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“Algebraic Groups over the Adeles - Criteria for Compactness, Finite Invariant Volume and Properness”

Verfasser
Christoph Harrach

angestrebter akademischer Grad
Magister der Naturwissenschaften (Mag.rer.nat)

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Betreuer:	Univ.-Prof. Dr. Joachim Schwermer

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Introduction

A highly interesting field in algebraic number theory is to gain structural information about matrix groups over number fields and their arithmetic subgroups. After trying to handle these objects with pure group theory, number theorists went over to consider groups with some additional structure. As a result, the whole Lie theory was developed; however, the analytic topology and structure on Lie groups did not really fit for the geometric aspect of interest. Therefore, groups objects in the category of algebraic varieties were considered, called algebraic groups. In particular, it was tried to receive information about groups $G \subset \mathrm{GL}_n(\mathcal{O}_k)$, where k is an algebraic number field and $\mathcal{O}_k$ denotes the ring of integers in k. It is well known that $\mathcal{O}_k$ is a free $\mathbb{Z}$-module, which means that the ring of integers can be viewed as a lattice in k. Now let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a Zariski-closed subset of the general linear group over the complex numbers. Now the integral points of G, i.e., the intersection $G \cap \mathrm{GL}_n(\mathcal{O}_k)$, is again a discrete subgroup of G_k.

In particular we can ask which structural properties does $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ have in $\mathrm{SL}_2(\mathbb{Q})$. For that, the natural action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half plane of the complex plane $\mathcal{H}$ by Moebius transformations can be considered. It became clear that the analysis of special functions on $\mathcal{H}$ lead to important results. These special functions, called modular forms, are meromorphic on $\mathcal{H}$, invariant under Γ and satisfy a special growth condition. This theory is strongly related to many other mathematical and physical fields, for example to the theory of elliptic curves. Clearly it was tried to generalize these functions to arbitrary Lie-groups G and arbitrary discrete subgroups Γ of G, which led to the definition of an automorphic form F. Here, the function $F\colon G \to \mathbb{C}$ is a Γ-invariant function which is an eigenfunction of special operators and satisfies a special growth condition. Due to the invariance property, it suffices to define automorphic forms on the quotient G/Γ.

Another important tool, developed in the 30's in the last century due to Chevalley, Artin and Weil, which has been very fruitful in the last years, are the adeles and ideles of an algebraic number field. It is possible to obtain a lot of arithmetic properties of the ring of integers $\mathcal{O}_k$ out of these constructions. Clearly, the question arises if the adeles are also a useful tool in the analysis of automorphic forms. In fact, it was shown that the automorphic forms can be viewed as functions defined on the adelic points of an algebraic group G, where the role of Γ is taken by G_k. Again, from the invariance property we conclude that an automorphic form is uniquely determined on the quotient $G_{\mathbb{A}}/G_k$. Thus, the latter quotient became an object of interest.

This diploma thesis is dedicated to develop a reduction theory for G_k in $G_{\mathbb{A}}$, i.e., to construct fundamental domains or sets for $G_{\mathbb{A}}$ with respect to the discrete subgroup G_k and to find structural information about them. We will proceed in the following way. First, we will answer the question how we can construct fundamental sets and domains. For that, we define the term arithmetic subgroup and construct fundamental sets for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$. After that, we will define the "adelization" of algebraic k-varieties and k-rational morphism between them and start to analyse adelic algebraic groups. We try to find a fundamental set for $\mathrm{GL}_n(k)$ in $\mathrm{GL}_n(\mathbb{A})$, for which we use a height function. We will see that $\mathrm{GL}_n(\mathbb{A})$ can be decomposed in an orthogonal, diagonal and unipotent part modulo the subgroup $\mathrm{GL}_n(k)$. Then we handle the general case and show that every fundamental set is a product of a fundamental set Σ for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$ and a compact subset of the restricted product $\prod_{v\in V_k^f}(k_v : O_v)$, where V_k^f denote the finite places, k_v the completion of k with respect to v and O_v the valuation ring of k_v.

Afterwards, we try to find criteria for the quotient $G_{\mathbb{A}}/G_k$ to be compact, to have finite invariant volume respectively. For that, we construct a measure on algebraic groups which is invariant under left translations for every place $v \in V_k$. After that we obtain a measure on $G_{\mathbb{A}}/G_k$ simply by multiplying the local measures. As an example, let $G = \mathrm{SL}_2$. We will compute the volume $\mathrm{vol}(\mathrm{SL}_2(\mathbb{Q}_v)/\mathrm{SL}_2(\mathbb{Z}_v))$ for every $v \in V_k$, which equals $\frac{\pi^2}{6}$ in the real case and $\frac{6}{\pi^2}$ for the finite part. So we will see that $\mathrm{vol}(\mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{Q})) = 1$. Our main goal will be the proof of the following Theorem:

Theorem. *Let G be an algebraic k-group. Then*

(i) $G_{\mathbb{A}}/G_k$ is compact if and only if every unipotent element of G_k belongs to the unipotent radical of G and $X(G^0)_k = 1$.

(ii) $G_{\mathbb{A}}/G_k$ has finite invariant volume if and only if $X(G^0)_k = 1$.

In the next step we will consider the quotient $G_{\mathbb{A}}^{(1)}/G_k$, where we define $G_{\mathbb{A}}^{(1)}$ as

$$G_{\mathbb{A}}^{(1)} = \bigcap_{\chi\in X(G)_k} \ker(g \mapsto \prod_{v\in V_k} |\chi(g_v)|_v),$$

where the intersection ranges over all k-rational characters of G. We will show that this object has always finite invariant volume. At last, we consider injective morphisms $\varphi\colon H \hookrightarrow G$ between reductive algebraic k-groups and show the following Theorem:

Theorem. *Let G, H be reductive algebraic k-groups. Let $\varphi\colon H \hookrightarrow G$ be an injective k-morphism and let*

$$\varphi_{\mathbb{A}}^{(1)}\colon H_{\mathbb{A}}^{(1)}/H_k \to G_{\mathbb{A}}^{(1)}/G_k$$

be the map induced by φ. Then the map $\varphi^{(1)}$ is proper.

In chapter 1, the basics of the theory of algebraic groups are introduced. First, we quickly review the basic results of algebraic varieties over algebraically closed fields and state some descending theory. After that, we introduce algebraic k-groups and show that they are obtained as closed subgroups of $\mathrm{GL}_n(k)$. In the third subsection, we give some structural properties and define some important subclasses of algebraic groups, as reductive and semisimple groups. At last, we mention a useful tool, the restriction of scalars, which plays an essential role in this setting.

The second chapter is dedicated to review some facts about topology and measure theory. In the first subsection the term of a proper map is introduced and basic results are given. Afterwards we construct a topological space out of infinitely many others, called the restricted topological product. We will also see that this topological product is locally compact under certain assumptions, where we will construct a Haar mesure on this product. Moreover, we will construct Haar measures explicitly and compute the volume of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.

In the third chapter we first review some basic results about algebraic number fields and valuations. Furthermore, we give some properties of the completion k_v of an algebraic number field k with respect to the valuation v. Out of these completions, we will construct the adele ring $\mathbb{A}_k$ and the idele group $\mathbb{J}_k$ of k as a restricted topological product and introduce some reduction theory for k in $\mathbb{A}_k$, k^* in $\mathbb{J}_k$ respectively. In the last part we introduce the term of an arithmetic subgroup Γ of an algebraic $\mathbb{R}$-group $G \subset \mathrm{GL}_n(\mathbb{R})$. Furthermore, we will see which conditions have to be satisfied to guarantee the quotient $G_{\mathbb{R}}/G_{\mathbb{Z}}$ to be compact or to have finite invariant volume.

The fourth section, which forms the main part of this thesis, examines algebraic k-groups G, where k is an algebraic number field, and their adelic points. In the first subsection we will associate to every k-variety X an adelic variety $X_{\mathbb{A}}$, i.e., a special subset of $\mathbb{A}_k^n$. Furthermore, we will "adelize" algebraic groups and give some basic properties about them. In particular, we will introduce the terms class number and strong approximation property. Afterwards, we try to construct a fundamental set for $\mathrm{GL}_n(\mathbb{Q})$ in $\mathrm{GL}_n(\mathbb{A})$, called Minkowski reduction. For that, we first decompose the general linear group at every place v and then obtain a general decomposition out of the local ones. After that, we improve the obtained result by using a height function on $\mathrm{GL}_n(\mathbb{A})$. In the third section, the case of an arbitrary group is considered, i.e., we try to find fundamental sets for $G_{\mathbb{A}}$ with respect to G_k. We will see that it suffices to focus on the case of an reductive group, where the desired set is obtained from that of $\mathrm{GL}_n(\mathbb{A})$. Furthermore, we investigate which conditions on G do we need that the quotient $G_{\mathbb{A}}/G_k$ is compact, have finite invariant volume respectively. After that, we consider the quotient $G_{\mathbb{A}}^{(1)}/G_k$, where $G_{\mathbb{A}}^{(1)}$ is the intersection of the kernels of all characters of G, and try to answer the same question.

In the last subsection we consider injective morphisms between reductive k-groups G, H and try to obtain a connection between the compactness of $G_{\mathbb{A}}^{(1)}/G_k$ and $H_{\mathbb{A}}^{(1)}/H_k$. Let $i\colon H \hookrightarrow G$ be such an injective morphism. First we will prove that the adelization $i_{\mathbb{A}}^{(1)}\colon H_{\mathbb{A}}^{(1)}/H_k \to G_{\mathbb{A}}^{(1)}/G_k$ of i remains injective. Our main goal is to show that the map $i_{\mathbb{A}}^{(1)}$ is proper. For that, we generalize the proof for the real case of Schwermer in [16, ch. 6.1] to the adeles.

At this point I want to thank Professor Schwermer for pointing out this interesting field of study to me and for granting me the utmost intellectual freedom and academic support throughout my studies.

1 Algebraic groups

As an initial step, we want to review some background material in the theory of algebraic k-groups, where k is a field. Although some of the results are essential in the later parts of this work, we will omit the proofs. More information about algebraic geometry can be found in [8, ch. 1], the main references for the theory of algebraic groups are [9] and [15].

In this first section a brief introduction to the theory of algebraic groups is given. The basic definitions and important results are stated.

1.1 Introduction to algebraic groups

Let Ω be an algebraically closed field. We denote by $\mathbf{A}^n_\Omega$ (or sometimes just $\mathbf{A}^n$) the n-dimensional affine Ω-space endowed with the Zariski topology. Let $\Omega[x_1, \ldots, x_n]$ be the polynomial ring in n variables. Let $X \subset \mathbf{A}^n_\Omega$ be an algebraic variety. We denote by $\Omega[X]$ the affine algebra of X. To every $a = (a_1, \ldots, a_n) \in X$ corresponds the maximal ideal $m_a = (x_1 - a_1, \ldots, x_n - a_n) \subset \Omega[X]$. Moreover, the stalk near a is denoted by $\mathcal{O}_a$, which is a local ring with unique maximal ideal $\mathfrak{m}_a$. Let $\varphi\colon X \to Y$ be a morphism between two algebraic varieties. Then we denote the comorphism of φ by $\varphi^*\colon \Omega[Y] \to \Omega[X]$.

Now let $k \subset \Omega$ be a subfield of Ω, let X be a Ω-variety. For an ideal $\mathfrak{a} \subset \Omega[X]$ we define

$$\mathfrak{a}_k := k[x_1, \ldots, x_n] \cap \mathfrak{a}.$$

If $\mathfrak{a} = \Omega[X]\mathfrak{a}_k$, then we say that $\mathfrak{a}$ is defined over k or $\mathfrak{a}$ is a k-ideal. For every k-ideal $\mathfrak{a}$ we can find generators $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$. A point $x \in X$ is said to be defined over k and will be called a k-point if the maximal ideal $m_x \subset \Omega[X]$ corresponding to x is defined over k. Moreover, a variety $X \subset \mathbf{A}^n_\Omega$ is said to be defined over k or a k-variety if $\Omega[X]\mathcal{I}(X)_k = \mathcal{I}(X)$. Now let X, Y be k-varieties, $\varphi\colon X \to Y$ be a morphism of varieties. Then we say that φ is defined over k if there are polynomials $f_1, \ldots, f_n \in k[X]$ with $\varphi = (f_1, \ldots, f_n)$. For a k-morphism between k-varieties we obtain that k-points are mapped to k-points.

Let X be a k-variety, let $x \in X$. Then a k-derivation $\delta\colon \mathcal{O}_x \to \Omega$ is a k-linear map from the stalk near x to Ω which satisfies

$$\delta(fg) = \delta(f)g(x) + f(x)\delta(g)$$

for all f, $g \in \mathcal{O}_x$. The set of all k-derivations of $\mathcal{O}_x$ form a k-vector space, called the tangent space $\mathcal{T}(X)_x$ of X in x. It can be shown that $\mathcal{T}(X)_x$ is canonically isomorphic to the dual space of $\mathfrak{m}_x/\mathfrak{m}_x^2$ viewed as a $\mathcal{O}_x/\mathfrak{m}_x$-vector space. We call an element $\xi \in \mathcal{T}(X)_x$ an tangent vector. Now let $\varphi\colon X \to Y$ be a k-morphism between k-varieties and let $x \in X$ and $y = \varphi(x) \in Y$. Then the comorphism φ^* induces a k-linear map $\mathrm{d}\,\varphi_x\colon \mathcal{T}(X)_x \to \mathcal{T}(Y)_y$, called the differential of φ at x.

Let G denote an affine algebraic group defined over k with multiplication $\mu\colon G \times G \to G$, inversion $\nu\colon G \to G$ and neutral element $e \in G_k$ For all g, $h \in G$ we will write gh (resp. $g + h$ in the commutative case) instead of $\mu(g, h)$ and also g^{-1} (resp. $-g$) instead of $\nu(g)$. Since only affine algebraic groups are considered in this thesis, the word "affine" will often be omitted.

Since an algebraic variety is uniquely determined by its affine algebra, we shall express the structure of an algebraic group G in terms of $k[G]$. We obtain induced comorphisms

$$\mu^*\colon k[G] \to k[G] \otimes k[G]$$
$$\nu^*\colon k[G] \to k[G]$$
$$e^*\colon k[G] \to k$$

which satisfy the dual group axioms, i.e., which make $k[G]$ into a Hopf algebra.

Let G be an algebraic k-group. Then we denote by G^0 the connected component of the identity. Moreover, for $g \in G$, let l_g be the left multiplication $h \mapsto \mu(g,h)$ with g, which is an automorphism of G. Analogously, the right multiplication with g is denoted by r_g.

As a major example we consider the general linear group GL_n as algebraic k-group for any $n \in \mathbb{N}$. It is realized as the zero set of the polynomial $\det(x_{ij})y - 1 \in k[x_{11}, \ldots, x_{nn}, y]$ in the $n^2 + 1$ dimensional affine space; the map μ (resp. ν) is the usual matrix multiplication (resp. inversion), with the unit matrix as the neutral element. For $n = 1$, we write G_m instead of GL_1 and call it the multiplicative group.

From the general linear group we can deduce many examples of algebraic groups by using the following Lemma:

Lemma 1.1. *Let G be an algebraic k-group, $H \subset G$ a k-closed subgroup. Then H is also an algebraic k-group.*

Proof. Since a k-closed subset of a k-variety is again a k-variety and H is a subgroup we obtain that H is also an algebraic k-group. □

Using the lemma, we can identify various classical matrix groups as algebraic groups, for example

(1) $\mathrm{SL}_n(k) = \{g \in \mathrm{GL}_n(k) \mid \det(g) = 1\}$

(2) $\mathrm{O}_n(k) = \{g \in \mathrm{GL}_n(k) \mid g^t = g^{-1}\}$

(3) $\mathrm{SO}_n(k) = \{g \in \mathrm{O}_n(k) \mid \det(g) = 1\}$

In fact, we obtain all algebraic k-groups in this way. This result is stated in the following Theorem, which is often called the Theorem of Chevalley.

Theorem 1.2. *Let G be an algebraic group defined over k. Then G is isomorphic to a k-closed subgroup of some $\mathrm{GL}_n(k)$.*

Proof. [9, 8.6] □

Although the last theorem could be used to reduce the study of algebraic groups completely to matrix groups, it is usually preferred to stay in the general context since the independence of the representation is complicated to handle. However, this theorem has a big technical value, because it allows to use special properties of matrices (e.g. the behaviour of Eigenvalues) for proofs in the setting of algebraic groups.

Now let us return to the general case. Let X be a k-variety, let G be an algebraic k-group which acts on X via the map $\varphi\colon G \times X \to X$. Usually, we will write $g.x$ (or simply gx if no confusion is possible) for $\varphi(g,x)$ for all $x \in X$, $g \in G$. The orbit of $x \in X$ under the action of G is denoted by Gx and its stabilizer by G_x. We obtain an isomorphism $G/G_x \to Gx$ of sets. In particular, for a k-group G and a normal k-subgroup H of G the set of cosets G/H can be endowed with the structure of an affine variety which makes G/H into an algebraic k-group.

Let G be an algebraic k-group which acts on an algebraic k-variety X. Then for all $g \in G$ let τ_g denote the comorphism corresponding to the morphism $X \to X$, $x \mapsto g.x$. We call τ_g the translation of functions by g. The maps τ_g, $g \in G$ induce a group homomorphism

$$\begin{aligned} \tau\colon G &\to GL(k[X]) \\ g &\mapsto \tau_g. \end{aligned}$$

In particular, the group G acts on itself by left translation. Then for $g \in G$, the induced representation ρ_g on $k[G]$ is called the right translation of functions by g.

Let G be a k-group which acts on a k-variety X. Then the affine algebra $k[X]$ of X consists completely of finite dimensional, translation-invariant subspaces, which is stated in the following Proposition.

Proposition 1.3. *Let G be an algebraic k-group acting on an affine $k-$variety X, let τ denote the induced comorphism of the action defined as above. Let F be a finite dimensional subspace of $k[X]$. Then there exists a finite dimensional subspace E of $k[X]$ including F which is stable under τ_g for all $g \in G$.*

Proof. [9, 8.6] □

Another important tool for the analysis of algebraic groups is the notion of a character. A k-character χ of G is a k-morphism $\chi\colon G \to G_m$ of algebraic k-groups. Let χ_1, χ_2 be k-characters of G. Then the morphisms

$$\chi_1 + \chi_2\colon g \mapsto \chi_1(g) \cdot \chi_2(g) \text{ and } -\chi_1\colon g \mapsto \chi_1(g)^{-1},$$

where multiplication and inversion are those of G_m, are again k-characters. Using this morphism we can endow the set $X(G)_k$ of all k-characters with a structure of an abelian group.

Now let G be an algebraic k-group, let $\varphi\colon G \to \mathrm{GL}_n(k)$ be a representation of G as a k-closed subgroup of some $\mathrm{GL}_n(k)$. Then we call an element $g \in G$ unipotent (respectively semisimple) if $\varphi(g)$ has this property. We can decompose every $g \in G$ into an unipotent and a semisimple part, which is stated in the following Theorem.

Theorem 1.4. *Let G be an algebraic group.*

(i) If $g \in G$, there exists unique elements g_s, $g_u \in G$ such that g_s is semisimple, g_u is unipotent and $g = g_s g_u = g_u g_s$.

(ii) If $\varphi\colon G \to G'$ is a morphism of algebraic groups then $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$.

Proof. [9, 15.3] □

We call g_s (resp. g_u) the semisimple (resp. unipotent) part of g. Let G_s (resp. G_u) denote the k-subgroup of G which consists of all semisimple (resp. unipotent) elements of G. We call an algebraic group G unipotent if $G = G_u$. The subgroup G_u is always closed, however, G_s is not closed in general. Moreover, if G is connected, so are G_s and G_u.

Let G be an algebraic k-group. Since the product of two closed solvable subgroups is again closed and solvable, we obtain the existence of a unique largest normal solvable subgroup, whose identity component is called the radical $R(G)$ of G. Moreover, the unipotent part of $R(G)$ is called the unipotent radical $R_u(G)$ of G. Both of these subgroups are defined over k. We call an algebraic k-group G semisimple (resp. reductive) if $R(G)$ (resp. $R_u(G)$) is trivial.

The following Proposition shows that it is often sufficient to study only unipotent and reductive groups.

Proposition 1.5. *Let G be a connected algebraic k-group, where k is a field of characteristic 0. Then there exists a reductive k-subgroup $H \subset G$ such that G can be written as semidirect product*

$$G = HR_u(G).$$

Moreover, any reductive k-subgroup $H' \subset G$ is conjugate by an element of $R_u(G)_k$ to a subgroup of H.

Proof. This result was proven by Mostow in [14]. □

The decomposition obtained in the previous Proposition is called the Levi-decomposition. In the next section we want to focus on reductive groups.

1.2 Reductive groups

The main example for a reductive group is the full general linear group. As a first result, we state the Iwasawa decomposition.

Proposition 1.6. *Let A denote the group of diagonal matrices with positive entries in $\mathbb{R}$ and U the group of real upper triangular unipotent matrices. Let $\mathrm{K} = \mathrm{O}_n(\mathbb{R})$. Then the natural map $\varphi\colon \mathrm{K} \times \mathrm{A} \times \mathrm{U} \to \mathrm{GL}_n(\mathbb{R})$, $\varphi(k, a, u) = kau$, is a homeomorphism of real groups.*

Proof. [5, ch. VII, §3, Prop.7] □

We can state another decomposition for the general linear group from which the existence of a maximal compact subgroup of reductive groups follows.

Proposition 1.7. *Let $\mathrm{S} \subset \mathrm{GL}_n(\mathbb{R})$ denote the set of positive definite symmetric matrices of dimension n and let $\mathrm{K} = \mathrm{O}_n(\mathbb{R})$. Then we have*

$$\mathrm{GL}_n(\mathbb{R}) = \mathrm{K}\,\mathrm{S},$$

and for each matrix $g \in \mathrm{GL}_n(\mathbb{R})$ its factorisation $g = ks$ with $k \in \mathrm{K}$ and $s \in \mathrm{S}$ is unique.

Proof. [15, Prop. 3.7] □

Proposition 1.8. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a reductive algebraic $\mathbb{R}$-group, let K and S be as before. Then we obtain* *(i)* $G_{\mathbb{R}} = (G \cap \mathrm{K})(G \cap \mathrm{S})$. *Furthermore,* $G \cap \mathrm{K}$ *is a maximal compact subgroup of* G.

(ii) Any compact subgroup of $G_{\mathbb{R}}$ *is contained in a maximal compact subgroup, and all maximal compact subgroups are conjugate.*

Proof. [15, Prop. 3.10] □

We need another result for reductive groups. It allows us in many cases to consider self-adjoint groups.

Definition. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be an algebraic $\mathbb{Q}$-group. For an element $g \in G_{\mathbb{R}}$ let g^t denote the transpose matrix of g. Then we call G self adjoint if $g^t \in G_{\mathbb{R}}$ for all $g \in G_{\mathbb{R}}$.

Proposition 1.9. *Let* $G \subset \mathrm{GL}_n(\mathbb{C})$ *be a reductive* $\mathbb{R}$*-group. There exists* $a \in GL_n(\mathbb{R})$ *such that* $H_{\mathbb{R}} = aG_{\mathbb{R}}a^{-1}$ *is self adjoint.*

Proof. [15, Thm 3.7] □

We have stated in chapter 1.1 that we obtain every algebraic k-group as a subgroup of the general linear group. For reductive groups we can proof a stronger version of this result. The following discussion follows [1, §7.6].

Let G be a connected group over k, where k is a field with characteristic 0. If G is reductive, then this is equivalent to the complete reducibility of G-representations, which is a fact following from the theory of reductive Lie algebras. Now let H be a reductive k-subgroup of G. Let Ω be an algebraic closure of k. Then the group G acts on the subspace $\Omega[G]^H$ of $\Omega[G]$ consisting of all rational functions $f \in \Omega[G]$ which are right-invariant under ρ_h for all $h \in H$, where ρ_h again denotes the right-translation of functions. We want to deduce a representation of G with useful properties out of that action. For that, we need to analyse the ring $I = \Omega[G]^H$ a bit further.

Lemma 1.10. *Let* G *be a reductive* k*-group, which acts on a* k*-vector space* W*, viewed as an affine* k*-space. Let* $X \subset W$ *be an irreducible subvariety of* W *which is stable under the* G*-action. Let* I *be the ring* $\Omega[X]^G$ *of rational function on* X *which are* G*-invariant. Then*

(i) there is a projection $\pi\colon \Omega[X] \to I$ *which is* I*-linear and stabilizes every* G*-invariant subspace of* $\Omega[X]$.

(ii) the ring I *separates the closed* G*-invariant subsets of* X.

(iii) the ring I *is finitely generated as an* Ω*-algebra.*

Proof. (i) Let N be the sum of all minimal subspaces of $\Omega[X]$ on which G does not operate trivially. Since G is reductive, the action of G on $\Omega[X]$ is completely reducible, so we can write $\Omega[X] = I \oplus N$. We now define the map $\pi\colon \Omega[X] \to I$ as the projection on I with respect to this decomposition. First, we show that π is I-linear. We have $I \cap N = \{0\}$, and since $1 \in I$, we obtain

$$I^2 \cap IN = I \cap IN = \{0\}.$$

From the complete reducibility and the above decomposition we conclude that $IN \subset N$. Now let $\varphi \in I$ and $\psi \in \Omega[X]$. Let $\psi = \psi_I + \psi_N$ be the decomposition with respect to the above direct sum. It follows that

$$\begin{aligned}\pi(\varphi \cdot \psi) &= \pi(\varphi \cdot \psi_I + \varphi \cdot \psi_N) \\ &= \pi(\varphi \cdot \psi_I) + \pi(\varphi \cdot \psi_N) \\ &= \varphi \cdot \psi_I \\ &= \varphi \cdot \pi(\psi),\end{aligned}$$

so π is I-linear. Now let $E \subset \Omega[X]$ be a G-stable subspace of $\Omega[X]$. Then we can write

$$E = (E \cap I) \oplus (E \cap N),$$

which shows that

$$\pi(E) = E \cap I \subset E.$$

(ii) Let A and B be two G-stable algebraic subsets of X with $A \cap B = \emptyset$. Let $\mathcal{I}(A)$ and $\mathcal{I}(B)$ the vanishing ideals of A respectively B in $\Omega[X]$. Since the two subsets are disjoint, we have $\mathcal{I}(A) + \mathcal{I}(B) = \Omega[X]$. Therefore, we can find elements $\alpha \in \mathcal{I}(A)$ and $\beta \in \mathcal{I}(B)$ with $\alpha + \beta = 1$. Since we have $1 \in I$, we obtain

$$1 = \pi(1) = \pi(\alpha) + \pi(\beta).$$

From the definition of the action of G on $\Omega[X]$ we conclude that also the ideals $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are G-stable. By (i) this implies that $\pi(\alpha) \in \mathcal{I}(A)$, $\pi(\beta) \in \mathcal{I}(A)$ and $\pi(\alpha)|_B = \pi(\beta)|_A = 0$.

