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[1] This chapter is a joint work with my colleague Emanuel Gasteiger.

Chapter 1

Search Frictions, Job Flows and Optimal Monetary Policy

1.1 Introduction

Recently, many authors have studies the impacts of monetary policy inn search frictional labour markets.[1] Blanchard and Gali (2008) and Sveen and Weinke (2008) illustrate the new Keynesian perspectives on labour market dynamics and find that adding new Keynesian elements (monetary friction) into search and matching labour market models does not change too much and confirm that the optimal monetary policy should be set to respond mainly to price changes instead of unemployment fluctuations. Some others examine the relationship between inflation and unemployment in an economy where exchanges in goods and labour markets both involve costly search, based on modern monetary search models.[2] Berentsen, Menzio and Wright (2008) introduce a labour market with search frictions into Lagos and

[1] For paradigm search and marching labour market model and the survey of its development, please refer to Pissarides (2000), Mortensen and Pissarides (1999) or Rogerson, Shimer and Wright (2005).

[2] Search monetary models are pioneered by Kiyotaki and Wright (1991, 1993) for the case of indivisible commodities and money. They impose the assumption of indivisible commodities and money to guarantee the tractability of the model, because otherwise the distribution of money holding becomes too complicated to get analytical results. The search monetary frameworks which are suitable for economic and monetary policy analysis, i.e., the models that allow divisible commodities and money but circumvent the aforementioned difficulties concerning the distribution of money holdings, are developed along there lines: i) to introduce the assumption of large families, like Shi (1997); ii) to introduce an extra centralized good market and the assumption of linear negative utility of labour input in this centralized good market, like Lagos and Wright (2005); iii) to introduce complete financial markets among groups like Faig (2006). For further details about the development of search monetary models, please refer to survey paper Shi (2006).

Wright (2005). Their main findings are that a rise of inflation cause a higher unemployment rate and that the long run Phillips curve slopes upwards. Shi (1998) and Kumar (2008b) introduce a labour market with search friction into another monetary search model, Shi (1997). Shi (1998) finds that an increase in the money growth rate increases the steady state employment rate and output when the money growth rate is relative low, but reduces steady state employment rate and output when the money growth rate is already high. Furthermore, Friedman rule is not optimal in general. However, Shi (1998) gets a negative correlation of inflation and the steady state employment rate when money growth rate is low (different from Berentsen, Menzio and Wright (2008)), because the endogenous search intensity of households in the goods market could be weakened by lower money growth rate. To summarize, all above studies focus on optimal monetary policy and the relationship between inflation and the unemployment rate.

The present study pursues a more detailed question: What are the impacts of monetary policy on job creations and job destructions? Furthermore, what is the implication on the optimal monetary policy in the long run if we introduce endogenous job destruction? Berentsen, Menzio and Wright (2008) (as well as Shi (1998)) assume the job destruction rate to be a constant, as in the standard search and matching labour market model, say, Pissarides (2000). Therefore, they cannot study the impacts of monetary policy on the job destruction rate. However, one cares about this question for at least two reasons. First, economists and the public not only care about the unemployment rate in the economy, but also about how many workers lost their jobs and the chance for unemployed workers to start a new job. Second, as will be shown later in this study, job creations and job destructions can have ambiguous effects on welfare and on the optimal long run monetary policy. Because we model job creations and job destructions in the sense that a firm chooses to enter into or quit the business, our model also relates to the literature which investigates the monetary influence on the economic welfare through the channel of endogenous firm entry and/or exit. Berentsen and Waller (2009) and Rocheteau and Wright (2005) study optimal long-run monetary policy in the presence of endogenous firm entry and monetary search frictions that make money essential in the economy:[3] Berentsen and Waller (2009) claim that the optimal policy would deviate from the Friedman rule if there is a congestion externality affecting firm entry; Rocheteau and Wright (2005) claim that the Friedman rule must be the optimal monetary policy in search equilibrium and competitive search equilibrium, but not necessarily in competitive equilibrium.[4]

[3] Lewis (2009), Bergin and Corsetti (2008) and Bilbiie et al. (2007) also study optimal monetary policy in the long run in the presence of endogenous firm entry, but they employ ingredients other than micro-founded search models to make money essential, say cash-in-advance constrains, money in utility function, wage rigidity, price rigidity and so on.

[4] Rocheteau and Wright (2005) define these three equilibria by their market structures

Endogenous job separation and firm heterogeneity along Mortensen and Pissarides (1994) are introduced into Berentsen, Menzio and Wright (2008) to study job market flows in a stationary equilibrium. We also follow Mortensen and Pissarides (1994) by assuming that new jobs have the highest productivity in the economy and old jobs experience idiosyncratic shocks which damage their productivity. We confirm the conclusion about the long run Phillips curve in Berentsen, Menzio and Wright (2008): a rise of inflation increases the unemployment rate; the long run Phillips curve slopes upwards; the economy reaches the lowest unemployment rate when the central bank applies the Friedman rule. We also show that the endogenous job destruction rate rises when the inflation rate rises, because a higher inflation rate reduce the profits of all firms and make the less productive firms more likely to quit the business. Therefore, high inflation encourages job destruction, which means that there are more jobs with highest productivity replacing jobs with lower productivity every period, at the cost of higher job losing rate and higher unemployment rate in the economy. Thus, in the steady state of the economy, the average productivity level of the economy is higher when inflation is higher, although the total employment becomes lower. This result shows that endogenous job separation and firm heterogeneity have major impacts on optimal monetary policy because the monetary authority can balance between average productivity level and unemployment rate by setting interest rates. We claim that the destruction of lower productivity jobs and the creation of higher productivity jobs might be too low under the Friedman rule, which in turn implies that the optimal long run monetary policy deviate from the Friedman rule.[5] We use numerical methods to verify this conjecture.

We then calibrate our theoretical model to the United States economy, using parameter values commonly choosen in the relevent macro-labour and monetary economics literature. Our numerical exercises show that the destruction of lower productivity jobs and the creation of higher productivity jobs are indeed too low under the Friedman rule from the perspective of a welfare maximizing monetary authority. The optimal interest rate implied by our numerical exercise is around 0.02 at quarterly level, which roughly equals the average quarterly interest rate of the US economy during 1955-2005. Moreover, we also report that the maximal welfare gain of deviating from the Friedman rule is worth less that 1 percent of consumption.

The remainder of the chapter is organized as follows. The basic model is

of money-goods exchange: search equilibrium corresponds to bargaining; competitive equilibrium corresponds to price taking; competitive search equilibrium corresponds to price posting.

[5] This channel is absent in Berentsen, Menzio and Wright (2008), because of its exogenous constant job separation rate and homogenous firms setting. A positive interest rate would only lead to a higher unemployment rate but not to productivity improvement. Therefore, the optimal monetary policy implied by their model is the Friedman rule.

presented in section 1.2. The description of stationary equilibrium and its properties are given in section 1.3. The numerical experiments of optimal long run monetary policy are performed in section 1.4, and a final section concludes.

1.2 The Basic Model

The present model is mostly based on the "price taking" version of Berentsen, Menzio and Wright (2008), except for the endogenous job separation and firm heterogeneity along Mortensen and Pissarides (1994).

1.2.1 The Environment

There are two types of private agents: firms and households. There exists a unit continuum of households; while the measure of firms are endogenously determined. Households work, consume, and enjoy utility; firms create jobs, maximize profits and pay out dividends to households.

Time is discrete and continues forever. In each period, there are three markets that open sequentially. Market 1 is a labour market, markets 2 is a decentralized goods market and market 3 is a centralized goods market. These three markets are indexed by $i = 1, 2, 3$, respectively.

The labour market is modeled in the spirit of Mortensen and Pissarides (1994), where firms create vacancies and households search for jobs. Firms and households meet each other bilaterally according to a matching technology. A pair consisting of a household and a firm then combine to create a job that produces a preliminary product, good x, and bargain over wages, w. Wages are chosen so as to share the surplus from a job match in fixed proportions at all times. The worker's share is $\eta \in (0, 1)$. Consequently, more productive jobs offer higher wages. We assume that the wage is paid in market 3, thus it does not matter whether wages are paid in cash or goods. Each job is characterized by its productivity y and newly created jobs represent the highest productivity $\bar{y}$, following Mortensen and Pissarides (1994). Idiosyncratic shocks change the productivity of jobs according to Poisson process with arrival rate δ. When a job is hit by such idiosyncratic shock, a new value of y is drown from the fixed distribution $F(y)$. y has finite upper support $\bar{y}$ and no mass points. Thus, the productivity of any given job is a stochastic process with initial condition of the upper support of the distribution and terminal state of the reservation productivity that leads to job destruction. Filled jobs do not always exit when they are hit by shocks, because there is a cost of maintaining a vacancy, k. Existing filled jobs are destroyed only if their productivity fall below some critical number y_d. Therefore, the endogenous rate at which existing jobs are destroyed is $\delta F(y_d)$.

Decentralized goods market is modeled in the spirit of the price-taking version of the decentralized goods market in Berentsen, Menzio and Wright (2008). Unlike the day time market in Lagos and Wright (2005), where goods are traded bilaterally by bargaining between pairs due to anonymous matching, there is a Walrasian auctioneer in the market so the participating buyers and sellers take the market price as given. However, the present goods market is not a centralized goods market because there are restrictions for the buyers (households) and sellers (firms) to participate in the market. We assume that the buyers and sellers must first meet bilaterally[6] according to a matching technology as a pair to enter into the decentralized market where goods q is traded. The decentralized market here is less decentralized relative to that of Lagos and Wright (2005), but it does not make money inessential as long as we maintain the anonymity.[7] The good q is formed by the preliminary product, good x without any labour input, according to a transforming technology which is specified below.[8] Unused good x is taken by the firm from market 2 to market 3, without any depreciation or cost.

Centralized goods market is the same as centralized goods market of Berentsen, Menzio and Wright (2008), which is in turn modeled in the spirit of night time market in Lagos and Wright (2005), where good x is traded multilaterally. All the private agents take the market price as given and there is no restriction of entering into this market. Unsold good x would vanish between two periods. Also, without loss of generality we assume that agents discount at rate β between market 3 and the next market 1, but not between the other markets.

We assume a central bank exists and controls the supply of fiat money. We denote the growth rate of the money supply by π, so that $\hat{M} = (1+\pi)M$, where M denotes the per capita money stock in market 3 and variables with a "hat" above indicate the values of variables next period. Therefore, in steady states, π is the inflation rate. The central bank implements its inflation goal by providing deterministic lump-sum injections of money or levying lump-sum taxes, πM, to the households at the end of each period. If $\pi > 0$, households receive lump-sum transfers of money; for $\pi < 0$, the central bank must be able to extract money via lump-sum taxes from the households.

[6] This assumption is made to make sure that the measure of sellers equals that of buyers, so there is only one term of trade (selling equals buying) in the economy. This is a slightly stronger restriction than other price-taking versions of the decentralized goods markets, such as, Rocheteau and Wright (2005) or Berentsen, Camera and Waller (2007). However, there is no loss of any generality and the analysis becomes easier.

[7] Rocheteau and Wright (2005), Berentsen, Camera and Waller (2007) and Berentsen, Menzio and Wright (2008) make similar points.

[8] Berentsen, Menzio and Wright (2008) assume that firms first produce the goods traded in market 2 while the goods traded in market 3 are made from the goods traded in market 2. Here we assume the transformation techonolodgy to be of a different type. The reason is explained in footnote 10.

We adopt Berentsen, Menzio and Wright (2008)'s convention for measuring real balances z, i.e. the nominal balances of households m in current market 3, next market 1 and market 2 are deflated by p, the price level in the current centralized goods market (market 3), which is the latest centralized goods market price known. For instance, when an agent brings $\hat{m}$ fiat money from market 3 to next market 1, we let $\hat{z} = \hat{m}/p$ denote his real balances. When he then takes this $\hat{m}$ fiat money to next market 2, his real balances is $\hat{z} = \hat{m}/p$, still deflated by p. If he were to bring $\hat{m}$ through the next market 2 and into the next market 3, his real balance is then given by $\hat{m}/\hat{p} = \hat{z}\hat{\rho}$, where $\hat{\rho} = p/\hat{p}$ converts $\hat{z}$ into the units of the numeraire in that market. Notice $\hat{\rho} = 1/(1+\pi)$, where π is the inflation rate between this and the next centralized goods market.

Throughout the discussion in the text, we assume policy and the productivity distribution to be constant, and focus on steady states.

1.2.2 Households

Let $s = e, u$ index employment status: e indicates that a household is employed; u indicates that a household is unemployed.

We now consider optimal decisions of the household, starting with market 3. A household, who is employed in a firm of productivity y and with real balances z, solves

$$\begin{aligned} W_{3,e}(z,y) &= \max_{x,\hat{z}}\{x + \beta W_{1,e}(\hat{z},y)\} \\ \text{s.t. } x + \hat{z} &= w(y) + \Delta + \frac{\pi M}{p} + z, \end{aligned} \tag{1.1}$$

where W denotes the value function of an employed household, x is the consumption in market 3, $w(y)$ is the wage paid by the firm of productivity y, and Δ is dividend income.[9] We here assume that utility is linear in x, as in Lagos and Wright (2005) and Berentsen, Menzio and Wright (2008), to ensure that all agents in the centralized market choose the same real balance to enter into next period. Substituting x from the budget constraint into (1.1) yields

$$W_{3,e}(z,y) = w(y) + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta W_{1,e}(\hat{z},y)\}. \tag{1.2}$$

Therefore, $W_{3,e}(z,y)$ is linear in z.

[9] We assume the representative household holds the representative portfolio and therefore receives same amount of dividend. So the equilibrium dividend Δ equals the average profit of all the firms.

Similarly, the problem for an unemployed household with real balances z reads

$$\begin{aligned} W_{3,u}(z) &= \max_{x,\hat{z}}\{x + \beta W_{1,u}(\hat{z})\} \\ \text{s.t. } x + \hat{z} &= b + \Delta + \frac{\pi M}{p} + z, \end{aligned} \tag{1.3}$$

where b is the home production of an unemployed household. We have $0 < b < \bar{y}$, which means that home production is lower than the highest productivity level. Substituting x from the budget constraint into (1.3) yields

$$W_{3,u}(z) = b + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta W_{1,u}(\hat{z})\}. \tag{1.4}$$

$W_{3,u}(z)$ is also linear in z.

We now move to market 2. The value functions of an employed household and an unemployed household, respectively, with real balance z, read

$$W_{2,e}(z, y) = \alpha_h \max_q\{v(q) + W_{3,e}[\rho(z - dq), y]\} + (1 - \alpha_h)W_{3,e}(\rho z, y) \tag{1.5}$$

$$W_{2,u}(z) = \alpha_h \max_q\{v(q) + \alpha_h W_{3,u}[\rho(z - dq)]\} + (1 - \alpha_h)W_{3,u}(\rho z) \tag{1.6}$$

where α_h is the probability for a household to trade in the decentralized goods market, and d is the real price of good q in second market and is taken as given by both households and firms. The matching technology in the goods market will be discussed later. $v(\cdot)$ is the household's utility function of consuming goods. $v(\cdot)$ is twice differentiable with $v(0) = 0$, $v' > 0$, $v'' < 0$, $\lim_{q\to 0} v'(q) = +\infty$, and $\lim_{q\to+\infty} v'(q) = 0$.

Using the linearity of W_3, implied by (1.2) and (1.4)), equations (1.5) and (1.6) become

$$W_{2,e}(z, y) = \alpha_h \max_q[v(q) - \rho dq] + W_{3,e}(\rho z, y), \tag{1.7}$$

$$W_{2,u}(z) = \alpha_h \max_q[v(q) - \rho dq] + W_{3,u}(\rho z). \tag{1.8}$$

The value function of an employed household with real balance z reads

$$W_{1,e}(z, y) = (1 - \delta)W_{2,e}(z, y) + \delta \int_{-\infty}^{\bar{y}} \max\{W_{2,e}(z, l), W_{2,u}(z)\}dF(l). \tag{1.9}$$

The second term of the right hand side of (1.9) shows that household would leave the firm (job separation) if the value of working in a firm with productivity l is smaller than the value of being unemployed.

The value function of an unemployed household with real balance z reads

$$W_{1,u}(z) = (1-\lambda_h)W_{2,u}(z) + \lambda_h W_{2,e}(z,\bar{y}), \tag{1.10}$$

where λ_h is the probability for an unemployed household to find a vacancy. The matching technology in the labour market will be discussed later. The second term on the right hand side of (1.10) shows that the job which an unemployed worker gets has the highest productivity. Substituting $W_{2,e}$ and $W_{2,u}$ from (1.7) and (1.8) into (1.9) and using the linearity of $W_{3,e}$ and $W_{3,u}$ yields

$$\begin{aligned} W_{1,e}(z,y) \quad &= \quad \alpha_h \max_q [v(q) - \rho dq] + \rho z + (1-\delta)W_{3,e}(0,y) \\ &\quad +\delta \int_{-\infty}^{\bar{y}} \max\{W_{3,e}(0,l), W_{3,u}(0)\} dF(l). \end{aligned} \tag{1.11}$$

Substituting $W_{2,e}$ and $W_{2,u}$ from (1.7) and (1.8) into (1.10) and using the linearity of $W_{3,e}$ and $W_{3,u}$, we get

$$W_{1,u}(z) = \alpha_h \max_q [v(q) - \rho dq] + \rho z + (1-\lambda_h)W_{3,u}(0) + \lambda_h W_{3,e}(0,\bar{y}). \tag{1.12}$$

Then, substituting $W_{1,e}$ and $W_{1,u}$ from (1.11) and (1.12) of next period into (1.2) yields our equation for the value function of households in market 3 only, namely

$$\begin{aligned} W_{3,e}(z,y) \quad &= \quad w(y) + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta\hat{\alpha}_h \max_{\hat{q}}[v(\hat{q}) - \hat{\rho}\hat{d}\hat{q}] + \beta\hat{\rho}\hat{z}\} \\ &\quad +\beta[\delta \int_{-\infty}^{\bar{y}} \max\{W_{3,e}(0,l), W_{3,u}(0)\} dF(l) + (1-\delta)W_{3,e}(0,y)]. \end{aligned} \tag{1.13}$$

Similarly, substituting $W_{1,e}$ and $W_{1,u}$ from (1.11) and (1.12) of next period into (1.4) yields

$$\begin{aligned} W_{3,u}(z) \quad &= \quad b + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta\hat{\alpha}_h \max_{\hat{q}}[v(\hat{q}) - \hat{\rho}\hat{d}\hat{q}] + \beta\hat{\rho}\hat{z}\} \\ &\quad +\beta[(1-\hat{\lambda}_h)W_{3,u}(0) + \hat{\lambda}_h W_{3,e}(0,\bar{y})]. \end{aligned} \tag{1.14}$$

This completes the description of households' problem.

1.2.3 Firms

We assume that at the beginning of each period, firms can post a vacancy at a fixed cost k. If the vacancy is matched with a household in the labour market, the firm enter into the decentralized market in this period; otherwise, the vacancy is destroyed. Let V denote the value function of a filled job and U the value function of a vacancy. We now consider optimal decisions of the firm, again starting with market 1. The value function of a job of productivity y reads,

$$V_1(y) = (1-\delta)V_2(y) + \delta \int_{-\infty}^{\bar{y}} \max\{V_2(l), 0\} dF(l). \tag{1.15}$$

The second term of the right hand side of (1.15) shows that firm terminate the job if its value after the productivity shock is smaller than zero. The value function for a vacancy reads

$$U = -k + \lambda_f V_2(\bar{y}), \tag{1.16}$$

where λ_f is the probability for a vacancy to be filled by an unemployed household.

The free entry condition reads

$$U = 0. \tag{1.17}$$

(1.16) and (1.17) then imply

$$k = \lambda_f V_2(\bar{y}). \tag{1.18}$$

We now consider market 2. We assume that the transforming technology from goods x into goods q is characterized by $c(q)$, which is assumed to be twice differentiable and to satisfy $c(0) = 0$, $c' > 0$, $c'' > 0$. We claim that $c(\cdot)$ is the opportunity cost of trade in the decentralized goods market in term of real balance in market 3.[10]

Then the value functions of a firm in market 2 reads,

$$V_2(y) = \alpha_f \max_q V_3[y, y - c(q), \rho q d] + (1-\alpha_f) V_3(y, y, 0), \tag{1.19}$$

where α_f is the probability for a firm to trade in market 2. $V_3(\cdot, \cdot, \cdot)$ is the value function of a firm in market 3, in which the first argument is the productivity of the firm, the second argument is good x that the firm takes

[10] We want the opportunity cost to be independant of the firm's productivity and to be identical across the economy to ensure tractability of the model. Therefore, unlike Berentsen, Menzio and Wright (2008), I assume that the preliminary product of the firm is traded in market 3 instead of market 2.

to market 3 and the third argument is the real money balance that the firm takes to market 3.

The value function of the firm in market 3 then reads,

$$V_3(y, x, z) = x + z - w(y) + \beta V_1(y). \tag{1.20}$$

Substituting V_3 from (1.20) into (1.19) yields

$$V_2(y) = y + \alpha_f \max_q[\rho q d - c(q)] - w(y) + \beta V_1(y). \tag{1.21}$$

Define $R \equiv \max_q[\rho q d - c(q)]$, which is a firm's profit in market 2 if it gets the chance to trade. Substituting V_1 from (1.15) into (1.21) yields

$$V_2(y) = y + \alpha_f R - w(y) + \beta(1-\delta)V_2(y) + \beta\delta \int_{-\infty}^{\bar{y}} \max\{V_2(l), 0\} dF(l). \tag{1.22}$$

This completes the description of the firms' problem.

1.3 Equilibrium

1.3.1 Goods Market

We first describe the matching technology in the decentralized goods market. We assume that the probability for household to trade in the decentralized goods market α_h is exogenous and constant.[11] Then, the probability for a firm to trade in market 2 α_f is determined in equilibrium. The measure of firms must equal to the measure of the employed households $1-u$, where u is the measure of the unemployed households, i.e. the unemployment rate. The measure of firms matched in decentralized goods market must equal the measure of households matched in the decentralized goods market, which yields $1 \cdot \alpha_h = \alpha_f(1-u)$. Therefore, we have

$$\alpha_f = \frac{\alpha_h}{1-u}. \tag{1.23}$$

(1.23) means that the more firms there are in the market, the harder it is for a firm to be matched in the decentralized goods market and profit. This is called a congestion externality in Berentsen and Waller (2009). We will

[11] I make this assumption for simplicity. My assumption here results in a unique equilibrium. Berensten, Menzio and Wright (2008) assume α_h to be endogenous and to depend on the unemployment rate, which brings much complexity into the analysis and results in the possibility of multiple equilibria. However, Berensten, Menzio and Wright (2008) only analyze the case of a unique equilibrium. Therefore, making α_h exogenous does not change the analytical conclusion for the properties of the equilibrium unemployment rate and the terms of trade.

return to (1.23) in section 1.4.3 to explain why the present study is different from Berentsen and Waller (2009).

We now establish the equilibrium conditions in market 2. The optimal selling problems are identical for all the firms in the economy and read[12]

$$\max_q [\rho q d - c(q)].$$

The FOC then reads

$$\rho d = c'(q). \tag{1.24}$$

The optimal buying problems are identical for all the households in the economy and read

$$\max_q [v(q) - \rho d q]$$
$$\text{s.t. } dq \leq z.$$

The FOC then reads

$$\begin{array}{rll} v'(q) = \rho d & \text{if} & z > z^* \\ dq = z & \text{if} & z \leq z^*, \end{array} \tag{1.25}$$

where z^* is defined so as to satisfy

$$v'(\frac{z^*}{d}) = \rho d.$$

Comparing (1.13) and (1.14) implies the problems of optimal real balance taken into next period are identical for all household in the economy and read:

$$\max_z \{-z + \beta\alpha_h \max_q [v(q) - \rho d q] + \beta \rho z\}. \tag{RB}$$

Assumption 1 *We have* $1 + \pi \geq \beta$, *i.e.* $\rho \leq \frac{1}{\beta}$.

Assumption 1 means that the money growth rate is higher than what the Friedman rule would require. To see its application more clearly, we use the Fisher equation, which links the nominal interest rate and inflation in the long run. The Fisher equation reads

$$1 + i = \frac{1+\pi}{\beta}, \tag{1.26}$$

where i is nominal interest rate. Therefore, Assumption 1 simply means that $i \geq 0$, while the Friedman rule requires $i = 0$, i.e. $\pi = \beta - 1$. Assumption 1 makes sure that optimal problem (RB) has a meaningful solution, as Lagos and Wright (2005) point out. Then we only consider the equilibrium of the economy either in the case $1 + \pi > \beta$, or the case $1 + \pi = \beta$ but equilibrium is the limit as $1 + \pi \to \beta$ from above.

[12] In fact, there is a feasibility constraint $q \leq y$. We assume that this conditions is always slack for simplicity, following Berensten, Menzio and Wright (2008).

Proposition 1 *Under Assumption 1, in equilibrium, q is the solution to*

$$\frac{v'(q)}{c'(q)} = \frac{i}{\alpha_h} + 1, \tag{1.27}$$

the households set their next period real balance at

$$z = c'(q)q, \tag{1.28}$$

and the firm's profit in market 2 (if they get the chance to trade) equals

$$R = c'(q)q - c(q). \tag{1.29}$$

Proof: See appendix.[13]

(1.27) fully characterizes the equilibrium in the decentralized goods market. We then have the following proposition for the goods market.

Proposition 2 *(1.27) is the equilibrium condition for the goods market. With Assumption 1, the solution exist and is unique. Furthermore, q is decreasing in i; and the firms' revenue in market 2, R, is increasing in q and decreasing in i.*

Proof: See appendix.

Because $R = c'(q)q - c(q)$ is a function of q, we will denote R as $R(q)$ later. Note that, when $i = 0$, q and $R(q)$ reach their maxima and $v'(q) = c'(q)$ holds.

1.3.2 Labour Market

We now describe the matching technology in the labour market. As in the standard labour market literature, the total match M is a constant return to scale function of the measure of unemployed households, u and of vacancies, ϕ,

$$M = m(\phi, u),$$

where $m(\phi, u)$ is a CRS function and increasing in both ϕ and u. M is also equal to the job destruction and job construction in equilibrium. Then, the probability for an unemployed household to find a job is

$$\lambda_h = \frac{m(\phi, u)}{u} = m(\mu, 1), \tag{1.30}$$

where $\mu = \phi/u$ is the labour market tightness. The probability for a vacancy to be filled is

[13] Similar calculations can also be found in Rocheteau and Wright (2005).

$$\lambda_f = \frac{m(\phi, u)}{\phi} = m(1, \frac{1}{\mu}). \tag{1.31}$$

Furthermore, the functional form of $m(\phi, u)$ is assumed to make sure that λ_h, $\lambda_f \in (0, 1)$.

We assume that wages are determined at the end of the labour market, although they are not paid until market 3, as mentioned earlier. Therefore, it does not matter whether wages are paid in money or goods. There is no commitment to wages across period, so wages are newly bargained in every period. We also use the generalized Nash solution, in which the household has bargaining power η and threat points are given by continuation values. The surplus for a household who meets a firm of productivity y is

$$S_h(y) = W_{2,e}(0, y) - W_{2,u}(0) = W_{3,e}(0, y) - W_{3,u}(0) \tag{1.32}$$

by the virtue of (1.7) and (1.8) and the linearity of W_2 in z. The surplus for a firm of productivity y is

$$S_f(y) = V_2(y). \tag{1.33}$$

The total surplus for such a job match $S(y)$ is

$$S(y) \equiv S_h(y) + S_f(y) = W_{3,e}(0, y) - W_{3,u}(0) + V_2(y) \tag{1.34}$$

by definition. Wage bargaining divides the surplus from a job match in fixed proportions, i.e.,

$$\frac{S_h(y)}{\eta} = \frac{S_f(y)}{1-\eta} = S(y). \tag{1.35}$$

Since $S(y)$ is monotonically increasing in y, job destruction satisfies the reservation property. (1.35) implies that there is a unique reservation productivity y_d that solves

$$\frac{S_h(y_d)}{\eta} = \frac{S_f(y_d)}{1-\eta} = S(y_d) = 0$$

such that jobs that get a shock $l < y_d$ are destroyed. Therefore, jobs are destroyed at rate $\delta F(y_d)$. There are two group of jobs: the first group includes newly created jobs and old jobs which have not experienced idiosyncratic shocks; the second group consists of old jobs which have experienced idiosyncratic shocks. Jobs in the first group have productivity $\bar{y}$ and we denote the measure of jobs in this group as γ; while the productivity of sjob in the second group follow a truncated distribution of $F(x)$, truncated at y_d, and we denote its distribution function as $G(x)$. In steady sate, we have that the unemployment rate, new job matching, job destruction, job destruction rate and the measures of two groups of jobs are all constant. Therefore, it required that, firstly, new job matching equals job destruction, i.e.

$$\lambda_h u = \delta F(y_d)(1-u), \tag{1.36}$$

and secondly, job flows into the first group equals job flows out of the first group, i.e.,

$$\lambda_h u = \delta\gamma. \tag{1.37}$$

We then have the following proposition for labour market.

Proposition 3 *The equilibrium condition in the labour market is characterized by the following two equations for the labour market tightness μ and the reservation productivity y_d:*

$$\frac{[1-\beta(1-\delta)]k}{(1-\eta)m(1,\frac{1}{\mu})} = \bar{y} - y_d \tag{1.38}$$

$$y_d + \alpha_h\left[1+\frac{\delta F(y_d)}{m(\mu,1)}\right]R(q) - b + \frac{\beta\delta}{1-\beta(1-\delta)}\int_{y_d}^{\bar{y}}[1-F(l)]dl - \frac{\eta\beta k}{1-\eta}\cdot\mu = 0. \tag{1.39}$$

The other labour market variables can be expressed by μ and y_d as follows:

$$u = \frac{\delta F(y_d)}{m(\mu,1)+\delta F(y_d)}, \tag{1.40}$$

$$\gamma = \frac{m(\mu,1)F(y_d)}{m(\mu,1)+\delta F(y_d)}, \tag{1.41}$$

$$M = \frac{\delta m(\mu,1)F(y_d)}{m(\mu,1)+\delta F(y_d)}. \tag{1.42}$$

Proof: See appendix.

1.3.3 Steady State Equilibrium

Proposition 1 and 2 show that, goods market equilibrium is solely defined by q and equilibrium condition (1.27), independantly of the situation in the labour market. Firm's revenue in market 2, $R(q)$ is the link between the labour market and the goods market. This leads to the following definition.

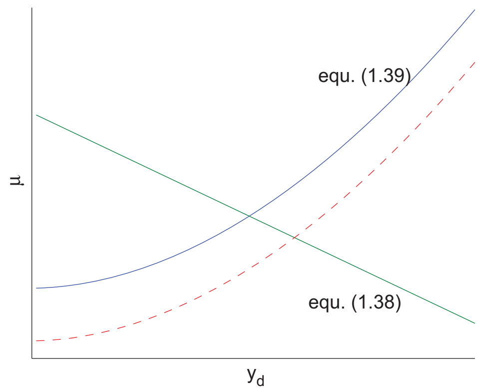
Definition 1 *A steady state equilibrium of the economy is a triple $\{q, \mu, y_d\}$ that satisfies equilibrium condition (1.27), (1.38) and (1.39).*

Before moving on to the existence and uniqueness of the equilibrium, we need to prove the following Lemma about the properties of the equilibrium conditions in the labour market.

Lemma 4 *(1.38) slopes downward in (y_d, μ) space. Given $R(q)$ constant, (1.39) slopes upward in (y_d, μ) space and (1.39) shifts down if $R(q)$ goes down.*

Proof: See appendix.

