot e_2 = \beta e_2$	$\mathfrak{r}_2(\mathbb{C})$	1 if $\beta \neq 1$, 2 if $\beta = 1$
W_3	$e_2 \cdot e_1 = -e_1, e_2 \cdot e_2 = e_1 - e_2$	$\mathfrak{r}_2(\mathbb{C})$	1
W_4	$e_1 \cdot e_1 = e_2, e_2 \cdot e_1 = -e_1,$ $e_2 \cdot e_2 = -2e_2$	$\mathfrak{r}_2(\mathbb{C})$	0
W_5	$e_1 \cdot e_2 = e_1, e_2 \cdot e_2 = e_1 + e_2$	$\mathfrak{r}_2(\mathbb{C})$	1

We exclude the value 0 for $W_2(\beta)$, because in this case $W_1(0) \cong W_2(0)$. Every pre-Lie algebra that has an abelian Lie algebra associated is commutative and associative and so are $U_1, \ldots, U_5$. From the list of pre-Lie algebras with Lie algebra $\mathfrak{r}_2(\mathbb{C})$, $W_1(-1)$ and $W_2(1)$ are associative, where not one of those is commutative. This is also easy to see, because any commutative pre-Lie algebra must have the abelian Lie algebra associated. Furthermore we observe that W_4 is the only simple algebra here.

One of our goals in this section is, to get the Hasse diagram for complex Novikov algebras of dimension two as a starting point for the 3-dimensional case. In addition, the study of all orbit closures of two-dimensional pre-Lie algebras is very interesting for itself. The pre-Lie algebras which are Novikov algebras are:

$$U_1, U_2, U_3, U_4, U_5, W_2(\beta)_{\beta\in\mathbb{C}}, W_5$$

To complete our observations about the structural properties of 2-dimensional pre-Lie algebras, we consider the invariants $\mathfrak{C}_{\mathrm{p,q}}(A)$ for $p, q \in \mathbb{N}$. A short calculation shows that $\mathfrak{C}_{\mathrm{p,q}}(U_2) = \mathfrak{C}_{\mathrm{p,q}}(W_5) = 1$, $\mathfrak{C}_{\mathrm{p,q}}(U_4) = \mathfrak{C}_{\mathrm{p,q}}(W_3) = 2$ for all $p, q \geq 1$, and

$$\mathfrak{C}_{\mathrm{p,q}}(W_1(\alpha)) = \frac{(\alpha^p + (-1)^p)(\alpha^q + (-1)^q)}{\alpha^{p+q} + (-1)^{p+q}}, \tag{4.1}$$

$$\mathfrak{C}_{\mathrm{p,q}}(W_2(\beta)) = \frac{(\beta^p + (\beta-1)^p)(\beta^q + (\beta-1)^q)}{\beta^{p+q} + (\beta-1)^{p+q}}, \tag{4.2}$$

$$\mathfrak{C}_{\mathrm{p,q}}(W_4)) = \frac{(2^p + 1)(2^q + 1)}{2^{p+q} + 1}. \tag{4.3}$$

The invariants $\mathfrak{C}_{\mathrm{p,q}}(U)$ do not exist for the algebras U_1, U_3, and U_5.

4.1.2 Degenerations of pre-Lie algebras of dimension two

By following the classification process of degenerations we start with the most powerful semi-invariant we have got, the orbit dimension or equally, the vector space dimension of derivations. If we order the set of 2-dimensional pre-Lie algebras by $\dim \mathrm{Der}(\lambda)$ we gain the following ordering, starting with the smallest dimension on the left:[1]

$$U_3, W_4; \qquad U_2, U_4, W_1(\alpha)_{\alpha\neq-1}, W_2(\beta)_{\beta\neq1}, W_3, W_5; \qquad U_5, W_1(-1), W_2(1); \qquad U_1.$$

[1] In this ordering, algebras with a different orbit dimension are separated by a semicolon.

A further refinement can be obtained by using the fact, that a pre-Lie algebra with abelian Lie algebra cannot degenerate to a pre-Lie algebra with non-abelian Lie algebra (see Lemma 3.11). We begin our study with those two algebras, which have the largest orbit dimension.

Lemma 4.1. *The orbit closure of U_3 in* $\mathrm{preLie}_2(\mathbb{C})$ *up to isomorphism consists only of the following algebras:*

$$U_1, U_2, U_3, U_4, U_5.$$

Proof. Because of the remark above, the commutative algebra U_3 cannot degenerate to a non-commutative algebra. Furthermore, if we look at the list where we ordered all algebras by $\dim \mathrm{Der}(\lambda)$ we find no restriction for U_3. Indeed, we have $U_3 \to_{\mathrm{deg}} U_2$ by $g_t^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ t^2 & t \end{smallmatrix}\right)$ and $U_3 \to_{\mathrm{deg}} U_4$ with $g_t^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & t \end{smallmatrix}\right)$. Finally, $U_3 \to_{\mathrm{deg}} U_5$ by $g_t^{-1} = \left(\begin{smallmatrix} 2t^2 & 2t \\ 0 & 1 \end{smallmatrix}\right)$. $\square$

Lemma 4.2. *The orbit closure of W_4 in* $\mathrm{preLie}_2(\mathbb{C})$ *up to isomorphism contains exactly the following algebras:*

$$U_1, U_5, W_1(-2), W_2(-1), W_4$$

Proof. Like in the case of U_3, the orbit dimension of W_4, which equals four, just excludes U_3 from lying in the orbit closure of W_4. Therefore the list of possible degenerations consists of the following algebras:

$$U_1, U_2, U_5, W_1(\alpha), W_2(\beta), W_3, W_5.$$

The reason why W_4 cannot degenerate to U_2, U_4, W_3, W_5 is because of the $\mathfrak{C}_{\mathrm{p,q}}(\lambda)$-invariant. For those algebras we have constant values for all p and q, namely $\mathfrak{C}_{\mathrm{p,q}}(U_2) = \mathfrak{C}_{\mathrm{p,q}}(W_5) = 1$, $\mathfrak{C}_{\mathrm{p,q}}(U_4) = \mathfrak{C}_{\mathrm{p,q}}(W_3) = 2$, while $\mathfrak{C}_{\mathrm{p,q}}(W_4) = \frac{(2^p+1)(2^q+1)}{2^{p+q}+1}$ depends on p and q. Computing the value for small p and q we immediately find $\mathfrak{C}_{1,1}(W_4) = \frac{9}{5}$, what would contradict a degeneration.

Now assume that $W_4 \to_{\mathrm{deg}} W_1(\alpha)$. Comparing the $\mathfrak{C}_{\mathrm{p,q}}(\lambda)$-invariants for the algebras W_4 and $W_1(\alpha)$ yields, that the condition $(\alpha+2)(2\alpha+1) = 0$ must be satisfied. The two solutions of this equation are the only possible values, for which a degeneration could be achieved. A short calculation shows that, indeed $W_4 \to_{\mathrm{deg}} W_1(-2)$ by $g_t^{-1} = \left(\begin{smallmatrix} t & 0 \\ t & 1 \end{smallmatrix}\right)$. If we try to compute a matrix g_t for $W_4 \to_{\mathrm{deg}} W_1(-1/2)$, our efforts will fail. The reason for this is seen by Lemma 3.46. If $x = x_1e_1 + x_2e_2$, $y = y_1e_1 + y_2e_2$ then the left and right-multiplications of W_4 are given by

$$\mathrm{L}^{W_4}(x) = \begin{pmatrix} -x_2 & 0 \\ x_1 & -2x_1 \end{pmatrix}, \quad \mathrm{R}^{W_4}(x) = \begin{pmatrix} 0 & -x_1 \\ x_1 & -2x_1 \end{pmatrix},$$
$$\mathrm{L}^{W_4}(y) = \begin{pmatrix} -y_2 & 0 \\ y_1 & -2y_1 \end{pmatrix}, \quad \mathrm{R}^{W_4}(y) = \begin{pmatrix} 0 & -y_1 \\ y_1 & -2y_1 \end{pmatrix}.$$

Searching for quadratic operator identities $T(x,y) = 0$ for all $x, y \in W_4$, we find that all $T_{r,s}(x,y) = 0$ for all $r, s \in \mathbb{C}$, where

$$\begin{aligned} T_{r,s}(x,y) = {} & r(\mathrm{L}^{W_4}(x)\mathrm{R}^{W_4}(y) - \mathrm{L}^{W_4}(y)\mathrm{R}(x)) + s(\mathrm{R}(x)\mathrm{L}^{W_4}(y) - \mathrm{R}(y)\mathrm{L}^{W_4}(x)) \\ & + (s-3r)[\mathrm{L}^{W_4}(x), \mathrm{L}^{W_4}(y)] + \frac{1}{2}(r-2s)[\mathrm{R}(x), \mathrm{R}(y)]. \end{aligned}$$

For $r = s = -2$ we obtain

$$T(x,y) = [2\mathrm{L}(x) - \mathrm{R}(x), 2\mathrm{L}(y) - \mathrm{R}(y)] = 0.$$

On the other side, if we take this operator identity with the left and right-multiplications of $W_1(\alpha)$ instead, we see that

$$T(x,y) = \begin{pmatrix} 0 & (\alpha+2)(x_2y_1 - x_1y_2) \\ 0 & 0 \end{pmatrix}.$$

Hence only $W_4 \to_{\deg} W_1(-2)$ is possible.
Next, assume that $W_4 \to_{\deg} W_2(\beta)$. We want to know for which values of $\beta \in \mathbb{C}$ a degeneration exists. We calculate the quadratic operator $T_{r,s}(x,y)$ for $W_2(\beta)$ and find:

$$T_{r,s}(x,y) = \begin{pmatrix} 0 & 2(\beta+1)(x_1y_2 - x_2y_1) \\ 0 & 0 \end{pmatrix}.$$

Obviously we must have $\beta = -1$, in which case a degeneration exists by $g_t^{-1} = \left(\begin{smallmatrix} 1/2 & 0 \\ -1/2 & t \end{smallmatrix}\right)$. Finally $W_4 \to_{\deg} U_5$ by $g_t^{-1} = \left(\begin{smallmatrix} 2t & 0 \\ t & 3t^2 \end{smallmatrix}\right)$. $\square$

Theorem 4.3. *The orbit closures of* $\mathrm{preLie}_2(\mathbb{C})$ *are given as follows.*

λ	$\overline{\mathrm{O}(\lambda)} \setminus \mathrm{O}(\lambda)$
U_3	U_1, U_2, U_4, U_5
W_4	$U_1, U_5, W_1(-2), W_2(-1)$
U_2	U_1, U_5
U_4	U_1, U_5
$W_1(\alpha)_{\alpha\neq -1}$	U_1, U_5
$W_2(\beta)_{\beta\neq 1}$	U_1, U_5
W_3	$U_1, U_5, W_1(-1)$
W_5	$U_1, U_5, W_2(1)$
U_5	U_1
$W_1(-1)$	U_1
$W_2(1)$	U_1

Proof. First of all, every pre-Lie algebra of dimension two degenerates to U_1 because of example 1.3. The orbit closures of U_3 and U_4 have already been treated in the two lemmas before. Next we conclude, that U_2 can only degenerate to commutative algebras of orbit dimension less than three, where we just have U_5 as an option. Indeed, $U_2 \to_{\deg} U_5$ by $g_t^{-1} = \left(\begin{smallmatrix} t & 0 \\ 1 & -t \end{smallmatrix}\right)$. The same argument applies to U_4 for which we have $U_4 \to_{\deg} U_5$ by $g_t^{-1} = \left(\begin{smallmatrix} t & 0 \\ 1 & t \end{smallmatrix}\right)$.
The orbit dimension of $W_1(\alpha)$ with $\alpha \neq -1$ equals three, hence possible algebras in the closure are $U_5, W_1(-1)$ and $W_2(1)$. There is a degeneration $W_1(\alpha)_{\alpha\neq -1} \to_{\deg} U_5$ by $g_t^{-1} = \left(\begin{smallmatrix} t & 0 \\ 1 & t^2(\alpha+1) \end{smallmatrix}\right)$. One has to notice that the degenerations matrix becomes singular for the value $\alpha = -1$, which fits to the fact that the degeneration $W_1(-1) \to_{\deg} U_5$ is excluded by orbit dimension.

For $\beta \neq 0, 1$ assume that $W_2(\beta) \to_{\deg} W_2(1)$. This assumption forces $\mathfrak{C}_{1,1}(W_2(\beta)) = \mathfrak{C}_{1,1}(W_2(1))$, which yields

$$\frac{(2\beta-1)^2}{\beta^2 + (2\beta-1)^2} = 1,$$

or equivalently $\beta(\beta-1)=0$. Using the same argument $W_2(\beta)$ with $\beta \neq -1$ cannot degenerate to $W_1(-1)$. There are degenerations to U_5 for every $\beta \neq 1$ by

$$g_t^{-1} = \begin{pmatrix} 1 & 0 \\ t & t^2(\beta-1) \end{pmatrix}.$$

The algebra W_3 degenerates to U_5 by $g_t^{-1} = \begin{pmatrix} t^{-2} & 0 \\ 0 & t^{-1} \end{pmatrix}$, and to $W_1(-1)$ by $g_t^{-1} = \begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix}$. Because $\mathfrak{C}_{1,1}(W_3) = 2$ and $\mathfrak{C}_{1,1}(W_2(1)) = 1$, there is no degeneration from W_3 to $W_2(1)$.
The algebra W_5 degenerates to U_5 by $g_t^{-1} = \begin{pmatrix} t^{-2} & 0 \\ 0 & t^{-1} \end{pmatrix}$, and to $W_2(1)$ by

$$g_t^{-1} = \begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix}.$$

Comparing $\mathfrak{C}_{1,1}(W_5) = 1$ with $\mathfrak{C}_{1,1}(W_1(-1)) = 2$, we conclude that there is no degeneration from W_5 to $W_1(-1)$. All the remaining degenerations in the list follow from transitivity. $\square$

Corollary 4.4. *The Hasse diagram of degenerations in* $\mathrm{preLie}_2(\mathbb{C})$ *is given as follows.*

$$\begin{tikzcd}
 & U_3 \arrow[dl] \arrow[d] & W_4 \arrow[d, "\alpha=-2"'] \arrow[dr, "\beta=-1"] & & & \\
U_2 \arrow[drrr] & U_4 \arrow[drr] & W_1(\alpha)_{\alpha\neq -1} \arrow[dr] & W_2(\beta)_{\beta\neq 1} \arrow[d] & W_3 \arrow[dl] \arrow[d] & W_5 \arrow[dll] \arrow[d] \\
 & & & U_5 \arrow[d] & W_1(-1) \arrow[dl] & W_2(1) \arrow[dll] \\
 & & & U_1 & &
\end{tikzcd}$$

Corollary 4.5. *The Hasse diagram for degenerations of Novikov algebra structures in* $\mathrm{preLie}_2(\mathbb{C})$ *is given as follows:*

$$\begin{tikzcd}
 & U_3 \arrow[dl] \arrow[d] & & \\
U_2 \arrow[dr] & U_4 \arrow[d] & W_2(\beta)_{\beta\neq 1} \arrow[dl] & W_5 \arrow[dll] \arrow[dl] \\
 & U_5 \arrow[d] & W_2(1) \arrow[dl] & \\
 & U_1 & &
\end{tikzcd}$$

4.2 Classification of Novikov algebras in dimension three

The Hasse diagram of all pre-Lie algebra degenerations gives a refinement of the corresponding Hasse diagram for Lie algebras (Lemma 3.11). Because Novikov algebras form a subclass of pre-Lie algebras this is especially true for the Hasse diagram of all Novikov algebras. The next subsection lists all Lie algebras in dimension 3. Afterwards we list all Novikov algebras in dimension three.

4.2.1 Complex Lie algebras of dimension three

The classification of 3-dimensional Lie algebras is well known, see for example [46].

$\mathfrak{g}$	Lie brackets
$\mathfrak{g}_1 = \mathbb{C}^3$	$-$
$\mathfrak{g}_2 = \mathfrak{n}_3(\mathbb{C})$	$[e_1, e_2] = e_3$
$\mathfrak{g}_3 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2$
$\mathfrak{g}_4 = \mathfrak{r}_3(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3$
$\mathfrak{g}_5(\lambda) = \mathfrak{r}_{3,\lambda}(\mathbb{C})$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3, \lambda \in \mathbb{C} \setminus \{0\}$
$\mathfrak{g}_6 = \mathfrak{sl}_2(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_4] = -2e_1, [e_2, e_3] = 2e_2$

The Lie algebras $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ and $\mathfrak{r}_{3,\mu}(\mathbb{C})$ are isomorphic if and only if $\mu = \lambda^{-1}$, or $\mu = \lambda$. The isomorphisms $\varphi\colon \mathfrak{r}_{3,\lambda}(\mathbb{C}) \to \mathfrak{r}_{3,1/\lambda}(\mathbb{C})$ are given by $\varphi(e_1) = \lambda e_1, \varphi(e_2) = e_3$ and $\varphi(e_3) = e_2$.

We consider the Hasse diagram of Lie algebra degenerations of dimension 3 ([46]).

$$\begin{tikzcd}
 & & \mathfrak{g}_6 \arrow[d] & \\
\mathfrak{g}_4 \arrow[dr] \arrow[drr] & \mathfrak{g}_5(\lambda) \arrow[dr] & \mathfrak{g}_5(-1) \arrow[d] & \mathfrak{g}_3 \arrow[dl] \\
 & \mathfrak{g}_5(1) \arrow[dr] & \mathfrak{g}_2 \arrow[d] & \\
 & & \mathfrak{g}_1 &
\end{tikzcd}$$

4.2.2 Classification of complex Novikov algebras of dimension three

Now we classify all Novikov algebras $(A, \cdot)$ according to their associated Lie algebra $\mathfrak{g}_A$. In that sense, the diagram of Novikov algebras gives a refinement of the Lie algebra diagram. To abbreviate the fact that a Novikov algebra has associated Lie algebra $\mathfrak{g}_1 = \mathbb{C}^3$ we speak of an A-class Novikov algebra or of a Novikov algebra of type A. Every algebra of this type is clearly commutative and in addition associative. Indeed, left-symmetry can be rewritten in the form:

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z).$$

If the associated Lie algebra is now abelian, the left-multiplication operators of the Novikov algebra commute. Therefore

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - y \cdot (x \cdot z) = [y \cdot z, x] = 0.$$

The classification of those algebras can be found for example in [3]. Similar we call a Novikov algebra with associated Lie algebra $\mathfrak{g}_2$ of type B. Further on, with increasing number of the index we sign to every Lie algebra a letter in alphabetical order. The list ends with the E-class Novikov algebras, having $\mathfrak{g}_5(\lambda)$ as their associated Lie algebras. If we want to emphasize the parameter λ that is associated to a specific E-class algebra we sometimes write $E(\lambda)$.[2] We also mention that we treat the classes $E(-1)$ and $E(1)$ separately in our classification. There is no Novikov structure with Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. This notation turns out to be very handy and is due to Burde ([9]). From there we also took the list of all 3-dimensional Novikov algebras that is given below. For a complete classification of all Novikov algebras published in an article we refer to [4]. Both classifications are equivalent.

Remark 4.6. The following table includes all non-isomorphic Novikov algebras of dimension three. We mention that there are non-trivial isomorphisms for only two families of Novikov algebras. We have $B_5(\beta) \cong B_5(\widetilde{\beta})$ if and only if $\widetilde{\beta} \in \{\beta, 1 - \beta\}$ and $E_{1,\lambda}(\alpha) \cong E_{1,\widetilde{\lambda}}(\widetilde{\alpha})$ if and only if $(\widetilde{\lambda}, \widetilde{\alpha}) \in \{(\lambda, \alpha), (\frac{1}{\lambda}, \frac{\alpha}{\lambda})\}$.

[2]Moreover we warn that this λ is written in the lower index of the algebras in class E.

A	Products	$\mathfrak{g}_A$	$\dim \mathrm{Der}(A)$
A_1	$-$	$\mathbb{C}^3$	9
A_2	$e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	4
A_3	$e_2 \cdot e_2 = e_2,\ e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	1
A_4	$e_1 \cdot e_1 = e_1,\ e_2 \cdot e_2 = e_2,$ $e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	0
A_5	$e_2 \cdot e_2 = e_1$	$\mathbb{C}^3$	5
A_6	$e_2 \cdot e_2 = e_1,\ e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	2
A_7	$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_2$	$\mathbb{C}^3$	2
A_8	$e_1 \cdot e_2 = e_1,\ e_2 \cdot e_1 = e_1,$ $e_2 \cdot e_2 = e_2,\ e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	1
A_9	$e_2 \cdot e_3 = e_1,\ e_3 \cdot e_2 = e_1$	$\mathbb{C}^3$	4
A_{10}	$e_2 \cdot e_3 = e_1,\ e_3 \cdot e_2 = e_1,$ $e_3 \cdot e_3 = e_2$	$\mathbb{C}^3$	3
A_{11}	$e_1 \cdot e_3 = e_1,\ e_2 \cdot e_3 = e_2,$ $e_3 \cdot e_1 = e_1,\ e_3 \cdot e_2 = e_2,$ $e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	4
A_{12}	$e_1 \cdot e_3 = e_1,\ e_2 \cdot e_2 = e_1,$ $e_2 \cdot e_3 = e_2,\ e_3 \cdot e_1 = e_1,$ $e_3 \cdot e_2 = e_2,\ e_3 \cdot e_3 = e_3$	$\mathbb{C}^3$	2
B_1	$e_1 \cdot e_1 = e_1,\ e_1 \cdot e_2 = e_2 + e_3,$ $e_1 \cdot e_3 = e_3,\ e_2 \cdot e_1 = e_2,$ $e_3 \cdot e_1 = e_3$	$\mathfrak{n}_3(\mathbb{C})$	2
B_2	$e_1 \cdot e_1 = e_1,\ e_1 \cdot e_2 = e_2 + e_3,$ $e_1 \cdot e_3 = e_3,\ e_2 \cdot e_1 = e_2,$ $e_2 \cdot e_2 = e_3,\ e_3 \cdot e_1 = e_3$	$\mathfrak{n}_3(\mathbb{C})$	1
B_3	$e_1 \cdot e_2 = \frac{1}{2}e_3,\ e_2 \cdot e_1 = -\frac{1}{2}e_3,$ $e_2 \cdot e_2 = e_3$	$\mathfrak{n}_3(\mathbb{C})$	4
$B_4(\alpha)$	$e_1 \cdot e_2 = \alpha e_3,\ e_2 \cdot e_1 = (\alpha - 1)e_3,$ $e_2 \cdot e_2 = e_1$	$\mathfrak{n}_3(\mathbb{C})$	3
$B_5(\beta)$ $\beta = x + iy$	$e_1 \cdot e_2 = \beta e_3,\ e_2 \cdot e_1 = (\beta - 1)e_3$ $x \leq 1/2$	$\mathfrak{n}_3(\mathbb{C})$	$4, \beta \neq \frac{1}{2}$ $6, \beta = \frac{1}{2}$
C_1	$e_1 \cdot e_1 = -e_1 + e_2,\ e_2 \cdot e_1 = -e_2,$ $e_3 \cdot e_3 = e_3$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	1
C_2	$e_1 \cdot e_1 = -e_1 + e_2,\ e_1 \cdot e_3 = -e_3,$ $e_2 \cdot e_1 = -e_2,\ e_3 \cdot e_1 = -e_3$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	2
C_3	$e_1 \cdot e_1 = -e_1 + e_2,\ e_2 \cdot e_1 = -e_2$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	2
C_4	$e_1 \cdot e_1 = e_3,\ e_1 \cdot e_2 = e_2$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	2

A	Products	$\mathfrak{g}_A$	$\dim \mathrm{Der}(A)$
$C_5(\alpha)$	$e_1 \cdot e_1 = \alpha e_1,\ e_1 \cdot e_2 = (\alpha+1)e_2,$ $e_2 \cdot e_1 = \alpha e_2$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$2, \alpha \neq 0, -1$ $3, \alpha = 0, -1$
$C_6(\beta)$	$e_1 \cdot e_1 = \beta e_1,\ e_1 \cdot e_2 = (\beta+1)e_2,$ $e_2 \cdot e_1 = \beta e_2,\ e_3 \cdot e_3 = e_3$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$1, \beta \neq -1$ $2, \beta = -1$
$C_7(\gamma)$ $\gamma \neq 0$	$e_1 \cdot e_1 = \gamma e_1,\ e_1 \cdot e_2 = (\gamma+1)e_2,$ $e_1 \cdot e_3 = \gamma e_3,\ e_2 \cdot e_1 = \gamma e_2,$ $e_3 \cdot e_1 = \gamma e_3$	$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$2, \gamma \neq -1$ $3, \gamma = -1$
D_1	$e_1 \cdot e_1 = -e_1 + e_3,\ e_1 \cdot e_3 = e_2,$ $e_2 \cdot e_1 = -e_2,\ e_3 \cdot e_1 = -e_3$	$\mathfrak{r}_3(\mathbb{C})$	2
$D_2(\alpha)$	$e_1 \cdot e_1 = \alpha e_1,\ e_1 \cdot e_2 = (\alpha+1)e_2,$ $e_1 \cdot e_3 = e_2 + (\alpha+1)e_3,\ e_2 \cdot e_1 = \alpha e_2,$ $e_3 \cdot e_1 = \alpha e_3$	$\mathfrak{r}_3(\mathbb{C})$	$2, \alpha \neq -1$ $3, \alpha = -1$
$E_{1,\lambda}(\alpha)$ $\lambda \neq 0$	$e_1 \cdot e_1 = \alpha e_1,\ e_1 \cdot e_2 = (\alpha+1)e_2,$ $e_1 \cdot e_3 = (\alpha+\lambda)e_3,\ e_2 \cdot e_1 = \alpha e_2,$ $e_3 \cdot e_1 = \alpha e_3$	$\mathfrak{r}_{3,\lambda}(\mathbb{C})$	$2, \lambda \neq 1,$ $\alpha \neq -1, -\lambda$ $3, \lambda \neq 1, \alpha = -\lambda$ $3, \lambda \neq 1, \alpha = -1$ $4, \lambda = 1, \alpha \neq -1$ $6, \lambda = 1, \alpha = -1$
$E_{2,\lambda}$ $\lambda \neq 0$	$e_1 \cdot e_1 = -e_1 + e_2,\ e_1 \cdot e_3 = (\lambda-1)e_3,$ $e_2 \cdot e_1 = -e_2,\ e_3 \cdot e_1 = -e_3$	$\mathfrak{r}_{3,\lambda}(\mathbb{C})$	$2, \lambda \neq 1$ $4, \lambda = 1$
E_3	$e_1 \cdot e_1 = -\frac{1}{2}e_1 + e_3,\ e_1 \cdot e_2 = \frac{1}{2}e_2,$ $e_1 \cdot e_3 = e_2,\ e_2 \cdot e_1 = -\frac{1}{2}e_2,$ $e_3 \cdot e_1 = e_2 - \frac{1}{2}e_3$	$\mathfrak{r}_{3,\frac{1}{2}}(\mathbb{C})$	1
E_4	$e_1 \cdot e_1 = -e_1 + e_2,\ e_1 \cdot e_3 = -\frac{1}{2}e_3,$ $e_2 \cdot e_1 = -e_2,\ e_3 \cdot e_1 = -e_3,$ $e_3 \cdot e_3 = e_2$	$\mathfrak{r}_{3,\frac{1}{2}}(\mathbb{C})$	1
$E_5(\beta)$	$e_1 \cdot e_1 = \beta e_1,\ e_1 \cdot e_2 = (\beta+1)e_2,$ $e_1 \cdot e_3 = (\beta+\frac{1}{2})e_3,\ e_2 \cdot e_1 = \beta e_2,$ $e_3 \cdot e_1 = \beta e_3,\ e_3 \cdot e_3 = e_2$	$\mathfrak{r}_{3,\frac{1}{2}}(\mathbb{C})$	$1, \beta \neq -1$ $2, \beta = -1$
E_6	$e_1 \cdot e_1 = -\frac{1}{2}e_1,\ e_1 \cdot e_2 = \frac{1}{2}e_2,$ $e_1 \cdot e_3 = e_2,\ e_2 \cdot e_1 = -\frac{1}{2}e_2,$ $e_3 \cdot e_1 = e_2 - \frac{1}{2}e_3$	$\mathfrak{r}_{3,\frac{1}{2}}(\mathbb{C})$	2

4.3 Degenerations of the classes A, D, and E(1)

Before we start we mention that all degenerations and non-degenerations are collected in tables in Appendix B. They have proved to be very handy in getting a quick overview of the orbit closure of a certain algebra.