(iii) We have an isomorphism $\Omega[X] \cong \Omega[W]/\mathcal{I}(X)$ of Ω-algebras, where $\mathcal{I}(X)$ denotes the vanishing ideal of X. Let $p\colon \Omega[W] \to \Omega[X]$ denote the canonical projection. Then for all $g \in G$, $f \in \Omega[W]$ and $x \in X$ we obtain

$$\rho_g(p(f)) = \rho_g(f + \mathcal{I}(X)) = \rho_g(f) + \rho_g(\mathcal{I}(X)) = \rho_g(f) + \mathcal{I}(X) = p(\rho_g(f)),$$

so p commutes with the G-action. Now let M (resp. N) be the sum of all minimal subspaces of $\Omega[W]$ (resp. $\Omega[X]$) on which G operates non-trivially. Then from complete reducibility we obtain that $\Omega[W] = \Omega[W]^G \oplus M$ and $\Omega[X] = I \oplus N$. This yields

$$I \oplus N = \Omega[X] = p(\Omega[W]) = p(\Omega[W]^G \oplus M) = p(\Omega[W]^G) \oplus p(M).$$

Since the G-action commutes with the map p, we obtain

$$p(\Omega[W]^G) = I.$$

Thus, it suffices to prove part (iii) in the case $X = W$. Let

$$J = \{f \in I \mid f(0) = 0\}.$$

Then J is an ideal in I generated by homogeneous, G-invariant polynomials. Now consider the ideal $\overline{J}$ of $\Omega[W]$ generated by J. Since $\Omega[W]$ is a Noetherian ring, we can find finitely many homogeneous polynomials $f_1, \ldots, f_s \in J$ with $\overline{J} = (f_1, \ldots, f_s)$. We now want to show that I is generated by $(f_1, \ldots, f_s)$ as an Ω-algebra. Since every polynomial is a linear combination of its homogeneous parts, it suffices to prove the assertion for homogeneous polynomials $f \in I$ with $\deg(f) > 0$. Now let $f \in I$ a homogeneous polynomial with $\deg(f) > 0$. Since $f(0) = 0$, we have $f \in J$, so $f \in \overline{J}$. Thus we can write

$$f = \sum_{i=1}^{s} a_i f_i$$

with $a_i \in \Omega[W]$. Since f and all f_i are homogeneous, so is a_i for all $i = 1, \dots, s$. Moreover, we have

$$\deg(f) = \deg(\sum_{i=1}^{s} a_i f_i) = \sum_{i=1}^{s} \deg(a_i f_i) = \sum_{i=1}^{s} (\deg(a_i) + \deg(f_i)),$$

and since f is homogeneous we obtain $\deg(f) = \deg(a_i) + \deg(f_i)$ for all $i = 1, \dots, s$. Thus, $\deg(a_i) = \deg(f) - \deg(f_i)$ for all $i = 1, \dots, s$ with $a_i \neq 0$. Now we use induction on $n = \deg(f)$. For $n = 1$ and $a_i \neq 0$, we have

$$\deg(a_i) = \deg(f) - \deg(f_i) = 1 - 1 = 0,$$

thus $a_i \in \Omega$.

If $n > 1$, then we can use the mapping π from part (i) to obtain

$$f = \pi(f) = \pi\left(\sum_{i=1}^{s} a_i f_i\right) = \sum_{i=1}^{s} \pi(a_i f_i) = \sum_{i=1}^{s} \pi(a_i) f_i,$$

so we can choose the polynomials a_i to lie in I. Moreover,

$$\deg(a_i) < \deg(f),$$

so from the induction hypothesis we get that $a_i \in (f_1, \dots, f_n)$ for all $i = 1, \dots, s$. Thus,

$$f = \sum_{i=1}^{s} a_i f_i \in (f_1, \dots, f_n). \qquad \square$$

Now we are able to prove a stronger version of the Theorem of Chevalley for connected reductive groups which will be useful at the end of this thesis.

Proposition 1.11. *Let G be a connected k-group, let H be a reductive k-subgroup of G. Then there is a finite dimensional k-vector space W, viewed as an affine space, an element $w \in W_k$ and a k-rational representation $\rho\colon W \times G \to W$ so that the orbit wG of w is closed in W and the stabilizer of w is exactly H.*

Proof. The proof follows [1, Prop. 7.7]. The group H operates from the right on the variety $G \subset \mathrm{GL}(V)$. From Lemma 1.10(iii) we know that the functions $I = \Omega[G]^H$ which are constant on the right cosets Hx of G modulo H are finitely generated as an Ω-algebra. Thus we can find generators $w_1, \dots, w_s \in \Omega[G]$ for I. Since G is a k-group, we can assume that the w_i are in $k[G]$ for $i = 1, \dots, s$. From Proposition 1.3 we conclude that we can find finite dimensional G-stable subspaces W_i of $\Omega[G]$ with $w_i \in W_i$ for $i = 1, \ldots, s$. Moreover, these subspaces are defined over k. We define the vector space W to be

$$W = \bigoplus_{i=1}^{s} W_i,$$

endowed with the G-action

$$\begin{aligned} W \times G &\to W \\ (v_1, \ldots, v_s) &\mapsto (\rho_g(v_1), \ldots, \rho_g(v_s)), \end{aligned}$$

and let $w = (w_1, \ldots, w_s) \in W_k$.

We first show that the isotropy group G_w of w is exactly H. For that, let $h \in H$. Since $w_i \in I$, we have $\rho_h(w_i) = w_i$ for all $i = 1, \ldots, s$, which implies $\rho_h(w) = w$. This shows that $H \subset G_w$. Conversely, let $h \in G_w$. This implies that $\rho_h(w_i) = w_i$ for all $i = 1, \ldots, s$. From the definition of the right translation this yields $w_i(h) = \rho_h(w_i)(e) = w_i(e)$ for $i = 1, \ldots, s$. Now I is generated by the w_i as an Ω-algebra, thus $f(h) = f(e)$ for all $f \in I$ and all $h \in G_w$. We have seen in Lemma 1.10(ii) that I separates the closed subsets of G which are H-stable, so in particular the right cosets of G modulo H. Thus, $G_w \subset H$.

Now it suffices to prove that the orbit $X = wG$ of w is closed in W. For that, let $\varphi\colon G \to W$ be the orbit map of w defined by $g \mapsto \rho_g(w)$. Then the corresponding comorphism φ^* has the form

$$\begin{aligned} \varphi^*\colon \Omega[\overline{X}] &\to \Omega[G] \\ f &\mapsto \varphi^*(f)\,(:g \mapsto f(\rho_g(w)))\,. \end{aligned}$$

The inclusion $H \subset G_w$ yields

$$\rho_h(\varphi^*(f))(g) = f(\rho_g(\rho_h(w))) = f(\rho_g(w)) = (\varphi^*(f))(g)$$

for every $f \in \Omega[\overline{X}]$, thus $\varphi^*(\Omega[\overline{X}]) \subset I$. We want to show that the comorphism φ^* induces an isomorphism of Ω-algebras

$$\varphi^*\colon \Omega[\overline{X}] \to I.$$

Since $\ker(\varphi^*) = \{0\}$, the map φ^* is injective. Now for a fixed $i \in \{1, \ldots, s\}$ let $\{z_1, \ldots, z_n\}$ be a base for W_i and let $\{a_1, \ldots, a_n\}$ be the corresponding dual base. Then we define rational functions $u_j \in \Omega[W]$, $j = 1, \ldots, n$ via

$$\begin{aligned} u_j\colon W &\to \Omega \\ (v_1, \ldots, v_n) &\mapsto a_j(v_i). \end{aligned}$$

Then we obtain for every $g \in G$ that

$$w_i(g) = (\rho_g(w_i))(e) = \sum_{j=1}^{n} a_j(\rho_g(w_i)) z_j(e) = \sum_{j=1}^{n} z_j(e)(\varphi^*(u_j))(g),$$

thus $w_i = \sum_{j=1}^{n} z_j(e)\varphi^*(u_j)$. So φ^* is an isomorphism from $\Omega[\overline{X}]$ to I.

To get the equality $X = \overline{X}$ we prove that for every $x \in \overline{X}$ the corresponding maximal ideal $m_x \in \Omega[\overline{X}]$ has a zero $y \in X$. Since φ^* is an isomorphism, this is equivalent to the existence of a zero $g \in G$ of $\varphi^*(m_x)$. For that, it suffices to prove that for every proper ideal $\mathfrak{a} \subset \Omega[\overline{X}]$ the ideal $\varphi^*(\mathfrak{a})\Omega[G]$ in $\Omega[G]$ is proper. If not, then there are elements $a_1, \dots, a_t \in \mathfrak{a}$ and $f_1, \dots, f_t \in \Omega[G]$ so that

$$\sum_{i=1}^{t} \varphi^*(a_i) f_i = 1.$$

Using the projection π from Lemma 1.10(i) and the inclusion $\varphi^*(\Omega[\overline{X}]) \subset I$ we obtain that

$$1 = \pi(1) = \pi(\sum_{i=1}^{t} \varphi^*(a_i) f_i) = \sum_{i=1}^{t} \pi(\varphi^*(a_i) f_i) = \sum_{i=1}^{t} \varphi^*(a_i) \pi(f_i).$$

Now φ^* is invertible, so

$$1 = \sum_{i=1}^{t} a_i((\varphi^*)^{-1} \circ \pi)(f_i),$$

which contradicts the properness of $\mathfrak{a}$. So X is closed in W. □

1.3 Restriction of scalars

In the theory of algebraic groups we often can restrict ourselves to the case of a closed subgroup of $\mathrm{GL}_n(k)$ for some k; however, the arbitrariness of the field of definition can be very hard to handle. Now we introduce a powerful tool which often allows us to consider algebraic groups defined over $\mathbb{Q}$. This is called the restriction of scalars.

Let k be an algebraic number field, let Ω be an algebraic closed field containing k. Let $G \subset \mathrm{GL}_n(\Omega)$ be an algebraic group defined over a finite separable extension l of k. We now want to construct an algebraic group G' whose k-points G'_k are naturally isomorphic to the l-points G_l of G. This can be done by considering the group $\mathbf{R}_{l/k}(G)$, the group which we obtain from G by restricting scalars from l to k. The construction of this group is now given; it can be found in [15, 2.1.2].

Choose a base $\theta_1, \dots, \theta_n$ of l over k and consider the left regular representation

$$\begin{aligned} \rho\colon l &\to \mathrm{M}_d(k) \\ x &\mapsto l_x(\colon y \mapsto xy) \end{aligned}$$

with respect to the chosen base.

Now let $P_\lambda(x_{ij})$ for $\lambda = 1, \dots, m$ be a finite set of generators of $\mathfrak{a}_l$, where $\mathfrak{a} \subset \Omega[x_{11}, \dots, x_{nn}, \det(x_{ij})^{-1}]$ is the ideal of functions vanishing on G, and let P_λ have the form

$$P_\lambda = \sum a_{\gamma_{11} \dots \gamma_{nn}} x_{11}^{\gamma_{11}} \dots x_{nn}^{\gamma_{nn}}.$$

We can associate to each polynomial the "matrix" polynomial

$$\tilde{P}_\lambda(y_{ij}) = \sum \rho(a_{\gamma_{11} \dots \gamma_{nn}}) y_{11}^{\gamma_{11}} \dots y_{nn}^{\gamma_{nn}},$$

where $y_{ij} = (y_{ij}^{\alpha\beta}), \alpha, \beta = 1, \dots, d$, are matrices of $\mathrm{M}_d(k)$, under the identification $\mathrm{M}_n(\mathrm{M}_d(k)) \cong \mathrm{M}_{nd}(k)$ given by

$$((y)_{ij})_{\alpha,\beta} \mapsto (y)_{(i-1)d+\alpha,(j-1)d+\beta}.$$

Furthermore, let $F_\mu(y^{\alpha\beta}) = 0$, $\alpha, \beta = 1, \dots, d$, $\mu = 1, \dots, r$ be a system of linear equations. Then we obtain that the image of G_l in $\mathrm{M}_{nd}(k)$ under ρ is defined by the equations

$$F_\mu(y_{ij}^{\alpha\beta}) = 0,\ i,j = 1, \dots, n,\ \mu = 1, \dots, r$$
$$\tilde{P}_\lambda(y_{ij}^{\alpha\beta}) = 0,\ \lambda = 1, \dots, m.$$

Let G' be the set of solutions of the above matrices in $\mathrm{GL}_n(\Omega)$. Then G' is the desired k-group.

We can interpret G' as the group of points of G in the $\bar{k}$-algebra $l \otimes_k \bar{k}$. Since l is separable over k we obtain that there exist $d = [l : k]$ distinct k-embeddings $\sigma_1, \dots, \sigma_d \colon l \to \bar{k}$. Then we obtain that $l \otimes_k \bar{k} \cong \bar{k}^d$, where the inclusion of l in $\bar{k}^d$ has the form $x \mapsto (\sigma_1(x), \dots, \sigma_d(x))$. Let G^{σ_i} denote the subgroup of $\mathrm{GL}_n(\Omega)$ which is determined by the solutions of $\sigma_i^*(f)$ for all $f \in \mathfrak{a}_l$. Then we conclude that there is a l-rational homomorphism $\mu \colon G' \to G$ such that

$$\mu^0 = (\mu^{\sigma_1}, \dots, \mu^{\sigma_d}) \colon G' \to G^{\sigma_1} \times \cdots \times G^{\sigma_d}$$

is an isomorphism over $\bar{l}$.

Now let $\varphi \colon G \to H$ be a l-morphism of algebraic l-groups G and H. Then there is a corresponding k-morphism $\mathbf{R}_{l/k}(\varphi) \colon \mathbf{R}_{l/k}(G) \to \mathbf{R}_{l/k}(H)$. Conversely, not every k-morphism can be written as $\mathbf{R}_{l/k}(\varphi)$ for a suitable l-morphism φ. This can be seen by considering a Galois extension l/k with Galois group $\mathcal{G}$. Then every $\sigma \in \mathcal{G}$ is a l-morphism, but the above representation of $\mathbf{R}_{l/k}(G)$ as the product of the G^σ for all $\sigma \in \mathcal{G}$ implies that $\mathbf{R}_{l/k}(\sigma) = \mathrm{id}_k$ for all $\sigma \in \mathcal{G}$. However, we have the following bijection:

Proposition 1.12. *Let l be an algebraic number field, let k be a subfield of l. Let G be an algebraic l-group. Then there is a bijection*

$$X(G)_l \cong X(G')_k$$

from the group of l-rational characters on G to the group of k-rational characters on $G' = \mathbf{R}_{l/k}(G)$.

Proof. [2, 1.5]. Without loss of generality let $\sigma_1 = id$. Then we define

$$G_1 = \bigcap_{i=2}^{d} \ker(\mu^{\sigma_i}).$$

Since this group is invariant under every isomorphism of $\mathbb{C}$ over l, it is defined over l. Moreover, the map μ maps G_1 isomorphically onto G. Now define the morphism $\nu \colon G \to G_1$ as the inverse of the restriction of μ onto G_1. Then we obtain that the homomorphism

$$\nu^0 = (\nu^{\sigma_1}, \dots, \nu^{\sigma_d}) \colon G^{\sigma_1} \times \cdots \times G^{\sigma_d} \to G'$$

is the inverse of μ^0.

Now let $\chi \in X_l(G)$. Then we clearly have that $\chi \circ \mu \in X_l(G')$, thus the product of the characters $(\chi \circ \mu)^\sigma$ in $X_l(G')$ is defined over k. So we obtain a map $\beta\colon X_l(G) \to X_k(G')$, $\chi \mapsto \prod_{i=1}^d (\chi \circ \mu)^{\sigma_i}$. Moreover, we have a map $\alpha\colon X_k(G') \to X_l(G)$ defined by $\chi \mapsto \chi \circ \mu$. We claim that α is a bijection with inverse β.

First, let $\chi \in X_l(G)$, $g \in G$. Then we have

$$\begin{aligned}(\alpha \circ \beta)(\chi)(g) &= \beta(\chi)(\nu(g)) \\ &= \prod_{i=1}^d (\chi \circ \mu)^{\sigma_i} \underbrace{(\nu(g))}_{\in G_1} \\ &= (\chi \circ \mu \circ \nu)(g) \\ &= \chi(g)\end{aligned}$$

Now let $\chi \in X_k(G')$, let $g \in G'$. Then we obtain

$$\begin{aligned}(\beta \circ \alpha)(\chi)(g) &= \alpha(\chi)\left(\prod_{i=1}^d \mu^{\sigma_i}(g)\right) \\ &= \chi\left(\nu\left(\prod_{i=1}^d \mu^{\sigma_i}(g)\right)\right) \\ &= \chi\left(\prod_{i=1}^d \nu^{\sigma_i}\mu^{\sigma_i}(g)\right) \\ &= \chi(g)\end{aligned}$$

So we obtain that

$$X_l(G) \cong X_k(G').$$

$\square$

2 Results from topology and measure theory

2.1 Properness

Let $f\colon X \to Y$ be a continuous map between topological spaces X, Y and let $K \subset Y$ be compact. It is well known that the preimage $f^{-1}(K) \subset X$ is not compact in general. Now we want to analyse continuous maps which satisfy the above condition. For further information see [4, §10].

Definition. Let X, Y be topological spaces, let $f\colon X \to Y$ be a continuous map. We call f proper if f is closed and for all points $y \in Y$ the preimage $f^{-1}(y) \subset X$ is compact.

So far we have defined a proper map only by considering preimages of points; however, the following Proposition shows that proper maps satisfy the desired property.

Proposition 2.1. *Let $f\colon X \to Y$ be a proper map between topological spaces X and Y. Then for every compact set $K \subset Y$ the preimage $f^{-1}(K) \subset X$ is compact.*

Proof. Let $K \subset Y$ be compact, let $\bigcup_{i\in I} U_i$ be an open covering of $f^{-1}(K)$. Since f is proper, it follows that for every $y \in K$ the preimage $f^{-1}(y) \subset f^{-1}(K)$ is compact. Thus, there exists a finite subcover $\bigcup_{i\in I_y} U_i$ of $\bigcup_{i\in I} U_i$ for $f^{-1}(y)$, where I_y is a finite subset of I. The set $X \setminus \bigcup_{i\in I_y} U_i$ is closed in X and f is closed, hence $f(X \setminus \bigcup_{i\in I_y} U_i)$ is closed in Y. Therefore, the set

$$V_y := Y \setminus f\left(X \setminus \bigcup_{i\in I_y} U_i\right)$$

is open in Y. Now we obtain

$$f^{-1}(y) \subset \bigcup_{i\in I_y} U_i \Rightarrow y \notin f\left(X \setminus \bigcup_{i\in I_y} U_i\right) \Rightarrow y \in Y \setminus f\left(X \setminus \bigcup_{i\in I_y} U_i\right),$$

so $y \in V_y$. Since $K = \bigcup_{y\in K}$, it follows that $K \subset \bigcup_{y\in K} V_y$. Now K is compact, so there exist elements $x_1, \dots, x_m \in K$ such that $K \subset \bigcup_{j=1}^{m} V_{x_j}$. Then

$$f^{-1}(K) \subset f^{-1}\left(\bigcup_{j=1}^{m} V_{x_j}\right) = f^{-1}\left(\bigcup_{j=1}^{m}\left(f\left(X \setminus \bigcup_{i\in I_{x_j}} U_i\right)\right)^{\complement}\right) = \bigcup_{j=1}^{m}\bigcup_{i\in I_{x_j}} U_i,$$

and the last union is a finite subcover of $\bigcup_{i\in I} U_i$. Thus, K is compact. $\square$

In chapter 4.5 we will be in the following situation: Given an injective map $i\colon H \to G$ between topological groups H and G, our aim will be to show that the map i is proper. As a first step, we will reformulate this due to the following Proposition.

Proposition 2.2. *Let $f\colon X \to Y$ be a continuous, injective map between topological spaces X and Y. Then the following are equivalent:*

(i) *f is proper.*

(ii) *f is closed.*

(iii) *f is a homeomorphism onto a closed subspace of Y.*

Proof. $(i) \Rightarrow (ii)$: follows from the definition.

$(ii) \Rightarrow (iii)$: The space X is closed, so $f(X) \subset Y$ is closed. From the assumptions we obtain that $\bar{f}\colon X \to f(X)$ is continuous and bijective. Now let $g\colon f(X) \to X$ denote the inverse function of $\bar{f}$ and let $A \subset X$ be an arbitrary closed set. Then we get $g^{-1}(A) = \bar{f}(A)$ is closed in Y, since f is closed. Thus, g is continuous.

$(iii) \Rightarrow (i)$: The map f is closed as a homeomorphism onto a closed subspace of Y. Now let $y \in Y$. If $y \in Y \setminus f(X)$, we have $f^{-1}(y) = \emptyset$ is compact. Now let $y \in f(X)$. Since f is bijective, there exists an element $x \in X$ with $f(x) = y$. Thus, $f^{-1}(y) = x$ is compact. □

So we will use part (iii) of the Proposition to show that the map $i\colon H \to G$ is proper. However, the topological groups H and G have further properties in our setting. Both of them are locally compact, σ-compact and Hausdorff, and the map i induces an action of H on the space G. In this case we are able to show that the map i is always a homeomorphism onto its image. Before that, we need a Lemma from topology which is usually known as the Baire Category Theorem.

Lemma 2.3. *Let M be a locally compact Hausdorff space. Let $\{M_i\}_{i=1}^{\infty}$ be a countable set of closed subsets of M with $M = \bigcup_{i=1}^{\infty} M_i$. Then there is a positive integer $i_0 \in \mathbb{N}$ such that M_{i_0} contains an open subset of M.*

Proof. Assume that M_i contains no open subset of M for all $i \in \mathbb{N}$. Since M is locally compact, we can find an open subset U_1 of M such that $\overline{U_1}$ is compact. Now $U_1 \not\subset M_1$, so the open set $U_1 \setminus M_1$ is non-empty. Now choose an element $a_1 \in U_1 \setminus M_1$. Since M is locally compact, we can find a neighbourhood U_2 of a_1 such that $\overline{U_2} \subset U_1 \setminus M_1$. Then $U_2 \cap M_1 = \emptyset$. Now $U_2 \not\subset M_2$, so the open set $U_2 \setminus M_2$ is non-empty. Then we choose an element $a_2 \in U_2 \setminus M_2$ and a neighbourhood U_3 of a_2 with $\overline{U_3} \subset U_2 \setminus M_2$, thus $U_3 \cap M_2 = \emptyset$. Inductively, we get a sequence of compact, non-empty sets

$$\overline{U_1} \subset \overline{U_2} \subset \cdots \subset \overline{U_n} \subset \cdots .$$

Since all $\overline{U_i}$, $i = 1, \ldots n$, are non-empty, there is an element $b \in \bigcap_{i=1}^{\infty} \overline{U_i}$. But then $b \notin \bigcup_{i=1}^{\infty} M_i$, which contradicts $M = \bigcup_{i=1}^{\infty} M_i$. □

Now we are able to prove the Theorem which allows us to simplify the proof of the properness of the map i.

Theorem 2.4. *Let G be a locally compact group, let X be a locally compact Hausdorff space. Suppose G acts on X continuously and transitively. For any $x \in X$ let G_x denote the isotropy group of x under the action of G. If G is σ-compact, then the map*

$$\omega\colon G/G_x \to X$$

is a homeomorphism.

Proof. Let $\pi\colon G \to G/G_x$ denote the canonical projection. By definition of the quotient topology on G/G_x, the map π is open. Let $\varphi\colon G \to X$, $g \mapsto g.x$ be the orbit map of x. Since G acts continuously, the map φ is continuous. Moreover $\omega \circ \pi = \varphi$, so it suffices to prove that φ is an open map.

Let U be an open subset of G, let $u \in U$. Since G is locally compact, we can choose a compact neighbourhood V of $1 \in G$ such that V is symmetric and $uVV \subset U$. Now G is σ-compact, so there is a countable set of compact sets $\{W_i\}_{i=1}^{\infty}$ such that $G = \bigcup_{i=1}^{\infty} W_i$. Each W_i is contained in the union $\bigcup_{w_i \in W_i} w_i \mathring{V}$ of open sets. Since W_i is compact, we can find finitely many elements $w_i^1, \ldots, w_i^{n_i}$ such that $W_i \subset \bigcup_{j=1}^{n_i} w_i^j \mathring{V}$ for all $i \in \mathbb{N}$. So if we choose $\{y_k\}_{k=1}^{\infty} = \{w_i^j\}_{i,j}$, then G is covered by the compact sets $\{y_k V\}_{k=1}^{\infty}$.