To establish the existence and uniqueness of the equilibrium, we only need to consider the labour market with a given $R(q)$. By virtue of the above Lemma, we know that y_d and μ is also uniquely determined by (1.38) and (1.39). Henceforth, we have established the existence and uniqueness of the steady state equilibrium. Now we check the property of equilibrium when the interest rate changes. The effect of interest rate changes on q is clear as we proved in Proposition 2, q and $R(q)$ become smaller when i goes up. By virtue of Lemma 4, we can draw the curve of (1.38) and (1.39) in a (y_d, μ) space.



Graph 1.1: The joint determination of y_d and μ

It has been shown in Lemma 4 that, when $R(q)$ becomes smaller because of i's increasing. Then curve (1.38) does not move and curve (1.39) goes down. We then conclude that μ becomes smaller, i.e. $\frac{\partial \mu}{\partial i} < 0$, and y_d becomes larger i.e. $\frac{\partial y_d}{\partial i} > 0$. It is also useful to show that the unemployment rate becomes larger. (1.40) implies that

$$\begin{aligned}\frac{\partial u}{\partial i} &= \frac{\partial u}{\partial \mu}\frac{\partial \mu}{\partial i} + \frac{\partial u}{\partial y_d}\frac{\partial y_d}{\partial i} \\ &= -\frac{\delta F(y_d)}{[m(\mu,1)+\delta F(y_d)]^2}\frac{dm(\mu,1)}{d\mu}\frac{\partial \mu}{\partial i} + \frac{\delta F'(y_d)m(\mu,1)}{[m(\mu,1)+\delta F(y_d)]^2}\frac{\partial y_d}{\partial i} > 0.\end{aligned}$$

So we confirm the conclusion from Berentsen, Menzio and Wright (2008) that a higher interest rate or a higher inflation rate leads to a higher unemployment rate in the long run; the Phillips Curve should slope upward.

Summarizing, we have established the following results:

Proposition 5 *With Assumption 1, the steady state equilibrium always exist and is unique. Furthermore, in this equilibrium the amount of goods traded in the decentralized goods market, q, and labour market tightness μ are decreasing in the interest rate i, while the reservation productivity level y_d and the unemployment rate u are increasing in i.*

The above proposition is the most important result of the present study. We could also study the job losing rate and job finding rate in the labour market. The job losing rate for the employed is $\delta F(y_d)$, which is also the job destruction rate and an increasing function of y_d; the job finding rate for the unemployed is $m(\mu, 1)$, which is an increasing function of μ. We then conclude from Proposition 5 that when the inflation rate increases, it is more likely for the employed to lose their job, because the job destruction rate $\delta F(y_d)$ will rise. The reason is that the job is less profitable for firms (R goes down) and the rise of trading opportunities (α_f goes up because there are less firms) can not compensate the profit loss. Furthermore, when the inflation rate increases, it is harder for the unemployed to find a new job, because the job finding rate $m(\mu, 1)$ will rise.

1.4 Optimal Monetary Policy: Numerical Experiments

We introduce endogenous job separation and firm heterogeneity into Berentsen, Menzio and Wright (2008) and find that a higher inflation causes more job separations and a higher unemployment rate. An important feature of the present model is that higher inflation cause also a higher job destruction rate $\delta F(y_d)$. Intuitively, higher inflation make firms less profitable in the decentralized goods market and also less profitable in general, so that firms with lower productivity are more likely to quit the market, i.e. the rise of y_d. Because job destructions are modeled as a process of higher productivity jobs replacing jobs with lower productivity every period, there is a role for monetary policy to affect welfare by adjusting the job destruction rate.

We notice that Berentsen, Menzio and Wright (2008) assume all firms to have the same productivity level and job destructions to take place at an exogenous constant rate. Thus job destruction in Berentsen, Menzio and Wright (2008) simply take the form of some firms replacing others with the very same productivity, and higher unemployment must consquently generate a welfare loss. A positive interest rate would only a incur higher unemployment rate but no productivity improvement. Therefore, the optimal monetary policy implied by their model is the Friedman rule.

However, in our model, higher inflation encourages job destruction, which means that more jobs with highest (higher) productivity replace jobs with lower productivity every period, at the cost of a higher job losing rate and higher unemployment rate. Thus, talking about the steady state of the economy, the average productivity level of the economy is higher when inflation is higher, although the total employment becomes lower. This result shows that endogenous job separation and firm heterogeneity may have a major impact on optimal monetary policy because the monetary authority can balance the average productivity level and the unemployment rate by setting interest rates.

In this section, the optimal monetary policy of the central bank is first formally depicted. The possibility of a positive interest rate is also illustrated by taking the derivative of the central bank's objective function with respect to the interest rate. However, we cannot give proper conditions with the exogenous parameters of the model to conform this possibility but just conjecture that the destruction of lower productivity jobs and the creation of higher productivity jobs might be too low under the Friedman rule. Therefore, we resort to numerical methods to justify our conjecture.

1.4.1 Optimal Monetary Policy

The total output of the economy Y is the sum of firm production and home production, i.e.

$$Y = \gamma \bar{y} + (1 - u - \gamma) \int_{y_d}^{\bar{y}} y dG(y) + ub. \tag{1.43}$$

We assume that the central bank treat all households equally when it designs the optimal monetary policy. Then, by virtue of the linearity of the households' utility function in market 3, we can define the periodic welfare function L as,

$$L = \alpha_h v(q) + Y - \alpha_h c(q) - k\phi, \tag{1.44}$$

where the first term is the total utility households get from market 2 every period and the remaining terms are the total utility households get from market 3. Because all the exogenous parameters are assumed to be constant, the central bank's optimal monetary policy design problem degenerates into a static problem, in which the central bank chooses the optimal interest rate i to maximize the periodic welfare function L subject to the equilibrium conditions (1.27), (1.38) and (1.39). The central bank's problem is,

$$\max_{i,q,u,\gamma,y_d,\mu} \alpha_h[v(q) - c(q)] + \gamma \bar{y} + (1 - u - \gamma) \int_{y_d}^{\bar{y}} y dG(y) + ub - k\mu u$$

subject to (1.27), (1.38) and (1.39), where u and γ are defined by (1.40) and (1.41).

We know that for a given interest rate, terms of trade in the decentralized good market, q is uniquely determined from Proposition 2; and the endogenous labour market variables $(\mu, y_d, u, \phi, \gamma)$ are functions of q from Proposition 3. So the central bank's optimal monetary policy design problem can be treated as a choosing decentralized goods market allocation q to maximize the periodic welfare function L subject to the labour market equilibrium conditions (1.38) and (1.39). So the central bank's problem is equivalent to

$$\max_q L(q) = \Psi(q) + \Phi(q)$$

where

$$\Psi(q) \equiv \alpha_h [v(q) - c(q)]$$

$$\Phi(q) \equiv \gamma(q)\bar{y} + [1 - u(q) - \gamma(q)] \int_{y_d(q)}^{\bar{y}} y dG(y) + u(q)b - k\mu(q)u(q).$$

$\mu(q)$ and $y_d(q)$ are solutions to (1.38) and (1.39); $u(q)$ and $\gamma(q)$ follow (1.40) and (1.41).

It is optimal for the central bank to set the interest rate at i_o such that the corresponding q_o, which is the solution to (1.27) for given i_o, satisfies

$$L'(q_o) = 0. \tag{cbo}$$

We also denote q_F to be the trade term in market 2 when the central bank applies the Friedman rule. Proposition 1 shows

$$\begin{array}{ll} v'(q_F) = c'(q_F) & \text{if} \quad i = 0 \\ v'(q) > c'(q) & \text{if} \quad i > 0, \end{array}$$

which is equivalent to

$$\begin{array}{ll} \Psi'(q_F) = 0 & \text{if} \quad i = 0 \\ \Psi'(q) > 0 & \text{if} \quad i > 0. \end{array}$$

We claim that the Friedman rule is optimal in the long run only if $\Phi'(q_F) \geq 0$. When $\Phi'(q_F) < 0$, we immediately have $L'(q_F) < 0$. Then it is optimal to set the allocation smaller than q_F. The monotonicity of $i \to q$ mapping asks the central bank to raise the interest rate above zero and deviate from the Friedman rule. Then, $i = 0$ is not an optimal monetary policy.

Now suppose $\Phi'(q) < 0$ holds when $i \geq 0$. We know the facts that $\Psi''(q) < 0$; $\lim_{q \to q_F} \Psi'(q) = 0$, becasue of the continuity of all functions and $\lim_{q \to 0} \Psi'(q) = +\infty$. Therefore, there exist at lest one $q_o \in (0, q_F)$, such that $\Phi'(q_o) = -\Psi'(q_o)$, i.e., $L'(q_o) = 0$. Then it is optimal for central bank choose the allocation $q = q_o$, if this the the unique solution for $L'(q_o) = 0$. The fact that $i \rightarrow q$ is a one-to-one monotonic mapping enable us to define the coresponding interest rate is i_o. Therefore, the optimal interest rate is i_o, which is the solution to (1.27) for given q_o. i_o must be above the zero, the interest rate required by the Friedman rule.

However, it is impossible to determine the signs of above terms analytically. Thus, we answer this question useing numerical methods in a reasonably calibrated model.

1.4.2 Parameters and Calibration Targets

As the calibration in Berentsen, Menzio and Wright (2008), we choose a quarter as one period and look at the United States economy during the period 1955-2005.

Some specific functional forms are firstly set;

i) The matching function in the labour market is standard,

$$m(\mu, 1) = Z\mu^{\alpha}$$

where the labour market matching efficiency is $Z > 0$ and the labour market matching elasticity is $\alpha \in (0, 1)$.

ii) The distribution from which the productivity level is drawn after idiosyncratic shock is a uniform distribution[14] within the support $[0, \bar{y}]$. Then the distribution function $F(y)$ reads

$$F(y) = \begin{cases} 0 & \text{if } y < 0, \\ \frac{y}{\bar{y}} & \text{if } 0 \leq y < \bar{y}, \\ 1 & \text{if } \bar{y} \leq y. \end{cases}$$

Thereafter, $G(y)$, the distribution function of a truncated distribution of $F(y)$ truncated at y_d becomes

$$G(y) = \begin{cases} 0 & \text{if } y < y_d, \\ \frac{F(y)-F(y_d)}{1-F(y_d)} = \frac{y-y_d}{\bar{y}-y_d} & \text{if } y_d \leq y < \bar{y}, \\ 1 & \text{if } \bar{y} \leq y. \end{cases}$$

iii) Household's utility function in market 2 is

$$v(q) = \frac{Aq^{1-a}}{1-a},$$

[14] A uniform productivity distribution is also used by Mortensen and Pissarides (1994).

where a the elasticity of utility function is a positive number belonging to $(0, 1)$ and A, the weight of market 2, is positive.

vi) Firm's cost function is assumed to be

$$c(q) = \frac{Aq^{1+\sigma}}{1+\sigma},$$

where σ, the elasticity of the cost function are positive. Therefore, the functional forms $c(\cdot)$ and $v(\cdot)$ satisfy the aforementioned properties. We set the cost function to have the same scale parameter as market 2's utility function. This is because we measure the total utility in term of market 3's goods, we want the sacle of the cost of market 3's goods to produce market 2's and its utility level to be comparable.[15]

Thus, there are twelve parameters to be set:

- preferences as described by β and a;
- technology as described by b, δ, $\bar{y}$, Z, α, α_h, σ and k;
- market structure as described by A, B and η.

We now calibrate this model such that the numerical prediction for the steady state equilibrium fits the 1955-2005 United States economy on average.

We set β to match the average quarterly real interest rate, measured as the difference between the nominal interest rate and inflation. We normalize the home production b to be unit. We also set $\bar{y} = 3$.[16] The job-specific technology shock arrival rate, δ, is chosen to be 0.081, as in Mortensen and Pissarides (1994). The parameters Z, α and k are fixed to be consistent with the US labour market feature which is studied in Shimer (2005). Thus, α is set to 0.028, the labour market matching elasticity with respect to vacancy. Z and k are chosen to match the average unemployment rate 0.06 and average UE (unemployment to employment) transition rate (λ_h), given the average vacancies are normalized to be 1 as in Berentsen, Menzio and Wright (2008). Notice that although Shimer (2005) claims that the monthly average UE transition rate is 0.45, we need to compute the quarterly rate: $\lambda_h = 1 - (1 - 0.45)^3 = 0.834$.

We then set A, B, a, α_h and σ as in the relevant monetary economics literature. First, we set $\alpha_h = 0.9$ without further explanation, because α_h matters little for our quantitative conclusions. We also set $\sigma = 1$ for simplicity reasons. We still need two conditions to pin down A and a. The

[15] In our comparable study, Berentsen, Menzio and Wright (2008) set the sclae of both the cost function and market 2's utility function to be 1.

[16] Berentsen, Menzio and Wright (2008) assume the average productivity to be twice as high as home production. So it is fairly reasonable to set the maximal productivity to be 3 here.

first condition is that the equilibrium condition (27), which links the goods and labour market, must hold numerically. Therefore, we have to use the numerical value of the average interest rate when computing the equilibrium condition (27). The average annal nominal interest rate is 0.074. Then we claim that the average quarterly nominal interest rate is $\frac{0.074}{4} = 0.019$. The second condition is that the money demand (real balance) predicted by our model must be consistant with the average US money demand, 0.179. This condition is commonly used in the monetary economics literature, say Lucas (2000) and Lagos and Wright (2005) among others. In our model, the money demand M/pY equals

$$\frac{M}{pY} = \frac{M/p}{Y} = \frac{c'(q)q}{\alpha_h[v(q) - c(q)] + \gamma\bar{y} + (1 - u - \gamma)\frac{\bar{y}+y_d}{2}}.$$

The targets discussed above are summarized in Table 1. These targets are sufficient to pin down all but one parameter, η, the wage bargaining power of the households. η is assumed to equal α, by the Hosios (1990) rule from the labour-macro literature, although it is not a necessary condition for reasonable calibrations.[17] We first set η to equal α and we will also check the robustness of our quantitative conclusions when η varies later.

Table 1.1: Calibration Targets

Description	Value
average unemployment u	0.060
average vacancies v (normalization)	1
average UE rate λ_h	0.834
elasticity of λ_h wrt μ	0.280
job-specific technology shock rate δ	0.081
household's trade probability in market 2 α_h	0.900
average money demand M/pY	0.179
elasticity σ of cost function	1
average nominal interest rate i	0.019
average real interest rate r	0.008
home production b (normalization)	1
ratio $\bar{y}/b$	3

Table 2 summarizes the calibrated parameter values. With , We first have to verify that check these values satisfy our assumption made in section 1.3.1. When determining the term of trade, we claim that the feasibility constraint $c(q) < y$ is always slack. The maximal q and minimal y_d occur when $i = 0$. With the functional form $c(q)$ and $v(q)$ we have $q^* = 1$ and the maximal $c(q)$ is $\frac{A(q^*)^{1+\sigma}}{1+\sigma} = \frac{1.826}{2} = 0.913$, while the minimal y is $y_d = 2.032$. Therefore, we confirm our assumption that the feasibility constraint $c(q) < y$ is always slack.

[17] See Shi (1998), among others.

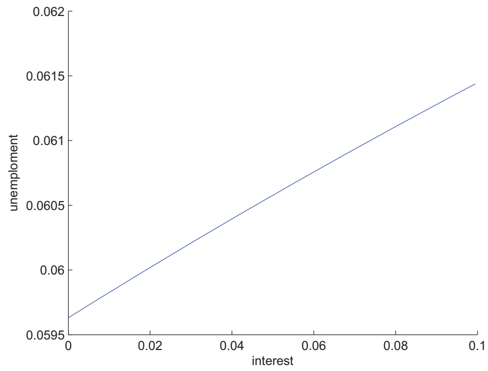


Figure 1.1: Interest rate and unemployment rate

Table 1.2: Parameter Values

	Description	Value
β	discount factor	0.008
b	home production	1
a	elasticity of utility function	0.799
δ	job-specific technology shock rate	0.081
$\bar{y}$	maximal productivity	3
Z	labour market matching efficiency	0.390
α	labour market matching elasticity	0.280
α_h	household's trade probability in market 2	0.900
σ	elasticity of cost function	1
k	vacancy posting cost	0.443
A	market 2's weight	1.826
η	wage bargaining power of household	0.280

1.4.3 Results

Using the calibrated parameters from above, we can compute the steady state equilibrium of the model for $i \in [0, 0.1]$ from our equilibrium conditions (1.27), (1.38) and (1.39). Figure 1 plots the steady state unemployment rate when the central bank sets different interest rate values. We confirm the conclusion of Berentsen, Menzio and Wright (2008) that higher inflation leads to a higher unemployment rate in the long run.

Figure 2 plots the steady state welfare level when the central bank sets different interest rate values. Normalizing the welfare at $i = 0$ to be 100, we can directly read off the relative loss or gain of welfare when the central

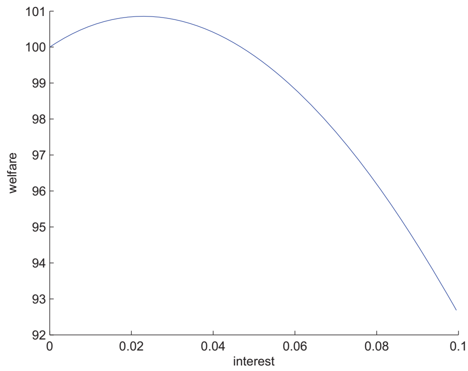


Figure 1.2: Interest rate and welfare ($\eta = 0.28$)

bank sets different interest rate levels. Figure 2 is the main quantitative finding in our numerical experiments. Our calibrations show that the highest welfare level is achieved when the central bank sets $i = 0.027$ (quarterly level), which is slightly above the long run interest rate level of the United States ($i = 0.019$ as mentioned before). The gain from deviating from the Friedman rule can also be seen from the plot. Figure 2 shows that the welfare improvement when the central bank sets $i = 0.027$ relative to the welfare level when central bank set $i = 0$ is less than one percent (0.5%, to be more precise). Because our welfare measure is in terms of consumption of good x and there is no disutility of work[18], we claim that the welfare gain of deviating from the Friedman Rule is worth 0.5 percent of consumption. Furthermore, when central bank sets i higher than the best interest rate level, there is much danger for the economy to experience significant welfare loss, given that the right side of the curve in Figure 2 becomes increasingly steeper.

Figure 3 plots the steady state job destruction when the central bank sets different interest rate values, because we are also interested in whether higher inflation can cause higher job destruction. Figure 3 confirms this relationship.

Therefore, Figure 2 and 3 jointly suggest that it is very likely that job destruction and the creation of higher productivity jobs are too low under the Friedman rule, which in turn causes that a zero interest rate is too low from a welfare maximizing point of view and drives the optimal long run monetary policy away from the Friedman rule. Then it is optimal for the central bank to set the interest rate strictly above zero.

[18] We assume that there is home production, which could be enjoyed as consuming goods x. That means we measure the utility of leisure in term of goods x in the model.

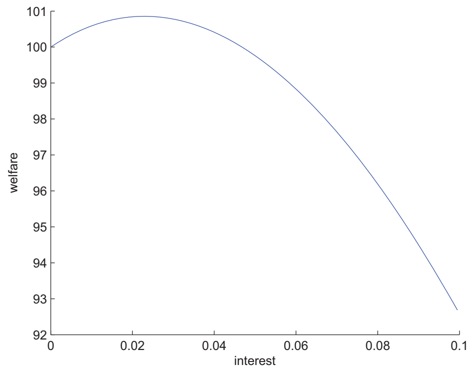


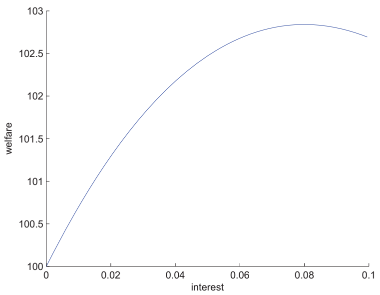
Figure 1.3: Interest rate and job destruction

To summarize, our numerical exercises confirm that the Friedman rule is not optimal in a calibrated model with endogenous job separation and firm heterogeneity.

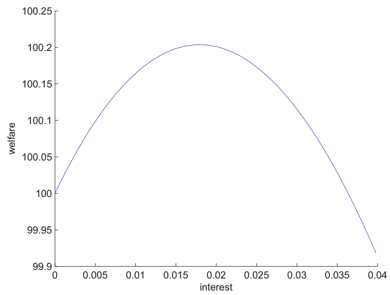
1.4.4 Robustness: η

Now we would like to check whether our quantitative conclusion that the optimal interest rate is above zero is robust for different η values. As mentioned earlier, there is no justified reason for assuming that wage bargaining power of households, η, satisfies the Hosios (1990) rule. We try four different values for η and the results are summarized in Figure 1.4 and Table 3, namely, $\{0.15; 0.5; 0.72; 0.8\}$.

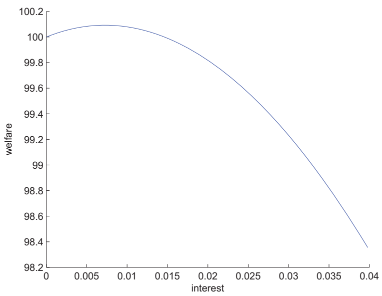
Figure 1.4(a) to 1.4(d) plots the steady state welfare level when central the bank sets different interest rates $i \in [0, 0.4]$ in four cases with different η, the wage bargaining power of households in the labour market. Firstly, it is clear that our conclusion that the optimal interest rate is above zero is fairly robust for different η values; secondly, the lower the wage bargaining power of the households, the more likely it is for the central bank to gain from setting the interest rate above zero and the more welfare gain from this derivation; thirdly, when η is sufficiently high, say $\eta \geq 0.80$, the Friedman rule is approximately or indeed the optimal policy; last but not least, the welfare improvements are all quite small, less than 1 percent. The optimal interest rate for each cases and its welfare improvement relative to the Friedman rule are shown in Table 1.3.



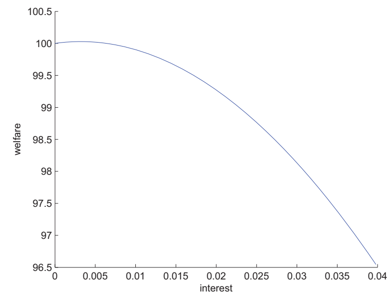
(a) $\eta = 0.15$



(b) $\eta = 0.50$



(c) $\eta = 0.72$



(d) $\eta = 80$

Figure 1.4: Interest rate and welfare with different η

Table 1.3: Optimal interest rate for different η

η	Optimal interest rate	Welfare improvement
0.15	0.080	2.03%
0.28	0.023	0.85%
0.50	0.018	0.20%
0.72	0.007	0.08%
0.80	0.002	less than 0.01%

1.4.5 Further Comments

As mentioned before, (16) shows that there is a congestion externality in the decentralized goods market, which means that the more firms there are in the market, the harder it is for a firm to be matched. Berentsen and Waller (2009) study optimal monetary policy in a modified version of Lagos and Wright (2005) with endogenous firm entry.[19] They claim that the congestion externality is a reason for the Friedman rule to be suboptimal, but their mechanism is different from mine. In their model, firms need to pay certain costs before entering into the market every period. They then show that there is too much firm entry in the market under the Friedman rule and the congestion externality makes entry less efficient. They suggest that the central bank should set the interest rate strictly above zero to reduce firm entry in equilibrium. In the present study, we emphasize the role of a positive interest rate in promoting less productive firms to quit by reducing their profit in decentralized market. However, a higher interest rate reduces firm's profit no matter with or without congestion externality. Furthermore, we claim that under the Friedman rule, the new job creations are too few instead of too many in the presence of this congestion externality, which is suggested by Berentsen and Waller (2009). Therefore, our explanations of why the Friedman rule is not optimal go beyond the congestion externality in decentralized goods market.

We also want to compare our results with Rocheteau and Wright (2005), who also reach the conclusion that the Friedman rule may not be the optimal long run monetary policy in a "competitive equilibrium"[20] (price-taking) in the presence of endogenous firm entry. The mechanism in Rocheteau and Wright (2005) is essentially the same as that in Berentsen and Waller (2009): due to the "congestion externality in decentralized goods market", firm entry is too high under the Friedman rule. Furthermore, both Rocheteau and Wright (2005) and Berentsen and Waller (2009) assume an exogenous firm

[19] The workhorse model that Berensten and Waller (2009) used is Berentsen, Camera and Waller (2007), which is a version of Lagos and Wright (2005) with a banking sector.

[20] We borrow the terminalogy "competitive equilibrium" from Rocheteau and Wright (2005) to show that the decentralized goods market pricing mechenism is price taking. Also see footnote 4.

exit rate equal to 1, while we study an endogenous firm exit rate.

1.5 Conclusion

We have presented a model with search frictional goods and labour markets while the job separate rate is endogenously determined. We confirm the conclusion from the related literature that the unemployment rate increases with the inflation rate and that the long run Phillips curve slopes upwards. The economy reaches the lowest unemployment rate when the central bank applies the Friedman rule. We also show that the endogenous job destruction rate becomes higher when the inflation rate rises, because higher inflation reduces the profits of all firms and makes the less productive firms more likely to quit the business. This study also points out the possibility of the optimal long run monetary policy to deviate from the Friedman rule if a higher inflation rate can promote enough high productivity job creation.

Our numerical exercises confirm the conjecture that the job destruction is too low under Friedman rule for a set of parameters calibrated from the United States economy. The optimal interest rate implied by our numerical exercise is around 0.023 at the quarterly level, which is slightly higher than the average quarterly interest rate of US economy from 1955-2005. Moreover, we also report that the maximal welfare gain of deviating from Friedman rule is less that 1 percent of consumption, which is a relatively small number.

Chapter 2

Technology, Wage Dispersion and Inflation

2.1 Introduction

This chapter studies the effect of inflation on the wage dispersions. The existing empirical literature that examines the relationship between the inflation rate and the dispersion of wages reports a negative correlation[1]. Kumar (2008a) develops a search-theoretic general equilibrium monetary model to explain that a reduction of the inflation rate would increases the pure wage dispersion due to on-the-job search.

The present study focuses on the effect of inflation on the wage dispersions due to firm heterogeneity as summarized by productivity differences in a search frictional labour market (on-the-job search is also allowed). We introduce a search labour market á la Postel-Vinay and Robin (2002a) into a micro-founded model of money, namely, Lagos-Wright (2005) search monetary framework[2]. We choose Lagos-Wright (2005) as our framework, because

[1] For instance, Hammermesh (1986) analyses the relationship between the dispersion in the relative wage changes and the inflation rate using the wage data from the period 1955–81 in the United States. He finds that higher inflation, especially unexpected inflation, reduced the relative wage dispersion. Erikson and Ichino (1995) study the effects of inflation on the wage differentials using the wage data from metal manufacture firms in Italy of the period 1976–90. They find that higher inflation rate significantly reduced the wage differentials.

[2] Search monetary models are pioneered by Kiyotaki and Wright (1991, 1993) for the case of indivisible commodities and money. They impose the assumption of indivisible commodities and money to guarantee the tractability of the model, because otherwise the distribution of money holding becomes too complicated to get analytical results. The search monetary frameworks which are suitable for economic and monetary policy analysis, i.e., the models that allow divisible commodities and money but circumvent the aforementioned difficulties concerning the distribution of money holdings, are developed along three lines: i) to introduce the assumption of large families like Shi (1997); ii) to introduce an extra centralized goods market and the assumption of linear disutility of labour input like Lagos and Wright (2005); iii) to introduce complete financial markets among groups like

it is easy to introduce Postel-Vinay and Robin (2002a) labour market and keep the tractability. The wage dispersions due to productivity differences in the context of a labour market with job posting firms and on-the-job searching workers are well studied by Mortensen (2003) and Postel-Vinay and Robin (2002a) among others. We choose to follow Postel-Vinay and Robin (2002a) to build the labour market, instead of Mortensen (2003), because, most of all, the productivity differences are essential for a continuous wage distribution in Postel-Vinay and Robin (2002a)[3]; while Mortensen (2003) could generate a continuous wage distribution without productivity differences, i.e., the pure wage dispersion. In addition, Mortensen (2003) models the endogenous productivity differences as the by-product of firms' optimal wage posting strategy, as the wage and productivity are decided at the same time and one wage level corresponds only to one productivity level. So the productivity differences in Mortensen (2003) are more like consequences rather than the reason of wage dispersion. On the contrary, in a Postel-Vinay and Robin (2002a) labour market with endogenous productivity distributions, the productivity investment decisions are made before the wage posting decision and independent of the wages paid by the firms.

One may ask why it is important to study the effect of inflation on the wage dispersions due to firm heterogeneity. There are a lot of theoretical explanations for why similar workers are paid differently. Mortensen (2003, Chapter 1.2) surveys these explanations within the search framework of labour market. The first kind of explanation is about on-the-job search, which is pioneered by Burdett and Mortensen (1989, 1998) and further exploited by many other wage posting models like Coles (2001) and Postel-Vinay and Robin (2002a)[4]. The second kind of explanation concerns the firm heterogeneity. Idson and Oi (1999) point out that labour productivity varies across industries and also across firms within industries and this is indeed the potential reasons for wage differentials. The third kind of explanation claims that the different bilateral bargaining powers over wages are a reason for wage differentials, which is a natural implication from the standard search and matching literature, say, Pissarides (2000). The last kind of explanation ascribes the compensation differentials as sources of wage dispersion (Hwang, Mortensen, and Reed (1998) and Lang and Mujumdar (2004)). As mentioned earlier, when Kumar (2008a) studies the effect of inflation on the wage dispersion, he only models one particular kind of wage dispersion: the pure wage dispersion due to on-the-job search.

Faig (2006). For further details about the development of search monetary models, please refer to the survey paper Shi (2006).

[3] In a Postel-Vinay and Robin (2002a) labour market without productivity differences, there are only two kinds of wages paid in the economy, the reservation wage and the highest wage that firms can afford, i.e., the zero profit wage.

[4] Postel-Vinay and Robin (2002a) study the wage dispersions when both on-the-job search and firm heterogeneity are present.

Then, some other important mechanisms listed above might be ignored in Kumar (2008a). In fact, the wage differentials caused by firm heterogeneity, especially by productivity differences of firms, is empirically very important. Postel-Vinay and Robin (2002b) construct an equilibrium search model with on-the-job search, worker and employer heterogeneity based on Postel-Vinay and Robin (2002a), and calibrate it using matched employer and employee French panel data of the 1990s (DADS, *Déclarations Annuelles des Données Sociales*). They use this structural estimation to provide a decomposition of cross-employee wage variance and find that i) the share of the wage variance that is explained by person effects is significant only among high-skilled white collars, and quickly decreases to 0% as the observed skill level decreases; ii) the contribution of market imperfections to wage dispersion is typically around 50%; iii) the contribution of firm differentials to wage dispersion is typically 30% to 40% except in the high-skilled white collars group. Therefore, we believe that modelling firm heterogeneity in the context of a labour market with on-the-job search[5] is quite important in wage dispersion studies.

