universität wien

DISSERTATION

Titel der Dissertation

A nonlinear theory of generalized tensor fields on Riemannian manifolds

Verfasser

Dipl.-Ing. Eduard Nigsch

angestrebter akademischer Grad

Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, im Oktober 2010

Studienkennzahl lt. Studienblatt:	A 091 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuer:	ao. Prof. Dr. Michael Kunzinger

Für Birgit, Valentin und Florentin.

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Acknowledgments

I have been depending on many colleagues, friends, and family members during the years I have been working on this thesis. This is the place to thank them all.

At the faculty of mathematics, the triumvirate formed by my thesis supervisor Michael Kunzinger and his colleagues Michael Grosser and Roland Steinbauer provided the ideal nest in which I could breed on my ideas. The working conditions there were incredibly friendly and supporting, as is the whole DIANA group. This work was financed by the Initiativkolleg "Differential Geometry and Lie Groups" (speaker Peter Michor) of the University of Vienna as well as projects P16742 "Geometric Theory of Generalized Functions" and START-project Y237 "Nonlinear Distributional Geometry" (Michael Kunzinger) of the Austrian Science Fund (FWF).

Extra thanks go to my supervisor for giving me the creative freedom to follow my inspirations in my own way; for having the willingness to painstakingly work through the minute details of my results; and for showing me not only how to do good mathematics, but what it takes to be a good mathematician.

The continuing support of my parents is invaluable, and has always been. In pressing times I can count on them. My brothers too, though far away, were at my side.

But those I owe the most are my beloved wife and our two sons, whose birth and life is more impressive than mathematics can ever be. You know how dear you are to me, even if it is not always easy to hold the balance between work and family.

Preface

I was introduced to Colombeau algebras when I was looking for a topic for my master's thesis back in 2006. After a stimulating lecture by Michael Grosser about Colombeau algebras and some conversations I found that the research direction of the Vienna branch of the DIANA group (DIfferential Algebras and Nonlinear Analysis) combined exactly the topics I was interested in, ranging from differential geometry over generalized functions to functional analysis. Furthermore, this is a relatively young field of research; there are many avenues to follow, and walking along established paths for another time can give many new and intriguing insights.

Later on, I was lucky to be able to stay at the faculty of mathematics for my Ph.D. thesis. A long-time research project of the DIANA group was the construction of a full diffeomorphism-invariant Colombeau-type algebra of nonlinear generalized tensor fields on manifolds. I eventually worked on three distinct but related topics focused in this area. They are presented in this thesis independently of each other.

My primary research assignment was to perform a construction of *a space of nonlinear generalized tensor fields* similar to the above, but based on Riemannian manifolds. This simplifies the whole construction. I took steps towards a new view on smoothing kernels, which lie at the basis of full Colombeau algebras on manifolds, and obtained results on the question of whether the embedding of tensor distributions commutes with pullback and Lie derivatives in my setting. This is presented in Part I.

Occasionally, in mathematical work questions arise about the basic mathematical building blocks one is using whenever some implicit details are missing. One can leave it then to good belief, or follow the urge to do it properly from scratch. The latter happened to me when I used well-known *isomorphisms of the space of tensor distributions*. My notes grew into a rather detailed treatment of the topological background of these isomorphisms in Part II.

Finally, it was tempting to extend the concept of *point values of generalized functions* – which has been available only in simpler settings before – to the diffeomorphism-invariant algebra. For the local case this was done in Part III.

Vienna, October 2010

Part I

The algebra of generalized tensor fields

Chapter 1

Introduction to Part I

While the theory of distributions developed by S. L. Sobolev and L. Schwartz as a generalization of classical analysis is a powerful tool for many applications, in particular in the field of linear partial differential equations, it is inherently linear and not well-suited for nonlinear operations. In particular, one cannot define a reasonable intrinsic multiplication of distributions ([Obe92]). Even more, if one aims at embedding the space of distributions $\mathcal{D}'(\Omega)$ (for some open set $\Omega \subseteq \mathbb{R}^n$) into a differential algebra one is limited by the Schwartz impossibility result [Sch54] which in effect states that there can be no associative commutative algebra $\mathcal{A}(\Omega)$ satisfying the following conditions:

(i) There is a linear embedding $\mathcal{D}'(\Omega) \to \mathcal{A}(\Omega)$ which maps the constant function 1 to the identity in $\mathcal{A}(\Omega)$.

(ii) $\mathcal{A}(\Omega)$ is a differential algebra with linear derivative operators satisfying the Leibniz rule.

(iii) The derivations on $\mathcal{A}(\Omega)$ extend the partial derivatives of $\mathcal{D}'(\Omega)$.

(iv) The product in $\mathcal{A}(\Omega)$ restricted to C^k-functions for some $k < \infty$ coincides with the usual pointwise product.

However, it was found that such a construction is indeed possible if one replaces condition (iv) by the stronger requirement

(iv') The product in $\mathcal{A}(\Omega)$ coincides with the pointwise product of smooth functions.

In the 1980s J. F. Colombeau developed a theory of generalized functions ([Col84, Col85, Obe92, GKOS01]) displaying maximal consistency with both the distributional and the smooth theory under the restrictions dictated by the Schwartz impossibility result. A Colombeau algebra thus has come to mean a differential algebra as above satisfying (i),(ii),(iii), and (iv'), i.e., containing the space of distributions as a linear subspace and the space of smooth functions as a faithful subalgebra.

The basic idea behind Colombeau algebras is to represent distributions as families of smooth functions obtained through a regularization procedure. The space of these families is then subjected to a quotient construction which ensures that the pointwise product of smooth functions is preserved. Once can distinguish two variants of Colombeau algebras, namely the full and the special variant. Full algebras possess a canonical embedding of distributions which allows for a more universal approach to physical models. Special algebras use a fixed mollifier for the embedding and thus are more restrictive but have a considerably simpler structure.

In the context of the special algebra on manifolds ([AB91, dRD91, GFKS01]) the development of generalized counterparts of elements of classical semi-Riemannian geometry was comparatively easy, leading to concepts like generalized sections of vector bundles (thus generalized tensor fields), point values, Lie and covariant derivatives, generalized vector bundle homomorphisms etc. ([KSV05, KS02b]). However, the embedding into the special algebra is not only non-canonical, it is essentially non-geometric ([GKOS01, Section 3.2.2]). Therefore the construction of a full variant was desired.

After several attempts and preliminary work by various authors in this direction ([CM94, VW98, Jel99]) the full diffeomorphism invariant algebra $\mathcal{G}^d(\Omega)$ of generalized functions on open subsets $\Omega \subseteq \mathbb{R}^n$ came to life in [GFKS01], which in turn led to the introduction of the full algebra $\hat{\mathcal{G}}(M)$ of generalized functions on a manifold M in intrinsic terms in [GKSV02].

The latest cornerstone in the development of geometric Colombeau algebras outlined here was the construction of a full Colombeau-type algebra of generalized tensor fields on a manifold as in [GKSV09]. Note that this is not possible by simply defining $\hat{\mathcal{G}}^r_s(M) := \hat{\mathcal{G}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)$ and using a coordinatewise embedding $\iota \otimes \mathrm{id}$ of $\mathcal{D}'^r_s(M) \cong \mathcal{D}'(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)$, as for the latter map to be well-defined one would require ι to be $C^\infty(M)$-linear, which cannot be the case; we refer to [GKSV09, Section 4] for a detailed discussion of the obstructions to tensorial extensions of generalized function algebras like $\hat{\mathcal{G}}(M)$.

The deeper reason for this (and also the key to the way forward) is that regularization of distributional tensor fields in a coordinate-invariant way requires some additional structure on the manifold in order to compare the values of a tensor field at different points, namely a connection. In [GKSV09] this connection is not assumed to be given on the manifold but – in order to obtain a *canonical* embedding – one introduces an additional parameter on which generalized objects depend instead, encoding all ways of transporting tensor fields as needed. This additional parameter further adds to the complexity of the theory; even more, one does not retain $\hat{\mathcal{G}}(M)$ as the space of scalars.

In this work we will take the other route and assume that a Riemannian metric is given on the manifold. This allows us to carry out a construction of a space of generalized tensor fields similar to [GKSV09], but instead of introducing an additional parameter for the generalized objects we use the Levi-Civita connection for embedding distributional tensor fields.

In Chapter 2 we will introduce necessary notation and the basic definitions of distributions on manifolds and local diffeomorphism-invariant Colombeau algebras.

Chapter 3 is devoted to smoothing kernels, the essential building blocks of full Colombeau algebras on manifolds. We introduce their local equivalent and study approximation properties of local smoothing kernels. This is not only useful in the construction to follow, but gives some new insights.

In Chapter 4 the space of generalized tensor fields on a Riemannian manifold is constructed. We establish algebraic isomorphisms and show localization and sheaf properties.

Chapter 5 will give the definition of the embedding of distributional tensor fields, using the background connection in an essential way.

In Chapter 6 we define pullback and Lie derivative of generalized tensor fields.

Chapter 7 finally studies commutation relations of pullback along diffeomorphisms and Lie derivatives with the embedding of tensor distributions. The main result is that these commute for isometries resp. Killing vector fields, but not in general.

Chapter 2

Preliminaries

In this chapter we will list the basic definitions and conventions we will be working with throughout. Additionally, some standard reference texts used are mentioned.

2.1 Notation

We write $A \subset\subset B$ if A is a compact subset of the interior of B. The identity mapping is denoted by id. We will frequently use the index set $I = (0, 1]$. The quotient map, assigning to an element of a set its class in a certain quotient space, is written as cl. The topological boundary of a set U is denoted by ∂U.

For any modules $M_1, \ldots, M_n, N$ over a commutative ring R we denote by $\mathrm{L}^n_R(M_1 \times \ldots \times M_n, N)$ the space of all R-multilinear maps from $M_1 \times \ldots \times M_n$ to N. We omit the subscript R whenever it is clear from the context, in particular in the case of linear maps between vector spaces. The subspace of all symmetric multilinear mappings is denoted by $\mathrm{L}^n_{\mathrm{sym}}(M_1 \times \ldots \times M_n, N)$. For any open set $V \subseteq \mathbb{R}^n$, $\Omega^n_c(V)$ denotes the space of compactly supported n-forms on V.

The space of smooth mappings between subsets U and V of finite-dimensional vector spaces (or manifolds) is $C^\infty(U, V)$, we write $C^\infty(U)$ if $V = \mathbb{R}$ or $\mathbb{C}$. We use the usual Landau notation $f(\varepsilon) = O(g(\varepsilon))$ $(\varepsilon \to 0)$ if there exist positive constants C and ε_0 such that $|f(\varepsilon)| \le Cg(\varepsilon)$ for all $\varepsilon \le \varepsilon_0$. $\mathcal{D}(\Omega)$ denotes the space of test functions on an open subset $\Omega \subseteq \mathbb{R}^n$. We use the usual multi-index notation.

For calculus on infinite-dimensional locally convex spaces we refer to [KM97] for a complete exposition of calculus on convenient vector spaces as we use it and to [GKSV09] for background information more specific to our setting. The differential $\mathrm{d}\colon C^\infty(U, F) \to C^\infty(U, L(E, F))$ is that of [KM97, Theorem 3.18]. Several smoothness arguments are identical to the corresponding ones in [GKSV09] and will only be referred to at the appropriate place.

Our basic references for differential geometry are [AMR88, Kli95]. A manifold will always mean an orientable second countable Hausdorff manifold M of finite dimension. This dimension will be denoted by n throughout if not otherwise stated. Charts are written as a pair (U, φ) with U an open subset of M and φ a homeomorphism from U to an open subset of $\mathbb{R}^n$. A vector bundle E with base M is denoted by $E \to M$, its fiber over the point $p \in M$ by E_p. The space of sections of E is denoted by $\Gamma(E)$, the space of sections with compact support by $\Gamma_c(E)$, and the space of sections with support in a compact set $L \subseteq M$ by $\Gamma_{c,L}(E)$. $\mathrm{T}M$ resp. T^*M is the tangent resp. cotangent bundle of M, $\Lambda^n\mathrm{T}^*M$ is the vector bundle of exterior n-forms on M. A particular vector bundle we will use is $\Gamma(\mathrm{pr}_2^* \mathrm{T}^r_s(M))$, the pullback of the tensor bundle $\mathrm{T}^r_s(M)$ along the projection of $M \times M$ onto the second factor. $\mathfrak{X}(M)$ resp. $\mathfrak{X}^*(M)$ is the space of vector resp. covector fields, $\Omega^n_c(M)$ denotes the space of n-forms and $\mathcal{T}^r_s(M)$ the space of (r,s)-tensor fields on M. $\mathcal{D}(M)$ is the space of test functions on M, i.e., the space of smooth functions with compact support. For a diffeomorphism $\mu\colon M \to N$ between manifolds M and N, μ^* denotes pullback of whatever object in question along μ, we set $\mu_* := (\mu^{-1})^*$. $\mathrm{T}\mu$ is the tangent map of μ, $(\mathrm{T}\mu)^r_s$ the corresponding map on the tensor bundle $\mathrm{T}^r_s(M)$. The result of the action of a tensor field $t \in \mathcal{T}^r_s(M)$ on a dual tensor field $u \in \mathcal{T}^s_r(M)$ is written as $t \cdot u$. L_X denotes the Lie derivative with respect to a vector field X.