The G-action on X is transitive, so we have

$$X = \bigcup_{k=1}^{\infty} \varphi(y_k V).$$

Now $\varphi(y_k V)$ is compact as the continuous image of the compact set $y_k V$ for all $k \in \mathbb{N}$. Moreover, since the map $X \to X$, $x \mapsto g.x$ is a homeomorphism for all $g \in G$, we obtain that $\varphi(y_k V)$ is homeomorphic to $\varphi(V)$ for all $k \in \mathbb{N}$. Now X is Hausdorff, so $\varphi(y_k V)$ is closed for all $k \in \mathbb{N}$. By Lemma 2.3 there is a positive integer $m \in \mathbb{N}$ such that $\varphi(y_m V)$ contains an open subset of X. Now $\varphi(y_m V)$ is homeomorphic to $\varphi(V)$, so also $\varphi(V)$ contains an open subset of X. Therefore, there is an element $u_1 \in V$ such that $\varphi(u_1)$ is an inner point of $\varphi(V)$. This implies that $\varphi(e)$ is an inner point of $\varphi(u_1^{-1}V)$ and thus $\varphi(u)$ is an inner point of $\varphi(uu_1^{-1}V)$. But we have

$$uu_1^{-1}V \subset uVV \subset U,$$

so $\varphi(u)$ is an inner point of $\varphi(U)$. Thus, every point $\varphi(u) \in \varphi(U)$ is inner, i.e. $\varphi(U)$ is open. □

Let G be a topological group, let X be a topological space. Assume that G acts on X continuously. Now we want to analyse the situation when the action of G on X is proper.

Definition. Let G be a topological group operating continuously on a topological space X. Then G is said to operate properly on X if the mapping

$$\begin{aligned} \theta\colon G \times X &\to X \times X \\ (g, x) &\mapsto (x, g.x) \end{aligned}$$

is proper.

Proposition 2.5. *Let G be a locally compact group operating continuously on a Hausdorff space X. Then G operates properly on X if and only if for each pair of points $(x, y) \in X \times X$ there is are neighbourhoods V_x of x and V_y of y such that the set*

$$K = \{s \in G \mid s.V_x \cap V_y \neq \emptyset\}$$

is contained in a compact set.

Proof. [4, §4.4, Prop. 7] □

In particular, let G be a discrete group operating continuously on a Hausdorff space X. Then G operates properly on X if and only if for each pair of points $(x, y) \in X \times X$ there is are neighbourhoods V_x of x and V_y of y such that the set $\{s \in G \mid s.V_x \cap V_y \neq \emptyset\}$ is finite.

Proposition 2.6. *Let G be a discrete group operating properly on a Hausdorff space X. Let x be a point of X and G_x be the stabilizer of x. Then:*

(i) The subgroup G_x is finite and there is an open subset $U \subset X$, containing x, which is stable under G_x, and on which the equivalence relation induced by the relation defined by G is the equivalence relation defined by G_x.

(ii) The canonical mapping

$$U/G_x \to X/G$$

is a homeomorphism of U/G_x onto an open neighbourhood of the class of x in X/G.

Proof. [4, §4.4., Prop. 8] □

2.2 Restricted topological product and Haar measure

In this section we want to introduce another definition from topology which is related to the product of topological spaces. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces, let $X = \prod_{\lambda \in \Lambda} X_\lambda$. Then from the Theorem of Tychonoff we know that X is compact if and only if X_λ is compact for all $\lambda \in \Lambda$. However, the infinite product of locally compact spaces is not locally compact in general. In particular, the usual infinite product $G = \prod_{\lambda \in \Lambda} G_\lambda$ of locally compact topological groups $\{G_\lambda\}_{\lambda \in \Lambda}$ is not necessarily locally compact, so we cannot define a Haar measure on G. Therefore we will introduce the notion of a "restricted topological product" of topological spaces, which remains locally compact if its factors are locally compact. For more information see [4, I, §4].

Definition. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of topological spaces indexed by an arbitrary index set Λ. Let $\Lambda_0 \subset \Lambda$ be a finite subset of Λ. For all $\lambda \notin \Lambda_0$ let X_λ contain an open, compact subspace $Y_\lambda \subset X_\lambda$. Then the restricted (topological) product $X = \prod_{\lambda \in \Lambda} (X_\lambda : Y_\lambda)$ of the X_λ with respect to the Y_λ is defined as the set

$$X = \{(x_\lambda)_{\lambda \in \Lambda} \mid x_\lambda \in Y_\lambda \text{ for all } \lambda \in \Lambda \setminus S, \text{ for all finite } S \subset \Lambda \text{ containing } \Lambda_0\},$$

endowed with a topology defined as follows: the open sets of X are of the form

$$U = \prod_{\lambda \in S} U_\lambda \times \prod_{\lambda \notin S} Y_\lambda,$$

where S is a finite subset of Λ containing Λ_0 and $U_\lambda \subset X_\lambda$ are open subsets of X_λ for all $\lambda \in S$. Otherwise said, an element $x = (x_\lambda)$ of the restricted product of X_λ with respect to the Y_λ is an element of $\prod_{\lambda \in \Lambda} X_\lambda$ with $x_\lambda \in Y_\lambda$ for almost all λ.

For each finite subset $S \subset \Lambda$ with $\Lambda_0 \subset S$ we define the set

$$X_S = \prod_{\lambda \in S} X_\lambda \times \prod_{\lambda \notin S} Y_\lambda,$$

endowed with the usual product topology. If S_1 and S_2 are such sets with $S_1 \subset S_2$, then we have $X_{S_1} \subset X_{S_2}$. Thus the sets $\{X_S\}$ is a directed system. Then it clearly follows that X is the direct limit

$$X = \varinjlim_S X_S,$$

where the limit is taken over all finite subsets S as above.

As we have said at the beginning of this section the reason for not considering the usual product $X = \prod_{\lambda \in \Lambda} X_\lambda$ is that for locally compact spaces X_λ the product space X is not locally compact in general. More precisely, we have the following Proposition.

Proposition 2.7. *(i) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of locally compact spaces such that X_λ is compact for all but a finite number of indices. Then the product space $X = \prod_{\lambda \in \Lambda} X_\lambda$ is locally compact.*

(ii) Conversely, if the product of a family $\{X_\lambda\}_{\lambda \in \Lambda}$ of non-empty topological spaces is locally compact, then the factors X_λ are compact for all but a finite number of indices, and the factors which are not compact are locally compact.

Proof. [4, I.§9.7, Prop.14] □

We will use this in the following situation. Let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a family of locally compact groups and H_λ be open compact subgroups for all $\lambda \in \Lambda \setminus \Lambda_0$, where Λ_0 is a finite subset of Λ. Then the restricted product

$$G = \prod_{\lambda \in \Lambda} (G_\lambda : H_\lambda)$$

itself becomes a locally compact topological group by component wise operations. Now we know that there exists a Haar measure on G, which can be constructed out of the factors G_λ resp. H_λ. For more information about Haar measures see the Appendix.

As a first step, we consider finite products of locally compact groups. Let G_i be such groups with Haar measures μ_i for $i = 1, 2$. Then G has a unique measure

$$\mu = \mu_1 \times \mu_2$$

such that for any μ_i-measurable subsets $M_i \subset G_i$, $i = 1, 2$, the set $M = M_1 \times M_2$ is μ-measurable and

$$\mu(M) = \mu_1(M_1)\mu_2(M_2).$$

Now let $\{G_\lambda\}_{\lambda \in \Lambda}$ be a family of locally compact groups with Haar measures μ_λ. Moreover, let $\Lambda_0 \subset \Lambda$ be a finite subset so that there is an open compact subgroup $H_\lambda \subset G_\lambda$ for all $\lambda \in \Lambda \setminus \Lambda_0$. Since the H_λ are compact, we can normalize μ_λ in a way so that $\mu_\lambda(H_\lambda) = 1$ for all $\lambda \in \Lambda \setminus \Lambda_0$. Then for every finite subset $S \subset \Lambda$ with $\Lambda_0 \subset S$ we obtain a measure μ_S on

$$G_S = \prod_{\lambda \in S} G_\lambda \times \prod_{\lambda \notin S} H_\lambda,$$

which is the infinite product of the μ_λ. It can easily be shown that if S_1, S_2 are finite subsets of Λ containing Λ_0 and $S_1 \subset S_2$, then we have also $G_{S_1} \subset G_{S_2}$ and thus $\mu_{S_2}|_{G_{S_1}} = \mu_{S_1}$. So using countable additivity of the Haar measure and the fact that G is the direct limit of the G_S with S as above, we obtain a left Haar measure μ on G. Note that we could also have forced the product

$$\prod_{\lambda \in \Lambda \setminus \Lambda_0} \mu_\lambda(H_\lambda)$$

to be absolutely convergent instead of using the condition $\mu_\lambda(H_\lambda) = 1$.

So we have constructed Haar measures for restricted products of locally compact groups. However, later we will analyse quotients of the form $X = G/H$, where G is a locally compact group and $H \subset G$ a closed subgroup, endowed with the quotient topology induced from G. In particular, we try to find criteria when this quotient has finite invariant volume, i.e., under which conditions there exists a G-invariant Borel measure β on X so that $\beta(X)$ is finite. The next Theorem gives a relation between the modulus of G and H and the existence of a left-invariant Borel measure on X.

Theorem 2.8. *A nonzero G-invariant Borel measure β on $X = G/H$ exists if and only if $\Delta_G|_H = \Delta_H$. It is uniquely determined up to a positive scalar.*

Proof. The proof can be found in [5, ch. VII, §2, n°6, Corr. 2] □

In particular, if H is discrete, then there exists a left-invariant measure on X if and only if G is unimodular. In general, we can find a connection between the measure β on the quotient G/H and the Haar measures μ and ν on G and H respectively. The rest of this chapter follows [15, ch. 3.5].

For that, let H be a closed subgroup of G and consider a function f which is integrable over G. For all $g \in G$ we put

$$\varphi(g) = \int_H f(gh)d\nu(h).$$

Then φ is a function on G which is constant on cosets modulo H, thus can be regarded as a function on G/H. We obtain the following formula:

$$\int_{G/H} \left(\int_H f(gh)d\nu(h) \right) d\beta(gH) = \int_G f(g)d\mu(g).$$

Another important ingredient for the analysis of the volume of $X = G/H$ is needed. We say that a subset $F \subset G$ is a fundamental domain with respect to H if $\pi|_F$ is bijective, where $\pi\colon G \to G/H$ denotes the canonical projection. In the given situation, we can always find a μ-measurable fundamental domain $F \subset G$. Then the above formula yields

$$\int_X f(x)d\beta(x) = \int_F f(\pi(g))d\mu(g),$$

for any function f integrable over $X = G/H$. It can be shown that this formula also holds if we use a more general notion of a fundamental domain. We say that $F \subset G$ is a fundamental domain for H in G if

(a) $G = FH$

(b) $F \cap Fh$ has measure 0 for all $h \in H \setminus \{e\}$

Such a situation is given if F is a closed subset of G with boundary of measure 0, covering G/H, and such that the H-translates of F have points in common only at the boundary. Now if we take $f \equiv 1$ in the above integral identity, we obtain that X has finite invariant measure if and only if there exists a measurable fundamental domain $F \subset G$ relative to H having finite measure. Since every measurable set covering X contains a fundamental domain, we obtain that X has finite measure if and only if it is covered by a set with finite measure. This will be very important in chapter 3.3.

We will see that it is not possible in general to construct fundamental domains explicitly; however, in many cases we can find a good approximation for it. A subset $\Omega \subset G$ is called a fundamental set of G with respect to H if

(a) $G = \Omega H$

(b) $\Omega^{-1}\Omega \cap H$ is finite.

If we can find such a fundamental set we may not be able to compute the volume $\mathrm{vol}(X)$ of $X = G/H$. But as above, it can be decided whether it is finite or not. Indeed, if Ω has finite invariant volume, then from above also the volume of X is finite. Conversely, let $\mathrm{vol}(X)$ be finite and let $F \subset \Omega$ be a measurable fundamental domain in the sense of the second definition. Then we have

$$\Omega \subset \cup_{h \in H_0} Fh,$$

where H_0 is a finite subset of H. Thus, $\mathrm{vol}(\Omega)$ is finite. So we obtain that G/H has finite volume if and only if there exists a fundamental set $\Omega \subset G$ of finite volume.

Our aim in the last part of this section is to describe explicitly Haar measures in the cases of our interest. First, we have the following Proposition which applies to the Iwasawa decomposition of real algebraic groups.

Proposition 2.9. *Let G be an unimodular locally compact group and let H, A and U be closed subgroups of G with left Haar measures ν, ρ and σ respectively such that the product morphism $H \times A \times U \to G$ is a homeomorphism. Assume that A normalizes U and that A and U are unimodular. Then*

$$\mu = \mathrm{mod}_U(\mathrm{inn}(a)|_U)\nu(h) \times \rho(a) \times \sigma(u)$$

is a left Haar measure on G.

Proof. The proof can be found in [11, Prop. 2.3] □

Another possibility to obtain Haar measures on locally compact groups is the use of differential forms. Let k be a field, let X be a smooth n-dimensional algebraic k-variety. Then a k-system of local parameters in the neighbourhood of x_0 in X is a system of k-rational functions $x_1, \dots, x_n$ defined at x_0 such that the differential $\mathrm{d}_{x_0} \varphi$ of the function

$$\begin{aligned} \varphi\colon X &\to \mathbf{A}^n \\ x &\mapsto (x_1(x), \dots, x_n(x)) \end{aligned}$$

is an isomorphism of tangent spaces. Now a k-differential form is an expression of the form $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$, where $f\colon X \to k$ is a k-rational function. It can be shown that for a connected k-group G with $n = \dim(G)$ there always exists a nonzero n-dimensional k-rational left-invariant differential form ω on G, which is unique up to a multiplicative constant.

Example. Let $G = \mathrm{SL}_2$ over $\mathbb{Q}$. As a local system of parameters of 1 we can use the coordinate functions $x,\ y,\ z\colon G \to \mathbf{A}^1$, which associates to a matrix $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G_{\mathbb{Q}}$, with $t = \frac{1+yz}{x}$, its corresponding coordinates. Now let

$$\omega = f(X)dx \wedge dy \wedge dz$$

be a $\mathbb{Q}$-rational left-invariant differential form, where $f\colon G_{\mathbb{Q}} \to \mathbb{Q}$. Let $A = \begin{pmatrix} a & b \\ c & \frac{1+bc}{a} \end{pmatrix} \in G_{\mathbb{Q}}$. Since ω is left-invariant, we conclude that

$$\begin{aligned}
&f(X)\ dx \wedge dy \wedge dz \\
&= f(AX)\ d(ax+bz) \wedge d(ay + b\frac{1+yz}{x}) \wedge d(cx + \frac{1+bc}{a}z) \\
&= f(AX) \left((a\ dx + b\ dz) \wedge (a\ dy + b\ d(\frac{1+yz}{x})) \wedge (c\ dx + \frac{1+bc}{a} dz) \right) \\
&= f(AX) dx \wedge dz \wedge (-a\ dy + \frac{b(1+yz)}{x^2}\ dx - \frac{bz}{x}\ dy - \frac{by}{x}\ dz) \\
&= f(AX)(\frac{ax+bz}{x})\ dx \wedge dy \wedge dz,
\end{aligned}$$

thus $f(AX)(ax+bz) = f(X)x$, where the multiplication is that of $\mathbb{Q}$. Since we have that $X_{11} = x$ and $(AX)_{11} = ax + bz$, we can write the last equation as

$$f(X) \cdot x(X) = f(AX) \cdot x(AX),$$

where the multiplication is again in $\mathbb{Q}$. Since this holds for all $A \in G_{\mathbb{Q}}$ we conclude that f has the form $f = \frac{\lambda}{x}$ for an element $\lambda \in \mathbb{Q}$.

Now consider the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$, which is obtained by intersecting the factors of the Iwasawa decomposition of $\mathrm{GL}_2(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{R})$. Then we can write every matrix $g \in G_{\mathbb{R}}$ as a product of matrices

$$g = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

with $\alpha > 0$. Now take α, φ and u as coordinates in $G_{\mathbb{R}}$. Then we see that

$$\begin{aligned}
x &= \alpha \cos(\varphi), \\
y &= \alpha u \cos(\varphi) - \alpha^{-1} \sin(\varphi), \\
z &= \alpha \sin(\varphi).
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
\omega &= \frac{\lambda}{x} dx \wedge dy \wedge dz \\
&= \frac{\lambda}{\alpha\cos(\varphi)} d(\alpha\cos(\varphi)) \wedge d(\alpha u\cos(\varphi) - \alpha^{-1}\sin(\varphi)) \wedge d(\alpha\sin(\varphi)) \\
&= -\frac{\lambda}{\alpha\cos(\varphi)} (\cos(\varphi)d\alpha - \alpha\sin(\varphi)d\varphi) \wedge (\sin(\varphi)d\alpha + \alpha\cos(\varphi)d\varphi)\wedge \\
&\quad d(\alpha u\cos(\varphi) - \alpha^{-1}\sin(\varphi)) \\
&= \frac{\lambda}{\alpha\cos(\varphi)} (\alpha\cos^2(\varphi) + \alpha\sin^2(\varphi)) d\varphi \wedge d\alpha \wedge d(\alpha u\cos(\varphi) - \alpha^{-1}\sin(\varphi)) \\
&= \frac{\lambda}{\alpha\cos(\varphi)} \alpha^2\cos(\varphi) d\varphi \wedge d\alpha \wedge du \\
&= \lambda\alpha d\varphi \wedge d\alpha \wedge du.
\end{aligned}
$$

We will use this $\mathbb{Q}$-differential form for SL_2 to compute the volume of the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$ in chapter 3.3 and for the adelic case in chapter 4.4.

3 Results from number theory

3.1 Algebraic number fields

In this first subsection we briefly want to review the theory of algebraic number fields. More information about the ring of integers and Dedekind rings can be found in [10, ch. X], for the theory of valuations see [12, ch. XII.4] and for the completions with respect to the valuations see [7].

Let k be an algebraic number field. Then the ring of integers in k is denoted by $\mathcal{O}_k$, which is free $\mathbb{Z}$-module of rank $[k : \mathbb{Q}]$ and a Dedekind ring. A (fractional) ideal $\mathfrak{a}$ of k is a $\mathcal{O}_k$-submodule of k so that there is an element $x \in \mathcal{O}_k$ with $x\mathfrak{a} \subset \mathcal{O}_k$. Every fractional ideal has a unique decomposition into a product of prime ideals. In particular, every ideal $\mathfrak{a}$ of $\mathcal{O}_k$ can be written as

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{r_{\mathfrak{p}}(\mathfrak{a})},$$

where the nonnegative integer $r_{\mathfrak{p}}(\mathfrak{a})$ is zero for almost all $\mathfrak{p}$ and where the product runs over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$. The set $\mathcal{J}_k$ of all fractional ideals of k form a group under ideal multiplication; the inverse of the ideal $\mathfrak{a}$ is given by $\mathfrak{a}^{-1} = \{x \in k \mid x\mathfrak{a} \subset \mathcal{O}_k\}$. Moreover, the set of all principal ideals (more explicitly, the $\mathcal{O}_k$-modules $x\mathcal{O}_k$ with $x \in k^*$), denoted by $\mathcal{P}_k$, is a normal subgroup of $\mathcal{J}_k$. The quotient group $\mathcal{C}_k = \mathcal{J}_k/\mathcal{P}_k$ is called the class group of k, which is always a finite group; its cardinality is called the class number h_k of k.

For any ideal $\mathfrak{a} \subset \mathcal{O}_k$ we denote by $N(\mathfrak{a})$ the cardinality of $\mathcal{O}_k/\mathfrak{a}$, which is always finite. For a prime ideal $\mathfrak{p}$, the resulting field is called the residue field of $\mathfrak{p}$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a $\mathbb{Q}$-base of k. Then $\mathcal{B}$ is an integral base if $v_i \in \mathcal{O}_k$ for all $i = 1, \ldots, n$. Since every $\mathbb{Z}$-base of $\mathcal{O}_k$ is also a $\mathbb{Q}$-base of k, we can always choose integral bases for algebraic number fields.

Now we want to obtain several arithmetic properties of the ring of integers $\mathcal{O}_k$. For that, it is useful to consider completions of k with respect to several valuations. For further information see [12, ch. XII.4].

Definition. A valuation v of an algebraic number field k is a map $v\colon k \to \mathbb{R}$ which satisfies the following conditions:

(1) $v(x) \geq 0$ for all $x \in k$ and $v(x) = 0 \Leftrightarrow x = 0$;

(2) $v(xy) = v(x)v(y)$ for all $x,\ y \in k$;

(3) $v(x + y) \leq v(x) + v(y)$ for all $x,\ y \in k$

If we replace condition (3) by the stronger condition

(3') $v(x + y) \leq \max\{v(x), v(y)\}$ for all $x,\ y \in k$,

then we call the valuation non-archimedean, otherwise archimedean.

Any valuation v on an algebraic number field k induces a metric d_v via $d_v(x, y) = v(x - y)$ for all $x,\ y \in k$. By the topology generated by v we mean the topology induced by the metric d_v. We say that two valuations are equivalent if they generate the same topology on k. We now want to construct (inequivalent) valuations for any algebraic number field k.

First, we have the usual archimedean value $|.|_\infty$ on $\mathbb{Q}$, which can be extended to k by setting $|x|_\infty = N_{k/\mathbb{Q}}(x)^n$ for all $x \in k$, where $n = [k : \mathbb{Q}]$ and $N_{k/\mathbb{Q}}(x)$ denotes the norm of x, i.e., the determinant of the left-multiplication with $x \in k$. Now let $\mathcal{G} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. By restricting every element $\sigma \in \mathcal{G}$ to k, we obtain n distinct embedding $\sigma_i : k \to \bar{\mathbb{Q}}$. Then the maps $|.|_i : k \to \mathbb{R}$, defined by $|x|_i := |\sigma_i(x)|_\infty$, are again archimedean valuations. A valuation $|.|_i$ is called real (resp. complex) if $\sigma_i(k) \subset \mathbb{R}$ (resp. $\sigma_i(k) \not\subset \mathbb{R}$). If σ_i is an embedding of k into $\mathbb{C}$, then $\overline{\sigma_i}$ is also a complex embedding. We order the archimedean valuations as follows: Let s (resp. $2t$) be the number of real (resp. complex) embeddings of k into $\mathbb{C}$, so $s + 2t = n$. We order the embeddings in such a way that σ_i is real for $1 \leq i \leq s$, $\sigma_i = \overline{\sigma_{i+t}}$ for $s + 1 \leq i \leq t$.

Non-archimedean valuations can be constructed as follows: Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_k$. For any $x \in k^*$ we consider the (fractional) ideal (x) (i.e., the $\mathcal{O}_k$-submodule $x\mathcal{O}_k$). Then this ideal has a unique factorization into prime ideals

$$(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{r_\mathfrak{p}(x)}$$

with suitable integers $r_\mathfrak{p}(x)$ which are zero for almost all $\mathfrak{p}$. Now define the map $|.|_\mathfrak{p} : k \to \mathbb{R}$ via $|x|_\mathfrak{p} = \mathrm{N}(\mathfrak{p})^{-r_\mathfrak{p}(x)}$, where $\mathrm{N}(\mathfrak{p}) = |\mathcal{O}_k/\mathfrak{p}|$. Then the map $|.|_\mathfrak{p}$ is a non-archimedean valuation. We identify a prime ideal $\mathfrak{p}$ with the valuation corresponding to it.

The valuations constructed above are exactly all non-trivial inequivalent valuations of k. We define the set V_k as the set of equivalence classes of valuations of k and call them the places of k. The archimedean (resp. non-archimedean) valuations are denoted by V_k^∞ (resp. V_k^f) and are called the infinite (resp. finite) places. Now we want to consider the completion k_v of k with respect to a valuation $v \in V_k$. If $v \in V_k^\infty$, we simply get $k_v = \mathbb{R}$, if v is a real place, or $k_v = \mathbb{C}$, if v is a complex place.

For $v \in V_k^f$, let $\mathfrak{p}$ denote the prime ideal corresponding to v. The field k_v is a locally compact, finite extension of $\mathbb{Q}_p$, the field of p-adic numbers, where $(p) = \mathfrak{p} \cap \mathbb{Z}$. The closure of the ring of integers $\mathcal{O}_k$ in k_v is the valuation ring

$$\mathcal{O}_v = \{x \in k_v \mid |x|_v \leq 1\},$$

called the ring of v-adic integers. The valuation ring is a free module over $\mathbb{Z}_p$, the ring of p-adic integers, whose rank equals the dimension $[k_v : \mathbb{Q}_p]$; thus, $\mathcal{O}_v$ is an open, compact subring of k_v. Furthermore, $\mathcal{O}_v$ is a local ring with maximal ideal

$$\mathfrak{p}_v = \{x \in k_v \mid |x|_v < 1\}$$

(called valuation ideal) and the group of units

$$U_v = \mathcal{O}_v \setminus \mathfrak{p}_v.$$

The valuation ideal is a principal ideal, any generator π of $\mathfrak{p}_v$ is called a uniformizing parameter. A uniformizing parameter has the property that $v(\pi)$ generates the value group $\Gamma = v(k_v^*)$, and any two uniformizing parameters differ by an element of U_v.

Example. As an important example, we want to look at the case $k = \mathbb{Q}$. The ring $\mathcal{O}_k$ is clearly $\mathbb{Z}$ itself. The only non-trivial archimedean valuation is the usual euclidean value $|.|_\infty$ on $\mathbb{Q}$, so $V_{\mathbb{Q}}^\infty$ consists of only one element. Since $\mathbb{Z}$ is a principal ideal domain, every prime ideal $\mathfrak{p}$ is of the form $\mathfrak{p} = (p)$ for a uniquely determined prime $p \in \mathbb{Z}$; thus the decomposition into prime ideals corresponds to the factorization into primes. Any element $x \in \mathbb{Q}$ can be uniquely written as $x = p^{r_p(x)}\frac{s}{t}$, where $(s,t) = (p,t) = (p,s) = 1$, for any prime p. The norm of the ideal (p) is the cardinality of the ring $\mathbb{Z}/p\mathbb{Z}$, thus equals p. Now the p-adic value on $\mathbb{Q}$ is defined by $|x|_p = N(p)^{-r_p(x)} = p^{-r_p(x)}$. The completition of $\mathbb{Q}$ with respect to $|.|_p$ is the field $\mathbb{Q}_p$, and the valuation ring $\mathcal{O}_p$ equals $\mathbb{Z}_p$. The valuation ideal is $p\mathbb{Z}_p$ and a uniformizing parameter is given by $\pi = p$. The group of units U_p are the elements in $\mathbb{Z}_p$ with p-adic value 1, thus the elements $x = \frac{s}{t} \in \mathbb{Q}$ whose unique prime factorization contains no power of p.