We confirm that a rise of inflation diminishes the wage dispersion due to firm heterogeneity and on-the-job search. Furthermore, we find an extra channel (productivity distribution) through which the inflation can affect the wage dispersion and this channel is absent from the existing literature. For tractability reasons, we only check the effects of inflation on the bounds of wages paid in the economy, as Kumar (2008a). We first study the effect of inflation on the wage dispersions due to firm heterogeneity in the case of an exogenous productivity distribution. We find that the rise of inflation diminishes the wage dispersion in the sense that the lower wage bound increases while the upper wage bound decreases. The intuitions are as follows. First, the upper wage bound decreases since higher inflation causes a profit loss of the firms. This is because the inflation is modelled as the cost of holding money in our micro-founded monetary exchange setting and a rise of inflation increases households' cost of holding money. Households then reduce their money holding in the decentralized market, which in turn reduces firms' selling and profits in this market. As being shown later, the upper wage bound is zero-profit wage level (highest wage) paid by the most productive firm. Therefore, higher inflation decreases the upper wage bound as the most productive firm pays less to keep its employees due to profit drop. Second, the decrease of the highest wage that the firm could

[5]The conclusion about the effect of inflation on the wage dispersion due to firms' productivity differences without on-the-job search are fairly easy to get in a version of Berentsen, Menzio and Wright (2008) with heterogeneously productive firms, see Chapter 1. Chapter 1 clams that the wage paid by firms equals the home products plus a fixed portion of its profit of hiring a worker, which is a linear function of its productivity. The profits become lower when inflation rises, while the home product is unchanged. Then the wage gap between any two workersis narrowed, i.e. the wages become less dispersed.

pay reduces the prospects of wage rising for workers and then makes the unemployed require a higher reservation wage (lowest wage) to compensate their home product. The second mechanism is not included Kumar (2008a) because of the different model chosen for the labour market. We then study the effect of inflation on the wage dispersion due to productivity differences in the case of an endogenous productivity distribution. We find that a rise of inflation first makes firms' productivity less dispersed in the sense that the lower bound of productivity rises while the upper bound of productivity is unchanged. Intuitively, the rise of inflation reduces firms' profit, which drives the less productive firms out of the economy. Therefore we observe a rise of the lower bound of productivity. The highest productivity in the economy is determined by the technology of productivity investment which is assumed to be unchanged. So the rise of inflation has no impact on the upper bound of productivity. Furthermore, the rise of inflation then diminishes the wage dispersion in the sense that the lower bound of wages rises while the upper bound of wages drops. We then identify three channels through which the inflation changes the wage dispersion due to productivity differences. Two of the channels are the same as in the case of an exogenous productivity distribution. The last one is the mechanism of how inflation effects the wage dispersion through its effect on the productivity distribution. This productivity distribution channel is our main contribution.

This study is also related to the literature incorporating goods and labour market search frictions in a single model. Berentsen, Menzio and Wright (2008) introduce search and matching into Lagos and Wright (2005).[6] Their main findings are that the rise of inflation causes higher unemployment and the long run Phillips curve slopes upwards. Shi (1998) introduces standard search friction of labour market into Shi (1997) and finds that an increase in the money growth rate increases steady state employment and output when the money growth rate is low; but reduces steady state employment and output when the money growth rate is already high. Furthermore, the Friedman rule is not optimal in general.

The remainder of this chapter is organized as follows. we present the basic model in section 2.2. The description of stationary equilibrium of the economy with an exogenous productivity distribution and the effects of inflation on the wage dispersion in this case are studied in section 2.3. Section 2.4 introduces endogenous productivity investment decision and defines the new stationary equilibrium in this case. Section 2.5 proves the main propositions about the effects of inflation on wage dispersion in the endogenous productivity distribution case. The final section concludes.

[6] For search and matching in the labour market and a suvery of its development, see Pissarides (2000) or Mortensen and Pissarides (1999).

2.2 The Basic Model

The basic structure of the economy in the present model is the same as in Berentsen, Menzio and Wright (2008), the goods market of which is based on Lagos and Wright (2005), although our labour market setting is in the spirit of Postel-Vinay and Robin (2002a).

2.2.1 The Environment

There are two types of private agents: firms and households. There is a unit continuum of households; while the measure of firms is N.[7] Households work, consume, and enjoy utility and all households are equally skilled; while firms enroll workers, maximize profits and pay out dividends to households.

Time is discrete and continues forever. In each period, there are three markets that open sequentially. Market 1 is a labour market, markets 2 is a decentralized goods market and market 3 is a centralized goods market. These three markets are indexed by $i = 1, 2, 3$, respectively.

Market 1 (labour market) is modeled in the spirit of Postel-Vinay and Robin (2002a), where households received job offers from the firms which operate constant returns to labour technologies to produce good q. The marginal productivity of labour (hereafter denoted by y) differ across firms. We first assume that y is exogenously given as being distributed over $[\underline{\mathrm{p}},\bar{\mathrm{p}}]$ according to the continuous distribution Γ (cdf). However, Γ will be endogenized in section 2.4. A firm is willing to employ any worker so long as he/she could make positive profit for the firm.[8] Firms send job offers to the labour market each period to enroll works. Workers can either be employed or unemployed, and the aggregate unemployment rate is denoted by u. The probability for an unemployed being contacted by firms is λ_h, which depends on the measure of firms, N. We also allow workers to search for a better job while employed, so firms make offers to employed workers as well. We follow Mortensen (2003) to assume that the arrival rate of offers to on-the-job searchers is also λ_h.[9] The matching process in the labour market will be discussed latter. Furthermore, we assume that layoffs occur

[7] In this section and section 2.3, the productivity distribution is exogenously given, so we assume that N is also exogenous for simplicity reason. However, we will assume N to be endogenous when we have an endogenous productivity distribution in sections 2.4 and 2.5.

[8] We also follow Postel-Vinay and Robin (2002a) to assume that there are no capacity constraints for firms, i.e., a firm could employ as many employees as it is willing to. As a result, no firm is ever induced to fire a given worker to replace him/her by a less costly unemployed in the context of this model.

[9] In Burdett and Mortensen (1998), Postel-Vinay and Robin (2002a) and most of the related literature, the offer arrive rate for the unemployed is usually allowed to differ from the arrive rate of the employed for empirical reasons. We abstract from this case because it is then simple to assume an endogenous λ_h in latter sections, which is crucial for our analysis.

at a constant rate δ. The wage setting mechanism is completely adopted from Postel-Vinay and Robin (2002a), that is, it is assumed that: a) firms have perfect information about the characteristics (reservation wage) of the workers applying for a job; b) instead of paying the same wage to all workers, firms could change their wage offers according to the characteristics of a particular worker; c) firms can counter the offers that their employees receive from another firms, instead of being utterly passive when facing from an outside competition.

Under these three assumptions, we can immediately get the following three basic rules about wage setting:

1. When a firm meets an unemployed worker, the firm would offer the reservation wage for the unemployed and the unemployed worker would take the job.

2. When two firms with same productivity level get into competition for a single worker, the wage will increase until it reaches the highest wage that firms can offer, i.e., the wage that yields zero marginal profit for the firms.

3. When two firms with different productivity get into competition for a single worker, then the more productive firm will keep/enroll the worker, because more productive firm can offer a more attractive wage than that which yields zero marginal profit for the less productive firm.

Also, as Postel-Vinay and Robin (2002a) do, we assume that there exist legal restrictions on the wage contracts, which can be renegotiated only by mutual agreement. Finally, to integrate Postel-Vinay and Robin (2002a)'s setting into the present model, we define that the working for a firm entails both the producing activity and participating in the matching process in the decentralized goods market.

Market 2 (decentralized goods market) is modeled in the spirit of day time market in Lagos and Wright (2005), where good q is traded bilaterally between pairs consisting of a household (buyer as well as consumer) and an employee of a firm (seller as well as producer), due to anonymous matching. All private agents are anonymous, which generates an essential role for a medium of exchange. Household and firm then bargain over the terms of trade, (q, d), where d is the real balance paid by the household to the firm. The terms of trade are chosen so as to share the surplus from a buyer-seller match in fixed proportions. The household's share is $\theta \in (0, 1]$. Unsold good q is transformed into another kind of good x for market 3, according to a linear transforming technology. Market 3 (centralized goods market) is modeled in the spirit of night time market in Lagos and Wright (2005), where good x is traded multilaterally. All the private agents take the market prices as given. Unsold good x then vanishes between two periods. Also, without loss of generality we assume that agents discount at rate β between market 3 and the next market 1, but not between the other markets.

We assume a central bank exists and controls the supply of fiat money. We denote the growth rate of the money supply by ϖ, so that $\hat{M} = (1+\varpi)M$, where M denotes the per capita money stock in market 3 and variable with a "hat" indicates the value of variables next period. Therefore, in steady states, ϖ is the inflation rate. The central bank implements its inflation goal by providing deterministic lump-sum injections of money, ϖM, to the household at the end of each period. If $\varpi > 0$, households receive lump-sum transfers of money; for $\varpi < 0$, the central bank must be able to extract money via lump-sum taxes from the households.

We adopt Berentsen, Menzio and Wright (2008)'s convention for measuring real balances z, i.e. the nominal balances of households in current market 3, next market 1 and market 2, m, are deflated by P, the price level in the current centralized goods market (market 3), which is the latest price known for that market. For instance, when an agent brings $\hat{m}$ fiat money from market 3 to next market 1, we let $\hat{z} = \hat{m}/P$ denote his real balances. When he then takes this $\hat{m}$ fiat money to next market 2, his real balances is $\hat{z} = \hat{m}/P$, still deflated by P. If he were to bring $\hat{m}$ through market 2 into the next market 3, his real balances is then given by $\hat{m}/\hat{P} = \hat{z}\hat{\rho}$, where $\hat{\rho} = P/\hat{P}$ converts $\hat{z}$ into the units of the numeraire in that market. Notice $\hat{\rho} = 1/(1+\varpi)$, where ϖ is the inflation rate between this and the next centralized goods market.

Throughout the discussion in the text, we assume policy and the productivity distribution to be constant, and focus only on steady states.

2.2.2 Households

Let $s = e, u$ index employment status: e indicates that a household is employed; u indicates that a household is unemployed.

We now consider optimal decisions of the household, starting with market 3. The household, who is employed in a firm of productivity y at wage level w and holds real balances z, solves

$$\begin{aligned} W_{3,e}(z,y,w) &= \max_{x,\hat{z}}\{x + \beta W_{1,e}(\hat{z},y,w)\}, \quad (2.1) \\ \text{s.t. } x + \hat{z} &= w + \Delta + \frac{\pi M}{p} + z, \end{aligned}$$

where W denotes the value function of an employed household, x is the consumption in market 3, Δ is dividend income.[10] Here we assume that

[10] We assume the representative household holds the representative portfolio to omit the optimal investment decision of the households in financial market. So the equilibrium dividend Δ equals the average profit acroos all the firms. We denote by $\pi(y)$ to be a firm's periodic profit in steady state. Therefore, $\Delta = N \int_{\underline{p}}^{\bar{p}} \pi(y) d\Gamma(y)$

utility is linear in x, as in Lagos and Wright (2005) and Berentsen, Menzio and Wright (2008), to make all agents in the centralized markets choose the same real balance to enter into next market. Substituting x from the budget constrain into (2.1) gets

$$W_{3,e}(z, y, w) = w + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta W_{1,e}(\hat{z}, y, w)\}. \tag{2.2}$$

Therefore, $W_{3,e}(z, y, w)$ is linear in z.

Analogously, the problem for an unemployment household with real balances z reads

$$\begin{aligned} W_{3,u}(z) &= \max_{x,\hat{z}}\{x + \beta W_{1,u}(\hat{z})\}, \\ \text{s.t. } x + \hat{z} &= b + \Delta + \frac{\pi M}{p} + z, \end{aligned} \tag{2.3}$$

where b is the home production of an unemployed household. Substituting x from the budget constrain in the (2.3) gets

$$W_{3,u}(z) = b + \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta W_{1,u}(\hat{z})\}. \tag{2.4}$$

$W_{3,u}(z)$ is also linear in z.

We now move to market 2. The value functions of an employed household and an unemployed household, respectively, with real balance z, read

$$W_{2,e}(z, y, w) = \alpha_h v(q) + \alpha_h W_{3,e}[\rho(z-d), y, w] + (1-\alpha_h) W_{3,e}(\rho z, y, w), \tag{2.5}$$

$$W_{2,u}(z) = \alpha_h v(q) + \alpha_h W_{3,u}[\rho(z - d)] + (1 - \alpha_h) W_{3,u}(\rho z), \tag{2.6}$$

where α_h is the probability for household to trade in the decentralized goods market. The matching process in the goods market will be discussed latter. $v(\cdot)$ is household's utility function of consuming good q. $v(\cdot)$ is twice differentiable with $v(0) = 0$, $v' > 0$, $v'' < 0$, $\lim_{q\to 0} v'(q) = +\infty$, $\lim_{q\to\infty} v'(q) < 1$. Define q^* as the solution to $v'(q^*) = 1$. Because v, q^* is unique and $q^* > 0$.

Because of the linearity of W_3 ((2.2) and (2.4)), (2.5) and (2.6) become

$$W_{2,e}(z, y, w) = \alpha_h[v(q) - \rho d] + W_{3,e}(\rho z, y, w), \tag{2.7}$$

$$W_{2,u}(z) = \alpha_h[v(q) - \rho d] + W_{3,u}(\rho z). \tag{2.8}$$

The value function of an employed household with real balance z in market 1, reads

$$\begin{aligned} W_{1,e}(z,y,w) &= \delta W_{2,u}(z) + \lambda_h \int_y^{\bar{\mathrm{p}}} W_{2,e}[z,p,\phi(y,p)]d\Gamma(p) \\ &+ \lambda_h \int_{Q(w,y)}^{y} W_{2,e}[z,y,\phi(p,y)]d\Gamma(p) \\ &+ \{1-\delta-\lambda_h+\lambda_h\Gamma[Q(w,y)]\}W_{2,e}(z,y,w), \quad (2.9) \end{aligned}$$

where $\phi(y,p)$ is the optimal wage offer of a firm with productivity p when two firms with different productivities (y, p and $y<p$) get into competition for a single worker. The threshold productivity level $Q(w,y)$ is defined as $\phi[Q(w,y),y]=w$. Therefore, the firm has to raise the household's wage to keep him if this household receives a offer from a firm with productivity higher than $Q(w,y)$. The first term of the right hand side of (2.9) indicates the case of layoff; the second term indicates the case where the current firm has to compete with a more productive firm; the third term indicates the case where the current firm compete with a less productive firm and the current firm has to a rise the current wage; the last term indicates otherwise.

The value functions of unemployed household with real balance z in market 1, reads

$$W_{1,u}(z) = (1-\lambda_h)W_{2,u}(z) + \lambda_h \int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} W_{2,e}[z,p,\phi_0(p)]d\Gamma(p), \tag{2.10}$$

where $\phi_0(p)$ is optimal wage offer of a firm with productivity p willing to hire an unemployed worker, i.e. the minimum wage that compensates this worker for the opportunity cost of employment. The second term of the right hand side of (2.10) indicates the case of receiving an job offer.

Substituting $W_{2,e}$ and $W_{2,u}$ from (2.7) and (2.8) into (2.9) and using the linearity of $W_{3,e}$ and $W_{3,u}$ yields

$$\begin{aligned} W_{1,e}(z,y,w) &= \alpha_h[v(q)-\rho d] + \rho z + \delta W_{3,u}(0) + \lambda_h \int_y^{\bar{\mathrm{p}}} W_{3,e}[0,p,\phi(y,p)]d\Gamma(p) \\ &+ \lambda_h \int_{Q(w,y)}^{y} W_{3,e}[0,y,\phi(p,y)]d\Gamma(p) \\ &+ \{1-\delta-\lambda_h+\lambda_h\Gamma[Q(w,y)]\}W_{3,e}(0,y,w). \quad (2.11) \end{aligned}$$

Substituting $W_{2,e}$ and $W_{2,u}$ from (2.7) and (2.8) into (2.10) and using the linearity of $W_{3,e}$ and $W_{3,u}$ yields

$$W_{1,u}(z) = \alpha_h[v(q)-\rho d]+\rho z+(1-\lambda_h)W_{3,u}(0)+\lambda_h \int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} W_{3,e}[0,p,\phi_0(p)]d\Gamma(p). \tag{2.12}$$

Then, substituting $W_{1,e}$ and $W_{1,u}$ from (2.11) and (2.12) of next period into (2.2), one gets an equation for the value function of households in market 3 only,

$$\begin{aligned} W_{3,e}(z,y,w) &= w+\Delta+\frac{\pi M}{p}+z+\max_{\hat{z}}\{-\hat{z}+\beta\hat{\alpha}_h[v(\hat{q})-\hat{\rho}\hat{d}]+\beta\hat{\rho}\hat{z}\} \\ &\quad +\beta\delta W_{3,u}(0)+\beta\hat{\lambda}_h\int_y^{\bar{\mathrm{p}}} W_{3,e}[0,p,\phi(y,p)]d\Gamma(p) \\ &\quad +\beta\hat{\lambda}_h\int_{Q(w,y)}^{y} W_{3,e}[0,y,\phi(p,y)]d\Gamma(p) \\ &\quad +\beta\{1-\delta-\hat{\lambda}_h+\hat{\lambda}_h\Gamma[Q(w,y)]\}W_{3,e}(0,y,w). \end{aligned} \tag{2.13}$$

Similarly, substituting $W_{1,e}$ and $W_{1,u}$ from (2.11) and (2.12) of next period into (2.4) yields

$$\begin{aligned} W_{3,u}(z) &= b+\Delta+\frac{\pi M}{p}+z+\max_{\hat{z}}\{-\hat{z}+\beta\hat{\alpha}_h[v(\hat{q})-\hat{\rho}\hat{d}]+\beta\hat{\rho}\hat{z}\} \\ &\quad +\beta(1-\hat{\lambda}_h)W_{3,u}(0)+\beta\hat{\lambda}_h\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} W_{3,e}[0,p,\phi_0(p)]d\Gamma(p). \end{aligned} \tag{2.14}$$

This completes the description of households' problem.

2.2.3 Firms

We assume that the wage offer is fixed at the end of the market 1, although the wage is paid in market 3. Therefore, it does not matter whether wages are paid in money or goods. Furthermore, We assume that the unsold goods q in market 2 are transformed into $y-q$ unit goods x when being taken into market 3.

It is useful to start with computing $S(y,w)$, the average periodic profit made by a firm with productivity y from employing a worker at wage w, in term of goods in market 3

$$\begin{aligned} S(y,w) &= \alpha_f(\rho d+y-q-w)+(1-\alpha_f)(y-w) && (2.15) \\ &= y+\alpha_f(\rho d-q)-w. && (2.16) \end{aligned}$$

The first term of the right hand side of (2.15) corresponds to the case of being matched in market 2; whereas the second term corresponds to the case of not being matched in market 2. Let $\phi_1(y)$ denote the highest wage that a firm with productivity level y could offer, i.e., the wage level that makes the firm obtain zero profit from hiring this employee. We have

$$S[y,\phi_1(y)]=0.$$

Combining the above equation with (2.16) gives us,

$$\phi_1(y) = y + \alpha_f(\rho d - q). \tag{2.17}$$

If the firm has to offer a wage level higher than $\phi_1(y)$ to win the competition over an employee, the firm would simply give up hiring this employee.

For $w \leq \phi_1(y)$, let $L(w \mid y)$ denote the number of employees who are paid wages not higher than w in a type y firm . We also define $L(y) \equiv L[\phi_1(y) \mid y]$ to be the total employment in a firm of type y. We also express a firm's periodic profit, $\pi(y)$, using the definitions we made above,

$$\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} S(y, w) dL(w \mid y). \tag{2.18}$$

Firms' profits maximization decisions have been implicitly expressed as firms' wage setting rules (1 to 3) in the section 2.2.1. That is because, there is no capacity constraints for firms, i.e., a firm could employ as many employees as it is willing to.

We now implement those three wage setting rules in turn. Wage setting rule 1 in section 2.2.1 implies that the wage paid by a firm of type y in order to hire an unemployed worker is the minimum wage $\phi_0(y)$ that compensates this worker for the opportunity cost of employment. $\phi_0(y)$ is then defined by

$$W_{2,e}[z, y, \phi_0(y)] = W_{2,u}(z). \tag{2.19}$$

Substituting $W_{2,e}$ and $W_{2,u}$ from (2.7) and (2.8) into (2.19) gives us

$$W_{3,e}[\rho z, y, \phi_0(y)] = W_{3,u}(\rho z). \tag{2.20}$$

Then substituting $W_{3,e}[z, y, \phi_0(y)]$ from the above equation into (2.14) and using the linearity of $W_{3,u}$ yields an expression for $W_{3,u}(0)$,

$$W_{3,u}(0) = \frac{b + \Omega}{1 - \beta}, \tag{2.21}$$

where

$$\Omega = \Delta + \frac{\pi M}{p} + z + \max_{\hat{z}}\{-\hat{z} + \beta\hat{\alpha}_h[v(\hat{q}) - \hat{\rho}\hat{d}] + \beta\hat{\rho}\hat{z}\}.$$

Wage setting rule 2 in the section 2.2.1 implies that

$$\phi(y, y) = \phi_1(y), \tag{2.22}$$

which could also be rewritten as, by definition,

$$Q[\phi_1(y), y] = y. \tag{2.23}$$

Wage setting rule 3 in the section 2.2.1 implies that when two firms with different productivity (y, p and $y < p$) get into competition for a single worker, the worker would feel indifferent between working in the firm with higher productivity at the optimal wage offer for this competition, $\phi(y, p)$, and working in the firm with lower productivity at the highest wage offer $\phi_1(y)$, i.e.,

$$W_{2,e}[z, p, \phi(y, p)] = W_{2,e}[z, y, \phi_1(y)]. \tag{2.24}$$

Substituting $W_{2,e}$ from (2.7) into (2.24) gives us

$$W_{3,e}[\rho z, p, \phi(y, p)] = W_{3,e}[\rho z, y, \phi_1(y)]. \tag{2.25}$$

Now we could derive 3 functions which explicitly depict firms wage offer posting behavior, namely, $Q(w, y)$, $\phi(y, p)$ and $\phi_0(y)$.

Evaluating (2.13) at $w = \phi_1(y)$ and using the result of (2.23) yields

$$\begin{aligned} W_{3,e}[z, y, \phi_1(y)] &= \phi_1(y) + z + \Omega + \beta\delta W_{3,u}(0) \\ &+\beta\hat{\lambda}_h \int_y^{\bar{\mathrm{p}}} W_{3,e}[0, p, \phi(y, p)] d\Gamma(p) \\ &+\beta[1 - \delta - \hat{\lambda}_h + \hat{\lambda}_h \Gamma(y)] W_{3,e}[z, y, \phi_1(y)]. \end{aligned} \tag{2.26}$$

Then substituting $W_{3,e}[z, p, \phi(y, p)]$ for $z = 0$ from (2.25) into (2.26) and using the linearity of $W_{3,u}$ yields the expression of $W_{3,e}[0, y, \phi_1(y)]$,

$$W_{3,e}[0, y, \phi_1(y)] = \frac{\phi_1(y) + \Omega + \beta\delta W_{3,u}(0)}{1 - \beta(1 - \delta)}. \tag{2.27}$$

Plugging (2.27) back into (2.13) yields the general expression of $W_{3,e}(0, y, w)$

$$\begin{aligned} W_{3,e}(0, y, w) &= w + \Omega + \beta\delta W_{3,u}(0) \\ &+\beta\hat{\lambda}_h \int_y^{\bar{\mathrm{p}}} W_{3,e}[0, y, \phi_1(y)] d\Gamma(p) \\ &+\beta\hat{\lambda}_h \int_{Q(w,y)}^{y} W_{3,e}[0, p, \phi_1(p)] d\Gamma(p) \\ &+\beta\{1 - \delta - \hat{\lambda}_h + \hat{\lambda}_h \Gamma[Q(w, y)]\} W_{3,e}(0, y, w) \\ &= w + \Omega + \beta\delta W_{3,u}(0) \\ &+\beta\hat{\lambda}_h \bar{\Gamma}(y) \frac{\phi_1(y) + \Omega + \beta\delta W_{3,u}(0)}{1 - \beta(1 - \delta)} \\ &+\beta\hat{\lambda}_h \int_{Q(w,y)}^{y} \frac{\phi_1(p) + \Omega + \beta\delta W_{3,u}(0)}{1 - \beta(1 - \delta)} d\Gamma(p) \\ &+\beta\{1 - \delta - \hat{\lambda}_h + \hat{\lambda}_h \Gamma[Q(w, y)]\} W_{3,e}(0, y, w), \end{aligned} \tag{2.28}$$

where $\bar{\Gamma}(y)$ is defined as $\bar{\Gamma}(y) \equiv 1 - \Gamma(y)$. Furthermore, by the definition of $Q(w, y)$,

$$W_{2,e}(z, y, w) = W_{2,e}\{z, Q(w, y), \phi_1[Q(w, y)]\} \tag{2.29}$$

Substituting $W_{2,e}$ and $W_{2,u}$ of (2.7) and (2.8) into (2.29) and using the (2.27) gives us

$$\begin{aligned} W_{3,e}(0, y, w) &= W_{3,e}\{0, Q(w, y), \phi_1[Q(w, y)]\} \\ &= \frac{\phi_1[Q(w, y)] + \Omega + \beta\delta W_{3,u}(0)}{1 - \beta(1-\delta)}. \end{aligned} \tag{2.30}$$

Combining (2.30) and (2.28) yields

$$\begin{aligned} \frac{\phi_1[Q(w, y)]}{1 - \beta(1-\delta)} &= w + \beta\hat{\lambda}_h\bar{\Gamma}(y)\frac{\phi_1(y)}{1 - \beta(1-\delta)} + \beta\hat{\lambda}_h \int_{Q(w,y)}^{y} \frac{\phi_1(p)}{1 - \beta(1-\delta)} d\Gamma(p) \\ &\quad + \beta\{1 - \delta - \hat{\lambda}_h + \hat{\lambda}_h\Gamma[Q(w, y)]\}\frac{\phi_1[Q(w, y)]}{1 - \beta(1-\delta)}. \end{aligned} \tag{2.31}$$

Note that $\int_{Q(w,y)}^{y} \frac{\phi_1(p)}{1-\beta(1-\delta)} d\Gamma(p)$ could be integrated by part, using the definition of $\phi_1(\cdot)$ in (2.17). (2.31) is then rearranged to become

$$Q(w, y) = w - \alpha_f(\rho d - q) + \frac{\beta\hat{\lambda}_h}{1 - \beta(1-\delta)} \int_{Q(w,y)}^{y} \bar{\Gamma}(p) d.p \tag{2.32}$$

Using the fact that

$$Q[\phi(y, p), p] = y \tag{2.33}$$

and (2.32), we have the expression of $\phi(y, p)$, when $y < p$,

$$\phi(y, p) = y + \alpha_f(\rho d - q) - \frac{\beta\hat{\lambda}_h}{1 - \beta(1-\delta)} \int_{y}^{p} \bar{\Gamma}(x) dx. \tag{2.34}$$

We now turn to the unemployed workers' reservation wages $\phi_0(y)$. Note that, evaluating (2.30) at $w = \phi_0(y)$ gives us

$$W_{3,e}[0, y, \phi_0(y)] = \frac{\phi_1\{Q[\phi_0(y), y]\} + \Omega + \beta\delta W_{3,u}(0)}{1 - \beta(1-\delta)}. \tag{2.35}$$

Then, plugging $W_{3,e}[0, y, \phi_0(y)]$ and $W_{3,u}(0)$ from (2.35) and (2.21) into (2.20) gives us,

$$\frac{b + \Omega}{1 - \beta} = \frac{\phi_1\{Q[\phi_0(y), y]\} + \Omega + \beta\delta\frac{b+\Omega}{1-\beta}}{1 - \beta(1-\delta)}$$

i.e.,

$$b = \phi_1\{Q[\phi_0(y), y]\}. \tag{2.36}$$

By the definition of ϕ_1 in (2.17), (2.36) also reads as

$$b = Q[\phi_0(y), y] + \alpha_f(\rho d - q). \tag{2.37}$$

Also, by definition $\phi[Q(w, y), y] = w$,

$$\phi\{Q[\phi_0(y), y], y\} = \phi_0(y). \tag{2.38}$$

Evaluating (2.32) at $w = \phi_0(y)$, gives us

$$Q[\phi_0(y), y] = \phi_0(y) - \alpha_f(\rho d - q) + \frac{\beta\hat{\lambda}_h}{1-\beta(1-\delta)} \int_{Q[\phi_0(y),y]}^{y} \bar{\Gamma}(p)dp. \tag{2.39}$$

Plugging (2.37) and (2.38) into (2.39), we obtain the expression of $\phi_0(y)$,

$$\phi_0(y) = b - \frac{\beta\hat{\lambda}_h}{1-\beta(1-\delta)} \int_{b-\alpha_f(\rho d-q)}^{y} \bar{\Gamma}(p)dp. \tag{2.40}$$

To summarize, (2.32), (2.34) and (2.40) explicitly give us firms' wage posting behavior. This completes the description of the firms' problem.

2.3 Equilibrium with Exogenous Productivity Distribution

2.3.1 Goods Market

We first describe the matching process in the decentralized goods market. We assume that the probability for a buyer (a household) to trade in the decentralized goods market α_h is exogenous and constant[11]. Then, the probability for a seller (an employee of a firm) to trade in market 2, α_f, is determined in equilibrium. The measure of all employees of firms must equal the measure of the employed households, $1-u$, where u is the measure of the unemployed households, i.e. the unemployment rate. The measure of employees of firms matched in the decentralized goods market must equal the measure of households matched in decentralized goods market, which is $1 \cdot \alpha_h = \alpha_f(1-u)$. Therefore, we have

$$\alpha_f = \frac{\alpha_h}{1-u}. \tag{2.41}$$

[11] I make this assumption for simplicity, because it results in a unique equilibrium. Berensten, Menzio and Wright (2008) assume that α_h depends endogenously on the unemployment rate, which opens the possibility of multiple equilibria. However, Berensten, Menzio and Wright (2008) analyse only the case of a unique equilibrium. Therefore, making α_h exogenous does not change the analytical conclusions for properties of the equilibrium unemployment rate and the terms of trade.

(2.41) means that the more employees of firms are in the market, the harder it is for a firm to be matched in decentralized goods market and to make profit.

We now use the generalized Nash solution[12] to determine the terms of trade (q, d) in market 2. The buyer has bargaining power θ and the threat points are given by continuation values. The surplus for an employed household is

$$v(q) + W_{3,e}[\rho(z-d), y, w] - W_{3,e}(\rho z, y, w) = v(q) - \rho d,$$

which is independent of y and w and equals to the surplus for an unemployed household

$$v(q) + W_{3,u}[\rho(z-d)] - W_{3,u}(\rho z) = v(q) - \rho d.$$

The surplus for firm in each matching is

$$\rho d - q,$$

which is also independent of y.[13] Hence, (q, d) solves,

$$\max_{q,d} [v(q) - \rho d]^{\theta} [\rho d - q]^{1-\theta} \tag{TT}$$

$$\text{s.t. } d \leq z, 0 \leq q.$$

It is obvious that Problem (TT) is independent of the firm's productivity level and household's employment status. Therefore, the problems of the households' optimal choice for $\hat{z}$, the real balance taken to next period, in (2.13) and (2.14) are identical and independent of z, y, w and the household's employment status. The problem is rewritten as

$$\max_{\hat{z}} \{-\hat{z} + \beta\hat{\alpha}_h[v(\hat{q}) - \hat{\rho}\hat{d}] + \beta\hat{\rho}\hat{z}\} \tag{RB}$$

We can conclude from problem (RB) that every household, irrespective of z (his real balance in market 3) and s (employment status), will choose the same real balance $\hat{z}$ to enter into next period. This conclusion, in turn, implies in that problems (TT) for all the meetings in the decentralized market are identical and the terms of trade (q, d) in the decentralized goods market are all the same.

[12] I employ Nash bargaining as the price determination process because it is standard in the search literature (say, Shi (1995,1997), Trejos and Wright (1995), Lagos and Wright (2005)). However, using different price determination mechanisms, like price taking, and price posting with directed search, does not change the conclusions in goods market as Berensten, Menzio and Wright (2008) have shown and in turn does not change my conclusion on labour markets.

[13] There is a feasibility constraint $q \leq y$. Following Berensten, Menzio and Wright (2008), we assume for simplicity that this conditions is always slack.