If M is endowed with a Riemannian metric g we speak of the Riemannian manifold (M, g). The action of g is denoted by $\langle \cdot, \cdot \rangle_g$ and the corresponding norm by $\|\cdot\|_g$. A metric ball of radius $r > 0$ about $p \in M$ with respect to g is denoted by $B^g_r(p)$. Following the notation of [Kli95, Definition 1.5.1] a covariant derivation is a mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ determined by a family of Christoffel symbols, which are smooth mappings

$$\Gamma\colon \varphi(U) \to \mathrm{L}^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$$

on each chart (U, φ) satisfying the appropriate transformation rule.

2.2 Distributions on manifolds

Our basic reference for distributions on manifolds is [GKOS01, Section 3.1]. For orientable manifolds we define the space of (scalar) distributions on M as

$$\mathcal{D}'(M) := (\Omega^n_c(M))'$$

and the space of (r,s)-tensor distributions on M as

$$\mathcal{D}'^r_s(M) := (\Gamma_c(\mathrm{T}^s_r M \otimes \Lambda^n \mathrm{T}^* M))'$$

where the spaces of sections carry the usual (LF)-topology (cf. Part III, Chapter 11). $\mathcal{D}'(M)$ and $\mathcal{D}'^r_s(M)$ are endowed with the strong dual topology ([Tre76, Chapter 19]). We will furthermore make use of the following isomorphic representations:

$$\begin{aligned}\mathcal{D}'^r_s(M) &\cong (\mathcal{T}^s_r(M) \otimes_{C^\infty(M)} \Omega^n_c(M))' \\ &\cong \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)) \\ &\cong \mathcal{T}^r_s(M) \otimes_{C^\infty(M)} \mathcal{D}'(M).\end{aligned}$$

Part II contains a detailed treatment of these isomorphisms. The action of a tensor distribution $T \in \mathcal{D}'^r_s(M)$ will accordingly be denoted by either of

$$\langle T, \xi \rangle = \langle T, s \otimes \omega \rangle = \langle T(s), \omega \rangle$$

for $\xi \in \Gamma_c(\mathrm{T}^s_r M \otimes \Lambda^n \mathrm{T}^* M)$, $s \in \mathcal{T}^s_r(M)$, and $\omega \in \Omega^n_c(M)$, with ξ corresponding to $s \otimes \omega$ under the isomorphism

$$\Gamma_c(\mathrm{T}^s_r(M) \otimes \Lambda^n \mathrm{T}^* M) \cong \mathcal{T}^s_r(M) \otimes_{C^\infty(M)} \Omega^n_c(M).$$

By $\mathcal{E}'(\Omega) \subseteq \mathcal{D}'(\Omega)$ we denote the space of distributions with compact support in $\Omega \subseteq \mathbb{R}^n$; this is only used in Chapter 7.

Given a chart (U, φ) on M, to each distribution $T \in \mathcal{D}'(U)$ there corresponds a unique distribution in $\mathcal{D}'(\varphi(U))$ also denoted by T such that for all $\omega \in \Omega^n_c(M)$ with support in U and local representation $\omega(x) = f(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ with $f \in \mathcal{D}(\varphi(U))$ the relation $\langle T, \omega \rangle = \langle T, f \rangle$ holds. More explicitly we may also write $\langle T(p), \omega(p) \rangle = \langle T(x), f(x) \rangle$.

For $T \in \mathcal{D}'^r_s(M)$ and $s \otimes \omega \in \mathcal{T}^s_r(M) \otimes_{C^\infty(M)} \Omega^n_c(M)$ with $\operatorname{supp} \xi \subseteq U$ we write

$$\langle T, s \otimes \omega \rangle = \langle T^\lambda, s_\lambda \cdot \omega \rangle$$

where the $T^\lambda \in \mathcal{D}'(M)$ are the coordinates of T and the $s_\lambda \in C^\infty(U)$ are the coordinates of s on U; we use the Einstein summation convention.

2.3 Full Colombeau algebras, the local theory

For the following definitions and for later use we need the spaces of mollifiers

$$\begin{aligned}\mathcal{A}_0(\Omega) &:= \{\, \varphi \in \mathcal{D}(\Omega) \mid \int \varphi(x)\,\mathrm{d}x = 1 \,\} \quad \text{and} \\ \mathcal{A}_q(\Omega) &:= \{\, \varphi \in \mathcal{A}_0(\Omega) \mid \int x^\alpha \varphi(x)\,\mathrm{d}x = 1 \ , 1 \le |\alpha| \le q,\ \alpha \in \mathbb{N}_0^n \,\},\end{aligned}$$

each endowed with the subspace topology. Furthermore, we need mappings for translating and scaling test functions, given for $\varepsilon \in (0, \infty)$ and $x \in \mathbb{R}^n$ by

$$\begin{aligned}\mathrm{T}_x \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n), &\quad (\mathrm{T}_x\varphi)(y) := \varphi(y - x) \\ \mathrm{S}_\varepsilon \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n), &\quad (\mathrm{S}_\varepsilon\varphi)(y) := \varepsilon^{-n}\varphi(y/\varepsilon) \\ \mathrm{T} \colon \mathcal{D}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{D}(\mathbb{R}^n), &\quad \mathrm{T}(\varphi, x)(y) := (\mathrm{T}_x\varphi)(y) \\ \mathrm{S} \colon (0, \infty) \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n), &\quad \mathrm{S}(\varepsilon, \varphi)(y) := (\mathrm{S}_\varepsilon\varphi)(y).\end{aligned}$$

The local diffeomorphism invariant algebra $\mathcal{G}^d(\Omega)$ on an open set $\Omega \subseteq \mathbb{R}^n$ can be given in two different but equivalent formalisms, called J-formalism and C-formalism after J. Jelínek and J. F. Colombeau, respectively (see [GFKS01, Section 5] for a detailed discussion). We will consider both. The corresponding basic spaces are given by

$$\mathcal{E}^J(\Omega) := C^\infty(\mathcal{A}_0(\Omega) \times \Omega) \quad \text{resp.} \quad \mathcal{E}^C(\Omega) := C^\infty(U(\Omega))$$

with

$$\begin{aligned} U(\Omega) &:= T^{-1}(\mathcal{A}_0(\Omega) \times \Omega) \\ &= \{\varphi \in \mathcal{A}_0(\mathbb{R}^n) \mid x + \operatorname{supp} \varphi \subseteq \Omega\}. \end{aligned}$$

Distributions $t \in \mathcal{D}'(\Omega)$ are embedded into $\mathcal{E}^J(\Omega)$ resp. $\mathcal{E}^C(\Omega)$ with the maps

$$(\iota^J t)(\varphi, x) := \langle t, \varphi \rangle \quad \text{resp.} \quad (\iota^C t)(\varphi, x) := \langle t, \mathrm{T}_x \varphi \rangle$$

and the embedding σ of smooth functions is given for both formalisms by

$$\sigma(f)(\varphi, x) = f(x).$$

By a procedure commonly called *testing* so-called moderate and negligible elements are singled out in order to perform a quotient construction that ensures equality of the two embeddings in the quotient. For this one needs suitable test objects. Set $I := (0, 1]$. The test objects for $\mathcal{G}^d(\Omega)$ are elements of the space $C^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$, which is defined as the set of all $\phi \in C^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$ such that $\forall K \subset\subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n$ the set $\{(\partial_x^\alpha \phi)(\varepsilon, x) \mid \varepsilon \in I, x \in K\} \subseteq \mathcal{D}(\mathbb{R}^n)$ is bounded. For partial derivatives of ϕ we use the notation $(\partial_x^\alpha \partial_y^\beta \phi)(\varepsilon, x_0)(y_0) = \partial^\beta(\partial^\alpha(\phi(\varepsilon, .))(x_0))(y_0)$.

An element $R \in \mathcal{E}^J(\Omega)$ then is called *moderate* if $\forall K \subset\subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n$ $\exists N \in \mathbb{N}$ $\forall \phi \in C^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$ we have $\sup_{x \in K} |\partial^\alpha(R(\mathrm{T}_x \mathrm{S}_\varepsilon \phi(\varepsilon, x), x))| = O(\varepsilon^{-N})$ for $\varepsilon \to 0$, the set of moderate elements is denoted by $\mathcal{E}^J_M(\Omega)$. $R \in \mathcal{E}^J_M(\Omega)$ is called *negligible* if $\forall K \subset\subset \Omega$ $\forall m \in \mathbb{N}$ $\exists q \in \mathbb{N}$ $\forall \phi \in C^\infty_b(I \times \Omega, \mathcal{A}_q(\mathbb{R}^n))$ we have $\sup_{x \in K} |R(\mathrm{T}_x \mathrm{S}_\varepsilon \phi(\varepsilon, x), x)| = O(\varepsilon^m)$ for $\varepsilon \to 0$, the set of negligible elements is denoted by $\mathcal{N}^J(\Omega)$.

In C-formalism one simple leaves away the T_x in the test and accordingly gets spaces $\mathcal{E}^C_M(\Omega)$ and $\mathcal{N}^C(\Omega)$. The bijective map $\mathrm{T}^* \colon \mathcal{E}^J(\Omega) \to \mathcal{E}^C(\Omega)$ allows to translate between the formalisms and preserves moderateness and negligibility.

The algebra of generalized functions $\mathcal{G}^d(\Omega)$ is then simply defined as the quotient $\mathcal{E}^J_M(\Omega)/\mathcal{N}^J(\Omega)$ resp. $\mathcal{E}^C_M(\Omega)/\mathcal{N}^C(\Omega)$.

The fact that in [GFKS01] for $\mathcal{G}^d(\Omega)$ the C-formalism was used has the following consequences:

- The class of test objects $C^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$ is not invariant under diffeomorphisms, which requires the introduction of a larger (but equivalent) class of test objects which are defined only on subsets of $I \times \Omega$ ([GFKS01, Section 7.4]).

- Smoothness on $U(\Omega)$ has to be handled carefully ([GFKS01, Section 6]).
- Because the full algebra on a manifold has to be constructed using J-formalism, any local calculation involving $\mathcal{G}^d(\Omega)$ invariably has to involve a change of formalism; even in case one uses J-formalism also for $\mathcal{G}^d(\Omega)$, the above test objects are still not well-behaved under diffeomorphisms.

We will see in Chapter 3 that one can replace these test objects by more suitable ones in order to evade these problems: the development of the full algebra $\hat{\mathcal{G}}(M)$ on a manifold M in [GKSV02] has shown that in some sense the right test objects for the diffeomorphism invariant setting are smoothing kernels, which will be treated next.

Chapter 3

Smoothing kernels

Although the full Colombeau algebra $\mathcal{G}^d(\Omega)$ is diffeomorphism invariant its formulation in [GFKS01] still uses the linear structure of $\mathbb{R}^n$: in C-formalism the domain of representatives of generalized functions is

$$U(\Omega) = \mathrm{T}^{-1}(\mathcal{A}_0(\Omega) \times \Omega)$$

and testing a representative R of a generalized function for moderateness or negligibility involves expressions of the form $R(\mathrm{S}_\varepsilon\phi(\varepsilon, x), x)$, but both translation T and scaling S have no direct counterpart on a manifold. In J-formalism the translation appears in the testing procedure instead of the basic space: the domain of representatives of generalized functions is $\mathcal{A}_0(\Omega) \times \Omega$, but tests now involve expressions of the form $R(\mathrm{T}_x\mathrm{S}_\varepsilon\phi(\varepsilon, x), x)$.

For the construction of the full algebra on a manifold replacing $\mathcal{A}_0(\Omega)$ by compactly supported n-forms with integral one gives a suitable basic space of generalized functions, $\hat{\mathcal{E}}(M) := C^\infty(\hat{\mathcal{A}}_0(M), C^\infty(M))$. Note that the author prefers this form to $C^\infty(\hat{\mathcal{A}}_0(M) \times M)$, and similarly for tensor case below. By the exponential law for spaces of smooth functions ([KM97, 27.17]) we have

$$C^\infty(\hat{\mathcal{A}}_0(M) \times M) \cong C^\infty(\hat{\mathcal{A}}_0(M), C^\infty(M))$$

so this amounts to a purely notational difference.

Going to a manifold, the test objects have to be adapted in the following way: one regards $\tilde{\phi}(\varepsilon, x) := \mathrm{T}_x\mathrm{S}_\varepsilon\phi(\varepsilon, x)$ as a test object (called smoothing kernel) depending on ε and x, infers its properties from those of ϕ, and in this way defines a new space of test objects in a coordinate-free way. This approach directly results in the global algebra $\hat{\mathcal{G}}(M)$ of [GKSV02], using smoothing kernels ([GKSV02, Section 3] or Definition 3.6 below) as direct equivalents of scaled and translated local test objects.

While [GKSV02] defines smoothing kernels only on manifolds we will introduce *local smoothing kernels* as immediate equivalents of their global version. This will serve two purposes.