Concerning the Hasse diagrams we remark that only algebras that are involved in a degeneration can occur. We use this convention to make the diagrams more readable.

4.3.1 Degenerations of Novikov algebras with abelian Lie algebra

Proposition 4.7. *The Hasse diagram of all* 3*-dimensional Novikov algebras with abelian Lie algebra, that arise from a* 2*-dimensional Novikov algebra by adding a* 1*-dimensional ideal, is given as follows.*

$$
\begin{tikzcd}
 & A_4 \arrow[dl] \arrow[d] \\
A_3 \arrow[d] \arrow[dr] & A_8 \arrow[dl] \arrow[d] \\
A_6 \arrow[d] & A_7 \arrow[dd] \\
A_2 \arrow[dr] & \\
 & A_5 \arrow[d] \\
 & A_1
\end{tikzcd}
$$

Proof. This Proposition is an immediate consequence of Corollary 3.3. We recall Corollary 4.5 and consider the following correspondences of algebras

$$
\begin{aligned}
A_1 &= U_1 \oplus \mathbb{C}, & A_2 &= U_1 \oplus P_2, \\
A_2 &= U_2 \oplus \mathbb{C}, & A_3 &= U_2 \oplus P_2, \\
A_3 &= U_3 \oplus \mathbb{C}, & A_4 &= U_3 \oplus P_2, \\
A_5 &= U_5 \oplus \mathbb{C}, & A_6 &= U_5 \oplus P_2, \\
A_7 &= U_4 \oplus \mathbb{C}, & A_8 &= U_4 \oplus P_2.
\end{aligned}
$$

□

Proposition 4.8. *The orbit closures of all complete algebras with abelian Lie algebra are given by the following diagram*

$$A_{10} \longrightarrow A_9 \longrightarrow A_5 \longrightarrow A_1$$

Proof. Because of Corollary 3.52 only degenerations to other complete algebras are possible. By transitivity we have $A_{10} \to_{\mathrm{deg}} A_5$ together with:

$$A_9 \to_{\mathrm{deg}} A_5 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & \frac{-1}{2t^2} & 1 \end{pmatrix} \text{ and}$$

$$A_{10} \to_{\mathrm{deg}} A_9 \text{ by } g_t = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Proposition 4.9. *The orbit closures of all* 3*-dimensional Novikov algebras with associated abelian Lie algebra are listed in the table below.*

A	$\overline{\mathrm{O}(A)} \setminus \mathrm{O}(A)$
A_1	$-$
A_2	A_1, A_5
A_3	$A_1, A_2, A_5, A_6, A_7, A_9, A_{10}$
A_4	$A_1, A_2, A_3, A_5, \ldots, A_{12}$
A_5	A_1
A_6	$A_1, A_2, A_5, A_9, A_{10}$
A_7	A_1, A_5, A_9, A_{10}
A_8	$A_1, A_2, A_5, A_6, A_7, A_9, \ldots, A_{12}$
A_9	A_1, A_5
A_{10}	A_1, A_5, A_9
A_{11}	A_1, A_5
A_{12}	$A_1, A_5, A_9, A_{10}, A_{11}$

Proof. A lot of possible degenerations are excluded by Theorem 1.16. We therefore order the algebras of class A by their dimensions of the orbit space:[3]

$$A_4; \quad A_3, A_8; \quad A_6, A_7, A_{12}; \quad A_{10}; \quad A_2, A_9, A_{11}; \quad A_5; \quad A_1.$$

If we consider this ordering all algebras can only degenerate from the left to the right, so we derive that for the algebras A_2, A_9, and A_{11} it is impossible to degenerate to any other algebra than A_5 (and trivially A_1). We have already seen that $A_2 \to_{\mathrm{deg}} A_5$ by Proposition 4.7 and $A_9 \to_{\mathrm{deg}} A_5$ by Proposition 4.8. We also have

$$A_{11} \to_{\mathrm{deg}} A_5 \text{ by } g_t = \begin{pmatrix} 1 & 0 & \frac{-1}{t^4} \\ 0 & 1 & 0 \\ 0 & \frac{1}{t} & \frac{1}{t^3} \end{pmatrix}.$$

The complete orbit closure of the algebra A_{10} is given by Proposition 4.8. We go on with the algebras A_6, A_7, and A_{12}. All these algebras degenerate to A_{10} and by

[3] The dimensions of the orbit space decrease from the left to the right. Algebras with different orbit dimensions are separated by a semicolon.

transitivity to A_9 and A_5:

$$A_6 \to_{\text{deg}} A_{10} \text{ by } g_t = \begin{pmatrix} -t & 1 & \frac{1}{t^3} \\ 0 & 1 & \frac{1}{t^2} \\ 0 & 0 & \frac{1}{t} \end{pmatrix},$$
$$A_7 \to_{\text{deg}} A_{10} \text{ by } g_t = \begin{pmatrix} 1 & \frac{2}{t^3} & \frac{1}{t^3} \\ t & \frac{3}{t^2} & \frac{2}{t} \\ 0 & 0 & 1 \end{pmatrix}, \text{ and}$$
$$A_{12} \to_{\text{deg}} A_{10} \text{ by } g_t = \begin{pmatrix} -it & 0 & -\frac{i}{t^3} \\ 0 & 1 & \frac{1}{t^2} \\ 0 & -it & 0 \end{pmatrix}.$$

In addition we have a degeneration from A_{12} to A_{11} by:

$$A_{12} \to_{\text{deg}} A_{11} \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To show that there are no other degenerations for the algebras A_6, A_7, and A_{12} we use the $\mathfrak{C}_{\mathrm{p,q}}$-invariant. We have:

$$\begin{aligned} &\mathfrak{C}_{\mathrm{p,q}}(A_2) = \mathfrak{C}_{\mathrm{p,q}}(A_6) = 1, \\ &\mathfrak{C}_{\mathrm{p,q}}(A_7) = 2, \text{ and} \\ &\mathfrak{C}_{\mathrm{p,q}}(A_{11}) = \mathfrak{C}_{\mathrm{p,q}}(A_{12}) = 3. \end{aligned}$$

Therefore A_6 can't degenerate to A_{11}, neither does A_7 to A_2 or A_{11}, and also for A_{12} a degeneration to A_2 is impossible.

The orbit closure of the algebra A_3 is nearly determined by Proposition 4.7. We only have to notice all the other degenerations which are already given by transitivity and the fact that A_3 has a non-trivial center, whereas A_{11} doesn't. Therefore A_3 can't degenerate to A_{11} by Lemma 3.30. By transitivity a degeneration from A_3 to A_{12} is also impossible.

There exist degenerations $A_8 \to_{\text{deg}} A_6$ and $A_8 \to_{\text{deg}} A_7$ by Proposition 4.7, furthermore:

$$A_8 \to_{\text{deg}} A_{12} \text{ by } g_t = \begin{pmatrix} 1 & -\frac{1}{t^2} & \frac{1}{t^2} \\ 0 & -\frac{1}{t} & \frac{1}{t} \\ 0 & 1 & 0 \end{pmatrix}.$$

Finally we mention Proposition 4.7 once more to see that the algebra A_4 degenerates to every algebra in class A. This completes the proof. □

Corollary 4.10. *The Hasse diagram of all commutative and associative Novikov algebras is given as follows.*

$$\begin{tikzcd}
& A_4 \arrow[dl] \arrow[d] & \\
A_3 \arrow[d] \arrow[dr] & A_8 \arrow[dl] \arrow[d] \arrow[dr] & \\
A_6 \arrow[dr] \arrow[dd] & A_7 \arrow[d] & A_{12} \arrow[dl] \arrow[dd] \\
& A_{10} \arrow[d] & \\
A_2 \arrow[dr] & A_9 \arrow[d] & A_{11} \arrow[dl] \\
& A_5 \arrow[d] & \\
& A_1 &
\end{tikzcd}$$

4.3.2 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_4$

Proposition 4.11. *The possible orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_4$ are listed in the table below.*

D	$\overline{\mathrm{O}(D)} \setminus \mathrm{O}(D)$
D_1	$D_2(-1)$
$D_2(-1)$	$-$
$D_2(\alpha)_{\alpha\neq 0,-1}$	$-$

The Hasse diagram consists of only one non-trivial degeneration.

$$\begin{tikzcd}
D_1 \arrow[d] \\
D_2(-1)
\end{tikzcd}$$

Proof. In matters of orbit dimension there can only be degenerations from D_1 and $D_2(\alpha)_{\alpha\neq -1}$ to $D_2(-1)$. In case of a degeneration $D_2(\alpha)_{\alpha\neq-1} \to_{\mathrm{deg}} D_2(-1)$, necessarily

$$\mathfrak{C}_{\mathrm{p,q}}(D_2(\alpha)) = \frac{(\alpha^p + 2(\alpha+1)^p)(\alpha^q + 2(\alpha+1)^q)}{\alpha^{p+q} + 2(\alpha+1)^{p+q}}$$

has to be equal to 1 for all $p, q \in \mathbb{N}$. Setting $p = q = 1$ leads to the condition

$$6\alpha^2 + 8\alpha + 2 = 0.$$

Therefore $\mathfrak{C}_{1,1}(D_2(\alpha)) = 1$ only for $\alpha = -1, -\frac{1}{3}$. While $\alpha = -1$ was already excluded at the beginning, $\mathfrak{C}_{\mathrm{p,q}}(D_2(-\frac{1}{3}))$ is clearly a non-constant function in p and q. For example we have $\mathfrak{C}_{2,2}(D_2(-\frac{1}{3})) = \frac{81}{32}$. There is a degeneration

$$D_1 \to_{\mathrm{deg}} D_2(-1) \text{ by } g_t^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}. \qquad \square$$

4.3.3 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(1)$

Proposition 4.12. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(1)$ are listed in the table below.*

E	$\overline{\mathrm{O}(E)} \setminus \mathrm{O}(E)$
$E_{1,1}(0)$	$-$
$E_{1,1}(-1)$	$-$
$E_{1,1}(\alpha)_{\alpha\neq 0,-1}$	$-$
$E_{2,1}$	$E_{1,1}(-1)$

The Hasse diagram consists of only one proper degeneration.

$$\begin{tikzcd} E_{2,1} \arrow[d] \\ E_{1,1}(-1) \end{tikzcd}$$

Proof. Regarding the dimension of derivations, the only possible degenerations are $E_{1,1}(\alpha)_{\alpha\neq -1}, E_{2,1} \to_{\mathrm{deg}} E_{1,1}(-1)$. First, suppose that $E_{1,1}(\alpha)_{\alpha\neq -1} \to_{\mathrm{deg}} E_{1,1}(-1)$. In this case we must have

$$\mathfrak{C}_{\mathrm{p,q}}(E_{1,1}(\alpha)_{\alpha\neq -1}) = \frac{(\alpha^p + 2(\alpha+1)^p)(\alpha^q + 2(\alpha+1)^q)}{\alpha^{p+q} + 2(\alpha+1)^{p+q})} = 1 = \mathfrak{C}_{\mathrm{p,q}}(E_{1,1}(-1)).$$

This is the same condition for α, that we had in the previous section. Therefore we conlude that there are no proper degenerations for any α. Finally, we find

$$E_{2,1} \to_{\mathrm{deg}} E_{1,1}(-1) \text{ by } g_t^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

4.3.4 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_4$ and $\mathfrak{g}_1$

Proposition 4.13. *The commutative and associative Novikov algebras A_9 and A_{10} don't lie in any orbit closure of the Novikov algebras D_1 and $D_2(\alpha)$ for any $\alpha \in \mathbb{C}$.*

Proof. To prove this proposition we use the technique developed in Theorem 3.8. Due to Proposition 3.6 we know that ideals have to be preserved under degeneration. If we look at the algebra D_1 we find a two-dimensional right ideal spanned by the basis vectors e_2 and e_3. Searching for a possible degeneration, Theorem 3.8 suggests to shift this ideal by a change of basis to form the first two basis vectors. We obtain such an isomorphic algebra $\widetilde{D_1} := g \cdot D_1$ by

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The multiplication laws of $\widetilde{D_1}$ are then given by

$$\langle e_1 \cdot e_3 = -e_1, e_2 \cdot e_3 = -e_2, e_3 \cdot e_3 = e_1 - e_3, \text{ and } e_3 \cdot e_1 = e_2 \rangle.$$

The matrix representation of $\widetilde{D_1}$ is then given as follows:

$$\mathrm{L}^{\widetilde{D_1}}(x) = \begin{pmatrix} 0 & 0 & x_3 - x_1 \\ x_3 & 0 & -x_2 \\ 0 & 0 & -x_3 \end{pmatrix}.$$

We now see[4] that $\langle e_1, e_2 \rangle$ spans a two-dimensional ideal of $\widetilde{D_1}$. Clearly,

$$\overline{\mathrm{GL}_n(\mathbb{C}) \cdot D_1} = \overline{\mathrm{GL}_n(\mathbb{C}) \cdot \widetilde{D_1}}$$

and from Theorem 1.17 we know that the orbit closure of any algebra up to an isomorphism can be done with an upper triangular matrix $B_\varepsilon \in \mathrm{B}_n(\mathbb{C}((\varepsilon)))$:

$$B_\varepsilon := \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix},$$

where the coefficients b_i for $1 \leqslant i \leqslant 6$ are dependent of the degeneration parameter ε.

We know that every algebra in the closure has to be isomorphic to some algebra in $\overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}$. What we are going to do is to determine the orbit $\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}$ in terms of the left-multiplication operators $\mathrm{L}^{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}(x)$ that depend on the coefficients $b_1, \ldots, b_6$. Because over the field of complex numbers the orbit closures of the Zariski topology coincides with the closures of the standard topology (Example 1.4), the process of orbit closure leads to the componentwise convergence of the matrices $\mathrm{L}^{B_\varepsilon \cdot \widetilde{D_1}}(x)$. Hence, due to the convergence of every coefficient in $\mathrm{L}^{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}(x)$ we can hopefully derive some assertions with respect to the limits of the coefficients b_i. In fact, because of the special way the coefficients $b_1, \ldots, b_6$ are arranged in the left-multiplication of an arbitrary algebra in $\overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}$ we can write the set $\overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}$ as the union of two disjoint subsets. We finally show that the algebras A_9 and A_{10} don't lie in any of these two subsets.

This is done in the following way: We first compute the left-multiplication of an arbitrary algebra in the $\mathrm{B}_n(\mathbb{C})$-orbit

$$\mathrm{O}_B(\widetilde{D_1}) := \mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}.$$

The left-multiplication by an element $x \in \mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}$ is given by the left-multiplication of the basis vectors: $\mathrm{L}^{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}(x) = \sum_{i=1}^{3} x_i \mathrm{L}^{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}(e_i)$. We have

$$\mathrm{L}^{\mathrm{O}_B(\widetilde{D_1})}(e_1) = \begin{pmatrix} 0 & 0 & -\frac{1}{b_6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{L}^{\mathrm{O}_B(\widetilde{D_1})}(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{b_6} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{L}^{\mathrm{O}_B(\widetilde{D_1})}(e_3) = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ 0 & 0 & -\frac{1}{b_6} \end{pmatrix},$$

[4] Compare with Lemma 2.39.

where

$$f_1 = \tfrac{b_2}{b_1 b_6}, f_2 = -\tfrac{b_2}{b_1 b_4 b_6}, f_3 = \tfrac{b_1^2 b_4 - b_2 b_3 b_4 + b_2^2 b_5}{b_1 b_4 b_6^2},$$
$$f_4 = \tfrac{b_4}{b_1 b_6}, f_5 = -\tfrac{b_2}{b_1 b_6}, \text{ and } f_6 = \tfrac{-b_3 b_4 + b_2 b_5}{b_1 b_6^2}.$$

Likewise, for the right-multiplication operator $\mathrm{R}^{\mathrm{B}_n(\mathbb{C})\cdot\widetilde{D_1}}(x) = \sum_{i=1}^3 x_i \mathrm{R}^{\mathrm{B}_n(\mathbb{C})\cdot\widetilde{D_1}}(e_i)$ we have

$$\mathrm{R}^{\mathrm{O}_B(\widetilde{D_1})}(e_1) = \begin{pmatrix} 0 & 0 & f_1 \\ 0 & 0 & f_4 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mathrm{R}^{\mathrm{O}_B(\widetilde{D_1})}(e_2) = \begin{pmatrix} 0 & 0 & f_2 \\ 0 & 0 & f_5 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mathrm{R}^{\mathrm{O}_B(\widetilde{D_1})}(e_3) = \begin{pmatrix} -\frac{1}{b_6} & 0 & f_3 \\ 0 & -\frac{1}{b_6} & f_6 \\ 0 & 0 & -\frac{1}{b_6} \end{pmatrix}.$$

Now, regarding a degeneration exists, for $\lim_{\varepsilon\to 0} \frac{1}{b_6}$ only two things can happen.

First, the limit of $\frac{1}{b_6}$ is non-zero under which circumstance the right-multiplication $\mathrm{R}^{\overline{\mathrm{B}_n(\mathbb{C})\cdot\widetilde{D_1}}}(x)$ by an arbitrary element $x \in \lim_{\varepsilon\to 0} B_\varepsilon \cdot \widetilde{D_1}$, understood as an operator is invertible. This property is an isomorphism invariant, hence every algebra in the orbit closure of $\widetilde{D_1}$ has this property. The operators $\mathrm{R}^{A_9}(x)$ and $\mathrm{R}^{A_{10}}(x)$ are not invertible, for which reason a degeneration with $\lim_{\varepsilon\to 0} \frac{1}{b_6} \neq 0$ can't lead to the algebras A_9 and A_{10}.

Second, the limit $\lim_{\varepsilon\to 0} \frac{1}{b_6}$ equals zero. In this case $L^{\overline{\mathrm{B}_n(\mathbb{C})\cdot\widetilde{D_1}}}(x_1 e_1 + x_2 e_2) = 0$ so $\langle e_1, e_2\rangle$ defines the right-annihilator of any algebra in the closure of $\widetilde{D_1}$ undertaken by an upper triangular matrix with the above condition on b_6. As can easily be seen the algebras A_9 and A_{10} don't have a two-dimensional right-annihilator and so a degeneration with $\lim_{\varepsilon\to 0} \frac{1}{b_6} = 0$ can't yield the algebras A_9 and A_{10}.

In conclusion, we decomposed the closure $\overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{D_1}}$ into two components, which are obviously disjoint. We took the isomorphy classes of those components and showed that the algebras A_9 and A_{10} don't lie in any of this classes. Therefore the algebras A_9 and A_{10} can't lie in the orbit closure of the algebra D_1.

For $D_2(\alpha)_{\alpha\neq 0}$ a similar argument can be carried out. A basis change with the same matrix g as above leads to the same conditions for a degeneration. In the case where $\alpha = 0$ the situation changes just a little bit. Now, any algebra in the orbit closure will be complete, but fortunately it will also have a two-dimensional right-annihilator. Therefore a degeneration from $D_2(\alpha)$ for any $\alpha \in \mathbb{C}$ to A_9 or A_{10} is impossible. □

Remark 4.14. We are going to use the argument we developed in the last proposition several times throughout this chapter. It is always useable when the $\mathrm{B}_n(\mathbb{C})$-orbit decomposes into a subset consisting only of non-complete algebras and a subset of algebras with a 2-dimensional right-annihilator. We will demonstrate this procedure once more in Proposition 4.18 but from there on we will simply refer back to this proposition so we don't have to bring the whole argument again.

Lemma 4.15. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_4$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_1$ are given as follows.*

D	$\overline{\mathrm{O}(D)}$
D_1	A_1, A_5
$D_2(0)$	A_1, A_5
$D_2(-1)$	A_1, A_5
$D_2(\alpha)_{\alpha\neq 0,-1}$	A_1, A_5

Including the degenerations we already got, the Hasse diagram looks as follows.

$$\begin{tikzcd}
D_2(0) \arrow[ddr] & D_1 \arrow[d] & D_2(\alpha)_{\alpha\neq 0,-1} \arrow[ddl] \\
 & D_2(-1) \arrow[d] & \\
 & A_5 \arrow[d] & \\
 & A_1 &
\end{tikzcd}$$

Proof. Regarding the dimensions of derivations and Proposition 4.13 only the following algebras can possibly lie in the orbit closure of a Novikov algebra from class D:

$$A_1, A_2, A_5, A_{11}.$$

The only exeption to mention here is A_{10} for $D_2(-1)$. The invariant $\mathfrak{C}_{\mathrm{p,q}}(A_{11}) = 3$ prevents A_{11} from lying in any orbit closure in this case here. Because $\mathfrak{C}_{\mathrm{p,q}}(D_1) = 1$ a degeneration from D_1 to A_{11} is excluded at once. For A_{11} lying in the orbit closure of $D_2(\alpha)$ the equation

$$\frac{(\alpha^p + 2(\alpha+1)^p)(\alpha^q + 2(\alpha+1)^q)}{\alpha^{p+q} + 2(\alpha+1)^{p+q}} = 3$$

would have a solution for some α. Instead, this is not possible for any $\alpha \in \mathbb{C}$.

Moreover $D_2(\alpha)_{\alpha\neq -1}$ cannot degenerate to A_2 because of the $\mathfrak{C}_{\mathrm{p,q}}$-invariant again. To exclude a degeneration from D_1 and $D_2(-1)$ to A_2 we use Lemma 3.14. If we can show that degenerations of the associated algebras $\mathfrak{j}_{D_1}$ and $\mathfrak{j}_{D_2(-1)}$ to $\mathfrak{j}_{A_2}$ are impossible, then degenerations of the corresponding Novikov algebras are impossible too. Therefore we compute:

$$\mathfrak{C}_{\mathrm{p,q}}(\mathfrak{j}_{D_1}) = \mathfrak{C}_{\mathrm{p,q}}(\mathfrak{j}_{D_2(-1)}) = \frac{((-2)^p + 2(-1)^p)((-2)^q + 2(-1)^q)}{(-2)^{p+q} + 2(-1)^{p+q}},$$

$$\mathfrak{C}_{\mathrm{p,q}}(\mathfrak{j}_{A_2}) = 1.$$

Clearly the first of the above expressions is a non-constant function in p and q and hence we are done.

Although we could use standard methods to exclude a degeneration from $D_2(-1)$ to A_9 and A_{10} we proof it by (α, β, γ)-derivations to demonstrate an application of

Lemma 3.34. For this we compute the dimensions of the vectorspaces $\mathrm{Der}_{(1,1,0)}(A) = \{D \in \mathrm{End}(A) \mid D(x \cdot y) = D(x) \cdot y\}$:

$$\begin{aligned}
&\dim \mathrm{Der}_{(1,1,0)}(D_2(-1)) = 5,\\
&\dim \mathrm{Der}_{(1,1,0)}(A_9) = 3,\\
&\dim \mathrm{Der}_{(1,1,0)}(A_{10}) = 3.
\end{aligned}$$

As Lemma 3.34 says, these dimensions have to equal or increase within the orbit closure. Our computation shows us, that the contrary is the case.