Assumption 2 *We have $1+\pi \geq \beta$, i.e. $\rho \leq \frac{1}{\beta}$.*

Assumption 2 means that the money growth rate is higher than what the Friedman rule would require. To see its application more clearly, we use the Fisher equation, which links the nominal interest rate and inflation in the long run. The Fisher equation reads,

$$1+i = \frac{1+\pi}{\beta} \tag{2.42}$$

where i is the nominal interest rate. Therefore, Assumption 2 simply means that $i \geq 0$. while the Friedman rule requires $i = 0$, i.e. $\pi = \beta - 1$. Assumption 1 makes sure that problem (RB) has meaningful solution as Lagos and Wright (2005) point out. Then, we only consider the equilibrium of the economy either in the case $1+\pi > \beta$ or the case $1+\pi = \beta$ but equilibrium is the limit as $1+\pi \to \beta$ from above.

Proposition 6 *Under Assumption 2, in stationary equilibrium $d = z$ is always satisfied; q is the solution to*

$$\frac{v'(q)}{g'(q)} = \frac{i}{\alpha_h} + 1 \tag{2.43}$$

where g is defined as

$$g(q) \equiv \frac{\theta q v'(q) + (1-\theta)v(q)}{\theta v'(q) + 1 - \theta}. \tag{2.44}$$

The household would set its next period real balance at

$$\hat{z} = \beta(1+i)g(q), \tag{2.45}$$

Proof:[14] see Appendix.

(2.43) fully characterizes the equilibrium in decentralized goods market.

Assumption 3 *For all $q \leq q^*$, we have that $\frac{v'(q)}{g'(q)}$ is strictly deceasing.*

The above assumption is simply imposed to make sure that the solution to (2.43) is unique. Lagos and Wright (2005) also establish some sufficient conditions for Assumption 3, like that $v'(\cdot)$ is log concave. We then have the following proposition for the goods market.

Proposition 7 *(2.43) is the equilibrium condition in the goods market. With Assumptions 2 and 3, the solution exists and is unique. Furthermore, q is decreasing in i; firm's revenue in market 2, $R = \rho d - q = g(q) - q$, which is clearly increasing in q, is also decreasing in i.*

Proof: see Appendix.

Because $R = g(q) - q$ is a function of q, we will denote R as $R(q)$ latter.

[14] The same exercise can also be found in Lagos and Wright (2005) and Rocheteau and Wright (2005).

2.3.2 Labour Market

We now describe the matching technology in the labour market. In the present model, where on-the-job search is allowed, the matching outcomes depend both on the measure of the total labour force and the measure of firms, the job offer suppliers. As we mentioned in section 2.2.1, the measure of the total labour force is always 1 and the measure of firms is exogenously given by N. Therefore, we may assume the arrival rate of offers to all searchers, λ_h, to be an exogenous constant in this section. However, we will endogenize λ_h and let it depend on N latter. In steady sate, we have that the unemployment rate is constant. Therefore, it is required that new job matchings equal total layoffs, i.e.

$$u\lambda_h = (1-u)\delta. \tag{2.46}$$

We then have the following proposition for the labour market.

Proposition 8 *The equilibrium condition in the labour market with exogenous productivity distribution is characterized by equation (2.46). Furthermore, the equilibrium unemployment rate is constant.*

2.3.3 Steady State Equilibrium

Proposition 6 and 7 show that goods market equilibrium is solely defined by q and equilibrium condition (2.43), despite of the situation in labour market. Similarly, Proposition 8 shows that, labour market equilibrium is solely defined by u and equilibrium condition (2.46), independently of the situation in goods market. This leads us to a very simple definition of equilibrium.

Definition 2 *A steady state equilibrium of the economy with exogenous productivity distribution is a pair $\{q, u\}$ that satisfies equilibrium conditions (2.43) and (2.46).*

Proposition 9 *With Assumption 2 and 3, the steady state equilibrium of the economy with exogenous productivity distribution always exists and is unique. Furthermore, q, the amount of goods traded in decentralized goods market, is decreasing in interest rate i, while unemployment rate u is constant and independent of interest rate i.*

Although u is constant and independent of interest rate i, the wage distribution depends on the interest rate. In next subsection, we will first express the wage distribution as a function of the distribution of productivity, and then we will check its dependence on the interest rate.

2.3.4 Wage Distribution

We begin by deriving the value of $L(w \mid y)$, the earning distribution within a firm, as a function of the distribution of productivity, $\Gamma(\cdot)$.

Consider a set of workers who are paid less than w by firms with productivity level y, i.e., $L(w \mid y)Nd\Gamma(y)$. Workers would leave this set either because they are laid off, which occurs at rate δ, or because they receive a more attractive offer, which grants them a wage increase. From previous sections we know that only those workers who receive an offer from a firm with productivity not less than $Q(w, y)$ will either see their wage raised above w, or leave their type y employer to a more productive firm. Such offers occur at rate $\lambda_h \bar{\Gamma}[Q(w, y)]$. Therefore, the total measure of the outflow of this set is $\{\delta + \lambda_h \bar{\Gamma}[Q(w, y)]\}L(w \mid y)Nd\Gamma(y)$. On the inflow side, workers enter the set either from a firm with productivity less than $Q(w, y)$ or out of unemployment. Also note that, when w is too small ($w \leq \phi(\underline{\mathrm{p}}, y)$), it can only attract the unemployed. When w is higher enough, it could also attract the workers who work for the firm with productivity lower than $Q(w, y)$. Then, the number of workers hired by such a firm from firms with productivity less than $Q(w, y)$ is $\lambda_h \cdot Nd\Gamma(y) \cdot \int_{\underline{\mathrm{p}}}^{Q(w,y)} L(p)d\Gamma(p)$, while the inflow of unemployed workers into this category is $\lambda_h \cdot u \cdot d\Gamma(y)$. Therefore, the stationary of $L(w \mid y)$ thus implies, for $\phi_0(y) \leq w \leq \phi_1(y)$,

$$\{\delta+\lambda_h\bar{\Gamma}[Q(w, y)]\}L(w \mid y)N = \begin{cases} \lambda_h u & \text{if } w \leq \phi(\underline{\mathrm{p}}, y), \\ \lambda_h u + \lambda_h N \int_{\underline{\mathrm{p}}}^{Q(w,y)} L(p)d\Gamma(p) & \text{if } w > \phi(\underline{\mathrm{p}}, y). \end{cases} \tag{2.47}$$

For small values of the wage w, specifically, for $w \leq \phi(\underline{\mathrm{p}}, y)$, the last integral term vanishes, because $Q\{\phi(\underline{\mathrm{p}}, y), y\} = \underline{\mathrm{p}}$. Note that only workers just coming out of unemployment will accept offers in this case. Workers who have already experienced at least one period in employment have a reservation wage greater than $\phi(\underline{\mathrm{p}}, y)$.

Note that to obtain an expression of $L(w \mid y)$ in the case $w > \phi(\underline{\mathrm{p}}, y)$, we first need to determine $L(y)$. This is done by considering the stock of workers employed at all firms with productivity levels less than y, which equals $N \int_{\underline{\mathrm{p}}}^{y} L(p)d\Gamma(p)$. This stock is also depleted at rate $\delta + \lambda_h \bar{\Gamma}(y)$, whereas it is fueled by hiring of unemployed workers. The flow of unemployed workers hired into firms with productivity less than y is given by $\lambda_h u \Gamma(y)$. Once again equating inflows and outflows for the stock of workers at hand leads to

$$N \int_{\underline{\mathrm{p}}}^{y} L(p)d\Gamma(p) = \frac{\lambda_h u \Gamma(y)}{\delta + \lambda_h \bar{\Gamma}(y)}. \tag{2.48}$$

Then, the expression for $L(y)$ is obtained by differentiating both sides of

(2.48) with respect to y, i.e., for $y \in [\underline{p}, \bar{p}]$,

$$L(y) = l(y) \equiv \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{[\delta + \lambda_h \bar{\Gamma}(y)]^2} \tag{2.49}$$

and for $y < \underline{p}$,

$$L(y) = 0.$$

Setting $y = Q(w, y)$, (2.48) implies, for $w > \phi(\underline{p}, y)$,

$$\int_{\underline{p}}^{Q(w,y)} L(p) d\Gamma(p) = \frac{1}{N} \frac{\lambda_h u \Gamma[Q(w, y)]}{\delta + \lambda_h \bar{\Gamma}[Q(w, y)]}. \tag{2.50}$$

.Furthermore, for $w \leq \phi(\underline{p}, y)$,

$$\Gamma[Q(w, y)] = 0. \tag{2.51}$$

Plugging (2.50) and (2.51) into (2.47) gives us, for $\phi_0(y) \leq w \leq \phi_1(y)$,

$$L(w \mid y) = \begin{cases} \frac{\lambda_h u}{N} \frac{1}{\delta + \lambda_h} & \text{if } w \leq \phi(\underline{p}, y), \\ \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \bar{\Gamma}[Q(w,y)]\}^2} & \text{if } w > \phi(\underline{p}, y). \end{cases} \tag{2.52}$$

Also note that, evaluating (2.49) at $l[Q(w, y)]$ gives us,

$$l[Q(w, y)] = \begin{cases} \frac{\lambda_h u}{N} \frac{1}{\delta + \lambda_h} & \text{if } w \leq \phi(\underline{p}, y) \\ \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \bar{\Gamma}[Q(w,y)]\}^2} & \text{if } w > \phi(\underline{p}, y) \end{cases} \tag{2.53}$$

Comparing (2.52) and (2.53), we have, for $\phi_0(y) \leq w \leq \phi_1(y)$,

$$L(w \mid y) = l[Q(w, y)] \tag{2.54}$$

always holds.

Let $G(w)$ denote the *cdf* of the aggregate earnings distribution. $G(w)$ hence denotes the proportion of workers earning less than w in the economy. We have the following proposition for the wage distribution.

Proposition 10 *Given fixed exogenous productivity distribution* $\Gamma(y)$ *over* $[\underline{p}, \bar{p}]$*, we have that the wage of all the workers comprise the segment* $[\phi_0(\bar{p}), \phi_1(\bar{p})]$. *Furthermore,*

$$G(w) =$$

$$\begin{cases} 0 & \textit{if } w < \phi_0(\bar{p}) \\ \delta \int_{\phi_0^{-1}(w)}^{\bar{p}} \frac{\delta+\lambda_h}{\{\delta+\lambda_h \bar{\Gamma}[Q(w,p)]\}^2} d\Gamma(p) & \textit{if } \phi_0(\bar{p}) \leq w < \phi_0(\underline{p}) \\ \delta \int_{\underline{p}}^{\bar{p}} \frac{\delta+\lambda_h}{\{\delta+\lambda_h \bar{\Gamma}[Q(w,p)]\}^2} d\Gamma(p) & \textit{if } \phi_0(\underline{p}) \leq w < \phi_1(\underline{p}) \\ \begin{array}{c} \delta \int_{\underline{p}}^{\phi_1^{-1}(w)} \frac{\delta+\lambda_h}{[\delta+\lambda_h \bar{\Gamma}(p)]^2} d\Gamma(p) \\ +\delta \int_{\phi_1^{-1}(w)}^{\bar{p}} \frac{\delta+\lambda_h}{\{\delta+\lambda_h \bar{\Gamma}[Q(w,p)]\}^2} d\Gamma(p) \end{array} & \textit{if } \phi_1(\underline{p}) \leq w < \phi_1(\bar{p}) \\ 1 & \textit{if } \phi_1(\bar{p}) \leq w \end{cases} \tag{2.55}$$

where $\phi_0(y)$ is defined in (2.40) and ϕ_0^{-1} is its inverse function; $\phi_1(y)$ is defined in (2.17) and ϕ_1^{-1} is its inverse function; $Q(w, y)$ is implicitly defined in (2.32).

Proof: see Appendix.

We have derived the wage accumulative distribution function $G(w)$. The probability density function of the wage distribution can be easily found by differentiating $G(w)$.

We now move to the effect of inflation on the wage dispersion. In statistics, the wage dispersion could be indicated by the variance of the wage, the 90th to 10th percentile ratio, or even the ratio of the highest wage to the lowest wage. Given the complexity of the wage distribution function $G(w)$, it is very hard to derive the the effect of inflation on the variance of wage analytically. However, assuming that the shape of the wage distribution does not change much when inflation changes, it makes sense to check the effect of inflation on the bounds of wages paid in the economy, as done by Kumar (2008a)[15][16]. Denote by $\underline{w}$ and $\bar{w}$ the lowest and the highest wages, respectively, paid in the economy. Proposition 10 shows that $\underline{w} = \phi_0(\bar{p})$ and $\bar{w} = \phi_1(\bar{p})$.

Proposition 11 *Given fixed exogenous productivity distribution $\Gamma(y)$, it holds that $\underline{w}$ increases and $\bar{w}$ decreases when the interest rate i increases.*

Proof: see Appendix.

In the present model, both the highest wage and the lowest wage are paid by the most productive firms. The highest wage decreases when the

[15] Kumar (2008a) only analyzes the gap between the highest wage and the lowest wage. The present model manages to analyze them separately.

[16] I leave the analysis of other wage dispersion indicators, such as variance of wage, to future numerical exercises.

interest rate i increases. The reason is as follows. We use a a micro-founded model of money to make the money essential in the economy, therefore the goods market reach the first optimal when central bank set the interest rate equals to zero, i.e., the Friedman rule. When the interest rate increases, there will be more distortion in the goods market, which means that the firms will suffer the profit loss. We know that the highest wage in the economy is the wage level which make the most productive firm to obtain zero profit and the profit loss reduces this zero-profit wage level. Therefore, a rise of the inflation rate reduces the highest wage level. The fact that the lowest wage increases when the interest rate i increases, shows that the reservation wage for the unemployed increases. That is because the decrease of the highest wage that the firm could pay reduce the prospects of wage rising and make the working opportunities less attractive for the unemployed. Therefore, the unemployed require a higher reservation wage to compensate them for home production. In fact, the same logic also holds for the wage dispersion with in any firm. We claim that the wage dispersion within a firm become smaller when the interest rate becomes higher. Then, given the exogenous productivity distribution, we reinforce the conclusion, (which is implied by Proposition 11 from an increasing lower bound and a decreasing upper bound) that higher inflation causes less wage dispersion in general.

Obviously, increasing $\underline{\mathrm{w}}$ and decreasing $\bar{\mathrm{w}}$ also implies some other dispersion indications, such as that $\bar{\mathrm{w}}$-$\underline{\mathrm{w}}$ or $\bar{\mathrm{w}}/\underline{\mathrm{w}}$, also decrease unambiguously.

We also want to compare our conclusion here with those in Kumar (2008a). In fact, our conclusion that an increase of inflation causes less dispersion within a firm is equivalent to that it causes smaller dispersion in general when there is no productivity dispersion in Kumar (2008a). However, our channels are not the same. The rise of inflation makes all the firms' producing activities less profitable relative to home production, which has two consequences in the present model. Firstly, it narrows the gap between the wage income and home production, which causes the highest wage to decrease in the present model, but results in an increase of the reservation wage in Kumar (2008a). Secondly, it reduces the room of wage increase and in turn increases the reservation wage of the unemployed in the present model. The second effect of rising inflation in the current study is a novelty.

When we assume the productivity distribution to be exogenous, a constant unemployment rate is reached. In the next two sections, We would like to endogenize the productivity distribution to study the effect of inflation on the productivity distribution and on wage dispersion. Moreover, as we will see in the next sections, endogenizing the productivity distribution makes the unemployment rate positively depending on the inflation rate, which is consistent with previous theoretical works about inflation and unemployment in the long run, such as Berentsen, Menzio and Wright (2008).

2.4 Equilibrium with Endogenous Productivity Distribution

We adopt the idea of Acemoglu and Shimer (2000) of endogenous productivity dispersion to our model. More specifically, we assume that a firm's productivity follows from its investment choices such that the equilibrium productivity dispersion arises from the firms' dispersed investment choices. Besides that, the optimization decisions of households as well as the wage setting rules for firms after investment choices are exactly the same as we have depicted in section 2.2.2 and 2.2.3. Finally, the equilibrium conditions for the goods market also hold as in section 2.3.1.

2.4.1 Investment and Productivity

We assume from now on that the productivity y of a firm depends on its investment decision, which is made before entering into the labour market, and that the productivity y is fixed during its operation. More specifically, a firm with productivity y must pay a periodic cost $cf(y)$ to keep its productivity level, where c is a positive constant and $f(y)$ is assumed to satisfy: $f(0) = f'(0) = 0$ and for all $y > 0$, $f(y) > 0$; $f'(y) > 0$; $f''(y) > 0$. The latter property of $f(\cdot)$ means that the cost of productivity investment is increasing with respect to productivity level and also convex; while the former property is imposed to ensure the existence of equilibrium without lose of generality.

Some explanations must be made concerning the above form of the productivity investment cost. Firstly, we assume that the costs are paid periodically instead of being paid once and for all when the firm is entering into the market, following Postel-Vinay and Robin (2002a). The reason is that employee dynamics within a firm is too complicated to trace in both models such that the profit function of a firm and the free entry condition, which is important to pin down the measure of firms latter, could only be computed and imposed in periodic form in steady state. Secondly, concerning the nature of such productivity investment cost, Postel-Vinay and Robin (2002a) ascribe it as the rental cost of capital used in the producing activity follows a Cobb-Douglas production function form. Therefore, their periodic productivity investment cost is of the form of $rg^{-1}(y)$, where r is a constant exogenous interest rate and $g^{-1}(\cdot)$ is the inverse function of Cobb-Douglas production function in per capita form. We would like to adopt their explanation on the source of productivity investment cost, but can not do so with an endogenous interest rate $r = \frac{1}{\beta}$ directly. Instead, we have to exclude capital from our economy because otherwise both capital and fiat money can serve as the medium of exchange in the goods market and then capital holdings, which could influence the term of trade in the goods market, make the analysis much more complicated[17]. Therefore, we simply assume that $cf(y)$ is the cost of using such a production technology and no private agent benefits from it. Thirdly, the productivity investment cost depends only on the productivity level but not on the scale of labour input of the firm, still following Postel-Vinay and Robin (2002a). Our explanation is that the technology modeled in the present study is merely a constant returns to labour technology. Once a firm gets the technology, it could be used for any employee within the firm at no cost.

Before moving on, let us derive the expression of $\pi(y)$. Note first, that the productivity investment decision is made before the firm enters the market. Therefore, our derivation of the wage distribution for a given productivity distribution is still valid for this section, i.e., (2.47) to (2.55) in section 2.3.4 also hold for the present case of an endogenous productivity distribution.

Lemma 12

$$\pi(y) = \int_{b-\alpha_f R}^{y} l(x)[1 + \frac{\beta\lambda_h\bar{\Gamma}(x)}{1-\beta(1-\delta)}]dx \tag{2.56}$$

Proof: see Appendix.

Given the specification of the productivity investment cost and $\pi(y)$, we can now begin to derive the distribution of productivity $\Gamma(y)$. Denote the periodic profit of a firm with productivity investment decision, y, as $\Pi(y)$, therefore,

$$\Pi(y) = \pi(y) - cf(y).$$

In equilibrium, all firms must make the same (maximal) profit, say Π^*. Thus, the following holds in equilibrium:

$$\pi(y) - cf(y) = \Pi^*, \quad \text{if } y \in [\underline{\mathrm{p}}, \bar{\mathrm{p}}], \tag{2.57}$$

$$\pi(y) - cf(y) < \Pi^*, \quad \text{otherwise.} \tag{2.58}$$

Since $\pi(y) - cf(y)$ is a constant over the support $[\underline{\mathrm{p}},\bar{\mathrm{p}}]$, and $\pi(y)$ is differentiable as Lemma 12 shows, it is therefore true that, when $y \in [\underline{\mathrm{p}},\bar{\mathrm{p}}]$,

$$\pi'(y) - cf'(y) = 0. \tag{2.59}$$

Combining (2.56) and (2.59) implies

$$l(y)[1 + \frac{\beta\lambda_h\bar{\Gamma}(y)}{1-\beta(1-\delta)}] = cf'(y). \tag{2.60}$$

Plugging $l(y)$ from (2.49) into (2.60) gives us,

$$\frac{\lambda_h u}{N}\frac{\delta+\lambda_h}{[\delta+\lambda_h\bar{\Gamma}(y)]^2}[1 + \frac{\beta\lambda_h\bar{\Gamma}(y)}{1-\beta(1-\delta)}] = cf'(y). \tag{2.61}$$

[17] See Lagos and Rocheteau (2008) and others for detailed discussion.

We need to know λ_h, u, and N to determine the shape of $\Gamma(\cdot)$. These variables are discussed in next subsection. We also need to impose two conditions concerning the definition of $\Gamma(\cdot)$ to determine the lower and upper bound of y. By definition,

$$\bar{\Gamma}(\underline{\mathrm{p}}) = 1,$$
$$\bar{\Gamma}(\bar{\mathrm{p}}) = 0.$$

When (2.61) is evaluated at $y = \underline{\mathrm{p}}$ and $y = \bar{\mathrm{p}}$, we get

$$\frac{\lambda_h u}{N}\frac{1}{\delta+\lambda_h}[1+\frac{\beta\lambda_h}{1-\beta(1-\delta)}] = cf'(\underline{\mathrm{p}}), \tag{2.62}$$

$$\frac{\lambda_h u}{N}\frac{\delta+\lambda_h}{\delta^2} = cf'(\bar{\mathrm{p}}). \tag{2.63}$$

2.4.2 Labour Market

We now describe the matching technology in the labour market. In this section, the measure of firms, N, is endogenously determined in equilibrium. In section 2.2.1, we assumed that λ_h depend on N. Therefore, from now on, λ_h becomes endogenous. Following Mortensen (2003), we assume[18] that,

$$\lambda_h = \lambda N, \tag{2.64}$$

where λ is a positive constant.

In steady state, the unemployment rate is constant. Therefore, it is required that new job matchings equal total layoffs and (39) still holds here.

The last condition required for labour market clearing is the free entry condition, which pins down the measure of firms. In the long run competitive economy, free entry and exit ensure that all the competing firms make zero profit, i.e.,

$$\Pi^* = 0. \tag{2.65}$$

We then have the following proposition for the labour market.

Proposition 13 *The equilibrium condition in labour market with endogenous productivity distribution can be characterized by two equations for the unemployment rate u and the lower bound of productivity $\underline{p}$,*

$$b - \frac{\alpha_h}{1-u}R = \underline{p} - \frac{f(\underline{p})}{f'(\underline{p})}, \tag{2.66}$$

[18] In a similar continuous time setting, Mortensen (2003) (Chapter2.2, page 38-39) clams that the arrival rate of offers for the unemployed follows a Poisson process with an arrival rate approximating to the firm-household ratio (it equals N in my case), when all the firms post their jobs offers, which are received by a particular worker pure randomly. In the current discrete time setting, I simply set the offer arriving probability is linear to the Poisson process arriving rate, just like other applications of labour market search and matching model in discrete time setting, say, Shi (1998), Berentsen, Menzio and Wright (2008) and so on.

$$\frac{(1-\beta)\lambda u^2}{[1-\beta(1-\delta)]\delta} + \frac{\beta\lambda u}{1-\beta(1-\delta)} = cf'(\underline{p}). \tag{2.67}$$

The other labour market variables can be expressed by u and $\underline{p}$:

$$N = \frac{\delta}{\lambda}(\frac{1}{u} - 1), \tag{2.68}$$

$$\lambda_h = \frac{\lambda}{u} - \lambda. \tag{2.69}$$

Furthermore, the upper bound of productivity $\bar{p}$ is the unique solution to,

$$\frac{\lambda}{\delta} = cf'(\bar{p}), \tag{2.70}$$

Proof: see Appendix.

2.4.3 Steady State Equilibrium

Proposition 6 and 7 show that goods market equilibrium is solely defined by q and equilibrium condition (2.43), independently of the situation in the labour market. Firms' revenue in market 2, $R(q)$ is the link between the labour market and the goods market. This suggests the following definition.

Definition 3 *A steady state equilibrium of the economy with endogenous productivity distribution is a triple $\{q, u, \underline{p}\}$ such that i) $\{q, u, \underline{p}\}$ satisfies equilibrium conditions (2.43), (2.66) and (2.67); ii) (2.58) holds, i.e., no firm has an incentive to enter into the market with $y \notin [\underline{p}, \bar{p}]$.*

Before moving on to the existence and uniqueness of the equilibrium, we need to prove the following Lemma about the properties of equilibrium conditions in the labour market.

Lemma 14 *Given that $R(q)$ is constant, the locus of points of $(u, \underline{p})$ defined by (2.66) is downward sloping and (2.66) shifts up if $R(q)$ goes down. The locus defined by (2.67) slopes upward in $(u, \underline{p})$ space and does not shift when $R(q)$ changes.*

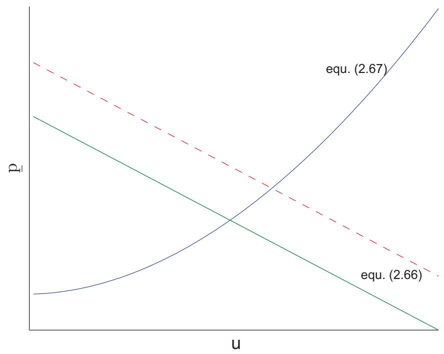
Proof: see Appendix.

Since q is uniquely determined by (2.43), and is independent of the situation in labour market, the above Lemma implies that u and $\underline{p}$ are also uniquely determined by (2.66) and (2.67). Therefore, to establish the existence and uniqueness of the equilibrium, we only need to consider the stability condition (2.58) to ensure that no firm would unilaterally deviate from the productivity investment strategy $y \in [\underline{p}, \bar{p}]$. To this end, we establish the following lemma.

Lemma 15 *Given the labour market equilibrium conditions hold, (2.58) always holds.*

Proof: see Appendix.

Henceforth, we have established the existence and uniqueness of the steady state equilibrium. Now we study the properties of equilibrium when the interest rate changes. The effect of interest rate changes on q is clear as we proved in Proposition 7, q and $R(q)$ become smaller when i goes up. By virtue of Lemma 14, we can draw the curves of (2.66) and (2.67) in a (u,$\underline{\mathrm{p}}$) space.



Graph 2.1: The joint determination of u and $\underline{\mathrm{p}}$

It has been shown in Lemma 14 that when $R(q)$ becomes smaller because of i's increasing, then, curve (2.67) does not move and curve (2.66) goes up. We then conclude that u becomes larger, i.e. $\frac{\partial u}{\partial i} > 0$, and $\underline{\mathrm{p}}$ becomes larger i.e. $\frac{\partial \underline{\mathrm{p}}}{\partial i} > 0$.

So with endogenous productivity distribution, we confirm the conclusion in Berentsen, Menzio and Wright (2008), namely that a higher interest rate or a higher inflation rate leads to a higher unemployment rate in the long run; the Phillips Curve slopes upward.

To summarize, we have established the following results.

Proposition 16 *With Assumption 2 and 3, the steady state equilibrium always exist and is unique. Furthermore, q, the amount of goods traded in the decentralized goods market, is decreasing in the interest rate i, while the lower bound of productivity $\underline{\mathrm{p}}$ and the unemployment rate u are increasing in i, in the steady state equilibrium.*

2.5 Technology, Wage Dispersion and Inflation

We will now study the effect of inflation on technology dispersion and its further effect on wage dispersion. Like in section 2.3.4, we only check the lower and upper bounds of productivity and wage distributions, while other indicators of dispersion are left to future numerical exercises.

Proposition 17 *In the steady state equilibrium with endogenous productivity distribution, a rise of inflation makes firms' productivity less dispersive in the sense that the lower bound of productivity* $\underline{p}$ *rises while the upper bound of productivity* $\bar{p}$ *is unchanged.*

The above proposition follows directly from Proposition 16 and the fact that no other endogenous variables appears in equation (2.70) determining the upper bound of productivity.

The intuition for Proposition 17 is fairly clear. When the interest rate increases, it is more costly for the buyers to hold money. Therefore, it is hard for the firms to sell products and a rise of inflation makes all the firms less profitable in the decentralized goods market. Thus, the firms with the lowest productivity are driven out of the economy as they make negative profit now and only the firms with higher productivity are left. Furthermore, the highest productivity depends on the investment cost form and is thus independent of the inflation rate and the profit earned in the decentralized goods market.

Proposition 18 *In the steady state equilibrium with endogenous productivity distribution, a rise of inflation diminishes the wage dispersion in the sense that the lower bound of wage rises while the upper bound of wages drops.*

Proof: see Appendix.

We claim that there are three effects of higher inflation on the equilibrium wage dispersion, which are represented by three different terms which show up in the proof of Proposition 18. First of all, the rise of inflation reduces the revenue of firms from the goods market and therefore reduces the highest (and also higher) wages. This effect is represented by (B.51) and it is also the reason for higher inflation diminishing the wage dispersion in the case of an exogenous productivity distribution. Second, the reduction of highest/higher wages means that there is less room for wage rises *within the firm*, i.e., it becomes less attractive to work for the firm then to be unemployed. So the reservation wages to compensate their home products, the unemployment benefit, go up. Then we would see the rise of reservation wages due to this reason, which is represented by second term in (B.58). This is also the reason for higher inflation diminishing the wage dispersion in the case of an exogenous productivity distribution case. Third, the rise of inflation increases the lower bound of productivity and the gap between the highest productivity and lowest productivity is narrowed as shown in Proposition 17. Therefore, it becomes relatively harder for the firms with the highest productivity to enroll new workers compared to the case before the inflation has risen. Then we would see the rise of reservation wages due to this reason, which is represented by term first term in (B.58). Furthermore, this effect does not exist in the case of an exogenous productivity distribution.

2.6 Conclusion

We present a tractable model to study the effects of inflation on the wage dispersion due to both productivity differences and on-the-job the search. The model contains explicit micro-foundations for money demand and for unemployment. We first study the case of an exogenous given productivity distribution and then the case where the productivity distribution is determined endogenously. We find that, for an exogenous productivity distribution, the wages become less dispersive when inflation goes up, in the sense that the lowest wage rises and the highest wage falls. In the case of an endogenous productivity distribution case, higher inflation reduces the dispersion of the productivity in the sense that the lowest productivity rises whereas the highest productivity remains unchanged; the wages also become less dispersive in the same sense as described above. To summarize, higher inflation diminishes the wage dispersion, which is consistent with the empirical evidence.

We believe that it is desirable and important to find empirical evidence on the link between inflation and productivity dispersion on the one hand and wage dispersion on the other. To our best knowledge, there is no such study. However, this task is beyond the focus of the current study and we aim to pursue this idea in subsequent research.

Chapter 3

Anticipation, Learning and Welfare: the Case of Distortionary Taxation[1]

3.1 Motivation

Nowadays, fiscal policy is usually accompanied by legislation and implementation lags. These lags create a non-negligible span of time between the announcement and effective date of a fiscal policy change. This gives individuals in the economy the opportunity to anticipate the tax changes. The economic literature denotes this aspect of fiscal policy either anticipated fiscal policy or fiscal foresight. From our reading, those two terms are equivalents and will be used as such.[2]

When agents anticipate, their resulting actions may to some extent depend on the way they form expectations about the future. The standard assumption of expectations in economics is perfect-foresight / rational expectations (RE). This assumption might be questioned. One prominent deviation of RE that imposes weaker requirements on the agent's information set when making his decisions, is the learning literature (see Evans and Honkapohja (2001) for the foundations of this approach). The main idea is that agents form expectations about future values of variables they cannot observe by engaging in a kind of statistical inference when making their economic choices.

Although the learning approach has gained significant popularity in some

[1] This chapter is a joint work with my colleague Emanuel Gasteiger.