3.2 Adeles and Ideles

Now we try to gain more information about the ring of integers $\mathcal{O}_k$ of an algebraic number field k by analysing the completions k_v for $v \in V_k$. By considering only one valuation we do not get much information; however, by looking at all completions at the same time we are able to prove several important results. This is the idea of the ring of adeles, which we now want to introduce. The proofs of the results in this chapter can be found in [7].

Definition. Let k be an algebraic number field. Then $\{k_v\}_{v\in V_k}$ is a family of locally compact topological rings. Moreover, for all $v \in V_k^f$ there is an open compact subring $\mathcal{O}_v$ of k_v. Then we define the ring of adeles $\mathbb{A}_k$ over k as the restricted product

$$\mathbb{A}_k = \prod_{v\in V_k} (k_v : \mathcal{O}_v),$$

together with component wise operations, where the sets from the definition of the restricted product are $\Lambda = V_k$ and $\Lambda_0 = V_k^\infty$.

By definition, $\mathbb{A}_k$ admits a topology with respect to which $\mathbb{A}_k$ is turned into a topological ring. Moreover, since the k_v are locally compact, the adele ring $\mathbb{A}_k$ is a locally compact ring. The open sets of $\mathbb{A}_k$ are of the form

$$W = \prod_{v\in S} W_v \times \prod_{v\notin S} \mathcal{O}_v,$$

where S is a finite subset of V_k containing V_k^∞ and W_v is an open subset of k_v for all $v \in S$. More generally, we can define for such a subset S the ring of S-adeles $\mathbb{A}_k(S)$ as the set

$$\mathbb{A}_k(S) = \prod_{v\in S} k_v \times \prod_{v\notin S} \mathcal{O}_v,$$

together with the component wise operations and the induced topology from the product topology on $\prod_{v\in V_k} k_v$.

Let $x \in k$. Since $x \in \mathcal{O}_v$ for almost all $v \in V_k$, we can embed the field k into $\mathbb{A}_k$ by the diagonal embedding $k \hookrightarrow \mathbb{A}_k$, $x \mapsto (x, x, \dots)$. For matters of simplification, we usually identify k with its image in $\mathbb{A}_k$. As a first result we can state the following Proposition.

Proposition 3.1. *k is a discrete subring of $\mathbb{A}_k$.*

Proof. [7, Theorem 3-2-3] $\square$

Now it is natural to ask whether we can find a "sufficiently nice" subset Ω so that the k-translates of Ω cover $\mathbb{A}_k$.

Definition. Let A be a commutative topological group, let Γ a discrete subgroup. Let $\Omega \subset A$ be a compact subset of A. Ω is called a fundamental set for Γ in A if

(1) $\Omega\Gamma = A$

(2) for all $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$ the intersection of the translates $\gamma_1\Omega \cap \gamma_2\Omega$ of Ω is finite.

If the intersection in (2) is empty, then Ω is called a fundamental domain.

So our aim is to find a fundamental domain (or at least a fundamental set) for k in $\mathbb{A}_k$ with respect to the additive structure. As a first step, we can state the following Lemma.

Lemma 3.2. *Let $S \subset V_k$ be a finite subset of k-primes containing V_k^∞. Then*

$$\mathbb{A}_k(S) + k = \mathbb{A}_k.$$

Proof. [7, Proposition 3-2-5] $\square$

So $\mathbb{A}_k(S)$ has the desired covering property of a fundamental set for every finite subset $S \subset V_k$ containing V_k^∞. However, the second property is not satisfied in general. After restricting the infinite part of the adeles we obtain the following result, which shows that $\{\sum_{i=1}^n t_i\theta_i \mid 0 \leq t_i < 1\} \times \prod_{v \in V_k^f} \mathcal{O}_v$ is a fundamental domain for k in $\mathbb{A}_k$, where $\theta_1, \ldots, \theta_n$ is an integral base for $\mathcal{O}_k$.

Proposition 3.3. *Every adele can be expressed uniquely in the form $s+t$, where $t \in k$, $s \in \mathbb{A}_k(\infty)$ and the infinite component of s is of the form*

$$\sum_{i=1}^n t_i\theta_i, \text{ for } 0 \leq t_i < 1,$$

where $\{\theta_1, \ldots, \theta_n\}$ is an integral basis for k.

Proof. [7, Proposition 3-2-6] $\square$

Furthermore, we have the following theorem.

Theorem 3.4. *The quotient $\mathbb{A}_k/k$ is compact.*

Proof. [7, Theorem 3-2-4] $\square$

So we have gained some topological information about the structure of k in $\mathbb{A}_k$.

Before we introduce another construction concerning the multiplicative group $\mathcal{O}_k^*$ we state an important result in the setting of adeles. It gives a connection between all the v-adic valuations of the elements of k^*.

Theorem 3.5 (Product Formula). *Let $x \in k^*$. Then*

$$\prod_{v \in V_k} |x|_v = 1.$$

Proof. [7, Theorem 3-2-7] □

So we have seen that the adele ring contains much information about the additive group $\mathcal{O}_k$ of k. It is natural to ask if we can achieve a similar construction for the multiplicative group $\mathcal{O}_k^*$. We define the idele group $\mathbb{J}_k$ over k as the restricted product

$$\mathbb{J}_k = \prod_{v \in V_k} (k_v^* : U_v),$$

where k_v^* is the multiplicative subgroup of k_v and U_v the groups of units in $\mathcal{O}_v$. Note that we again have $\Lambda = V_k$ and $\Lambda_0 = V_k^\infty$ in the notation of chapter 2.2. The ideles are exactly the invertible elements of the adele ring; however, the topology on $\mathbb{J}_k$ is not the induced topology from $\mathbb{A}_k$, it is finer than that of $\mathbb{A}_k$. But there is a connection between $\mathbb{A}_k$ and $\mathbb{J}_k$; namely, the idele group $\mathbb{J}_k$ can be viewed as the general linear group $\mathrm{GL}_1(\mathbb{A}_k)$ with entries in the ring of adeles. This fact will be made clearer in chapter 4.1.

As in the adelic case, the subgroup k^* can be embedded into $\mathbb{J}_k$ diagonally. In this way we get the analogous result as before:

Proposition 3.6. *k^* is a discrete subgroup of $\mathbb{J}_k$.*

Proof. [7, Theorem 3-3-1] □

So the group k^* plays the same role in $\mathbb{J}_k$ as k does in $\mathbb{A}_k$; however, it can be shown that the quotient $\mathbb{J}_k/k^*$ is not compact. Therefore, our next aim will be to obtain the structural description of this quotient. For that, we need the notion of the idelic norm.

Definition. The idelic norm is the map $|.|\colon \mathbb{J}_k \to \mathbb{R}_{>0}$ defined by

$$a = (a_v)_v \mapsto |a| = \prod_{v \in V_k} |a_v|_v.$$

Since $a_v \in U_v$ for almost all $v \in V$, the v-adic value $|a_v|_v$ equals 1 for almost all $v \in V$. Thus, the idelic norm is well defined. In addition, for a, $b \in \mathbb{J}_k$ it follows immediately from the definition that $|ab| = |a||b|$ and $|1| = 1$, so the idelic norm is a group homomorphism from $\mathbb{J}_k$ into the multiplicative group $\mathbb{R}^*$. This in fact is a homomorphism of topological groups, which follows from the next result:

Lemma 3.7. *The idelic norm map is continuous.*

Proof. [7, Proposition 3-3-2] □

Now let $\mathbb{J}_k^{(1)} = \{a \in \mathbb{J}_k \mid |a| = 1\}$ be the subgroup of $\mathbb{J}_k$ of ideles with norm 1. Then we have the exact sequence

$$1 \to \mathbb{J}_k^{(1)} \to \mathbb{J}_k \to \mathbb{R}_{>0} \to 1$$

of topological groups. From the product formula it follows that $k^* \subset \mathbb{J}_k^{(1)}$. Now we obtain the following result:

Theorem 3.8. *The quotient $\mathbb{J}_k^{(1)}/k^*$ is compact.*

Proof. [7, Theorem 3-3-5] □

Another fact which is strongly related to the above theorem is that we can decompose the ideles as $\mathbb{J}_k = k^*(\mathbb{R}_{>0} \times \prod_{v \in V_k^f} U_v)$.

The reason why we have to reduce to the subgroup $\mathbb{J}_k^{(1)}$ in this case will be made clear at the beginning of chapter 4.5.

3.3 Arithmetic subgroups and reduction theory

One aim of this diploma thesis is to obtain information about the topological structure of the quotient $G_{\mathbb{A}}/G_k$ for an algebraic number field k. In particular, we try to find criteria for which this quotient is compact, has finite invariant volume respectively. We will see in chapter 4 that this is strongly connected to the real case, i.e., we have to analyse the quotient $G_{\mathbb{R}}/G_{\mathbb{Z}}$. The following chapter is dedicated to the definition of the term arithmetic group and the development of some reduction theory for $G_{\mathbb{R}}$ with respect to $G_{\mathbb{Z}}$. The stated results can be found in [15, ch. 4].

First, we give the definition of an arithmetic subgroup of an algebraic k-group, where k is an algebraic number field. For that, let $G_{\mathcal{O}_k}$ denote the intersection $G_k \cap \mathrm{GL}_n(\mathcal{O}_k)$.

Definition. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be an algebraic k-group. A subgroup $\Gamma \subset G$ is called arithmetic if it is commensurable with $G_{\mathcal{O}_k}$, i.e., if $\Gamma \cap G_{\mathcal{O}_k}$ has finite index in both Γ and $G_{\mathcal{O}_k}$.

It is notable that the definition of the $\mathcal{O}_k$-points of G depends on the realisation of G as a subgroup of $\mathrm{GL}_n(k)$. More precisely, let V be a k-vector space on which G operates. We must fix a base $e_1, \ldots, e_n$ of $V_{\mathcal{O}_k}$, or equivalently, a lattice $L = \mathcal{O}_k e_1 + \cdots + \mathcal{O}_k e_n$. Then $G_{\mathcal{O}_k}$ is the stabilizer $G^L_{\mathcal{O}_k}$ of L, i.e.,

$$G^L_{\mathcal{O}_k} = \{g \in G_k \mid g(L) = L\}.$$

The following Proposition guarantees that every choice of a lattice delivers an arithmetic group.

Proposition 3.9. *Let $\varphi\colon G \to G'$ be a k-isomorphism of k-groups. If Γ is an arithmetic subgroup of G, then $\varphi(\Gamma)$ is an arithmetic subgroup of G'.*

Proof. [15, Prop. 4.1] □

So we obtain that arithmetic subgroups are mapped onto arithmetic subgroups by k-isomorphisms. As a corollary, we can deduce that for an arithmetic subgroup $\Gamma \subset G$ and for every $g \in G_k$ also the group $g\Gamma g^{-1}$ is an arithmetic group. We have also seen that every $\mathcal{O}_k$-lattice in V_k induces an arithmetic subgroup of G_k. This in fact gives rise to a bijective correspondence, which is stated in the next Proposition.

Proposition 3.10. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be an algebraic k-group and $\Gamma \subset G_k$ an arithmetic subgroup. Then there exists a Γ-invariant lattice $L \subset k^n$. Moreover,*

$$[G^L_{\mathcal{O}_k} : \Gamma] < \infty.$$

Proof. [15, Prop. 4.2] □

It should be noted that we often can restrict ourselves to the case $k = \mathbb{Q}$. To see that let $\theta_1, \dots, \theta_n$ be an integral base for $\mathcal{O}_k$. Applying restriction of scalars with the chosen base we obtain a $\mathbb{Q}$-group $G' = \mathbf{R}_{k/\mathbb{Q}}(G)$ with $G_{\mathcal{O}_k} \cong G'_{\mathbb{Z}}$. Thus we will develop a reduction theory first for the case $G_{\mathbb{R}}/G_{\mathbb{Z}}$.

Definition. Let Γ be an arithmetic subgroup of $G_{\mathbb{Q}}$, where G is an algebraic $\mathbb{Q}$-group. An open subset $\Omega \subset G_{\mathbb{R}}$ is called a fundamental domain for Γ in $G_{\mathbb{R}}$ if

(1) $\mathrm{K}\,\Omega = \Omega$, where K is a suitable maximal compact subgroup of G;

(2) $\bar{\Omega}\Gamma = G_{\mathbb{R}}$;

(3) $\Omega \cap \Omega\gamma = \emptyset$ for all $\gamma \in \Gamma$, $\gamma \neq e$.

A subset $\Omega \subset G_{\mathbb{R}}$ is called a fundamental set for Γ in $G_{\mathbb{R}}$ if

(1) $\mathrm{K}\,\Omega = \Omega$, where K is a suitable maximal compact subgroup of G;

(2) $\Omega\Gamma = G_{\mathbb{R}}$;

(3) $\Omega^{-1}\Omega \cap (xG_{\mathbb{Z}}y)$ is finite for all x, $y \in G_{\mathbb{Q}}$.

It follows immediately from the definitions that every fundamental domain is also a fundamental set. Now we want to find a fundamental domain (or at least a fundamental set) for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$. For that, we could first consider the cases of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ and try to apply the following Lemma.

Lemma 3.11. *Let $G = \Sigma\Gamma$ be a decomposition of an abstract group G as a product of some subset Σ and a subgroup Γ. Furthermore, given a right action of G on some set X, let $H = G_x$ denote the stabilizer of a point $x \in X$. Assume that for a suitable $a \in G$ the intersection $(xa\Sigma) \cap x\Gamma$ is finite, let us say equal to $\{xb_1, \dots, xb_r\}$ with $b_i \in \Gamma$. Then $H = \Omega(\Gamma \cap H)$, where $\Omega = (\bigcup_{i=1}^{r} a\Sigma b_i^{-1}) \cap H$. If, in addition, there is some subgroup D, $\Gamma \subset D \subset G$, for which $\Sigma^{-1}\Sigma \cap g\Gamma h$ is finite for any g, $h \in D$, then $\Omega^{-1}\Omega \cap g(\Gamma \cap H)h$ is also finite for any g, $h \in D \cap H$.*

Proof. It is clear that $H \supset \Omega(\Gamma \cap H)$. Now let $h \in H$. Since $a^{-1}h \in G$, we can find elements $\sigma \in \Sigma$, $\gamma \in \Gamma$ with

$$a^{-1}h = \sigma\gamma. \tag{1}$$

Since H is the stabilizer of x it follows that

$$x = xh = xaa^{-1}h = xa\sigma\gamma,$$

thus $x\gamma^{-1} = xa\sigma$. Now the left side of this equation is in $x\Gamma$ and the right side in $xa\Sigma$, so from the assumptions we can find an integer $i \in \{1, \dots, r\}$ with

$$x\gamma^{-1} = xb_i. \tag{2}$$

Put $\gamma' = b_i\gamma \in \Gamma$. Then from (2) we obtain that $\gamma' \in H$, thus $\gamma' \in \Gamma \cap H$. Now (1) yields

$$h = a\sigma\gamma = a\sigma b_i^{-1}\gamma'$$

and $a\sigma b_i^{-1} = h\gamma'^{-1} \in H$, so the first claim follows.

Now we prove the second part. For d_1, $d_2 \in D$ let

$$\Sigma^{-1}\Sigma \cap d_1\Gamma d_2 = \{d_1\gamma_1 d_2, \ldots, d_1\gamma_n d_2\}.$$

Let g, $h \in D \cap H$ be arbitrary. Let $\theta \in (a\Sigma b_{i_1})^{-1}(a\Sigma b_{i_2}) \cap g(\Gamma \cap H)h$. Then we can find elements σ_1, $\sigma_2 \in \Sigma$, $\gamma \in \Gamma$ with

$$\theta = g\gamma h = (a\sigma_1 b_{i_1})^{-1}(a\sigma_2 b_{i_2}) = b_{i_1}^{-1}\sigma_1^{-1}\sigma_2 b_{i_2}.$$

This is equivalent to

$$\sigma_1^{-1}\sigma_2 = b_{i_1} g\gamma h b_{i_2}^{-1} \in \Sigma^{-1}\Sigma \cap d_1\Gamma d_2,$$

thus we can find an element $b_{i_1} g\gamma_j h b_{i_2}^{-1} \in \{b_{i_1} g\gamma_1 h b_{i_2}^{-1}, \ldots, b_{i_1} g\gamma_n h b_{i_2}^{-1}\}$ with

$$\sigma_1^{-1}\sigma_2 = b_{i_1} g\gamma_j h b_{i_2}^{-1}.$$

This yields

$$\theta = g\gamma_j h,$$

so for every i_1, $i_2 \in \{1, \ldots, r\}$ the intersection $(a\Sigma b_{i_1})^{-1}(a\Sigma b_{i_2}) \cap g(\Gamma \cap H)h$ is finite. Since there are only finitely many choices for i_1 and i_2, the claim follows. ☐

So as a first step, we want to construct a fundamental set for $G = \mathrm{GL}_n(\mathbb{R})$ with respect to $\Gamma = \mathrm{GL}_n(\mathbb{Z})$. For that, let K, A, U denote the subgroups of orthogonal matrices, diagonal matrices with positive entries, and unipotent matrices respectively.

Definition. A Siegel set $\Sigma_{t,u}$ of G is a set of the form $\Sigma_{t,u} = \mathrm{K}\,\mathrm{A}_t\,\mathrm{U}_u$ with t, $u > 0$, where

$$\begin{aligned} A_t &= \{a \in A \mid \frac{a_i}{a_{i+1}} \leq t, i = 1, \ldots, n-1\}, \\ U_u &= \{u \in U \mid |u_{ij}| \leq u \text{ for all } 1 \leq i < j \leq n\}. \end{aligned}$$

This definition is the essential ingredient for the reduction theory in the real case. Put $\Gamma = \mathrm{GL}_n(\mathbb{Z})$. Then we obtain

Theorem 3.12. *We have $G = \Sigma_{t,u}\Gamma$ for $t \geq \frac{2}{\sqrt{3}}$ and $u \geq \frac{1}{2}$.*

Proof. The proof uses the height function Ψ which associates to every $g \in G$ the number $\Psi(g) = |ge_1|$, where $|.|$ denotes the usual Euclidean norm on $\mathbb{R}^n$. It is shown that the minimum of Ψ on $g\Gamma$ has the desired properties for $n = 2$. The general case is obtained by induction. The complete proof can be found in [15, Thm 4.4]. ☐

The proof of the last Theorem is not very difficult; However, it is omitted since we consider the adelic case in section 4.2 where the proof is analogous. From 3.12 we immediately obtain a fundamental set for $\mathrm{SL}_n(\mathbb{R})$.

Corollary 3.13. *Put* $\Sigma^{(1)}_{t,u} = \Sigma_{t,u} \cap \mathrm{SL}_n(\mathbb{R})$. *Then* $\mathrm{SL}_n(\mathbb{R}) = \Sigma^{(1)}_{t,u}\,\mathrm{SL}_n(\mathbb{Z})$ *for* $t \geq \frac{2}{\sqrt{3}}$ *and* $u \geq \frac{1}{2}$. *Moreover,*

$$\Sigma^{(1)}_{t,u} = (\mathrm{K} \cap \mathrm{SL}_n(\mathbb{R}))(\mathrm{A}_t \cap \mathrm{SL}_n(\mathbb{R}))U_u.$$

Proof. [15, p.180, Corollary] □

So we obtained a fundamental set for the cases $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$. Unfortunately, the interior of the Siegel set $\Sigma = \Sigma^0_{\frac{2}{\sqrt{3}},\frac{1}{2}}$ is not a fundamental domain. This can be seen if we consider the case $\mathrm{SL}_2(\mathbb{R})$. Let $\mathcal{H}$ denote upper half plane of the complex plane, i.e., $\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. Then the group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ from the right by Moebius transformations. Moreover, the stabilizer of i is given by the subgroup $\mathrm{SO}(2)$; thus, we can identify $\mathcal{H}$ with the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. If we consider the image of the Siegel set in this setting, it can be shown that the intersection of Γ-translates of Σ is not always empty. Thus, Σ is not a fundamental domain.

The next step is to establish the existence of a fundamental set for an arbitrary $\mathbb{Q}$-group G. Here we can restrict ourselves to connected groups since the quotient G/G^0 is finite. Moreover, the following Lemma allows us to consider only reductive groups.

Lemma 3.14. *(i) Let N be a unipotent $\mathbb{Q}$-group. Then there is an open, relatively compact subset $U \subset N_{\mathbb{R}}$ (i.e., the closure of U is compact) such that*

(a) $N_{\mathbb{R}} = UN_{\mathbb{Z}}$,

(b) $U^{-1}U \cap (nN_{\mathbb{Z}}m)$ *is finite for any* n, $m \in N_{\mathbb{Q}}$.

(ii) Let G be a connected $\mathbb{Q}$-group, let $G = HN$ be its Levi-decomposition, where H is a maximal reductive $\mathbb{Q}$-group of G and $N = R_u(G)$ its unipotent radical. Suppose $\Sigma \subset H_{\mathbb{R}}$ satisfies $H_{\mathbb{R}} = \Sigma H_{\mathbb{Z}}$ and the intersection $\Sigma^{-1}\Sigma \cap (gH_{\mathbb{Z}}h)$ is finite for any g, $h \in H_{\mathbb{Q}}$. If $U \subset N_{\mathbb{R}}$ is as in (i), then the set $\Omega = \Sigma U$ satisfies

(a) $G_{\mathbb{R}} = \Omega G_{\mathbb{Z}}$,

(b) $\Omega^{-1}\Omega \cap (xG_{\mathbb{Z}}y)$ *is finite for any* x, $y \in G_{\mathbb{Q}}$.

Proof. [15, Thm 4.9] □

Thus we only need to construct a fundamental set for connected reductive groups G, for which we want to use Lemma 3.11. For that, the following points have to be satisfied:

(i) Define a right action of GL_n on some set X such that G is the stabilizer of a suitable point $x \in X$;

(ii) Find an element $a \in \mathrm{GL}_n(\mathbb{R})$ for which $xa\Sigma \cap x\,\mathrm{GL}_n(\mathbb{Z})$ is finite, where Σ is a Siegel set in $\mathrm{GL}_n(\mathbb{R})$.

Since G is a reductive group, we see from Proposition 1.11 that there is a $\mathbb{Q}$-rational representation $\rho\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$, where V is a $\mathbb{Q}$-vector space, and a vector $v \in V_{\mathbb{Q}}$ so that G is the stabilizer of v and the orbit of v under ρ is closed. Now choose an element $a \in \mathrm{GL}_n(\mathbb{R})$ so that $a^{-1}Ga$ is self-adjoint. The finiteness of the desired intersection now follows from

Proposition 3.15. *Let $\rho\colon \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(V)$ be a $\mathbb{Q}$-representation and let L be a lattice in $V_{\mathbb{Q}}$. If $v \in V_{\mathbb{R}}$ is a point whose stabilizer*

$$G = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid v\rho(g) = v\}$$

is a self-adjoint group and $v\rho(\mathrm{GL}_n(\mathbb{C}))$ is closed in the Zariski-topology, then $v\rho(\Sigma) \cap L$ is finite for any Siegel set $\Sigma \subset \mathrm{GL}_n(\mathbb{R})$.

Proof. [15, Prop 4.5] □

Then we obtain the following Theorem:

Theorem 3.16. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a reductive algebraic $\mathbb{Q}$-group, and let $\Sigma = \Sigma_{t,u}$ for $t \geq \frac{2}{\sqrt{3}}$, $u \geq \frac{1}{2}$, be a Siegel set of $\mathrm{GL}_n(\mathbb{R})$. Then we can find an element $a \in \mathrm{GL}_n(\mathbb{R})$ and elements $b_1, \ldots, b_r \in \mathrm{GL}_n(\mathbb{Z})$ such that the set*

$$\Omega = (\bigcup_{i=1}^{r} a\Sigma b_i) \cap G$$

is a fundamental set for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$.

Proof. The original proof can be found in [3]. □

As a corollary, we obtain that for every arithmetic subgroup of a connected $\mathbb{Q}$-group there exists an open fundamental set. The next question, which naturally arises, is about the topological structure of the quotient $G_{\mathbb{R}}/G_{\mathbb{Z}}$, or equivalently, of the fundamental set developed above. As a first result, we want to give a criterion for compactness of a subset of $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$, which is due to Mahler.

Proposition 3.17 (Mahler's Criterion)**.** *A subset $\Omega \subset \mathrm{GL}_n(\mathbb{R})$ is relatively compact modulo $\mathrm{GL}_n(\mathbb{Z})$ if and only if*

(i) $\det(g)$ *is bounded for all $g \in \Omega$;*

(ii) $\Omega(\mathbb{Z}^n \setminus \{0\}) \cap U = \emptyset$ *for a suitable neighborhood U of 0 in $\mathbb{R}^n$.*

Proof. The original proof can be found in [13]. □

For the general case of an arbitrary $\mathbb{Q}$-group G we need some preparation. First, we can clearly reduce to the case of connected groups. Let $G = HR_u(G)$ be the Levi-decomposition of G, where H is a maximal reductive subgroup of G. We want to lead the case back to the reductive part of G, for which the following Lemma is needed.

Lemma 3.18. *Let $H \subset G$ be a reductive subgroup of a connected group G, both defined over $\mathbb{Q}$. Then $H_{\mathbb{R}}/H_{\mathbb{Z}}$ is closed in $G_{\mathbb{R}}/G_{\mathbb{Z}}$.*

Proof. [15, Lemma 4.15] □

From that, we can deduce that the compactness of $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is equivalent to that of $H_{\mathbb{R}}/H_{\mathbb{Z}}$. Indeed, if the first quotient is compact, then clearly also the latter as a closed subset. Conversely, let $H_{\mathbb{R}}/H_{\mathbb{Z}}$ be compact. Then we can find a compact fundamental set Σ such that $H_{\mathbb{R}} = \Sigma H_{\mathbb{Z}}$. Then from Proposition 3.14 we obtain that $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is compact. So it suffices to consider connected reductive groups. We will omit the rest of the proof of the following Theorem which states an important criteria for compactness.