The only non-trivial degenerations from class D to class A concern the algebra A_5. We note first that we have degenerations from D_1 and $D_2(-1)$ to A_5 by transitivity. To anticipate the statements of Lemma 4.16 and Lemma 4.17 we remark that $E_{2,1} \to_{\mathrm{deg}} A_5$ and $D_2(-1) \to_{\mathrm{deg}} E_{2,1}$. Therefore we can build the sequence:

$$D_1 \to_{\mathrm{deg}} D_2(-1) \to_{\mathrm{deg}} E_{2,1} \to_{\mathrm{deg}} A_5.$$

Furthermore we have a degeneration from $D_2(\alpha)_{\alpha \neq -1}$ to A_5 by the matrix

$$g_t = \begin{pmatrix} -\frac{\alpha+1}{t^2} & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One has to notice, that because of the parameter α there are infinitely many algebras and therefore infinitely many degenerations. We see that in this degeneration matrix α can take any value except $\alpha = -1$. □

4.3.5 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_5(1)$ and $\mathfrak{g}_1$

Lemma 4.16. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(1)$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_1$ are listed in the table below.*

E	$\overline{\mathrm{O}(E)}$
$E_{1,1}(-1)$	A_1
$E_{1,1}(\alpha)_{\alpha\neq-1}$	A_1, A_5
$E_{2,1}$	A_1, A_5

Including the degenerations we already got, the Hasse diagram looks as follows.

$$\begin{tikzcd}
E_{1,1}(\alpha) \arrow[d] & E_{2,1} \arrow[dl] \arrow[dd] \\
A_5 \arrow[dd] & \\
 & E_{1,1}(-1) \arrow[dl] \\
A_1 &
\end{tikzcd}$$

Proof. Because of the small orbit dimensions of the algebras of type E, the classification of degenerations from this class to the A-class is an easy matter. We know that every Novikov algebra degenerates to the trivial algebra A_1. The algebras that could contain A_5 in their orbit closure are $E_{1,1}(\alpha)$ and $E_{2,1}$. Both is possible:

$$E_{1,1}(\alpha) \rightarrow_{\text{deg}} A_5 \text{ by } g_t^{-1} = \begin{pmatrix} -\frac{\alpha+1}{t^2} & t & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_{2,1} \rightarrow_{\text{deg}} A_5 \text{ by } g_t^{-1} = \begin{pmatrix} -\frac{1}{t^2} & \frac{1}{t^2} & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.3.6 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_4$ and $\mathfrak{g}_5(1)$

Lemma 4.17. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_4$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_5(1)$ are as follows.*

D	$\overline{\mathrm{O}(D)}$
D_1	$E_{1,1}(-1), E_{2,1}$
$D_2(-1)$	$E_{1,1}(-1), E_{2,1}$
$D_2(\alpha)_{\alpha\neq-1}$	$E_{1,1}(\alpha)$

Including the degenerations we already got, the Hasse diagram looks as follows.

$$\begin{tikzcd} D_1 \arrow[d] & D_2(\alpha)_{\alpha\neq-1} \arrow[dd] \\ D_2(-1) \arrow[d] & \\ E_{2,1} \arrow[d] & E_{1,1}(\alpha)_{\alpha\neq-1} \\ E_{1,1}(-1) & \end{tikzcd}$$

Proof. The dimension of derivations are of no use here, because the orbit dimension of any algebra in class E is less than the orbit dimension of every algebra of class D. Fortunately the $\mathfrak{C}_{\mathrm{p,q}}$-invariant is doing all the work. Suppose that $D_2(\alpha)_{\alpha\neq-1}$ degenerates to $E_{2,1}$. In this case we must have

$$\mathfrak{C}_{\mathrm{p,q}}(D_2(\alpha)_{\alpha\neq-1}) = \frac{(\alpha^p + 2(\alpha+1)^p)(\alpha^q + 2(\alpha+1)^q)}{\alpha^{p+q} + 2(\alpha+1)^{p+q}} = 1 = \mathfrak{C}_{\mathrm{p,q}}(E_{2,1}).$$

This case has already been treated in section 3.4 and 3.5, where we saw, that the above equation only holds for the values $\alpha = -1, -\frac{1}{3}$. Further on we found that for

$\alpha = -\frac{1}{3}$ the value of $\mathfrak{C}_{\mathrm{p,q}}(D)$ is not constant for p and q, for which reason $D_2(-1)$ is the only possibility for a degeneration. Indeed, we have

$$D_2(-1) \to_{\deg} E_{2,1} \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{t^2} & t & 0 \\ 0 & t^2 & 1 \end{pmatrix}.$$

To exclude degenerations from $D_2(\alpha)_{\alpha \neq -1}$ to $E_{1,1}(\bar{\alpha})_{\alpha \neq -1}$ for $\alpha \neq \bar{\alpha}$ we have to use the computer. Furthermore, transitivity forces $D_2(-1) \to_{\deg} E_{1,1}(-1)$. For the algebra $E_{1,1}(-1)$ we have

$$\mathfrak{C}_{\mathrm{p,q}}(E_{1,1}(-1)) = \mathfrak{C}_{\mathrm{p,q}}(E_{2,1}) = 1$$

hence $D_2(\alpha)_{\alpha \neq -1}$ cannot degenerate to $E_{1,1}(-1)$ as well. There are infinitely many proper degenerations, that can't be reached by transitivity:

$$D_2(-1) \to_{\deg} E_{2,1} \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{t^2} & t & 0 \\ 0 & t^2 & 1 \end{pmatrix}, \text{ and}$$

$$D_2(\alpha)_{\alpha \neq -1} \to_{\deg} E_{1,1}(\alpha)_{\alpha \neq -1} \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

4.4 Degenerations of the classes A, B, and D

4.4.1 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_2$

Proposition 4.18. *The Novikov algebra B_1 degenerates properly within the class B only to the algebras $B_4(0)$ and $B_5(0) \cong B_5(1)$.*

Proof. Because $\dim \mathrm{O}(B_1) < \dim \mathrm{O}(B_2)$ the algebra B_1 cannot degenerate to B_2. To exclude a degeneration from B_1 to B_3 we use a similar argument as in Proposition 4.13.[5] We change the basis of B_1 by the matrix

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In terms of the left-multiplication operator the new algebra, which we will denote by $\overline{B_1}$, then takes the form

$$\mathrm{L}^{\overline{B_1}}(x) := \begin{pmatrix} x_3 & x_3 & x_1 \\ 0 & x_3 & x_2 \\ 0 & 0 & x_3 \end{pmatrix},$$

where $x \in A$. We form the orbit closure again with an upper triangular matrix

$$B_\varepsilon := \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix}$$

[5] See Remark 4.14.

and conclude that every algebra in it must satisfy

$$\mathrm{L}^{\overline{B_\varepsilon \cdot B_1}}(x_1e_1 + x_2e_2) = \begin{pmatrix} 0 & 0 & -\frac{x_1}{b_6} \\ 0 & 0 & -\frac{x_2}{b_6} \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that for $b_6 \to \infty$ we have $\dim \mathrm{Ann}_\mathrm{R}(\mathrm{B}_1) = 2 \neq 1 = \dim \mathrm{Ann}_\mathrm{R}(\mathrm{B}_3)$. Because

$$\dim \mathrm{Ann}_\mathrm{R}(\mathrm{B}_4(\alpha)) = \dim \mathrm{Ann}_\mathrm{R}(\mathrm{B}_5(\beta)) = 1$$

for all $\alpha, \beta \neq 0$ the algebras $B_4(\alpha)_{\alpha\neq 0}$ and $B_5(\beta)_{\beta\neq 0}$ cannot lie in this specific orbit closure (taken with $b_6 \to \infty$) of the algebra B_1 too.

The condition $\lim \frac{1}{b_6} \neq 0$ forces that only non-complete algebras can lie in this closure, which is not the case for the algebras B_3, $B_4(\alpha)$, and $B_5(\beta)$ for all $\alpha, \beta \in \mathbb{C}$. We indeed have a degeneration by

$$B_1 \to_{\mathrm{deg}} B_4(0) \text{ by } g_t = \begin{pmatrix} -\frac{1}{t^2} & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ -\frac{1}{t^3} & 0 & -\frac{1}{t} \end{pmatrix}$$

and to $B_5(0) \cong B_5(1)$ by transitivity (which follows from the next Proposition). □

Proposition 4.19. *The Novikov algebra $B_4(\alpha)$ degenerates to the Novikov algebra $B_5(\beta)$ if and only if $\alpha = \beta$ or $\alpha = 1 - \beta$.*

Proof. Similar like in the proposition before we bring the left-multiplication operator of $B_4(\alpha)$ in upper triangular form by:

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and denote the new algebra by $\widetilde{B_4}(\alpha)$. We then take the orbit closure by an arbitrary upper triangular matrix

$$B_\varepsilon := \begin{pmatrix} b_1(\varepsilon) & b_2(\varepsilon) & b_3(\varepsilon) \\ 0 & b_4(\varepsilon) & b_5(\varepsilon) \\ 0 & 0 & b_6(\varepsilon) \end{pmatrix}.$$

All the $b_i's$ and thus B_ε are dependend of the degeneration parameter ε. The resulting left-multiplication operator for any algebra in the $\mathrm{B}_n(\mathbb{C})$-closure then takes the form

$$\mathrm{L}^{\mathrm{O}_B(\widetilde{B_4}(\alpha))}(x) = \begin{pmatrix} 0 & (\alpha-1)\lambda x_3 & \alpha\lambda x_2 + ((1-2\alpha)\lambda\mu + \nu)x_3 \\ 0 & 0 & \xi x_3 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\lambda = \frac{b_1(\varepsilon)}{b_4(\varepsilon)b_6(\varepsilon)}$, $\mu = \frac{b_5(\varepsilon)}{b_6(\varepsilon)}$, $\nu = \frac{b_2(\varepsilon)}{b_6^2(\varepsilon)}$, and $\xi = \frac{b_4(\varepsilon)}{b_6^2(\varepsilon)}$. By the same matrix g as above the left-multiplication operator of $B_5(\beta)$ can be brought to upper triangular form denoted by $\mathrm{L}^{\widetilde{B_5}(\beta)}(x)$:

$$\mathrm{L}^{\widetilde{B_5}(\beta)}(x) = \begin{pmatrix} 0 & (\beta-1)x_3 & \beta x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So if the algebra $B_5(\beta)$ would lie in the orbit closure of $B_4(\alpha)$ we would have an isomorphism $h \in \mathrm{GL}_n(\mathbb{C})$ sending $\mathrm{L}^{\widetilde{B_5}(\beta)}(x)$ to $\mathrm{L}^{\overline{\mathrm{O}_B(\widetilde{B_4}(\alpha))}}(x)$. By the Bruhat decomposition ([56, p. 74]) we know that every matrix $h \in \mathrm{GL}_n(\mathbb{C})$ can be written in the form $h = B_1PB_2$, where $B_1, B_2 \in \mathrm{B}_n(\mathbb{C})$ and P is a permutation matrix. Hence we have

$$\begin{aligned} h \cdot \widetilde{B_5}(\beta) \in \overline{\mathrm{O}_B(\widetilde{B_4}(\alpha))} &\Leftrightarrow (B_1PB_2) \cdot \widetilde{B_5}(\beta) \in \overline{\mathrm{O}_B(\widetilde{B_4}(\alpha))} \\ &\Leftrightarrow P \cdot (B_2 \cdot \widetilde{B_5}(\beta)) \in \overline{\mathrm{O}_B(\widetilde{B_4}(\alpha))}. \end{aligned}$$

Because the matrices $\mathrm{L}^{\mathrm{O}_B(\widetilde{B_4}(\alpha))}(x)$ and $\mathrm{L}^{\mathrm{O}_B(\widetilde{B_5}(\beta))}(x)$ are in upper triangular form the permutation matrix P must leave $\mathrm{L}^{B_2 \cdot \widetilde{B_5}(\beta)}(x)$ in upper triangular form. Besides the unity matrix I_n there is only one permutation matrix that keeps $B_2 \cdot \widetilde{B_5}(\beta)$ in that way, namely:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In other words we have decomposed the orbit closure of $B_4(\alpha)$ into:

$$\begin{aligned} \overline{\mathrm{GL}_n(\mathbb{C}) \cdot B_4(\alpha)} &= \overline{\mathrm{GL}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)} = \mathrm{GL}_n(\mathbb{C}) \cdot \overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)} \\ &= \mathrm{B}_n(\mathbb{C})(I_n \cup P)\mathrm{B}_n(\mathbb{C}) \cdot \overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)} \\ &= \overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)} \cup \mathrm{B}_n(\mathbb{C}) \cdot P \cdot \overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)}. \end{aligned}$$

The question if $\widetilde{B_5}(\beta) \in \overline{\mathrm{GL}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)}$ therefore leads to the question if

$$\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_5}(\beta) \cap \left(\overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)} \cup P \cdot \overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)}\right) \neq \emptyset.$$

We will answer this question in the following way. First of all we have to see what the $\mathrm{B}_n(\mathbb{C})$-orbit of $\widetilde{B_5}(\beta)$ looks like. Let $B \in \mathrm{B}_n(\mathbb{C})$ be an upper triangular matrix (that is not dependend of a degeneration parameter):

$$B := \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix}.$$

$$B \cdot \mathrm{L}^{\widetilde{B_5}(\beta)}(x) = \begin{pmatrix} 0 & (\beta-1)\varrho x_3 & \beta\varrho x_2 + (1-2\beta)\varrho\tau x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\varrho = \frac{b_1}{b_4 b_6}$, $\tau = \frac{b_5}{b_6}$. Now, on the whole $\mathrm{B}_n(\mathbb{C})$-orbit of $\widetilde{B_4}(\alpha)$ the following polynomial function is zero:

$$f(X_{ij}^k) = (\alpha-1)X_{23}^1 - \alpha X_{32}^1.$$

We write for this fact $f(\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)) = 0$. By definition of the Zariski topology this polynomial function has to be zero for every algebra in the $\mathrm{B}_n(\mathbb{C})$-orbit closure of $\widetilde{B_4}(\alpha)$. However, for the $\mathrm{B}_n(\mathbb{C})$-orbit of $\widetilde{B_5}(\beta)$ we have:

$$f(\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_5}(\beta)) = \varrho((\alpha-1)\beta - \alpha(\beta-1)).$$

Hence, $f(\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_5}(\beta)) = 0$ if and only if $\frac{\alpha}{\alpha-1} = \frac{\beta}{\beta-1}$, which has a solution in β if and only if $\alpha = \beta$.

In the same way as before, assuming $\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_5}(\beta)$ having nonzero intersection with $P \cdot \mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)$ we must have $\bar{f}(\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_5}(\beta)) = 0$ where

$$\bar{f}(X_{ij}^k) = (\alpha - 1)X_{32}^1 - \alpha X_{23}^1.$$

Now the parameters α and β have to satisfy $\frac{\alpha}{\alpha-1} = \frac{\beta-1}{\beta}$, which can be done if and only if $\alpha = 1 - \beta$. We see that the solutions for one α correspond to the isomorphic algebras $B_5(\alpha)$ and $B_5(1-\alpha)$. We indeed have degenerations

$$B_4(\alpha) \to_{\deg} B_5(\alpha) \text{ by } g_t = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Remark 4.20. Similar like for the $\mathfrak{C}_{\mathrm{p,q}}$-invariant we can associate to the $\mathrm{B}_n(\mathbb{C})$-orbit of $\widetilde{B_4}(\alpha)$ the invariant $c_f := \frac{c_{23}^1}{c_{32}^1} = \frac{\alpha}{\alpha-1}$ whenever it is defined.[6] Every algebra that lies in this orbit must have the same value for $\frac{c_{23}^1}{c_{32}^1}$. By $\bar{f}(\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)) = 0$ we can also associate an invariant to the set $P \cdot (\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha))$. We denote it by $c_{\bar{f}} := \frac{c_{32}^1}{c_{23}^1}$.

Lemma 4.21. *The orbit closures of all* 3*-dimensional Novikov algebras with associated Lie algebra* $\mathfrak{g}_2$ *are as follows.*

B	$\overline{\mathrm{O}(B)} \setminus \mathrm{O}(B)$
B_1	$B_4(0), B_5(0)$
B_2	$B_1, B_3, B_4(\alpha), B_5(\beta)$
B_3	$B_5(\frac{1}{2})$
$B_4(\alpha)$	$B_5(\alpha)$
$B_4(\frac{1}{2})$	$B_3, B_5(\frac{1}{2})$
$B_5(\beta)$	$-$

The Hasse diagram of all Novikov algebras with associated Lie algebra $\mathfrak{g}_2$ *looks as follows.*

$$\begin{tikzcd}
& B_2 \arrow[dl] \arrow[ddl] \arrow[ddd] \\
B_1 \arrow[d, "\alpha=0"'] & \\
B_4(\alpha) \arrow[d, "\alpha=\frac{1}{2}"'] \arrow[dr, "\alpha=\beta"] & \\
B_3 \arrow[d] & B_5(\beta) \\
B_5(\frac{1}{2}) &
\end{tikzcd}$$

[6] We mean that there is no division by zero.

Proof. We start our proof with the algebras $B_4(\alpha)$. As we have seen in the last proposition we can decompose the orbit closure of $B_4(\alpha)$ for every $\alpha \in \mathbb{C}$ into $\overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)} \cup \mathrm{B}_n(\mathbb{C}) \cdot P \cdot \overline{\mathrm{B}_n(\mathbb{C}) \cdot \widetilde{B_4}(\alpha)}$. We can therefore find all algebras that can possibly lie in the orbit closure of $B_4(\alpha)$ by computing the invariants c_f and $c_{\bar{f}}$. For[7] $\widetilde{B_3}$ we have $c_f = c_{\bar{f}} = -1$ which forces $\alpha = \frac{1}{2}$ and indeed

$$B_4(\tfrac{1}{2}) \to_{\text{deg}} B_3 \text{ by } g_t = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

The classification of all degenerations $B_4(\alpha) \to_{\text{deg}} B_5(\beta)$ was already treated in Proposition 4.19 and $B_4(\alpha)$ cannot degenerate to B_1 and B_2 because of the orbit dimension.

The algebras $B_5(\beta)$ have a very similar structure like $B_4(\alpha)$. The quotient $\frac{c_{23}^1}{c_{32}^1}$ defines an invariant on $\overline{B \cdot \widetilde{B_5}(\beta)}$ again. For $\widetilde{B_5}(\frac{1}{2})$ we have $\frac{c_{23}^1}{c_{32}^1} = -1$ and so $\beta = \frac{1}{2}$, yielding only the trivial degeneration. All the other possible degenerations are excluded by the dimensions of the orbits.

Regarding the orbit dimensions the algebra B_3 can only degenerate to $B_5(\frac{1}{2})$, which is also possible by the invariant $\frac{c_{23}^1}{c_{32}^1}$. Indeed, we have $B_3 \to_{\text{deg}} B_5(\frac{1}{2})$ by

$$g_t = \begin{pmatrix} t & 0 & 0 \\ t^2 & 1 & 0 \\ 0 & 0 & \frac{t}{1-2t^2} \end{pmatrix}.$$

The algebra B_2 degenerates to every algebra in class B. We list the corresponding degeneration matrices:

$$B_2 \to_{\text{deg}} B_1 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{t} \end{pmatrix},$$

$$B_2 \to_{\text{deg}} B_3 \text{ by } g_t = \begin{pmatrix} \frac{1}{2t^3} & \frac{1}{t^3} & 0 \\ \frac{1}{t^2} & 0 & 0 \\ \frac{1}{4t^5} - \frac{1}{t^4} & 0 & -\frac{1}{t^5} \end{pmatrix},$$

$$B_2 \to_{\text{deg}} B_4(\alpha) \text{ by } g_t = \begin{pmatrix} -\frac{1}{t^2} & -\frac{1}{\alpha t^2} & 0 \\ \frac{1}{t} & 0 & 0 \\ \frac{\alpha-1}{t^3} & 0 & \frac{1}{\alpha t^3} \end{pmatrix} \text{ for } \alpha \neq 0,$$

$$B_2 \to_{\text{deg}} B_5(\beta) \text{ by } g_t = \begin{pmatrix} \frac{1}{t^2} & 0 & 0 \\ \frac{\beta-1}{t^3} & \frac{1}{t^3} & 0 \\ \frac{\beta(\beta-1)}{t^5} & 0 & \frac{1}{\alpha t^5} \end{pmatrix}.$$

The degeneration $B_2 \to_{\text{deg}} B_4(0)$ is done by transitivity. The orbit closure of the algebra B_1 in the class B has already been studied in Proposition 4.18. This completes the proof. □

[7] The algebra $\widetilde{B_3}$ is isomorphic to B_3 with the isomorphism g from Proposition 4.19, where the left-multiplication operator is then given in an upper triangular form.

4.4.2 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_4$ and $\mathfrak{g}_2$

Lemma 4.22. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_4$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_2$ are as follows.*

D	$\overline{\mathrm{O}(D)}$
D_1	$B_4(0), B_5(0)$
$D_2(-1)$	$B_4(0), B_5(0)$
$D_2(\alpha)_{\alpha\neq -1}$	$B_4(0), B_5(0)$

Including the degenerations we already got, the Hasse diagram looks as follows.

$$\begin{tikzcd}
D_2(0) \arrow[ddr] & D_1 \arrow[d] & D_2(\alpha)_{\alpha\neq 0,-1} \arrow[ddl] \\
 & D_2(-1) \arrow[d] & \\
 & B_4(0) \arrow[d] & \\
 & B_5(0) &
\end{tikzcd}$$

Proof. Regarding the dimensions of the vectorspace of derivations the only possible algebras in an orbit closure of a D-class algebra are:

$$B_3, B_4(\alpha), B_5(\beta).$$

The same technique, which we derived in Proposition 4.13 applies here too, because the algebras $B_3, B_4(\alpha), B_5(\beta)$ are all complete and have a 1-dimensional right-annihilator (except for $\alpha = \beta = 0$). Indeed we have degenerations $D_1 \to_{\deg} B_4(0)$ and $D_2(\alpha)_{\alpha\neq -1} \to_{\deg} B_4(0)$ by

$$g_t = \begin{pmatrix} \frac{1}{t} & \frac{1}{t} & -\frac{1}{t^3} \\ 0 & 1 & -1-\frac{1}{t^2} \\ 0 & 0 & \frac{1}{t} \end{pmatrix} \text{ and } g_t = \begin{pmatrix} \frac{1+\alpha t^2}{(\alpha+1)t} & \frac{1}{t} & -\frac{(1+\alpha t^2)^2}{(\alpha+1)^2 t^3} \\ 0 & -\frac{(\alpha+1)}{1+\alpha t^2} & 0 \\ 0 & 0 & \frac{1}{t} \end{pmatrix}.$$

□

4.4.3 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_2$ and $\mathfrak{g}_1$

Lemma 4.23. *The orbit closures of all Novikov algebras of dimension three with associated Lie algebra $\mathfrak{g}_2$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_1$ are as follows.*

B	$\overline{\mathrm{O}(B)}$
B_1	A_1, A_5, A_{11}
B_2	$A_1, A_5, A_9, A_{10}, A_{11}, A_{12}$
B_3	A_1, A_5
$B_4(\alpha)$	A_1, A_5
$B_5(\beta)_{\beta\neq\frac{1}{2}}$	A_1, A_5
$B_5(\frac{1}{2})$	$-$

Including the degenerations we already got, the Hasse diagram looks as follows.

$$\begin{tikzcd}
 & & B_2 \arrow[dll] \arrow[dl] \arrow[dd] \arrow[dddr] & \\
A_{12} \arrow[d] \arrow[ddr] & B_1 \arrow[dd] \arrow[dr, "\alpha=0"'] & & \\
A_{10} \arrow[d] & & B_4(\alpha) \arrow[d, "\alpha=\frac{1}{2}"'] \arrow[dr, "\alpha=\beta"] & \\
A_9 \arrow[dr] & A_{11} \arrow[d] & B_3 \arrow[dl] & B_5(\beta) \arrow[dll, "\beta\neq\frac{1}{2}"] \\
 & A_5 \arrow[d] & & \\
 & A_1 & &
\end{tikzcd}$$

Proof. Because we will use transitivity several times in this proof we begin with the orbit closure of the algebra $B_5(\beta)$. For $\beta = \frac{1}{2}$ the orbit dimension is too small for any A-class algebra to lie in that closure. For arbitrary β not equal to $\frac{1}{2}$ we can only have $B_5(\beta) \to_{\deg} A_5$ which indeed is true by:

$$g_t = \begin{pmatrix} 1 & \frac{1}{t^2} & -\frac{t^2}{2\beta-1} \\ 0 & 1 & 0 \\ 0 & 0 & t^3 \end{pmatrix}.$$

The algebra B_3 has the same orbit dimension as the algebra $B_5(\beta)_{\beta\neq\frac{1}{2}}$ and so it can also degenerate only to A_5. This is done by the matrix

$$g_t = \begin{pmatrix} 1 & 0 & t^4 \\ 0 & t^2 & 0 \\ 0 & 0 & t^5 \end{pmatrix}.$$

Now we look at the algebras $B_4(\alpha)$. We immediately see that by transitivity for all $\alpha \neq \frac{1}{2}$ the algebras $B_4(\alpha)$ degenerate to A_5 because $B_5(\beta)$ does for every $\beta \neq \frac{1}{2}$ and $B_4(\alpha) \to_{\deg} B_5(\alpha)$ for all α. But also for $\alpha = \frac{1}{2}$ we get a degeneration by transitivity because $B_4(\frac{1}{2}) \to_{\deg} B_3 \to_{\deg} A_5$. Regarding the orbit dimensions, the only algebras left that can possibly lie in the orbit closure of $B_4(\alpha)$ are A_2, A_9, and A_{11}. As a consequence of Lemma 3.50 and in particular example 3.51, the complete algebra $B_4(\alpha)$ cannot degenerate to A_2 and A_{11} by Corollary 3.52. The algebra A_9 can also not lie in the orbit closure of the algebra $B_4(\alpha)$ because of the same argument brought in Proposition 4.19. We know (Remark 4.20) that the quotient

$$c_f(B_4(\alpha)) = \frac{\alpha}{\alpha - 1}$$

is a B-invariant whenever it is defined. For A_9 we must have

$$c_f(B_4(\alpha)) = c_f(A_9) = c_{\bar{f}}(A_9) = 1,$$

yielding $\frac{\alpha}{\alpha-1} = 1$ which is impossible except for the values $\alpha = 0, 1$. But in this cases the structure constants c^1_{23} and c^1_{32}, respectively, are zero and therefore define invariants by themselves.[8]

The algebra B_2 degenerates to the algebra A_{12} by:

$$g_t = \begin{pmatrix} 0 & 0 & t^2 \\ 0 & t & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and by transitivity to A_5, A_9, A_{10} and A_{11}. We compute the invariant $\mathfrak{C}_{\mathrm{p,q}}(B_2) = 3$. Because $\mathfrak{C}_{\mathrm{p,q}}(A_2) = \mathfrak{C}_{\mathrm{p,q}}(A_6) = 1$ and $\mathfrak{C}_{\mathrm{p,q}}(A_7) = 2$ it is impossible for the algebras A_2, A_6, and A_7 to lie in the orbit closure of B_2. The algebra B_2 cannot degenerate to the algebras A_3, A_4, and A_8 because their orbit dimensions are to high.

Finally, the algebra B_1 degenerates to the algebra A_{11} by:

$$g_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and by transitivity to A_5. Regarding orbit dimensions again we see that the only possible algebras in the orbit closure of B_1 are A_2, A_9, and A_{10}. We have

$$\mathfrak{C}_{\mathrm{p,q}}(B_1) = 3 \neq 1 = \mathfrak{C}_{\mathrm{p,q}}(A_2).$$

Therefore the algebra B_1 does not degenerate to the algebra A_2. The algebras A_9 and A_{10} are complete and so the same argument as in Proposition 4.18 can be carried out. □

4.5 Degenerations of the classes A, B, and E(-1)

4.5.1 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(-1)$

Lemma 4.24. *There are no proper degenerations for Novikov algebras with associated Lie algebra* $\mathfrak{g}_5(-1)$.

Proof. There are no proper degenerations from $E_{1,-1}(1)$ and $E_{1,-1}(-1)$ to any other algebra because of the orbit dimensions. The algebras $E_{1,-1}(\alpha)$ with $\alpha \neq \pm 1$ can

[8] In fact we have the Zariski equations $c^1_{23} = 0$ and $c^1_{32} = 0$ which are, as one could say, of the most simple form.

only have the algebras $E_{1,-1}(1)$ and $E_{1,-1}(-1)$ in their closure. Both is impossible because of the invariant $\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(\alpha))$:

$$\begin{aligned}
\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(\alpha)) &= \frac{[\alpha^p + (\alpha+1)^p + (\alpha-1)^p][\alpha^q + (\alpha+1)^q + (\alpha-1)^q]}{\alpha^{p+q} + (\alpha+1)^{p+q} + (\alpha-1)^{p+q}}, \\
\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(1)) &= \frac{(1+2^p)(1+2^q)}{1+2^{p+q}}, \\
\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(-1)) &= \frac{((-1)^p + (-2)^p)((-1)^q + (-2)^q)}{(-1)^{p+q} + (-2)^{p+q}}.
\end{aligned}$$

We want to decide wether $E_{1,-1}(\alpha)$ degenerates to $E_{1,-1}(1)$ and therefore compute the values of the above invariants for $p = q = 1$. In this case the equality $\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(\alpha)) = \mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(1))$ holds if and only if $a = \pm 1$, but we already excluded these two cases. The same is true for the algebra $E_{1,-1}(1)$ and so for every α the set

$$\overline{\mathrm{O}(E_{1,-1}(\alpha))} \setminus \mathrm{O}(E_{1,-1}(\alpha))$$

contains no algebra with associated Lie algebra $\mathfrak{g}_5(-1)$.