[2] Recently (2009, p.11) has listed empirical evidence for fiscal foresight and reemphasized the relevance of expectations for sound fiscal policy. Furthermore, Leeper et al. (2009) is another good example of empirical evidence of fiscal foresight. Therein they also demonstrate the challenges for econometricians that aim to quantify the impact of fiscal policy actions and at the same time account adequately for fiscal foresight.

areas of macroeconomics, anticipated fiscal policy has, until recently, been neglected. A pioneering contribution to studying the consequences of anticipated fiscal policy when agents learn factor prices, has been made by Evans et al. (2009). They demonstrate the adaptive constant gain learning approach in several deterministic economic environments, taking changes in lump-sum taxation as an example. The choice of a constant gain therein is motivated by the fact that fiscal policy moves may state structural change. First Evans et al. (2009, p.932) consider permanent, temporary and repeated tax changes in an endowment economy with a balanced-budget policy. The core message of their results is that under learning, anticipated fiscal policy changes have instant effects on key variables as in the perfect foresight case, but the transition paths are remarkably different from the latter. This result, at least with regard to the volatility of key variables' time paths may not come as a surprise. It is well known that constant gain learning causes excess volatility compared to the case of RE (see Evans and Honkapohja (2001, p.49) for an illustration). Thereafter, Evans et al. (2009, p.941) turn attention to the scenario of debt financing of anticipated fiscal policy changes and find that, given agents understand the structure of government financing, the so-called "near Ricardian equivalence" holds under learning. Finally, Evans et al. (2009, p.944) introduce the adaptive learning approach to the basic Ramsey model. For an anticipated balanced-budget permanent tax change they once more confirm that under learning the time paths of key variables are strikingly different from their perfect foresight counterparts.

In subsequent work, Evans et al. (2010) focus on Ricardian equivalence in the basic Ramsey model with anticipated fiscal policy under learning. Most important, Evans et al. (2010, p.8) formally proof that the assumption of RE is not necessary for the classic Ricardian equivalence result. Furthermore, Evans et al. (2010, p.10) provide new departures from the Ricardian equivalence proposition. First, if government expenditures are endogenous, i.e. depend on a fiscal rule, then Ricardian equivalence holds only under RE but fails under learning. Second, Ricardian equivalence breaks down, if the expected interest rates depend on changes in the level of public debt.

Building on the contribution of Evans et al. (2009), we aim to generalize their analysis of anticipated fiscal policy under learning into an economy featuring distortionary taxes and elastic labour supply. More specifically, we derive the dynamic paths of key variables for permanent changes in distortionary taxes in a deterministic version of the prominent Ramsey model. In particular we consider permanent changes in distortionary labour income, capital income and consumption tax in turn. In addition, we examine more sophisticated fiscal policy reforms, in the presence of several tax instruments. There are fundamental differences between lump-sum taxation and distortionary taxation: a labour income tax under inelastic labour supply does not affect household margins and therefore causes no distortion, but under elastic labour supply the labour income tax affects the intra-temporal choice between consumption and leisure of the household and may cause an intra-temporal distortion. Next, a capital income tax has the potential to cause up to two types of distortion. First, the capital income tax in any case affects the inter-temporal household Euler equation. In case of elastic labour supply, the capital income tax also affects the intra-temporal choice between consumption and leisure of the household due to its distortion of the consumption choice. Finally, a consumption tax may also cause an inter-temporal distortion by affecting the household Euler equation, but there is an important difference compared to capital income taxation. The consumption tax affects the price of consumption in both periods considered in the household Euler equation whereas the capital income tax always affects only the price of next period's consumption in the household Euler equation. Loosely speaking, a consumption tax can distort consumption and investment decision via the household's Euler equation, only when it is changed, i.e. time-varying, whereas a capital income tax always causes distortions in the Ramsey economy. Thus, we may expect that the dynamics of the economy for a capital income tax reform may be fundamentally different from the economic dynamics for a consumption tax reform.[3]

Furthermore, the assumption of elastic labour supply implies that endogenous variables such as factor prices as well as employment and consumption are not predetermined as in Evans et al. (2009, p.943) or in Evans et al. (2010), but determined simultaneously in each period.

Next to the analytical derivations, we also calibrate our model and calculate welfare consequences for several policy experiments under perfect foresight as well as under learning. For this purpose, we make use of the welfare measure proposed by Lucas (1990) and also applied by Cooley and Hansen (1992) (for discrete time), which takes into account the whole transition path between the initial and new steady-states associated with initial and changed tax rate. Thus, putting it differently, we ask, to what extent the excess volatility caused by constant gain learning affects the well-being of households compared to the perfect foresight case. Using such a measure of welfare consequences, may even allow comparison of results for learning dynamics to previous studies such as Cooley and Hansen (1992), Cooley and Hansen (1992) or Garcia-Milà et al. (2010). All these studies evaluate and rank various distortionary tax reforms according to their welfare consequences under perfect foresight.

Our main results are as follows. When we assume that agents use adaptive learning rules to forecast factor prices, our model predicts oscillatory dynamic responses to anticipated permanent tax changes. Unfortunately we cannot isolate an exclusive source of the oscillatory dynamics. Sensitivity analyses suggest that there are at least two sources. In addition, policy ex-

[3] Note that a consumption tax may also be a desirable subject of study, as it has special stability properties. See Giannitsarou (2007) for the details.

periments indicate that these volatile responses may have a major impact on the welfare consequences of tax reforms. In particular we consider experiments that improve welfare but do so to a much lower extent under learning compared to perfect foresight.

Note that our approach links the learning literature to that part of the public finance literature that is concerned with the welfare consequences of different types of taxation. See Chamley (1981) for an example of a comparative statics analysis or Judd (1987) for differences in unanticipated and anticipated changes in factor taxes. In addition, there have been studies in stochastic set-ups, like Cooley and Hansen (1992). With regard to the implementation of anticipated optimal fiscal policy an example is Domeij and Klein (2005) or its extension for public goods and capital by Trabandt (2007). Moreover, Garcia-Milà et al. (2010) have recently conducted research on welfare consequences of fiscal policy experiments in the spirit of Cooley and Hansen (1992) in a heterogeneous agents model.

The remainder of the chapter is organized as follows. In Section 3.2 we outline the economic model, derive optimality conditions and detail our approach of learning. Section 3.3 compares the dynamics with and without elastic labour supply for the case of lump-sum tax changes. This section also provides sensitivity analysis for some structural parameters. In Section 3.4 we consider changes in distortionary taxation and compare the resulting dynamics to the case of lump-sum taxation. The last part of this section contains the welfare analysis of selected policy experiments. Section 3.5 concludes and points out directions for further research.

3.2 The Model

Our economy is a version of the Ramsey economy outlined in detail in Ljungqvist and Sargent (2000, p.305). The capital stock k_t evolves according to the economy-wide resource constraint

$$k_{t+1} = F(k_t, n_t) - c_t - g_t + (1 - \delta)k_t, \tag{3.1}$$

where $F(k_t, n_t)$ is the economy's production function (equalling output) showing that the firm sector uses capital k_t and labour n_t as inputs to produce the single good of the economy (see Section 3.2.2 for the details). Output can either be consumed by households (c_t) or the government (g_t) or added to the capital stock. Capital is assumed to depreciate at a constant rate δ.

3.2.1 Households

With regard to the household sector, we assume a continuum of households, where we normalize the size of the economy to unity and each household faces the problem

$$\max_{c_t,n_t} E_t^* \left\{ \sum_{t=0}^{\infty} \beta^t \left[\log(c_t) + \eta \log(\bar{L} - n_t) \right] \right\} \tag{3.2}$$

s.t.

$$\begin{aligned} k_{t+1} + \frac{b_{t+1}}{R_t} + (1+\tau_t^c)c_t &= (1-\tau_t^l)w_t n_t + (1-\tau_t^k)r_t k_t + (1-\delta)k_t \\ &\quad + b_t - \tau_t + \pi_t, \end{aligned} \tag{3.3}$$

where all variables are in per capita terms. Thus, the variable k_{t+1} denotes the level of capital in period $t+1$ and b_{t+1} is the level of government debt holdings chosen in period t. Furthermore, r_t is the rental rate of capital and R_t is the gross real interest rate in period t. The level of consumption chosen in period t is indicated by c_t. Next, $\tau_t^{\bullet}$ denotes a distortionary tax either on consumption, labour income or capital income[4]. The real wage in period t is given by w_t and $l_t = \bar{L} - n_t$ denotes leisure. In consequence, n_t is labour supply of the household. τ_t is a per capita lump-sum tax and $\pi_t = 0$ is the profit under perfect competition. Furthermore, the parameter $\eta \geq 0$ measures the elasticity of labour supply.

$E_t^*\{\bullet\}$ denotes subjective period t expectations for future values of variables. Households apply this operator, if they do not have perfect foresight. This assumption is commonly used in the learning literature. Furthermore, note that we abstract from aggregate uncertainty, i.e. we conduct our analysis in a deterministic economy. Thus, if households do not have perfect foresight, their expectations are so-called point expectations, i.e. agents base their economic choices on the mean of their expectations, see Evans and Honkapohja (2001, p.61). In Section 3.2.4 below we outline our concept of learning. An important aspect of this concept is that forecasts of single variables are independent of each other. In consequence, we can assume that for any two variables X and Y it is true that $E_t^*\{XY\} = E_t^*\{X\}E_t^*\{Y\}$ holds.

Now, we detail the household's decisions. Each household solves the Lagrangian

$$\begin{aligned} \mathcal{L} \;=\; & E_t^* \sum_{t=0}^{\infty} \beta^t \{ \log(c_t) + \eta \log(\bar{L} - n_t) \\ & -\lambda_t [k_{t+1} + \frac{b_{t+1}}{R_t} + (1+\tau_t^c)c_t - (1-\tau_t^l)w_t n_t - (1-\tau_t^k)r_t k_t - (1-\delta)k_t \\ & -b_t + \tau_t]\} \end{aligned}$$

[4] We use the symbol $\bullet$ as a placeholder throughout our analysis.

with first-order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} \quad : \quad \beta^t \left\{ c_t^{-1} - \lambda_t (1 + \tau_t^c) \right\} = 0 \tag{3.4}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} \quad : \quad \beta^t \left\{ -\lambda_t \right\} + \beta^{t+1} E_t^* \left\{ \lambda_{t+1} \left[(1-\delta) + (1 - \tau_{t+1}^k) r_{t+1} \right] \right\} = 0 \tag{3.5}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} \quad : \quad \beta^t \left\{ -\lambda_t R_t^{-1} \right\} + \beta^{t+1} E_t^* \left\{ \lambda_{t+1} \right\} = 0 \tag{3.6}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} \quad : \quad \beta^t \left\{ -\eta (\bar{L} - n_t)^{-1} - \lambda_t [-(1 - \tau_t^l) w_t] \right\} = 0. \tag{3.7}$$

From (3.4) and (3.6) we get the household Euler condition

$$c_t^{-1} = \beta R_t E_t^* \left\{ c_{t+1}^{-1} \frac{(1 + \tau_t^c)}{(1 + \tau_{t+1}^c)} \right\}, \tag{3.8}$$

(3.5) and (3.6) yield the no-arbitrage condition for capital and bonds

$$R_t = \left[(1 - \delta) + (1 - E_t^* \left\{ \tau_{t+1}^k \right\}) E_t^* \left\{ r_{t+1} \right\} \right], \tag{3.9}$$

and from (3.4) and (3.7) we get the consumption leisure trade-off

$$n_t = \bar{L} - \frac{\eta (1 + \tau_t^c) c_t}{(1 - \tau_t^l) w_t}. \tag{3.10}$$

3.2.2 Firms

In our economy, there is a unit continuum of firms who compete perfectly. Each firm in each period t rents capital at given price r_t and labour at given price w_t and produces the numeraire good with constant returns to scale production function

$$\begin{aligned} y_t &= F(k_t, n_t) \\ y_t &= A k_t^{\alpha} n_t^{(1-\alpha)}, \end{aligned} \tag{3.11}$$

where $\alpha \in (0, 1)$. The optimal firm behaviour requires that

$$r_t = \frac{\partial y_t}{\partial k_t} = A \alpha k_t^{\alpha - 1} n_t^{1-\alpha}, \tag{3.12}$$

as well as

$$w_t = \frac{\partial y_t}{\partial n_t} = A(1 - \alpha) k_t^{\alpha} n_t^{-\alpha}, \tag{3.13}$$

i.e. each production factor earns its marginal product. Finally, we have the per capita national income identity

$$\begin{aligned} y_t &= r_t k_t + w_t n_t, \\ \pi_t &= y_t - r_t k_t - w_t n_t = 0, \end{aligned} \tag{3.14}$$

which means zero profits, as one can expect from perfect competition.

3.2.3 Government

The government finances its expenses on goods and debt repayment by tax revenues and the issuance of new bonds in each period t

$$g_t + b_t = \tau_t^c c_t + \tau_t^l w_t n_t + \tau_t^k r_t k_t + \tau_t + \frac{b_{t+1}}{R_t}.$$

For the remainder, we will assume that the government operates a balanced-budget rule in each period t, thus tax revenues will fully cover expenses such that bonds are in zero net supply as a direct consequence. Thus the government sets $g_t, \tau_t^c, \tau_t^l, \tau_t^k$ and τ_t constrained by

$$g_t = \tau_t^c c_t + \tau_t^l w_t n_t + \tau_t^k r_t k_t + \tau_t \tag{3.15}$$

in each period t.

3.2.4 Learning

Now, we aim to detail our concept of learning that was elaborated in Evans et al. (2009, p.943). For completeness we restate the crucial assumptions on learning. Under learning, households are supposed to know the entire history of endogenous variables. They observe the current period value of exogenous variables and they know the state variables. Furthermore, they know the structure of the economy with regard to the fiscal policy sector. Agents understand the implications of the announced policy change for the government budget constraint. They are also convinced that the intertemporal government budget constraint will always hold (see Evans et al. (2009, p.944)). Agents then forecast factor prices such as interest rates and wages $r_{t+j}^e(t), w_{t+j}^e(t), j \geq 1$, by making use of constant-gain steady-state adaptive learning rules[5]

$$r_{t+j}^e(t) = r^e(t) \quad and \quad w_{t+j}^e(t) = w^e(t), \tag{3.16}$$

where

$$\begin{aligned} r^e(t) &= r^e(t-1) + \gamma(r_{t-1} - r^e(t-1)) \\ w^e(t) &= w^e(t-1) + \gamma(w_{t-1} - w^e(t-1)), \end{aligned} \tag{3.17}$$

[5] Here we apply the same short-hand notation as Evans et al. (2009). Thus for any variable say z, its period t expected future value in period $t+j$ derived by a learning rule may either be denoted $E_t^*\{z_{t+j}\}$ or equivalently $z_{t+j}^e(t)$. An additional notation we introduce is $z_{t+j}^p(t)$ which denotes the agent's planned choice of the variable z in period $t+j$ based on expected values formed via the learning rule in period t.

where $0 < \gamma \leq 1$ is the gain parameter.[6] Our choice of this specific learning rule is motivated by two well known arguments in the learning literature. First, as Evans and Honkapohja (2001, p.332) outline, choosing a constant gain learning rule is the appropriate choice for agents, when they are aware of structural change, as in such a learning rule agents discount past data exponentially. Note that rules (3.17) are equivalent to $r^e(t) = \gamma \sum_{i=0}^{\infty}(1-\gamma)^i r_{t-i-1}$ and $w^e(t) = \gamma \sum_{i=0}^{\infty}(1-\gamma)^i w_{t-i-1}$. Second, the timing of the learning rule, i.e. that agents' update in period t uses data up to period $t-1$, is chosen in order to avoid simultaneity between $r^e(t)$ and r_t as well as $w^e(t)$ and w_t (see for example Evans and Honkapohja (2001, p.51)). Think of simultaneity in this context as a situation in which agents' expectations affect current values of aggregate endogenous variables and vice versa, which may potentially introduce some strategic behaviour.

Such a learning rule yields a sequence of so-called *temporary equilibria*, which consist of sequences of (planned) time paths for all endogenous variables. These sequences satisfy the learning rule above, the expectation history, household and firm optimality conditions, the government budget constraint and the economy-wide resource constraint given the exogenous variables as well as the current stock of capital in each period. These plans are revisited and potentially altered in each period after expectations have been updated.

3.3 Base Case: Lump-Sum Tax

Before pursuing our core issue, i.e. the case of distortionary taxation, we would like to illustrate the applied methodology for the case of lump-sum taxation for two reasons: first, we want to illustrate the consequences of the introduction of elastic labour supply compared to the case of inelastic labour supply as assumed in Evans et al. (2009, p.943) and its effect on the dynamic paths of the key variables such as consumption and capital, given their calibration (see Table 3.1 below); second, below in Subsection 3.3.2, we aim to present a sensitivity analysis for the very basic version of the model under examination.

Let us now derive the dynamic paths under learning for an anticipated lump-sum tax change. Consequently we assume all other types of taxation away, i.e. $\tau_t^c = \tau_t^l = \tau_t^k = 0$. The Euler equation (3.8) is standard

$$c_t^{-1} = \beta(c_{t+1}^p(t))^{-1}\left[(1-\delta) + r_{t+1}^e(t)\right]$$

[6] The gain parameter measures the responsiveness of the forecast to new observations, see Evans and Honkapohja (2001, p.18). Be aware that in our model the gain parameter is exogenous. See Branch and Evans (2007) for a recent example where agents can choose the gain parameter.

and forward substitution of this yields

$$c^p_{t+j}(t) = \beta^j D^e_{t,t+j}(t) c_t, \tag{3.18}$$

where we define $D^e_{t,t+j}(t) \equiv \Pi^j_{i=1}[(1-\delta) + r^e_{t+i}(t)]$. One can think of this term as "expectations of the interest rate factor $D_{t,t+j}$ at time t" (see Evans et al. (2009, p.933)). Next, we notice that the consumption leisure trade-off in this case is

$$n_t = \bar{L} - \frac{\eta c_t}{w_t}. \tag{3.19}$$

Given the adequate transversality condition for capital

$$\lim_{T \mapsto \infty} \left(D^e_{t,t+T}(t)\right)^{-1} k^p_{t+T+1}(t) = 0, \tag{3.20}$$

the inter-temporal budget constraint of the consumer is

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} c^p_{t+j}(t) &= [(1-\delta) + r_t]k_t + w_t n_t - \tau_t \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \left[w^e_{t+j}(t) n^p_{t+j}(t) - \tau^e_{t+j}(t) \right], \end{aligned}$$

which by the virtue of (3.18) and (3.19) yields

$$\begin{aligned} c_t \frac{(1+\eta)}{(1-\beta)} &= [(1-\delta) + r_t]k_t + w_t \bar{L} - \tau_t \\ &\quad + \underbrace{\sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} w^e_{t+j}(t) \bar{L}}_{\equiv SW_1} - \underbrace{\sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau^e_{t+j}(t)}_{\equiv ST_1}. \end{aligned} \tag{3.21}$$

Equations (3.12) and (3.13) hold for firms. Finally, government faces the constraint

$$g_t = \tau_t \tag{3.22}$$

in each period t and the economy-wide resource constraint is given by (3.1).

We now need to think about the policy experiment we will study. We are looking at a scenario of a credible permanent change in taxes announced at the outset of period $t = 1$ and effective from period $t = T_p$ onwards. In particular a tax change from τ_0 to τ_1 at some point in time T_p. The dynamics under perfect foresight are standard.[7] Under learning we can directly follow Evans et al. (2009, p.943). The crucial step is to calculate the infinite sums

[7] Ljungqvist and Sargent (2000, p.305) illustrate the analytical derivations and numerical simulation alternatives for the perfect foresight case. We will simply make use of the DYNARE toolbox throughout all calculations to compute dynamics under perfect foresight. Note that this toolbox employs linearization methods.

on the right-hand side of (3.21), i.e. SW_1 and ST_1. Directly following the appendix in Evans et al. (2009, p.951) we calculate

$$SW_1 = \frac{w^e(t)\bar{L}}{r^e(t)-\delta}. \tag{3.23}$$

With regard to ST_1, we have[8]

$$ST_1 = \frac{\tau_0}{r^e(t)-\delta} + (\tau_1 - \tau_0)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}} \tag{3.24}$$

for $1 \leq t < T_p$ and

$$ST_1 = \frac{\tau_1}{r^e(t)-\delta}. \tag{3.25}$$

for $t \geq T_p$. From (3.21) follows that we have

$$\begin{aligned} c_t &= \frac{(1-\beta)}{(1+\eta)}\{[(1-\delta)+r_t]k_t + w_t\bar{L} - \tau_0 + \frac{w^e(t)\bar{L}}{r^e(t)-\delta} \\ &-\frac{\tau_0}{r^e(t)-\delta} - (\tau_1-\tau_0)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}\} \end{aligned} \tag{3.26}$$

for $1 \leq t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)}\left[[(1-\delta)+r_t]k_t + w_t\bar{L} - \tau_1 + \frac{w^e(t)\bar{L}}{r^e(t)-\delta} - \frac{\tau_1}{r^e(t)-\delta}\right] \tag{3.27}$$

for $t \geq T_p$. Given a calibration, we can then compute the dynamics of consumption and other endogenous variables.

3.3.1 Inelastic Labour Supply vs. Elastic Labour Supply

We believe that it is of importance to use a model that features elastic labour supply in order to calculate welfare implications of fiscal policy reforms adequately. Completely inelastic labour supply is a quite unrealistic assumption itself and at least some moderately elastic labour supply should be considered. Moreover, it implies that agents' choices of current period endogenous variables are in fact predetermined as is pointed out in Evans et al. (2009, p.944). In order to illustrate differences in the dynamics of endogenous variables based on the assumption of inelastic and elastic labour supply, we return to the simulation exercise of Evans et al. (2009, p.943). Note that $\tau_t^c = \tau_t^l = \tau_t^k = \delta = 0$ and $\eta = 0$ imply that $n_t = \bar{L}$ (i.e. inelastic labour supply) for all t (see equation (3.19)). Therefore, we are exactly in the same scenario as in Evans et al. (2009, p.943). Although we do not fully agree with the calibration of Evans et al. (2009), we will stick to their calibration

[8] See Appendix C.1 for details on derivations.

in this subsection to keep our results comparable. We will indicate, when we deviate from their calibration. The basic reason for this disagreement is the combination of parameters $\beta = 0.95$ and $T_p = 20$. These parameter choices imply that a government, which in reality is usually in charge of a legislation period of four to six years, may announce a tax policy change that will be effective in 20 years' time. From our perception of political execution and our confidence in fiscal policy makers' ability to commit, this appears to be unrealistic in most cases. Nevertheless, we would like to mention, that in all the subsequent numerical illustrations of our analytical derivations, we experienced severe difficulties in finding calibrations that could yield convergence for the dynamics under learning. Our experience is, that it is quite a difficult task, to perfectly calibrate the model to empirically estimated structural parameters and achieve convergence, at least with the numerical methods, we have at our disposal.

For the moment, we calibrate the model according to Table 3.1 below. The policy experiment considered in Evans et al. (2009, p.943) is a perma-

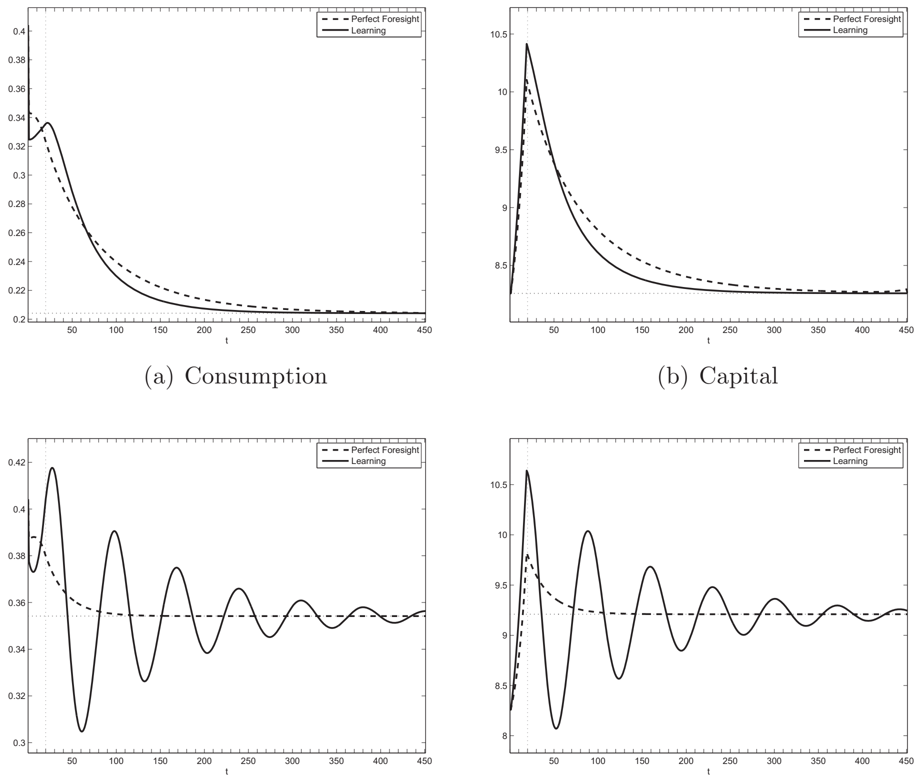
Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	20
β	0.95	γ	0.10

Table 3.1: Parameters similar as in Evans et al. (2009, p.945)

nent increase in government purchases from $g_0 = \tau_0 = 0.9$ to $g_1 = \tau_1 = 1.1$ that is announced credibly in period $t = 1$ and will be effective from period $T_p = 20$ onwards. It is assumed that the economy is in steady-state in period $t = 0$. Simulations in Evans et al. (2009, p.943) for consumption and capital are recalculated (with $\eta = 0, \bar{L} = 0.5182$) and displayed in Figures 3.1(a) and 3.1(b) below. Furthermore, Figures 3.1(c) and 3.1(d) exhibit the dynamics for elastic labour supply with $\eta = 2.00$ and $\bar{L} = 1.00$, values that match $n_0 = 0.5182$ and $g_0 = 0.9$ in this set-up.[9]Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve) with inelastic labour supply as in Evans et al. (2009) as well as consumption (c) and capital (d) dynamics under learning (solid curve) and perfect foresight (dashed curve) with elastic labour supply. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

Two distinct features emerge from Figure 3.1. First, when we compare the dynamic paths of consumption (as well as capital) under perfect foresight and learning, they are different from each other no matter with or

[9]Note that $n_0 = 0.5182$ corresponds to 12.44 hours per day. This appears to be quite unrealistic, but we choose those numbers in order to achieve both comparable magnitudes in Figure 3.1 below as well as convergence under learning.



(a) Consumption

(b) Capital

(c) Consumption with elastic n_t

(d) Capital with elastic n_t

Figure 3.1: Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve) with inelastic labour supply as in Evans et al. (2009, p.943) as well as consumption (c) and capital (d) dynamics under learning (solid curve) and perfect foresight (dashed curve) with elastic labour supply. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

without elastic labour supply. Therefore, it may be quite important to consider learning when evaluating fiscal policies as learning is a more realistic assumption of human behaviour from our point of view.[10] Second, obviously the learning paths in Figures 3.1(a) and 3.1(b) for inelastic labour supply are strikingly different to the ones under elastic labour supply in Figures 3.1(c) and 3.1(d). In particular, elastic labour supply yields much more volatility in the time paths of consumption and capital (as well as other variables in the model) compared to the inelastic labour supply case. In fact, the variables oscillate around their steady-state until they converge to it. This implies, that the tax reforms may have different welfare implications in an economy with elastic labour supply, when one compares the case of perfect foresight against the case of learning.

From our point of view, possible reasons for the significant differences in the dynamics under learning between elastic and inelastic labour supply could be as follows. Consider agents' behaviour under perfect foresight. Agents fix their current and future choices once and for all. They do not form expectations about current and future factor prices. Second, agents without perfect foresight forecast current period factor prices in each period. Therefore, they make an update of their expectations of factor prices. Thereby agents also make an expectational error. Based on their updated expectations of factor prices they revise current and planned future choices of variables in each period. In addition, actual factor prices in that period are determined based on the agents updated expectations of factor prices. Be aware that the first and the second point above are true for inelastic labour supply as well as elastic labour supply. So the learning itself cannot explain the differences in the dynamics. Furthermore, note that with inelastic labour supply, factor prices are predetermined, whereas with elastic labour supply factor prices are free variables. Moreover, with elastic labour supply, households can react to structural changes by substitution of consumption for leisure or vice versa in order to sustain a certain level of utility. For agents with perfect foresight nothing really changes when factor prices are no longer predetermined. Now, as labour supply is elastic, they choose a plan for leisure in addition to their plan for consumption, but they still do that in a once and for all manner. Transition paths should be smooth as before. But for agents that use adaptive learning, it might make a difference. In particular, we suspect that the expectational error could be larger in the case in which factor prices are no longer predetermined. This could lead to more volatility in the expectations of factor prices which translates into higher volatility of actual factor prices as well as consumption and leisure choices. We suggest that the correction of the expectational error in each period could explain the oscillations.

[10] This is the core result of Evans et al. (2009).

3.3.2 Sensitivity Analysis

Compared to the previous literature on welfare evaluation of tax reforms, our learning approach introduces two additional structural parameters. One is γ, the gain parameter and a second one is T_p, the period, in which the pre-announced tax change becomes effective. Therefore, we are interested in how these two parameters affect the dynamic properties of the model.

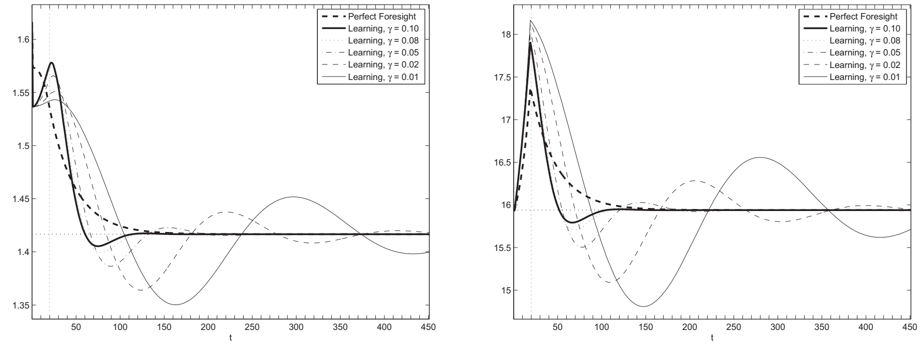
Sensitivity Analysis for the Gain Parameter

No matter what calibration, one always has to choose a gain parameter γ in the adaptive learning literature. In this subsection we would therefore like to illustrate the consequences of different choices of the gain parameter. The sole empirical estimate we are aware of is provided by Milani (2007, p.2074) for quarterly frequency and is $\gamma = 0.0183$. This number indicates that agents use approximately $1/\gamma \approx 55$ quarters of data. But a reason to be cautious to use the estimate of Milani (2007, p.2074) is that it is based on a data set containing output, inflation and the nominal interest rate, whereas in our setting agents forecast the rental rate of capital and the real wage. Next, Milani (2007, p.2074) mentions that for constant gain learning a range of $\gamma \in [0.01, 0.03]$ is commonly used. Evans and Honkapohja (2009, p.154) note a range of $\gamma \in [0.01, 0.06]$ as known estimates.

Below we will present sensitivity of the dynamics under learning for $\gamma \in \{0.01, 0.02, 0.05, 0.08, 0.10\}$. We do so for the original numerical analysis of Evans et al. (2009, p.943) ($\bar{L} = 1.00$, $\eta = 0.00$), as in this case, there is inelastic labour supply and we can focus solely on the possible fluctuations introduced by varying the gain parameter γ. Note that the two thick lines in Figures 3.2(a) and 3.2(b) exactly replicate the Figures 8 and 9 in Evans et al. (2009, p.943).

In Figure 3.2(a) we observe that the smaller the gain γ, the smaller the increase in consumption until the period of the tax change T_p (after the initial drop). Furthermore, as we recognize from Figure 3.2(b), the smaller the gain γ, the larger the increase in capital accumulation until the period of the tax change T_p. However, in both Figure 3.2(a) and 3.2(b), we observe that with decreasing γ the dynamics fluctuate around the steady-state with increasing amplitude and it takes an increasing number of periods to converge to the steady-state. These observations are partly at odds with what Evans and Honkapohja (2001, p.332) report: "a larger gain is better at tracking changes but at the cost of a larger variance". In our case it holds, that, the smaller the gain, the larger the volatility.