Theorem 3.19. *Let G be an algebraic $\mathbb{Q}$-group. Then the following are equivalent:*

(i) $G_{\mathbb{R}}/G_{\mathbb{Z}}$ *is compact*

(ii) Every unipotent element of $G_{\mathbb{Q}}$ *belongs to the unipotent radical of* G *and* $X(G^0)_{\mathbb{Q}} = 1$.

Proof. [15, Theorem 4.12] □

Another question of interest is under which conditions the quotient $G_{\mathbb{R}}/G_{\mathbb{Z}}$ has finite invariant volume. Here the result, which is proven in [15, ch. 4.6], is stated without any further preparation.

Theorem 3.20. *Let G be an algebraic $\mathbb{Q}$-group. Then the following are equivalent:*

(i) $G_{\mathbb{R}}/G_{\mathbb{Z}}$ *has finite invariant volume*

(ii) G^0 *does not have non-trivial* $\mathbb{Q}$*-characters.*

Proof. [15, Theorem 4.13] □

It is clearly interesting to compute the volume of $G_{\mathbb{R}}/G_{\mathbb{Z}}$ (if it exists) with respect to a canonical Haar measure. Due to Langlands, the solution of this problem is strongly related to the theory of Eisenstein series and other analytic techniques which are not treated in this diploma thesis. However, we want to consider the case $G = \mathrm{SL}_2$, for which the computation can be made directly.

Example. Let $G = \mathrm{SL}_2$. We have seen at the end of chapter 2.2 that a $\mathbb{Q}$-rational left-invariant differential form is given by

$$\omega = \alpha d\varphi \wedge d\alpha \wedge du,$$

where the coordinates φ, α and u can be computed for any $x \in G_{\mathbb{R}}$ by considering the Iwasawa decomposition

$$x = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

So the volume of $G_{\mathbb{R}}/G_{\mathbb{Z}}$ can be expressed as $\int_F \alpha d\varphi d\alpha du$, where $F \subset G_{\mathbb{R}}$ is a fundamental domain with respect to $G_{\mathbb{Z}}$. So our aim is to construct such a fundamental domain F. For that, let $\mathcal{H} = \mathrm{SO}_2(\mathbb{R})\backslash \mathrm{SL}_2(\mathbb{R})$ denote the upper half plane of the complex plane. We have seen that we have a right action of $\mathrm{SL}_2(\mathbb{R})$ on P via

$$\rho\colon \left(\begin{pmatrix} x & y \\ u & t \end{pmatrix}, z \right) \mapsto \frac{tz + y}{uz + x}.$$

It can be shown that the closed set

$$\overline{D} = \{z \in \mathcal{H} \mid |\Re(z)| \leq \frac{1}{2}, |z| \geq 1\},$$

where $\Re(z)$ denotes the real part of z, is a fundamental domain for the induced action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{H}$. Now define the subsets

$$\mathrm{K}_0 = \left\{ \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \mid \varphi \in [0,\pi] \right\},$$
$$\Omega = \left\{ (\alpha, u) \in \mathbb{R}_{>0} \times \mathbb{R} \mid \rho \begin{pmatrix} \alpha & \alpha u \\ 0 & \alpha^{-1} \end{pmatrix} \in \overline{D} \right\},$$
$$\mathrm{D}_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid (\alpha, u) \in \Omega \right\}.$$

Then it is easy to see that we can define $F = K_0 D_0$. For the computation of the integral we need to write the set Ω in another way. For that, let $(\alpha, u) \in \Omega$. Then we have

$$\begin{aligned} (\alpha, u) \in \Omega &\Leftrightarrow \rho \begin{pmatrix} \alpha & \alpha u \\ 0 & \alpha^{-1} \end{pmatrix} \in \overline{D} \\ &\Leftrightarrow u + \frac{1}{\alpha^2} i \in \overline{D} \\ &\Leftrightarrow |u| \leq \frac{1}{2} \wedge |u + \frac{1}{\alpha^2} i| = \sqrt{u^2 + \frac{1}{\alpha^4}} \geq 1 \\ &\Leftrightarrow |u| \leq \frac{1}{2} \wedge 0 \leq \alpha \leq \frac{1}{\sqrt[4]{1-u^2}}, \end{aligned}$$

so it follows that

$$\Omega = \left\{ (\alpha, u) \in \mathbb{R}_{>0} \times \mathbb{R} \mid |u| \leq \frac{1}{2} \wedge 0 \leq \alpha \leq \frac{1}{\sqrt[4]{1-u^2}} \right\}.$$

So we obtain

$$\begin{aligned} \mathrm{vol}(G_{\mathbb{R}}/G_{\mathbb{Z}}) &= \int_0^\pi d\varphi \int_\Omega \alpha d\alpha du \\ &= \int_0^\pi d\varphi \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\frac{1}{\sqrt[4]{1-u^2}}} \alpha d\alpha du \\ &= \frac{\pi}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{\pi}{2} \arcsin(u)|_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{\pi}{2} \left(\frac{\pi}{6} + \frac{\pi}{6} \right) \\ &= \frac{\pi^2}{6} = \zeta(2). \end{aligned}$$

Here, ζ denotes the Riemannian zeta-function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for all $s \in \mathbb{C}$ with $\Re(s) > 1$. There is a strong relation to number theory which is reflected by the following formula:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}.$$

In the last part of this section we will generalize some of the definitions and results to arbitrary number fields. Let k be an algebraic number field, G a k-group and let $\mathcal{O}_k$ denote the ring of integers of k. Then a $\mathcal{O}_k$-arithmetic subgroup of G is a subgroup Γ of G which is commensurable with the group $G_{\mathcal{O}_k}$. The group $G_{\mathcal{O}_k}$ is a discrete subgroup of $G_\infty = \prod_{v \in V_k^\infty} G_{k_v}$, so this definition is the direct analogue of $G_\mathbb{Z}$ in $G_\mathbb{R}$. So we are also interested in developing a reduction theory in this case. The following theorem gives the most important results, whose proofs are tied to the case of $k = \mathbb{Q}$ by restriction of scalars.

Theorem 3.21. *Let G be an algebraic k-group, where k is an algebraic number field. Then the following hold:*

(i) there exists an open fundamental set $\Omega \subset G_\infty$ relative to $G_{\mathcal{O}_k}$, i.e.

- *(a) $\mathrm{K}\,\Omega = \Omega$ for a suitable maximal compact subgroup $\mathrm{K} \subset G_\infty$,*
- *(b) $\Omega G_{\mathcal{O}_k} = G_\infty$,*
- *(c) $\Omega^{-1}\Omega \cap x G_{\mathcal{O}_k} y$ is finite for all $x, y \in G_k$;*

(ii) $G_\infty / G_{\mathcal{O}_k}$ is compact if and only if every unipotent element of G_k belongs to the unipotent radical of G and $X(G^0)_k = 1$;

(iii) $G_\infty / G_{\mathcal{O}_k}$ has finite invariant volume if and only if $X(G^0)_k = 1$.

Proof. [15, Thm 4.17]. Choose an integral base of $\mathcal{O}_k$ over $\mathbb{Z}$ and use it to construct $H = \mathbf{R}_{k/\mathbb{Q}}(G)$. Then we obtain that $G_{\mathcal{O}_k} \cong H_\mathbb{Z}$ and $G_\infty \cong H_\mathbb{R}$, so (i) immediately follows from the case $k = \mathbb{Q}$. In addition, it can be shown that if $U = R_u(G)$ is the unipotent radical of G, then $U' = \mathbf{R}_{k/\mathbb{Q}}(U)$ is that of H. Furthermore, every unipotent element of G_k is in U if and only if every unipotent element of $H_\mathbb{Q}$ is in U'. Since we know from Proposition 1.12 that $X(H^0)_\mathbb{Q} = X(G^0)_k$, the claims (ii) and (iii) follow from Theorem 3.19 respectively Theorem 3.20. $\square$

4 Adelic algebraic groups

As we have seen in section 3.2 the adele ring and the idele group of an algebraic number field k contain a lot of information about the arithmetic properties of the ring of integers $\mathcal{O}_k$ respectively of its multiplicative group $\mathcal{O}_k^*$. Now we are interested in the $\mathcal{O}_k$-points of an algebraic group G defined over k. We could try to gain some structural properties out of $G_{\mathcal{O}_k}$ itself; however, from the discussion above we can expect to receive more information by considering the "adelic points" of G. The aim of this chapter is to introduce the adelic points of an algebraic group G. Although these points do not form an algebraic variety, we call G in this context an "adelic algebraic group" by abuse of language. Moreover, we want to obtain several result in analogy to the adele ring and the idele group. In particular, a reduction theory for adelic algebraic groups is built up.

4.1 Adelization of affine varieties

Throughout this section, let k be an algebraic number field and let X be an affine k-variety. Moreover, fix an embedding $\varphi\colon X \hookrightarrow \mathbf{A}_k^n$, so X is presented as a k-closed subset of $\mathbf{A}_k^n$. We want to associate to X an "adelic" variety $X_{\mathbb{A}}$, i.e., a subset of $\mathbb{A}^n$ corresponding to X. More details can be found in [15, ch. 5].

Definition. The space $X_{\mathbb{A}}$ is defined as the base change of X to the k-adeles $\mathbb{A}_k$. In other words,

$$X_{\mathbb{A}} = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in \mathcal{I}(X)\},$$

where $\mathcal{I}(X)$ denotes the vanishing ideal of X, endowed with the topology induced from the product topology.

Note that this definition is equivalent to defining $X_{\mathbb{A}}$ by means of the restricted product. In fact, for $\Lambda = V_k$ and $\Lambda_0 = V_k^\infty$ we have

$$X_{\mathbb{A}} = \prod_{v \in V_k} (X_{K_v} : X_{\mathcal{O}_v}).$$

In analogy to the construction of the ring of adeles, we can define the space of the S-integral adeles. To do so, let S be a finite subset of V_k containing V_k^∞. Now we can define the S-integral adele space $X_{\mathbb{A}(S)}$ as

$$X_{\mathbb{A}(S)} = \prod_{v \in S} X_{k_v} \times \prod_{v \notin S} X_{\mathcal{O}_v}.$$

In particular, for $S = V_k^\infty$, we write $X_{\mathbb{A}(\infty)}$ and call it the space of integral adeles. The topology on $X_{\mathbb{A}(S)}$ is the usual product topology. It follows from the definition of the adelization that $X_{\mathbb{A}}$ can be written as

$$X_{\mathbb{A}} = \bigcup_S X_{\mathbb{A}(S)}.$$

It can easily be seen that for two k-varieties X and Y we have

$$(X \times Y)_{\mathbb{A}} = X_{\mathbb{A}} \times Y_{\mathbb{A}}.$$

The diagonal embedding $k \hookrightarrow \mathbb{A}$ induces an embedding $k^n \hookrightarrow \mathbb{A}^n$, thus also $X_k \hookrightarrow X_{\mathbb{A}}$. The image of this embedding is called the space of principal adeles. Since k is discrete in $\mathbb{A}$, it follows that the k-points of X (more exactly, their image under the diagonal mapping) are also discrete (and closed) in $X_{\mathbb{A}}$. For simplification of notation, we usually identify X_k with its image in $X_{\mathbb{A}}$.

We could now define an adelic algebraic group simply as the adelization of its underlying variety (what we will do later). However, we first have to prove that the construction above is independent of the embedding of X in an affine space $\mathbf{A}_k^n$. For that, we need the notion of functions between adelic spaces.

Let X, Y be k-varieties, let $f\colon X \to Y$ be a k-rational map. Then f induces a continuous map $f_{k_v}: X_{k_v} \to Y_{k_v}$ for any $v \in V_k$. Now we define the adelization $f_{\mathbb{A}}$ of the function f to be the restriction of the product $\prod_{v \in V_k} f_{k_v}$ to $X_{\mathbb{A}}$. Clearly, we expect the function $f_{\mathbb{A}}$ to map into $Y_{\mathbb{A}}$ continuously.

Lemma 4.1. *Let $X \subset \mathbf{A}^n$, $Y \subset \mathbf{A}^m$ be k-varieties, let $f\colon X \to Y$ be a k-rational map. Then $f_{\mathbb{A}}(X_{\mathbb{A}}) \subset Y_{\mathbb{A}}$ and the map $f_{\mathbb{A}}\colon X_{\mathbb{A}} \to Y_{\mathbb{A}}$ is continuous.*

Proof. Since f is a k-rational map between X and Y we know that f is of the form

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

with polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$. Now let S_0 be a finite subset of V_k containing V_k^∞, so that the coefficients of all the f_i's are v-integers for $v \notin S_0$. Then it follows that $f_{K_v}(X_{\mathcal{O}_v}) \subset Y_{\mathcal{O}_v}$ for any $v \notin S_0$, hence $f_{\mathbb{A}}(X_{\mathbb{A}(S)}) \subset Y_{\mathbb{A}(S)}$ for any subset S of V_k containing S_0. Since $X_{\mathbb{A}(S)}$ inherits the direct product topology, we obtain that $f_{\mathbb{A}}|_{X_{\mathbb{A}(S)}}$ is continuous. Now $X_{\mathbb{A}}$ (resp. $Y_{\mathbb{A}}$) can be written as the union of all $X_{\mathbb{A}(S)}$ (resp. $Y_{\mathbb{A}(S)}$), so the Lemma follows. □

Now we want to prove the independence of the construction of the adelic space from the representation of the variety at the beginning of this section. Since any two presentations of the k-variety X are biregular k-isomorphic, we have to ensure that these presentations remain homeomorph when passing to the adelization.

Proposition 4.2. *Let $f\colon X \to Y$ be a biregular k-isomorphism between two k-closed subsets $X \subset \mathbf{A}^n$, $Y \subset \mathbf{A}^m$. Then the adelization $f_{\mathbb{A}}\colon X_{\mathbb{A}} \to Y_{\mathbb{A}}$ is a homeomorphism.*

Proof. Let $g\colon Y \to X$ be the inverse of f, then g is also a k-regular map. So by Lemma 4.1 we obtain that $g_{\mathbb{A}}\colon Y_{\mathbb{A}} \to X_{\mathbb{A}}$ is continuous. Now g_{k_v} is the inverse of f_{k_v} for all $v \in V_k$, thus $g_{\mathbb{A}}$ is the inverse of $f_{\mathbb{A}}$. □

Now the time has come to define an adelic algebraic group. For that, we have seen in chapter 1.1 that every algebraic k-group can be realized as a k-closed subgroup of $\mathrm{GL}_n(k)$ for a positive integer n. Now we want to use that fact to reduce the definition of adelization of an algebraic group to that of the general linear group. However, we have to ensure that for an arbitrary algebraic k-group G the induced topology from $\mathrm{GL}_n(k)$ and the topology of the adelization of the underlying space coincide. But this is an easy fact which we state in the following Lemma:

Lemma 4.3. *Let Y be a closed k-subvariety of X. Then $Y_{\mathbb{A}} = X_{\mathbb{A}} \cap \prod_v Y_{k_v}$. Moreover, the topology on $Y_{\mathbb{A}}$ is induced from that on $X_{\mathbb{A}}$.*

Proof. We clearly have $Y_{\mathbb{A}} \subset X_{\mathbb{A}} \cap \prod_v Y_{k_v}$. Conversely, let $x \in X_{\mathbb{A}} \cap \prod_v Y_{k_v}$. Then $x_v \in X_{\mathcal{O}_v}$ for almost all $v \in V_k^f$, so in particular $x_v \in \mathcal{O}_v^n$. Moreover, $f(x_v) = 0$ for all $f \in \mathcal{I}(Y)$, where $\mathcal{I}(Y)$ denotes the vanishing ideal of Y. Thus,

$$x_v \in X_{\mathcal{O}_v} \cap Y_{k_v} = Y_{\mathcal{O}_v} \text{ for almost all } v \in V_k^f.$$

So we obtain that $x \in \prod_{v \in V_k} Y_{k_v}$ and $x_v \in Y_{\mathcal{O}_v}$ for almost all $v \in V_k^f$, i.e., $x \in Y_{\mathbb{A}}$. Since the topology on Y_{k_v} is induced form X_{k_v} for all $v \in V_k$ the topology on $Y_{\mathbb{A}}$ is induced from that on $X_{\mathbb{A}}$. □

Now we want to describe $\mathrm{GL}_{n_{\mathbb{A}}}$. For this, we consider the standard realization of GL_n as a hypersurface in $\mathbf{A}^{n^2+1}$:

$$\mathrm{GL}_n = \{(x_{11}, \dots, x_{nn}, y) \in \mathbf{A}^{n^2+1} \mid y \det(x_{ij}) - 1 = 0\}.$$

From the construction of the adelization it follows that $\mathrm{GL}_{n_{\mathbb{A}}}$ consists of all matrices in $\mathrm{M}_n(\mathbb{A})$ whose determinant is invertible in $\mathbb{A}$, i.e., $\mathrm{GL}_{n_{\mathbb{A}}} = \mathrm{GL}_n(\mathbb{A})$. We can also view $\mathrm{GL}_{n_{\mathbb{A}}}$ as restricted product. Since also $\mathrm{GL}_{n_{\mathcal{O}_v}} = \mathrm{GL}_n(\mathcal{O}_v)$, we obtain that

$$\mathrm{GL}_n(\mathbb{A}) = \prod_{v \in V_k} (\mathrm{GL}_n(k_v) : \mathrm{GL}_n(\mathcal{O}_v)).$$

A base of the topology on $\mathrm{GL}_n(\mathbb{A})$ is formed by defining the sets

$$U = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_v),$$

to be open, where S is a finite subset of V_k containing V_k^∞ and where the U_v are open subsets of $\mathrm{GL}_n(k_v)$. The subgroup

$$\mathrm{GL}_{n_{\mathbb{A}(S)}} = \mathrm{GL}_n(\mathbb{A}(S)) = \prod_{v \in S} \mathrm{GL}_n(k_v) \times \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_v)$$

is called the group of S-integral adeles. For $S = V_\infty^k$, we write $\mathrm{GL}_{n_{\mathbb{A}(\infty)}}$ instead of $\mathrm{GL}_{n_{\mathbb{A}(S)}}$ and call it the group of integral adeles. Again, the k-points of GL_n can be embedded diagonally into $\mathrm{GL}_{n_{\mathbb{A}}}$, where it is a discrete subset. We will identify GL_{n_k} with its image under this embedding, and call them the subgroup of principal adeles.

These concepts can be easily extended to arbitrary algebraic groups G, where we restrict ourselves to the case of a closed subgroup of $\mathrm{GL}_n(k)$. We define the adele group $G_{\mathbb{A}}$ as the restricted product

$$G_{\mathbb{A}} = \prod_{v \in V_k} (G_{k_v} : G_{\mathcal{O}_v}),$$

where $G_{\mathcal{O}_v} = G \cap \mathrm{GL}_n(\mathcal{O}_v)$ for $v \in V_k^f$, together with the induced topology from $\mathrm{GL}_n(\mathbb{A})$. Therefore a base for the topology is given by the sets

$$W = \prod_{v \in S} W_v \times \prod_{v \notin S} G_{\mathcal{O}_v},$$

where S is a finite subset of V_k containing V_k^∞ and $W_v \subset G_{k_v}$ are open sets. The adele group is a locally compact topological group and the k-points G_k form a discrete subgroup of $G_{\mathbb{A}}$, where we again identify G_k with its image under the diagonal embedding. For any finite subset $S \subset V_k$ containing V_k^∞ we can define the group of S-integral adeles

$$G_{\mathbb{A}(S)} = \prod_{v \in S} G_{k_v} \times \prod_{v \notin S} G_{\mathcal{O}_v};$$

we obviously have

$$G_{\mathbb{A}} = \bigcup_S G_{\mathbb{A}(S)}.$$

For $S = V_k^\infty$ we write $G_{\mathbb{A}(\infty)}$ instead of $G_{\mathbb{A}(V_k^\infty)}$. Note that the definition of $G_{\mathcal{O}_v}$ depends on the matrix realization of G as a closed subgroup of $\mathrm{GL}_n(k)$.

Another important term in the setting of adelic points is the concept of truncated adeles. For an arbitrary subset $S \subset V_k$ we define the group of S-adeles $G_{\mathbb{A}_S}$ as the image of $G_{\mathbb{A}}$ under the projection of $\prod_v G_{k_v}$ onto $\prod_{v \notin S} G_{k_v}$. Otherwise said, the group of S-adeles is the restricted product

$$G_{\mathbb{A}_S} = \prod_{v \in V_k \setminus S} (G_{k_v} : G_{\mathcal{O}_v}).$$

We can again consider the embedding $G_k \hookrightarrow G_{\mathbb{A}_S}$, whose image is called the group of principal S-adeles. Moreover, we can define the group of T-integral S-adeles. Let $S \subset V_k$ be an arbitrary subset of V_k, let T be a subset of V_k containing S. Then the T-integral S-adeles are defined as

$$G_{\mathbb{A}_S(T)} = \prod_{v \in T \setminus S} G_{k_v} \times \prod_{v \notin T \cup V_k^\infty} G_{\mathcal{O}_v}.$$

For $G_{\mathbb{A}_{V_k^\infty}}$ (resp $G_{\mathbb{A}_{V_k^\infty}}(V_k^\infty)$) we write $G_{\mathbb{A}_f}$ (resp. $G_{\mathbb{A}_f(\infty)}$). We need to introduce another notation. For $S \subset V_k$, we define $G_S = \prod_{v \in S} G_{k_v}$.

As in the non-adelic case we want to have the possibility to restrict ourselves to the case of the $\mathbb{Q}$-adeles. Let G be an algebraic k-group, let $H = \mathbf{R}_{k/\mathbb{Q}}(G)$ be the group obtained by restriction of scalars. Then it can be shown that we have an isomorphism $H_{\mathbb{A}_{\mathbb{Q}}} \cong G_{\mathbb{A}_k}$ of adelic algebraic groups.

As in the case of the adeles, we now want to get some topological information about the discrete subgroup G_k in $G_{\mathbb{A}}$. For that, we introduce the notion of strong approximation which will be important in the following section.

Definition. Let G be an algebraic k-group, let $S \subset V_k$ be a finite subset of k-primes containing V_k^∞. Then G is said to satisfy the strong approximation property relative to S if the image under the diagonal embedding $G_k \hookrightarrow G_{\mathbb{A}_S}$ is dense in $G_{\mathbb{A}_S}$. For $S = V_k^\infty$ we say that G has the absolute strong approximation property.

For an algebraic group G which satisfy the strong approximation property, this means that the k-points of G are dense in the set $\prod_{v \notin S}(G_{k_v} : G_{\mathcal{O}_v})$. In other words, the product $G_S G_k$, where G_S and G_k are viewed as subgroups in $G_{\mathbb{A}}$, is dense in the full adele group $G_{\mathbb{A}}$.

It is clearly possible to define the S-adelic space for an arbitrary k-variety X, just by adapting the definition of the adelic space. Moreover, there is also a notion of (absolute) strong approximation property for varieties which can be used to simplify several proofs; however, it is not as much known as in the setting of algebraic groups. Further information can be found in [15, ch.7].

We could also ask which role the subgroups of integral and principal adeles play in the structure of the whole adele group. For that, consider a decomposition

$$G_{\mathbb{A}} = \bigcup_{\lambda \in \Lambda} G_{\mathbb{A}(\infty)} x_\lambda G_k$$

of $G_{\mathbb{A}}$ into double cosets modulo $G_{\mathbb{A}(\infty)}$ and G_k. We define the class number of G as the cardinality of Λ, denoted by $\mathrm{cl}(G)$. As we will see later the class number of an arbitrary algebraic group is always finite. For that, we will use the Levi decomposition to reduce to the case of a reductive group.

First, we need a Lemma.

Lemma 4.4. *Let G be an algebraic k-group, let $G = HN$ be a semidirect product, where H, N are k-subgroups of G. Assume that N is normal in G. Then the adelization $N_{\mathbb{A}}$ of N is normal in $G_{\mathbb{A}}$ and $G_{\mathbb{A}} = H_{\mathbb{A}} N_{\mathbb{A}}$ is again a semidirect product.*

Proof. Since G is isomorphic to $H \times N$ as a variety, we get that

$$G_{\mathbb{A}} = (H \times N)_{\mathbb{A}} = H_{\mathbb{A}} \times N_{\mathbb{A}},$$

i.e., $G_{\mathbb{A}} = H_{\mathbb{A}} N_{\mathbb{A}}$. The normality of $N_{\mathbb{A}}$ in $G_{\mathbb{A}}$ follows from the normality of N_{k_v} in G_{k_v} for every $v \in V_k$. □

The next Proposition allows us in a special situation to reduce the computation of the class number of an algebraic group G to that of a subgroup H.