The algebra $E_{2,-1}$ cannot contain the algebras $E_{1,-1}(\alpha)$ in its closure except for $\alpha = \pm 1$. But also for these values a degeneration is impossible because $\mathfrak{C}_{\mathrm{p,q}}(E_{2,-1}) = 1$, whereas $\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(1))$ and $\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(-1))$ are non-constant funcions in p and q. □

4.5.2 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(-1)$ and $\mathfrak{g}_2$

Lemma 4.25. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(-1)$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_2$ are as follows.*

E	$\overline{\mathrm{O}(E)}$
$E_{1,-1}(1)$	$B_5(0)$
$E_{1,-1}(-1)$	$B_5(0)$
$E_{1,-1}(\alpha)_{\alpha\neq\pm1}$	$B_4(0), B_5(0)$
$E_{2,-1}$	$B_4(0), B_5(0)$

Including the degenerations we already got, the Hasse diagram looks as follows.

$$\begin{tikzcd}
 & & E_{1,-1}(\alpha) \arrow[d, "\alpha\neq\pm1"'] & E_{2,-1} \arrow[dl] \\
E_{1,-1}(1) \arrow[drr] & E_{1,-1}(-1) \arrow[dr] & B_4(0) \arrow[d] & \\
 & & B_5(0) &
\end{tikzcd}$$

Proof. An algebra with associated Lie algebra $\mathfrak{g}_5(-1)$ cannot degenerate to the Novikov algebras B_1 and B_2 because of the orbit dimensions. The algebras $E_{1,-1}(\alpha)$ cannot degenerate to the algebras B_3, $B_4(\overline{\alpha})$ ($\overline{\alpha} \neq 0$), and $B_5(\beta)$ ($\beta \neq 0$) because of an argument first brought in Proposition 4.13. We frequently used similar techniques over the last few sections and therefore we will not explain the whole procedure in detail once again. An algebra in the orbit closure of the algebra $E_{1,-1}(\alpha)$ either is not complete or has a two-dimensional right annihilator. Both conditions do not hold for $B_4(\overline{\alpha})$ for all $\overline{\alpha}$ except $\overline{\alpha} = 0$. We indeed have $E_{1,-1}(\alpha) \to_{\deg} B_4(0)$ by:

$$g_t = \begin{pmatrix} -\frac{1}{(\alpha+1)t^2} & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ \frac{1}{(\alpha+1)t^3} & \frac{\alpha-1}{2t} & 1 \end{pmatrix},$$

and $E_{1,-1}(\alpha) \to_{\deg} B_5(0)$ by transitivity. For the special cases $E_{1,-1}(1)$ and $E_{1,-1}(-1)$ the same argument holds to exclude a degeneration to B_3 and $B_5(\beta)$ $(\beta \neq 0)$. A degeneration to $B_4(\overline{\alpha})$ is not possible for any $\overline{\alpha}$ because of the orbit dimension. Nevertheless we have $E_{1,-1}(1), E_{1,-1}(-1) \to_{\deg} B_5(0)$ by:

$$g_t = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ 0 & -\frac{1}{2t} & 1 \end{pmatrix}.$$

This degeneration matrix even performs $E_{1,-1}(\alpha) \to_{\deg} B_5(0)$ for every α.

It is left to determine the orbit closure of $E_{2,-1}$. Concerning a degeneration to B_3, $B_4(\alpha)$ $(\alpha \neq 0)$, and $B_5(\beta)$ $(\beta \neq 0)$ a similar argument as for $E_{1,-1}(\alpha)$ can be carried out. Therefore a degeneration to these algebras is not possible. We have $E_{2,-1} \to_{\deg} B_4(0)$ by

$$g_t = \begin{pmatrix} 0 & \frac{1}{t^2} & 0 \\ \frac{1}{t} & 0 & 0 \\ \frac{1}{4t^3} & -\frac{1}{2t^3} & 1 \end{pmatrix},$$

and $E_{2,-1} \to_{\deg} B_5(0)$ by transitivity. Hence, we are done. □

4.5.3 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(-1)$ and $\mathfrak{g}_1$

Lemma 4.26. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(-1)$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_1$ are as follows.*

E	$\overline{\mathrm{O}(E)}$
$E_{1,-1}(1)$	A_1, A_5
$E_{1,-1}(-1)$	A_1, A_5
$E_{1,-1}(\alpha)_{\alpha\neq\pm1}$	A_1, A_5
$E_{2,-1}$	A_1, A_5

Including the degenerations we already got, the Hasse diagram looks as follows.[9]

[9]We remark that in this diagram the algebras with associated Lie algebra $\mathfrak{g}_5(-1)$ are not ordered by orbit dimension.

$$\begin{tikzcd}
E_{1,-1}(1) \arrow[drr] & E_{1,-1}(-1) \arrow[dr] & E_{1,-1}(\alpha) \arrow[d, "\alpha\neq\pm1"] & E_{2,-1} \arrow[dl] \\
& & A_5 \arrow[d] & \\
& & A_1 &
\end{tikzcd}$$

Proof. We order the algebras of class E and class A by orbit dimensions. As an overview we give a table where for the algebras on the left we list all algebras that can possibly lie in the orbit closure on the right.

E	$\overline{\mathrm{O}(E)}$
$E_{1,-1}(\alpha)_{\alpha\neq\pm1}, E_{2,-1}$	$A_1, A_2, A_5, A_9, A_{10}, A_{11}$
$E_{1,-1}(1), E_{1,-1}(-1)$	$A_1, A_2, A_5, A_9, A_{11}$

We start our proof with the algebra $E_{1,-1}(\alpha)$. For every algebra in the orbit closure of $E_{1,-1}(\alpha)$ it is true that it has a two-dimensional right annihilator or is incomplete by a method worked out in Proposition 4.13. Both is not the case for the algebras A_9 and A_{10}, so these algebras can't lie in the closure of $E_{1,-1}(\alpha)$. From Lemma 4.24 we know that $\mathfrak{C}_{\mathrm{p,q}}(E_{1,-1}(\alpha))$ is a non-constant function in p and q for any α, whereas $\mathfrak{C}_{\mathrm{p,q}}(A_2) = 1$ and $\mathfrak{C}_{\mathrm{p,q}}(A_{11}) = 3$, which are both constant in p and q. Hence, a degeneration from $E_{1,-1}(\alpha)$ to A_2 and A_{11} is impossible. We have degenerations $E_{1,-1}(\alpha) \to_{\deg} A_5$ for any $\alpha \neq 1$ by

$$g_t = \begin{pmatrix} 1 & 0 & -\frac{1}{(\alpha-1)t^2} \\ 0 & 0 & \frac{1}{t} \\ 0 & 1 & 0 \end{pmatrix}$$

and $E_{1,-1}(1) \to_{\deg} A_5$ by:

$$g_t = \begin{pmatrix} 1 & -\frac{1}{2t} & 0 \\ 0 & 0 & \frac{1}{t^2} \\ 0 & 1 & \frac{1}{t^3} \end{pmatrix}.$$

The algebras that can possibly lie in the orbit closure of $E_{2,-1}$ are the same as the ones for $E_{1,-1}(\alpha)$. We exclude a degeneration to A_2 and A_{11} in the same way as before, namely by the invariant $\mathfrak{C}_{\mathrm{p,q}}(E_{2,-1})$. Furthermore, the algebra $E_{2,-1}$ has a 2-dimensional left annihilator. Because $\dim \mathrm{Ann}_L(A_9) = \dim \mathrm{Ann}_L(A_{10}) = 1$ a degeneration from $E_{2,-1}$ to A_9 and A_{10} is impossible by Lemma 3.29. We once again have a degeneration $E_{2,-1} \to_{\deg} A_5$ by

$$g_t = \begin{pmatrix} 1 & 0 & \frac{1}{t^2} \\ 0 & 0 & \frac{1}{t} \\ 0 & 1 & 0 \end{pmatrix}.$$

This completes the proof. $\square$

4.6 Degenerations of the classes A, B, and C

4.6.1 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_3$

Proposition 4.27. *The orbit closures of the Novikov algebras $C_6(\beta)$ are listed in the table below.*

C	$\overline{\mathrm{O}(C)} \setminus \mathrm{O}(C)$
$C_6(0)$	$C_4, C_5(0)$
$C_6(-1)$	$C_5(-1), C_7(-1)$
$C_6(\beta)$	$C_5(\beta), C_7(\gamma)$

Proof. The algebra with the lowest orbit dimension is $C_6(-1)$. All the other algebras $C_6(\beta)$ have the same orbit dimension and so the only possible degenerations are from $C_6(\beta)$ with $\beta \neq -1$ to $C_6(-1)$.

We start our proof with the special case $C_6(0)$. We compute the dimension of the first element in the derived series of $C_6(0)$ and find $\dim \delta^{(1)}(C_6(0)) = 2$. From Corollary 3.24 we know that the dimension of every element in the derived series has to be equal or smaller in the orbit closure. Therefore we immediately exclude the algebras C_1, C_2, $C_6(\beta)_{\beta\neq -1}$, and $C_7(\gamma)$, for all γ, of lying in the orbit closure of $C_6(0)$. They all have $\dim \delta^{(l)} = 3$ for all $l \geqslant 1$. Moreover, for the algebras $C_3, C_5(\alpha)_{\alpha\neq 0}$ the dimension of $\delta^{(l)}$ equals 2 for all $l \geqslant 1$, which would contradict $\dim \delta^{(l)}(C_6(0)) = 1$ for all $l \geqslant 2$ in case of a degeneration. The only algebras that remain are C_4 and $C_5(0)$. We have $C_6(0) \to_{\deg} C_5(0)$ as a special case of $C_6(\alpha) \to_{\deg} C_5(\alpha)$, which we will treat later, and

$$C_6(0) \to_{\deg} C_4 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{t} & 0 & \frac{1}{t^2} \end{pmatrix}.$$

The next special case is $C_6(-1)$. Because of the orbit dimension only degenerations to the algebras $C_5(0), C_5(-1)$, and $C_7(-1)$ are possible. Looking for quadratic operator identities $T(x,y) = 0$ for all $x, y \in C_6(-1)$ we find among others that $T^{C_6(-1)}(x) = 0$, where

$$T(x) := \mathrm{L}(x)^2 - \mathrm{L}(x)\mathrm{R}(x).$$

Because of Lemma 3.46 this identity has to vanish for every algebra in the orbit closure. In the case of $C_5(0)$ we have[10]

$$T^{C_5(0)}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A degeneration is therefore impossible. For $C_6(-1) \to_{\deg} C_5(-1)$ we again refer to the case $C_6(\alpha) \to_{\deg} C_5(\alpha)$, which is coming soon. There is also a degeneration $C_6(-1) \to_{\deg} C_7(-1)$ by

$$g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{t} & 0 & \frac{1}{t} \end{pmatrix}.$$

[10] We set $x = \sum_{i=1}^{3} x_i e_i$ and for y likewise.

Finally we consider the case $C_6(\beta)$ for every $\beta \neq 0, -1$. The algebras C_1 and $C_6(0)$ do not lie in the orbit closure of $C_6(\beta)$ for any $\beta \in \mathbb{C}$ by reasons of the respective orbit dimensions. To show that all the other algebras of class C are also not lying in the orbit closure of any $C_6(\beta)$ we use the following quadratic operator identity:

$$S(x) := \mathrm{L}(x)^2 - \mathrm{L}(x)\mathrm{R}(x) - \frac{\beta+1}{\beta}\mathrm{R}(x)\mathrm{L}(x) + \frac{\beta+1}{\beta}\mathrm{R}(x)^2.$$

We have $S^{C_6(\beta)}(x) = 0$ for every $\beta \neq 0, -1$. Hence, for all algebras $\overline{C}$ in the orbit closure of $C_6(\beta)$ we must have $S^{\overline{C}}(x) = 0$. Instead we have

$$S^{\overline{C}}(x) = \frac{\beta+1}{\beta}\begin{pmatrix} 0 & 0 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for } \overline{C} = C_2,\, C_3,\, C_4,\, C_6(-1),$$

$$S^{C_5(\alpha)}(x) = \frac{\beta-\alpha}{\beta}\begin{pmatrix} 0 & 0 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad .$$

$$S^{C_7(\gamma)}(x) = \frac{\beta-\gamma}{\beta}\begin{pmatrix} 0 & 0 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the operator identity $S(x)$ equals zero for $C_5(\alpha)$ and $C_7(\gamma)$ if and only if $\alpha = \beta$ and $\gamma = \beta$, respectively. Indeed, in these cases we have degenerations for all $\beta \in \mathbb{C}$:

$$C_6(\beta) \to_{\deg} C_5(\beta) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t} \end{pmatrix} \text{ and}$$

$$C_6(\beta) \to_{\deg} C_7(\beta) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\beta}{t} & 0 & \frac{1}{t} \end{pmatrix}.$$

□

Lemma 4.28. *The orbit closures of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_3$ are listed in the table below.*

C	$\overline{\mathrm{O}(C)} \setminus \mathrm{O}(C)$
C_1	$C_2, C_3, C_5(-1), C_6(-1), C_7(-1)$
C_2	$C_7(-1)$
C_3	$C_5(-1)$
C_4	$C_5(0)$
$C_5(0)$	–
$C_5(-1)$	–
$C_5(\alpha)$	–
$C_6(0)$	$C_4, C_5(0)$
$C_6(-1)$	$C_5(-1), C_7(-1)$
$C_6(\beta)$	$C_5(\beta), C_7(\gamma)$
$C_7(-1)$	–
$C_7(\gamma)$	–

Proof. We order all the algebras of class C by their orbit dimensions, beginning with the highest on the left:

$C_1, C_6(\beta)_{\beta\neq -1};\quad C_2, C_3, C_4, C_5(\alpha)_{\alpha\neq 0,-1}, C_6(-1), C_7(\gamma)_{\gamma\neq -1};\quad C_5(0), C_5(-1), C_7(-1).$

Because of transitivity arguments we begin with the algebras C_2 and C_3. Considering the orbit dimensions the algebra C_2 can only degenerate to the algebras $C_5(0)$, $C_5(-1)$ and $C_7(-1)$. However, only a degeneration to $C_7(-1)$ is possible because the $\mathfrak{C}_{\mathrm{p,q}}$-invariant of C_2 equals two, but for $C_5(0)$ and $C_5(-1)$ it equals one. The degeneration matrix for $C_2 \to_{\deg} C_7(-1)$ is given by

$$g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A similar situation arises for the algebra C_3 which has the same orbit dimension as C_2. This time a degeneration to $C_7(-1)$ is impossible because of

$$\mathfrak{C}_{\mathrm{p,q}}(C_3) = 1 \neq 2 = \mathfrak{C}_{\mathrm{p,q}}(C_7(-1)).$$

Assuming $C_1 \to_{\deg} C_3$ a degeneration $C_2 \to_{\deg} C_5(0)$ is impossible by transitivity with $C_5(0) \notin \overline{\mathrm{O}(C_1)}$. We will prove this assumption immediately. Finally we have

$$C_3 \to_{\deg} C_5(-1) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, regarding orbit dimensions, the algebra C_1 can't degenerate to $C_6(\beta)$ for any $\beta \neq -1$. For the value $\beta = -1$ and the algebras C_2 and C_3 we have degenerations:

$$C_1 \to_{\deg} C_2 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{t} & t & \frac{1}{t} \end{pmatrix},$$

$$C_1 \to_{\deg} C_3 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & \frac{1}{t} \end{pmatrix},$$

$$C_1 \to_{\deg} C_6(-1) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Like in the Proposition before we look at the operator identity $T^C(x) = 0$ which holds for every $C \in \mathrm{O}(C_1)$, where

$$T(x) = \mathrm{L}(x)^2 - \mathrm{L}(x)\mathrm{R}(x).$$

On the other hand one can easily compute that

$$T^{C_4}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -x_1x_2 & x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^{C_5(\alpha)}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -(\alpha+1)x_1x_2 & (\alpha+1)x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^{C_7(\gamma)}(x) = \begin{pmatrix} 0 & 0 & 0 \\ -(\gamma+1)x_1x_2 & (\gamma+1)x_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that only in the cases $C_5(-1)$ and $C_7(-1)$ the operator identity $T(x) = 0$ is consistent with respect to a degeneration. In fact we have $C_1 \to_{\deg} C_5(-1)$ and $C_1 \to_{\deg} C_7(-1)$ by transitivity with $C_1 \to_{\deg} C_2$, $C_2 \to_{\deg} C_7(-1)$, and $C_3 \to_{\deg} C_5(-1)$.

Because of $\dim \mathrm{O}(C_4) = 7$ we only have to check possible degenerations from C_4 to $C_5(0)$, $C_5(-1)$ and $C_7(-1)$. The algebra C_4 has the same $\mathfrak{C}_{\mathrm{p,q}}$-invariant as C_3 and so we are done for $C_7(-1)$. While a degeneration to $C_5(-1)$ is impossible by

$$\dim \delta^{(2)}(C_4) = 0 \ngeq 2 = \dim \delta^{(2)}(C_5(-1))$$

using Corollary 3.21, we have

$$C_4 \to_{\deg} C_5(0) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

The next orbit closures we study are that of the algebras $C_5(\alpha)$. We have two special cases, namely $C_5(0)$ and $C_5(-1)$. Both of them can't degenerate to any algebra in class C because their orbit dimensions are to small. For our further considerations we therefore exclude the values $\alpha = 0, -1$ when we speak about $C_5(\alpha)$. Yet we still have to show that $C_5(\alpha)$ can't degenerate to $C_5(0)$ or $C_5(-1)$. To see this we compute

$$\begin{aligned} \mathfrak{C}_{\mathrm{p,q}}(C_5(\alpha)) &= \frac{(\alpha^p + (\alpha+1)^p)(\alpha^q + (\alpha+1)^q)}{\alpha^{p+q} + (\alpha+1)^{p+q}}, \text{ and} \\ \mathfrak{C}_{\mathrm{p,q}}(C_5(0)) &= \mathfrak{C}_{\mathrm{p,q}}(C_5(-1)) = 1. \end{aligned}$$

If we calculate the first expression for the values $p = q = 1$ we find that $\mathfrak{C}_{1,1}(C_5(\alpha)) = 1$ if and only if $\alpha = 0, -1$. Furthermore (by Example 3.54)

$$\mathfrak{R}_{\mathrm{p,q}}(C_5(\alpha)) = 2 \neq 3 = \mathfrak{R}_{\mathrm{p,q}}(C_7(\gamma)),$$

for every α and γ not equal to zero. Hence, there are no proper degenerations from $C_5(\alpha)$ to any other algebra in class C.

The same statement as for $C_5(\alpha)$ is also true for $C_7(\gamma)$. We use the same arguments as before and are therefore done. □

Corollary 4.29. *The Hasse diagram of all 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_3$ is given as follows.*

$$\begin{tikzcd}
C_6(0) \arrow[d] & C_6(\beta) \arrow[d, "\beta=\alpha"'] \arrow[dr, "\beta=\gamma"] & & & C_1 \arrow[dl] \arrow[d] \arrow[dr] & \\
C_4 \arrow[d] & C_5(\alpha) & C_7(\gamma) & C_3 \arrow[d] & C_6(-1) \arrow[dl] \arrow[dr] & C_2 \arrow[d] \\
C_5(0) & & & C_5(-1) & & C_7(-1)
\end{tikzcd}$$

4.6.2 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_3$ and $\mathfrak{g}_2$

Proposition 4.30. *The orbit closures of the algebras $C_5(\alpha)$ in class B are given as follows.*

C	$\overline{\mathrm{O}(C)}$
$C_5(0)$	$B_5(0)$
$C_5(-\frac{1}{2})$	B_3
$C_5(-1)$	$B_5(0)$
$C_5(\alpha)_{\alpha\neq 0,-\frac{1}{2},-1}$	$B_4(-\alpha), B_5(-\alpha)$

Proof. All the algebras $C_5(\alpha)$ with arbitrary $\alpha \in \mathbb{C}$ cannot degenerate to the algebras B_1 and B_2 because of their orbit dimensions.

We want to know what degenerations are possible from the algebra $C_5(\alpha)$ with a fixed $\alpha \in \mathbb{C}$ to the algebras $B_5(\beta)$. Therefor we consider the following polynomials:

$$\begin{aligned} f_1(x_{ij}^k) &= x_{32}^1 x_{23}^1 - x_{22}^1 x_{33}^1 - \alpha(\alpha+1)(x_{32}^1 - x_{23}^1)^2, \\ f_2(x_{ij}^k) &= x_{31}^2 x_{13}^2 - x_{11}^2 x_{33}^2 - \alpha(\alpha+1)(x_{31}^2 - x_{13}^2)^2, \\ f_3(x_{ij}^k) &= x_{21}^3 x_{12}^3 - x_{11}^3 x_{22}^3 - \alpha(\alpha+1)(x_{21}^3 - x_{12}^3)^2. \end{aligned}$$

The corresponding polynomial functions f_1, f_2, and f_3 in the 27 variables x_{ij}^k with $i, j, k = 1, 2, 3$ are defined on the affine variety $\mathrm{Alg}_3(\mathbb{C})$.[11] Now, by computations one can show that the functions f_1, f_2, and f_3 are zero on the whole orbit of $C_5(\alpha)$. By definition of the Zariski topology the corresponding equations must hold on the orbit closure of $C_5(\alpha)$. Let $(c_{ij}^k)_{ij,k}^{B_5(\beta)}$ be the vector of structure constants of the algebra $B_5(\beta)$. In this notation, if we would have $B_5(\beta) \in \overline{\mathrm{O}(C_5(\alpha))}$ then the following equations must hold:

$$f_l((c_{ij}^k)_{ij,k}^{B_5(\beta)}) = 0 \quad \text{for } l = 1, 2, 3.$$

This is true for f_1 and f_2, however we have:

$$f_3((c_{ij}^k)_{ij,k}^{B_5(\beta)}) = \beta(\beta-1) - \alpha(\alpha+1).$$

This equation is zero if and only if $\beta = -\alpha$ or $\beta = 1+\alpha$. These two different solutions for β correspond to the isomorphic algebras $B_5(-\alpha) \cong B_5(1+\alpha)$. By transitivity, using Proposition 4.19, we can exclude a degeneration to the algebras $B_4(\bar{\alpha})$ with $\bar{\alpha} \neq -\alpha, 1+\alpha$ as well. Finally there are the following degenerations:

$$C_5(\alpha) \to_{\deg} B_4(-\alpha) \text{ by } g_t = \begin{pmatrix} -\frac{1}{(\alpha+1)t^2} & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ \frac{1-2\alpha}{(\alpha+1)\alpha t^3} & -\frac{1}{t} & 1 \end{pmatrix} \quad \text{for all } \alpha \neq 0, -1,$$

and therefore by transitivity, using Lemma 4.21 we have $C_5(-\frac{1}{2}) \to_{\deg} B_3$ and $C_5(\alpha) \to_{\deg} B_5(-\alpha)$.

Nevertheless $C_5(\alpha)$ cannot degenerate to the algebra $B_4(\alpha+1)$ because of the following equation:

$$f_4(x_{ij}^k) = \alpha(x_{32}^1 x_{33}^2 - x_{33}^1 x_{32}^2) + (\alpha+1)(x_{33}^1 x_{32}^2 - x_{23}^1 x_{33}^2).$$

This equation is again zero on the whole orbit of $C_5(\alpha)$, however for[12] $\widetilde{B_4}(\alpha)$ we have

$$f_4((c_{ij}^k)_{ij,k}^{\widetilde{B_4}(\bar{\alpha})}) = -\alpha - \bar{\alpha}.$$

[11] We wrote the arguments x_{ij}^k with upper and lower indices to emphasize that the values we consider here are given by vectors of structure constants. In this sense the indices of x_{ij}^k and c_{ij}^k correspond, which makes an evaluation at the point $(c_{ij}^k)_{ij,k}$ much more easier.

[12] We defined the algebra $\widetilde{B_4}(\alpha)$ in Proposition 4.19.

For the algebra B_3 we have[13] $f_3(B_3) = -\frac{1}{4} - \alpha(\alpha+1)$. This leads to $\alpha = -\frac{1}{2}$ as only solution.

For the special cases $\alpha = 0, -1$ we can't have degenerations from $C_5(\alpha)$ to the algebras $B_4(\bar{\alpha})$ for any $\bar{\alpha}$ because of their orbit dimensions. Furthermore the same is true for the algebras B_3 and $B_5(\beta)$ for all $\beta \neq 0$. We use the dimension of the left-annihilator to see this:

$$\begin{aligned}
&\dim \operatorname{Ann}_L(C_5(0)) = \dim \operatorname{Ann}_L(C_5(-1)) = 2\\
&\dim \operatorname{Ann}_L(B_3) = \dim \operatorname{Ann}_L(B_5(\beta)) = 1 \quad \text{for all } \beta \neq 0, 1.
\end{aligned}$$

We know that the dimension of the left-annihilator has to increase with respect to a degeneration (Lemma 3.29). However, to $B_5(0) \cong B_5(1)$ degenerations do exist:

$$\begin{aligned}
C_5(0) \to_{\text{deg}} B_5(1) \cong B_5(0) \text{ by } g_t &= \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\\
C_5(-1) \to_{\text{deg}} B_5(0) \cong B_5(1) \text{ by } g_t &= \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & t & 0 \\ 0 & 1 & 1 \end{pmatrix}.
\end{aligned}$$

□

Remark 4.31. In the last proof it would have been sufficient to take only the polynomial f_3 to exclude a degeneration from $C_5(\alpha)$ to $B_5(\beta)$ for all $\beta \neq -\alpha, 1+\alpha$. However, there are algebras isomorphic to $B_5(\beta)$ for which f_3 is zero. The system of equations $\{f_1, f_2, f_3\}$ will always be non-zero for some f_i in the orbit of $B_5(\beta)$ with $\beta \neq -\alpha, 1+\alpha$.

Lemma 4.32. *All possible degenerations of 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_3$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_2$ are listed in the table below.*

C	$\overline{\mathrm{O}(C)}$
C_1	$B_3, B_4(\alpha), B_5(\beta)$
C_2	$B_4(0), B_5(0)$
C_3	$B_4(1), B_5(1)$
C_4	$B_4(0), B_5(0)$
$C_5(0)$	$B_5(0)$
$C_5(-\frac{1}{2})$	$B_3, B_4(\frac{1}{2}), B_5(\frac{1}{2})$
$C_5(-1)$	$B_5(0)$
$C_5(\alpha)_{\alpha\neq 0,-\frac{1}{2},-1}$	$B_4(-\alpha), B_5(-\alpha)$
$C_6(0)$	$B_3, B_4(\alpha), B_5(\beta)$
$C_6(-1)$	$B_5(0)$
$C_6(\beta)_{\beta\neq 0,-1}$	$B_3, B_4(\alpha), B_5(\bar{\beta})$
$C_7(-1)$	$B_5(0)$
$C_7(\gamma)_{\gamma\neq 0,-1}$	$B_4(0), B_5(0)$

[13] We write $f_i(A)$ for the polynomial funcion applied to the vector of structure constants of the algebra A.

Proof. For the algebras in class C we treat all the orbit closures that lie in class B consecutively. Before we start doing so we remark that regarding orbit dimensions only the following degenerations to the algebras B_1 and B_2 are possible. Namely $C_1 \to_{\deg} B_1$ and $C_6(\beta)_{\beta\neq -1} \to_{\deg} B_1$. We treat these cases first, so we can then focus on degenerations to the algebras B_3, $B_4(\alpha)$, and $B_5(\beta)$ only.