Summing up, we find that for the parameter range considered in this sensitivity analysis, the choice of the gain parameter γ is not crucial for the shape of the dynamic response.



(a) Consumption (b) Capital

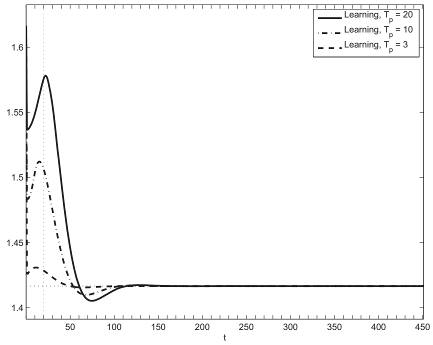
Figure 3.2: Consumption (a) and capital (b) dynamics under learning and perfect foresight with inelastic labour supply as in Evans et al. (2009, p.943) for alternating values of γ. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

Sensitivity Analysis for the Implementation Date

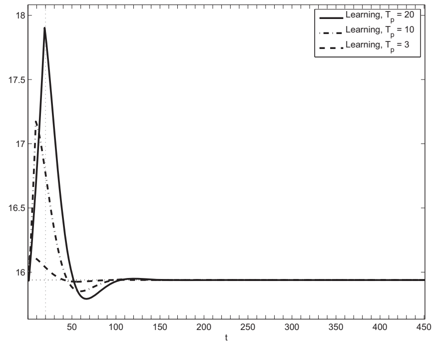
Another issue that may be of interest is the implementation date T_p. As mentioned above a tax policy change that is going to be effective in 20 years time appears to be unrealistic from our point of view. Therefore, we examine sensitivity of dynamics under learning for various implementation dates, in particular $T_p \in \{3, 10, 20\}$. Figures 3.3(a) and 3.3(b) below display the results.

In Figure 3.3(a) we observe that the shorter the distance between the announcement date and implementation date of the tax change, the higher the initial drop in consumption and the lower the increase in consumption until the implementation date thereafter. Focusing on capital, in Figure 3.3(b) we observe that with decreasing distance between the announcement date and implementation date of the tax change, the level that capital reaches until the implementation date, is also lower. Finally, for implementation in three years time, i.e. $T_p = 3$, learning dynamics are not significantly different from $T_p \in \{10, 20\}$, but lower in scale. Overall, we observe that the shorter the distance between announcement date and implementation date of the tax change, the earlier the learning dynamics approach the steady-state, but, at least for the parameter range considered herein, the nature of dynamics is not seriously affected.

Thus, we learn that in the subsequent numerical analysis, next to the elasticity of labour supply η (and the commonly known candidate parameters β and δ), the choice of the gain parameter γ as well as the implementation date T_p may also be crucial in achieving convergence on the one hand and determining the magnitude of volatility of the dynamics on the other



(a) Consumption



(b) Capital

Figure 3.3: Consumption (a) and capital (b) dynamics under learning with inelastic labour supply as in Evans et al. (2009, p.943) for alternating values of T_p. The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period $T_p = 20$.

hand. But these choices may not affect the general nature of the dynamics. Furthermore, our experience with β and δ suggests that they strongly affect the scale of results, next to their impact on convergence.

In order to summarize, there are three important insights from the analysis above. First, there are at least qualitative differences between the case of inelastic labour supply ($\eta = 0$) and elastic labour supply ($\eta > 0$). Therefore, if one regards the latter assumption as more realistic, a model that allows for elastic labour supply is a more appropriate framework to study anticipated fiscal policy under learning. Second, our sensitivity analysis suggests that the choice of the gain parameters γ and the implementation date T_p does not affect the nature of transition paths so we consider ourselves free to choose any of the values considered in the sensitivity analysis.[11] Finally and most notably, we observed at least a qualitative difference in the dynamics under learning compared to the dynamics under perfect foresight. The former appear to be much more volatile than the latter. This stylized fact, from our point of view, justifies the quantification and comparison of welfare cost of anticipated fiscal policy reforms under learning and under perfect foresight. In order to be able to mimic, at least to some extent, a realistic fiscal policy reform, we will introduce distortionary taxes. Before we look at complex fiscal policy reforms, we qualitatively inspect isolated changes in distortionary taxes and the resulting dynamics for each type of tax. Thereafter, we analyze more sophisticated fiscal policy reforms with regard to their welfare costs in a realistic calibration.

[11] In particular, in the subsequent analysis, we will choose $\gamma = 0.08$ and $T_p = 8$, which will correspond to 8 quarters.

3.4 The Case of Distortionary Taxation

After the base case of lump-sum taxation, we now study the case of distortionary taxes. In the remainder, we will assume elastic labour supply. We first characterize the dynamics for a permanent change in a single distortionary tax. This follows closely from Evans et al. (2009, p.943) similar to the last section. Next, we simulate the dynamic paths of the economy for a change in each type of distortionary tax in turn, given there are no other tax instruments. We inspect the associated dynamics for each distortionary tax with regard to qualitative differences compared to the lump-sum tax and the other distortionary taxes. Thereafter, in Section 3.4.4 below, we derive the dynamic paths of the economy in presence of all types of taxes considered in this economy. Moreover, we evaluate some specific tax reforms with regard to welfare, given our suggested calibration.

3.4.1 Labour Income Tax

Let us now assume that $\tau_t^c = \tau_t^k = \tau_t = 0$ for all t and $\tau_t^l \in [0, 1]$. The Euler equation (3.8) is standard and forward substitution again yields (3.18). Next we notice that the consumption leisure trade-off in this case is

$$n_t = \bar{L} - \frac{\eta c_t}{(1 - \tau_t^l) w_t}. \tag{3.28}$$

Given the adequate transversality condition for capital (3.20), the intertemporal budget constraint of the consumer is

$$\begin{aligned} c_t + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^e(t)} c_{t+j}^p(t) \quad &= \quad [(1-\delta) + r_t] k_t + (1 - \tau_t^l) w_t n_t \\ &+ \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^e(t)} \left[(1 - \tau_{t+j}^{l,e}(t)) w_{t+j}^e(t) n_{t+j}^p(t) \right]. \end{aligned}$$

Given (3.18) and (3.28) we can rewrite the latter as

$$\begin{aligned} c_t \frac{(1+\eta)}{(1-\beta)} \quad &= \quad [(1-\delta) + r_t] k_t + (1 - \tau_t^l) w_t \bar{L} \\ &+ \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^e(t)} \left[(1 - \tau_{t+j}^{l,e}(t)) w_{t+j}^e(t) \bar{L} \right]. \end{aligned} \tag{3.29}$$

For firms, nothing changes compared to the base case in Section 3.3. Finally, the government now faces the constraint

$$g_t = \tau_t^l w_t n_t \tag{3.30}$$

in each period t and the economy-wide resource constraint is given by (3.1).

We now once more consider the scenario of a credible permanent change in the tax rate announced in period $t = 1$ and effective from period $t = T_p$ onwards. In particular, the labour income tax is changed from τ_0^l to τ_1^l at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we can directly follow Evans et al. (2009, p.943). The crucial step is to calculate the infinite sums on the right-hand side of (3.29). That is

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{1}{D^e_{t,t+j}(t)}\left[(1-\tau^{l,e}_{t+j}(t))w^e_{t+j}(t)\bar{L}\right] &= \sum_{j=1}^{\infty}\frac{1}{D^e_{t,t+j}(t)}w^e_{t+j}(t)\bar{L} \\ &\quad -\sum_{j=1}^{\infty}\frac{1}{D^e_{t,t+j}(t)}\tau^{l,e}_{t+j}(t)w^e_{t+j}(t)\bar{L} \\ &= SW_1 - ST_2.\end{aligned}$$

Given (3.16) and (3.17), we get the term $SW_1 = \frac{w^e(t)\bar{L}}{r^e(t)-\delta}$ as before in Section 3.3. With regard to ST_2, for $1 \le t < T_p$ we calculate[12]

$$ST_2 = w^e(t)\bar{L}\left[\frac{\tau_0^l}{r^e(t)-\delta} + \left(\tau_1^l - \tau_0^l\right)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}\right] \tag{3.31}$$

and for $t \ge T_p$ we calculate

$$ST_2 = \frac{\tau_1^l\, w^e(t)\bar{L}}{r^e(t)-\delta}. \tag{3.32}$$

Given (3.29), it follows that we have

$$\begin{aligned}c_t &= \frac{(1-\beta)}{(1+\eta)}[[(1-\delta)+r_t]k_t + (1-\tau_0^l)w_t\bar{L} + (1-\tau_0^l)\frac{w^e(t)\bar{L}}{r^e(t)-\delta} \\ &\quad -w^e(t)\bar{L}(\tau_1^l-\tau_0^l)\frac{[(1-\delta)+r^e(t)]^{t-T_p}}{1-[(1-\delta)+r^e(t)]^{-1}}]\end{aligned} \tag{3.33}$$

for $1 \le t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)}\left[[(1-\delta)+r_t]k_t + (1-\tau_1^l)w_t\bar{L} + (1-\tau_1^l)\frac{w^e(t)\bar{L}}{r^e(t)-\delta}\right]. \tag{3.34}$$

for $t \ge T_p$. Given a calibration, we are then able to compute the dynamics of consumption and other endogenous variables.

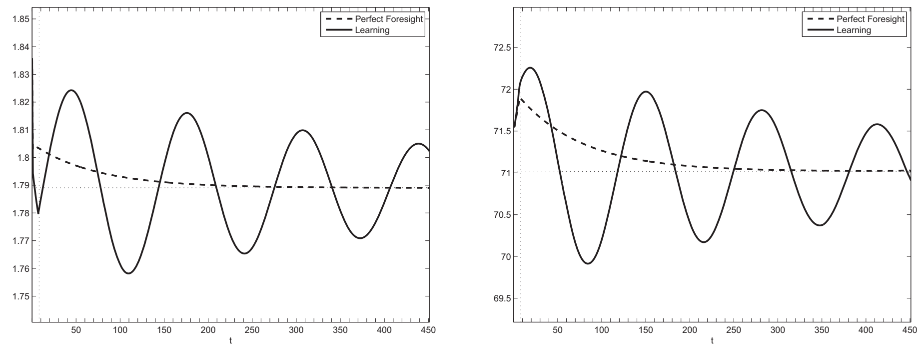
Now let us return to the numerical example. Here we calibrate the model according to Table 3.2 below.

We choose the initial labour income tax rate to be $\tau_0^l = 0.23$ as in Cooley and Hansen (1992, p.305) and assume a credible pre-announced permanent

[12] See Appendix C.1 for details on derivations.

Parameter	Value	Parameter	Value
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	1.00	$\bar{L}$	1.00

Table 3.2: Calibration for the case with labour income tax only.



(a) Consumption

(b) Capital

Figure 3.4: Consumption (a) and capital dynamics under learning (solid curve) and perfect foresight (dashed curve). The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

increase by 10% to $\tau_1^l = 0.2530$. These parameter choices yield initial steady-state employment of $n_0 = 0.3774$, which corresponds to 9.06 hours per day. Simulations for the first 450 periods are displayed in Figures 3.4(a) to 3.4(b) below.

We find that the only qualitative difference in the dynamics compared to the case of lump-sum taxation with elastic labour supply, is the remarkably slower convergence. We conjecture that this is due to the different calibration of key parameters such as β, η and γ.

3.4.2 Capital Income Tax

Let us now assume that $\tau_t^c = \tau_t^l = \tau_t = 0$ for all t and $\tau_t^k \in [0, 1]$. The Euler equation (3.8) now changes to

$$c_t^{-1} = \beta(c_{t+1}^p(t))^{-1}[(1-\delta) + (1-\tau_{t+1}^{k,e}(t))r_{t+1}^e(t)]$$

and forward substitution of this equation yields

$$c_{t+j}^p(t) = \beta^j(D_{t,t+j}^{k,e}(t))c_t, \tag{3.35}$$

where we define $D^{k,e}_{t,t+j}(t) \equiv \Pi^{j}_{i=1}[(1-\delta)+(1-\tau^{k,e}_{t+i}(t))r^{e}_{t+i}(t)]$. Furthermore, notice that the consumption leisure trade-off is again given by (3.19). Given the adequate transversality condition for capital

$$\lim_{T\mapsto\infty}\left(D^{k,e}_{t,t+T}(t)\right)^{-1}k^{p}_{t+T+1}(t)=0, \tag{3.36}$$

the inter-temporal budget constraint of the consumer is given by

$$\begin{aligned} c_t+\sum_{j=1}^{\infty}\frac{1}{D^{k,e}_{t,t+j}(t)}c^{p}_{t+j}(t) &= [(1-\delta)+(1-\tau^{k}_{t}(t))r_t]k_t+w_t n_t \\ &\quad +\sum_{j=1}^{\infty}\frac{1}{D^{k,e}_{t,t+j}(t)}[w^{e}_{t+j}(t)\bar{L}-\eta c^{p}_{t+j}(t)]. \end{aligned}$$

By the virtue of (3.35) as well as (3.19) we can rewrite the latter as

$$\frac{(1+\eta)}{(1-\beta)}c_t=[(1-\delta)+(1-\tau^{k}_{t}(t))r_t]k_t+w_t\bar{L}+\sum_{j=1}^{\infty}\frac{1}{D^{k,e}_{t,t+j}(t)}w^{e}_{t+j}(t)\bar{L}. \tag{3.37}$$

For firms again nothing changes compared to the base case in Section 3.3. Finally, the government now faces the constraint

$$g_t=\tau^{k}_{t}r_t k_t \tag{3.38}$$

in each period t. The economy-wide resource constraint is again given by (3.1).

We now consider the scenario of a permanent change in the capital income tax rate. The rate is changed from τ^{k}_{0} to τ^{k}_{1} at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we follow the approach of Evans et al. (2009, p.943). The infinite sum on the right-hand side of (3.37) is

$$SW_2=\sum_{j=1}^{\infty}\frac{1}{D^{k,e}_{t,t+j}(t)}w^{e}_{t+j}(t)\bar{L}. \tag{3.39}$$

Given (3.16) and (3.17), for $1\leq t<T_p$ we calculate[13]

$$\begin{aligned} SW_2 &= \frac{w^{e}(t)\bar{L}}{[(1-\tau^{k}_{0})r^{e}(t)-\delta]}+w^{e}(t)\bar{L}\times \\ &\quad \left[\frac{[(1-\delta)+(1-\tau^{k}_{1})r^{e}(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau^{k}_{1})r^{e}(t)]^{-1}}-\frac{[(1-\delta)+(1-\tau^{k}_{0})r^{e}(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau^{k}_{0})r^{e}(t)]^{-1}}\right] \end{aligned} \tag{3.40}$$

[13] See Appendix C.1 for details on derivations.

and for $t \geq T_p$ we calculate

$$SW_2 = \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{3.41}$$

From (3.37) follows that we have

$$\begin{aligned} c_t \quad &= \quad \frac{(1-\beta)}{(1+\eta)}\{[(1-\delta)+(1-\tau_0^k)r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times \\ &\left[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right]\} \end{aligned} \tag{3.42}$$

for $1 \leq t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)}\left[[(1-\delta)+(1-\tau_1^k)r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}\right] \tag{3.43}$$

for $t \geq T_p$. Given a calibration, we can compute the dynamics of consumption and other endogenous variables.

Now let's return to the numerical example. Here we calibrate the model according to Table 3.3 below.

Parameter	Value	Parameter	Value
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	0.85	$\bar{L}$	1.00

Table 3.3: Calibration for the case with capital income tax only.

We choose the initial capital income tax rate to be $\tau_0^k = 0.5000$ as in Cooley and Hansen (1992, p.305) and assume a credible pre-announced permanent increase by 10% to $\tau_1^k = 0.5500$. These parameter choices yield initial steady-state employment of $n_0 = 0.4848$, which corresponds to 11.6 hours per day.

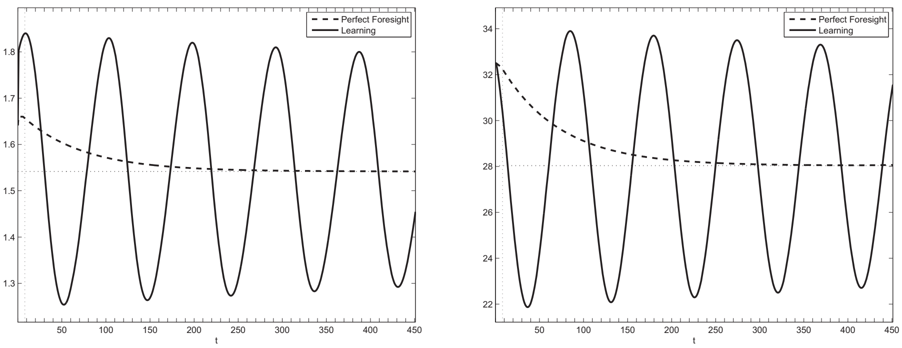
Simulation results are displayed in Figures 3.5(a) to 3.5(b) below.

The only qualitative difference we can find in the dynamics, is the larger size of fluctuations and higher frequency of them. We can also observe that under learning time paths need more periods to converge to the steady-state compared to the case of lump-sum tax or labour income tax.

3.4.3 Consumption Tax

Let us now assume that $\tau_t^l = \tau_t^k = \tau_t = 0$ for all t and $\tau_t^c \in [0,1]$. The Euler equation (3.8) now changes to

$$c_t^{-1} = \beta(c_{t+1}^p(t))^{-1}\frac{(1+\tau_t^c)}{(1+\tau_{t+1}^{c,e}(t))}[(1-\delta)+r_{t+1}^e(t)]$$



(a) Consumption (b) Capital

Figure 3.5: Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve). The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

and forward substitution of this expression yields

$$c^p_{t+j}(t) = \beta^j D^{c,e}_{t,t+j}(t) c_t, \tag{3.44}$$

where we define $D^{c,e}_{t,t+j}(t) \equiv \left[\frac{(1+\tau^c_t)}{(1+\tau^{c,e}_{t+j}(t))}\right] D^e_{t,t+j}(t)$. Next we notice that the consumption leisure trade-off in this case is

$$n_t = \bar{L} - \frac{\eta(1+\tau^c_t)c_t}{w_t}. \tag{3.45}$$

Given the adequate transversality condition for capital (3.18), the intertemporal budget constraint of the consumer is given by

$$\begin{aligned}(1+\tau^c_t)c_t + \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)}(1+\tau^{c,e}_{t+j}(t))c^p_{t+j}(t) &= [(1-\delta)+r_t]k_t + w_t n_t \\ &+ \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} w^e_{t+j}(t) n^p_{t+j}(t).\end{aligned}$$

which by the virtue of (3.44) as well as (3.45) yields

$$\frac{(1+\eta)}{(1-\beta)}(1+\tau^c_t)c_t = [(1-\delta)+r_t]k_t + w_t\bar{L} + SW_1. \tag{3.46}$$

For firms nothing changes compared to the base case in Section 3.3. Finally government now faces the constraint

$$g_t = \tau^c_t c_t \tag{3.47}$$

in each period t. The economy-wide resource constraint again is given by (3.1).

We now consider the scenario of a permanent change in the consumption tax rate from τ_0^c to τ_1^c at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we follow again the methodology of Evans et al. (2009, p.943). The infinite sum on the right-hand side of (3.46) is equal to $SW_1 = \frac{w^e(t)\bar{L}}{r^e(t)-\delta}$ as in Section 3.3. Obviously from (3.46) follows that we have

$$c_t = \frac{(1-\beta)}{(1+\eta)(1+\tau_0^c)}\left[[(1-\delta)+r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{r^e(t)-\delta}\right] \tag{3.48}$$

for $1 \leq t < T_p$ and

$$c_t = \frac{(1-\beta)}{(1+\eta)(1+\tau_1^c)}\left[[(1-\delta)+r_t]k_t + w_t\bar{L} + \frac{w^e(t)\bar{L}}{r^e(t)-\delta}\right] \tag{3.49}$$

for $t \geq T_p$. Given a calibration, we have everything at hands to compute the dynamics of consumption and the other endogenous variables.

Note that inspection of (3.48) and (3.49) makes clear that in the case of a consumption tax the dynamics of consumption are independent of the implementation date T_p. At least, this is true in our economy.[14] This fact may have a major impact on the dynamics.

In order to illustrate the dynamics numerically we calibrate the model according to Table 3.4 below.

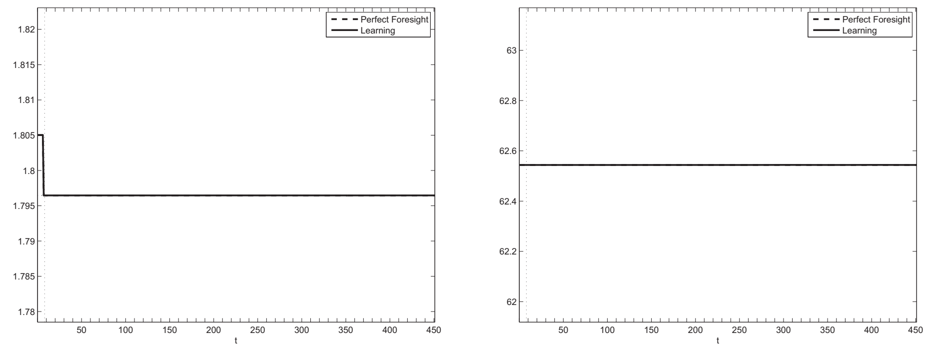
Parameter	**Value**	**Parameter**	**Value**
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	1.25	$\bar{L}$	1.00

Table 3.4: Calibration for the case with consumption tax only.

Initial consumption tax rate is $\tau_0^c = 0.0500$ as in Giannitsarou (2007, p.1424) and assume a credible pre-announced permanent increase by 10% to $\tau_1^c = 0.0550$. These parameter choices yield initial steady-state employment of $n_0 = 0.3299$, which approximately corresponds to 8.0 hours per day. Simulation results are displayed in Figures 3.6(a) to 3.6(b) below.

We observe that the dynamics of the consumption tax reform coincide for perfect foresight and learning. It appears, that in both cases, the consumption tax only matters in the period when it is changed, as suspected in our motivation above. We presume that this result depends on our utility specification with regard to consumption, that is log-utility. Ljungqvist and

[14] Compare (3.42) for the case of the capital income tax or for the labour income tax.



(a) Consumption (b) Capital

Figure 3.6: Consumption (a) and capital (b) dynamics under learning (solid curve) and perfect foresight (dashed curve). The dotted horizontal line indicates the (new) steady state, the dotted vertical line indicates period T_p.

Sargent (2000, p.318) present results for a power utility function under perfect foresight. The different perfect foresight dynamics therein suggest that the specification of utility may be the source of our result.

Nevertheless it would be an interesting subject of study to check, whether one can formally prove that the dynamics in response to a change in the consumption tax under learning and perfect foresight are similar in general, but we leave that to future research.

3.4.4 Policy Experiments

Let us now assume that $\tau_t^c, \tau_t^l, \tau_t^k \in [0,1]$ and $\tau_t \neq 0$ for all t. The Euler equation (3.8) now changes to

$$c_t^{-1} = \beta(c_{t+1}^p(t))^{-1} \left[\frac{(1+\tau_t^c)}{(1+\tau_{t+1}^{c,e}(t))}\right] [(1-\delta) + (1-\tau_{t+1}^{k,e}(t)) r_{t+1}^e(t)]$$

and forward substitution of this expression yields

$$c_{t+j}^p(t) = \beta^j D_{t,t+j}^{k,e}(t) \left[\frac{(1+\tau_t^c)}{(1+\tau_{t+j}^{c,e}(t))}\right] c_t. \qquad (3.50)$$

Furthermore, notice that the consumption leisure trade-off is now given by (3.10). Given the adequate transversality condition for capital (3.36), the inter-temporal budget constraint of the consumer is

$$\begin{aligned}
& (1+\tau_t^c)c_t + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} (1+\tau_{t+j}^{c,e}(t))c_{t+j}^p(t) \\
= \ & [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t n_t - \tau_t \\
& + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} [(1-\tau_{t+j}^{l,e}(t))w_{t+j}^e(t)n_{t+j}^p(t) - \tau_{t+j}^e(t)],
\end{aligned}$$

which by the virtue of (3.50) as well as (3.10) yields

$$\begin{aligned}
\frac{(1+\eta)}{(1-\beta)}(1+\tau_t^c)c_t &= [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t\bar{L} - \tau_t \\
& \quad + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} [(1-\tau_{t+j}^{l,e}(t))w_{t+j}^e(t)\bar{L} - \tau_{t+j}^e(t)] \\
&= [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t\bar{L} - \tau_t \\
& \quad + \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} [w_{t+j}^e(t)\bar{L} - \tau_{t+j}^{l,e}(t)w_{t+j}^e(t)\bar{L} - \tau_{t+j}^e(t)] \\
&= [(1-\delta)+(1-\tau_t^k)r_t]k_t + (1-\tau_t^l)w_t\bar{L} - \tau_t \\
& \quad + SW_2 - ST_3 - ST_4. \qquad (3.51)
\end{aligned}$$

For firms nothing changes compared to the base case in Section 3.3. Finally government now faces the constraint (3.15) in each period t. The economy-wide resource constraint is again given by (3.1).

We now consider the scenario of a permanent (simultaneous) change in (some of the) taxes at some point in time T_p. The dynamics under perfect foresight are again standard. Under learning we again follow the approach Evans et al. (2009, p.943). The infinite sum SW_2 on the right-hand side of (3.51) is already known to be

$$\begin{aligned}
SW_2 \ = \ & \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times \\
& \left[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}} \right] \qquad (3.52)
\end{aligned}$$

for $1 \leq t < T_p$ and

$$SW_2 = \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]} \qquad (3.53)$$

for $t \geq T_p$ from Section 3.4.2. ST_3 on the right-hand side of (3.51) is

$$ST_3 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^{l,e}(t)w_{t+j}^e(t)\bar{L}. \qquad (3.54)$$

Given (3.16) and (3.17), for $1 \leq t < T_p$ we calculate[15]

$$\begin{aligned} ST_3 \;=\; & \frac{\tau_0^l \; w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times \\ & \left[\frac{\tau_1^l \; [(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{\tau_0^l \; [(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right] \end{aligned} \tag{3.55}$$

and for $t \geq T_p$ we calculate

$$ST_3 = \frac{\tau_1^l \; w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{3.56}$$

Finally, ST_4 on the right-hand side of (3.51) is

$$ST_4 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^e(t). \tag{3.57}$$

Given (3.16) and (3.17), for $1 \leq t < T_p$ we calculate

$$\begin{aligned} ST_4 \;=\; & \frac{\tau_0}{[(1-\tau_0^k)r^e(t)-\delta]} + [\; \frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} \;\tau_1 \\ & -\frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}} \;\tau_0 \;] \end{aligned} \tag{3.58}$$

and for $t \geq T_p$ we calculate

$$ST_4 = \frac{\tau_1}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{3.59}$$

Given (3.51) we can then compute the dynamics responses for consumption and the other endogenous variables as before. Now, we will conduct several policy experiments numerically and compute welfare measures following the approach of Cooley and Hansen (1992, p.301).[16] Intuitively speaking, we compute the increase in consumption that an individual would require to be as well off as under the equilibrium allocation without taxes. We express that number in percentage of output. First, we will do so for our initial choice of tax levels (see line 1 in Table 3.6 below). Thereafter, we carry out policy reforms, where we change taxes in a certain way and each time recalculate welfare measure both for learning and perfect foresight. As a result we can then compare the welfare implications for a tax change under perfect foresight against the case under learning. Note that we use the

[15] See appendices C.1 and C.1 for the details on derivations of ST_3 and ST_4.

[16] We detail the computation in Appendix C.2.

measure of Cooley and Hansen (1992, p.301) for the transition paths. We do so because their measure for static comparison would lead to the same number for perfect foresight and learning, as in both cases the initial and new steady-states are identical.

An additional parameter needs to be chosen. That is the evaluation horizon T. Cooley and Hansen (1992, p.301) choose a horizon $T \geq 2000$ and give no further detail on the motivation of that choice. Garcia-Milà et al. (2010) use $T = 200$ and give no motivation either. We will choose the latter in our welfare evaluations. For the series of experiments in Table 3.6 below, our calibration of the model is according to Table 3.5 below.

Parameter	Value	Parameter	Value
A	1.00	δ	0.00
α	0.33	T_p	8
β	0.99	γ	0.08
η	0.99	$\bar{L}$	1.00

Table 3.5: Model calibration for policy experiments $1 - 4$

We choose the initial tax rates to be $\tau_0 = 0.0000$, $\tau_0^l = 0.2300$, $\tau_0^k = 0.5000$ and $\tau_0^c = 0.0500$. These non-zero tax rates lead to distortions. The first row in Table 3.6 reveals the welfare loss between the steady-state of the economy without taxes and the steady-state of the economy with our initially chosen tax rates amounts to 73.72%. This number tells us the change in consumption (in percentage of output) which is required so that households in the economy with initial tax levels are as well off as in the case with zero taxes is 73.72%. Be aware that Table 3.6 also indicates that without taxes our calibration yields a first best steady-state employment of $n_{FB} = 0.4024$, which implies 9.66 hours. With the initial taxes in place, the steady-state employment is $n_0 = 0.4326$, which implies 10.38 hours.

Now we assume a credible pre-announced permanent tax reform that favours capital accumulation, i.e. we lower the capital income tax to a level of $\tau_1^k = 0.2500$. As suggested by Judd (1987), Lucas (1990) and Cooley and Hansen (1992), this is expected to reduce the welfare costs of distortionary taxation. In each experiment reported lines 2 to 4 in Table 3.6 below, one of the other tax instruments, $\tau_\bullet$, $\tau_\bullet^l$ or $\tau_\bullet^c$ will be raised to a level that ensures that the periodic tax revenue in the new steady-state is the same as in the initial steady-state.[17] The second row of Table 3.6 indicates that compensating the cut in the capital income tax to τ_1^k by an increase in

[17] Note, that as long as the dynamics under learning and perfect foresight differ, one is not able to equalize present values of tax revenues under learning and perfect foresight to the present value of tax revenues in the initial steady-state by manipulating tax rates in the same way. This approach was used in the analysis of Cooley and Hansen (1992) for perfect foresight only, but is not feasible in our case. In addition, we believe that keeping present values constant is not the kind of fiscal policy change that governments conduct in reality. Moreover, we believe that our comparison of welfare costs under learning to

the labour income tax to τ_1^l leads to a welfare improvement under perfect foresight as well as under learning as both welfare measures decrease. But the numbers also reveal that the magnitude of the improvement differs. Whereas under learning the welfare measure goes down from 73.72% to 72.12%, under perfect foresight it decreases much more to 64.47%.[18] We can also observe that the new steady-state employment n_1 is lower than the initial steady-state employment n_0.

The pattern just described is also true, if we compensate the cut in $\tau_\bullet^k$ by an increase in $\tau_\bullet^c$ or $\tau_\bullet$ as the third and fourth row in Table 3.6 indicate. It is noteworthy that using the the lump-sum tax to compensate for the cut in the capital income tax yields the largest welfare improvement and keeps steady-state employment at the highest level independent of the assumption about expectations.

Thus, experiments 2 to 4 indicate that the resulting welfare improvements of an anticipated tax reform might be much smaller in magnitude under learning compared to its improvements under perfect foresight.

welfare costs under perfect foresight is valid even without equalizing present values of the tax revenue.