Proposition 4.5. *Let $G = HN$ be an algebraic k-group, let H, N be k-subgroups of G. Assume that N is normal in G and N satisfies the absolute strong approximation property. Then it follows that*

$$\mathrm{cl}(G) \leq \mathrm{cl}(H).$$

Proof. Since translation with an element $x \in G_{\mathbb{A}}$ is a homeomorphism and $N_{\mathbb{A}(\infty)}$ is normal in $G_{\mathbb{A}}$, the subgroup $x^{-1} N_{\mathbb{A}(\infty)} x$ is open in $N_{\mathbb{A}}$ for all $x \in G_{\mathbb{A}}$ as the image of the open subset $N_{\mathbb{A}(\infty)}$. Furthermore, N satisfies the absolute strong approximation property, hence the set $N_\infty N_k$ is dense in $N_{\mathbb{A}}$. Thus, the two open subsets $(x^{-1} N_{\mathbb{A}(\infty)} x) y$ and $N_\infty N_k$ must intersect for all $y \in N_{\mathbb{A}}$. Now N_∞ is a normal subgroup of $G_{\mathbb{A}}$ contained in $N_{\mathbb{A}(\infty)}$, so we get

$$N_{\mathbb{A}} = (x^{-1} N_{\mathbb{A}(\infty)} x) N_\infty N_k = (x^{-1} N_{\mathbb{A}(\infty)} N_\infty x) N_k = (x^{-1} N_{\mathbb{A}(\infty)} x) N_k.$$

In particular, for $x = 1$ this yields $N_{\mathbb{A}} = N_{\mathbb{A}(\infty)} N_k$. From the above equation we deduce

$$x N_{\mathbb{A}(\infty)} N_k = N_{\mathbb{A}(\infty)} x N_k$$

for any $x \in G_{\mathbb{A}}$. Now using Lemma 4.4 we obtain

$$G_{\mathbb{A}} = H_{\mathbb{A}} N_{\mathbb{A}} = H_{\mathbb{A}} N_{\mathbb{A}(\infty)} N_k = N_{\mathbb{A}(\infty)} H_{\mathbb{A}} N_k.$$

Now let $H_{\mathbb{A}} = \bigcup_i H_{\mathbb{A}(\infty)} x_i H_k$ be a decomposition into double cosets modulo the principal and integral adeles, then we get a decomposition of $G_{\mathbb{A}}$ via

$$G_{\mathbb{A}} = N_{\mathbb{A}(\infty)} H_{\mathbb{A}} N_k = \bigcup_i N_{\mathbb{A}(\infty)} H_{\mathbb{A}(\infty)} x_i H_k N_k = \bigcup_i G_{\mathbb{A}(\infty)} x_i G_k.$$

Now in the last union some of the cosets could coincide, thus we get the required result. □

The Proposition clearly implies that every algebraic group G which satisfies the absolute strong approximation property has class number equal to one. Moreover, we have the following Lemma:

Lemma 4.6. *Let U be a unipotent group defined over k. Then U satisfies the strong approximation property for any nonempty subset S.*

Proof. This proof is based on a consideration about birational morphisms between algebraic k-varieties and can be found in [15, Lemma 5.1]. □

The Lemma implies that every unipotent group has trivial class number. This will in fact be important if we use the Levi decomposition of an algebraic group and Proposition 4.5. So we can restrict ourselves to the case of a reductive group.

To prove the finiteness of the class number we need some more preparation. Since G_k is a discrete subgroup of $G_{\mathbb{A}}$, it is natural to ask whether a reduction theory can be developed. In the next section, we will handle the case $G = \mathrm{GL}_n$.

4.2 Minkowski reduction of $\mathrm{GL}_n(\mathbb{A})$

Since for an algebraic k-group G the k-points G_k are discrete in $G_{\mathbb{A}}$, we could ask if we can find a suitable "nice" subset $\Omega \subset G_{\mathbb{A}}$ so that the G_k-translates of Ω cover the whole adele group in a suitable approximative way. More precisely, we have the following definition.

Definition. A subset Ω of $G_{\mathbb{A}}$ is called a fundamental set for G_k if

(1) $\Omega G_k = G_{\mathbb{A}}$

(2) $\Omega^{-1}\Omega \cap G_k$ is finite.

If the intersection in (2) is empty, then we say that Ω is a fundamental domain for G_k.

In other words, condition (2) means that the intersection of two G_k-translates of the subset Ω is finite (resp. empty) if Ω is a fundamental set (resp. domain).

In this subsection, we want to develop a reduction theory for the case $\mathrm{GL}_n(\mathbb{A})$, i.e., we want to find a suitable fundamental set for $\mathrm{GL}_n(\mathbb{Q})$ in $\mathrm{GL}_n(\mathbb{A})$. This procedure is called Minkowski reduction. The general case of an arbitrary algebraic group G is handled in section 4.3. The discussion mainly follows [6].

From Proposition 1.8 we know that the group $\mathrm{GL}_n(\mathbb{R})$ contains a maximal compact subgroup, namely the group $\mathrm{O}(n)$ of orthogonal matrices. Moreover, it can be shown that also the general linear groups $\mathrm{GL}_n(\mathbb{Q}_v)$ with $v \in V_{\mathbb{Q}}^f$ has the same property. In this setting, the maximal compact subgroup is $\mathrm{GL}_n(\mathbb{Z}_p)$. Now we have the following definition.

Definition. For all $v \in V_\mathbb{Q}$ we define K_v as the maximal compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_v)$; namely

$$\mathrm{K}_v = \begin{cases} \mathrm{O}(n) & \text{if } v \in V_\mathbb{Q}^\infty \\ \mathrm{GL}_n(\mathbb{Z}_p) & \text{if } v \in V_\mathbb{Q}^f \end{cases} .$$

As a first step, we want to locally decompose a matrix into a compact and an upper triangular part. This result is delivered by the following Proposition.

Proposition 4.7. *Let* $v \in V_\mathbb{Q}$, $\mathrm{U}_v = \{g \in \mathrm{GL}_n(\mathbb{Q}_v) \mid g \text{ is upper triangular }\}$. *Then*

$$\mathrm{GL}_n(\mathbb{Q}_v) = \mathrm{K}_v\, \mathrm{U}_v,$$

to be called Iwasawa decomposition.

Proof. First, let $v \in V_\mathbb{Q}^\infty$ and let $g \in \mathrm{GL}_n(\mathbb{Q}_v) = \mathrm{GL}_n(\mathbb{R})$. Let g^i denote the i-th column of g for $i = 1, \dots, n$. Applying the Gram-Schmidt process on these vectors, we obtain an orthonormal family $\{v^1, \dots, v^n\}$ of vectors, where every v^i is a linear combination of the vectors g^j for $j = 1, \dots, i$. By viewing the vectors v^i, $i = 1, \dots, n$ as the columns of a matrix $k \in \mathrm{O}(n)$ we obtain a decomposition

$$k = gb,$$

where the i-th column of b contains the coefficients of v^i with respect to the basis $\{g^1, \dots, g^n\}$. From the Gram-Schmidt process we obtain that $b \in U_v$. Since the inverse of an upper triangular matrix is again upper triangular, we can write $g = k\tilde{b}$ with $\tilde{b} \in U_v$ and $k \in \mathrm{O}(n)$. Thus,

$$\mathrm{GL}_n(\mathbb{R}) = \mathrm{O}(n)\, \mathrm{U}_v\, .$$

Now let $v \in V_\mathbb{Q}^f$, $g \in \mathrm{GL}_n(\mathbb{Q}_v)$. We have to find an element $k \in \mathrm{GL}_n(\mathbb{Z}_v)$ such that $kg \in U_v$. This will be done by induction on n.

For $n = 1$, there is nothing to show. So let $n > 1$. Since $g \in \mathrm{GL}_n(\mathbb{Q}_v)$, the first column $g^{(1)}$ of g is not the zero vector. So there is an entry of $g^{(1)}$ with maximal v-adic norm. Now we can choose a permutation matrix P so that $Pg = \tilde{g}$ and that $\tilde{g}_{11}$ has maximal v-adic norm in the first column. Define $\lambda_i = -\frac{\tilde{g}_{i1}}{\tilde{g}_{11}}$; then we have $\lambda_i \in \mathbb{Z}_v$. Let $T \in \mathrm{GL}_n(\mathbb{Z}_v)$ be the matrix

$$T = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \lambda_2 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \lambda_n & 0 & \dots & 1 \end{pmatrix}.$$

Then by multiplication we obtain

$$T\tilde{g} = \begin{pmatrix} \tilde{g}_{11} & * \\ 0 & g' \end{pmatrix},$$

where g' is an element of $\mathrm{GL}_{n-1}(\mathbb{Q}_v)$. By the induction hypothesis, we can find an element $k' \in \mathrm{GL}_{n-1}(\mathbb{Z}_v)$ such that $k'g' \in U_v^{(n-1)}$. Now define

$$k = \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} TP.$$

Then it follows that

$$kg = \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} TPg = \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} T\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} \begin{pmatrix} \tilde{g}_{11} & * \\ 0 & g' \end{pmatrix} = \begin{pmatrix} \tilde{g}_{11} & * \\ 0 & k'g' \end{pmatrix} \in U_p^{(n)}.$$

□

It is clear that we could improve the above decomposition by splitting the upper triangular part into a diagonal and an unipotent part. Furthermore, we can use this local result to obtain a global decomposition. For that, let $\Delta_n \subset \mathrm{GL}_n$ denote the algebraic $\mathbb{Q}$-group of n-dimensional invertible diagonal matrices.

Proposition 4.8. *Define the following subsets of* $\mathrm{GL}_n(\mathbb{A})$*:*

$$\begin{gathered} \mathrm{K} = \textstyle\prod_{v\in V_{\mathbb{Q}}} K_v \\ \mathrm{A}_{\mathbb{R}} = \{g = (g_{ij}) \in \Delta_n(\mathbb{R}) \mid g_{ii} > 0 \textit{ for all } i = 1, \dots, n\} \\ \mathrm{A} = \mathrm{A}_{\mathbb{R}} \times \textstyle\prod_{v\in V_{\mathbb{Q}}} \{\mathrm{id}\} \\ \mathrm{D} = \Delta_n(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{Q}) \\ \mathrm{N} = \{(n_{ij}) \in \mathrm{GL}_n(\mathbb{A}) \mid n_{ij} = 0 \textit{ if } i > j \textit{ and } n_{ii} = 1 \textit{ for all } i\}. \end{gathered}$$

Then we have

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{K}\,\mathrm{A}\,\mathrm{D}\,\mathrm{N}\,.$$

Proof. Let $g \in \mathrm{GL}_n(\mathbb{A})$. By using the local Iwasawa decomposition we can write g at every place $v \in V_{\mathbb{Q}}$ as $g^{(v)} = k^{(v)}u^{(v)}$ with $k^{(v)} \in \mathrm{K}_v$, $u^{(v)} \in \mathrm{U}_v$. Since $g^{(v)} \in \mathrm{K}_v$ for almost all $v \in V_{\mathbb{Q}}$, it follows that $u^{(v)} = \mathrm{id}_{\mathbb{Q}_v^n}$ for almost all v. Now define $k = (k^{(v)})_v \in \mathrm{K}$, $u = (u^{(v)})_v \in \mathrm{GL}_n(\mathbb{A})$. Since u is upper triangular, we can find $n \in N$, and a diagonal matrix $\tilde{d} \in \mathrm{GL}_n(\mathbb{A})$ such that $u = \tilde{d}n$. Note that $u^{(v)} = \mathrm{id}$ for almost all v imply that $\tilde{d}^{(v)} = \mathrm{id}$ for almost all v. From the fact that $\tilde{d}_i^{(v)} = 1$ for almost all $v \in V_k$ it follows that $\tilde{d}_i \in \mathbb{J}$. Now we have

$$\mathbb{J} = \mathbb{Q}^* \left(\mathbb{R}_{>0} \times \prod_{v \in V_{\mathbb{Q}}^f} \mathbb{Z}_v^* \right),$$

thus we can choose $q_i \in \mathbb{Q}^*$ such that $q_i\tilde{d}_i \in \mathbb{R}_{>0} \times \prod_{v\in V_{\mathbb{Q}}^f} \mathbb{Z}_v^*$. Now set $d = \mathrm{diag}(q_1, \dots, q_n) \in \mathrm{D}$. Then

$$d\tilde{d} = \begin{pmatrix} q_1\tilde{d}_1 & 0 & \dots & 0 \\ 0 & \ddots & \vdots & \\ 0 & \dots & 0 & q_n\tilde{d}_n \end{pmatrix} = ra,$$

where $a \in \mathrm{A}$ and $r \in \{\mathrm{id}\} \times \prod_{v\in V_{\mathbb{Q}}^f} \Delta_n(\mathbb{Z}_v^*) \subset \mathrm{K}$. This yields

$$g = ku = k\tilde{d}n = kdd^{-1}\tilde{d}n = kd\tilde{d}d^{-1}n = \underbrace{kr}_{\in \mathrm{K}} ad^{-1}n \in \mathrm{K}\,\mathrm{A}\,\mathrm{D}\,\mathrm{N}\,.$$

□

The above result provides a nice characterization of the full adele group in terms of subgroups; however, since we have a right action of the $\mathbb{Q}$-points of GL_n on $\mathrm{GL}_n(\mathbb{A})$, we would prefer the discrete part (i.e. the subgroup D) to be on the right side. For that, let $g = dn \in \mathrm{GL}_n(\mathbb{A})$ be an upper triangular matrix with $d \in \mathrm{D}$ and $n \in \mathrm{N}$. Then we recursively can find a matrix $\tilde{n} \in \mathrm{N}$ so that $g = \tilde{n}d$. Thus, $\mathrm{N\,D} = \mathrm{D\,N}$, so the above decomposition can be written as

$$\mathrm{GL}_n(\mathbb{A}) = \mathrm{K\,A\,N\,D}\,.$$

Our next aim will be the decomposition of N into a subset $\mathrm{N}_{\mathbb{Q}} \subset \mathrm{GL}_n(\mathbb{Q})$ and a subset consisting of unipotent matrices with bounded off-diagonal entries.

Lemma 4.9. *Let* $\mathrm{N}_{\mathbb{Q}} = \mathrm{N} \cap \mathrm{GL}_n(\mathbb{Q})$, $\mathrm{N}_{\frac{1}{2}} = \{n \in \mathrm{N} \mid n_{ij} \in \left[-\frac{1}{2}, \frac{1}{2}\right] \times \prod_p \mathbb{Z}_p\}$. *Then*

$$\mathrm{N} = \mathrm{N}_{\frac{1}{2}}\, \mathrm{N}_{\mathbb{Q}}\,.$$

Proof. Let $x = (x_{ij}) \in \mathrm{N}$. From chapter 3.2 we know that

$$\mathbb{A}_k = k + \mathbb{A}(\infty) = \mathbb{Q} + \left(\mathbb{R} \times \prod_{v \in V_k^f} \mathbb{Z}_v\right).$$

Now for any $a \in \mathbb{A}_k$ we can find an element $z \in \mathbb{Z}$ so that the infinite part of $a + z$ lies in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Moreover, for any $v \in V_{\mathbb{Q}}^f$ we have $(a+z)_v \in \mathbb{Z}_v$, so we obtain

$$\mathbb{A}_k = \mathbb{Q} + \left(\left[-\frac{1}{2}, \frac{1}{2}\right] \times \prod_{v \in V_k^f} \mathbb{Z}_v\right).$$

This implies that we can recursively find an element $y = (y_{ij}) \in \mathrm{N}_{\mathbb{Q}}$ so that $xy \in \mathrm{N}_{\frac{1}{2}}$. □

Corollary 4.10. $\mathrm{GL}_n(\mathbb{A}) = \mathrm{K\,A\,N}_{\frac{1}{2}}\,\mathrm{N}_{\mathbb{Q}}\,\mathrm{D}$ *and for each* $g \in \mathrm{GL}_n(\mathbb{A})$ *there is an element* $\gamma \in \mathrm{GL}_n(\mathbb{Q})$ *with* $g\gamma \in \mathrm{K\,A\,N}_{\frac{1}{2}}$.

Note that a decomposition of the form $g = kand$ with $k \in \mathrm{K}$, $a \in \mathrm{A}$, $n \in \mathrm{N}_{\frac{1}{2}}$, $d \in \mathrm{GL}_n(\mathbb{Q})$ is not necessarily unique.

Thus we have found a subset of $\mathrm{GL}_n(\mathbb{A})$ which satisfies condition (1) of definition 4.2. However, the intersection of two $\mathrm{GL}_n(\mathbb{Q})$-translates of the subset $\mathrm{K\,A\,N}_{\frac{1}{2}}$ is not finite in general. The reason for that is the size of A. Therefore we want to improve this decomposition of $\mathrm{GL}_n(\mathbb{A})$ by shrinking the set A to a smaller subset. For the analysis of this right action, we define a height function as follows:

Definition. For each $v \in V_{\mathbb{Q}}$ we define a local height function $\eta_v \colon \mathbb{Q}_v^n \to \mathbb{R}_{\geq 0}$ by

$$\eta_v(x) = \begin{cases} \sqrt{x_1^2 + \cdots + x_n^2} & \text{if } v \in V_{\mathbb{Q}}^{\infty} \\ \sup_i |x_i|_v & \text{if } v \in V_{\mathbb{Q}}^f \end{cases},$$

where $|.|_v$ denotes the v-adic valuation on $\mathbb{Q}_v$ for all $v \in V_{\mathbb{Q}}^f$.

So for an element $x = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ the height function has a large value at a finite place $v \in V_{\mathbb{Q}}^f$ if the denominator of any x_i contains a large power of p_v in its prime decomposition (where p_v is the prime number associated to v) .

Since η_v is continuously defined on a dense subspace of $\mathbb{Q}_v$ for all $v \in V_{\mathbb{Q}}$, we can extend the local heights to the domain $\mathbb{Q}_v$. Now we could define a height function on $\mathbb{A}^n$ simply by multiplying the local heights for all $\mathbb{Q}$-primes v; however, this product may not converge. So we restrict ourselves to a subset of $\mathbb{A}^n$ on which this product has a well-defined value.

Definition. (1) Let $P = \{x \in \mathbb{A}^n \mid \eta_v(x^{(v)}) = 1 \text{ for almost all } v \in V_{\mathbb{Q}}\}$. Then the elements $x \in P$ are called primitive.

(2) Let $x = (x_1, \ldots, x_n) \in \mathbb{A}^n$ be a primitive vector. Then define the (global) height function as

$$\eta(x) = \prod_{v \in V_{\mathbb{Q}}} \eta_v(x^{(v)}),$$

where $x^{(v)} = (x_1^{(v)}, \ldots, x_n^{(v)})$ for all $v \in V_{\mathbb{Q}}$.

In the next Lemma some basic properties of the height function are given.

Lemma 4.11. *(i) If $t \in \mathbb{J}$, then it follows for all $x \in P$ that $tx \in P$ and $\eta(tx) = |t|\eta(x)$, where $|.|$ denotes the idelic norm.*

(ii) For all $g \in \mathrm{GL}_n(\mathbb{A})$, $x \in P$ we have $gx \in P$.

(iii) The set $\mathbb{Q}^n \setminus 0$ is a subset of P.

(iv) $\mathrm{GL}_n(\mathbb{A})(\mathbb{Q}^n \setminus 0) = P^* := \{x \in P \mid \eta(x) \neq 0\}$.

(v) If $x \in P^*$, then there is an element $q \in \mathbb{Q}^*$ such that $qx \in \mathbb{R}^n \times \prod_p S^{n-1}(\mathbb{Q}_p)$ and $\eta(x) = \eta_\infty(qx)$. Here $S^{n-1}(\mathbb{Q}_p) = \{x \in \mathbb{Q}_p^n \mid \eta_p(x) = 1\}$ denotes the $(n-1)$-dimensional unit sphere in $\mathbb{Q}_p$.

(vi) If $g \in \mathrm{GL}_n(\mathbb{A})$, then $g(\mathbb{Q}^n \setminus 0)$ is discrete and closed in $\mathbb{A}^n$.

Proof. (i) Since $|t^{(v)}|_v = 1$ for almost all v it follows that

$$\eta_v((tx)^{(v)}) = \eta_v(t^{(v)}x^{(v)}) = |t^{(v)}|\eta_v(x^{(v)}) = 1$$

for almost all v and thus

$$\eta(tx) = \prod_v \eta_v((tx)^{(v)}) = \prod_v |t^{(v)}|_v \eta_v(x^{(v)}) = \prod_v |t^{(v)}|_v \prod_v \eta_v(x^{(v)}) = |t|\eta(x).$$

(ii) Let $g \in \mathrm{GL}_n(\mathbb{A})$, $x \in P$. Then $g^{(v)} \in \mathrm{K}_v$ for almost all v. From the definition of the height function it follows immediately that for $k \in \mathrm{K}_v$ we have $\eta_v(kh) = \eta_v(h)$ for all $h \in \mathrm{GL}_n(\mathbb{A}_v)$. This yields

$$\eta_v((gx)^{(v)}) = \eta_v((g^{(v)}x^{(v)}) = \eta_v(x^{(v)}) \text{ for almost all } v.$$

Since $x \in P$, we have $\eta_v((gx)^{(v)}) = 1$ for almost all v.

(iii) Let $x \in \mathbb{Q} \setminus \{0\}$. Then in the coordinates of x only finitely many primes appear, i.e., $x \in (\mathbb{Z}_v^*)^n$ for almost all v, thus $\eta_v(x) = 1$ for almost all v.

(iv) One inclusion is clear by combining part (ii) and (iii). So let $x \in P^*$ and let $S = V_\infty \cup \{v \mid \eta_v(x^{(v)}) \neq 1\}$. For $v \in S$, we can simply choose an element $g^{(v)} \in \mathrm{GL}_n(\mathbb{Q}_v)$ such that $g^{(v)}x^{(v)} = e_1$. Now let $v \notin S$. Then we can choose $h^{(v)} \in \mathrm{GL}_n(\mathbb{Q}_v)$ such that $h^{(v)}e_1 = x^{(v)}$. Using the Iwasawa decomposition, we can write $h^{(v)} = k^{(v)}u^{(v)}$, so $u^{(v)}e_1 = (k^{(v)})^{-1}x^{(v)}$. Now we have $\eta_v((k^{(v)})^{-1}x^{(v)}) = 1 = |u_{11}^{(v)}| \in \mathbb{Z}_v^*$, thus $e_1 = \frac{1}{u_{11}^{(v)}}(k^{(v)})^{-1}x^{(v)}$.

(v) Let $x \in P^*$. Then there are only finitely many primes $p_1, \ldots, p_k$ with $\eta_{p_i}(x) \neq 1$ for all i. Now choose $e_i \in \mathbb{Z}$ with $\eta_{p_i}(p_i^{e_i}x) = 1$ for all i and put $q := \prod_{i=1}^k p_i^{e_i}$. Then we obtain

$$\eta(x) = \eta(qx) = \eta_\infty(qx).$$

(vi) Since $\mathbb{Q} \subset \mathbb{A}$ is discrete and closed we obtain the same for $\mathbb{Q}^n \subset \mathbb{A}^n$. Now $\mathbb{A}$ is a topological ring and for all $g \in \mathrm{GL}_n(\mathbb{A})$ the map

$$\begin{aligned} l_g\colon \quad \mathbb{A}^n &\to \mathbb{A}^n \\ x &\mapsto gx \end{aligned}$$

is a polynomial in x, thus a homeomorphism with inverse $l_g^{-1} = l_{g^{-1}}$. Therefore, $g(\mathbb{Q} \setminus \{0\})$ is discrete and closed.

□

From the Lemma it immediately follows:

Corollary 4.12. *The function η is well defined on the n-dimensional projective space $\mathbb{P}^n_{\mathbb{Q}} := (\mathbb{Q} \setminus \{0\})/\mathbb{Q}^*$.*

The next Proposition shows that the set of projective points, whose image under $g \in \mathrm{GL}_n(\mathbb{A})$ has bounded height, is finite. This in fact will be needed for the construction of a fundamental set, or more exactly, for the proof of Proposition 4.16.

Proposition 4.13. *Let $c > 0$ and $g \in \mathrm{GL}_n(\mathbb{A})$. Then the set*

$$\{x \in \mathbb{P}^n_{\mathbb{Q}} \mid \eta(gx) \leq c\}$$

is finite. In particular, there is a vector $x_0 \in \mathbb{Q}^n \setminus \{0\}$ such that $\eta(gx_0)$ is minimal.

Proof. Let $x \in \mathbb{Q}^n \setminus \{0\}$. By 4.11(v) there is an element $q \in \mathbb{Q}^*$ such that

$$qgx \in \mathbb{R}^n \times \prod_p S^{n-1}(\mathbb{Q}_p).$$

Now let $c > 0$. If $\eta(gx) \leq c$, then by 4.11(v) we see that

$$g(qx) = q(gx) \in F := \overline{B_c(0)} \times \prod_p S^{n-1}(\mathbb{Q}_p).$$

The set F is compact in $\mathbb{A}^n$. Now we know from 4.11(vi) that $g(\mathbb{Q}^n \setminus \{0\})$ is discrete and closed in $\mathbb{A}^n$, thus the set

$$g(\mathbb{Q}^n \setminus \{0\}) \cap \overline{B_c(0)} \times \prod_p S^{n-1}(\mathbb{Q}_p)$$

is finite. This means that for all $x \in \mathbb{Q}^n \setminus \{0\}$ with $\eta(x) \leq c$ we can find an element $q \in \mathbb{Q}^*$ so that $g(qx) \in F$. Thus, the set $g^{-1}F$ is finite. Now the classes $[x]$ and $[qx]$ coincide in $\mathbb{P}^n_{\mathbb{Q}}$, so the claim follows. $\square$

Definition. Define the function $\Phi\colon \mathrm{GL}_n(\mathbb{A}) \to \mathbb{R}$ by $g \mapsto \eta(ge_1)$.

By definition, Φ associates with any element $g \in \mathrm{GL}_n(\mathbb{A})$ the height of the image of the first base vector in $\mathbb{Q}^n$ under g. Let $g = kand$ with $k \in \mathrm{K}$, $a \in \mathrm{A}$, $n \in \mathrm{N}$ and $d \in \mathrm{D}$ the decomposition of g as in Proposition 4.8. The next Lemma will show that the values this function takes only depend on the matrix a. This in fact will be useful to improve the decomposition by reducing A to a smaller subset.

Lemma 4.14. *Let $g \in \mathrm{GL}_n(\mathbb{A})$, $k \in \mathrm{K}$, $n \in \mathrm{N}$, $a \in \mathrm{A}$ and $d \in \mathrm{D}$. Then we get*

(i) $\Phi(gn) = \Phi(g)$.

(ii) $\Phi(kg) = \Phi(g)$.

(iii) $\Phi(gd) = \Phi(g)$.