To exclude degenerations from C_1 and $C_6(0)$ to B_1 we use Lemma 3.50 and the fact that

$$\det \mathrm{L}^{C_1}(x) = \det \mathrm{L}^{C_6(0)}(x) = 0$$

for all $x \in C_1$ and $x \in C_6(0)$, respectively. This is a Zariski closed condition which does not hold for the algebra B_1 since $\det \mathrm{L}^{B_1}(e_1) = 1$. Degenerations from $C_6(\beta)_{\beta\neq -1,0}$ to B_1 are also impossible because the quadratic operator identities

$$T(x) = \mathrm{L}^2(x) - \mathrm{L}(x)\mathrm{R}(x) - \frac{\beta+1}{\beta}\mathrm{R}(x)\mathrm{L}(x)$$

are zero for all $\beta \neq -1, 0$ and

$$T^{B_1}(x) = \frac{1}{\beta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1x_2 & x_1^2 & 0 \end{pmatrix} \neq 0.$$

Now we start our study of degenerations from algebras in class C to algebras in class B step by step. We begin with C_2 for reasons of transitivity. The argument of Proposition 4.13 excludes a degeneration to B_3, $B_4(\alpha)$ for all $\alpha \neq 0$, and $B_5(\beta)$ for all $\beta \neq 0$. For the exceptional values $\alpha = \beta = 0$ we have degenerations:

$$C_2 \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} \frac{1}{t^2} & \frac{1}{t^2} & 0 \\ \frac{1}{t} & 1 & 0 \\ 0 & -\frac{1}{t^3} & 1 \end{pmatrix},$$

and $C_2 \to_{\deg} B_5(0)$ by transitivity.

The algebra C_3 has only the algebras $B_4(1)$ and $B_5(1)$ in its closure. To see this we use $\dim \mathrm{Ann}_L(C_3)$ which equals two. By Lemma 3.29 we know that for every algebra in $\overline{\mathrm{O}(C_3)}$ the dimension of the left-annihilator has to be equal or higher than that of C_3. Beneath the algebras $\{B_3, B_4(\alpha), B_5(\beta)\}$ this is only possible for the algebras $B_4(1)$ and $B_5(1)$ for which we have indeed degenerations

$$C_2 \to_{\deg} B_4(1) \text{ by } g_t = \begin{pmatrix} 0 & \frac{1}{t^2} & 0 \\ \frac{1}{t} & 0 & 0 \\ -\frac{1}{t^3} & -\frac{1}{t^3} & 1 \end{pmatrix},$$

and $C_3 \to_{\deg} B_5(1)$ by transitivity.

Now we continue with the algebra C_1. By Lemma 4.28 we know that C_1 degenerates to C_2 and C_3. Therefore we have $C_1 \to_{\deg} B_4(0), B_4(1)$. Moreover we can find degenerations for all the other values of $\alpha \neq 0, 1$

$$C_1 \to_{\deg} B_4(\alpha) \text{ by } g_t = \begin{pmatrix} 0 & \frac{1}{t^2} & 0 \\ \frac{1}{t} & 0 & 0 \\ -\frac{1}{\alpha t^3} & -\frac{1}{t^3} & \frac{1}{\alpha(\alpha-1)t^3} \end{pmatrix}.$$

By Proposition 4.19 and transitivity there are degenerations from C_1 to B_3 and $B_5(\beta)$ for every $\beta \in \mathbb{C}$.

The situation for the algebra C_4 is slightly different than that for C_3. We now regard the dimension of the right-annihilator, which equals two for C_4. We use again Lemma 3.29 to exclude degenerations to the algebras B_3, $B_4(\alpha)$ for all $\alpha \neq 0$, and $B_5(\beta)$ for all $\beta \neq 0$, which all have right-annihilators with dimension equal to one. We however have degenerations:

$$C_4 \rightarrow_{\text{deg}} B_4(0) \text{ by } g_t = \begin{pmatrix} 0 & 0 & \frac{1}{t^2} \\ \frac{1}{t} & 0 & 0 \\ \frac{1}{t^3} & 1 & \frac{1}{t^3} \end{pmatrix},$$

and $C_4 \rightarrow_{\text{deg}} B_5(0)$ by transitivity.

Considering the family of algebras $C_6(\beta)_{\beta \neq -1}$ we can find, for now, only degenerations for the values $\beta \neq -1, -\alpha$:

$$C_6(\beta) \rightarrow_{\text{deg}} B_4(\alpha) \text{ by } g_t = \begin{pmatrix} -\frac{1}{(\beta+1)t^2} & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ -\frac{\alpha-\beta+1}{\alpha(\beta+1)t^3} & -\frac{1}{t} & \frac{1}{\alpha(\alpha+\beta)t^3} \end{pmatrix},$$

where α must not be equal to 0. Additionally we have the following:

$$C_6(-\alpha) \rightarrow_{\text{deg}} B_4(\alpha) \text{ by } g_t = \begin{pmatrix} \frac{1}{t} & 0 & \frac{1+\sqrt{\frac{t^3-4\alpha}{t^5}}}{2\alpha t} \\ 0 & 0 & \frac{1}{t^2} \\ \frac{2\alpha(1-2\alpha)}{(\alpha-1)t^3(1+t\sqrt{\frac{t^3-4\alpha}{t^5}})} & 1 & -\frac{1}{t^3} \end{pmatrix},$$

where α can't take the values 0 and 1. There is only one value of α that is not covered by the last two degeneration matrices, namely $\alpha = 0$. However, also in this case degenerations are possible for all $\beta \neq -1$:

$$C_6(\beta) \rightarrow_{\text{deg}} B_4(0) \text{ by } g_t = \begin{pmatrix} \frac{1}{t^3} & 0 & \frac{1}{(t-\beta)t^3} \\ \frac{1}{t} & 0 & 0 \\ \frac{t+\beta+1}{(\beta+1)(t+1)t^4} & 1 & \frac{1}{(1+t)(t-\beta)t^4} \end{pmatrix}.$$

In conclusion we have $C_6(\beta)_{\beta \neq -1} \rightarrow_{\text{deg}} B_4(\alpha)$ for all $\alpha \in \mathbb{C}$. Hence, by transitivity with $B_4(\frac{1}{2}) \rightarrow_{\text{deg}} B_3$ and $B_4(\alpha) \rightarrow_{\text{deg}} B_5(\alpha)$ we have $C_6(\beta)_{\beta \neq -1} \rightarrow_{\text{deg}} B_3, B_5(\bar{\beta})$ for all $\bar{\beta} \in \mathbb{C}$.

For the exceptional value $\beta = -1$ we can't have degenerations to B_3, $B_4(\alpha)$ for all $\alpha \in \mathbb{C}$, and $B_5(\bar{\beta})$ for all $\bar{\beta} \neq 0$, because of the following dimensions of vector spaces:

$$\begin{aligned} &\dim \operatorname{Der}_{(1,1,0)}(C_6(-1)) = 5, \\ &\dim \operatorname{Der}_{(1,1,0)}(B_3) = 3, \\ &\dim \operatorname{Der}_{(1,1,0)}(B_4(\alpha)) = 3 \text{ if } \alpha \neq \tfrac{1}{2}, \\ &\dim \operatorname{Der}_{(1,1,0)}(B_4(\tfrac{1}{2})) = 4, \\ &\dim \operatorname{Der}_{(1,1,0)}(B_5(\bar{\beta})) = 3 \text{ if } \bar{\beta} \neq 0, 1. \end{aligned}$$

To lie in the orbit closure of $C_6(-1)$ an algebra B has to satisfy $\dim \operatorname{Der}_{(1,1,0)}(B) \geqslant 5$ by Lemma 3.34. This is not valid for either of these algebras except $B_5(1) \cong B_5(0)$ for which we indeed have a degeneration

$$C_6(-1) \rightarrow_{\text{deg}} B_5(1) \text{ by } g_t = \begin{pmatrix} 0 & 0 & \frac{1}{t^2} \\ \frac{1}{t} & 0 & 0 \\ 1 & 1 & \frac{1}{t^3} \end{pmatrix}.$$

Finally we consider the family of algebras $C_7(\gamma)$ where $\gamma \neq 0$ by definition. We start with the special case $C_7(-1)$ which has a different orbit dimension as $C_7(\gamma)$ for any γ other than -1. Using this fact we derive that $C_7(-1)$ cannot degenerate to $B_4(\alpha)$ for any $\alpha \in \mathbb{C}$. By transitivity a degeneration to B_3 is also impossible. Next we compute the following dimensions

$$\begin{aligned}&\dim \mathrm{Der}_{(1,1,0)}(B_5(\beta)) = 3 \text{ for all } \beta \neq 0, 1 \text{ and}\\ &\dim \mathrm{Der}_{(1,1,0)}(C_7(-1)) = 5.\end{aligned}$$

Therefore, by Lemma 3.34, we have no degenerations from $C_7(-1)$ to $B_5(\beta)$ except

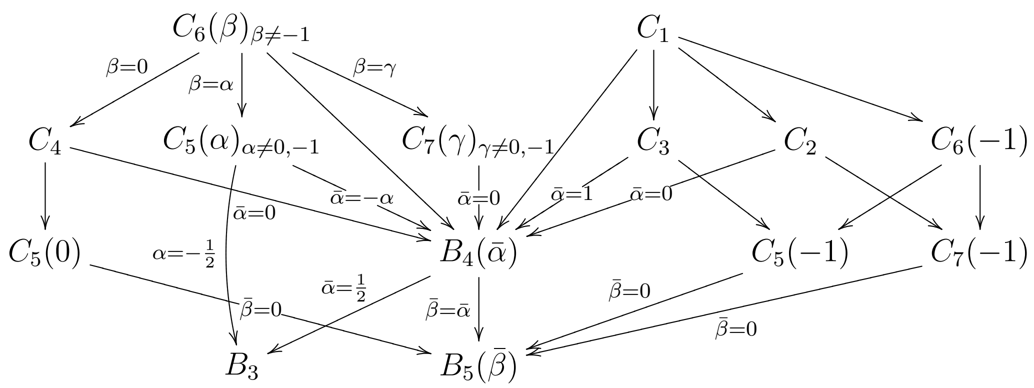
$$C_7(-1) \to_{\deg} B_5(1) \cong B_5(0) \text{ by } g_t = \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & t & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

For the algebras $C_7(\gamma)$ with $\gamma \neq 0, -1$ the exact same argument as for the algebra C_2 holds. We have a degenerations

$$C_7(\gamma) \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} -\frac{1}{(\gamma+1)t^2} & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ -\frac{1}{\gamma(\gamma+1)t^3} & -\frac{1}{t} & 1 \end{pmatrix}$$

and $C_7(\gamma) \to_{\deg} B_5(0)$ by transitivity. $\square$

Corollary 4.33. *The Hasse diagram of all* 3*-dimensional Novikov algebras with associated Lie algebras* $\mathfrak{g}_3$ *and* $\mathfrak{g}_2$ *is given as follows.*



4.6.3 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_3$ and $\mathfrak{g}_1$

Proposition 4.34. *The orbit closures of the algebras* $C_5(\alpha)$ *in class* A *are given as follows.*

C	$\overline{\mathrm{O}(C)}$
$C_5(0)$	A_1, A_5
$C_5(-1)$	A_1, A_5
$C_5(\alpha)_{\alpha\neq 0,-1}$	A_1, A_5

Proof. We show that degenerations from $C_5(\alpha)$ to A_2, A_9, and A_{11} are impossible for every $\alpha \in \mathbb{C}$. After this we conclude with transitivity that degenerations to any other algebras than A_5 and A_1 are impossible too. We begin with the algebras A_2 and A_{11}, for which the $\mathfrak{R}_{\mathrm{p,q}}$-invariant equals one and three, respectively. We have $\mathfrak{R}_{\mathrm{p,q}}(C_5(\alpha)) = 2$ for all $\alpha \neq 0$ which contradicts a degeneration for those algebras. For the value $\alpha = 0$ we find $\dim C_5(0)^2 = 0$, whereas $\dim A_2^2 = 1$ and $\dim A_{11}^2 = 3$. Hence a degeneration is impossible by Lemma 3.23.

The algebra A_9 can't lie in the orbit closure of $C_5(\alpha)$ for any α. To see this we regard the polynomials f_1, f_2, and f_3 from Proposition 4.30. We have $f_3((c_{ij}^k)_{ij,k}^{A_9}) = 1$.

Finally we have degenerations from $C_5(\alpha)$ to A_5 for every α because of Proposition 4.30 and Lemma 4.23. □

Lemma 4.35. *All possible degenerations of* 3*-dimensional Novikov algebras with associated Lie algebra* $\mathfrak{g}_3$ *to Novikov algebras with associated Lie algebra* $\mathfrak{g}_1$ *are listed in the table below.*

C	$\overline{\mathrm{O}(C)}$
C_1	$A_1, A_2, A_5, A_6, A_9, A_{10}$
C_2	A_1, A_5
C_3	A_1, A_5
C_4	A_1, A_5
$C_5(\alpha)$	A_1, A_5
$C_6(-1)$	A_1, A_2, A_5
$C_6(\beta)_{\beta\neq -1}$	$A_1, A_2, A_5, A_6, A_9, A_{10}$
$C_7(\gamma)_{\gamma\neq 0}$	A_1, A_5

Proof. A lot of possible degenerations are excluded by the dimensions of the orbit space. As an overview we give a table where for the algebras on the left we list all algebras that can possibly lie in the orbit closure on the right.

C	$\overline{\mathrm{O}(C)}$
$C_6(0), C_1, C_6(\beta)_{\beta\neq -1}$	$A_1, A_2, A_5, A_6, A_7, A_9, A_{10}, A_{11}, A_{12}$
$C_2, C_3, C_4, C_5(\alpha)_{\alpha\neq 0,-1}, C_6(-1), C_7(\gamma)_{\gamma\neq 0,-1}$	$A_1, A_2, A_5, A_9, A_{10}, A_{11}$
$C_5(0), C_5(-1), C_7(-1)$	$A_1, A_2, A_5, A_9, A_{11}$

We start our proof with the algebra C_1. From the table above we conclude that degenerations to the algebras A_3, A_4, and A_8 are impossible. Also A_{11} can't lie in the orbit closure of C_1. Indeed, $\det \mathrm{L}^{C_1}(x) = 0$ for all $x \in C_1$ and because of Lemma 3.50 this must hold for every algebra in the orbit closure of C_1. However, for the algebra A_{11} we have $\det \mathrm{L}^{A_{11}}(e_3) = 1$. By transitivity the algebra A_{12} can't lie in the orbit closure of the algebra C_1 as well. Furthermore there is no degeneration to the algebra A_7. To see this we use Theorem 3.8 and its corollary. The third basis vector e_3 generates a 1-dimensional ideal in C_1. Therefore we can take the factor $C_1/\langle e_3\rangle$ which is isomorphic to the 2-dimensional Novikov algebra W_5. By Corollary 3.9 we know that in case we have a degeneration $C_1 \to_{\deg} A_7$ there must exist a 1-dimensional ideal I in A_7 such that

$$C_1/\langle e_3\rangle \cong W_5 \to_{\deg} A_7/I.$$

There are exactly two 1-dimensional ideals in A_7, namely $\langle e_1\rangle$ and $\langle e_3\rangle$. We have

$$A_7/\langle e_1\rangle \cong U_2 \quad \text{and} \quad A_7/\langle e_3\rangle \cong U_4.$$

However, by Corollary 4.5 neither U_2 nor U_4 lies in the orbit closure of W_5. Hence a degeneration of C_1 to A_7 is impossible. Finally we have

$$C_1 \to_{\text{deg}} A_6 \text{ by } g_t = \begin{pmatrix} 0 & \frac{1}{t^2} & 0 \\ \frac{1}{t} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by transitivity $C_1 \to_{\text{deg}} A_2, A_5, A_9, A_{10}$.

We continue with the orbit closure of the algebra C_2, which can only degenerate to the algebras A_2, A_5, A_9, A_{10}, and A_{11}. We consider the $\mathfrak{C}_{\text{p,q}}$-invariants of the algebras C_2, A_2, and A_{11}: $\mathfrak{C}_{\text{p,q}}(C_2) = 2$, $\mathfrak{C}_{\text{p,q}}(A_2) = 1$, and $\mathfrak{C}_{\text{p,q}}(A_{11}) = 3$. Therefore a degeneration from C_2 to A_2 and A_{11} is impossible. Furthermore, by the same argument worked out in Proposition 4.13 the algebra C_2 can't degenerate to the algebras A_9 and A_{10}. We use transitivity regarding $C_2 \to_{\text{deg}} B_4(0) \to_{\text{deg}} A_5$ (Lemma 4.46 and Lemma 4.23) to conclude that $C_2 \to_{\text{deg}} A_5$.

By the above diagram the algebra C_3 can't degenerate to the algebras A_3, A_4, A_6, A_7, A_8, and A_{12}. We exclude degenerations to the algebras A_9 and A_{10} by the dimension of the left-annihilator. We have $\dim \operatorname{Ann}_L(C_3) = 2$, but $\dim \operatorname{Ann}_L(A_9) = \dim \operatorname{Ann}_L(A_{10}) = 1$ which contradicts Lemma 3.29 in case of a degeneration. It is easily seen by the $\mathfrak{R}_{\text{p,q}}$-invariants that there are no degenerations to the algebras A_2 and A_{11}. We have $\mathfrak{R}_{\text{p,q}}(C_3) = 2$, $\mathfrak{R}_{\text{p,q}}(A_2) = 1$, and $\mathfrak{R}_{\text{p,q}}(A_{11}) = 3$. A degeneration from C_3 to A_5 is accomplished by transitivity via $C_3 \to_{\text{deg}} B_5(1) \to_{\text{deg}} A_5$.

The orbit dimension of the algebra C_4 is the same as for C_3. Therefore we have the same algebras that possibly lie in the orbit closure of C_4 as for C_3. We start with the algebra A_9 and consider this time the dimension of the right-annihilator $\dim \operatorname{Ann}_R(A_9) = 1$. Conversely, we have $\dim \operatorname{Ann}_R(C_4) = 2$ which would contradict a degeneration. By transitivity A_{10} can't lie in the orbit closure of C_4 as well. To exclude degenerations to the algebras A_2 and A_{11} we regard Corollary 3.21 and compute $\dim \delta^{(2)}(C_4) = 0$, $\dim \delta^{(2)}(A_2) = 1$, and $\dim \delta^{(2)}(A_{11}) = 3$. Like in the case of C_3 there is a degeneration to A_5 by transitivity.

We already treated the case $C_5(\alpha)$ in Proposition 4.34 and therefore continue with the orbit closures of the algebras $C_6(\beta)$. In this case we have two exceptional values for β, namely $\beta = 0, -1$. The cases $\beta \neq 0, -1$ and $\beta = 0$ are nearly the same. For these we will show that degenerations to A_7 and A_{11} are impossible and so are, by transitivity, degenerations to A_3, A_4, A_8, and A_{12}. For this we regard the factor of $C_6(\beta)$ by the 1-dimensional ideal $\langle e_3 \rangle$: $C_6(\beta)/\langle e_3 \rangle$ which is isomorphic to the 2-dimensional algebra $W_2(-\beta)$ via $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. However, we have only the following factors by 1-dimensional ideals for the algebras A_7 and A_{11}:

$$\begin{aligned} &A_7/\langle e_1 \rangle \cong U_2 \\ &A_7/\langle e_3 \rangle \cong U_4 \\ &A_{11}/\langle k_1 e_1 + k_2 e_2 \rangle \cong U_4 \quad \text{for all } k_1, k_2 \in \mathbb{C}. \end{aligned}$$

From Corollary 4.5 we know that $W_2(\beta)$ does neither degenerate to U_2 nor to U_4 if and only if $\beta \neq -1$. This shows that a degeneration from $C_6(\beta)$ to A_7 and A_{11} is impossible for every $\beta \neq -1$. Nevertheless there are the following degenerations for all $\beta \neq -1$:

$$C_6(\beta) \to_{\text{deg}} A_6 \text{ by } g_t = \begin{pmatrix} \frac{1}{(1+\beta)t^2} & 1 & 0 \\ -\frac{1}{t} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $C_6(\beta) \to_{\deg} A_2, A_5, A_9, A_{10}$ by transitivity using Corollary 4.10.

Now we study the case $\beta = -1$. It is immediately seen that $\det \mathrm{L}^{C_6(-1)}(x) = 0$ for every $x \in C_6(-1)$ which is a Zariski-closed condition on $\mathrm{Alg}_3(\mathbb{C})$. Hence $C_6(-1)$ can't degenerate to A_{11}. The same is true for A_9, if we regard the following dimensions:

$$\begin{aligned} \dim \mathrm{Der}_{(1,1,0)}(C_6(-1)) &= 5, \\ \dim \mathrm{Der}_{(1,1,0)}(A_9) &= 3. \end{aligned}$$

By Lemma 3.34 the dimension of $\mathrm{Der}_{(1,1,0)}(C_6(-1))$ has to increase. Hence the algebra A_9 can't lie in the orbit closure of the algebra $C_6(-1)$.

A non-degeneration to the algebras A_9 and A_{11} excludes already all algebras, except A_1, A_2, and A_5, of lying in the orbit closure of $C_6(-1)$. We indeed have a degeneration

$$C_6(-1) \to_{\deg} A_2 \text{ by } g_t = \begin{pmatrix} 0 & 0 & \frac{1}{t} \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

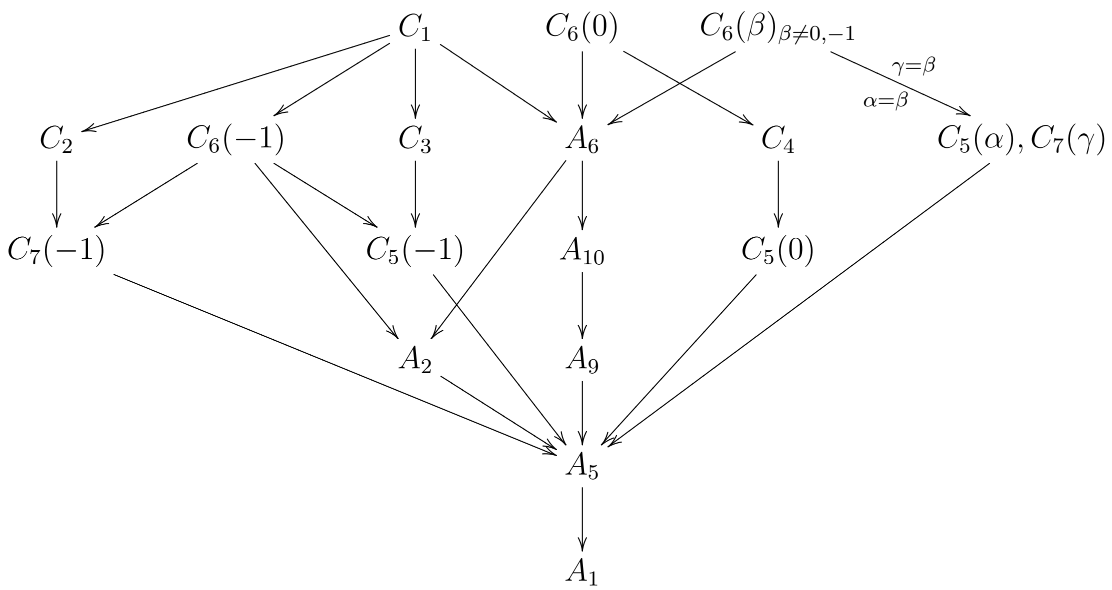
and therefore by transitivity $C_6(-1) \to_{\deg} A_5$.

Finally we observe the closures of the algebras $C_7(\gamma)$ for all $\gamma \neq 0$ in class A. The $\mathfrak{R}_{\mathrm{p,q}}$-invariant of $C_7(\gamma)$ equals three for which reason a degeneration to the algebra A_2 is impossble by $\mathfrak{R}_{\mathrm{p,q}}(A_2) = 1$. In order to exclude a degeneration to the algebra A_{11} we have to show that

$$\mathfrak{C}_{\mathrm{p,q}}(C_7(\gamma)) = \frac{(2\gamma^p + (\gamma+1)^p)(2\gamma^q + (\gamma+1)^q)}{2\gamma^{p+q} + (\gamma+1)^{p+q}}$$

can't take the value three for some p and q in $\mathbb{N}$. This can easily be shown with a computer. Moreover a degeneration to A_9 is impossible because of an argument brought in Proposition 4.13. By transitivity, no algebras except A_1 and A_5 can lie in the orbit closure of $C_7(\gamma)$. We have $C_7(\gamma) \to_{\deg} A_5 \to_{\deg} A_1$ by transitivity with $C_7(\gamma) \to_{\deg} B_5(0) \to_{\deg} A_5$ for all nonzero $\gamma \in \mathbb{C}$. This completes the proof. □

Corollary 4.36. *The Hasse diagram of all* 3*-dimensional Novikov algebras with associated Lie algebras* $\mathfrak{g}_3$ *and* $\mathfrak{g}_1$ *is given as follows. Lack of space forced us to omit the restrictions* $\alpha, \gamma \neq 0, -1$ *for the algebras* $C_5(\alpha)$ *and* $C_7(\gamma)$*.*



4.7 Degenerations of the classes A, B, and E(λ)

We have already treated the special classes $E(1)$ and $E(-1)$ in the sections 4.3 and 4.5. Therefore, in this section it is always assumed that the parameter λ can't take the values ± 1. Sometimes we don't mention this restriction in the text.