- First, they eliminate the need for a change of formalism in local calculations as in [GKSV02, Lemma 4.2]. Having established approximation properties of local smoothing kernels, proofs like injectivity of the embedding of distributions or results related to the concept of association (in the sense it is usually used in Colombeau algebras) can be obtained more easily.
- Second, the J-setting together with smoothing kernels as test objects apparently seems to be the natural way for describing the diffeomorphism invariant algebra, which suggest that smoothing kernels will make possible a clearer formulation also of the local diffeomorphism invariant theory. Most notable, diffeomorphism invariance is seen very easily with smoothing kernels (see Chapter 6). In comparison, the use of C-formalism in [GFKS01] entails considerable technical difficulties because the space of test objects $C_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$ is not invariant under diffeomorphisms; one needs to introduce a larger (but ultimately equivalent, cf. [GFKS01, Section 7.4]) class of test objects having smaller domains of definition in order to prove diffeomorphism invariance of $\mathcal{G}^d(\Omega)$ in the C-setting.

We will begin with defining smoothing kernels on a manifold before we study their local equivalent.

3.1 Global smoothing kernels

Smoothing kernels basically are n-forms depending on $\varepsilon \in I$ and an additional space variable, satisfying certain properties needed for the construction of Colombeau algebras. As a preliminary we will define such n-forms on a manifold as well as their Lie derivative in both variables and pullback. We will only be concerned with compactly supported n-forms throughout. All subsequent results remain valid if Φ additionally depends on $\varepsilon \in I$, as it will for smoothing kernels.

Lemma 3.1. *The Lie derivative* $\mathrm{L}_X \colon \Omega_c^n(M) \to \Omega_c^n(M)$ *is smooth with respect to the (LF)-topology.*

Proof. By [GFKS01, Theorem 4.1] it suffices to verify that for each compact set $K \subset\subset M$ the mapping $\mathrm{L}_X \colon \Omega_{c,K}^n(M) \to \Omega_c^n(M)$ is bounded, which follows from [GKSV09, Proposition A.2 (2) (i)]. □

Definition 3.2. On $C^\infty(M, \Omega_c^n(M))$ we define two Lie derivatives

$$\mathrm{L}_X \Phi := \mathrm{L}_X \circ \Phi$$
$$(\mathrm{L}'_X \Phi)(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\mathrm{Fl}_t^X\, p)$$

for $\Phi \in C^\infty(M, \Omega^n_c(M))$, $X \in \mathfrak{X}(M)$, and $p \in M$.

Proposition 3.3. L_X *and* L'_X *are smooth linear maps from* $C^\infty(M, \Omega^n_c(M))$ *into itself.*

Proof. The case of L_X is clear from Lemma 3.1. For L'_X, Φ is an element of $C^\infty(M, \Omega^n_c(M))$ if and only if for each chart (U, φ) of an atlas $\Phi \circ \varphi^{-1}$ is in $C^\infty(\varphi(U), \Omega^n_c(M))$. Denote by $\alpha(t, x)$ the local flow of X in the chart. Then for fixed p and t in a neighborhood of zero $\alpha(t, \varphi(p))$ exists and (denoting the local expression of X by the same letter)

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\mathrm{Fl}^X_t\, p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi \circ \varphi^{-1})(\varphi \circ \mathrm{Fl}^X_t\, p) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi \circ \varphi^{-1})(\alpha(t, \varphi(p))) \\
&= \mathrm{d}(\Phi \circ \varphi^{-1})(\varphi(p)) \cdot X(\varphi(p)).
\end{aligned}$$

From this we see that the limit exists and is smooth. □

Definition 3.4. Given a smooth map $\mu\colon M \to N$ and $\Phi \in C^\infty(N, \Omega^n_c(N))$, the *pullback* of Φ along μ is defined as

$$\mu^*\Phi := \mu^* \circ \Phi \circ \mu \in C^\infty(M, \Omega^n_c(M)).$$

Now we will examine how the Lie derivatives defined above translate under pullbacks.

Lemma 3.5. *Let* $\Phi \in C^\infty(N, \Omega^n_c(N))$. *Then for any diffeomorphism* $\mu\colon M \to N$ *we have* $\mathrm{L}_X(\mu^*\Phi) = \mu^*(\mathrm{L}_{\mu_* X}\Phi)$ *and* $\mathrm{L}'_X(\mu^*\Phi) = \mu^*(\mathrm{L}_{\mu_* X}\Phi)$.

Proof. First, we have $\mathrm{L}_X(\mu^*\Phi) = \mathrm{L}_X \circ \mu^* \circ \Phi \circ \mu = \mu^* \circ \mathrm{L}_{\mu_* X} \circ \Phi \circ \mu = \mu^*(\mathrm{L}_{\mu_* X}\Phi)$. Second, because $\mu^*\colon \Omega^n_c(M) \to \Omega^n_c(M)$ is linear and smooth we have

$$\begin{aligned}
\mathrm{L}'_X(\mu^*\Phi)(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mu^*\Phi)(\mathrm{Fl}^X_t\, p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mu^* \circ \Phi \circ \mu \circ \mathrm{Fl}^X_t)(p) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mu^* \circ \Phi \circ \mathrm{Fl}^{\mu_* X}_t \circ \mu)(p) = \mu^*\Big(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\mathrm{Fl}^{\mu_* X}_t(\mu(p)))\Big) \\
&= \mu^*((\mathrm{L}'_{\mu_* X}\Phi)(\mu(p))) = \mu^*(\mathrm{L}'_{\mu_* X}\Phi).
\end{aligned}$$ □

With ε-dependence added we can now give the definition of (global) smoothing kernels ([GKSV02, Definition 3.3]).

Definition 3.6. A map $\Phi \in C^\infty(I \times M, \hat{\mathcal{A}}_0(M))$ is called a *smoothing kernel* if it satisfies the following conditions for any Riemannian metric g on M:

(i) $\forall K \subset\subset M\ \exists \varepsilon_0, C > 0\ \forall p \in K\ \forall \varepsilon \leq \varepsilon_0$: $\operatorname{supp} \Phi(\varepsilon, p) \subseteq B^g_{\varepsilon C}(p)$,

(ii) $\forall K \subset\subset M\ \forall l, m \in \mathbb{N}_0\ \forall \theta_1, \dots, \theta_m, \zeta_1, \dots, \zeta_l \in \mathfrak{X}(M)$ we have

$$\sup_{\substack{p\in K\\ q\in M}} \left\|(\mathrm{L}_{\theta_1} \dots \mathrm{L}_{\theta_m}(\mathrm{L}'_{\zeta_1} + \mathrm{L}_{\zeta_1}) \dots (\mathrm{L}'_{\zeta_l} + \mathrm{L}_{\zeta_l})\Phi)(\varepsilon, p)(q)\right\|_g = O(\varepsilon^{-n-m}).$$

The space of all smoothing kernels is denoted by $\widetilde{\mathcal{A}}_0(M)$.
For each $k \in \mathbb{N}$ denote by $\widetilde{\mathcal{A}}_k(M)$ the set of all $\Phi \in \widetilde{\mathcal{A}}_0(M)$ such that $\forall f \in C^\infty(M)$ and $K \subset\subset M$ we have the approximation property

$$\sup_{p\in K} \left| f(p) - \int_M f \cdot \Phi(\varepsilon, p) \right| = O(\varepsilon^{k+1}). \tag{3.1}$$

Remark 3.7. Note that elements of $\widetilde{\mathcal{A}}_0(M)$ satisfy (3.1) for $k = 0$. One furthermore even has

$$\sup_{p\in K} \left| f(p,p) - \int_M f(p,.) \cdot \Phi(\varepsilon, p) \right| = O(\varepsilon^{k+1})$$

for $\Phi \in \widetilde{\mathcal{A}}_k(M)$ ([GKSV09, Lemma 3.6]).

That Definition 3.6 indeed is independent of the metric follows from the next lemma.

Lemma 3.8. *Let (M, g) and (N, h) be Riemannian manifolds. Given a diffeomorphism $\mu\colon M \to N$ and a compact set $K \subset\subset M$ there exists a constant $C > 0$ such that*

(i) $\|(\mu^* t)(p)\|_g \leq C \|t(\mu(p))\|_h\ \forall t \in \mathcal{T}^r_s(N)\ \forall p \in K$.

(ii) $\|(\mu^* \omega)(p)\|_g \leq C \|\omega(\mu(p))\|_h\ \forall \omega \in \Omega^n_c(N)\ \forall p \in K$.

(iii) $B^g_r(p) \subseteq \mu^{-1}(B^h_{rC}(\mu(p))) = B^{\mu^* h}_{rC}(p)$ *for all small* $r > 0$ *and* $\forall p \in K$.

Proof. First, we note that for $t \in \mathcal{T}^r_s(N)$ we have

$$\|(\mu^* t)(p)\|_{\mu^* h} = \left\|((\mathrm{T}\mu^{-1})^r_s \circ t \circ \mu)(p)\right\|_{\mu^* h} = \|t(\mu(p))\|_h .$$

Second, we can assume without limitation of generality that K is contained in a chart (U, φ) where U is strongly convex (as defined in [Kli95, Definition 1.9.9]). Then for any $\omega = f\, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$ in $\Omega^n_c(U)$ with $f \in C^\infty(U)$ we have

$$\begin{aligned}
\|(\mu^* \omega)(p)\|_{\mu^* h} &= \left\|(f \circ \mu)(p)(\mu^*(\mathrm{d}x^1) \wedge \dots \wedge \mu^*(\mathrm{d}x^n))(p)\right\|_{\mu^* h} \\
&= |f(\mu(p))| \cdot \left|\det(\langle \mu^*(\mathrm{d}x^i)(p), \mu^*(\mathrm{d}x^j)(p)\rangle_{\mu^* h})_{i,j}\right|^{1/2} \\
&= |f(\mu(p))| \cdot \left|\det(\langle \mathrm{d}x^i(\mu(p)), \mathrm{d}x^j(\mu(p))\rangle_h)_{i,j}\right|^{1/2} \\
&= \left\|(f\, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n)(\mu(p))\right\|_h = \|\omega(\mu(p))\|_h .
\end{aligned}$$

Third, $\mu^{-1}(B_r^h(\mu(p))) = B_r^{\mu^* h}(p)$ is clear for small r because the isometry $\mu\colon (M, \mu^* h) \to (N, h)$ preserves geodesics in both directions.

Denote the extensions of g, h to $\mathrm{T}^r_s(M)$ resp. $\Lambda^n \mathrm{T}^* M$ by $\tilde{g}, \tilde{h}$. Let $\tilde{g}_U, \tilde{h}_U \subseteq \mathrm{L}^2_{sym}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R})$ be local representatives of $\tilde{g}, \tilde{h}$ where m is the dimension of the respective chart. Then all claims follow directly from

$$\sup_{\substack{v \in \mathbb{R}^n \setminus \{0\} \\ x \in \varphi(K)}} \frac{\tilde{g}_U(v,v)}{\tilde{h}_U(v,v)} = \sup_{\substack{\|v\|=1 \\ x \in \varphi(K)}} \frac{\tilde{g}_U(v,v)}{\tilde{h}_U(v,v)} < \infty. \qquad \square$$

Now we use the pullback from Definition 3.4 for smoothing kernels.

Proposition 3.9. *Given $\Phi \in \widetilde{\mathcal{A}}_k(N)$ with $k \in \mathbb{N}_0$ and a diffeomorphism $\mu\colon M \to N$ the map $\Psi\colon I \times M \to \Omega^n_c(M)$ defined by $\Psi(\varepsilon, p) := \mu^*(\Phi(\varepsilon, \mu(p)))$ is in $\widetilde{\mathcal{A}}_k(M)$.*

Proof. For (i) of Definition 3.6, let $K \subset\subset M$ be given. For any Riemannian metric h on N there are constants $\varepsilon_0 > 0$ and $C > 0$ such that for all $p \in K$ and $\varepsilon \leq \varepsilon_0$ the support of $\mu^*(\Phi(\varepsilon, \mu(p)))$ is contained in $\mu^{-1}(B^h_{\varepsilon C}(\mu(p)))$. By Lemma 3.8 (iii) for any Riemannian metric g on M there is a constant $L > 0$ such that $\mu^{-1}(B^h_{\varepsilon C}(\mu(p))) \subseteq B^g_{\varepsilon L C}(p)$ for all $p \in K$ and small ε.