(iv) $\Phi(a) = |a_1|_\infty$, *where a_1 is the first column of a.*

In particular, for $g \in \mathrm{GL}_n(\mathbb{A})$, $g = kand$, we have $\Phi(g) = |a_1|_\infty$.

Proof. (i) $\Phi(gn) = \eta(gne_1) = \eta(ge_1) = \Phi(g)$.

(ii) $\Phi(kg) = \eta(kge_1) = \eta(ge_1) = \Phi(g)$.

(iii) The element d_{11} is in $\mathbb{Q}^*$ embedded diagonally into the ideles $\mathbb{J}_{\mathbb{Q}}$. Therefore, it commutes with every $g \in \mathrm{GL}_n(\mathbb{A})$. Using 4.11(i) and the product formula, we obtain that

$$\Phi(gd) = \eta(gde_1) = \eta(gd_{11}e_1) = \eta(d_{11}ge_1) = |d_{11}|\eta(ge_1) = \eta(ge_1) = \Phi(g).$$

(iv) $\Phi(a) = \eta(ae_1) = \eta(a_1) = |a_1|_\infty$.

$\square$

We are now ready to give the central Lemma from which the desired decomposition follows. It states that in each coset of $\mathrm{GL}_n(\mathbb{A})$ modulo $\mathrm{GL}_n(\mathbb{Q})$, the diagonal elements of the representative with smallest value under Φ satisfy an inequality.

Lemma 4.15. *Let $g \in \mathrm{GL}_n(\mathbb{A})$, $g = kan_{\frac{1}{2}}n_{\mathbb{Q}}d$, with $\Phi(g\gamma) \geq \Phi(g)$ for all $\gamma \in \mathrm{GL}_n(\mathbb{Q})$. Then*

$$\frac{a_1}{a_2} \leq \frac{2}{\sqrt{3}}.$$

Proof. Define

$$\gamma := (n_{\mathbb{Q}}d)^{-1} \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1_{n-2}\right) \in \mathrm{GL}_n(\mathbb{Q}).$$

Then it follows that

$$\begin{aligned}
\Phi(g\gamma) &= \Phi\left(kan_{\frac{1}{2}}n_{\mathbb{Q}}d\,(n_{\mathbb{Q}}d)^{-1}\operatorname{diag}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},1_{n-2}\right)\right)\\
&= \Phi\left(kan_{\frac{1}{2}}\operatorname{diag}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},1_{n-2}\right)\right)\\
&= \eta\left(kan_{\frac{1}{2}}\operatorname{diag}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},1_{n-2}\right)e_1\right)\\
&= \eta\left(kan_{\frac{1}{2}}e_2\right)\\
&= \eta\left(a((n_{\frac{1}{2}})_{12}e_1+e_2)\right)\\
&= \eta\left(a_1(n_{\frac{1}{2}})_{12}e_1+a_2e_2\right) \geq \Phi(g)
\end{aligned}$$

Now $t=(n_{\frac{1}{2}})_{12}\in N_{\frac{1}{2}}$, hence

$$\eta(a_1n_{12}e_1+a_2e_2)=\sqrt{a_1^2t_\infty^2+a_2^2}\geq a_1.$$

Since $t_\infty\in\left[-\frac{1}{2},\frac{1}{2}\right]$, we obtain $a_1^2\cdot\frac{1}{4}+a_2^2\geq a_1^2$, thus $\frac{a_1}{a_2}\leq\frac{2}{\sqrt{3}}$. □

Definition. Let $\Omega(t)=\{a\in A\mid \frac{a_i}{a_{i+1}}\leq t\}$, $\mathrm{N}_u=\{n\in N\mid |n_{ij}|\leq u\}$. Then we define the so called Siegel set $\Sigma_{t,u}^{\mathbb{A}}$ as

$$\Sigma_{t,u}^{\mathbb{A}}=\mathrm{K}\,\Omega(t)\,\mathrm{N}_u\,.$$

Now we want to prove the main Proposition of this section, namely that every orbit of the right action of $\mathrm{GL}_n(\mathbb{Q})$ on $\mathrm{GL}_n(\mathbb{A})$ has a minimal element (with respect to Φ) in $\Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$.

Proposition 4.16. *Let $g\in\mathrm{GL}_n(\mathbb{A})$. Then there is an element $\gamma_0\in\mathrm{GL}_n(\mathbb{Q})$ such that*

$$\Phi(g\gamma_0)=\min_{\gamma\in\mathrm{GL}_n(\mathbb{Q})}\Phi(g\gamma)$$

and $g\gamma_0\in\Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$.

Proof. Let $g\in\mathrm{GL}_n(\mathbb{A})$. We can choose an element $\gamma'\in\mathrm{GL}_n(\mathbb{Q})$ with $\Phi(g\gamma')$ minimal. Let

$$g\gamma'=kan_{\frac{1}{2}}n_{\mathbb{Q}}d\in\mathrm{K\,A\,N}_{\frac{1}{2}}\,\mathrm{N}_{\mathbb{Q}}\,\mathrm{D}\,.$$

Now define $\gamma''=(n_{\mathbb{Q}}d)^{-1}\gamma'$. Then we obtain

$$\Phi(g\gamma'')=\Phi(kan_{\frac{1}{2}}n_{\mathbb{Q}}d(n_{\mathbb{Q}}d)^{-1}\gamma')=\Phi(kan_{\frac{1}{2}}\gamma')=\Phi(g\gamma').$$

Now we want to proof the claim by induction on n. The case $n=2$ follows from Lemma 4.15.

Let $n>2$. Set $\tilde{g}:=k^{-1}g\gamma''=\begin{pmatrix}a_1&*\\0&b\end{pmatrix}$, with $b\in\mathrm{GL}_{n-1}(\mathbb{A})$. Then by induction hypothesis, we can choose an element $x\in\mathrm{GL}_{n-1}(\mathbb{Q})$ such that

$$bx=k'a'n'\in\Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}.$$

Now let $\tilde{x}$ be the matrix $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Q})$. Then we obtain

$$\tilde{g}\tilde{x} = \begin{pmatrix} a_1 & * \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_1 & * \\ 0 & bx \end{pmatrix} = \begin{pmatrix} a_1 & * \\ 0 & k'a'n' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a' \end{pmatrix} \underbrace{\begin{pmatrix} 1 & * \\ 0 & n' \end{pmatrix}}_{=:\tilde{n}=\tilde{n}_{\frac{1}{2}}\tilde{n}_{\mathbb{Q}}}.$$

Since $\Phi(\tilde{g}) = \Phi(k^{-1}g\gamma'') = \Phi(g\gamma'')$ it follows that $\Phi(\tilde{g})$ is minimal. By Lemma 4.15 we get that $\frac{a_1}{a_1'} \leq \frac{2}{\sqrt{3}}$. Moreover,

$$g\gamma''\tilde{x}\tilde{n}_{\mathbb{Q}}^{-1} = k \begin{pmatrix} 1 & 0 \\ 0 & k' \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a' \end{pmatrix} \tilde{n}_{\frac{1}{2}} \in \Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}.$$

Now define $\gamma_0 = \gamma''\tilde{x}\tilde{n}_{\mathbb{Q}}^{-1}$. Then we obtain

$$\Phi(g\gamma_0) = \Phi(g\gamma''\tilde{x}\tilde{n}_{\mathbb{Q}}^{-1}) = \Phi(g\gamma''),$$

so $\Phi(g\gamma_0)$ has minimal value. □

Theorem 4.17. $\mathrm{GL}_n(\mathbb{A}) = \mathrm{K}\,\Omega(c)\,\mathrm{N}_{\frac{1}{2}}\,\mathrm{GL}_n(\mathbb{Q})$ *for* $c \geq \frac{2}{\sqrt{3}}$.

Proof. Follows immediately from Proposition 4.16. □

So we obtain that the $\mathrm{GL}_n(\mathbb{Q})$-translates of the Siegel set $\Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$ cover the group $\mathrm{GL}_n(\mathbb{A})$. So clearly the question arises if this set is a fundamental set or even a fundamental domain. Let $\gamma = -\mathrm{id} \in \mathrm{GL}_n(\mathbb{Q})$, let $g \in \Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$ be arbitrary. Since γ is also in K and commutes with every element in $\mathrm{GL}_n(\mathbb{A})$, we obtain that

$$g\gamma \in \mathrm{K}\,\Omega(c)\,\mathrm{N}_{\frac{1}{2}} \cap \mathrm{K}\,\Omega(c)\,\mathrm{N}_{\frac{1}{2}}\,\gamma,$$

thus $\Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$ is not a fundamental domain. However, it can be shown that the intersection between two $\mathrm{GL}_n(\mathbb{Q})$-translates of $\Sigma^{\mathbb{A}}_{\frac{2}{\sqrt{3}},\frac{1}{2}}$ is always finite, so it is a fundamental set for $\mathrm{GL}_n(\mathbb{Q})$ in $\mathrm{GL}_n(\mathbb{A})$.

4.3 Fundamental sets

Now we try to handle the general case of an arbitrary algebraic group G defined over k. Although G_k is discrete in $G_{\mathbb{A}}$, we cannot expect the quotient $G_{\mathbb{A}}/G_k$ to be compact. Our first aim in this chapter is to construct fundamental sets for G_k in $G_{\mathbb{A}}$. Since every algebraic group can be realized as a closed subgroup of $\mathrm{GL}_n(k)$ it is convenient to look at the special case of $G = \mathrm{GL}_n$ over $\mathbb{Q}$ first.

Proposition 4.18. *Let* $G = \mathrm{GL}_n$ *over* $\mathbb{Q}$ *and let* Σ *be a fundamental set for* $G_{\mathbb{Z}}$ *in* $G_{\mathbb{R}}$. *Then*

$$\Omega = \Sigma \times \prod_{v \in V_k^f} G_{\mathbb{Z}_v}$$

is a fundamental set for $G_{\mathbb{Q}}$ *in* $G_{\mathbb{A}}$.

Proof. [15, Prop. 5.7] □

As we have seen in 4.17, the $\mathrm{GL}_n(\mathbb{Q})$-translates of the set $\Sigma^{\mathbb{A}}_{t,\frac{1}{2}}$ cover $G_{\mathbb{A}}$. From the definitions it follows that we can write

$$\Sigma^{\mathbb{A}}_{t,\frac{1}{2}} = \Sigma_{t,\frac{1}{2}} \times \prod_{v \in V_k^f} \mathrm{GL}_n(\mathbb{Z}_v)$$

where $\Sigma_{t,\frac{1}{2}}$ is the Siegel set from chapter 3.3. Thus, $\Sigma^{\mathbb{A}}_{t,\frac{1}{2}}$ a fundamental set for $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{GL}_n(\mathbb{R})$, so Theorem 4.17 would also follow from Proposition 4.18.

Now we want to construct fundamental sets for arbitrary adelic groups G over $\mathbb{Q}$. For that, we use the notion of an action of $G_{\mathbb{A}}$. Recall that an action of a k-group G on a k-variety X is a continuous k-morphism $\rho\colon G \times X \to X$ such that the induced morphism $\breve{\rho}\colon G \to \mathrm{Aut}(X)$ is a group homomorphism. Now any action ρ of G on X induces an action

$$\rho_{\mathbb{A}}\colon (G \times X)_{\mathbb{A}} = G_{\mathbb{A}} \times X_{\mathbb{A}} \to X_{\mathbb{A}}$$

on the adelization $X_{\mathbb{A}}$. This will be needed for the following Proposition.

Proposition 4.19. *Let $G \subset \mathrm{GL}_n$ be a reductive $\mathbb{Q}$-group and let Ω be the fundamental set for $G_{\mathbb{Q}}$ in $G_{\mathbb{A}}$ corresponding to the Siegel domain $\Sigma = \Sigma_{t,u}$ for $t \geq \frac{2}{\sqrt{3}}$, $u \geq \frac{1}{2}$ (in other words, $\Omega = \Sigma^{\mathbb{A}}_{t,u}$). For $a \in \mathrm{GL}_n(\mathbb{R})$ let a^∞ be the embedding of a into $\mathrm{GL}_n(\mathbb{A})$, i.e., $a^\infty = (a, \mathbb{I}_n, \mathbb{I}_n, \dots)$, where $\mathbb{I}_n$ is the n-dimensional unit matrix. Then there is an element $a \in \mathrm{GL}_n(\mathbb{R})$ and elements $b_1, \dots, b_r \in \mathrm{GL}_n(\mathbb{Q})$ such that*

$$\Delta = \left(\bigcup_{i=1}^{r} a^\infty \Omega b_i \right) \cap G_{\mathbb{A}}$$

is a fundamental set for $G_{\mathbb{Q}}$ in $G_{\mathbb{A}}$.

Proof. By Proposition 1.11 there exists a $\mathbb{Q}$-representation $\rho\colon \mathrm{GL}_n \to \mathrm{GL}_m$ and a vector $v \in \mathbb{Q}^n$ such that the GL_n-Orbit of v is closed in GL_m and the isotropy group is G. Using Proposition 1.9 we can choose an element $a \in \mathrm{GL}_n(\mathbb{R})$ such that the group $a^{-1}Ga$ is self-adjoint. Now from the definition of Ω and the continuousness of the adelization $\rho_{\mathbb{A}}$ it follows that the projection of all elements of the set $v(a^\infty\Omega)$ onto $\mathbb{A}^m_\infty$ lie in a suitable compact set. So we obtain that the set $M = v(a^\infty) \cap \mathbb{Q}^m$ is contained in the lattice $\frac{1}{l}\mathbb{Z}$ for a suitable integer l. This yields that $lM \subset (lv)\Sigma \cap \mathbb{Z}^m$, which is finite from Proposition 3.15. So we obtain that the subset $v(a^\infty\Omega) \cap v\,\mathrm{GL}_n(\mathbb{Q})$ is finite, let us say $v(a^\infty\Omega) \cap v\,\mathrm{GL}_n(\mathbb{Q}) = \{vb_1 \dots vb_r\}$ for $b_i \in \mathrm{GL}_n(\mathbb{Q})$. Now from Lemma 3.11 and Proposition 4.18 we conclude that Δ is a fundamental set for $G_{\mathbb{Q}}$ in $G_{\mathbb{A}}$. $\square$

So we have obtained a way to construct a fundamental set for $G_{\mathbb{Q}}$ in $G_{\mathbb{A}}$ for any reductive group. Note that the projection on the non-Archimedean part of the set Δ is compact. Now the general case of an reductive group over an arbitrary number field simply follows by considering the restriction of scalars. More precisely, we have the following

Theorem 4.20. *Let G be a reductive algebraic group defined over an algebraic number field k. Then there exists a fundamental set for G_k in $G_{\mathbb{A}}$ having compact projection onto the non-Archimedean part.*

Proof. Let $H = \mathbf{R}_{k/\mathbb{Q}}(G)$ be the group obtained from G by restriction of scalars. Then by proposition 4.19 we obtain that $H_{\mathbb{Q}}$ has a fundamental set in $H_{\mathbb{A}_{\mathbb{Q}}}$ which has compact projection on the non-Archimedean part. By carring over this set with respect to the isomorphism $H_{\mathbb{A}_{\mathbb{Q}}} \cong G_{\mathbb{A}}$, we get the desired result. $\square$

So we have obtained fundamental sets for reductive k-groups for any algebraic number field k. Now we are interested in the properties of these fundamental sets. We can clearly ask if they are compact, or if the have finite volume with respect to the Haar measure on the product space $\mathbb{A}^n$. Before that, we will obtain the fundamental result for the class number of an arbitrary algebraic group. For that, we first need two lemmas.

Lemma 4.21. *Let G be an algebraic k-group. Then $G_{\mathbb{A}(\infty)}$ and $yG_{\mathbb{A}(\infty)}y^{-1}$ are commensurable, for any $y \in G_{\mathbb{A}}$.*

Proof. Let $U = \prod_{v \in V_k^f} G_{\mathcal{O}_v}$. Then we have $G_{\mathbb{A}(\infty)} = G_\infty \times U$ and

$$yG_{\mathbb{A}(\infty)}y^{-1} = (y_\infty G_\infty y_\infty^{-1}) \times (y_f U y_f^{-1}) = G_\infty \times (y_f U y_f^{-1}),$$

where y_∞ (resp. y_f) denotes the infinite (resp. finite) part of $y \in G_{\mathbb{A}}$. But U and $y_f U y_f^{-1}$ are open compact subgroups of $G_{\mathbb{A}_f}$, and therefore are commensurable. Thus, $G_{\mathbb{A}(\infty)}$ and $yG_{\mathbb{A}(\infty)}y^{-1}$ are also commensurable. $\square$

Lemma 4.22. *Let G be an algebraic k-group, G^0 its component of the identity. Then the quotient $G_{\mathbb{A}}/G^0_{\mathbb{A}}$ is compact.*

Proof. [15, Prop. 5.5] $\square$

Theorem 4.23. *Let G be an algebraic k-group. Then the class number* $\mathrm{cl}(G)$ *of G is finite.*

Proof. We first treat the case of a connected group G. Let $G = HU$ be the Levi-decomposition of G, where $U = R_u(G)$ is the unipotent radical of G and H is a reductive k-group. Then from Proposition 4.5 we get that the finiteness of $\mathrm{cl}(H)$ is sufficient to show. For that, we use the fundamental set $\Delta \subset H_{\mathbb{A}}$ which we constructed in Theorem 4.19. Since Δ has compact projection on the non-Archimedean part and the subset $H_{\mathbb{A}(\infty)}x$ is open in $H_{\mathbb{A}}$ for any $x \in H_{\mathbb{A}}$, we can find finitely many $x_i \in H_{\mathbb{A}}$ so that

$$\Delta \subset \bigcup_{i=1}^{r} H_{\mathbb{A}(\infty)} x_i.$$

Thus,

$$H_{\mathbb{A}} = \Delta H_k = \bigcup_{i=1}^{r} H_{\mathbb{A}(\infty)} x_i H_k,$$

which means that the class number of H is finite.

Now let G be arbitrary, let G^0 denote the connected component of the identity. So from above we can find elements $x_1, \ldots, x_n \in G^0_{\mathbb{A}}$ such that

$$G^0_{\mathbb{A}} = \bigcup_{i=1}^{r} G^0_{\mathbb{A}(\infty)} x_i G^0_k.$$

From Lemma 4.22 we know that the quotient $G_{\mathbb{A}}/G^0_{\mathbb{A}}$ is compact, thus we can find a compact set $D \subset G_{\mathbb{A}}$ so that $G_{\mathbb{A}} = DG^0_{\mathbb{A}}$. By the same argument as above we can find elements $y_j \in G_{\mathbb{A}}$ so that $D \subset \bigcup_{j=1}^{s} G_{\mathbb{A}(\infty)}y_j$. This yields

$$G_A = DG^0_A = \bigcup_{i=1}^{r}\bigcup_{j=1}^{s} G_{\mathbb{A}(\infty)}y_j G^0_{\mathbb{A}(\infty)}x_i G^0_k.$$

Now from Lemma 4.21 we know that $yG_{\mathbb{A}(\infty)}y^{-1}$ is commensurable with $G_{\mathbb{A}(\infty)}$, i.e., there exists elements $z_1, \ldots, z_t \in G_{\mathbb{A}}$ such that $yG_{\mathbb{A}(\infty)}y^{-1} \subset \bigcup_{l=1}^{t} G_{\mathbb{A}(\infty)}z_l$. Then

$$G_{\mathbb{A}(\infty)}yG_{\mathbb{A}(\infty)}xG_k = G_{\mathbb{A}(\infty)}(yG_{\mathbb{A}(\infty)}y^{-1})yxG_k \subset \bigcup_{l=1}^{t} G_{\mathbb{A}(\infty)}z_l yxG_k,$$

from which we deduce the desired decomposition. □

By reviewing what we have done so far one question arises. Namely, is the fundamental set from proposition 4.18 universal? The answer is given by the following Proposition.

Proposition 4.24. *Let G be an k-group. If B is a fundamental set in G_∞ relative to $G_{\mathcal{O}_k}$, then there is a compact subset C of $G_{\mathbb{A}_f}$ such that $B \times C$ is a fundamental set in $G_{\mathbb{A}}$ relative to G_k.*

Proof. [15, Proposition 5.9] □

4.4 Criteria for compactness and finite invariant volume of $G_{\mathbb{A}}/G_k$

In this section we want to obtain criteria for the compactness and the finiteness of the volume of $G_{\mathbb{A}}/G_k$. For that, we need to construct an invariant measure on the adelic points of an algebraic group G. Since there exists a Haar measure on $\mathbb{A}_k$ the product space $\mathbb{A}_k^n$ can be viewed as a measurable space. Now we can define an invariant measure μ on $G_{\mathbb{A}}$ simply by restriction of the Haar measure of $\mathbb{A}_k^n$. However, we want to describe μ in terms of differential form, as we have seen at the end of chapter 2.2. This construction follows [15, ch. 5.3].

So let G be a connected algebraic group, let ω be a left-invariant rational differential k-form on G of degree $n = \dim(G)$. Then we have seen in section 2.2 that ω induces a left-invariant measure μ on G_k, thus also a left-invariant measure μ_v on G_{k_v} for each $v \in V_k$. The idea to obtain a measure on the adelic points of G is simply to multiply the local measures; however, this product do not need to converge. Choose numbers λ_v for $v \in V_k^f$, which we call convergence coefficients, such that the product

$$\prod_{v \in V_k} \lambda_v \mu_v(G_{\mathcal{O}_v})$$

converges absolutely. Then we define a Haar measure τ on $G_{\mathbb{A}}$ as the infinite product of the local measures. The absolute convergence of the above product guarantees that it is well-defined. Now let ω' another left-invariant rational differential k-form on G. Then we can find an element $c \in k^*$ so that $\omega' = c\omega$. This yields $\mu'_v = \|c\|_v^n \mu_v$, thus by using the product formula we obtain

$$\tau' = (\prod_{v \in V_k} \|c\|_v^n)\tau = \tau.$$

This implies that the definition of the measure τ is independent of the left-invariant rational differential k-form on G.

Definition. The Haar measure τ constructed above is called the Tamagawa measure corresponding to the set of convergence coefficients $\lambda = (\lambda_v)$.

So we have constructed an invariant measure on $G_{\mathbb{A}}$; however, we are interested in the volume of the set $G_{\mathbb{A}}/G_k$. Now since the subgroup G_k is discrete in $G_{\mathbb{A}}$, an invariant measure on $G_{\mathbb{A}}/G_k$ exists if and only if $G_{\mathbb{A}}$ is unimodular, as we have seen in Theorem 2.8. As a first step, we want to show that the last condition is equivalent to the unimodularity of G_∞.

Lemma 4.25. *Let G be an algebraic group. Then $G_{\mathbb{A}}$ is unimodular if and only if G_∞ is unimodular.*

Proof. As a first step, we want to reduce the proof to the case of a connected algebraic group. Since the topological group $G_{\mathbb{A}}/G^0_{\mathbb{A}}$ is compact, it has finite $G_{\mathbb{A}}$-invariant measure. This implies that the restriction of the module function $\Delta_{G_{\mathbb{A}}}$ to $G^0_{\mathbb{A}}$ is $\Delta_{G^0_{\mathbb{A}}}$. In particular, if $\Delta_{G_{\mathbb{A}}} \equiv 1$, then $\Delta_{G^0_{\mathbb{A}}} \equiv 1$. Conversely, let $\Delta_{G^0_{\mathbb{A}}} \equiv 1$. Then the connected component of the identity $G^0_{\mathbb{A}}$ of the adelic points is contained in the kernel of $\Delta_{G_{\mathbb{A}}}$. Thus we get a continuous homomorphism

$$\Delta \colon G_{\mathbb{A}}/G^0_{\mathbb{A}} \to \mathbb{R}_{>0}.$$

Now the image of Δ is compact as the continuous image of a compact set, thus a compact subgroup of $\mathbb{R}_{>0}$. However, there are no nontrivial compact subgroups of $\mathbb{R}_{>0}$, thus $\Delta_{G_{\mathbb{A}}} \equiv 1$. So we can restrict ourselves to the case of a connected algebraic group. Since the quotient G_∞/G^0_∞ is finite (as the number of components of G), we obtain the analogue result by using the same argumentation as above.

So let G be connected. From the construction of the Tamagawa measure we obtain that $G_{\mathbb{A}}$ is unimodular if and only if all G_{k_v} are unimodular. Now if G_∞ is unimodular, then ω is also right invariant, where ω is a left-invariant rational differential k-form used to construct the Tamagawa measure. Thus,

$$\Delta_{G_\infty} \equiv 1 \Leftrightarrow \Delta_{G_{\mathbb{A}}} \equiv 1.$$

□

Now one of the main results of this thesis is proved. It states that the criteria for the compactness respectively the finiteness of the volume of $G_{\mathbb{A}}/G_k$ are the same as in the infinite case.

Theorem 4.26. *Let G be an algebraic k-group. Then*

(i) $G_{\mathbb{A}}/G_k$ is compact if and only if every unipotent element in G_k is contained in the unipotent radical of G and $X(G^0)_k = 1$.

(ii) $G_{\mathbb{A}}/G_k$ has finite invariant volume if and only if $X(G^0)_k = 1$.

Proof. [15, Theorem 5.5]. From Proposition 4.24 we know that there exists a fundamental set of G_k in $G_{\mathbb{A}}$ of the form $\Omega = B \times C$, where $B \subset G_\infty$ is a closed fundamental set with respect to $G_{\mathcal{O}}$ and C is a compact open set of $G_{\mathbb{A}(\infty)}$. Now if Ω is compact (respectively has finite invariant volume), then also $G_{\mathbb{A}}/G_k$ has this property. Conversely, let $G_{\mathbb{A}}/G_k$ be compact. Since the G_k-translates of Ω have finite intersection, we get that Ω is compact. Since $\Omega = B \times C$ and C is compact, the last result is equivalent to B being compact. Furthermore, since C has finite volume, the finite volume of Ω is equivalent to B having finite volume. Now from Lemma 4.25 we conclude that $G_{\mathbb{A}}/G_k$ is compact (respectively has finite invariant volume) if and only if $G_\infty/G_{\mathcal{O}}$ is compact (resp. has finite invariant volume). So the claim follows from Theorem 3.21. □

So we have obtained criteria for the compactness and the finiteness of the volume of the quotient $G_{\mathbb{A}}/G_k$ for every Tamagawa measure τ. It can be shown that the convergence coefficients can be chosen canonically for an algebraic group G, see e.g. [17, p.115]. In particular, if G is semisimple, we can choose $\lambda_v = 1$ for all $v \in V$. With respect to the Tamagawa measure obtained in this way, we can define the following important term.