[18] We would like to emphasize that we set the rate of depreciation to $\delta = 0$ in order to achieve convergence for the dynamics under learning. That might be the reason, why the scale of $\mathcal{W}$ both under learning and perfect foresight is approximately twice the scale as the results in Cooley and Hansen (1992).

no.	$\tau_\bullet$		$\tau^l_\bullet$		$\tau^k_\bullet$		$\tau^c_\bullet$		$\mathcal{W}$		n_{FB}	$n_\bullet$	
	0	1	0	1	0	1	0	1	P	L		0	1
1.	0.0		0.2300		0.5000		0.0500		73.72%		0.4024	0.4326	
2.	0.0		0.2300	0.2931	0.5000	0.2500	0.0500		64.47%	72.12%	0.4024	0.4326	0.3976
3.	0.0		0.2300		0.5000	0.2500	0.0500	0.1027	63.71%	71.28%	0.4024	0.4326	0.4045
4.	0.0	0.0600	0.2300		0.5000	0.2500	0.0500		63.49%	70.23%	0.4024	0.4326	0.4139

0: Value before the tax change 1: Value after the tax change
P: Value under perfect foresight L: Value under learning

Table 3.6: Simulation results of various policy experiments.

3.5 Conclusion

We demonstrate that under the assumption of elastic labour supply the responses to anticipated permanent lump-sum tax changes when agents learn are remarkably different compared to their counterparts under perfect foresight. The dynamics under learning appear to oscillate around the steady-state to which they converge slowly. Thus, there is more volatility under learning.

However sensitivity analyses show that even under inelastic labour supply these oscillations may be present for some choices of the gain parameter. We also find that a smaller gain parameter leads to higher volatility in our framework. This result is at odds with conventional wisdom about the link between the gain parameter and the dynamic responses in the learning literature. Overall, we detect two sources that may lead to oscillatory dynamics under learning given an anticipated permanent lump-sum tax change. These are the assumption of elastic labour supply and the choice of the gain parameter in the learning rule.

In the subsequent analysis we derive the dynamics for several distortionary taxes and illustrate that the dynamics for labour income as well as capital income tax rate changes are quite similar to changes of the lump-sum tax given elastic labour supply. Again we observe oscillating time paths. In case of a consumption tax, there is no oscillatory behaviour for the dynamics under learning, at least when agents have a log-utility function.

Moreover, policy experiments in the presence of multiple tax instruments indicate that the magnitude of welfare improvements due to the tax reform considered herein appears to be substantially lower under the assumption of learning compared to the case of perfect foresight. The reason may be the oscillatory behaviour of the dynamics under learning.

Form our point of view these results raise two major issues. First, oscillatory dynamic responses to exogenous shocks are rarely found in actual economic data. This fact questions the suitability of the model herein for policy analysis. Second, given that this model would be suitable for policy analysis, our results indicate that permanent tax changes may lead to lower welfare improvements under learning compared to perfect foresight.

We believe that future research in this area needs to come up with convincing empirical evidence on whether or how agents learn about fiscal policy. In addition, we also need to clarify from actual economic data, how the dynamic responses to anticipated permanent tax changes look like. Are they smooth or oscillatory?

With regard to theoretical considerations, it would also be desirable to derive a version of the model that allows for changing different tax rates at different points in time and therefore allows for public debt accumulation. But this task is beyond the focus of this study and we aim to pursue that idea in subsequent research.

Furthermore, we think that perfect foresight and the implied once and for all choices of agents on the one hand and learning which implies periodic revision of current and future choices of agents on the other hand are extreme cases. One could also imagine agents that use adaptive learning, but infrequently and with differing interval length update their expectations and revise their current and future choices. Alternatively, agents randomly receive a signal to update their expectations.

In addition, more sophisticated computational methods may allow to calibrate the rate of depreciation different from zero or more realistic values of the elasticity of labour supply and still ensure convergence for the dynamics under learning on the other side. This could facilitate numerical results that are directly comparable to the existing literature in public finance.

Appendix A

Proofs for Chapter 1

Proof of Proposition 1

Proof. Denote the objective function as

$$O(z) \equiv -z + \beta\alpha_h \max_q [v(q) - \rho dq] + \beta\rho z$$

Case I, $\rho < \frac{1}{\beta}$.
(1.25) implies

$$\text{if } z > z^*, \frac{\partial q}{\partial z} = 0 \tag{A.1}$$

$$\text{if } z < z^*, \frac{\partial q}{\partial z} = \frac{1}{d}. \tag{A.2}$$

Then (A.1) implies,

$$\text{if } z > z^*, O'(z) = \beta\rho - 1 < 0, \tag{A.3}$$

which means that the objective function is decreasing in z for all $z > z^*$. Furthermore, the second condition in (1.25) implies,

$$\text{if } z < z^*, O(z) = -z + \beta\alpha_h[v(q) - \rho z] + \beta\rho z. \tag{A.4}$$

Computation then shows that

$$\text{if } z < z^*, O'(z) = \beta\rho - 1 + \beta\alpha_h[v'(q)\frac{1}{d} - \rho]. \tag{A.5}$$

Furthermore,

$$\lim_{z \to z^{*-}} O'(z) = \beta\rho - 1 + \beta\alpha_h[v'(\frac{z^*}{d})\frac{1}{d} - \rho] = \beta\rho - 1 < 0. \tag{A.6}$$

Therefore, (A.3) and (A.6) imply that the optimal $O(z)$ is reached when $z < z^*$ and z satisfies

$$\beta\rho - 1 + \beta\alpha_h[v'(q)\frac{1}{d} - \rho] = 0. \tag{A.7}$$

Substituting d from (1.24) into (A.7) gets

$$\beta\rho - 1 + \beta\alpha_h[\frac{\rho v'(q)}{c'(q)} - \rho] = 0. \tag{A.8}$$

Substituting ρ from $\rho = \frac{1}{1+\pi}$ and the Fisher equation (1.26) into (A.3) and rearranging yields (1.27). Given the knowledge of q, (1.24) and the FOC of (1.25) for $z \leq z^*$ implies (1.28). Similarly, substituting (1.24) into the definition of R yields (1.29).

Case II, $\rho = \frac{1}{\beta}$.

As mentioned above, we only consider the equilibrium of the economy in the case $1+\pi = \beta$ but as a limit as $1+\pi \to \beta$ from above. By continuity of all functions, this case can be proved using the same arguments as above. ■

Proof of Proposition 2

Proof. Computation shows that

$$\frac{d[\frac{v'(q)}{c'(q)}]}{dq} = \frac{v''(q)c'(q) - v'(q)c''(q)}{[c'(q)]^2} < 0,$$

$$\lim_{q\to 0}\frac{v'(q)}{c'(q)} = +\infty,$$

$$\lim_{q\to +\infty}\frac{v'(q)}{c'(q)} = 0.$$

Then, given $i \geq 0$, the solution of q for (1.27) exist and is unique. Furthermore, q is decreasing in i, i.e., $\frac{\partial q}{\partial i} < 0$. To see that R is also decreasing in i, we have, by the virtue of (1.29) in Proposition 1,

$$\frac{\partial R}{\partial i} = \frac{\partial R}{\partial q}\frac{\partial q}{\partial i} = c''(q)q\frac{\partial q}{\partial i} < 0.$$

■

Proof of Proposition 3

Proof. Firstly, (1.36) implies (1.40) directly. Then (1.40) and (1.37) give us (1.41). The total matching amount M is $\lambda_h u$ by definition, which is (1.42).

Subtracting (1.14) from (1.13) and using the steady state condition yields

$$[1-\beta(1-\delta)]S_h(y) = w(y) - b - \beta\lambda_h S_h(\bar{y}) + \beta\delta\int_{-\infty}^{\bar{y}} \max\{S_h(l), 0\}dF(l). \tag{A.9}$$

(1.22) can be rewritten as

$$[1-\beta(1-\delta)]S_f(y) = y+\alpha_f R(q)-w(y)+\beta\delta \int_{-\infty}^{\bar{y}} \max\{S_f(l), 0\}dF(l). \quad (A.10)$$

Adding (A.9) and (A.10) and using the definition of y_d implies

$$[1-\beta(1-\delta)]S(y) = y+\alpha_f R(q) - b - \beta\lambda_h S_h(\bar{y}) + \beta\delta \int_{y_d}^{\bar{y}} S(l)dF(l). \quad (A.11)$$

Using the fact that $S_h(y) = \eta S(y)$, (A.11) becomes

$$[1-\beta(1-\delta)]S(y) = y+\alpha_f R(q) - b - \eta\beta\lambda_h S(\bar{y}) + \beta\delta \int_{y_d}^{\bar{y}} S(l)dF(l). \quad (A.12)$$

Taking the derivative of (A.12) with respect to y yields

$$S'(y) = \frac{1}{1-\beta(1-\delta)} \quad (A.13)$$

(A.12) and (A.13) imply after integration by parts that

$$\begin{aligned} [1-\beta(1-\delta)]S(y) &= y+\alpha_f R(q) - b - \eta\beta\lambda_h S(\bar{y}) + \beta\delta \int_{y_d}^{\bar{y}} S'(l)[1-F(l)]dl \\ &= y+\alpha_f R(q) - b - \eta\beta\lambda_h S(\bar{y}) \\ &\quad + \frac{\beta\delta}{1-\beta(1-\delta)} \int_{y_d}^{\bar{y}} [1-F(l)]dl. \end{aligned} \quad (A.14)$$

Setting $y = y_d$ and $y = \bar{y}$ in (A.14) implies

$$\eta\beta\lambda_h S(\bar{y}) = y_d + \alpha_f R(q) - b + \frac{\beta\delta}{1-\beta(1-\delta)} \int_{y_d}^{\bar{y}} [1-F(l)]dl, \quad (A.15)$$

$$[1-\beta(1-\delta)]S(\bar{y}) + \eta\beta\lambda_h S(\bar{y}) = \bar{y} + \alpha_f R(q) - b + \frac{\beta\delta}{1-\beta(1-\delta)} \int_{y_d}^{\bar{y}} [1-F(l)]dl. \quad (A.16)$$

Using (1.18) and (1.35) to eliminate $S(\bar{y})$ from (A.15) yields

$$\frac{\eta\beta k}{1-\eta} \cdot \mu = y_d + \alpha_f R(q) - b + \frac{\beta\delta}{1-\beta(1-\delta)} \int_{y_d}^{\bar{y}} [1-F(l)]dl. \quad (A.17)$$

Substituting α_f from (1.23) into (A.17) and using the expression of u in (1.40), we get (1.39). Subtracting (A.15) from (A.16) yields

$$[1-\beta(1-\delta)]S(\bar{y}) = \bar{y} - y_d \quad (A.18)$$

Substituting $S(\bar{y})$ from (1.18) into (A.18) implies (1.38) ■

Proof of Lemma 4

Proof. It is obvious that the left hand side of (1.38) is an increasing function of μ and independent of y_d; while the right hand side of (1.38) is a decreasing function of y_d and independent of μ. Therefore, (1.38) slopes downward in (y_d, μ) space.

Denote the left hand side of (1.39) by $\Gamma(y_d, \mu, R)$. Computation shows that

$$\frac{\partial \Gamma}{\partial y_d} = 1 - \frac{\beta\delta}{1-\beta+\beta\delta} + \frac{\alpha_h \delta R(q)}{m(\mu,1)} F'(y_d) + \frac{\beta\delta}{1-\beta(1-\delta)} F(y_d) > 0,$$

$$\frac{\partial \Gamma}{\partial \mu} = -\frac{\eta\beta k}{1-\eta} - \frac{\alpha_h \delta F(y_d)}{[m(\mu,1)]^2} R(q) \frac{dm(\mu,1)}{d\mu} < 0.$$

This implies,

$$-\frac{\frac{\partial \Gamma}{\partial y_d}}{\frac{\partial \Gamma}{\partial \mu}} > 0.$$

Therefore, (1.39) slopes upward in (y_d, μ) space. Furthermore, computation shows that

$$\frac{\partial \Gamma}{\partial R} = \alpha_h \left[1 + \frac{\delta F(y_d)}{m(\mu,1)} \right] > 0$$

such that

$$-\frac{\frac{\partial \Gamma}{\partial R}}{\frac{\partial \Gamma}{\partial \mu}} > 0.$$

This implies that, for any given y_d, μ become smaller when $R(q)$ goes down, i.e. (1.39) shifts down if $R(q)$ goes down. ∎

Appendix B

Proofs for Chapter 2

Proof of Proposition 6

Proof. If the constrain $d \leq z$ does not bind, the first order condition for problem (TT) reads,

$$v'(q) = 1 \tag{B.1}$$

$$\rho d = (1-\theta)v(q) + \theta q. \tag{B.2}$$

With the definition of q^*, (B.1) and (B.2) become

$$q = q^* \tag{B.3}$$

$$d = z^* \equiv \frac{(1-\theta)v(q^*) + \theta q^*}{\rho}. \tag{B.4}$$

If the constrain $d \leq z$ does bind, problem (TT) becomes

$$\max_{q,d} [v(q) - \rho z]^\theta [\rho z - q]^{1-\theta}.$$

The first order condition for above problem reads,

$$\rho z = \frac{\theta q v'(q) + (1-\theta)v(q)}{\theta v'(q) + 1 - \theta} = g(q). \tag{B.5}$$

(B.3) to (B.5) imply that

$$\text{if } z > z^*, \frac{\partial q}{\partial z} = \frac{\partial d}{\partial z} = 0 \tag{B.6}$$

$$\text{if } z < z^*, \frac{\partial q}{\partial z} = \frac{\rho}{g'(q)}, \frac{\partial d}{\partial z} = 1. \tag{B.7}$$

Problem (RB) of last period reads,

$$\max_z \{\beta \alpha_h [v(q) - \rho d] + (\beta \rho - 1) z\}.$$

Denote the objective function as $O(z) \equiv \beta\alpha_h[v(q(z)) - \rho d] + (\beta\rho - 1)z$.
Case I, $\rho < \frac{1}{\beta}$.
(B.6) implies

$$\text{if } z > z^*, O'(z) = \beta\rho - 1 < 0, \tag{B.8}$$

which means that the object function is decreasing in z for all $z > z^*$. (B.7) implies

$$\text{if } z < z^*, O'(z) = \beta\alpha_h\rho\frac{v'(q)}{g'(q)} - \beta\alpha_h\rho + \beta\rho - 1. \tag{B.9}$$

Furthermore,

$$\frac{v'(q)}{g'(q)} = \frac{v'(q)[\theta v'(q) + 1 - \theta]^2}{v'(q)[\theta v'(q) + 1 - \theta] - \theta(1-\theta)[v(q) - q]v''(q)} \tag{B.10}$$

$$\lim_{z \to z^{*-}} \frac{v'(q)}{g'(q)} = \frac{1}{1 - \theta(1-\theta)[v(q^*) - q^*]v''(q^*)} \leq 1. \tag{B.11}$$

The inequality is strict except for $\theta = 1$; (B.11) uses the facts that $v'' < 0$ and $v(q^*) > q^*$ (this is true because firstly $v'' < 0$ and $v'(q^*) = 1$ imply $v'(q) > 1$ whenever $q < q^*$; secondly $v(0) = 0$ and above results imply $v(q^*) \equiv \int_0^{q^*} v'(q)dq > \int_0^{q^*} 1dq \equiv q^*$). Therefore,

$$\lim_{z \to z^{*-}} O'(z) = \beta\alpha_h\rho(\lim_{z \to z^{*-}} \frac{v'(q)}{g'(q)} - 1) + \beta\rho - 1 < 0. \tag{B.12}$$

(B.8) and (B.12) imply that the optimal z is reached when $z < z^*$ and satisfies

$$O'(z) = \beta\alpha_h\rho\frac{v'(q)}{g'(q)} - \beta\alpha_h\rho + \beta\rho - 1 = 0. \tag{B.13}$$

The real balance choice satisfies (B.5), and $d = z$ always holds. It is a natural conclusion because it is costly to carry cash when the Friedman rule does not hold.

Substituting ρ from $\rho = \frac{1}{1+\pi}$ and the Fisher equation (2.42) into (B.13) and rearranging, we get (2.43). Similarly, substituting ρ into (B.5) and using the stationary condition ($z = \hat{z}$ for there is no economic growth in the present model), we get (2.45).

Case II, $\rho = \frac{1}{\beta}$.

As mentioned in the text, we only consider the equilibrium of the economy in the case $1+\pi = \beta$ when the equilibrium is the limit as $1+\pi \to \beta$ from above. The proof of this case follows therefore from the same arguments as in the previous case and continuity. ■

Proof of Proposition 7

Proof. Given that $\frac{v'(q)}{g'(q)}$ is strictly deceasing for all $q \leq q^*$ and $\lim_{q\to 0}\frac{v'(q)}{g'(q)} = +\infty$, it is obvious that the solution of q in (2.43) exist and is unique. Furthermore, q is decreasing in i, i.e., in equilibrium,

$$\frac{\partial q}{\partial i} < 0. \tag{B.14}$$

(2.43) implies that in equilibrium

$$\frac{v'(q)}{g'(q)} \geq 1. \tag{B.15}$$

(B.10) implies that,

$$\frac{v'(q^*)}{g'(q^*)} \leq 1. \tag{B.16}$$

Therefore, given that $\frac{v'(q)}{g'(q)}$ is strictly deceasing, (B.15) and (B.16) imply that in equilibrium,

$$q \leq q^* \tag{B.17}$$

must hold and that the equality holds if and only if $i = 0$ and $\theta = 1$.

To see that R is also decreasing in i, we have, by the virtue of Proposition 6,

$$R = \rho d - q = g(q) - q = \frac{(1-\theta)[v(q)-q]}{\theta v'(q) + 1 - \theta}. \tag{B.18}$$

The computation then shows that

$$\frac{\partial R}{\partial i} = \frac{\partial R}{\partial q}\frac{\partial q}{\partial i} = (1-\theta)\frac{[v'(q)-1][\theta v'(q)+1-\theta] - \theta[v(q)-q]v''(q)}{[\theta v'(q)+1-\theta]^2}\frac{\partial q}{\partial i}. \tag{B.19}$$

(B.17) implies that in equilibrium,

$$v'(q) \geq 1 \tag{B.20}$$

and

$$v(q) > q \tag{B.21}$$

for the same reason as we established in last proof. Combining (B.14), (B.19), (B.20), (B.21), and the fact $v'' < 0$, gives us that in equilibrium,

$$\frac{\partial R}{\partial i} \leq 0$$

must always hold and the equality holds only when $i = 0$ and $\theta = 1$. ■

Proof of Proposition 10

Proof. We first look at the support of $G(w)$. (2.40) implies that the slope of $\phi_0(y)$ is

$$\frac{d[\phi_0(y)]}{dy} = -\frac{\bar{\Gamma}(y)\beta\lambda_h}{1-\beta(1-\delta)} \leq 0,$$

which implies that the lowest wage which workers get to exit unemployment is $\phi_0(\bar{\mathrm{p}})$. Similarly, (2.17) implies that the slope of $\phi_1(y)$ is

$$\frac{d[\phi_1(y)]}{dy} = 1 > 0,$$

which implies that the highest possible wage is $\phi_1(\bar{\mathrm{p}})$. The support of $G(w)$ is the interval $[\phi_0(\bar{\mathrm{p}}), \phi_1(\bar{\mathrm{p}})]$.

To determine $G(w)$, we now count the measure of workers earning less than w in any firm, and then integrate over the relevant set of firms. Therefore, we are going to use the expression of $L(w \mid y)$ and $L(y)$ when deriving $G(w)$. Note that the functional forms of $L(w \mid y)$ and $L(y)$ we got in section 2.3.4 are restricted in their meaningful segments. So we have to partition the support of $\Gamma(y)$ to get an accurate expression of $G(w)$.

Case I: $\phi_0(\bar{\mathrm{p}}) \leq w < \phi_0(\underline{\mathrm{p}})$. Here the wage is so low that the least productive firms are even hardly attractive to unemployed worker. Following the definition, firms with productivity greater than $\phi_0^{-1}(w)$ can hire workers for less than w. And with w in the range $\phi_0(\bar{\mathrm{p}}) \leq w < \phi_0(\underline{\mathrm{p}})$, $\phi_0^{-1}(w) > \underline{\mathrm{p}}$, so not all firms will actually be able to have employees paid less than w. Accordingly, $G(w)$ is given by,

$$(1-u)G(w) = N\int_{\phi_0^{-1}(w)}^{\bar{\mathrm{p}}} L(w \mid p)d\Gamma(p). \tag{B.22}$$

Then we substitute $L(w \mid p)$ from (2.54) into (B.22) and use the expression of $L(\cdot)$ in (2.49) to get

$$G(w) = \frac{\lambda_h u}{1-u}\int_{\phi_0^{-1}(w)}^{\bar{\mathrm{p}}} \frac{\delta+\lambda_h}{\{\delta+\lambda_h\bar{\Gamma}[Q(w,p)]\}^2}d\Gamma(p). \tag{B.23}$$

Then, substituting u from the labour market steady state condition (2.46) into (B.23) yields the equality in the second line of (2.55). Note that the continuity of $G(w)$ at $w = \phi_0(\bar{\mathrm{p}})$, is ensured by $\phi_0^{-1}[\phi_0(\bar{\mathrm{p}})] = \bar{\mathrm{p}}$.

Case II: $\phi_0(\underline{\mathrm{p}}) \leq w < \phi_1(\underline{\mathrm{p}})$. In this case, all firms are productive enough to attract at least some workers by an offer of w. $G(w)$ is thus simply given by

$$(1-u)G(w) = N\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} L(w \mid p)d\Gamma(p). \tag{B.24}$$

Then we substitute $L(w \mid p)$ from (2.54) into (B.24) and use the expression of $L(\cdot)$ in (2.49) to get

$$G(w) = \frac{\lambda_h u}{1-u} \int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \bar{\Gamma}[Q(w,p)]\}^2} d\Gamma(p). \tag{B.25}$$

Similarly, substituting u from the labour market steady state condition (2.46) into (B.25) gets the equality in the third line of (2.55). Note that the continuity of $G(w)$ at $w = \phi_0(\underline{\mathrm{p}})$, is ensured by $\phi_0^{-1}[\phi_0(\underline{\mathrm{p}})] = \underline{\mathrm{p}}$.

Case III: $\phi_1(\underline{\mathrm{p}}) \le w < \phi_1(\bar{\mathrm{p}})$. In this case, all firms have employees paid less than w, but only those more productive than $\phi_1(w)$ also have employees paid more than w. We thus have to distinguish between those two categories of firms to define $G(w)$:

$$(1-u)G(w) = N \int_{\underline{\mathrm{p}}}^{\phi_1^{-1}(w)} L(p) d\Gamma(p) + N \int_{\phi_1^{-1}(w)}^{\bar{\mathrm{p}}} L(w \mid p) d\Gamma(p). \tag{B.26}$$

Then we substitute $L(w \mid p)$ from (2.54) into (B.24) and use the expression of $L(\cdot)$ in (2.49) to get

$$G(w) = \frac{\lambda_h u}{1-u} [\int_{\underline{\mathrm{p}}}^{\phi_1^{-1}(w)} \frac{\delta + \lambda_h}{[\delta + \lambda_h \bar{\Gamma}(p)]^2} d\Gamma(p) + \int_{\phi_1^{-1}(w)}^{\bar{\mathrm{p}}} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \bar{\Gamma}[Q(w,p)]\}^2} d\Gamma(p)]. \tag{B.27}$$

Similarly, substituting u from the labour market steady state condition (2.46) into (B.27) gets the equality in the fourth line of (2.55). Note that the continuity of $G(w)$ at $w = \phi_1(\underline{\mathrm{p}})$, is ensured by $\phi_1^{-1}[\phi_1(\underline{\mathrm{p}})] = \underline{\mathrm{p}}$ and thus

$$\int_{\underline{\mathrm{p}}}^{\phi_1^{-1}[\phi_1(\underline{\mathrm{p}})]} \frac{\delta + \lambda_h}{[\delta + \lambda_h \bar{\Gamma}(p)]^2} d\Gamma(p) = 0,$$

while the continuity of $G(w)$ at $w = \phi_1(\bar{\mathrm{p}})$, is ensured by $\phi_1^{-1}[\phi_1(\bar{\mathrm{p}})] = \bar{\mathrm{p}}$ and thus

$$\int_{\phi_1^{-1}[\phi_1(\bar{p})]}^{\bar{\mathrm{p}}} \frac{\delta + \lambda_h}{\{\delta + \lambda_h \bar{\Gamma}[Q(w,p)]\}^2} d\Gamma(p) = 0.$$

$$\begin{aligned}
\delta \int_{\underline{\mathrm{p}}}^{\phi_1^{-1}[\phi_1(\bar{p})]} \frac{\delta + \lambda_h}{[\delta + \lambda_h \bar{\Gamma}(p)]^2} d\Gamma(p) &= \delta \int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} \frac{\delta + \lambda_h}{[\delta + \lambda_h \bar{\Gamma}(p)]^2} d\Gamma(p) \\
&= \delta \frac{N}{\lambda_h u} \int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} L(p) d\Gamma(p) \\
&= \delta \frac{N}{\lambda_h u} \cdot \frac{1}{N} \frac{\lambda_h u \Gamma(\bar{\mathrm{p}})}{\delta + \lambda_h \bar{\Gamma}(\bar{\mathrm{p}})} \\
&= 1,
\end{aligned} \tag{B.28}$$

where the second equality of (B.28) employs the expression of $L(\cdot)$ in (2.49); the third equality employs the evaluation of (2.48) at $y = \bar{\mathrm{p}}$; the last equality employs $\Gamma(\bar{\mathrm{p}}) = 1$, $\bar{\Gamma}(\bar{\mathrm{p}}) = 0$. ■

Proof of Proposition 11

Proof. It is obvious that (2.40) implies

$$\frac{\partial \underline{\mathrm{w}}}{\partial i} = \frac{\partial \phi_0(\bar{\mathrm{p}})}{\partial i} = \frac{\partial \phi_0(\bar{\mathrm{p}})}{\partial R}\frac{\partial R}{\partial i} = \frac{\beta\lambda_h\bar{\Gamma}[b-\alpha_f(\rho d - q)]}{1-\beta(1-\delta)} \cdot (-\alpha_f)\frac{\partial R}{\partial i} > 0, \tag{B.29}$$

where the last inequality uses $\bar{\Gamma}(\cdot) > 0$ and the conclusion in Proposition 7 that $\frac{\partial R}{\partial i} < 0$.

Similarly, (2.17) implies that

$$\frac{\partial \bar{\mathrm{w}}}{\partial i} = \frac{\partial \phi_1(\bar{\mathrm{p}})}{\partial i} = \frac{\partial \phi_1(\bar{\mathrm{p}})}{\partial R}\frac{\partial R}{\partial i} = \alpha_f\frac{\partial R}{\partial i} < 0. \tag{B.30}$$

■

Proof of Lemma 12

Proof. Given y, define $\Lambda_y(\cdot)$ as $\Lambda_y[Q(w,y)] = w$. Then,

$$\begin{aligned} \frac{d\Lambda_y(x)}{dx}|_{x=Q(w,y)} &= \frac{d\Lambda_y[Q(w,y)]}{dQ(w,y)} \\ &= \frac{d\Lambda_y[Q(w,y)]}{dw}\frac{\partial w}{\partial Q(w,y)} = 1 \cdot [\frac{\partial Q(w,y)}{\partial w}]^{-1} \end{aligned} \tag{B.31}$$

Differentiating both side of (2.32) with respect to w gives us

$$\frac{\partial Q(w,y)}{\partial w} = 1 + \frac{\beta\lambda_h}{1-\beta(1-\delta)}(-1)\bar{\Gamma}[Q(w,y)]\frac{\partial Q(w,y)}{\partial w},$$

which could be rearranged to get

$$\frac{\partial Q(w,y)}{\partial w} = [1 + \frac{\beta\lambda_h\bar{\Gamma}[Q(w,y)]}{1-\beta(1-\delta)}]^{-1}. \tag{B.32}$$

(B.31) and (B.32) then implies

$$\frac{d\Lambda_y(x)}{dx}|_{x=Q(w,y)} = \frac{d\Lambda_y[Q(w,y)]}{dQ(w,y)} = 1 + \frac{\beta\lambda_h\bar{\Gamma}[Q(w,y)]}{1-\beta(1-\delta)}. \tag{B.33}$$

Henceforth, (2.18), the definition of $\pi(y)$, and (2.16) imply,

$$\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} S(y,w)dL(w \mid y) = \int_{\phi_0(y)}^{\phi_1(y)} (y + \alpha_f R - w)dL(w \mid y). \tag{B.34}$$

Integrating (B.34) by parts gives us,

$$\begin{aligned} \pi(y) &= (y+\alpha_f R - w)L(w \mid y)|_{\phi_0(y)}^{\phi_1(y)} - \int_{\phi_0(y)}^{\phi_1(y)} L(w \mid y)d(y+\alpha_f R - w) \\ &= \int_{\phi_0(y)}^{\phi_1(y)} L(w \mid y)dw, \end{aligned} \tag{B.35}$$

where we use the results

$$L(w \mid y)|_{w=\phi_0(y)} = 0,$$

$$(y + \alpha_f R - w)|_{w=\phi_1(y)} = 0,$$

$$\frac{d(y + \alpha_f R - w)}{dw} = -1,$$

Equation (2.54) shows that (B.35) is equivalent to

$$\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} l[Q(w,y)]dw, \tag{B.36}$$

With the definition of $\Lambda_y(\cdot)$, (B.36) could be rewritten as

$$\pi(y) = \int_{\phi_0(y)}^{\phi_1(y)} l[Q(w,y)]d\Lambda_y[Q(w,y)], \tag{B.37}$$

Now changing $Q(w,y)$ into x in the integral in (B.37) yields the following equivalent expression:

$$\pi(y) = \int_{b-\alpha_f R}^{y} l(x)[1 + \frac{\beta\lambda_h\bar{\Gamma}(x)}{1-\beta(1-\delta)}]dx,$$

where (B.33) and the following facts, which are rearrangement of (2.23) (2.36), are used,

$$Q[\phi_1(y), y] = y,$$

$$Q[\phi_0(y), y] = b - \alpha_f R.$$

■

Proof of Proposition 13

Proof. (2.65) holds for all $y \in [\underline{p}, \bar{p}]$. Therefore,

$$\pi(\underline{p}) - cf(\underline{p}) = 0. \tag{B.38}$$

Plugging $\pi(\cdot)$ from (2.56) into (B.38) gives us,

$$\int_{b-\alpha_f R}^{\underline{p}} l(x)[1 + \frac{\beta\lambda_h\bar{\Gamma}(x)}{1-\beta(1-\delta)}]dx = cf(\underline{p}). \tag{B.39}$$

Then with the expression of $l(\cdot)$ from (2.49), (B.39) becomes

$$\int_{b-\alpha_f R}^{\underline{p}} \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{[\delta + \lambda_h\bar{\Gamma}(x)]^2}[1 + \frac{\beta\lambda_h\bar{\Gamma}(x)}{1-\beta(1-\delta)}]dx = cf(\underline{p}). \tag{B.40}$$

We know that $\bar{\Gamma}(x) = 1$, when $x \leq \underline{p}$. Then simplifying the integrate in (B.40) yields

$$\frac{\lambda_h u}{N}\frac{1}{\delta+\lambda_h}[1+\frac{\beta\lambda_h}{1-\beta(1-\delta)}](\underline{p}-b+\alpha_f R) = cf(\underline{p}). \tag{B.41}$$

Plugging λ_h from (2.64) into (2.46) gives us

$$\lambda N u = (1-u)\delta. \tag{B.42}$$

Solving (B.42) N yields (2.68). Then plugging N from (2.68) into (2.64) gets (2.69). Now dividing (B.41) by (2.62) gives us

$$\underline{p} - b + \alpha_f R = \frac{f(\underline{p})}{f'(\underline{p})} \tag{B.43}$$

Substituting α_f from (2.41) into (B.43) and rearranging give us (2.66). Substituting N and λ_h from (2.68) and (2.69) into (2.62) and rearranging give us (2.67). Substituting N and λ_h from (2.68) and (2.69) into (2.63) and rearranging give us (2.70). Furthermore, given the property of $f(\cdot)$ that $f'(0) = 0$ and $f'' > 0$, there is a unique positive solution of $\bar{p}$ for (2.70). ■

Proof of Lemma 14

Proof. It is obvious that the left hand side of (2.66) is a decreasing function of u and independent of $\underline{p}$, for any fixed value of R. Taking the derivative of right hand side of (2.66) with respect of $\underline{p}$ yields that

$$\frac{d[\underline{p} - \frac{f(\underline{p})}{f'(\underline{p})}]}{d\underline{p}} = 1 - \frac{f'(\underline{p})f'(\underline{p}) - f(\underline{p})f''(\underline{p})}{[f'(\underline{p})]^2} = \frac{f(\underline{p})f''(\underline{p})}{[f'(\underline{p})]^2} > 0. \tag{B.44}$$

Consequently, (2.66) slopes downward in $(u, \underline{p})$ space. Furthermore, the left hand side of (2.66) is a decreasing function of R and independent of $\underline{p}$. This implies that (2.66) shifts up if $R(q)$ goes down.