For (ii), given any vector fields $\zeta_1, \ldots, \zeta_l, \theta_1, \ldots, \theta_m \in \mathfrak{X}(M)$ we see that

$$\mathrm{L}_{\theta_1} \ldots \mathrm{L}_{\theta_m} (\mathrm{L}'_{\zeta_1} + \mathrm{L}_{\zeta_1}) \ldots (\mathrm{L}'_{\zeta_l} + \mathrm{L}_{\zeta_l}) \Psi$$

equals (by Lemma 3.5)

$$\mu^*(\mathrm{L}_{\mu_* \theta_1} \ldots \mathrm{L}_{\mu_* \theta_m} (\mathrm{L}'_{\mu_* \zeta_1} + \mathrm{L}_{\mu_* \zeta_1}) \ldots (\mathrm{L}'_{\mu_* \zeta_l} + \mathrm{L}_{\mu_* \zeta_l}) \Phi)$$

whence by Lemma 3.8 (i) the assertion on the derivatives of Ψ follows from the defining properties of Φ. Finally, the approximation property (3.1) follows directly from writing down the corresponding integral. $\square$

Definition 3.10. Given $\Phi \in \widetilde{\mathcal{A}}_k(N)$ for $k \in \mathbb{N}_0$ and a diffeomorphism $\mu\colon M \to N$, the map $\mu^* \Phi \in \widetilde{\mathcal{A}}_k(M)$ defined by $(\mu^* \Phi)(\varepsilon, p) := \mu^*(\Phi(\varepsilon, \mu(p))$ is called the *pullback* of Φ along μ.

3.2 Local smoothing kernels

In this section we will introduce local versions of the spaces of smoothing kernels $\widetilde{\mathcal{A}}_k(M)$.

Locally, compactly supported n-forms can be identified with test functions, which is made precise by the vector space isomorphism $\hat{\lambda}\colon \Omega^n_c(\Omega) \to \mathcal{D}(\Omega)$ assigning to $\omega \in \Omega^n_c(\Omega)$ the function $x \mapsto \omega(x)(e_1, \ldots, e_n)$ in $\mathcal{D}(\Omega)$, where

$e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$; its inverse is the mapping $f \mapsto f\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$ and both assignments are continuous. The local equivalent of the Lie derivative is simply the directional derivative, defined as the smooth map

$$\begin{aligned} \mathrm{L}_X \colon \mathcal{D}(\Omega) &\to \mathcal{D}(\Omega) \\ (\mathrm{L}_X f)(x) &:= (\mathrm{d}f)(x) \cdot X(x) \end{aligned}$$

for $f \in \mathcal{D}(\Omega)$, $X \in C^\infty(\Omega, \mathbb{R}^n)$, and $x \in \Omega$. It also satisfies the relation $\mathrm{L}_X(\mu^* f) = (\mathrm{L}_{\mu_* X} f) \circ \mu$ with the pushforward of X along a diffeomorphism μ defined by

$$(\mu_* X)(x) := \mathrm{d}\mu(x) \cdot X(\mu^{-1}(x)).$$

For any n-form $\omega \in \Omega^n_c(\Omega)$ we have the identities $\omega = \hat\lambda(\omega)\,\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n$, $\int \hat\lambda(\omega)(x)\,\mathrm{d}x = \int \omega$, $\operatorname{supp} \hat\lambda(\omega) = \operatorname{supp} \omega$, and $\hat\lambda(\mathrm{L}_X \omega) = \mathrm{L}_X(\hat\lambda(\omega))$. We furthermore have a vector space isomorphism $C^\infty(\Omega, \Omega^n_c(\Omega)) \cong C^\infty(\Omega, \mathcal{D}(\Omega))$ realized by the mapping $\hat\lambda_* \colon \tilde\phi \mapsto \hat\lambda \circ \tilde\phi$.

Definition 3.11. On $C^\infty(\Omega, \mathcal{D}(\Omega))$ we define two Lie derivatives,

$$\mathrm{L}_X \tilde\phi := \mathrm{L}_X \circ \tilde\phi \quad \text{and} \quad (\mathrm{L}'_X \tilde\phi)(x) := (\mathrm{d}\tilde\phi)(x) \cdot X(x)$$

for $\tilde\phi \in C^\infty(\Omega, \mathcal{D}(\Omega))$, $x \in \Omega$, and $X \in C^\infty(\Omega, \mathbb{R}^n)$.

From $\mathrm{L}_X(\hat\lambda_* \tilde\phi) = \mathrm{L}_X \circ \hat\lambda \circ \tilde\phi = \hat\lambda \circ \mathrm{L}_X \circ \tilde\phi$ and $\mathrm{L}'_X(\hat\lambda_* \tilde\phi)(x) = \mathrm{d}(\hat\lambda \circ \tilde\phi)(x) \cdot X(x) = \hat\lambda((\mathrm{d}\tilde\phi)(x) \cdot X(x)) = \hat\lambda((\mathrm{L}'_X \tilde\phi)(x))$ we see that $\hat\lambda_*$ commutes with both L_X and L'_X, thus we have the following lemma.

Lemma 3.12. $\hat\lambda_* \colon C^\infty(I \times \Omega, \Omega^n_c(\Omega)) \to C^\infty(I \times \Omega, \mathcal{D}(\Omega))$ *is a vector space isomorphism with inverse* $(\hat\lambda^{-1})_*$ *and commutes with* L_X *and* L'_X.

We will see in Proposition 3.15 that the defining properties of a smoothing kernel $\Phi \in C^\infty(I \times U, \hat{\mathcal{A}}_0(U))$, namely shrinking support, growth estimates for all derivatives, and approximation properties, translate verbatim to its *local expression*, which is defined as

$$\tilde\phi := \hat\lambda_*(\varphi_* \Phi) \in C^\infty(I \times \varphi(U), \mathcal{A}_0(\varphi(U)))$$

for a chart (U, φ). We thus define local smoothing kernels as follows.

Definition 3.13. A mapping $\tilde\phi \in C^\infty(I \times \Omega, \mathcal{A}_0(\Omega))$ is called a *local smoothing kernel* (on Ω) if it satisfies the following conditions:

(i) $\forall K \subset\subset \Omega\ \exists \varepsilon_0, C > 0\ \forall x \in K\ \forall \varepsilon \le \varepsilon_0$: $\operatorname{supp} \tilde\phi(\varepsilon, x) \subseteq B_{\varepsilon C}(x)$.

(ii) $\forall K \subset\subset \Omega\ \forall \alpha, \beta \in \mathbb{N}^n_0$ we have

$$\sup_{\substack{x \in K \\ y \in \Omega}} \left| (\partial^\beta_y \partial^\alpha_{x+y} \tilde\phi)(\varepsilon, x)(y) \right| = O(\varepsilon^{-n-|\beta|}).$$

The space of all local smoothing kernels is denoted by $\widetilde{\mathcal{A}}_0(\Omega)$.

For each $k \in \mathbb{N}$ denote by $\widetilde{\mathcal{A}}_k(\Omega)$ the set of all $\tilde{\phi} \in \widetilde{\mathcal{A}}_0(\Omega)$ such that for all $f \in C^\infty(\Omega)$ and $K \subset\subset \Omega$ we have the approximation property

$$\sup_{x \in K} \left| f(x) - \int_\Omega f(y)\tilde{\phi}(\varepsilon, x)(y)\,\mathrm{d}y \right| = O(\varepsilon^{k+1}). \tag{3.2}$$

Remark 3.14. Again, elements of $\widetilde{\mathcal{A}}_0(\Omega)$ satisfy (3.2) for $k = 0$. By the usual methods (Taylor expansion of f) one even has

$$\sup_{x \in K} \left| f(x,x) - \int_\Omega f(x,y)\tilde{\phi}(\varepsilon, x)(y)\,\mathrm{d}y \right| = O(\varepsilon^{k+1})$$

for $\tilde{\phi} \in \widetilde{\mathcal{A}}_k(\Omega)$.

Proposition 3.15. *For any chart (U, φ) on M there is a vector space isomorphism $\widetilde{\mathcal{A}}_k(U) \cong \widetilde{\mathcal{A}}_k(\varphi(U))$ given by $\Phi \mapsto \hat{\lambda}_*(\varphi_*\Phi)$.*

Proof. Let $\Phi \in \widetilde{\mathcal{A}}_k(U)$ and set $\tilde{\phi} := \hat{\lambda}_*(\varphi_*\Phi)$. For (i) of Definition 3.13 fix $K \subset\subset \Omega$, then there are $\varepsilon_0 > 0$ and $C > 0$ such that $\operatorname{supp}\Phi(\varepsilon, p) \subseteq B_{\varepsilon C}(p)$ for all $p \in \varphi^{-1}(K)$ and $\varepsilon \le \varepsilon_0$. We may assume that $\varepsilon_0 C < \operatorname{dist}(\varphi^{-1}(K), \partial U)$. Then

$$\begin{aligned} \operatorname{supp}\tilde{\phi}(\varepsilon, x) &= \operatorname{supp}(\varphi_*\Phi)(\varepsilon, x) = \operatorname{supp}\varphi_*(\Phi(\varepsilon, \varphi^{-1}(x))) \\ &\subseteq \varphi(B_{\varepsilon C}(\varphi^{-1}(x))) \subseteq B_{\varepsilon C'}(x) \end{aligned}$$

for some $C' > 0$ by Lemma 3.8 (iii). (ii) of Definition 3.13 is a consequence of Lemmata 3.12, 3.5 and 3.8 (i), while (3.2) is immediate from the definitions. The other direction works analogously. ☐

Finally, we state a result showing that local smoothing kernels are suitable test objects for the local diffeomorphism invariant algebra $\mathcal{G}^d(\Omega)$.

Proposition 3.16. *(i) $R \in \mathcal{E}^J(\Omega)$ is moderate if and only if $\forall K \subset\subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n$ $\exists N \in \mathbb{N}$ $\forall \tilde{\phi} \in \widetilde{\mathcal{A}}_0(\Omega)$: $\sup_{x \in K} \left| \partial^\alpha(R(\tilde{\phi}(\varepsilon, x), x)) \right| = O(\varepsilon^{-N})$.*

(ii) $R \in \mathcal{E}_M^J(\Omega)$ is negligible if and only if $\forall K \subset\subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n$ $\forall m \in \mathbb{N}$ $\exists q \in \mathbb{N}$ $\forall \tilde{\phi} \in \widetilde{\mathcal{A}}_q(\Omega)$: $\sup_{x \in K} \left| \partial^\alpha(R(\tilde{\phi}(\varepsilon, x), x)) \right| = O(\varepsilon^m)$.

Proof. One can directly use [GKSV02, Theorems 4.3 and 4.4] which state that for a chart (U, φ) on M, $R \in \mathcal{E}^J(\varphi(U))$ is moderate resp. negligible if and only if the mapping $(\omega, p) \mapsto R((\hat{\lambda} \circ \varphi_*)(\omega), \varphi(p)) \in \hat{\mathcal{E}}(U)$ is so; using Proposition 3.15 this immediately translates into the conditions stated. As moderateness and negligibility can be tested locally this gives the claim. ☐

3.3 Approximation properties of smoothing kernels

The practical importance of smoothing kernels lies in their approximation properties as in (3.1) and (3.2). We will now consider expressions of the form $\int f(y)\tilde{\phi}(\varepsilon,x)(y)\,\mathrm{d}y$ with variants involving derivatives of $\tilde{\phi}$ and integration over x instead of y, which appears for example in the proof of Proposition 5.3.

In the following we write $\tilde{\phi}(\varepsilon,x,y)$ instead of $\tilde{\phi}(\varepsilon,x)(y)$ where it is convenient.

One can intuit the behavior of the integrals just mentioned by considering the simple example

$$\tilde{\phi}(\varepsilon,x)(y) := (\mathrm{T}_x\mathrm{S}_\varepsilon\varphi)(y) = \frac{1}{\varepsilon}\varphi(\frac{y-x}{\varepsilon^n})$$

for some mollifier $\varphi \in \mathcal{A}_0(\Omega)$. In this case the following convergences are easily obtained by Taylor expansion of f, partial integration, and the fact that $\partial_{x+y}\tilde{\phi} = 0$:

(i) $\int f(x,y)\tilde{\phi}(\varepsilon,x,y)\,\mathrm{d}y \to f(x,x)$,

(ii) $\int f(x,y)\tilde{\phi}(\varepsilon,x,y)\,\mathrm{d}x \to f(y,y)$,

(iii) $\int f(x,y)(\partial_{y_i}\tilde{\phi})(\varepsilon,x,y)\,\mathrm{d}y \to -(\partial_{y_i}f)(x,x)$,

(iv) $\int f(x,y)(\partial_{x_i}\tilde{\phi})(\varepsilon,x,y)\,\mathrm{d}x \to -(\partial_{x_i}f)(y,y)$,

(v) $\int f(x,y)(\partial_{x_i}\tilde{\phi})(\varepsilon,x,y)\,\mathrm{d}y \to (\partial_{y_i}f)(x,x)$, and

(vi) $\int f(x,y)(\partial_{y_i}\tilde{\phi})(\varepsilon,x,y)\,\mathrm{d}x \to (\partial_{x_i}f)(y,y)$.

Here convergence is like $O(\varepsilon)$ uniformly for x resp. y in compact sets and analogous statements are valid for higher derivatives. We will see that the same results can be obtained for arbitrary smoothing kernels (for the integral over x we will have to assume f to have compact support): from (i) and (ii) (remark after Definition 3.13 and Proposition 3.18 (i)) partial integration gives (iii) and (iv), while (v) and (vi) result from Corollary 3.19.