Definition. Let G be an algebraic k-group, let τ be the Tamagawa measure obtained canonically. Then we define the Tamagawa number $\tau(G)$ of G as the invariant volume of $G_{\mathbb{A}}/G_k$, if it exists.

As we have seen in Theorem 4.26 the Tamagawa number of a semisimple group is finite. As an example, we want to compute $\tau(\mathrm{SL}_2)$.

Example. Let $G = \mathrm{SL}_2$ over $\mathbb{Q}$. As we have seen in chapter 2.2 a $\mathbb{Q}$-rational differential form for G is given by $\omega = \frac{1}{x} dx \wedge dy \wedge dz$, where $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G_{\mathbb{Q}}$. Let μ_∞ denote the corresponding measure on the infinite place, let F be a fundamental domain for $G_{\mathbb{R}}/G_{\mathbb{Z}}$. Then we have seen in chapter 3.3 that

$$\mathrm{vol}(G_{\mathbb{R}}/G_{\mathbb{Z}}) = \mu_\infty(F) = \frac{\pi^2}{6}.$$

Now we have to compute the volume of $G_{\mathbb{Q}_p}/G_{\mathbb{Z}_p}$ for any prime p. Let μ_p denote the measure on $G_{\mathbb{Q}_p}$ corresponding to ω. Let $\mathrm{SL}_2(\mathbb{Z}_p, p)$ be the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ modulo p, i.e.,

$$\mathrm{SL}_2(\mathbb{Z}_p, p) = \{g \in \mathrm{SL}_2(\mathbb{Z}_p) \mid g \equiv \mathbb{I}_2 \mod p\},$$

where the reduction modulo p is meant to be componentwise. It is clear that

$$\mu_p(\mathrm{SL}_2(\mathbb{Z}_p)) = |\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})| \mu_p(\mathrm{SL}_2(\mathbb{Z}_p, p)),$$

and an easy combinatorical argument shows that $|\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2-1)$. Thus it remains to compute $\Gamma = \mu_p(\mathrm{SL}_2(\mathbb{Z}_p, p))$.

The set Γ is mapped by x, y, z onto $p\mathbb{Z}_p \times p\mathbb{Z}_p \times p\mathbb{Z}_p$ and we have $\left|\frac{1}{x}\right|_p = 1$ on Γ. Thus $\mu_p(\Gamma) = (\nu_p(p\mathbb{Z}_p))^3$, where ν_p is the Haar measure on $\mathbb{Q}_p$ normalized by $\nu_p(\mathbb{Z}_p) = 1$. Let $\sigma_p \colon \mathbb{Q}_p \to \mathbb{Q}_p$ denote the left-multiplication with p. Then it can be shown that $\mathrm{mod}_{\mathbb{Q}_p}(\sigma_p) = |p|_p = \frac{1}{p}$, where $|.|_p$ denotes the p-adic valuation. This yields

$$\mu_p(\mathrm{SL}_2(\mathbb{Z}_p)) = |\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})| \mu_p(\mathrm{SL}_2(\mathbb{Z}_p, p)) = p(p^2-1)p^{-3} = 1 - \frac{1}{p^2}.$$

Using the analogous result of Proposition 4.18 we see that $\Omega = F \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$ is a fundamental set for $\mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{Q})$. Thus

$$\tau(G) = \mu_\infty(F) \times \prod_p \mu_p(\mathrm{SL}_2(\mathbb{Z}_p)) = \frac{\pi^2}{6} \prod_p \left(1 - \frac{1}{p^2}\right) = \zeta(2)\zeta(2)^{-1} = 1.$$

For the last example we used that we have an explicit description of the fundamental domain for $\mathrm{SL}_2(\mathbb{A})/\mathrm{SL}_2(\mathbb{Q})$. Therefore, this computation cannot be used to compute the Tamagawa number of an arbitrary algebraic group G defined over a number field k.

4.5 Properness of injective morphisms

We have seen in the previous chapter that the quotient $G_\mathbb{A}/G_k$ cannot be expected to be compact or to have finite invariant volume. This result is not really surprising, since we have seen this in chapter 3.2 for the case of the idele group $\mathbb{J}/k^* \cong \mathrm{GL}_1(\mathbb{A})/\mathrm{GL}_1(k)$. But after restricting to the special ideles $\mathbb{J}_k^{(1)}$ we again obtained compactness. Now for arbitrary groups G the question arises if we can always restrict to a subgroup $G_\mathbb{A}^{(1)}$ of $G_\mathbb{A}$ such that the quotient $G_\mathbb{A}^{(1)}/G_k$ is compact, has finite invariant volume respectively.

Let $\chi \in X(G)_k$. Since the k-points of G are dense in G_{k_v} for all $v \in V_k$, we can extend the character χ uniquely to all of G_{k_v}. Now we associate to any χ the continuous homomorphism

$$\begin{aligned} c_k(\chi): \ G_\mathbb{A} &\to \mathbb{R}_{>0} \\ (g_v)_v &\mapsto \prod_v |\chi(g_v)|_v. \end{aligned}$$

Since χ is k-rational, we know that χ maps $G_{\mathcal{O}_v}$ into $\mathcal{O}_v$ for almost all $v \in V_k^f$. Therefore for all $g \in G_\mathbb{A}$ we have $|\chi(g_v)|_v \leq 1$ for almost all $v \in V_k^f$, so the map $c_k(\chi)$ is well defined. Now define the subgroup $G_\mathbb{A}^{(1)}$ of $G_\mathbb{A}$ via

$$G_\mathbb{A}^{(1)} = \bigcap_{\chi \in X(G)_k} \ker(c_k(\chi)).$$

From the product formula it follows that $G_k \subset G_\mathbb{A}^{(1)}$.

The following Proposition delivers structural information about the quotient $G_\mathbb{A}^{(1)}/G_k$.

Theorem 4.27. *Let G be a connected algebraic k-group. Then $G_\mathbb{A}^{(1)}$ is unimodular and $G_\mathbb{A}^{(1)}/G_k$ has finite invariant volume. Moreover, the quotient $G_\mathbb{A}^{(1)}/G_k$ is compact if and only if every unipotent element of G_k is contained in the unipotent radical of G.*

Proof. [15, Theorem 5.6] □

Now in the rest of this work we try to analyse the connection between reductive algebraic groups G and their subgroups in view of the compactness of $G_\mathbb{A}^{(1)}/G_k$. The following discussion is based on the ideas in [16, ch. 6.1].

For that, let G, H be reductive algebraic k-groups. Let $i : H \hookrightarrow G$ be an injective k-homomorphism. Then it follows that the map $i\colon H_{\mathbb{A}} \to G_{\mathbb{A}}$ is injective as well as the restriction $i_{\mathbb{A}}|_{H_{\mathbb{A}}^{(1)}}\colon H_{\mathbb{A}}^{(1)} \to G_{\mathbb{A}}^{(1)}$. Since $i(H_k) \subset G_k$, we get a map

$$i_{\mathbb{A}}^{(1)}\colon H_{\mathbb{A}}^{(1)}/H_k \to G_{\mathbb{A}}^{(1)}/G_k,$$

which is easily seen to be injective.

We now want to show that this map is proper.

Proposition 4.28. *The map $i_{\mathbb{A}}^{(1)}$ is proper.*

Proof. Since $i_{\mathbb{A}}^{(1)}$ is injective, we have seen in Proposition 2.2 that is suffices to show that $i_{\mathbb{A}}^{(1)}$ is a homeomorphism onto a closed subspace of $G_{\mathbb{A}}^{(1)}/G_k$. By Theorem 2.4 it suffices to show that the image of $i_{\mathbb{A}}^{(1)}$ is closed in $G_{\mathbb{A}}^{(1)}/G_k$. By the definition of the quotient topology this means that $i_{\mathbb{A}}^{(1)}(H_{\mathbb{A}}^{(1)}/H_k)G_k = i_{\mathbb{A}}(H_{\mathbb{A}}^{(1)})G_k$ is closed in $G_{\mathbb{A}}^{(1)}$. This in fact is equivalent to G_k being closed in $i_{\mathbb{A}}(H_{\mathbb{A}}^{(1)})\backslash G_{\mathbb{A}}^{(1)}$. From 1.11 we know that there are a finite dimensional k-vector space W, viewed as affine k-variety, a k-representation $\varphi\colon W \times G \to W$ and a k-point $w \in W_k$ such that the orbit of w is closed in W and the isotropy group of w is $i(H)$. Let $\check{\varphi}\colon G \to \mathrm{GL}(W)$ denote the induced k-homomorphism. Now by Lemma 4.1 we obtain that the adelization $\check{\varphi}_{\mathbb{A}}\colon G_{\mathbb{A}} \to \mathrm{GL}(W_{\mathbb{A}})$ is continuous, so we get a representation of $G_{\mathbb{A}}$ and after restriction a representation

$$\varphi_{\mathbb{A}}^{(1)}\colon W_{\mathbb{A}} \times G_{\mathbb{A}}^{(1)} \to W_{\mathbb{A}}$$

of $G_{\mathbb{A}}^{(1)}$. Since $i(H) = \mathrm{pr}_2(\varphi^{-1})(w)$ and adelization preserves continuity, we deduce that $i_{\mathbb{A}}(H_{\mathbb{A}}^{(1)}) = \mathrm{pr}_2((\varphi_{\mathbb{A}}^{(1)})^{-1})(w)$, so it is still the stabilizer of w. Now φ is a k-rational representation, so we have $\check{\varphi}(g)(w) \in W_k$ for all $g \in G_k$. Now define $\Gamma := W_k$. Then it follows that Γ is a G_k-stable lattice in $W_{\mathbb{A}}$ and $w \in \Gamma$. Since the k-points of W are discrete in $W_{\mathbb{A}}$ and φ maps k-points onto k-points, we get that the G_k-orbit of w is discrete in $W_{\mathbb{A}}$ and also closed since wG_k is closed in W_k and W_k is closed in $W_{\mathbb{A}}$. Now let $p\colon G_{\mathbb{A}}^{(1)} \to i_{\mathbb{A}}(H_{\mathbb{A}}^{(1)})\backslash G_{\mathbb{A}}^{(1)}$ denote the canonical projection. Then we have an identification $p(G_k) \to wG_k$ by sending $i_{\mathbb{A}}(H_{\mathbb{A}}^{(1)})g$ onto $wH_{\mathbb{A}}^{(1)}g$, so $p(G_k)$ is closed in $i_{\mathbb{A}}(H_{\mathbb{A}}^{(1)})/G_{\mathbb{A}}^{(1)}$. $\square$

As a corollary, we obtain that if G is a reductive algebraic k-group with $G_{\mathbb{A}}^{(1)}/G_k$ is compact, then also $H_{\mathbb{A}}^{(1)}/H_k$ is compact for every reductive k-subgroup H of G. Moreover, since every algebraic k-group can be embedded into some $\mathrm{GL}_n(k)$, it suffices to prove the compactness of the quotient for "big" closed subgroups of the general linear group.

Appendix

Haar measure

In this section we want to introduce a left-invariant measure on locally compact topological groups, called Haar-measure. Since the proof of the existence of this measure is rather technical, it will be omitted. For more information about Haar measures, see [5, ch. VII, §1].

Proposition. *Let G be a locally compact group. Then there exists a nonzero measure μ on G such that*

(i) all continuous functions $f\colon G \to \mathbb{C}$ with compact support are μ-integrable.

(ii) μ is invariant under left translations, i.e., for all $h \in G$ and all μ-integrable functions f we have

$$\int_G f(g)d\mu(g) = \int_G f(hg)d\mu(g).$$

Proof. [5, ch. VII, §1, Théorème 1] □

Definition. Such a measure is called a left Haar measure on G.

Let μ, μ' be two left Haar measures on G. Then there exists a nonzero constant $\lambda \in \mathbb{C}^*$ with $\mu = \lambda\mu'$, so the Haar measure is unique up to a constant multiple. Let $A \subset G$ be a μ-measurable subset of G. Then for each $g \in G$ we have $\mu(gA) = \mu(A)$. Every Borel set of G is μ-measurable. Moreover, if $A \subset G$ is open (resp. compact), then $\mu(A)$ is positive (resp. finite).

We can analogously define a right Haar measure on a locally compact group to be a nonzero measure on a locally compact group G which satisfies condition (i) of the above Proposition and is invariant under right translations. Every left Haar measure μ on G induces a right Haar measure μ' by defining

$$\mu'(Ag) = \mu(gA)$$

for all $g \in G$ and all Borel sets $A \subset G$.

Now let σ be an isomorphism of topological groups. Then σ maps μ-measurable sets to μ-measurable sets. Therefore we can define a measure μ' on G by $\mu'(A) = \mu(\sigma(A))$ for any μ-measurable set A. It can be shown that μ' is again a left Haar measure on G, thus there exists a nonzero constant $\mathrm{mod}_G(\sigma)$ (or just $\mathrm{mod}(\sigma)$) with $\mu' = \mathrm{mod}(\sigma)\mu$. Clearly this constant is independent of the original choice of μ.

Definition. Let σ be an isomorphism of topological groups. Then the non-zero constant $\mathrm{mod}(\sigma) \in \mathbb{C}^*$ is called the modulus of the automorphism σ.

We now want to establish some properties of the modulus of an automorphism. For that, let $\mathrm{Aut}(G)$ denote the group of automorphisms of G as a topological group and fix a left Haar measure μ on G. Then we have the following Lemma.

Lemma. *Let G be a locally compact group.*

(i) If G is compact then $\operatorname{mod}(\sigma) = 1$ *for all* $\sigma \in \operatorname{Aut}(G)$.

(ii) If G is discrete, then $\operatorname{mod}(\sigma) = 1$ *for all* $\sigma \in \operatorname{Aut}(G)$.

Proof. Let μ be a left Haar measure on G, let $\sigma \in \operatorname{Aut}(G)$.

(i) If G is compact, then $\mu(G) < \infty$. Then we obtain

$$\mu(\sigma(G)) = \int_G d\mu(\sigma(G)) = \int_{\sigma^{-1}(G)} d\mu(G) = \int_G d\mu(G) = \mu(G),$$

thus $\operatorname{mod}(\sigma) = 1$.

(ii) If G is discrete, then the set $\{e\}$ is open, hence has positive measure. This yields

$$\mu(\sigma(e)) = \int_{\{e\}} d\mu(\sigma(e)) = \int_{\{\sigma^{-1}(e)\}} d\mu(e) = \int_{\{e\}} d\mu(e) = \mu(e),$$

so $\operatorname{mod}(\sigma) = 1$.

□

For every $g \in G$ is the map $\operatorname{inn}_g \colon G \to G$, $h \mapsto ghg^{-1}$ an automorphism of topological groups. Let $\Delta_G(g)$ denote the corresponding modulus. This induces a continuous homomorphism

$$\Delta_G \colon G \to \mathbb{R}_{>0},$$

called the modulus of G. We say that a locally compact group G is unimodular if $\Delta_G \equiv 1$. Let μ be a left Haar measure on an unimodular group G. Then μ is a also a right Haar measure, i.e., μ is also invariant under right translations. Moreover, $\mu(X) = \mu(X^{-1})$ for all measurable sets $X \subset G$. Now we want to give some examples of unimodular groups. From the definition of the modulus it immediately follows that any abelian locally compact group G satisfies $\Delta_G \equiv 1$. Moreover, we have the following Corollary of the Lemma.

Corollary. *Any compact or discrete locally compact group is unimodular.*

Summary

This diploma thesis deals with algebraic groups G defined over algebraic number fields k. To gain information about the k-points of G we can consider the adelic points $G_{\mathbb{A}}$ of G. One aim of this thesis is to construct fundamental domains respectively sets for G_k in $G_{\mathbb{A}}$. In addition, criteria for compactness and for the existence of a finite invariant volume for the quotient $G_{\mathbb{A}}/G_k$ shall be found. Furthermore, we consider the group $G_{\mathbb{A}}^{(1)}$ defined by the intersection of the kernels of all k-characters of G and again try to find conditions which guarantee that the quotient $G_{\mathbb{A}}^{(1)}/G_k$ is compact, has finite invariant volume respectively. At the end of this thesis we analyse inclusions $i\colon H \hookrightarrow G$ of reductive algebraic k-groups and show that the induced morphism $i_{\mathbb{A}}^{(1)}\colon H_{\mathbb{A}}^{(1)}/H_k \to G_{\mathbb{A}}^{(1)}/G_k$ is proper.

After a short introduction to the theory of algebraic groups the first important result in the first chapter is that every affine algebraic k-group is isomorphic to a closed subgroup of $\mathrm{GL}(V)$ for a suitable k-vector space V. In the third subsection we improve this representation for the case of a connected algebraic group G and a reductive subgroup $H \subset G$.

In the second chapter we introduce the notion of properness for continuous maps. In particular, we show the connection between such maps and the compactness of preimages of compact sets. In the second subsection we consider the restricted topological product of topological spaces. Furthermore, we construct measures on such products and give conditions when a measure on an algebraic group induces a measure on a quotient of that group. At the end, we compute a rational left-invariant differential form on SL_2 over $\mathbb{Q}$.

The third chapter is dedicated to a review of the theory of algebraic number fields and their completions with respect to valuations. After that, the adeles and ideles of an algebraic number field are defined and important results in this context are stated. In the third subsection we consider arithmetic subgroups of algebraic groups over $\mathbb{Q}$ and construct fundamental sets for $G_{\mathbb{R}}$ with respect to $G_{\mathbb{Z}}$. In addition, we give conditions which are equivalent to the compactness, the finite invariant volume of $G_{\mathbb{R}}/G_{\mathbb{Z}}$ respectively. At the end, we generalize these results to arbitrary number fields.

The main part of this diploma thesis is formed by the fourth chapter. First, we associate to every algebraic variety X defined over k an "adelic" variety. Afterwards we construct fundamental sets respectively domains for G_k in $G_{\mathbb{A}}$, where $G_{\mathbb{A}}$ denote the adelic points of G. Here we reduce the general case to that of GL_n. In the fourth subsection we consider $G_{\mathbb{A}}/G_k$ and analyse under which condition this quotient is compact, has finite invariant volume respectively. We show, that this question can be reduced to the infinite case. After that, we restrict to the subgroup $G_{\mathbb{A}}^{(1)} \subset G_{\mathbb{A}}$ defined by the intersection of the kernels of all k-characters of G, and clear which criteria are equivalent to the compactness respectively to the finite invariant volume of $G_{\mathbb{A}}^{(1)}/G_k$ respectively. In the last chapter we consider injective k-morphisms $i\colon H \hookrightarrow G$ and show, that the induced map

$$i_{\mathbb{A}}^{(1)}\colon H_{\mathbb{A}}^{(1)}/H_k \hookrightarrow G_{\mathbb{A}}^{(1)}/G_k$$

is proper.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit algebraischen Gruppen G definiert über einem algebraischen Zahlkörper k. Um Informationen über die k-Punkte von G zu erhalten können wir die adelischen Punkte $G_{\mathbb{A}}$ von G betrachten. Ein Ziel dieser Arbeit ist die Konstruktion von Fundamentalbereichen bzw. -mengen für G_k in $G_{\mathbb{A}}$. Weiters sollen Kriterien für Kompaktheit sowie für endliches invarinates Volumen des Quotienten $G_{\mathbb{A}}/G_k$ gefunden werden. Ferner betrachten wir die Gruppe $G_{\mathbb{A}}^{(1)}$, definiert als der Schnitt über die Kerne aller k-Charaktere von G, und wollen auch hier Bedingungen finden, die garantieren, dass der Quotient $G_{\mathbb{A}}^{(1)}/G_k$ kompakt ist bzw. endliches invariantes Volumen hat. Am Ende dieser Arbeit betrachten wir Inklusionen $i\colon H \to G$ von reduktiven algebraischen k-Gruppen und zeigen, dass der induzierte Morphismus $i_{\mathbb{A}}^{(1)}: H_{\mathbb{A}}^{(1)}/H_k \to G_{\mathbb{A}}^{(1)}/G_k$ eigentlich ist.

Nach einer kurzen Einführung in die Theorie der algebraischen Gruppen ist das erste wichtige Resultat im ersten Kapitel die Darstellung einer beliebigen (affinen) algebraischen k-Gruppe als abgeschlossene Untergruppe von $\mathrm{GL}(V)$ für einen k-Vektorraum V. Im 3. Abschnitt verfeinern wir diese Darstellung im Fall einer zusammenhängenden algebraischen Gruppe G und einer reduktiven Untergruppe $H \subset G$.

Im zweiten Kapitel führen wir den Begriff der Eigentlichkeit von stetigen Abbildungen ein. Insbesondere zeigen wir den Zusammenhang solcher Abbildungen zur Kompaktheit von Urbildern von kompakten Mengen. Im zweiten Unterkapitel betrachten wir das verschränkte topologische Produkt von topologischen Räumen. Weiters konstruieren wir Maße auf solchen Produkten und geben Bedingungen, unter welchen das Maß auf einer algebraischen Gruppe eines auf dem Quotienten induziert. Am Schluss berechnen wir eine rationale Differentialform auf der Gruppe SL_2 über $\mathbb{Q}$.

Das dritte Kapitel widmet sich zunächst der Wiederholung der Theorie der algebraischen Zahlkörper und deren Vervollständigungen durch Bewertungen. Danach werden die Adele und Idele eines algebraischen Zahlkörpers definiert und die wichtigsten Resultate zitiert. Im dritten Unterabschnitt betrachten wir arithmetische Untergruppen von algebraischen Gruppen über $\mathbb{Q}$ und konstruieren Fundamentalmengen für $G_{\mathbb{R}}$ bezüglich $G_{\mathbb{Z}}$. Weiters geben wir Bedingungen an, die äquivalent sind zur Kompaktheit bzw. zum endlichen Volumen von $G_{\mathbb{R}}/G_{\mathbb{Z}}$. Am Schluss geben wir die Verallgemeinerungen auf beliebige algebraische Zahlkörper.

Der Hauptteil dieser Arbeit wird vom vierten Kapitel gebildet. Zunächst assoziieren wir zu jeder algebraischen k-Varietät X eine "adelische" Varietät. Danach konstruieren wir Fundamentalbereiche bzw. -mengen für G_k in $G_{\mathbb{A}}$, wo $G_{\mathbb{A}}$ die adelischen Punkte von G bezeichnen, wobei wir zunächst den Fall $G = \mathrm{GL}_n$ betrachten und den allgemeinen Fall darauf zurückführen. Im vierten Unterkapitel betrachten wir den Quotient $G_{\mathbb{A}}/G_k$ und analysieren, unter welchen Bedingungen dieser kompakt ist bzw. endliches invariantes Volumen hat. Danach beschränken wir uns auf die Untergruppe $G_{\mathbb{A}}^{(1)} \subset G_{\mathbb{A}}$, definiert durch den Durchschnitt über die Kerne aller k-Charaktere von G, und klären wieder welche Kriterien äquivalent zur Kompaktheit bzw. zum endlichen Volumen von $G_{\mathbb{A}}^{(1)}/G_k$ sind. Im letzten Kapitel betrachten wir injektive k-Morphismen $i\colon H \hookrightarrow G$ und

zeigen, dass die induzierte Abbildung

$$i_{\mathbb{A}}^{(1)} \colon H_{\mathbb{A}}^{(1)}/H_k \hookrightarrow G_{\mathbb{A}}^{(1)}/G_k$$

eigentlich ist.

Notation

$\mathbb{Z}$	Ring of integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
V_k	Places of an algebraic number field k
$\mathcal{O}_k$	Ring of integers in an algebraic number field k
k_v	Completion of an algebraic number k with respect to a valuation v
$\mathcal{O}_v$	Valuation ring in k_v
$\mathbb{A}_k$	Ring of adeles over k
$\mathrm{GL}_n(k)$	Group of n-dimensional invertible matrices with entries in k
$\mathrm{SL}_n(k)$	Group of n-dimensional matrices with entries in k and determinant equal to 1
$\Sigma_{t,u}$	Siegel set for $G_{\mathbb{Z}}$ in $G_{\mathbb{R}}$ with the parameters t and u
$\Sigma_{t,u}^{\mathbb{A}}$	Siegel set for G_k in $G_{\mathbb{A}}$ corresponding to $\Sigma_{t,u}$
$\mathrm{cl}(G)$	Class number of an algebraic group G
$X(G)_k$	Group of characters of G defined over k
$G_{\mathbb{A}}$	Points of G in the adeles $\mathbb{A}$
$G_{\mathbb{A}}^{(1)}$	Subgroup of $G_{\mathbb{A}}$ defined by the intersection of the kernels of all k-characters of G
$\tau(G)$	Tamagawa number of the algebraic group G

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Curriculum Vitae

Personal

Name	Christoph Harrach
Date of Birth	17.10.1987
Nationality	Austria
E-Mail	charrach@gmx.at

Education

Since 10/2007	Bachelor Studies in Informatics at the University of Vienna
Since 10/2006	Diploma Studies in Mathematics at the University of Vienna
06/2006	Matura passed with honours
2000 - 2006	HAK Frauenkirchen
1996 - 2000	Hauptschule Frauenkirchen
1992 - 1996	Volksschule Frauenkirchen

Employment

10/08 - 03/09	Tutor at the Faculty of Informatics, University of Vienna
10/09 - 06/10	Tutor at the Faculty of Informatics, University of Vienna

Stipends

2010	Leistungsstipendium (merit scholarship) by the University of Vienna for the academic year 2009/2010
2009	Leistungsstipendium (merit scholarship) by the University of Vienna for the academic year 2008/2009
2008	Leistungsstipendium (merit scholarship) by the University of Vienna for the calendar year 2007

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