4.7.1 Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(\lambda)$

Proposition 4.37. *For a given $\lambda \in \mathbb{C}$ the algebra $E_{2,\lambda}$ degenerates properly only to the algebra $E_{1,\lambda}(-1)$ within class $E(\lambda)$.*

Proof. Regarding orbit dimensions, the algebras $E_{2,\lambda}$ can possibly degenerate only to the algebras $E_{1,\bar{\lambda}}(-1)$ and $E_{1,\bar{\lambda}}(-\bar{\lambda})$. To restrict these possibilities a little bit more we introduce a polynomial identity, linear in the structure constants. We set:

$$F(x_{ij}^k) = \sum_{i,j,k}^{3} r_{ij}^k \cdot x_{ij}^k,$$

where $r_{ij}^k \in \mathbb{C}$ for all $i, j, k = 1, 2, 3$. We write $F(E_{2,\lambda})$ to evaluate the function F at the point $(c_{ij}^k)_{ij,k}^{E_{2,\lambda}}$, the vector of structure constants of the algebra $E_{2,\lambda}$. If we want to have F zero on the whole $\mathrm{GL}_3(\mathbb{C})$-orbit of the algebra $E_{2,\lambda}$, we have to solve the equation

$$F(g \cdot E_{2,\lambda}) = 0,$$

in the coefficients $r_{ij}^k \in \mathbb{C}$, where $g \in \mathrm{GL}_3(\mathbb{C})$. For $\lambda \neq 2$ this solution is given by the relations

$$\begin{aligned}
&r_{11}^2 = r_{11}^3 = r_{12}^3 = r_{13}^2 = r_{21}^3 = r_{21}^1 = r_{22}^3 = r_{23}^1 = r_{31}^2 = r_{32}^1 = r_{33}^1 = r_{33}^2 = 0,\\
&r_{12}^2 = \tfrac{3}{\lambda} r_{11}^1,\ r_{13}^3 = 3r_{11}^1 - \lambda r_{12}^2,\ r_{21}^1 = \tfrac{3}{\lambda-2} r_{12}^1,\ r_{21}^2 = r_{11}^1 - r_{12}^2\\
&r_{22}^2 = r_{12}^1 + r_{21}^1,\ r_{23}^2 = r_{13}^1,\ r_{23}^3 = 3r_{12}^1 + (3-\lambda) r_{21}^1,\ r_{31}^1 = \tfrac{1}{\lambda-2}(2r_{13}^1 + r_{13}^1)\\
&r_{31}^3 = -2r_{11}^1 + \lambda r_{12}^2,\ r_{32}^2 = r_{13}^1 - r_{23}^2 + r_{31}^1,\ r_{32}^3 = -2r_{12}^1 + (\lambda-2) r_{21}^1,\ r_{33}^3 = r_{13}^1 + r_{31}^1.
\end{aligned}$$

If we take the polynomial function F with these values for r_{ij}^k we find that

$$F(E_{1,\bar{\lambda}}(\alpha)) = \tfrac{3}{\lambda-2}(1 + \alpha + \alpha\lambda + \bar{\lambda}) r_{13}^1.$$

So if the algebra $E_{1,\bar{\lambda}}(\alpha)$ would lie in the orbit closure of $E_{2,\lambda}$ we must have[14]

$$1 + \alpha + \alpha\lambda + \bar{\lambda} = 0$$

for some $\alpha, \bar{\lambda} \in \mathbb{C}$. If we set $\alpha = -1$ we get $\bar{\lambda} = \lambda$ and for $\alpha = -\bar{\lambda}$ we get $\bar{\lambda} = \frac{1}{\lambda}$, corresponding to the isomorphic algebras $E_{1,\bar{\lambda}}(-1) \cong E_{1,\frac{1}{\lambda}}(-\bar{\lambda})$. Indeed there are degenerations

$$E_{2,\lambda} \to_{\deg} E_{1,\lambda}(-1) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

[14] The solutions for the equation $F(g \cdot E_{2,2} = 0$ leads to the same condition as for the general case.

Remark 4.38. In the next proposition and in the following lemma we are confronted with the situation of using the $\mathfrak{C}_{\mathrm{p,q}}$-invariant frequently. On the one hand we are lucky that it is definable in the most cases, on the other hand, however, because of the various parameters that occur especially in this class we obtain highly complicated identities when we compare these invariants. It seems to be no other way of using the computer to solve the equations that arise when we compare the $\mathfrak{C}_{\mathrm{p,q}}$-invariants of two different algebras. To keep the proofs at a readable level we are forced to abbreviate the arguments and present only the outcome of our calculations. As far as it is possible we give the algorithms in the appendix.

Proposition 4.39. *For a given $\beta \in \mathbb{C}$ the algebra $E_5(\beta)$ degenerates properly only to the algebra $E_{1,\frac{1}{2}}(\beta)$ within class $E(\lambda)$, except for the value $\beta = -\frac{1}{2}$ where we have $E_5(-\frac{1}{2}) \to_{\deg} E_6$.*

Proof. There are two special values for $\beta \in \mathbb{C}$ which we want to treat seperately. These are $\beta = -\frac{1}{2}, -1$. We start with the case $\beta = -\frac{1}{2}$. Using the $\mathfrak{C}_{\mathrm{p,q}}$-invariant we find that there is only one possible degeneration to the algebras $E_{1,\lambda}(\alpha)$, namely: $E_5(-\frac{1}{2}) \to_{\deg} E_{1,\frac{1}{2}}(-\frac{1}{2})$. This degeneration indeed exists and is a special case of $E_5(\beta) \to_{\deg} E_{1,\frac{1}{2}}(\beta)$, what we will show later. Neither of the algebras $E_{2,\bar{\lambda}}$ with $\bar{\lambda} \neq 2$ can lie in the orbit closure of the algebra $E_5(-\frac{1}{2})$ because of the $\mathfrak{C}_{\mathrm{p,q}}$-invariant again. Unfortunately we have

$$\mathfrak{C}_{\mathrm{p,q}}(E_5(-\tfrac{1}{2})) = \mathfrak{C}_{\mathrm{p,q}}(E_{2,2}).$$

We use Theorem 3.8 to show that a degeneration doesn't exist for the value $\bar{\lambda} = 2$ too. Therefor we regard the factor $E_5(-\frac{1}{2})/\langle e_2 \rangle$ which is isomorphic to the 2-dimensional Novikov algebra $W_2(1)$. We remember that for the algebra $W_2(1)$ we have only trivial and improper degenerations (Corollary 4.5). However, there is no 1-dimensional ideal in the algebra $E_{2,2}$ that has a factor which is abelian or isomorphic to $W_2(1)$. Hence, by the above mentioned theorem a degeneration from $E_5(-\frac{1}{2})$ to $E_{2,2}$ is impossible. The algebras E_3, E_4, and $E_5(\beta)$ for every $\beta \in \mathbb{C}$ other than $-\frac{1}{2}$ are immediately excluded of lying in the orbit closure of $E_5(-\frac{1}{2})$ by reasons of orbit dimension. The $\mathfrak{C}_{\mathrm{p,q}}$-invariant applies once more, preventing the algebra $E_5(-1)$ of lying in the orbit closure of $E_5(-\frac{1}{2})$. We finally have

$$E_5(-\tfrac{1}{2}) \to_{\deg} E_6 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{t} & -t & 0 \\ \frac{1}{t} & 0 & 1 \end{pmatrix}.$$

Regarding orbit dimensions the algebra $E_5(-1)$ cannot degenerate to any algebras except $E_{1,\bar{\lambda}}(-1)$ and $E_{1,\bar{\lambda}}(-\bar{\lambda})$. The $\mathfrak{C}_{\mathrm{p,q}}$-invariant excludes degenerations for all values of $\bar{\lambda} \in \mathbb{C}$ except $\bar{\lambda} = \frac{1}{2}$ for $E_{1,\bar{\lambda}}(-1)$ and $\bar{\lambda} = 2$ for $E_{1,\bar{\lambda}}(-\bar{\lambda})$. Considering $E_{1,\frac{1}{2}}(-1) \cong E_{1,2}(-2)$ we have only one degeneration for the algebra $E_5(-1)$, which is a special case of $E_5(\beta) \to_{\deg} E_{1,\frac{1}{2}}(\beta)$.

The orbit dimension of the algebra $E_5(\beta)$ equals that of $E_5(-\frac{1}{2})$ for every $\beta \neq -1$ for which reason we exclude a degeneration to the algebras E_3, E_4, and $E_5(\beta)$ for every $\beta \neq -1$ at once. It is easily seen that the $\mathfrak{C}_{\mathrm{p,q}}$-invariant prevents the algebras $E_5(-1)$ and E_6 of lying in the orbit closure of $E_5(\beta)$ for every $\beta \neq -1$.

We now want to know for which pairs $(\lambda, \alpha) \in \mathbb{C}^2$ the algebra $E_{1,\lambda}(\alpha)$ lies in the orbit closure of the algebra $E_5(\beta)$. Because of the three parameters $\lambda, \alpha, \beta \in \mathbb{C}$ involved this is a difficult question. To answer it we use a combination of two methods:

1. A poynomial $F(x_{ij}^k) = \sum_{i,j,k=1}^{3} r_{ij}^k \cdot x_{ij}^k$ that defines a Zariski-equation on the orbit of $E_5(\beta)$.

2. The $\mathfrak{C}_{\mathrm{p,q}}$-invariant.

The values of the coefficients r_{ij}^k such that $F(E_5(\beta)) = 0$ are listed in the appendix. Evaluating the function F at the point $(c_{ij}^k)_{ij,k}^{E_{1,\lambda}(\alpha)}$ we find that[15]

$$\begin{aligned} F(E_{1,\lambda}(\alpha)) &= \tfrac{3\alpha-2\beta(1+\lambda)}{1+2\beta} \cdot r_{13}^1 \quad \text{for all } \beta \neq -\frac{1}{2} \text{ and} \\ F(E_{1,\lambda}(\alpha)) &= (1+3\alpha+\lambda) \cdot r_{31}^1 \quad \text{for } \beta = -\frac{1}{2}. \end{aligned}$$

In both cases $F(E_{1,\lambda}(\alpha)) = 0$ if and only if $\alpha = \frac{2}{3}\beta(1+\lambda)$. We set for these values

$$\mathfrak{C}_{1,1}(E_5(\beta)) = \mathfrak{C}_{1,1}(E_{1,\lambda}(\tfrac{2}{3}\beta(1+\lambda)))$$

and derive the following three solutions in λ:

1. Every $\lambda \in \mathbb{C}$ with $1 - \lambda + \lambda^2 \neq 0$ if $\beta = -\frac{1}{2}$,

2. $\lambda = \frac{1}{2}$ where $5 + 12\beta + 12\beta^2 \neq 0$, and

3. $\lambda = 2$ where $5 + 12\beta + 12\beta^2 \neq 0$.

The first solution was already checked when we classified the orbit closures of the algebra $E_5(-\frac{1}{2})$. The second solution enables the algebra $E_{1,\frac{1}{2}}(\beta)$ of lying in the orbit closure $\overline{\mathrm{O}(E_5(\beta))}$. Finally the third solution corresponds to the algebra $E_{1,2}(2\beta)$. Regarding $E_{1,\frac{1}{2}}(\beta) \cong E_{1,2}(2\beta)$ we have only the following degeneration:

$$E_5(\beta) \to_{\deg} E_{1,\frac{1}{2}}(\beta) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally we want to show that neither of the algebras $E_{2,\lambda}$ lies in the orbit closure of the algebra $E_5(\beta)$ for every $\beta \in \mathbb{C}$. To see this we use the polynomial F we introduced before and evaluate it at the point $(c_{ij}^k)_{ij,k}^{E_{2,\lambda}}$:

$$\begin{aligned} F(E_{2,\lambda}) &= -\tfrac{3+2\beta(1+\lambda)}{1+2\beta} \cdot r_{13}^1 \quad \text{for all } \beta \neq -\tfrac{1}{2} \text{ and} \\ F(E_{2,\lambda}) &= (\lambda - 2) \cdot r_{31}^1 \quad \text{for } \beta = -\tfrac{1}{2}. \end{aligned}$$

We already investigated the orbit closures of the algebra $E_5(-\frac{1}{2})$ for which reason we shall continue with the general case. The polynomial F applied to the structure constants of the algebra $E_{2,\lambda}$ equals zero if and only if $\lambda = -\frac{3+2\beta}{2\beta}$ and so we set

$$\mathfrak{C}_{1,1}(E_5(\beta)) = \mathfrak{C}_{1,1}(E_{2,-\frac{3+2\beta}{2\beta}}).$$

This equation has the solutions $\beta = -\frac{1}{2}, -1$ which correspond to the two special values of β we have treated at the beginning of this proof. $\square$

[15] For the sake of completeness we mention the case $\beta = -\frac{1}{2}$ once again here.

Lemma 4.40. *All degenerations of 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(\lambda)$ are listed in the table below.*

E	$\overline{\mathrm{O}(E)}$
$E_{1,\lambda}(\alpha)$ $\lambda \neq 0, \pm 1$ $\lambda \neq -1, -\lambda$	$-$
$E_{1,\lambda}(-1)$	$-$
$E_{1,\lambda}(-\lambda)$	$-$
$E_{2,\lambda}$ $\lambda \neq 0, \pm 1$	$E_{1,\lambda}(-1)$
E_3	$E_{1,2}(-1), E_{2,2}, E_6$
E_4	$E_{1,\frac{1}{2}}(-1), E_{2,\frac{1}{2}}, E_5(-1)$
$E_5(\beta)_{\beta \neq -\frac{1}{2}, -1}$	$E_{1,\frac{1}{2}}(\beta)$
$E_5(-\frac{1}{2})$	$E_{1,2}(-1), E_6$
$E_5(-1)$	$E_{1,\frac{1}{2}}(-1)$
E_6	$E_{1,2}(-1)$

Proof. A lot of possible degenerations can be excluded by the dimensions of the orbit spaces. We therefore order the algebras in class $E(\lambda)$ by their orbit dimension, beginning with the highest on the left:

$$E_3, E_4, E_5(\beta)_{\beta \neq -1}; \quad E_{1,\lambda}(\alpha)_{\lambda \neq -1, -\lambda}, E_{2,\lambda}, E_5(-1), E_6; \quad E_{1,\lambda}(-1), E_{1,\lambda}(-\lambda).$$

In this ordering, by Theorem 1.16, degenerations can go only from the left to the right. In fact all non-degenerations in this proof can be handled by the dimensions of the orbit space or the $\mathfrak{C}_{\mathrm{p,q}}$-invariant. However, as already explained in Remark 4.38 this has to be done in a computational way. Nevertheless we give a list of all the $\mathfrak{C}_{\mathrm{p,q}}$-invariants in use:

$$\mathfrak{C}_{\mathrm{p,q}}(E_{1,\lambda}(\alpha)) = \frac{(\alpha^p + (\alpha+1)^p + (\alpha+\lambda)^p)(\alpha^q + (\alpha+1)^q + (\alpha+\lambda)^q)}{\alpha^{p+q} + (\alpha+1)^{p+q} + (\alpha+\lambda)^{p+q}},$$

$$\mathfrak{C}_{\mathrm{p,q}}(E_{1,\lambda}(-\lambda)) = \frac{(\lambda^p + (\lambda-1)^p)(\lambda^q + (\lambda-1)^q)}{\lambda^{p+q} + (\lambda-1)^{p+q}},$$

$$\mathfrak{C}_{\mathrm{p,q}}(E_{2,\lambda}) = \mathfrak{C}_{\mathrm{p,q}}(E_{1,\lambda}(-1)) = \frac{((-1)^p + (\lambda-1)^p)((-1)^q + (\lambda-1)^q)}{(-1)^{p+q} + (\lambda-1)^{p+q}},$$

$$\mathfrak{C}_{\mathrm{p,q}}(E_3) = \mathfrak{C}_{\mathrm{p,q}}(E_5(-1/2)) = \mathfrak{C}_{\mathrm{p,q}}(E_6) = \frac{((-1)^p + 1)((-1)^q + 1)}{(-1)^{p+q} + 1},$$

$$\mathfrak{C}_{\mathrm{p,q}}(E_4) = \mathfrak{C}_{\mathrm{p,q}}(E_5(-1)) = \frac{((1/2)^p + 1)((1/2)^q + 1)}{(1/2)^{p+q} + 1},$$

$$\mathfrak{C}_{\mathrm{p,q}}(E_5(\beta)) = \frac{(\beta^p + (\beta+1)^p + (\beta+1/2)^p)(\beta^q + (\beta+1)^q + (\beta+1/2)^q)}{\beta^{p+q} + (\beta+1)^{p+q} + (\beta+1/2)^{p+q}}.$$

We are therefore done by listing all the degenerations in this class:

$$E_3 \to_{\deg} E_{2,2} \text{ by } g_t = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & t & 0 \end{pmatrix},$$

$$E_3 \to_{\deg} E_6 \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix},$$

$$E_4 \to_{\deg} E_{2,\frac{1}{2}} \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t} \end{pmatrix},$$

$$E_4 \to_{\deg} E_5(-1) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2 & 0 \\ t & 0 & t \end{pmatrix},$$

$$E_5(\beta) \to_{\deg} E_{1,\frac{1}{2}}(\beta) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_6 \to_{\deg} E_{1,\frac{1}{2}}(-\tfrac{1}{2}) \text{ by } g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t} \end{pmatrix}.$$

The degenerations $E_3 \to_{\deg} E_{1,2}(-1) \cong E_{1,\frac{1}{2}}(-\frac{1}{2})$ and $E_4 \to_{\deg} E_{1,\frac{1}{2}}(-1) \cong E_{1,2}(-2)$ follow by transitivity. □

Corollary 4.41. *The Hasse diagram of all* 3*-dimensional Novikov algebras with associated Lie algebra* $\mathfrak{g}_5(\lambda)$ *is given as follows.*

$$\begin{tikzcd}
E_5(-\frac{1}{2}) \arrow[d] & E_3 \arrow[dl] \arrow[d, "\lambda=2"'] & E_4 \arrow[dl, "\lambda=\frac{1}{2}"'] \arrow[d] & E_5(\beta)_{\beta\neq-\frac{1}{2},-1} \arrow[d, "\lambda=\frac{1}{2},\,\alpha=\beta"'] \\
E_6 \arrow[dr, "\bar\lambda=2"'] & E_{2,\lambda} \arrow[d, "\lambda=\bar\lambda"'] & E_5(-1) \arrow[dl, "\bar\lambda=\frac{1}{2}"] & E_{1,\lambda}(\alpha) \\
 & E_{1,\bar\lambda}(-1) & &
\end{tikzcd}$$

4.7.2 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_5(\lambda)$ and $\mathfrak{g}_2$

Proposition 4.42. *The orbit closures of the algebras* $E_5(\beta)$ *in class* B *are given as follows.*

E	$\overline{\mathrm{O}(E)}$
$E_5(-\frac{1}{2})$	$B_3, B_5(\bar\beta)$
$E_5(-1)$	$B_4(1), B_5(1)$
$E_5(\beta)_{\alpha\neq-\frac{1}{2},-1}$	$B_3, B_4(\bar\alpha), B_5(\bar\beta)$

Proof. By Lemma 4.40 we know that $E_5(-\frac{1}{2}) \to_{\deg} E_6$. Furthermore, in the next Lemma we prove that $E_6 \to_{\deg} B_3, B_5(\bar\beta)$ for all $\beta \in \mathbb{C}$. By transitivity we have

$E_5(-\frac{1}{2}) \to_{\deg} B_3, B_5(\bar{\beta})$ for all $\bar{\beta} \in \mathbb{C}$. The algebra $E_5(-\frac{1}{2})$ doesn't degenerate to the algebra $B_4(\bar{\alpha})$ for any $\bar{\alpha}$. To see this we consider the factor of $E_5(-\frac{1}{2})$ by the ideal $\langle e_2 \rangle$ which is the 2-dimensional Novikov algebra $W_2(1)$. There exists only a single 1-dimensional ideal in $B_4(\bar{\alpha})$, namely $\langle e_3 \rangle$. Its factor is given by $B_4(\bar{\alpha})/\langle e_3 \rangle \cong U_5$ which does not lie in the orbit closure of the algebra $W_2(1)$ by Corollary 4.5. Therefore, by Theorem 3.8 a degeneration from $E_5(-\frac{1}{2})$ to $B_4(\bar{\alpha})$ is impossible for all $\bar{\alpha}$.

Next we show that a degeneration from $E_5(-1)$ to $B_5(\bar{\beta})$ for all $\bar{\beta} \neq 0, 1$ is impossible. We consider the following set of polynomials:

$$\begin{aligned} f_1(x_{ij}^k) &= x_{32}^1 x_{23}^1 - x_{22}^1 x_{33}^1, \\ f_2(x_{ij}^k) &= x_{31}^2 x_{13}^2 - x_{11}^2 x_{33}^2, \\ f_3(x_{ij}^k) &= x_{21}^3 x_{12}^3 - x_{11}^3 x_{22}^3. \end{aligned}$$

Now, by computations one can show that the functions f_1, f_2, and f_3 are zero on the whole orbit of $E_5(-1)$. By definition of the Zariski topology the corresponding equations must hold on the orbit closure of $E_5(-1)$. In this notation, if $B_5(\bar{\beta})$ would lie in $\overline{\mathrm{O}(E_5(-1))}$ the functions f_l for $l = 1, 2, 3$ need to be zero at the vector of structure constants of $B_5(\bar{\beta})$. Instead we have

$$f_3(B_5(\bar{\beta})) = \bar{\beta}(\bar{\beta} - 1).$$

We find that only for the values $\bar{\beta} = 0, 1$ the above set of polynomials would be consistent with respect to a degeneration. However, regarding the polynomial

$$f_4(x_{ij}^k) = x_{33}^1 x_{32}^2 - x_{32}^1 x_{33}^2$$

which is zero on the whole orbit of $E_5(-1)$, we find[16] that $f_4(\widetilde{B_4}(0)) = 1$. Hence a degeneration from $E_5(-1)$ to $B_4(0)$ is impossible. Nevertheless we have

$$E_5(-1) \to_{\deg} B_4(1) \text{ by } g_t = \begin{pmatrix} \frac{2}{t^2} & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ 0 & -\frac{t}{2} & \frac{2}{t} \end{pmatrix}$$

and by transitivity $E_5(-1) \to_{\deg} B_5(1) \cong B_5(0)$. Also by transitivity degenerations of $E_5(-1)$ to the algebras B_3 and $B_4(\bar{\alpha})$ for all $\bar{\alpha} \neq 0, 1$ are impossible by Lemma 4.21.

Finally we treat the case $E_5(\beta)$ with $\beta \neq -\frac{1}{2}, -1$. There are the following degenerations for all $\bar{\alpha} \neq 0$:

$$E_5(\beta) \to_{\deg} B_4(\bar{\alpha}) \text{ by } g_t = \begin{pmatrix} -\frac{2}{(2\beta+1)t^2} & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ \frac{2(\bar{\alpha}-1)}{(2\beta^2+3\beta+1)t^3} & \frac{1}{2}\bar{\alpha}(2\beta+1)t & \frac{2}{t} \end{pmatrix}.$$

Furthermore we have:

$$E_5(\beta) \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} -\frac{2}{(2\beta+1)t^2} & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ \frac{-2(2\beta+1)+4t}{(\beta+1)(2\beta+1)^2 t^3} & t^2 & \frac{2}{t} \end{pmatrix}$$

and by transitivity with Lemma 4.21 we get $E_5(\beta) \to_{\deg} B_3$ and $E_5(\beta) \to_{\deg} B_5(\bar{\beta})$ for all $\bar{\beta} \in \mathbb{C}$. □

[16] The algebra $\widetilde{B_4}(\alpha)$ was introduced in Proposition 4.19.

Lemma 4.43. *All possible degenerations of 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(\lambda)$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_2$ are listed in the table below.*

E	$\overline{\mathrm{O}(E)}$
$E_{1,\lambda}(\alpha)$ $\lambda \neq 0, \pm 1$ $\alpha \neq -1, -\lambda$	$B_4(0), B_5(0)$
$E_{1,\lambda}(-1)$	$B_5(0)$
$E_{1,\lambda}(-\lambda)$	$B_5(0)$
$E_{2,\lambda}$ $\lambda \neq 0, \pm 1$	$B_4(0), B_5(0)$
E_3	$B_3, B_4(\bar{\alpha}), B_5(\bar{\beta})$
E_4	$B_3, B_4(\bar{\alpha}), B_5(\bar{\beta})$
$E_5(\beta)_{\beta \neq -\frac{1}{2}, -1}$	$B_3, B_4(\bar{\alpha}), B_5(\bar{\beta})$
$E_5(-\frac{1}{2})$	$B_3, B_5(\bar{\beta})$
$E_5(-1)$	$B_3, B_4(1), B_5(1)$
E_6	$B_3, B_5(\bar{\beta})$

Proof. Because of the orbit dimension neither of the algebras in class E can degenerate to B_1 and B_2 except the algebras E_3, E_4, and $E_5(\beta)$ for $\beta \neq -1$ which all can possibly have B_1 in their closure. We show first that this can't happen so we have only to take degenerations to the algebras B_3, $B_4(\bar{\alpha})$, and $B_5(\bar{\beta})$ into account. The orbit closures of the algebras $E_5(\beta)$ have already been studied in Proposition 4.42. For E_3 and E_4 we consider the $\mathfrak{C}_{\mathrm{p,q}}$-invariants

$$\mathfrak{C}_{\mathrm{p,q}}(E_3) = \mathfrak{C}_{\mathrm{p,q}}(E_4) = \frac{((-1)^p + 1)((-1)^q + 1)}{(-1)^{p+q} + 1} = 2 \quad \text{for } p \text{ and } q \text{ even.}$$

Because of $\mathfrak{C}_{\mathrm{p,q}}(B_1) = 3$ for all $p, q \in \mathbb{N}$ and Proposition 3.47 a degeneration to B_1 is impossible.

We start the study of degenerations from class E to B now case for case, beginning with the family of algebras $E_{1,\lambda}(\alpha)$ where $\lambda \neq 0, \pm 1$. We have different orbit dimensions for the values $\alpha = -1, -\lambda$, which prevents the algebras $B_4(\bar{\alpha})$ for every $\bar{\alpha} \in \mathbb{C}$ of lying in the closure. However, this is the only difference between this exceptional values and the general case. Degenerations from $E_{1,\lambda}(\alpha)$, $\lambda \neq 0, \pm 1$, to B_3, $B_4(\bar{\alpha})$ for all $\bar{\alpha} \neq 0$, and $B_5(\bar{\beta})$ for all $\bar{\beta} \neq 0$ are excluded by the argument brought in Proposition 4.13. There are degenerations for all $\lambda \neq 0, \pm 1$ and $\alpha = -1, -\lambda$:

$$E_{1,\lambda}(\alpha) \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} -\frac{1}{(\alpha+1)t^2} & 1 & 0 \\ \frac{1}{t} & 0 & 0 \\ -\frac{1}{(\alpha+1)(\alpha+\lambda)t^3} & \frac{1}{(\lambda-1)t} & 1 \end{pmatrix}$$

and $E_{1,\lambda}(\alpha) \to_{\deg} B_5(0)$ by transitivity. We additionally have for all $\lambda \neq 1$:

$$E_{1,\lambda}(-1) \to_{\deg} B_5(1) \text{ and } E_{1,\lambda}(-\lambda) \to_{\deg} B_5(1) \text{ by } g_t = \begin{pmatrix} \frac{1}{t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{(\lambda-1)t} & 1 \end{pmatrix}.$$

Algebras of the family $E_{2,\lambda}$ cannot degenerate to the algebras B_3, $B_4(\bar{\alpha})$ for all $\bar{\alpha} \neq 0$, and $B_5(\bar{\beta})$ for all $\bar{\beta} \neq 0$, for any $\lambda \in \mathbb{C}$. Like in the case of the family $E_{1,\lambda}(\alpha)$

this is because of a similar argument brought in Proposition 4.13. However we have degenerations for all $\lambda \neq 1$:

$$E_{2,\lambda} \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} -\frac{1}{(\lambda-1)t^2} & \frac{1}{t^2} & 0 \\ \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{(\lambda-1)t^3} & 1 \end{pmatrix}$$

and $E_{2,\lambda} \to_{\deg} B_5(0)$ by transitivity.