Lemma 3.17. *Let $\tilde{\phi} \in \widetilde{\mathcal{A}}_0(\Omega)$ be a local smoothing kernel. Then $\forall K \subset\subset \Omega$ $\exists \varepsilon_0, C > 0$ such that $\operatorname{supp}(\partial_x^\alpha\partial_y^\beta\tilde{\phi})(\varepsilon,x) \subseteq B_{C\varepsilon}(x)$ for all $x \in K$, $\varepsilon \le \varepsilon_0$, and $\alpha,\beta \in \mathbb{N}_0^n$.*

Proof. As ∂_y preserves the support we can set $\beta = 0$. Given any $\delta > 0$ with $\overline{B}_\delta(K) \subseteq \Omega$ we know that there are $\varepsilon_0 > 0$ and $C > 0$ such that $\operatorname{supp}\tilde{\phi}(\varepsilon,x) \subseteq B_{\varepsilon C}(x)$ for all $x \in \overline{B}_\delta(K)$ and $\varepsilon \le \varepsilon_0$. Choose any $C' > C$ and suppose $\alpha = e_{i_1} + \ldots + e_{i_k}$ with $k = |\alpha|$, then $\partial_x^\alpha\tilde{\phi}(\varepsilon,x)$ is given by derivatives at $t_1,\ldots,t_k = 0$ of $\tilde{\phi}(\varepsilon, x + t_1e_{i_1} + \ldots + t_ke_{i_k})$; for small t_i the support of each difference quotient is in $B_{\varepsilon C}(x + t_1e_{i_1} + \ldots + t_ke_{i_k}) \cup B_{\varepsilon C}(x)$ which is a subset of $B_{\varepsilon C'}(x)$. □

For any $f \in C^\infty(\Omega \times \Omega)$ and $\tilde{\phi} \in \widetilde{\mathcal{A}}_0(\Omega)$ simple Taylor expansion gives, for any $K \subset\subset \Omega$ and $|\alpha| > 0$,

$$\sup_{x \in K} \left| \int_\Omega f(x,y)\tilde{\phi}(\varepsilon,x,y)\,\mathrm{d}y - f(x,x) \right| = O(\varepsilon)$$

and

$$\sup_{x \in K} \left| \int_\Omega f(x,y)(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,x,y)\,\mathrm{d}y \right| = O(\varepsilon).$$

We will now show the analog statement for the integral over x; the idea behind the following proof is that for $\tilde{\phi}(\varepsilon,x) = \mathrm{T}_x\mathrm{S}_\varepsilon\varphi$ as above we have the identity $\tilde{\phi}(\varepsilon,x,y) = \tilde{\phi}(\varepsilon,y,2y-x)$.

Proposition 3.18. *Let $\tilde{\phi} \in \widetilde{\mathcal{A}}_0(\Omega)$ be a local smoothing kernel. Given a function $f \in C^\infty(\Omega \times \Omega)$ such that there is $K \subset\subset \Omega$ with* $\operatorname{supp} f(.,y) \subseteq K$ *for all $y \in \Omega$ we have*

$$(i) \quad \sup_{y \in \Omega} \left| \int_\Omega f(x,y)\tilde{\phi}(\varepsilon,x,y)\,\mathrm{d}x - f(y,y) \right| = O(\varepsilon) \qquad \textit{and}$$

$$(ii) \quad \sup_{y \in \Omega} \left| \int_\Omega f(x,y)(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,x,y)\,\mathrm{d}x \right| = O(\varepsilon) \qquad \textit{for } |\alpha| > 0.$$

Proof. Without limitation of generality we can assume that there are $r > 0$ and $a \in \Omega$ such that $K \subseteq \overline{B}_r(a) \subseteq \overline{B}_{4r}(a) \subseteq \Omega$. In fact, any $K \subset\subset \Omega$ can be written as the union of finitely many compact sets contained in suitable closed balls which lie in Ω. If the result holds for each of these, it holds for K.

The integral then is over $x \in B_r(a)$. By Lemma 3.17 there exist $\varepsilon_0, C > 0$ such that $\operatorname{supp}(\partial^\beta_{x+y}\tilde{\phi})(\varepsilon,x) \subseteq B_{C\varepsilon}(x)$ for all $\beta \in \mathbb{N}_0^n$, $x \in \overline{B}_{4r}(a)$, and $\varepsilon < \varepsilon_0$. For $\varepsilon < \varepsilon_0$ and $x \in B_r(a)$ this implies $\operatorname{supp}(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,x) \subseteq B_{r+C\varepsilon}(a)$ thus we only have to consider y in this set, as for $y \notin B_{r+C\varepsilon}(a)$ the integral vanishes. We furthermore note that for $\varepsilon < \min(\varepsilon_0, (r/(4C)))$ and $y \in B_{r+C\varepsilon}(a)$ we have

$$\begin{aligned} \operatorname{supp}(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,y) &\subseteq B_{C\varepsilon}(y) = 2y - B_{C\varepsilon}(y) \subseteq 2y - B_{r+2C\varepsilon}(a) \\ &\subseteq B_{3r+4C\varepsilon}(a) \subseteq B_{4r}(a) \subseteq \Omega \end{aligned}$$

hence the above integral equals

$$\int_{B_{r+2C\varepsilon}(a)} f(x,y)(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,y,2y-x)\,\mathrm{d}x$$

and we can rewrite it as

$$\int_{B_{r+2C\varepsilon}(a)} f(x,y)\big((\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,x,y) - (\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,y,2y-x)\big)\,\mathrm{d}x$$
$$+ \int_{B_{r+2C\varepsilon}(a)} f(x,y)(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon,y,2y-x)\,\mathrm{d}x. \quad (3.3)$$

For $x \in B_{r+2C\varepsilon}(a) \subseteq B_{4r}(a)$ the Taylor formula [KM97, Theorem 5.12] gives (due to $2y - x \in B_{3r+4C\varepsilon}(a) \subseteq B_{4r}(a) \subseteq \Omega$)

$$\begin{aligned}(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, x, y) &= (\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y, 2y - x) \\ &+ \int_0^1 \mathrm{d}(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y + t(x - y), 2y - x + t(x - y)) \cdot (x - y, x - y)\,\mathrm{d}t\end{aligned}$$

where the differential d is with respect to the pair of variables (x, y). Then the first summand of (3.3) is given by (substituting $x = y + \varepsilon z$)

$$\int_{B_{r/\varepsilon+2C}(a-y)} f(y + \varepsilon z, y) \cdot \int_0^1 \mathrm{d}(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y + t\varepsilon z, y + (t - 1)\varepsilon z) \cdot (\varepsilon z, \varepsilon z)\,\mathrm{d}t\,\varepsilon^n\,\mathrm{d}z \quad (3.4)$$

By linearity of the differential the inner integrand equals

$$\varepsilon \sum_{i=1}^n (\partial^{\alpha+e_i}_{x+y}\tilde{\phi})(\varepsilon, y + t\varepsilon z, y + (t - 1)\varepsilon z)z^i$$

where $z = (z^1, \ldots, z^n)$. From the properties of local smoothing kernels we have that $\left|(\partial^{\alpha+e_i}_{x+y}\tilde{\phi})(\varepsilon, x, y)\right| = O(\varepsilon^{-n})$ uniformly for $x \in \overline{B}_{r+2C\varepsilon}(a)$ and $y \in \Omega$ and also $\operatorname{supp}(\partial^{\alpha+e_i}_{x+y}\tilde{\phi})(\varepsilon, x) \subseteq B_{C\varepsilon}(x)$ for $x \in \overline{B}_{4r}(a)$ and all $\varepsilon < \varepsilon_0$. In (3.4) we only need to integrate over those z such that $y + (t - 1)\varepsilon z \in B_{C\varepsilon}(y + t\varepsilon z)$, i.e., $|y + (t - 1)\varepsilon z - y - t\varepsilon z| = |\varepsilon z| < C\varepsilon$ which is implied by $|z| < C$, so this expression is given by

$$\varepsilon \cdot \int_{B_{r/\varepsilon+2C}(a-y)\cap B_C(0)} f(y + \varepsilon z, y) \cdot \int_0^1 \sum_{i=1}^n (\partial^{\alpha+e_i}_{x+y}\tilde{\phi})(\varepsilon, y + t\varepsilon z, y + (t - 1)\varepsilon z)z^i\,\mathrm{d}t\,\varepsilon^n\,\mathrm{d}z$$

which can be estimated by $O(\varepsilon)$ uniformly for $y \in B_{4r}(a)$ and thus for $y \in \Omega$. It remains to examine the second term of (3.3) for $y \in B_{r+C\varepsilon}(a)$. With Taylor expansion in the first slot of f this is

$$\begin{aligned}&f(y, y) \cdot \int_{B_{r+2C\varepsilon}(a)} (\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y, 2y - x)\,\mathrm{d}x \\ &+ \int_{B_{r+2C\varepsilon}(a)} \int_0^1 (\mathrm{d}_1 f)(y + t(x - y), y) \cdot (x - y)\,\mathrm{d}t\,(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y, 2y - x)\,\mathrm{d}x.\end{aligned}$$

Substituting $2y - x = z$, because

$$\begin{aligned}\operatorname{supp}\tilde{\phi}(\varepsilon, y) \subseteq B_{C\varepsilon}(y) = 2y - B_{C\varepsilon}(y) &\subseteq 2y - B_{r+2C\varepsilon}(a) \\ &\subseteq B_{3r+4C\varepsilon}(a) \subseteq B_{4r}(a)\end{aligned}$$

for $y \in B_{r+C\varepsilon}(a)$ and ε small as above the first integral is given by

$$\int_\Omega (\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y, z)\,\mathrm{d}z$$

which is 1 for $\alpha = 0$ and 0 for $|\alpha| > 0$. In the second integral we substitute $x = y + \varepsilon z$ and obtain

$$\varepsilon \int_{B_{r/\varepsilon+2C}(a-y)} \int_0^1 (\mathrm{d}_1 f)(y + t\varepsilon z, y)z\,\mathrm{d}t\,(\partial^\alpha_{x+y}\tilde{\phi})(\varepsilon, y, y - \varepsilon z)\varepsilon^n\,\mathrm{d}z.$$

By the support property of smoothing kernels (Definition 3.13 (i)) we only have to integrate over a bounded set and the integrand is uniformly bounded on all x and y in question, so this integral is $O(\varepsilon)$. □

Corollary 3.19. *From Proposition 3.18 we obtain the following.*

(i) For any $f \in C^\infty(\Omega \times \Omega)$, $K \subset\subset \Omega$, and $\alpha \in \mathbb{N}_0^n$ we have

$$\sup_{y\in K}\left| \int f(x, y)(\partial^\alpha_x\tilde{\phi})(\varepsilon, x, y)\,\mathrm{d}y - (\partial^\alpha_y f)(x, x)\right| = O(\varepsilon).$$

(ii) For any $f \in C^\infty(\Omega \times \Omega)$ with $\operatorname{supp} f(.,y) \subset\subset \Omega$ we have

$$\sup_{y\in\Omega}\left| \int f(x, y)(\partial^\alpha_y\tilde{\phi})(\varepsilon, x, y)\,\mathrm{d}x - (\partial^\alpha_x f)(y, y)\right| = O(\varepsilon).$$

Proof. We perform induction on $|\alpha|$. The case $\alpha = 0$ was mentioned after Definition 3.13 (resp. handled in Proposition (3.18) for (ii)). For $|\alpha| > 0$ we have (by induction or combinatorically) the identity

$$\partial^\alpha_x = \partial^\alpha_{x+y} - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta_y \partial^{\alpha-\beta}_x$$

where $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$. Using partial integration the integral

$$\int f(x, y)(\partial^\alpha_x\tilde{\phi})(\varepsilon, x, y)\,\mathrm{d}y$$

is given by

$$O(\varepsilon) - \sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta} (-1)^\beta \int (\partial^\beta_y f)(x, y)(\partial^{\alpha-\beta}_x\tilde{\phi})(\varepsilon, x, y)\,\mathrm{d}y.$$

From the result for $|\alpha| - 1$ we see that the integral converges to $(\partial^\alpha_y f)(x, x)$ uniformly and of order $O(\varepsilon)$, so (i) follows because $\sum_{0<\beta\leq\alpha} \binom{\alpha}{\beta}(-1)^{|\beta|} = -1$. (ii) is done in the same way. □

Remark 3.20. In a certain sense smoothing kernels are asymmetric. Let us consider again the most simple smoothing kernel, given by $\tilde{\phi}(\varepsilon, x)(y) := \mathrm{T}_x \mathrm{S}_\varepsilon \varphi$ for some test function φ with integral 1. It obviously has the properties

(i) $\int \tilde{\phi}(\varepsilon, x)(y) \,\mathrm{d}x = 1$,

(ii) $\tilde{\phi}(\varepsilon, x)(y) - \tilde{\phi}(\varepsilon, y)(2y - x) = 0$, and

(iii) $(\partial_x + \partial_y)\tilde{\phi} = 0$.

but for an arbitrary smoothing kernel these only hold asymptotically. Even more, there is a problem with vanishing moments when integrating over x: if φ has vanishing moments up to order q we have for $\tilde{\phi}$ as above

$$\int f(y)\tilde{\phi}(\varepsilon, x)(y) \,\mathrm{d}y - f(x) = O(\varepsilon^{q+1})$$

$$\int f(x)\tilde{\phi}(\varepsilon, x)(y) \,\mathrm{d}x - f(y) = O(\varepsilon^{q+1})$$

but for a general smoothing kernel $\tilde{\phi}$ we can only obtain

$$\int f(x)\tilde{\phi}(\varepsilon, x)(y) \,\mathrm{d}x - f(y) = O(\varepsilon).$$

Higher rates of convergence can be obtained through a rather circumstantial procedure of (recursively) estimating the first derivative by the function itself and the second derivative, as is referred to before [GKSV09, Proposition 9.10] and performed in [GFKS01, Lemma 16.6]. Conceivably, it could be more natural to also have $O(\varepsilon^{q+1})$ in the last integral by a modified definition of smoothing kernels. At least with the motivation here this seems to be the case, and it would remove the technicalities just referred to. We leave this as an open issue.