It is obvious that the left hand side of (2.67) is an increasing function of u and independent of $\underline{p}$; while the right hand side of (2.67) is an increasing function of $\underline{p}$ and independent of u. Therefore, (2.67) slopes upward in $(u,\underline{p})$ space. ■

Proof of Lemma 15

Proof. We first consider the case where the productivity investment decision yields $y > \bar{p}$. Suppose there is a firm with $y > \bar{p}$ in the economy, then its labour force situation is as good as the firm with productivity $\bar{p}$, in term of employees measure and their wage distribution. The firm's profit can be expressed as

$$\Pi(y) = \pi(y) - cf(y) = \pi(\bar{p}) + L(\bar{p})(y - \bar{p}) - cf(y).$$

Computation then shows that

$$\Pi'(y) = L(\bar{p}) - cf'(y),$$
$$\Pi''(y) = -cf''(y) < 0,$$
$$\Pi'(\bar{p}) = 0.$$

So when $y > \bar{p}$, it holds that

$$\Pi'(y) < 0,$$

which means that when $y > \bar{p}$

$$\Pi(y) < \Pi(\bar{p}) = 0.$$

We now consider a productivity investment decision yielding $y < \underline{p}$. Suppose there is a firm with $y < \underline{p}$ in the economy. We claim that for $y < \underline{p}$,

$$\Pi(y) < \pi(y) - cf(y) \equiv \Xi(y), \tag{B.45}$$

where we denote by $\Xi(y)$ the profit of firms with productivity y in a economy with exogenous continuous productivity distribution over $[y, \bar{p}]$. That is because the firm is the least attractive firm in the economy for workers and then its profit $\pi(y)$ (without considering the productivity investment cost) is smaller than that of the firms of y in a economy with exogenous continuous productivity distribution over $[y, \bar{p}]$. It is obvious that $\Xi(\underline{p}) = \Pi(\underline{p})$. Plugging the expression of $\pi(y)$ gives us,

$$\begin{aligned}
\Xi(y) &= \int_{b-\alpha_f R}^{y} \frac{\lambda_h u}{N} \frac{\delta + \lambda_h}{[\delta + \lambda_h \bar{\Gamma}(y)]^2} [1 + \frac{\beta \lambda_h \bar{\Gamma}(x)}{1 - \beta(1-\delta)}] dx - cf(y) \\
&= \frac{\lambda_h u}{N} \frac{1}{(\delta + \lambda_h)} [1 + \frac{\beta \lambda_h}{1 - \beta(1-\delta)}] \int_{b-\alpha_f R}^{y} 1 dx - cf(y),
\end{aligned} \tag{B.46}$$

Note that being less productive than $b - \alpha_f R$ means being unable to attract any worker and therefore cannot be optimal. We thus focus on values of $y \geq b - \alpha_f R$. $\Xi(y)$ is continuously differentiable, and such that

$$\Xi'(y) = \frac{\lambda_h u}{N} \frac{1}{(\delta + \lambda_h)} [1 + \frac{\beta \lambda_h}{1 - \beta(1-\delta)}] - cf'(y),$$
$$\Xi'(\underline{p}) = \frac{\lambda_h u}{N} \frac{1}{(\delta + \lambda_h)} [1 + \frac{\beta \lambda_h}{1 - \beta(1-\delta)}] - cf'(\underline{p}) = 0,$$
$$\Xi''(y) = -cf''(y) < 0.$$

So when $y < \underline{p}$

$$\Xi'(y) > 0$$

which means that when $y < \underline{p}$

$$\Xi(y) < \Xi(\underline{p}) = \Pi(\underline{p}) = 0. \tag{B.47}$$

Combining (B.45) and (B.47), we get for $y < \underline{p}$ that

$$\Pi(y) < 0.$$

■

Proof of Proposition 18

Proof. Like in section 2.3.4, $\underline{\mathrm{w}}= \phi_0(\bar{\mathrm{p}})$ and $\bar{\mathrm{w}}= \phi_1(\bar{\mathrm{p}})$. Rewriting (2.40) and (2.17) down and substituting α_f and λ_h from (2.41) and (2.64) yield,

$$\underline{\mathrm{w}} = \phi_0(\bar{\mathrm{p}}) = b - \frac{\beta\lambda N}{1-\beta(1-\delta)} \int_{b-\frac{\alpha_h R}{1-u}}^{\bar{\mathrm{p}}} \bar{\Gamma}(p)dp, \tag{B.48}$$

$$\bar{\mathrm{w}} = \phi_1(\bar{\mathrm{p}}) = \bar{\mathrm{p}} + \frac{\alpha_h R}{1-u}, \tag{B.49}$$

where u, $\underline{\mathrm{p}}$, N and $\bar{\Gamma}(\cdot)$ are determined by (2.66), (2.67), (2.68), and (2.61).

We first determine $\frac{\partial \bar{\mathrm{w}}}{\partial i}$. The derivative of (B.49) with respect to i reads

$$\frac{\partial \bar{\mathrm{w}}}{\partial i} = \frac{\partial(\frac{\alpha_h R}{1-u})}{\partial i}. \tag{B.50}$$

We know that $\frac{\partial \underline{\mathrm{p}}}{\partial i} > 0$ from Proposition 16 and $[\underline{\mathrm{p}} - \frac{f(\underline{\mathrm{p}})}{f'(\underline{\mathrm{p}})}]' > 0$ from (B.44), then (2.66) implies that

$$\frac{\partial(\frac{\alpha_h R}{1-u})}{\partial i} < 0. \tag{B.51}$$

Combining this with (B.50) yields

$$\frac{\partial \bar{\mathrm{w}}}{\partial i} < 0, \tag{B.52}$$

i.e., the upper bound of wages $\underline{\mathrm{w}}$ drops when the inflation rises.

We then determine $\frac{\partial \underline{\mathrm{w}}}{\partial i}$. Note that $\bar{\Gamma}(\cdot)$ is always continuously differentiable within the integration interval of (B.48). Define $\phi = \frac{\beta\lambda}{1-\beta(1-\delta)}$ and rewrite (B.48) as

$$\begin{aligned}
\underline{\mathrm{w}} &= b - \phi N[\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} \bar{\Gamma}(p)dp + \int_{b-\frac{\alpha_h R}{1-u}}^{\underline{\mathrm{p}}} \bar{\Gamma}(p)dp] \\
&= b - \phi N[\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} \bar{\Gamma}(p)dp + 1 \cdot (\underline{\mathrm{p}} - b + \frac{\alpha_h R}{1-u})] \\
&= b - \phi[\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} N\bar{\Gamma}(p)dp + N(\underline{\mathrm{p}} - b + \frac{\alpha_h R}{1-u})]
\end{aligned} \tag{B.53}$$

The derivative of (B.53) with respect to i reads

$$\begin{aligned}
\frac{\partial \underline{\mathrm{w}}}{\partial i} &= -\phi[\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} \frac{\partial N\bar{\Gamma}(p)}{\partial i}dp - N\bar{\Gamma}(\underline{\mathrm{p}})\frac{\partial \underline{\mathrm{p}}}{\partial i}] \\
&\quad -\phi[\frac{\partial N}{\partial i}(\underline{\mathrm{p}} - b + \frac{\alpha_h R}{1-u}) + N(\frac{\partial \underline{\mathrm{p}}}{\partial i} + \frac{\partial(\frac{\alpha_h R}{1-u})}{\partial i})] \\
&= -\phi[\int_{\underline{\mathrm{p}}}^{\bar{\mathrm{p}}} \frac{\partial N\bar{\Gamma}(p)}{\partial i}dp + \frac{\partial N}{\partial i}(\underline{\mathrm{p}} - b + \frac{\alpha_h R}{1-u}) + N\frac{\partial(\frac{\alpha_h R}{1-u})}{\partial i}]
\end{aligned} \tag{B.54}$$

When $y \in [\underline{\mathrm{p}},\bar{\mathrm{p}}]$, $\bar{\Gamma}(y)$ is determined in (2.61). Solving u out of (2.46) gets

$$u = \frac{\delta}{\delta + \lambda_h}. \tag{B.55}$$

Substituting u from (B.55) into (2.61) and then expressing λ_h with λN yields

$$\frac{\lambda\delta}{[\delta + \lambda N\bar{\Gamma}(y)]^2}[1 + \frac{\beta\lambda N\bar{\Gamma}(y)}{1-\beta(1-\delta)}] = cf'(y). \tag{B.56}$$

Observation of (B.56) implies that $N\bar{\Gamma}(y)$ is solely determined by (B.56) for any given $y \in [\underline{\mathrm{p}},\bar{\mathrm{p}}]$, i.e.,

$$\frac{\partial N\bar{\Gamma}(p)}{\partial i} = 0. \tag{B.57}$$

Plugging (B.57) into (B.54) yields

$$\frac{\partial \underline{\mathrm{w}}}{\partial i} = -\phi\frac{\partial N}{\partial i}(\underline{\mathrm{p}} - b + \frac{\alpha_h R}{1-u}) - \theta N\frac{\partial(\frac{\alpha_h R}{1-u})}{\partial i}. \tag{B.58}$$

The conclusion of $\frac{\partial u}{\partial i} > 0$ from Proposition 16 and (2.68) imply that

$$\frac{\partial N}{\partial i} < 0. \tag{B.59}$$

(B.43) implies that

$$\underline{\mathrm{p}} - b + \frac{\alpha_h R}{1-u} > 0 \tag{B.60}$$

always hold. From (B.51) (B.58) - (B.60), it follows that

$$\frac{\partial \underline{\mathrm{w}}}{\partial i} > 0$$

i.e., the lower bound of wage $\underline{\mathrm{w}}$ rises when the inflation rises. ■

Appendix C

Proofs for Chapter 3

C.1 Model Derivations

Timing

We believe that the understanding of the timing is crucial to follow the derivations. For time periods indexed by t, discounting periods indexed by j, and an implementation date T_p announced in $t = 1$ and $T \equiv T_p - t$ denoting the number of periods until T_p we got the following picture:

$$\begin{aligned} t &= 1, 2, 3, 4, 5, 6, ... \\ j &= 0, 1, 2, 3, 4, 5, ... \\ T \equiv T_p - t &= 4, 3, 2, 1, 0, -1, ..., \end{aligned}$$

thus for the infinite sum over index j

$$\sum_{j=1}^{T-1} \{\bullet\} + \sum_{j=T}^{\infty} \{\bullet\} \tag{C.1.1}$$

from period $t = 1$ perspective, given exemplary $T_p = 5$ on the line $1 \leq t \leq T_p - 1$, until $j = 3 = T - 1$ we have the old tax rate. Furthermore, on the line $t \geq T_p$ from $j = 4 = T$ onwards we have the new tax rate. Equivalently for the infinite sum

$$\sum_{j=0}^{T-2} \{\bullet\} + \sum_{j=T-1}^{\infty} \{\bullet\} \tag{C.1.2}$$

from period $t = 1$ perspective, given exemplary $T_p = 5$ on the line $1 \leq t \leq T_p - 1$, until $j = 2 = T - 2$ we have the old tax rate. Furthermore, on the line $t \geq T_p$ from $j = 3 = T - 1$ onwards we have the new tax. This allows us later on to replace T with $T_p - t$ for $1 \leq t \leq T_p - 1$ and $T - 1$ with 0 for $t \geq T_p$.

Derivation of ST_1

Here we want to illustrate the methodology we apply in all derivations under learning for the example of ST_1. Starting from

$$ST_1 = \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau^e_{t+j}(t)$$

we split this infinite sum into

$$ST_1 = \left[\sum_{j=1}^{T-1} \frac{1}{D^e_{t,t+j}(t)} \tau_0 + \sum_{j=T}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau_1 \right].$$

Next we go back to the definition of $D^e_{t,t+j}(t)$. Given the learning rules (3.16) and (3.17) we get

$$D^e_{t,t+j}(t) = \Pi^j_{i=1} \left[(1-\delta) + r^e(t)\right] = \left[(1-\delta) + r^e(t)\right]^j . \tag{C.1.3}$$

Consequently we get

$$ST_1 = \left[\sum_{j=1}^{T-1} \left(\left[(1-\delta) + r^e(t)\right]^{-1} \right)^j \tau_0 + \sum_{j=T}^{\infty} \left(\left[(1-\delta) + r^e(t)\right]^{-1} \right)^j \tau_1 \right],$$

or

$$\begin{aligned} ST_1 &= \left[(1-\delta) + r^e(t)\right]^{-1} \times \\ &\quad \left[\sum_{j=0}^{T-2} \left(\left[(1-\delta) + r^e(t)\right]^{-1} \right)^j \tau_0 + \sum_{j=T-1}^{\infty} \left(\left[(1-\delta) + r^e(t)\right]^{-1} \right)^j \tau_1 \right]. \end{aligned}$$

Given the property of a finite geometric series $\sum_{j=m}^{n} f^j = \frac{f^{n+1} - f^m}{f-1}$ for some constant f, we get

$$\begin{aligned} ST_1 &= \left[(1-\delta) + r^e(t)\right]^{-1} \times \\ &\quad \left[\left(\frac{\left[(1-\delta) + r^e(t)\right]^{1-T} - 1}{\left[(1-\delta) + r^e(t)\right]^{-1} - 1} \right) \tau_0 + \left(\frac{- \left[(1-\delta) + r^e(t)\right]^{1-T}}{\left[(1-\delta) + r^e(t)\right]^{-1} - 1} \right) \tau_1 \right], \end{aligned}$$

which can be rewritten as

$$ST_1 = \frac{\tau_0}{r^e(t) - \delta} + \frac{(\tau_1 - \tau_0)}{\left[(1-\delta) + r^e(t)\right]} \frac{\left[(1-\delta) + r^e(t)\right]^{1-T}}{1 - \left[(1-\delta) + r^e(t)\right]^{-1}}. \tag{C.1.4}$$

Now, considering the timing outlined in Appendix C.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.24)

$$ST_1 = \frac{\tau_0}{r^e(t) - \delta} + (\tau_1 - \tau_0)\, \frac{[(1-\delta) + r^e(t)]^{t-T_p}}{1 - [(1-\delta) + r^e(t)]^{-1}}, \tag{C.1.5}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.25)

$$ST_1 = \frac{\tau_1}{r^e(t) - \delta}. \tag{C.1.6}$$

Derivation of ST_2

Starting from

$$ST_2 = \sum_{j=1}^{\infty} \frac{1}{D^e_{t,t+j}(t)} \tau^{l,e}_{t+j}(t) w^e_{t+j}(t) \bar{L}$$

and given the learning rules (3.16) and (3.17) as well as (C.1.3) from above and $\tau^{l,e}_{t+j}$ being either τ^l_0 or τ^l_1, we may split the infinite sum above into

$$ST_2 = w^e(t)\bar{L} \left[\sum_{j=1}^{T-1} \left([(1-\delta) + r^e(t)]^j \right)^{-1} \tau^l_0 + \sum_{j=T}^{\infty} \left([(1-\delta) + r^e(t)]^j \right)^{-1} \tau^l_1 \right]$$

or

$$\begin{aligned} ST_2 &= \frac{\tau^l_0 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \sum_{j=0}^{T-2} \left([(1-\delta) + r^e(t)]^{-1} \right)^j \\ &\quad + \frac{\tau^l_1 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \sum_{j=T-1}^{\infty} \left([(1-\delta) + r^e(t)]^{-1} \right)^j . \end{aligned}$$

Now, as above, the properties of the geometric series allow us to rewrite this as

$$\begin{aligned} ST_2 &= \frac{\tau^l_0 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \left(\frac{[(1-\delta) + r^e(t)]^{1-T} - 1}{[(1-\delta) + r^e(t)]^{-1} - 1} \right) \\ &\quad + \frac{\tau^l_1 \; w^e(t)\bar{L}}{[(1-\delta) + r^e(t)]} \left(\frac{-[(1-\delta) + r^e(t)]^{1-T}}{[(1-\delta) + r^e(t)]^{-1} - 1} \right). \end{aligned}$$

For the timing outlined in Appendix C.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.31)

$$ST_2 = w^e(t)\bar{L} \left[\frac{\tau^l_0}{r^e(t) - \delta} + \left(\tau^l_1 - \tau^l_0 \right) \frac{[(1-\delta) + r^e(t)]^{t-T_p}}{1 - [(1-\delta) + r^e(t)]^{-1}} \right] \tag{C.1.7}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.32)

$$ST_2 = \frac{\tau^l_1 \; w^e(t)\bar{L}}{r^e(t) - \delta}. \tag{C.1.8}$$

Derivation of SW_2

We start from (3.39)

$$SW_2 = \sum_{j=1}^{\infty} \frac{1}{D^{k,e}_{t,t+j}(t)} w^e_{t+j}(t)\bar{L}.$$

Next, we recall the definition of $D^{k,e}_{t,t+j}(t)$. Given the learning rules (3.16) and (3.17) we get

$$D^{k,e}_{t,t+j}(t) = \Pi^j_{i=1}\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right] = \left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^j \tag{C.1.9}$$

for $\tau^{k,e}_{t+j}(t) = \tau^k_0$ and

$$D^{k,e}_{t,t+j}(t) = \Pi^j_{i=1}\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right] = \left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^j \tag{C.1.10}$$

for $\tau^{k,e}_{t+j}(t) = \tau^k_1$. Thereafter, we split this infinite sum into

$$\begin{aligned} SW_2 &= \bar{L}\left[\sum_{j=1}^{T-1} \frac{1}{D^{k,e}_{t,t+j}(t)} w^e(t) + \sum_{j=T}^{\infty} \frac{1}{D^{k,e}_{t,t+j}(t)} w^e(t)\right] \\ &= \bar{L}[\sum_{j=1}^{T-1}(\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^j)^{-1} w^e(t) + \\ &\quad \sum_{j=T}^{\infty}(\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^j)^{-1} w^e(t)\ ], \end{aligned}$$

or

$$\begin{aligned} SW_2 = &\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]} \sum_{j=0}^{T-2}\left(\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^{-1}\right)^j \\ &+\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]} \sum_{j=T-1}^{\infty}\left(\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^{-1}\right)^j. \end{aligned}$$

As in Section C.1 above, we exploit the properties of geometric series and derive

$$\begin{aligned} SW_2 = &\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]}\left(\frac{1-\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^{1-T}}{1-\left[(1-\delta)+(1-\tau^k_0)r^e(t)\right]^{-1}}\right) \\ &+\frac{w^e(t)\bar{L}}{\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]}\left(\frac{\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^{1-T}}{1-\left[(1-\delta)+(1-\tau^k_1)r^e(t)\right]^{-1}}\right). \end{aligned}$$

Now we get back to the timing outlined in Appendix C.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.40)

$$\begin{aligned} SW_2 &= \frac{w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t) - \delta]} + w^e(t)\bar{L} \times \\ & \left[\frac{[(1-\delta) + (1-\tau_1^k)r^e(t)]^{t-T_p}}{1 - [(1-\delta) + (1-\tau_1^k)r^e(t)]^{-1}} - \frac{[(1-\delta) + (1-\tau_0^k)r^e(t)]^{t-T_p}}{1 - [(1-\delta) + (1-\tau_0^k)r^e(t)]^{-1}} \right] \end{aligned} \tag{C.1.11}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.41)

$$SW_2 = \frac{w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t) - \delta]}. \tag{C.1.12}$$

Derivation of ST_3

Starting from (3.54)

$$ST_3 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^{l,e}(t) w_{t+j}^e(t)\bar{L}$$

for (C.1.9) and (C.1.10) and $\tau_{t+j}^{l,e}(t)$ is either given by τ_0^l or τ_1^l, we may once more split the infinite sum into

$$\begin{aligned} ST_3 &= w^e(t)\bar{L} \times \\ & [\sum_{j=1}^{T-1} \left(\left[(1-\delta) + (1-\tau_0^k)r^e(t) \right]^j \right)^{-1} \tau_0^l \\ & + \sum_{j=T}^{\infty} \left(\left[(1-\delta) + (1-\tau_1^k)r^e(t) \right]^j \right)^{-1} \tau_1^l \ ], \end{aligned}$$

or

$$\begin{aligned} ST_3 &= \frac{\tau_0^l \ w^e(t)\bar{L}}{\left[(1-\delta) + (1-\tau_0^k)r^e(t) \right]} \sum_{j=0}^{T-2} \left(\left[(1-\delta) + (1-\tau_0^k)r^e(t) \right]^{-1} \right)^j \\ & + \frac{\tau_1^l \ w^e(t)\bar{L}}{\left[(1-\delta) + (1-\tau_1^k)r^e(t) \right]} \sum_{j=T-1}^{\infty} \left(\left[(1-\delta) + (1-\tau_1^k)r^e(t) \right]^{-1} \right)^j. \end{aligned}$$

Now, the properties of the geometric series allow us to rewrite this as

$$\begin{aligned} ST_3 &= \frac{\tau_0^l \ w^e(t)\bar{L}}{\left[(1-\delta) + (1-\tau_0^k)r^e(t) \right]} \left(\frac{\left[(1-\delta) + (1-\tau_0^k)r^e(t) \right]^{1-T} - 1}{\left[(1-\delta) + (1-\tau_0^k)r^e(t) \right]^{-1} - 1} \right) \\ & + \frac{\tau_1^l \ w^e(t)\bar{L}}{\left[(1-\delta) + (1-\tau_1^k)r^e(t) \right]} \left(\frac{-\left[(1-\delta) + (1-\tau_1^k)r^e(t) \right]^{1-T}}{\left[(1-\delta) + (1-\tau_1^k)r^e(t) \right]^{-1} - 1} \right). \end{aligned}$$

For the timing outlined in Appendix C.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.55)

$$\begin{aligned} ST_3 &= \frac{\tau_0^l \; w^e(t)\bar{L}}{[(1-\tau_0^k)r^e(t)-\delta]} + w^e(t)\bar{L} \times \\ &\left[\frac{\tau_1^l \; [(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}} - \frac{\tau_0^l \; [(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\right] \end{aligned} \tag{C.1.13}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.56)

$$ST_3 = \frac{\tau_1^l \; w^e(t)\bar{L}}{[(1-\tau_1^k)r^e(t)-\delta]}. \tag{C.1.14}$$

Derivation of ST_4

Starting from (3.57)

$$ST_4 = \sum_{j=1}^{\infty} \frac{1}{D_{t,t+j}^{k,e}(t)} \tau_{t+j}^e(t)$$

given (C.1.9) and (C.1.10) are true and $\tau_{t+j}^e(t)$ is either τ_0 or τ_1, we again split the infinite sum into

$$\begin{aligned} ST_4 &= [\sum_{j=1}^{T-1} \left(\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^j\right)^{-1} \tau_0 \\ &+ \sum_{j=T}^{\infty} \left(\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^j\right)^{-1} \tau_1 \;], \end{aligned}$$

or

$$\begin{aligned} ST_4 &= \left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1} \left[\sum_{j=0}^{T-2} \left(\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1}\right)^j \tau_0\right] \\ &+ \left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1} \left[\sum_{j=T-1}^{\infty} \left(\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1}\right)^j \tau_1\right]. \end{aligned}$$

Given the properties of geometric series we can rewrite the latter as

$$\begin{aligned} ST_4 &= \left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1} \left(\frac{\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{1-T} - 1}{\left[(1-\delta)+(1-\tau_0^k)r^e(t)\right]^{-1} - 1}\tau_0\right) \\ &+ \left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1} \left(\frac{-\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{1-T}}{\left[(1-\delta)+(1-\tau_1^k)r^e(t)\right]^{-1} - 1}\tau_1\right). \end{aligned}$$

Now given the timing outlined in Appendix C.1 above, for $1 \leq t \leq T_p - 1$ we plug in $T_p - t$ for T and get (3.58)

$$\begin{aligned} ST_4 &= \frac{\tau_0}{[(1-\tau_0^k)r^e(t) - \delta]} \\ &\quad +[\frac{[(1-\delta)+(1-\tau_1^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_1^k)r^e(t)]^{-1}}\,\tau_1 \\ &\quad -\frac{[(1-\delta)+(1-\tau_0^k)r^e(t)]^{t-T_p}}{1-[(1-\delta)+(1-\tau_0^k)r^e(t)]^{-1}}\,\tau_0\,] \end{aligned} \tag{C.1.15}$$

and for $t \geq T_p$ we have $T - 1 = 0$, thus we get (3.59)

$$ST_4 = \frac{\tau_1}{[(1-\tau_1^k)r^e(t) - \delta]}. \tag{C.1.16}$$

C.2 Computing Welfare Changes

Comparative Statics

We follow the approach of Cooley and Hansen (1992, p.301) based on Lucas (1990). Their measure of welfare change for a given policy change is derived by solving

$$U_0 = \log[c_1(1+x^\bullet)] + \eta \log[1-n_1] \tag{C.2.1}$$

for x in our case.[1] U_0 is the utility a household obtains in the steady-state without any tax and c_1 and n_1 are the values of consumption and employment at the new steady-state after the tax change either under perfect foresight or learning. It follows that

$$x^\bullet = \frac{\exp(U_0)}{c_1(1-n_1)^\eta} - 1. \tag{C.2.2}$$

Thus, in general, we need to solve for x for the perfect foresight dynamics and another x^* for the dynamics under learning.[2] Given $x^\bullet$ we can calculate

$$\overline{\mathcal{W}} = \frac{\triangle C}{y_1} = \frac{x^\bullet c_1}{y_1}, \tag{C.2.3}$$

where $\triangle C$ is the restoration value of consumption, which in our case may be interpreted as the total change in consumption required to restore a household to the level of utility obtained under the allocation associated with zero taxes. y_1 is the level of output at the new steady-state.

[1] $x^\bullet$ is either x under perfect foresight or x^* under learning.

[2] Of course we are aware that this must yield the same $x = x^*$ both under perfect-foresight and under learning, but this number may be useful to compare different policy experiments.

Transition Measure

Again we follow the approach of Cooley and Hansen (1992, p.301) based on Lucas (1990). Their measure of welfare change accounting for transition given a policy change is derived by solving

$$\sum_{t=1}^{T} \beta^t \{\log[c_t(1+x^\bullet)] + \eta \log[1-n_t] - U_0\} = 0 \tag{C.2.4}$$

for x under perfect foresight and x^* under learning. T is the terminal period, c_t is period t consumption either under perfect foresight or learning and y_t is period t output either under perfect foresight or learning.

$$x^\bullet = \left[\frac{\exp\left(U_0\left[\beta^1 + ... + \beta^T\right]\right)}{\left(c_1^{\beta^1} ... \, c_T^{\beta^T}\right) \times \left[(1-n_1)^{\eta\beta^1} ... \, (1-n_T)^{\eta\beta^T}\right]}\right]^{\frac{1}{[\beta^1+...+\beta^T]}} - 1.$$

$$x^\bullet = \left[\frac{\exp\left(U_0 \sum_{t=1}^{T} \beta^t\right)}{\Pi_{t=1}^{T} c_t^{\beta^t} \times \Pi_{t=1}^{T} (1-n_t)^{\eta\beta^t}}\right]^{\frac{1}{\sum_{t=1}^{T}\beta^t}} - 1. \tag{C.2.5}$$

Given $x^\bullet$ we can calculate

$$\mathcal{W}^\bullet = \frac{\sum_{t=1}^{T} \beta^t \{xc_t\}}{\sum_{t=1}^{T} \beta^t \{y_t\}}, \tag{C.2.6}$$

which will be reported as $\mathcal{W}$ for the perfect foresight dynamics and as $\mathcal{W}^*$ for the dynamics under learning.

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Abstract

We set up macroeconomics models with labour market frictions to evaluate several public policies.

In the first model, job creation and job destruction are investigated in the presence of search frictions in both labour and goods markets as well as firm heterogeneity. We show that both the unemployment rate and the endogenous job destruction rate increase when the inflation rate rises. Our numerical exercises suggest that the destruction of lower productivity jobs and the creation of higher productivity jobs may be inefficiently low under the Friedman rule, which in turn causes the deviation of optimal long run monetary policy from the Friedman rule.

In the second model, we study the effect of inflation on the wage dispersions due to firm heterogeneity and on-the-job search, in the context of a labour market á la Postel-Vinay and Robin (*International Economic Review* 43, 2002) and micro-founded money demand. The productivity (distribution) of firms is first assumed to be exogenously given. We find that a rise of inflation diminishes the wage dispersion. We then allow the firms to adjust their productivity level by investment. We then find that a rise of inflation first makes firms' productivity less dispersed; and furthermore also diminishes the wage dispersion.

In the third model, we study the impact of anticipated fiscal policy changes in the Ramsey economy when agents form expectations of average wage and interest rate using adaptive learning. We extend the existing framework by distortionary taxes as well as elastic labour supply, which makes agents' decisions non-predetermined but more realistic. We detect that the dynamic responses to anticipated tax changes under learning have oscillatory behavior. Moreover, we demonstrate that this behavior can have important implications for the welfare consequences of fiscal reforms.

Zusammenfassung

In der Dissertation werden makroökonomische Modelle mit Arbeitsmarktsbeschränkung verwendet, um einige Public Policies zu evaluieren.

In dem ersten Modell werden Job Creation und Job Destruction in Anwesenheit von sowohl Search Friction als auch Unternehmensheterogenität im Arbeitsmarkt und Gütermarkt untersucht. Es wird gezeigt, dass nicht nur die Arbeitslosenquote sondern auch die Rate der endogenen Job Creation ansteigen, wenn die Inflationsrate zunimmt. Die numerischen Ergebnisse zeigen, dass die Zerstörung der Arbeitsplätze von geringerer Produktivität und die Schaffung der Arbeitsplätze von höherer Produktivität unter der Friedman Rule wirkungslos niedrig sein können. Das führt darüber hinaus die Deviation der optimalen Geldpolitik in der langen Frist herbei.

In dem zweiten Modell werden der Effekt der Inflation auf Lohnspreizung aufgrund der Unternehmensheterogenität und on-the-job Search im Rahmen eines Arbeitsmarktes à la Postel-Vinay und Robin (*International Economic Review* 43, 2002) und der mikrofundierten Geldnachfrage untersucht. Die Produktivität (Distribution) der Firmen ist zunächst als exogen angenommen. Es wird gefunden, dass ein Anstieg der Inflation die Lohnspreizung vermindert. Lassen danach die Firmen ihres Produktivitätsniveau durch Investition anpassen, macht ein Anstieg der Inflation zuerst die Produktivitätsniveau der Firmen weniger dispergiert (verteilt); weiterhin vermindert es auch die Lohnspreizung.

In dem dritten Modell wird die Auswirkung der Veränderungen der antizipierten Fiskalpolitik im Rahmen einer Ramsey Ökonomie analysiert, wo Agenten Erwartungen über durchschnittlichen Lohn und Zinssatz durch adaptives Lernen bilden. Der bestehende Rahmen wird mit verzerrenden Steuern und elastisches Arbeitsangebot erweitert, was die Entscheidungen der Agenten nicht vorher determiniert, sondern realistisch macht. Es wird festgestellt, dass die dynamischen Reaktionen auf antizipierte Veränderungen der Steuern unter Lernen einen oszillatorischen Charakter haben. Des Weiteren wird es demonstriert, dass dieser Charakter Wohlfahrtsauswirkung für Fiskalreform haben kann.

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