For the algebra E_3 all possible degenerations do exist:

$$E_3 \to_{\deg} B_4(\bar{\alpha}) \text{ by } g_t = \begin{pmatrix} 0 & 0 & \frac{1}{t^2} \\ \frac{1}{t} & 0 & 0 \\ \frac{4}{t^3} & \frac{\alpha}{t^3} & \frac{2}{t^3} \end{pmatrix} \quad \text{for all } \alpha \neq 0 \text{ and}$$

$$E_3 \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} 0 & 0 & \frac{1}{t^2} \\ \frac{1}{t} & 0 & 0 \\ \frac{4}{t^3} & 1 & \frac{2}{t^3} \end{pmatrix}.$$

In view of Lemma 4.21 we have $E_3 \to_{\deg} B_5(\bar{\beta})$ and $E_3 \to_{\deg} B_3$ by transitivity.

For the algebra E_4 the same situation as for the algebra E_3 occurs:

$$E_4 \to_{\deg} B_4(\bar{\alpha}) \text{ by } g_t = \begin{pmatrix} \frac{2}{t^2} & 0 & \sqrt{\frac{1-\alpha}{\alpha t^4}} \\ \frac{1}{t} & 0 & 0 \\ 0 & \frac{2(\alpha-1)}{t^3} & \sqrt[4]{\frac{1-\alpha}{\alpha t^6}} \end{pmatrix} \quad \text{for all } \bar{\alpha} \neq 0, 1,$$

$$E_4 \to_{\deg} B_4(0) \text{ by } g_t = \begin{pmatrix} \frac{2}{t^2} & 0 & \frac{1}{t^3} \\ \frac{1}{t} & 0 & 0 \\ 0 & -\frac{2}{t^3+4t^5} & \frac{2}{t^4} \end{pmatrix}, \quad \text{and}$$

$$E_4 \to_{\deg} B_4(1) \text{ by } g_t = \begin{pmatrix} \frac{2}{t^2} & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ 0 & -\frac{t}{2} & \frac{2}{t} \end{pmatrix}.$$

Again, using Lemma 4.21 we have $E_3 \to_{\deg} B_5(\bar{\beta})$ and $E_3 \to_{\deg} B_3$ by transitivity.

Finally we look at the algebra E_6. This time a degeneration to any of the algebras $B_4(\bar{\alpha})$ is impossible because of the following argument. We know that $E_5(-1/2) \to_{\deg} E_6$ by Lemma 4.40. If a degeneration from E_6 to $B_4(\bar{\alpha})$ for some $\bar{\alpha}$ would exist, transitivity forces $E_5(-1/2)$ degenerating to $B_4(\bar{\alpha})$ for the same $\bar{\alpha}$. However, this is impossible by Proposition 4.42. Nevertheless we have the following degenerations:

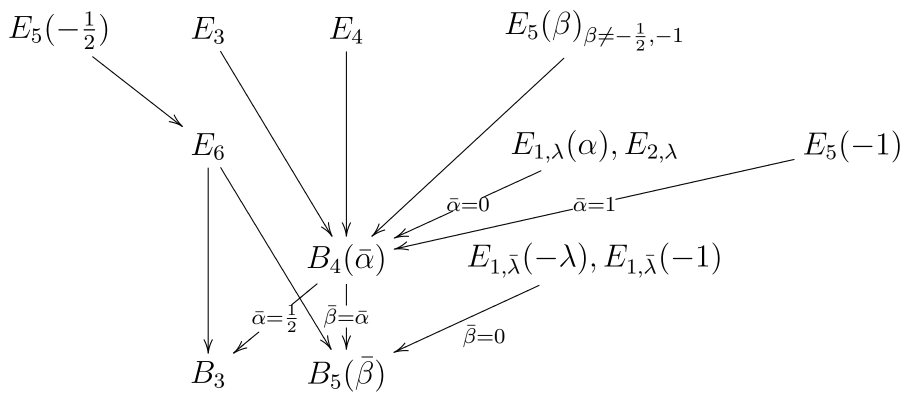
$$E_6 \to_{\deg} B_3 \text{ by } g_t = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ -\frac{2}{t^2} & \frac{1}{2t} & \frac{2}{t} \end{pmatrix},$$

$$E_4 \to_{\deg} B_5(\bar{\beta}) \text{ by } g_t = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ 0 & \frac{\bar{\beta}}{t} & \frac{2}{t} \end{pmatrix} \quad \text{for all } \bar{\beta} \neq 0, \text{ and}$$

$$E_4 \to_{\deg} B_4(1) \text{ by } g_t = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{t} & 0 & 0 \\ 0 & 1 & \frac{2}{t} \end{pmatrix}.$$

□

Corollary 4.44. *The Hasse diagram of all* 3*-dimensional Novikov algebras with associated Lie algebra* $\mathfrak{g}_5(\lambda)$ *and* $\mathfrak{g}_2$ *is given as follows. For a better readability we omitted in this diagram the restrictions on* λ *and* α *for the algebras* $E_{1,\lambda}(\alpha)$ *and* $E_{2,\lambda}$*.*



4.7.3 Degenerations of Novikov algebras with Lie algebras $\mathfrak{g}_5(\lambda)$ and $\mathfrak{g}_1$

Proposition 4.45. *The orbit closures of the algebras* $E_5(\beta)$ *in class* A *are given as follows.*

E	$\overline{\mathrm{O}(E)}$
$E_5(-\frac{1}{2})$	A_1, A_5, A_9
$E_5(-1)$	A_1, A_5
$E_5(\beta)_{\alpha\neq-\frac{1}{2},-1}$	A_1, A_5, A_9, A_{10}

Proof. We consider the $\mathfrak{R}_{\mathrm{p,q}}$-invariant of the algebra $E_5(\beta)$ which equals 3 for every $\beta \in \mathbb{C}$. On the other hand $\mathfrak{R}_{\mathrm{p,q}}(A_2) = \mathfrak{R}_{\mathrm{p,q}}(A_7) = 1$ and hence a degeneration from $E_5(\beta)$ to A_2 and A_7 is impossible for any $\beta \in \mathbb{C}$. By transitivity using Corollary 4.10 we find that degenerations to the algebras A_3, A_4, A_6, and A_8 are impossible too.

We regard the $\mathfrak{C}_{\mathrm{p,q}}$-invariant of A_{11} which is equal to 3 and therefore can't coincide with $\mathfrak{C}_{\mathrm{p,q}}(E_5(\beta))$ for every $\beta \in \mathbb{C}$. This prevents A_{11}, and by transitivity A_{12} of lying in the orbit closure of $E_5(\beta)$ for every $\beta \in \mathbb{C}$.

However, for all $\beta \neq -\frac{1}{2}, -1$ we have the following:

$$E_5(\beta) \to_{\mathrm{deg}} A_{10} \text{ by } g_t = \begin{pmatrix} \frac{2}{(1+\beta)(1+2\beta)t^3} & \frac{1}{2}(1+2\beta)t & 0 \\ -\frac{2}{(1+2\beta)t^2} & 0 & 1 \\ \frac{1}{t} & 0 & 0 \end{pmatrix}.$$

We first treat the exceptional value $\beta = -\frac{1}{2}$. A degeneration from $E_5(-\frac{1}{2})$ to A_{10} is indeed impossible. To see this we regard Theorem 3.8. Looking for 1-dimensional ideals in $E_5(-\frac{1}{2})$ we find that $E_5(-\frac{1}{2})/\langle e_2\rangle \cong W_2(1)$. The only ideal in A_{10}, however, is $\langle e_1\rangle$ with the factor $A_{10}/\langle e_1\rangle \cong U_5$. The algebra $W_2(1)$ does not degenerate to the algebra U_5 as can be seen by Corollary 4.5 and so $E_5(-\frac{1}{2})$ does not degenerate to A_{10}. Nevertheless we have a degeneration:

$$E_5(-\tfrac{1}{2}) \to_{\mathrm{deg}} A_9 \text{ by } g_t = \begin{pmatrix} \frac{2}{t^5} & t & 0 \\ \frac{1}{t^3} & 0 & 1 \\ -\frac{1}{t^2} & 0 & 0 \end{pmatrix}.$$

Finally we regard the case $\beta = -1$. We consider the following polynomials:

$$f_1(x_{ij}^k) = x_{32}^1 x_{23}^1 - x_{22}^1 x_{33}^1,$$
$$f_2(x_{ij}^k) = x_{31}^2 x_{13}^2 - x_{11}^2 x_{33}^2,$$
$$f_3(x_{ij}^k) = x_{12}^3 x_{21}^3 - x_{11}^3 x_{22}^3.$$

It is true, as one might check with a computer, that $f_i(E_5(-1)) = 0$ for $i = 1, 2, 3$.[17] So these three functions have to be zero on the whole orbit closure. However, for the algebras A_9 and A_{10} we have:

$$f_3(A_9) = f_3(A_{10}) = 1.$$

This makes a degeneration from $E_5(-1)$ to A_9 and A_{10} impossible and completes the proof. □

Lemma 4.46. *All degenerations of 3-dimensional Novikov algebras with associated Lie algebra $\mathfrak{g}_5(\lambda)$ to Novikov algebras with associated Lie algebra $\mathfrak{g}_2$ are listed in the table below.*

E	$\overline{\mathrm{O}(E)}$
$E_{1,\lambda}(\alpha)$ $\lambda \neq 0, \pm 1$	A_1, A_5
$E_{2,\lambda}$ $\lambda \neq 0, \pm 1$	A_1, A_5
E_3	A_1, A_5, A_9, A_{10}
E_4	A_1, A_5, A_9, A_{10}
$E_5(\beta)_{\beta \neq -1, -\frac{1}{2}}$	A_1, A_5, A_9, A_{10}
$E_5(-\frac{1}{2})$	A_1, A_5, A_9
$E_5(-1)$	A_1, A_5
E_6	A_1, A_5, A_9

Proof. We will organize this proof a little bit different than the others before. Because of Corollary 4.10 every algebra of class A with orbit dimension higher than five degenerates to at least one of the algebras A_2, A_9, and A_{11}. If we can show that a given algebra of class E has neither of these three algebras in its closure, then a degeneration to any other algebra of class A is impossible by transitivity. Therefore we will start in each case with the algebras A_2, A_9, and A_{11}.

First of all, we remark that every algebra of class E has the algebra A_5 in its closure using transitivity. This is because every algebra of class B except $B_5(\frac{1}{2})$ has A_5 in its closure (Lemma 4.23) and furthermore every algebra of class E degenerates to some algebra of class B (Lemma 4.43).

Second, not one of the E-class algebras has A_2 and A_7 in its closure. The $\mathfrak{R}_{\mathrm{p,q}}$-invariant of every E-class algebra equals three, whereas $\mathfrak{R}_{\mathrm{p,q}}(A_2) = \mathfrak{R}_{\mathrm{p,q}}(A_7) = 1$, making degenerations impossible. Using the transitivity argument from the beginning of the proof we immediately find that also A_3, A_4, A_6, and A_8 can't lie in the orbit closure of any E-class algebra.

[17] We write $f_i(A)$ for the polynomial funcion applied to the vector of structure constants of the algebra A.

Moreover there are no degenerations of any E-class algebras to A_{11}. To see this we use the $\mathfrak{C}_{\mathrm{p,q}}$-invariant of A_{11} which equals three and therefore is constant with respect to the parameters p and q. This value can't be taken by the $\mathfrak{C}_{\mathrm{p,q}}$-invariant of any E-class algebra as one can easily show by computations. By transitivity again, we can exclude degenerations from class E to A_{12}.

In conclusion we have degenerations from any E-class algebra to A_5 (and trivially to A_1), whereas there are no degenerations to the algebras A_3, A_4, A_6, A_7, A_8, A_{11}, and A_{12}. We have two algebras left that can possibly lie in the orbit closure of some E-class algebra, namely A_9 and A_{10}, for which we note $A_{10} \to_{\deg} A_9$.

The algebra A_9 can't lie in the closure of any $E_{1,\lambda}(\alpha)$ and any $E_{2,\lambda}$ because of an argument similar to that of Proposition 4.13. By transitivity neither algebra of the families $E_{1,\lambda}(\alpha)$ and $E_{2,\lambda}$ can have A_{10} in its closure.

We have the following degenerations for the algebras E_3 and E_4:

$$E_3 \to_{\deg} A_{10} \text{ by } g_t = \begin{pmatrix} 0 & \frac{1}{t^3} & 0 \\ 0 & 0 & \frac{1}{t^2} \\ \frac{1}{t} & 0 & 0 \end{pmatrix},$$

$$E_4 \to_{\deg} A_{10} \text{ by } g_t = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}t^3} & 0 \\ \frac{1}{t^2} & 0 & \frac{i}{t^2} \\ \frac{1}{\sqrt{2}t} & 0 & 0 \end{pmatrix},$$

and $E_3, E_4 \to_{\deg} A_9$ by transitivity.

The orbit closures of the algebras $E_5(\beta)$ have already been studied in Proposition 4.45 so we continue with the algebra E_6. We have a degeneration

$$E_6 \to_{\deg} A_9 \text{ by } g_t = \begin{pmatrix} 0 & \frac{1}{t} & 0 \\ 0 & 0 & 1 \\ \frac{1}{t} & 0 & 0 \end{pmatrix}.$$

However, there is no degeneration from E_6 to A_{10}. We consider $E_6/\langle e_2 \rangle$ which is isomorphic to the 2-dimensional Novikov algebra $W_2(1)$. With respect to Corollary 4.5 the algebra $W_2(1)$ can only degenerate to U_1, the abelian algebra in dimension two. In A_{10} there exists only a single 1-dimensional ideal, namely $\langle e_1 \rangle$. We have $A_{10}/\langle e_1 \rangle \cong U_5$ and therefore, by Theorem 3.8, a degeneration from E_6 to A_{10} is impossible. □

Corollary 4.47. *The Hasse diagram of all* 3*-dimensional Novikov algebras with associated Lie algebra* $\mathfrak{g}_5(\lambda)$ *and* $\mathfrak{g}_1$ *is given as follows. In this diagram we omitted* *degenerations within the classes.*

$$
\begin{tikzcd}
E_5(-\frac{1}{2}) \arrow[ddrr] & E_3 \arrow[dr] & & E_4 \arrow[dl] & E_5(\beta)_{\beta\neq-\frac{1}{2},-1} \arrow[dll] \\
E_5(-1), E_{1,\lambda}(\alpha)_{\lambda\neq 0,\pm 1} \arrow[ddrr] & & A_{10} \arrow[d] & E_6 \arrow[dl] & \\
E_{2,\lambda} \arrow[drr] & & A_9 \arrow[d] & E_{1,\lambda}(-\lambda) \arrow[dl] & E_{1,\lambda}(-1) \arrow[dll] \\
& & A_5 \arrow[d] & & \\
& & A_1 & &
\end{tikzcd}
$$

5 Appendix

A Preliminaries from algebraic geometry

In this subsection we shall give a short overview of some basic concepts from algebraic geometry necessary for the definition of a degeneration. This exposition is not supposed to be a selfcontained introduction to algebraic geometry. In fact it's a collection of the most important terms, to make the reading more comfortable.

Definition 5.1. The set $\mathbb{K}^n = \mathbb{K} \times \cdots \times \mathbb{K}$ (n-times) will be called affine n-space and denoted by $\mathbb{A}$. For the purposes in this work it sufficies to consider affine varieties, by which we mean subsets of $\mathbb{A}^n$ defined by the common zeros of a finite collection of polynomials.

Definition 5.2. Defining closed sets to be the affine varieties we can establish a topology on $\mathbb{A}^n$, called the Zariski topology.

For the correspondence between closed sets and ideals, and the notion of the coordinate ring we refer to [37].

Example 5.3. Let A be an n-dimensional (not necessarily associative) algebra over an algebraically closed field $\mathbb{K}$. Let $(e_1, \ldots, e_n)$ be a basis of the underlying vector space, then $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ defines a vector $(c_{ij}^k) \in \mathbb{K}^{n^3}$, which is called the representing vector of structure constants of the algebra A. In this way, the set of all algebra structures becomes an affine subvariety of $\mathbb{K}^{n^3}$, denoted by $\mathrm{Alg}_n(\mathbb{K})$. Relations like commutativity, associativity, skew-symmetry, and the Jacobi identity can be expressed by polynomials in the structure constants and therefore define subvarieties $\mathrm{Comm}_n(\mathbb{K})$, $\mathrm{Assoc}_n(\mathbb{K})$, and $\mathrm{Lie}_n(\mathbb{K})$ of $\mathrm{Alg}_n(\mathbb{K})$. The perspective of identifying an algebra structure with a point in $\mathbb{K}^{n^3}$, allows us to study $\mathrm{Alg}_n(\mathbb{K})$ and its various subvarieties in terms of algebraic geometry.

Definition 5.4. Let X be a topological space. Then X is said to be irreducible if X cannot be written as a union of two proper, non-empty, closed subsets of X.

If X is a noetherian topological space, then it can be shown that X has only finitely many maximal irreducible subspaces, which are called the irreducible components or simply components (if the meaning is clear).

Definition 5.5. Let X be an irreducible variety, then the dimension $\dim X$ of X as a variety is defined to be the transcendence degree over $\mathbb{K}$ of the quotient field $\mathbb{K}(X)$ of the coordinate ring $\mathbb{K}[X]$. If X is not irreducible, splitting up into $X = X_1 \cup \cdots \cup X_m$, we define $\dim X := \max \dim X_i$.

We consider the notion of the tangent space of a variety X to be given (see [8]). A point $x \in X$ is called simple if the dimension of the tangent space in the point x equals the dimension of X as a variety. A variety X is called smooth if every point $x \in X$ is simple.

Definition 5.6. Let G be a variety endowed with the structure of a group. Consider the two maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$, where $\mu(x,y) = xy$ and $\iota(x) = x^{-1}$. If μ and ι are morphisms of varieties, we call G an algebraic group.

Example 5.7. The most important example of an algebraic group in our case here is that of the general linear group $\mathrm{GL}_n(\mathbb{K})$. In fact, $\mathrm{GL}_n(\mathbb{C})$ and some of its subgroups are the only algebraic groups we need for our calculations in chapter four of this work. This is one of the reasons why we restricted ourselves to affine varieties, although we gave the definition of an algebraic group in a more general form.

Definition 5.8. Let G be an algebraic group and X a variety. We say that G acts on X morphically or regularly if G acts on X as a group via $\varphi\colon G \times X \to X$ such that φ is a morphism of varieties. We denote by $\mathrm{O}_G(x)$ (or simply $\mathrm{O}(x)$) the orbit of the point $x \in X$.

Example 5.9. If we choose G to be the general linear group $\mathrm{GL}_n(\mathbb{K})$ acting on the affine variety $\mathrm{Alg}_n(\mathbb{K})$ the orbit of a point $A \in \mathrm{Alg}_n(\mathbb{K})$ consists of all algebras isomorphic to A.

A very important result for studying degenerations is the so called *Borel's closed orbit lemma*, which can be found in [8, p. 53].

Theorem 5.10. *Let G be an algebraic group acting morphically on a non-empty variety V. Then every orbit is a smooth variety which is open in its closure in V. Its boundary is a union of orbits of strictly lower dimension. In particular, orbits with minimal dimension are closed.*

B Tables of orbit closures

We give tables that summarize wether a degeneration is possible or not. In the latter case an abbreviation will indicate what kind of argument was used to exclude that degeneration. This provides a comfortable tool to point out and trail back a certain method more quickly. Legend of the diagram:

$\to$ ... a degeneration exists.

$\to_t$ ... a degeneration exists by transitivity.

t ... a degeneration is impossible by transitivity.

d ... a degeneration is impossible because of the dimension of the derivation space.

n ... a degeneration is impossible because of the dimensions of the lower central series or the derived series.

c ... a degeneration is impossible because of the $\mathfrak{C}_{\mathrm{p,q}}$-invariant.

r ... a degeneration is impossible because of the $\mathfrak{R}_{\mathrm{p,q}}$-invariant.

Ann ... a degeneration is impossible because of the dimension of the annihilator (left, right or both).

FT ... a degeneration is impossible because of Theorem 3.8 (factor theorem).

j ... a degeneration is impossible of because Lemma 3.14.

f ... a degeneration is impossible of because the argument brought in Proposition 4.13.

td ... a degeneration is impossible of because the dimension of an (α, β, γ)-derivation.

bi ... a degeneration is impossible because of the argument brought in Proposition 4.19.

op ... a degeneration is impossible because of quadratic operator identities.

z ... a degeneration is impossible because of a polynomial identity in the structure constants (Zariski-equation).

det ... a degeneration is impossible because the determinant defines a Zariski-equation.

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_1$

$\to_{\text{deg}}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
A_1	$\to$	d	d	d	d	d	d	d	d	d	d	d
A_2	$\to$	$\to$	d	d	d	d	d	d	d	d	d	d
A_3	$\to$	$\to_t$	$\to$	d	$\to_t$	$\to$	$\to$	d	$\to_t$	$\to_t$	n	n
A_4	$\to$	$\to_t$	$\to$	$\to$	$\to_t$	$\to_t$	$\to_t$	$\to$	$\to_t$	$\to_t$	$\to_t$	$\to_t$
A_5	$\to$	d	d	d	$\to$	d	d	d	d	d	d	d
A_6	$\to$	$\to$	d	d	$\to_t$	$\to$	d	d	$\to_t$	$\to$	c	d
A_7	$\to$	c	d	d	$\to_t$	d	$\to$	d	$\to_t$	$\to$	c	d
A_8	$\to$	$\to_t$	d	d	$\to_t$	$\to$	$\to$	$\to$	$\to_t$	$\to_t$	$\to_t$	$\to$
A_9	$\to$	d	d	d	$\to$	d	d	d	$\to$	d	d	d
A_{10}	$\to$	c	d	d	$\to_t$	d	d	d	$\to$	$\to$	c	d
A_{11}	$\to$	d	d	d	$\to$	d	d	d	d	d	$\to$	d
A_{12}	$\to$	c	d	d	$\to_t$	d	d	d	$\to_t$	$\to$	$\to$	$\to$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_4$

$\to_{\text{deg}}$	D_1	$D_2(0)$	$D_2(-1)$	$D_2(\alpha)_{\alpha\neq 0,-1}$
D_1	$\to$	d	$\to$	d
$D_2(0)$	d	$\to$	n	n
$D_2(-1)$	d	d	$\to$	d
$D_2(\alpha)_{\alpha\neq 0,-1}$	d	d	c	$\to$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(1)$

$\to_{\deg}$	$E_{1,1}(0)$	$E_{1,1}(-1)$	$E_{1,1}(\alpha)_{\alpha\neq 0,-1}$	$E_{2,1}$
$E_{1,1}(0)$	$\to$	n	d	d
$E_{1,1}(-1)$	d	$\to$	d	d
$E_{1,1}(\alpha)_{\alpha\neq 0,-1}$	d	c	$\to$	d
$E_{2,1}$	d	$\to$	d	$\to$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_4$ and $\mathfrak{g}_1$

$\to_{\deg}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
D_1	$\to$	j	d	d	$\to_t$	d	d	d	f	f	c	d
$D_2(0)$	$\to$	c	d	d	$\to$	d	d	d	f	f	c	d
$D_2(-1)$	$\to$	j	d	d	$\to_t$	d	d	d	td	d	c	d
$D_2(\alpha)_{\alpha\neq 0,-1}$	$\to$	c	d	d	$\to$	d	d	d	f	f	c	d

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(1)$ and $\mathfrak{g}_1$

$\to_{\deg}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
$E_{1,1}(0)$	$\to$	d	d	d	$\to$	d	d	d	d	d	d	d
$E_{1,1}(-1)$	$\to$	d	d	d	d	d	d	d	d	d	d	d
$E_{1,1}(\alpha)_{\alpha\neq 0,-1}$	$\to$	d	d	d	$\to$	d	d	d	d	d	d	d
$E_{2,1}$	$\to$	d	d	d	$\to$	d	d	d	d	d	d	d

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_4$ and $\mathfrak{g}_5(1)$

$\to_{\deg}$	$E_{1,1}(0)$	$E_{1,1}(-1)$	$E_{1,1}(\alpha)$	$E_{1,1}(\bar{\alpha})_{\bar{\alpha}\neq\alpha}$	$E_{2,1}$
D_1	c	$\to_t$	c	c	$\to_t$
$D_2(0)$	$\to$	c	c	c	c
$D_2(-1)$	c	$\to_t$	c	c	$\to$
$D_2(\alpha)_{\alpha\neq 0,-1}$	c	c	$\to$	c	c

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_2$

$\to_{\deg}$	B_1	B_2	B_3	$B_4(0)$	$B_4(\bar{\alpha})_{\bar{\alpha}\neq 0}$	$B_5(0)$	$B_5(\bar{\beta})_{\bar{\beta}\neq 0}$
B_1	$\to$	d	f	$\to$	f	$\to_t$	$-_f$
B_2	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$-_f$
B_3	d	d	$\to$	d	d	d	$-_f$
$B_4(\frac{1}{2})$	d	d	$\to$	d	d	bi	$-_f$
$B_4(\alpha)_{\alpha\neq\frac{1}{2}}$	d	d	td	d	$\to_{\alpha=\bar{\alpha}}$	$\to_{\alpha=\bar{\beta}}$	$-_f$
	d	d	d	d	$bi_{\alpha\neq\bar{\alpha}}$	$bi_{\alpha\neq\bar{\alpha}}$	$-_f$
$B_5(\beta)$	d	d	d	d	d	d	$-_f$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_4$ and $\mathfrak{g}_2$

$\to_{\deg}$	B_1	B_2	B_3	$B_4(0)$	$B_4(\alpha)_{\alpha\neq 0}$	$B_5(0)$	$B_5(\beta)_{\beta\neq 0}$
D_1	d	d	f	$\to_t$	f	$\to_t$	f
$D_2(0)$	d	d	f	$\to$	f	$\to_t$	f
$D_2(-1)$	d	d	f	$\to$	d	$\to_t$	f
$D_2(\alpha)_{\alpha\neq 0,-1}$	d	d	f	$\to$	f	$\to_t$	f

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_2$ and $\mathfrak{g}_1$

$\to_{\deg}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
B_1	$\to$	c	d	d	$\to_t$	d	d	d	f	f	$\to$	d
B_2	$\to$	c	d	d	$\to_t$	c	c	d	$\to_t$	$\to_t$	$\to_t$	$\to$
B_3	$\to$	d	d	d	$\to$	d	d	d	d	d	d	d
$B_4(\alpha)$	$\to$	c	d	d	$\to_t$	d	d	d	f	d	c	d
$B_5(\frac{1}{2})$	$\to$	d	d	d	d	d	d	d	d	d	d	d
$B_5(\beta)$	$\to$	d	d	d	$\to$	d	d	d	d	d	d	d

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(-1)$

$\to_{\deg}$	$E_{1,-1}(1)$	$E_{1,-1}(-1)$	$E_{1,-1}(\alpha)_{\alpha\neq\pm 1}$	$E_{2,-1}$
$E_{1,-1}(1)$	$\to$	d	d	d
$E_{1,-1}(-1)$	d	$\to$	d	d
$E_{1,-1}(\alpha)_{\alpha\neq\pm 1}$	c	c	$\to$	d
$E_{2,-1}$	c	c	d	$\to$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(-1)$ and $\mathfrak{g}_1$

$\to_{\text{deg}}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
$E_{1,-1}(1)$	$\to$	c	d	d	$\to$	d	d	d	Ann	d	c	d
$E_{1,-1}(-1)$	$\to$	c	d	d	$\to$	d	d	d	Ann	d	c	d
$E_{1,-1}(\alpha)_{\alpha\neq\pm1}$	$\to$	c	d	d	$\to$	d	d	d	Ann	Ann	c	d
$E_{2,-1}$	$\to$	c	d	d	$\to$	d	d	d	f	f	c	d