Chapter 4

Construction of the Algebra

In this chapter we will detail the construction of an algebra of generalized tensor fields on a Riemannian manifold. The basic idea is, as in other variants of Colombeau algebras, that generalized objects are families of their smooth counterparts indexed by some parameters which are required for regularizing the corresponding distributional objects. Our case is a direct extension of the full algebra $\hat{\mathcal{G}}(M)$ and contains it as the special case of $r = s = 0$. Scalar distributions are regularized using n-forms in $\Omega^n_c(M)$; as seen in Chapter 5, on a Riemannian manifold the Levi-Civita connection provides the means to also regularize tensor distributions, hence the indexing set for the basic space remains the same: instead of $C^\infty(\hat{\mathcal{A}}_0(M), C^\infty(M))$ from the scalar theory we simply take $C^\infty(\hat{\mathcal{A}}_0(M), \mathcal{T}^r_s(M))$. The norm induced by the Riemannian metric on the tensor bundles enables us to use the same notion of moderateness resp. negligibility as in the scalar case (cf. [GKSV02, Definitions 3.10 and 3.11]).

4.1 The basic spaces

For the remainder of this section let (M, g) be a Riemannian manifold with fixed metric g.

We first introduce the basic space and appropriate moderateness and negligibility tests.

Definition 4.1. The *basic space* for the algebra of generalized (r, s)-tensor fields on M is defined as

$$\hat{\mathcal{E}}^r_s(M) := C^\infty(\hat{\mathcal{A}}_0(M), \mathcal{T}^r_s(M)).$$

An element $R \in \hat{\mathcal{E}}^r_s(M)$ is called *moderate* if it satisfies

$$\forall K \subset\subset M\ \forall l \in \mathbb{N}_0\ \exists N \in \mathbb{N}\ \forall X_1, \dots, X_l \in \mathfrak{X}(M)$$
$$\forall\ \Phi \in \widetilde{\mathcal{A}}_0(M) : \sup_{p \in K} \left\| \mathrm{L}_{X_1} \dots \mathrm{L}_{X_l} R(\Phi(\varepsilon, p))(p) \right\|_g = O(\varepsilon^{-N}).$$

The space of moderate generalized (r,s)-tensor fields on M is denoted by $(\hat{\mathcal{E}}^r_s)_m(M)$. $R \in (\hat{\mathcal{E}}^r_s)_m(M)$ is called *negligible* if it satisfies

$$\forall K \subset\subset M\ \forall l,m \in \mathbb{N}_0\ \exists k \in \mathbb{N}\ \forall X_1,\dots,X_l \in \mathfrak{X}(M)$$
$$\forall \Phi \in \widetilde{\mathcal{A}}_k(M) : \sup_{p\in K} \left\| \mathrm{L}_{X_1}\dots\mathrm{L}_{X_l} R(\Phi(\varepsilon,p))(p)\right\|_g = O(\varepsilon^m).$$

The space of negligible generalized (r,s)-tensor fields on M is denoted by $\hat{\mathcal{N}}^r_s(M)$.

Remark 4.2. (i) The expression inside the norm has to be read as

$$\mathrm{L}_{X_1}\dots\mathrm{L}_{X_l}[p' \mapsto R(\Phi(\varepsilon,p'))(p')](p).$$

(ii) By Lemma 3.8 this Definition is independent of the metric used, which will only enter in the embedding of distributions later on.

(iii) For $r = s = 0$ this reproduces (up to the exponential law applied to the basic space) exactly the global algebra $\hat{\mathcal{G}}(M)$ and the related spaces $\hat{\mathcal{E}}(M)$, $\hat{\mathcal{E}}_m(M)$, and $\hat{\mathcal{N}}(M)$ of [GKSV02].

While in the local algebra $\mathcal{G}^d(\Omega)$ as well as the special and the elementary local algebras ([Col84, Col85]) the spaces of test objects (resp. the mollifier in case of the special algebra, to be precise) depend only on the dimension of Ω and the same test objects can be used for all $\Omega \subseteq \mathbb{R}^n$, this feature was lost during the construction of the global algebras $\hat{\mathcal{G}}(M)$ and $\hat{\mathcal{G}}^r_s(M)$ of [GKSV02, GKSV09]. There, moderateness resp. negligibility tests employ the spaces $\widetilde{\mathcal{A}}_k(M)$ of smoothing kernels. At several points where questions of localization or sheaf properties are treated one thus has to restrict or extend smoothing kernels using cutoff functions in order to relate between smoothing kernels defined on different sets (see, for example, [GKSV02, Lemma 4.2 and Theorem 4.3] and [GKSV09, Proposition 8.10]). The following Lemma will partially recover this locality of test objects, alleviating the need to use further constructions with cut-off functions.

Lemma 4.3. *In Definition 4.1 one can replace "$\forall \Phi \in \widetilde{\mathcal{A}}_0(M)$" resp. "$\forall \Phi \in \widetilde{\mathcal{A}}_k(M)$" by "$\exists U \supseteq K$ open $\forall \Phi \in \widetilde{\mathcal{A}}_0(U)$" resp. "$\exists U \supseteq K$ open $\forall \Phi \in \widetilde{\mathcal{A}}_k(U)$". Furthermore, one can instead of "$\exists U \supseteq K$ open" demand "$\forall U \supseteq K$ open".*

Proof. The nontrivial part is to show that "$\exists U \supseteq K$ open" implies "$\forall U \supseteq K$ open". Let $U, V \subseteq M$ both be open subsets of M and let $R \in \hat{\mathcal{E}}^r_s(M)$ satisfy the moderateness resp. negligibility test for all $\Phi \in \widetilde{\mathcal{A}}_k(U)$. Let $K \subset\subset U \cap V$. Given $\Psi \in \widetilde{\mathcal{A}}_k(V)$, let $0 < \delta < \operatorname{dist}(K, \partial(U \cap V))$. Choose $\theta \in \mathcal{D}(M)$ with $\operatorname{supp}\theta \subseteq B_\delta(K)$ and $\theta = 1$ on $\overline{B}_{\delta/2}(K)$. Let $\varepsilon_0 > 0$ such that $\operatorname{supp}\Psi(\varepsilon,p) \subseteq B_\delta(K)$ for all $\varepsilon < \varepsilon_0$ and $p \in \operatorname{supp}\theta$. With $\lambda \in C^\infty(\mathbb{R})$ such that $\lambda = 1$ on $(-\infty, \varepsilon_0/2]$ and $\lambda = 1$ on $[\varepsilon_0, \infty)$ define $\Phi \in \widetilde{\mathcal{A}}_k(U)$ by

$$\Phi(\varepsilon,p) := (1 - \lambda(\varepsilon)\theta(p))\Psi_0(\varepsilon,p) + \lambda(\varepsilon)\theta(p)\Psi(\varepsilon,p)$$

where $\Phi_0 \in \widetilde{\mathcal{A}}_k(U)$ is arbitrary. Then for $\varepsilon \le \varepsilon_0/2$ and $p \in B_{\delta/2}(K)$, $R(\Psi(\varepsilon, p))$ equals $R(\Phi(\varepsilon, p))$ which satisfies the respective test. □

Remark 4.4. In the preceding proof one implicitly uses the fact that an n-form with compact support in an open set $U \subseteq M$ can be regarded as an element of $\Omega^n_c(U)$ regardless of its domain of definition. We will use this without further notice, especially in the definition of restriction and the proof of Theorem 4.13 later on.

$\hat{\mathcal{E}}^r_s(M)$, $(\hat{\mathcal{E}}^r_s)_m(M)$, and $\hat{\mathcal{N}}^r_s(M)$ are $C^\infty(M)$-modules and $\hat{\mathcal{N}}^r_s(M)$ a submodule of $(\hat{\mathcal{E}}^r_s)_m(M)$, so we can form the quotient space.

Definition 4.5. The space of generalized (r, s)-tensor fields is defined as the quotient $C^\infty(M)$-module

$$\hat{\mathcal{G}}^r_s(M) := (\hat{\mathcal{E}}^r_s)_m(M)/\hat{\mathcal{N}}^r_s(M).$$

$\hat{\mathcal{G}}^r_s(M)$ then is easily verified to be a $\hat{\mathcal{G}}(M)$-module with multiplication

$$(R + \hat{\mathcal{N}}(M)) \cdot (T + \hat{\mathcal{N}}^r_s(M)) := RT + \hat{\mathcal{N}}^r_s(M),$$

where $(RT)(\omega) := R(\omega) \cdot T(\omega)$ for $R \in \hat{\mathcal{E}}(M)$ and $T \in \hat{\mathcal{E}}^r_s(M)$.

The mapping $\sigma^r_s \colon \mathcal{T}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$, $\sigma^r_s(t)(\omega) := t$ is $C^\infty(M)$-linear and has moderate image; the corresponding map into $\hat{\mathcal{G}}^r_s(M)$ evidently is injective, thus we have the following embedding.

Definition 4.6. Smooth tensor fields are embedded into $\hat{\mathcal{G}}^r_s(M)$ via the mapping

$$\begin{aligned} \sigma^r_s \colon \mathcal{T}^r_s(M) &\to \hat{\mathcal{G}}^r_s(M), \\ t &\mapsto \mathrm{cl}[\omega \mapsto t]. \end{aligned}$$

4.2 Algebraic description

We will now give isomorphic descriptions of the spaces just introduced and show that the respective isomorphisms are smooth. Before we state the theorem we specify suitable topologies on the spaces involved.

Let E, F be locally convex modules over a locally convex commutative ring R injectively containing $\mathbb{K}$ as a subring. Then E and F also are vector spaces over $\mathbb{K}$. $\mathrm{L}^b_R(E, F)$ denotes the space of all bounded R-linear mappings from E into F; this is a subset of $\mathrm{L}^b(E, F)$, the set of bounded linear mappings $E \to F$ with the topology of uniform convergence on bounded sets (denoted $\mathrm{L}(E, F)$ in [KM97, Section 5]) and we equip it with the subspace topology. Clearly the embedding $\mathrm{L}^b_R(E, F) \to \mathrm{L}^b(E, F)$ is bornological, i.e., a subset of the former space is bounded if and only if it is bounded in the latter.

The module tensor product $E \otimes_R F$ is endowed with the finest locally convex Hausdorff topology such that the canonical bilinear map $\otimes\colon E \times F \to E \otimes_R F$ is bounded (called the bornological tensor product in [KM97, Section 5.7]). A map f from $E \otimes_R F$ into any locally convex space then is bounded if and only if $f \circ \otimes$ is bounded.

For locally convex spaces E,F let an affine subspace $E_1 = G(E_0) \subseteq E$ be given by the affine image of a subspace $E_0 \subseteq E_1$, where $G\colon x \mapsto x_0 + U(x)$ is such that $x \in E_1$ and U is a linear bounded automorphism of E_0 (i.e., bounded with bounded inverse). Supplementing the remark about smooth functions defined on affine subspaces in [GFKS01, Section 4], we define $C^\infty(E_1, F)$ as the space of all maps $f\colon E_1 \to F$ such that G^*f is in $C^\infty(E_0, F)$. We endow $C^\infty(E_1, F)$ with the projective topology with respect to $G^*\colon C^\infty(E_1, F) \to C^\infty(E_0, F)$, which is easily seen to be independent of the particular choice of G. For a linear subspace E this definition agrees with the one we already use ([KM97, Definition 3.11]). In the terminology of [FK88] E_1 carries the final smooth structure with respect to G. In the following we will have the case $E = \mathcal{D}(M)$, $E_0 = \hat{\mathcal{A}}_{00}(M)$, and $E_1 = \hat{\mathcal{A}}_0(M)$ for some suitable G.

Lemma 4.7. *The following mappings are smooth:*

(i) Tensor product $\mathcal{T}^r_s(M) \times \mathcal{T}^p_q(M) \to \mathcal{T}^{r+p}_{s+q}(M)$,

(ii) multiplication $\hat{m}\colon \hat{\mathcal{E}}(M) \times \hat{\mathcal{E}}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$,

(iii) the isomorphism $\zeta\colon \mathcal{T}^r_s(M) \to (\mathcal{T}^s_r(M))^*$ *as well as its inverse and the associated mapping* $(t, u) \mapsto \zeta(t)(u)$, *and*

(iv) the embedding $\sigma^r_s\colon \mathcal{T}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$.

Proof. (i) and (iii) are clear by writing down the respective seminorms. For (ii), $\hat{m}$ is smooth if and only if $(G^*)_*\hat{m} = G^* \circ \hat{m}$ is smooth as an element of $C^\infty(\hat{\mathcal{E}}(M) \times \hat{\mathcal{E}}^r_s(M), C^\infty(\hat{\mathcal{A}}_{00}(M), \mathcal{T}^r_s(M)))$. This is a bilinear mapping from convenient vector spaces into the space $C^\infty(\hat{\mathcal{A}}_{00}(M), \mathcal{T}^r_s(M))$ so it is bounded if and only if it is separately bounded ([KM97, Theorem 5.19]). By the uniform boundedness principle [KM97, Theorem 5.26] this reduces to verifying that $\mathrm{ev}_\omega \circ (G^*)_*\hat{m}$ is separately bounded from $\hat{\mathcal{E}}(M) \times \hat{\mathcal{E}}^r_s(M)$ into $\mathcal{T}^r_s(M)$ for all $\omega \in \hat{\mathcal{A}}_{00}(M)$ (where ev_ω is point evaluation at ω) which holds by (i). For (iv), σ^r_s is smooth if and only if

$$(G^*)_*\sigma \in C^\infty(\mathcal{T}^r_s(M), C^\infty(\hat{\mathcal{A}}_{00}(M), \mathcal{T}^r_s(M))),$$

which by the exponential law is the case because $((G^*)_*\sigma)^\wedge$ is the projection on the second factor, which is smooth. $\square$

The following is called the "saturation principle" in [GKSV09, Proposition 8.8]. It states that moderateness and negligibility of a generalized tensor field can be tested for by saturating it with dual smooth tensor fields and testing the resulting generalized functions.

Theorem 4.8. *One has the following smooth isomorphisms:*

$$\begin{aligned}\hat{\mathcal{E}}^r_s(M) &\cong \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M)) \cong \hat{\mathcal{E}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)\\ (\hat{\mathcal{E}}^r_s)_m(M) &\cong \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}_m(M)) \cong \hat{\mathcal{E}}_m(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)\\ \hat{\mathcal{N}}^r_s(M) &\cong \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{N}}(M)) \cong \hat{\mathcal{N}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)\end{aligned}$$

Proof. We start with

$$\begin{aligned}\varphi\colon \hat{\mathcal{E}}^r_s(M) &\to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M)),\\ (\varphi R)(v)(\omega) &:= \zeta(R(\omega))(v)\end{aligned}$$

where ζ is the map from Lemma 4.7. As to smoothness of φ, we note that by definition $R \in \hat{\mathcal{E}}^r_s(M)$ means $G^* R \in C^\infty(\hat{\mathcal{A}}_{00}(M), \mathcal{T}^r_s(M))$, so

$$\begin{aligned}\zeta_*(G^*R) &\in C^\infty(\hat{\mathcal{A}}_{00}(M), \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), C^\infty(M)))\\ &\subseteq C^\infty(\hat{\mathcal{A}}_{00}(M), C^\infty(\mathcal{T}^s_r(M), C^\infty(M))).\end{aligned}$$

Denoting by flip the map $(x, y) \mapsto (y, x)$ we consequently obtain

$$((\zeta_*(G^*R))^\wedge \circ \mathrm{flip})^\vee \in C^\infty(\mathcal{T}^s_r(M), C^\infty(\hat{\mathcal{A}}_{00}(M), C^\infty(M)))$$

and finally

$$\varphi R = ((G^{-1})^*)_*((\zeta_*(G^*R))^\wedge \circ \mathrm{flip})^\vee \in C^\infty(\mathcal{T}^s_r(M), C^\infty(\hat{\mathcal{A}}_0(M), C^\infty(M))).$$

As φR is $C^\infty(M)$-linear φ has values in $\mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M))$ and also is smooth into that space. Similarly one sees that the inverse mapping given by

$$\varphi^{-1}\colon S \mapsto (G^{-1})^*(\zeta^{-1})_*(((G^*)_*S)^\wedge \circ \mathrm{flip})^\vee$$

is smooth. Smoothness of the map $\psi\colon \hat{\mathcal{E}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$ induced by the $C^\infty(M)$-bilinear map

$$\begin{aligned}\tilde{\psi}\colon \hat{\mathcal{E}}(M) \times \mathcal{T}^r_s(M) &\to \hat{\mathcal{E}}^r_s(M),\\ (R, t) &\mapsto R \cdot \sigma^r_s(t)\end{aligned}$$

is equivalent to boundedness of $\tilde{\psi}$, which is the composition of $\mathrm{id} \times \sigma$ and multiplication $\hat{\mathcal{E}}(M) \times \hat{\mathcal{E}}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$; Lemma 4.7 (ii) and (iv) thus give smoothness of ψ.

Now to $\theta\colon \hat{\mathcal{E}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M))$ which is induced by the mapping

$$\begin{aligned}\tilde{\theta}\colon \hat{\mathcal{E}}(M) \times \mathcal{T}^r_s(M) &\to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M))\\ \tilde{\theta}(R,t)(u) &:= R \cdot \sigma(\zeta(t)(u)).\end{aligned}$$

By the exponential law smoothness of $\tilde{\theta}$ into $C^\infty(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M))$ (and thus into $\mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M)))$ is equivalent to smoothness of the map

$$\begin{array}{ccccc}\hat{\mathcal{E}}(M) \times \mathcal{T}^r_s(M) \times \mathcal{T}^s_r(M) & \longrightarrow & \hat{\mathcal{E}}(M) \times C^\infty(M) & \longrightarrow & \hat{\mathcal{E}}(M)\\ (R,t,u) & \longrightarrow & (R, \zeta(t)(u)) & \longrightarrow & R \cdot \sigma(\zeta(t)(u))\end{array}$$

which is a composition of smooth functions by Lemma 4.7 (ii), (iii), and (iv).

Because $\mathcal{T}^r_s(M)$ is finitely generated and projective we know that θ is an isomorphism ([Bou70, Chapter II §4.2]). We can even give the inverse explicitly, first locally. Let $U \subseteq M$ be a coordinate neighborhood with bases $(b_\lambda)_\lambda$ and $(b^\lambda)_\lambda$ of $\mathcal{T}^r_s(U)$ and $\mathcal{T}^s_r(U)$ such that $\zeta(b_\lambda)(b^\mu) = \delta^\mu_\lambda$ (Kronecker delta). Then for any $\chi \in C^\infty(M)$ with support in U, $T \in \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M))$, and $u \in \mathcal{T}^s_r(M)$ we have that

$$\begin{aligned}\theta(T(\chi b^\lambda) \otimes \chi b_\lambda)(u) &= T(\chi b^\lambda) \cdot \sigma(\zeta(\chi b_\lambda)(u)) = T(\chi b^\lambda) \cdot \sigma(\chi u_\lambda)\\ &= T(\chi^2 u) = \chi^2 T(u)\end{aligned}$$

where $u|_U = u_\lambda b^\lambda$ defines the coordinates u_λ of u on U and we sum over λ. For a global inverse choose a partition of unity $(\chi_i)_i$ subordinate to a finite atlas of M, which exists by [GHV72, Chapter I §1]. Set $\tilde{\chi}_i = \chi_i / \sum_i \chi_i^2$, such that we have $\sum_i \tilde{\chi}_i^2 = 1$ on M. Then $\theta^{-1}(T) = \sum_i T(\chi_i b^\lambda) \otimes (\chi_i b_\lambda)$. A bounded set $B \subseteq \mathrm{L}_{C^\infty(M)}(\mathcal{T}^r_s(M), \hat{\mathcal{E}}(M))$ is by definition uniformly bounded on bounded sets, so $\{T(\chi_i b_\lambda) \mid T \in B\}$ is bounded, whence boundedness of $\theta^{-1}(B)$ follows because the bornology of $\hat{\mathcal{E}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)$ is generated by all sets of the form $B_1 \otimes B_2$ with $B_1 \subseteq \hat{\mathcal{E}}(M)$ and $B_2 \subseteq \mathcal{T}^r_s(M)$ both bounded, so θ^{-1} is smooth.

Furthermore, $\varphi \circ \psi = \theta$ as for $R \otimes t \in \hat{\mathcal{E}}(M) \otimes \mathcal{T}^r_s(M)$ we have

$$\begin{aligned}(\varphi \circ \psi)(R \otimes t)(u)(\omega) &= \varphi(R \cdot \sigma^r_s(t))(u)(\omega) = \zeta((R \cdot \sigma^r_s(t))(\omega))(u)\\ &= R(\omega) \cdot \zeta(t)(u) = \theta(R \otimes t)(u)(\omega).\end{aligned}$$

This implies that also $\psi = \varphi^{-1} \circ \theta$ is a smooth isomorphism.

Finally, it is verified without effort that the maps φ, ψ, and θ preserve moderateness and negligibility. Their restrictions to the corresponding spaces of moderate resp. negligible functions map into the appropriate subspaces; the latter are closed, so these restrictions also are smooth ([KM97, Lemma 3.8]). □

Now let $\mathrm{pr}_2^*(\mathrm{T}^r_s(M))$ denote the pullback bundle of $\mathrm{T}^r_s(M)$ along the map $\mathrm{pr}_2\colon \hat{\mathcal{A}}_0(M) \times M \to M$, $(\omega, x) \mapsto x$, which is given by the set of all $((\omega, x), v)$ in $(\hat{\mathcal{A}}_0(M) \times M) \times \mathrm{T}^r_s(M)$ such that v is in the fiber over x. Then we have an isomorphism of $C^\infty(M)$-modules $\hat{\mathcal{E}}^r_s(M) \cong \Gamma(\mathrm{pr}_2^*(\mathrm{T}^r_s(M)))$: to any $R \in \hat{\mathcal{E}}^r_s(M)$ corresponds the mapping

$$\begin{aligned} s\colon \hat{\mathcal{A}}_0(M) \times M &\to \mathrm{pr}_2^*(\mathrm{T}^r_s(M)) \\ (\omega, x) &\mapsto ((\omega, x), R(\omega)(x)). \end{aligned}$$

Conversely, given $s = (s_1, s_2) \in \Gamma(\mathrm{pr}_2^*(\mathrm{T}^r_s(M)))$ we define $R(\omega)(x) := s_2(\omega, x)$, which is the second coordinate of $s(\omega, x)$. These two assignments obviously are inverse to each other, $C^\infty(M)$-linear, and smooth.

As tensor products of sections and sections of the tensor product can be identified with each other ([GHV72, Chapter II §5 Proposition XIV]) we consequently obtain the isomorphism

$$\begin{aligned} \hat{\mathcal{E}}^r_s(M) \otimes_{C^\infty(M)} \hat{\mathcal{E}}^p_q(M) &\cong \Gamma(\mathrm{pr}_2^*(\mathrm{T}^r_s(M))) \otimes_{C^\infty(M)} \Gamma(\mathrm{pr}_2^*(\mathrm{T}^p_q(M))) \\ &\cong \Gamma(\mathrm{pr}_2^*(\mathrm{T}^r_s(M)) \otimes_{C^\infty(M)} \mathrm{pr}_2^*(\mathrm{T}^p_q(M))) \\ &\cong \Gamma(\mathrm{pr}_2^*(\mathrm{T}^{r+p}_{s+q}(M))) \cong \hat{\mathcal{E}}^{r+p}_{s+q}(M) \end{aligned}$$

where for $R \in \hat{\mathcal{E}}^r_s(M)$ and $S \in \hat{\mathcal{E}}^p_q(M)$ the canonical image of $R \otimes S$ in $\hat{\mathcal{E}}^{r+p}_{s+q}(M)$ is given by $(R \otimes S)(\omega) = R(\omega) \otimes S(\omega)$, i.e., $R \otimes S = \otimes_*(R \times S)$ which also is a smooth map by Lemma 4.7 (i). As the bilinear mapping

$$\begin{aligned} \hat{\mathcal{E}}^r_s(M) \times \hat{\mathcal{E}}^p_q(M) &\to \hat{\mathcal{E}}^{r+p}_{s+q}(M) \\ (R, S) &\mapsto R \otimes S \end{aligned}$$

preserves moderateness and negligibility it induces an isomorphism

$$\hat{\mathcal{G}}^r_s(M) \otimes_{\hat{\mathcal{G}}(M)} \hat{\mathcal{G}}^p_q(M) \cong \hat{\mathcal{G}}^{r+p}_{s+q}(M).$$

Proposition 4.9. *As $C^\infty(M)$-modules,*

$$\begin{aligned} \hat{\mathcal{G}}^r_s(M) &\cong \hat{\mathcal{G}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) &(4.1)\\ &\cong \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{G}}(M)) \cong \mathrm{L}_{C^\infty(M)}(\mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s; \hat{\mathcal{G}}(M)) &(4.2)\\ &\cong \mathrm{L}_{\hat{\mathcal{G}}(M)}(\hat{\mathcal{G}}^s_r(M), \hat{\mathcal{G}}(M)) \cong \mathrm{L}_{\hat{\mathcal{G}}(M)}(\hat{\mathcal{G}}^0_1(M)^r \times \hat{\mathcal{G}}^1_0(M)^s; \hat{\mathcal{G}}(M)). &(4.3) \end{aligned}$$