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(-1)$ and $\mathfrak{g}_2$

$\to_{\text{deg}}$	B_1	B_2	B_3	$B_4(\bar\alpha)$	$B_5(\beta)_{\beta\neq0}$
$E_{1,-1}(1)$	d	d	f	f	f
$E_{1,-1}(-1)$	d	d	f	f	f
$E_{1,-1}(\alpha)_{\alpha\neq\pm1}$	d	d	f	$\to_{\bar\alpha=0}$ $f_{\bar\alpha\neq0}$	$\to_{\bar\beta=0}$ $f_{\bar\beta\neq0}$
$E_{2,-1}$	d	d	f	$\to_{\bar\alpha=1}$	$\to_{\bar\beta=1}$
				$f_{\bar\alpha\neq1}$	$f_{\bar\beta\neq1}$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_3$

$\to_{\text{deg}}$	C_1	C_2	C_3	C_4	$C_5(0)$	$C_5(-1)$	$C_5(\alpha)$	$C_6(0)$	$C_6(-1)$	$C_6(\bar\beta)$	$C_7(-1)$	$C_7(\gamma)$
C_1	$\to$	$\to$	$\to$	op	op	$\to_t$	op	d	$\to$	d	$\to_t$	op
C_2	d	$\to$	d	d	c	c	d	d	d	d	$\to$	d
C_3	d	d	$\to$	d	t	$\to$	d	d	d	d	c	d
C_4	d	d	d	$\to$	$\to$	n	d	d	d	d	c	d
$C_5(0)$	d	d	d	d	$\to$	d	d	d	d	d	d	d
$C_5(-1)$	d	d	d	d	d	$\to$	d	d	d	d	d	d
$C_5(\alpha)$	d	d	d	d	c	c	$\to$	d	d	d	n	d
$C_6(0)$	d	n	n	$\to$	$\to_t$	n	n	$\to$	n	d	n	n
$C_6(-1)$	d	d	d	d	op	$\to$	d	d	$\to$	d	$\to$	d
$C_6(\beta)$	d	op	op	op	op	op	$\to_{\alpha=\beta}$	d	op	$\to_{\beta=\bar\beta}$	op	$\to_{\gamma=\beta}$
							$op_{\alpha\neq\beta}$			$d_{\beta\neq\bar\beta}$		$op_{\gamma\neq\beta}$
$C_7(-1)$	d	d	d	d	d	d	d	d	d	d	$\to$	d
$C_7(\gamma)$	d	d	d	d	c	c	d	d	d	d	c	$\to$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_3$ and $\mathfrak{g}_1$

$\to_{\text{deg}}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
C_1	$\to$	$\to_t$	d	d	$\to_t$	$\to$	FS	d	$\to_t$	$\to_t$	det	t
C_2	$\to$	c	t	t	$\to_t$	t	t	t	f	t	c	t
C_3	$\to$	r	t	t	$\to_t$	t	t	t	Ann	t	n	t
C_4	$\to$	n	t	t	$\to_t$	t	t	t	Ann	t	n	t
$C_5(0)$	$\to$	n	t	t	$\to_t$	t	t	t	td	t	t	t
$C_5(-1)$	$\to$	r	t	t	$\to_t$	t	t	t	td	t	t	t
$C_5(\alpha)$	$\to$	c	t	t	$\to$	t	t	t	z	t	n	t
$C_6(0)$	$\to$	$\to_t$	t	t	$\to_t$	$\to$	n	t	$\to_t$	$\to_t$	n	t
$C_6(-1)$	$\to$	$\to$	t	t	$\to_t$	t	t	t	td	t	det	t
$C_6(\beta)$	$\to$	$\to_t$	t	t	$\to_t$	$\to$	FS	t	$\to_t$	$\to_t$	FS	t
$C_7(-1)$	$\to$	c	t	t	$\to_t$	t	t	t	td	t	c	t
$C_7(\gamma)$	$\to$	c	t	t	$\to$	t	t	t	f	t	c	t

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_3$ and $\mathfrak{g}_2$

$\to_{\text{deg}}$	B_1	B_2	B_3	$B_4(\bar{\alpha})$	$B_5(\bar{\beta})$
C_1	op	d	$\to_t$	$\to$	$\to_t$
C_2	d	d	f	$\to_{\bar{\alpha}=0}$ $f_{\bar{\alpha}\neq 0}$	$\to_{t,\bar{\beta}=0}$ $f_{\bar{\beta}\neq 0}$
C_3	d	d	Ann_L	$\to_{\bar{\alpha}=1}$ $t_{\bar{\alpha}\neq 1}$	$\to_{\bar{\beta}=0}$ $(\mathrm{Ann}_L)_{\bar{\beta}\neq 0}$
C_4	d	d	Ann_R	$\to_{\bar{\alpha}=0}$ $(\mathrm{Ann}_R)_{\bar{\alpha}\neq 0}$	$\to_{\bar{\beta}=0}$ $(\mathrm{Ann}_R)_{\bar{\beta}\neq 0}$
$C_5(0)$	d	d	Ann_L	d	$\to_{\bar{\beta}=0}$ $(\mathrm{Ann}_L)_{\bar{\beta}\neq 0}$
$C_5(-1)$	d	d	Ann_L	d	$\to_{\bar{\beta}=0}$ $(\mathrm{Ann}_L)_{\bar{\beta}\neq 0}$
$C_5(\alpha)$	d	d	$\to_t$	$\to_{\bar{\alpha}=-\alpha}$ $z_{\bar{\alpha}\neq -\alpha}$	$\to_{\bar{\beta}=-\alpha}$ $z_{\bar{\beta}\neq -\alpha}$
$C_6(0)$	n	d	$\to_t$	$\to$	$\to_t$
$C_6(-1)$	d	d	td	td	$(\to_t)_{\bar{\beta}=0}$ $td_{\bar{\beta}\neq 0}$
$C_6(\beta)$	op	d	$\to_t$	$\to$	$\to_t$
$C_7(-1)$	d	d	t	d	$\to_{\bar{\beta}=0}$ $td_{\bar{\beta}\neq 0}$
$C_7(\gamma)$	d	d	f	$\to_{\bar{\alpha}=0}$ $f_{\bar{\alpha}\neq 0}$	$\to_{\bar{\beta}=0}$ $f_{\bar{\beta}\neq 0}$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(\lambda)$

$\to_{\text{deg}}$	$E_{1,\bar\lambda}(\bar\alpha)$	$E_{1,\bar\lambda}(-1)$	$E_{1,\bar\lambda}(-\bar\lambda)$	$E_{2,\bar\lambda}$	E_3	E_4	$E_5(\bar\beta)$	$E_5(-1/2)$	$E_5(-1)$	E_6
$E_{1,\lambda}(\alpha)$ $\lambda \neq 0, \pm 1$ $\lambda \neq -1, -\lambda$	$\to^{\bar\alpha=\alpha}_{\bar\lambda=\lambda}$ $d^{\bar\alpha\neq\alpha}_{\bar\lambda\neq\lambda}$	c	c	d	d	d	d	d	d	d
$E_{1,\lambda}(-1)$	d	$\to_{\bar\lambda=\lambda}$ $d_{\bar\lambda\neq\lambda}$	d	d	d	d	d	d	d	d
$E_{1,\lambda}(-\lambda)$	d	d	$\to_{\bar\lambda=\lambda}$ $d_{\bar\lambda\neq\lambda}$	d	d	d	d	d	d	d
$E_{2,\lambda}$ $\lambda \neq 0, \pm 1$	d	$\to_{\bar\lambda=\lambda}$ $z_{\bar\lambda\neq\lambda}$	$\to_{\bar\lambda=\frac{1}{\lambda}}$ $z_{\bar\lambda\neq\frac{1}{\lambda}}$	$\to_{\bar\lambda=\lambda}$ $d_{\bar\lambda\neq\lambda}$	d	d	d	d	d	d
E_3	c	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	$\to_{\bar\lambda=\frac12}$ $c_{\bar\lambda\neq\frac12}$	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	$\to$	d	d	d	c	$\to$
E_4	c	$\to_{\bar\lambda=\frac12}$ $c_{\bar\lambda\neq\frac12}$	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	d	$\to$	d	d	$\to$	c
$E_5(\beta)$ $\beta \neq -\frac12, -1$	$\to^{\bar\alpha=\beta}_{\bar\lambda=\frac12}$ $z+c^{\bar\alpha\neq\beta}_{\bar\lambda\neq\frac12}$	$z+c$	$z+c$	$z+c$	d	d	$\to_{\bar\beta=\beta}$ $d_{\bar\beta\neq\beta}$	d	c	c
$E_5(-1/2)$	c	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	$\to_{\bar\lambda=\frac12}$ $c_{\bar\lambda\neq\frac12}$	$FS_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	d	d	d	$\to$	c	$\to$
$E_5(-1)$	d	$\to_{\bar\lambda=\frac12}$ $c_{\bar\lambda\neq\frac12}$	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	d	d	d	d	d	$\to$	d
E_6	c	$\to_{\bar\lambda=2}$ $c_{\bar\lambda\neq 2}$	$\to_{\bar\lambda=\frac12}$ $c_{\bar\lambda\neq\frac12}$	d	d	d	d	d	d	$\to$

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(\lambda)$ and $\mathfrak{g}_1$

$\to_{\text{deg}}$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
$E_{1,\lambda}(\alpha)$	$\to$	r	t	t	$\to$	t	d	t	f	t	c	t
$E_{1,\lambda}(-1)$	$\to$	r	t	t	$\to$	t	d	t	f	t	c	t
$E_{1,\lambda}(-\lambda)$	$\to$	r	t	t	$\to$	t	d	t	f	t	c	t
$E_{2,\lambda}(\lambda \neq 0, \pm 1)$	$\to$	r	t	t	$\to_t$	t	d	t	f	t	c	t
E_3	$\to$	r	t	t	$\to_t$	t	c	t	$\to_t$	$\to$	c	t
E_4	$\to$	r	t	t	$\to_t$	t	r	t	$\to_t$	$\to$	c	t
$E_5(\beta)_{\beta\neq -1}$	$\to$	r	t	t	$\to$	t	r	t	$\to_t$	$\to$	c	t
$E_5(-1/2)$	$\to$	r	t	t	$\to$	t	r	t	$\to$	FS	c	t
$E_5(-1)$	$\to$	r	t	t	$\to$	t	c	t	z	z	c	t
E_6	$\to$	r	t	t	$\to$	t	d	t	$\to$	FS	c	t

Degenerations of Novikov algebras with Lie algebra $\mathfrak{g}_5(\lambda)$ and $\mathfrak{g}_2$

$\to_{\text{deg}}$	B_1	B_2	B_3	$B_4(\bar\alpha)$	$B_5(\bar\beta)$
$E_{1,\lambda}(\alpha)$ $\lambda \neq 0, \pm 1$ $\lambda \neq -1, -\lambda$	d	d	t	$\to_{\bar\alpha=0}$ $f_{\bar\alpha\neq 0,1}$	$\to_{t,\bar\beta=0}$ $f_{\bar\beta\neq 0}$
$E_{1,\lambda}(-1)$	d	d	t	d	$\to_{\bar\beta=0}$ $f_{\bar\beta\neq 0}$
$E_{1,\lambda}(-\lambda)$	d	d	t	d	$\to_{\bar\beta=0}$ $f_{\bar\beta\neq 0}$
$E_{2,\lambda}$ $\lambda \neq 0, \pm 1$	d	d	t	d $f_{\bar\alpha\neq 0,1}$	$\to_{\bar\beta=0}$ $f_{\bar\beta\neq 0}$
E_3	c	d	$\to_t$	$\to$	$\to_t$
E_4	c	d	$\to_t$	$\to$	$\to_t$
$E_5(\beta)_{\beta\neq -1}$	FS	d	$\to_t$	$\to$	$\to_t$
$E_5(-1/2)$	FS	d	$\to$	FS	$\to$
$E_5(-1)$	d	d	t	$\to_{\bar\alpha=1}$ $t_{\bar\beta\neq 1}$	$\to_t$ $z_{\bar\beta\neq 0}$
E_6	d	d	$\to$	t	$\to$

C Tables of semi-invariants

A	$\dim \operatorname{Ann}_R(A)$	$\dim \operatorname{Ann}_L(A)$	$\dim \operatorname{Ann}(A)$
A_1	3	3	3
A_2	2	2	2
A_3	1	1	1
A_4	0	0	0
A_5	2	2	2
A_6	1	1	1
A_7	1	1	1
A_8	0	0	0
A_9	1	1	1
A_{10}	1	1	1
A_{11}	0	0	0
A_{12}	0	0	0

B	$\dim \mathrm{Ann}_R(B)$	$\dim \mathrm{Ann}_L(B)$	$\dim \mathrm{Ann}(B)$
B_1	0	0	0
B_2	0	0	0
B_3	1	1	1
$B_4(0)$	2	1	1
$B_4(1)$	1	2	1
$B_4(\alpha)$	1	1	1
$B_5(0)$	2	2	2
$B_5(\frac{1}{2})$	1	1	1
$B_5(\beta)$	1	1	1

C	$\dim \mathrm{Ann}_R(C)$	$\dim \mathrm{Ann}_L(C)$	$\dim \mathrm{Ann}(C)$
C_1	0	1	0
C_2	0	1	0
C_3	1	2	1
C_4	2	1	1
$C_5(-1)$	1	2	1
$C_5(0)$	2	2	1
$C_5(\alpha)$	1	1	1
$C_6(-1$	0	1	0
$C_6(0)$	1	1	0
$C_6(\beta)$	0	0	0
$C_7(-1)$	0	1	0
$C_7(\gamma)$	0	0	0

D	$\dim \mathrm{Ann}_R(D)$	$\dim \mathrm{Ann}_L(D)$	$\dim \mathrm{Ann}(D)$
D_1	0	1	0
$D_2(-1)$	0	1	0
$D_2(0)$	2	1	0
$D_2(\alpha)$	0	0	0

E	$\dim \mathrm{Ann}_R(E)$	$\dim \mathrm{Ann}_L(E)$	$\dim \mathrm{Ann}(E)$
$E_{1,1}(-1)$	0	2	0
$E_{1,1}(0)$	2	1	0
$E_{1,1}(\alpha)$	0	0	0
$E_{2,1}$	0	2	0
$E_{1,-1}(-1)$	0	1	0
$E_{1,-1}(1)$	0	1	0
$E_{1,-1}(\alpha)$	0	0	0
$E_{2,-1}$	0	1	0

E	$\dim \mathrm{Ann}_R(E)$	$\dim \mathrm{Ann}_L(E)$	$\dim \mathrm{Ann}(E)$
$E_{1,\lambda}(-1)$	0	1	0
$E_{1,\lambda}(-\lambda)$	0	1	0
$E_{1,\lambda}(0)$	2	1	0
$E_{1,\lambda}(\alpha)$	0	0	0
$E_{2,\lambda}$	0	1	0
E_3	0	1	0
E_4	0	1	0
$E_5(-1)$	0	1	0
$E_5(-\frac{1}{2})$	0	0	0
$E_5(0)$	1	1	0
$E_5(\beta)$	0	0	0
E_6	0	1	0

D Algorithms and computations

All computations were arranged with the program Wolfram Mathematica 7.

Algorithm to calculate a degeneration matrix

We used the following algorithm to calculate a certain degeneration matrix. Thereby the input li corresponds to the left-multiplication operator of the basis vector e_i.

```
 g = {{g1, g2, g3}, {g4, g5, g6}, {g7, g8, g9}}

(* A_4  Novikov dim 3 *)

l1 = {{1, 0, 0}, {0, 0, 0}, {0, 0, 0}}
l2 = {{0, 0, 0}, {0, 1, 0}, {0, 0, 0}}
l3 = {{0, 0, 0}, {0, 0, 0}, {0, 0, 1}}

Clear[i, j, k, l, p]

p = {l1, l2, l3}

m1 = Transpose[{g.(Sum[
      Inverse[g][[k]][[1]]*
       Inverse[g][[l]][[1]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
       3}]), g.(Sum[
      Inverse[g][[k]][[1]]*
       Inverse[g][[l]][[2]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
       3}]), g.(Sum[
      Inverse[g][[k]][[1]]*
       Inverse[g][[l]][[3]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
       3}])}]

M1 = {{FullSimplify[m1[[1]][[1]]], FullSimplify[m1[[1]][[2]]],
```

```
   FullSimplify[m1[[1]][[3]]]}, {FullSimplify[m1[[2]][[1]]],
   FullSimplify[m1[[2]][[2]]],
   FullSimplify[m1[[2]][[3]]]}, {FullSimplify[m1[[3]][[1]]],
   FullSimplify[m1[[3]][[2]]], FullSimplify[m1[[3]][[3]]]}}

m2 = Transpose[{g.Sum[
     Inverse[g][[k]][[2]]*
      Inverse[g][[l]][[1]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
      3}], g.Sum[
     Inverse[g][[k]][[2]]*
      Inverse[g][[l]][[2]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
      3}], g.Sum[
     Inverse[g][[k]][[2]]*
      Inverse[g][[l]][[3]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
      3}]}]

M2 = {{FullSimplify[m2[[1]][[1]]], FullSimplify[m2[[1]][[2]]],
   FullSimplify[m2[[1]][[3]]]}, {FullSimplify[m2[[2]][[1]]],
   FullSimplify[m2[[2]][[2]]],
   FullSimplify[m2[[2]][[3]]]}, {FullSimplify[m2[[3]][[1]]],
   FullSimplify[m2[[3]][[2]]], FullSimplify[m2[[3]][[3]]]}}

m3 = Transpose[{g.Sum[
     Inverse[g][[k]][[3]]*
      Inverse[g][[l]][[1]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
      3}], g.Sum[
     Inverse[g][[k]][[3]]*
      Inverse[g][[l]][[2]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
      3}], g.Sum[
     Inverse[g][[k]][[3]]*
      Inverse[g][[l]][[3]] Transpose[p[[k]]][[l]], {l, 1, 3}, {k, 1,
      3}]}]

M3 = {{FullSimplify[m3[[1]][[1]]], FullSimplify[m3[[1]][[2]]],
   FullSimplify[m3[[1]][[3]]]}, {FullSimplify[m3[[2]][[1]]],
   FullSimplify[m3[[2]][[2]]],
   FullSimplify[m3[[2]][[3]]]}, {FullSimplify[m3[[3]][[1]]],
   FullSimplify[m3[[3]][[2]]], FullSimplify[m3[[3]][[3]]]}}
```

Algorithm to calculate the derivations of an algebra

If we run the following program we get the space $\mathrm{Der}_{(r,s,t)}(A_4)$.

```
 M = {{a, b, c}, {d, e, f}, {g, h, j}}

(* Novikov A_4 *)

l1 = {{1, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

```
l2 = {{0, 0, 0}, {0, 1, 0}, {0, 0, 0}}

l3 = {{0, 0, 0}, {0, 0, 0}, {0, 0, 1}}

equ1 = r *M.l1 == t* l1.M + s*(a*l1 + d*l2 + g*l3)

equ2 = r*M.l2 == t*l2.M + s*(b*l1 + e*l2 + h*l3)

equ3 = r*M.l3 == t*l3.M + s*(c*l1 + f*l2 + j*l3)

Reduce[{equ1, equ2, equ3}, {a, b, c, d, e, f, g, h, j, r, t}]
```

Algorithm to calculate the polynomial identities linear in the structure constants

We used the following program in section 4.7 several times to compute polynomial identities in the structure constants of the form

$$F(x_{ij}^k) = \sum_{i,j,k}^{3} r_{ij}^k \cdot x_{ij}^k.$$

We have to mention that in this program the input of the functions li[1], li[2], li[3] usually consists of an algebra with which a basis change was undertaken. For this basis change we used the matrix $g = \{\{g_1, g_2, g_3\}, \{g_4, g_5, g_6\}, \{g_7, g_8, g_9\}\}$. That's why we solve with the "reduce"-command in the variables $g_1, \ldots, g_9$.

```
li[1] = {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
li[2] = {{0, 0, 1}, {0, 0, 0}, {0, 0, 0}}
li[3] = {{0, 1, 0}, {0, 0, 0}, {0, 0, 0}}

t = Function[li[#]]

w = Function[t[#1][[#3]][[#2]]]

F = Sum[r[i][j][k]*w[i, j, k], {i, 1, 3}, {j, 1, 3}, {k, 1, 3}]

Reduce[CoefficientList[
   Numerator[Together[F]], {g1, g2, g3, g4, g5, g6, g7, g8, g9}] ==
  0, {r[1][1][1], r[1][1][2], r[1][1][3], r[1][2][1], r[1][2][2],
  r[1][2][3], r[1][3][1], r[1][3][2], r[1][3][3], r[2][1][1],
  r[2][1][2], r[2][1][3], r[2][2][1], r[2][2][2], r[2][2][3],
  r[2][3][1], r[2][3][2], r[2][3][3], r[3][1][1], r[3][1][2],
  r[3][1][3], r[3][2][1], r[3][2][2], r[3][2][3], r[3][3][1],
  r[3][3][2], r[3][3][3]}]
```

Furthermore we used the functions t and w to compute Zariski-equations. In this notation we have for example (Proposition 4.30):

```
 equ1 = w[2, 3, 1]*w[3, 2, 1] - w[2, 2, 1]*w[3, 3, 1] -
  a (a + 1) (w[3, 2, 1] - w[2, 3, 1])^2

equ2 = w[1, 3, 2]*w[3, 1, 2] - w[1, 1, 2]*w[3, 3, 2] -
  a (a + 1) (w[3, 1, 2] - w[1, 3, 2])^2

equ3 = w[2, 1, 3]*w[1, 2, 3] - w[2, 2, 3]*w[1, 1, 3] -
  a (a + 1) (w[2, 1, 3] - w[1, 2, 3])^2
```

Lists of the coefficients r_{ij}^k

We list here the solution of the equation $F(g \cdot E_5(\beta)) = 0$ in r_{ij}^k that occured in Proposition 4.39.

```
r[1][1][2] = 0
r[1][1][3] = 0
r[1][2][3] = 0
r[1][3][2] = 0
r[1][3][3] = r[1][2][2]
r[2][1][2] = r[1][1][1] - r[1][2][2]
r[2][1][3] = 0
r[2][2][1] = 0
r[2][2][2] = r[1][2][1] + r[2][1][1]
r[2][2][3] = 0
r[2][3][1] = 0
r[2][3][2] = r[1][3][1]
r[2][3][3] = r[2][1][1]
r[3][1][2] = 0
r[3][1][3] = r[1][1][1] - r[1][2][2]
r[3][2][1] = 0
r[3][2][2] = r[3][1][1]
r[3][2][3] = r[1][2][1]
r[3][3][1] = 0
r[3][3][2] = 0
r[3][3][3] = r[1][3][1] + r[3][1][1]
```

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Abstract

In this thesis we are concerned with the orbit closure problem for algebras in algebraic transformation group theory. The general linear group $\mathrm{GL}(V)$ over a field $\mathbb{K}$ acts on the vector space $V^* \otimes V^* \otimes V$, the space of $\mathbb{K}$-algebra structures, by the change of basis. For two $\mathbb{K}$-algebra structures λ and μ we say that μ is a degeneration of λ if μ lies in the orbit closure of λ with respect to the Zariski topology. For this we write $\lambda \to_{\mathrm{deg}} \mu$. The orbit closure problem in this form is about the classification of all degenerations of a certain algebra structure in a fixed dimension.

The main result in this work is the classification of all degenerations of Novikov algebras over $\mathbb{C}$ in dimension three. Such algebras form a subclass of left-symmetric algebras, so called pre-Lie algebras. Approaching this we also give the complete classification of 2-dimensional pre-Lie algebras. This is surprisingly complicated. For example in dimension two there are only two non-isomorphic Lie algebras. However, we have already infinitely many 2-dimensional pre-Lie algebras. Both classifications turn out to be very extensive.

To reach these goals we generalize and enlarge methods that were applied in the case of Lie algebra degenerations. For example the $\mathfrak{C}_{\mathrm{p,q}}$-invariant and semi-invariants like the dimension of the center of an algebra are of that kind. Thereby we establish semi-invariants that are characteristic for the type of pre-Lie and Novikov algebras. Furthermore we bring new results that show the relation between degenerations in different dimensions. A substantial statement in this direction is that in case of a degeneration of two given algebras $A \to_{\mathrm{deg}} B$ also all factors A/I formed by an arbitrary ideal $I \subset A$ have to degenerate to corresponding factors of the algebra B.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit dem Orbitabschlussproblem für Algebren in der Theorie der algebraischen Transformationsgruppen. Die allgemeine lineare Gruppe $\mathrm{GL}(V)$ über einem Körper $\mathbb{K}$ operiert auf dem Vektorraum $V^* \otimes V^* \otimes V$, dem Raum aller $\mathbb{K}$-Algebra-Strukturen, durch Basiswechsel. Liegt bezüglich der Zariski-Topologie eine $\mathbb{K}$-Algebra-Struktur μ im Orbitabschluss einer $\mathbb{K}$-Algebra-Struktur λ so spricht man von einer Degeneration $\lambda \to_{\mathrm{deg}} \mu$. Das Orbitabschlussproblem in dieser Form stellt die Frage nach der Klassifikation aller Degenerationen einer bestimmten Algebra-Struktur in einer fixen Dimension.

In der vorliegenden Arbeit werden alle Degenerationen von Novikovalgebren über $\mathbb{C}$ in der Dimension drei klassifiziert. Jene Algebren bilden eine Unterklasse linkssymmetrischer Algebren, sogenannter pre-Liealgebren, deren sämtliche Degenerationen wir in der Dimension zwei bestimmen. Überraschenderweise ist dies bereits sehr aufwendig. Gibt es in dieser Dimension lediglich zwei nicht isomorphe Liealgebren so haben wir unendlich viele nicht isomorphe 2-dimensionale pre-Liealgebren. Wegen der großen Anzahl an Algebren und damit verbunden eine noch größere Anzahl möglicher Degenerationen haben sich beide Klassifikationen als äußerst umfangreich erwiesen.

Um diese Ziele zu erreichen werden bekannte Methoden zum Studium von Liealgebra-Degenerationen auf die Klasse der pre-Liealgebren verallgemeinert und erweitert. Bei diesen handelt es sich beispielsweise um die $\mathfrak{C}_{\mathrm{p,q}}$-Invariante und um Semi-Invarianten wie etwa die Dimension des Zentrums einer Algebra. Weiters werden Semi-Invarianten

eingeführt, die speziell auf den Fall von pre-Lie- bzw. Novikov-Algebren anwendbar sind. Darüber hinaus werden neue Resultate bewiesen, welche Degenerationen unterschiedlicher Dimension in Zusammenhang setzen. Es konnte beispielsweise gezeigt werden, dass im Falle einer Degeneration zweier gegebener Algebren $A \to_{\mathrm{deg}} B$ auch alle Faktoren A/I mit einem beliebigen Ideal $I \subset A$ gegen entsprechende Faktoren der Algebra B degenerieren müssen.