Proof. Considering $\hat{\mathcal{N}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \cong \hat{\mathcal{N}}^r_s(M)$ to be a submodule of $\hat{\mathcal{E}}_m(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \cong (\hat{\mathcal{E}}^r_s)_m(M)$ we can form the quotient $C^\infty(M)$-module, which is isomorphic to $\hat{\mathcal{G}}(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)$ via $\mathrm{cl}[x \otimes t] \mapsto \mathrm{cl}[x] \otimes t$, which gives (4.1). (4.2) follows from [Bou70, Chapter II §4.2] because $\mathcal{T}^r_s(M)$ is projective and finitely generated ([GHV72, Chapter II §5 Lemma II]); the second part of (4.2) and (4.3) follow from [Bou70, Chapter II §3.9 (36)], using

$\mathcal{T}_1^0(M)^{\otimes r} \otimes_{C^\infty(M)} \mathcal{T}_0^1(M)^{\otimes s} \cong \mathcal{T}_r^s(M)$ and $\hat{\mathcal{G}}_1^0(M)^{\otimes r} \otimes_{\hat{\mathcal{G}}(M)} \hat{\mathcal{G}}_0^1(M)^{\otimes s} \cong \hat{\mathcal{G}}_r^s(M)$, respectively. Finally, by [Bou70, Chapter II §2.3 Proposition 5 and §4.2 Proposition 1 (b)] (4.3) follows from

$$\begin{aligned} \mathrm{L}_{C^\infty(M)}(\mathcal{T}_r^s(M), \hat{\mathcal{G}}(M)) &\cong \mathrm{L}_{C^\infty(M)}(\mathcal{T}_r^s(M), \mathrm{L}_{\hat{\mathcal{G}}(M)}(\hat{\mathcal{G}}(M), \hat{\mathcal{G}}(M))) \\ &\cong \mathrm{L}_{\hat{\mathcal{G}}(M)}(\mathcal{T}_r^s(M) \otimes_{C^\infty(M)} \hat{\mathcal{G}}(M), \hat{\mathcal{G}}(M)) \\ &\cong \mathrm{L}_{\hat{\mathcal{G}}(M)}(\hat{\mathcal{G}}_r^s M, \hat{\mathcal{G}}(M)). \end{aligned}$$

□

As negligibility of elements of $\hat{\mathcal{E}}_m(M)$ can be tested without resorting to derivatives ([GKSV02, Corollary 4.5]) this result carries over to the present setting at once.

Corollary 4.10. *For an element $R \in (\hat{\mathcal{E}}_s^r)_m(M)$ to be negligible is suffices to have the respective test of Definition 4.1 be satisfied for $l = 0$.*

Now we will examine coordinates in $\hat{\mathcal{G}}_s^r(M)$. Let the open set $U \subseteq M$ be such that $\mathcal{T}_s^r(U)$ has a basis $(b_\lambda)_\lambda$ with dual basis $(b^\lambda)_\lambda$ of $\mathcal{T}_r^s(U)$ where λ runs through some index set. From

$$\begin{aligned} \varphi(\varphi(R)(b^\lambda) \cdot \sigma(b_\lambda))(u)(\omega) &= \zeta(\varphi(R)(b^\lambda)(\omega) \cdot b_\lambda)(u_\mu b^\mu) \\ &= \varphi(R)(b^\lambda)(\omega) u_\mu \zeta(b_\lambda)(b^\mu) = \varphi(R)(u)(\omega) \end{aligned}$$

we see that $R = \varphi(R)(b^\lambda) \cdot \sigma(b_\lambda)$, i.e., the $\sigma(b_\lambda)$ form a basis of $\hat{\mathcal{E}}_s^r(U)$ resp. of $\hat{\mathcal{G}}_s^r(U)$. It follows that we can define the coordinates of $\hat{R} = \mathrm{cl}[R] \in \hat{\mathcal{G}}_s^r(M)$ on U as $\hat{R}^\lambda := \mathrm{cl}[\varphi(R)(b^\lambda)]$.

4.3 Localization and sheaf properties

Assigning to each open subset $U \subseteq M$ the $\hat{\mathcal{G}}(U)$-module $\hat{\mathcal{G}}_s^r(U)$ of generalized tensor fields on the submanifold U we obtain a presheaf $\hat{\mathcal{G}}_s^r$ of $\hat{\mathcal{G}}$-modules (note that $\hat{\mathcal{G}}$ is a sheaf by [GKSV02, Theorem 4.8]). The corresponding restriction mapping which will turn it into a sheaf is given as follows.

Definition 4.11. For any open subset $U \subseteq M$ we define the restriction of $R \in \hat{\mathcal{E}}_s^r(M)$ to U as the element of $\hat{\mathcal{E}}_s^r(U)$ given by the map

$$\begin{aligned} R|_U \colon \hat{\mathcal{A}}_0(U) &\to \mathcal{T}_s^r(U) \\ \omega &\mapsto R(\omega)|_U. \end{aligned}$$

The next proposition establishes essential localization properties.

Proposition 4.12. *Let $R \in \hat{\mathcal{E}}_s^r(M)$.*

(i) Given an open subset $U \subseteq M$, $R|_U$ is moderate resp. negligible if R is.

(ii) Let $(U_\lambda)_\lambda$ be an open covering of M. If each $R|_{U_\lambda}$ is moderate resp. negligible then so is R.

Proof. (i) is immediate from Lemma 4.3. (ii) Given $K \subset\subset M$ for testing we can write $K = \bigcup_i K_{\lambda_i}$ with finitely many $K_{\lambda_i} \subset\subset U_{\lambda_i}$, thus we can assume that K is contained in U_λ for some fixed λ and the result also follows directly from Lemma 4.3. □

Restriction is compatible with the module structure: for open sets U and V in M with $U \subseteq V$, $R \in \hat{\mathcal{E}}(M)$, and $T \in \hat{\mathcal{E}}^r_s(V)$ we have

$$\begin{aligned}(RT)|_U(\omega) &= (RT)(\omega)|_U = (R(\omega) \cdot T(\omega))|_U \\ &= R(\omega)|_U \cdot T(\omega)|_U = R|_U(\omega) \cdot T|_U(\omega).\end{aligned}$$

The analogue for the product $\hat{\mathcal{G}}(V) \times \hat{\mathcal{G}}^r_s(V) \to \hat{\mathcal{G}}^r_s(V)$ also holds.

Theorem 4.13. *$\hat{\mathcal{G}}^r_s$ is a fine sheaf of $\hat{\mathcal{G}}$-modules.*

Proof. Let an open subset $U \subseteq M$ be given and fix an open cover $\{U_\lambda\}_\lambda$ of U. First, note that for any open subsets U, V of M with $U \subseteq V$ and $T \in \hat{\mathcal{E}}^r_s(M)$ we have $(T|_V)|_U = T|_U$.

Second, we note that Proposition 4.12 already gives one property required from a sheaf: given $\hat{S} = \mathrm{cl}[S]$ and $\hat{T} = \mathrm{cl}[T]$ in $\hat{\mathcal{G}}^r_s(M)$, $\hat{S}|_{U_\lambda} = \hat{T}|_{U_\lambda}$ means $S|_{U_\lambda} - T|_{U_\lambda} = (S - T)|_{U_\lambda} \in \hat{\mathcal{N}}^r_s(M)$. If this holds for all λ then $S - T$ is negligible and $\hat{S}$ equals $\hat{T}$ in $\hat{\mathcal{G}}^r_s(M)$.

Third, we will show how to glue together global objects from local ones. Suppose that for each λ we are given an element of $\hat{\mathcal{G}}^r_s(U_\lambda)$ represented by $T_\lambda \in (\hat{\mathcal{E}}^r_s)_m(U_\lambda)$ such that $(T_\lambda - T_\mu)|_{U_\lambda \cap U_\mu}$ is negligible for all λ, μ.

Choose a locally finite open covering $\{W_j\}_{j \in \mathbb{N}}$ of U such that each W_j is relatively compact in $U_{\lambda(j)}$ for some $\lambda(j)$. This may be done in the following way: as M is locally compact ([BC70, Proposition 3.3.2]) each point $p \in M$ has an open neighborhood U_p which is relatively compact in some U_λ ([Eng89, Theorem 3.3.2]). The U_p clearly cover U which inherits the property of being second countable and thus Lindelöf from M, thus $\{U_p\}_p$ has a countable subcover. Being Lindelöf U is paracompact ([Eng89, Theorem 5.1.2]), thus this subcover has a locally finite open refinement $\{W_j\}_{j \in \mathbb{N}}$ satisfying our requirement.

Let $\{\chi_j\}_j$ be a smooth partition of unity on U subordinate to $\{W_j\}_j$ as in [Spi99, Chapter 2 Theorem 15], i.e., the χ_j are smooth positive functions on U with $\operatorname{supp} \chi_j \subseteq W_j$.

Choose bump functions $\theta_j \in \mathcal{D}(U_{\lambda(j)})$ which are 1 on each $\overline{W_j}$, respectively. Fixing some arbitrary $\omega_j \in \hat{\mathcal{A}}_0(U_{\lambda(j)})$ for each j, we define the mappings $\pi_j \in C^\infty(\hat{\mathcal{A}}_0(U), \hat{\mathcal{A}}_0(U_{\lambda(j)}))$ by

$$\pi_j(\omega) := \theta_j\omega - (\int_{U_{\lambda(j)}} \theta_j\omega - 1)\omega_j \in \hat{\mathcal{A}}_0(U_{\lambda(j)}) \qquad \forall \omega \in \hat{\mathcal{A}}_0(U).$$

Note that $\operatorname{supp}\omega \subseteq \overline{W_j}$ implies $\pi_j(\omega) = \omega$.

We will now glue together the T_λ to a mapping $T_j \in \hat{\mathcal{E}}^r_s(U)$ by defining $T(\omega) := \sum_{j\in\mathbb{N}} \chi_j \cdot T_{\lambda(j)}(\pi_j(\omega))$. Because the family $\{W_j\}_{j\in\mathbb{N}}$ and thus $\{\operatorname{supp}\chi_j\}_{j\in\mathbb{N}}$ is locally finite this sum is well-defined and smooth.

For testing moderateness of T we have to form Lie derivatives of the mapping $p \mapsto T(\Phi(\varepsilon, p))(p)$ on a compact set $K \subset\subset U$, where Φ is in $\widetilde{\mathcal{A}}_0(U)$. Because the family $\{W_j\}_j$ is locally finite K has an open neighborhood intersecting only finitely many sets W_j. It therefore suffices to establish moderateness of each summand of T individually, which amounts to estimating the modulus of

$$\mathrm{L}_{X_1} \dots \mathrm{L}_{X_l}[p \mapsto \chi_j(p)T_{\lambda(j)}(\pi_j(\Phi(\varepsilon, p)))(p)] \tag{4.4}$$

on $K \cap \operatorname{supp}\chi_j$, where $X_1, \dots, X_l$ are vector fields on M.

Choose an open neighborhood L of $K \cap \operatorname{supp}\chi_j$ which is relatively compact in W_j and a bump function $\theta \in \mathcal{D}(U_{\lambda(j)})$ which is 1 on $\overline{L}$ and has support in W_j. Then there exists $\varepsilon_0 > 0$ such that $\operatorname{supp}\Phi(\varepsilon, p) \subseteq W_j$ and thus $\pi_j(\Phi(\varepsilon, p)) = \Phi(\varepsilon, p)$ for all $\varepsilon < \varepsilon_0$ and $p \in \operatorname{supp}\theta$. Moderateness of (4.4) now follows directly from moderateness of $T_{\lambda(j)}$ using Lemma 4.3.

Fourth, we establish $\mathrm{cl}[T]|_{U_\lambda} = \mathrm{cl}[T_\lambda]$. By the second point above we only have to show $\mathrm{cl}[T]|_{U_\lambda\cap W_k} = \mathrm{cl}[T_\lambda]|_{U_\lambda\cap W_k}$ for all k. Because $T_\lambda|_{U_\lambda\cap U_{\lambda(k)}} - T_{\lambda(k)}|_{U_\lambda\cap U_{\lambda(k)}}$ is negligible and $W_k \subseteq U_{\lambda(k)}$ it suffices to show negligibility of $T|_{U_\lambda\cap W_k} - T_{\lambda(k)}|_{U_\lambda\cap W_k}$, which is given at $\omega \in \hat{\mathcal{A}}_0(U_\lambda \cap W_k)$ by

$$\sum_{j\in F} \chi_j \left(T_{\lambda(j)}(\pi_j(\omega)) - T_{\lambda(k)}(\omega)\right) \in \mathcal{T}^r_s(U_\lambda \cap W_k)$$

where the set $F := \{j \in \mathbb{N} : \operatorname{supp}\chi_j \cap U_\lambda \cap W_k \neq \emptyset\}$ is finite because $U_\lambda \cap W_k$ is relatively compact. We will show negligibility for a single summand. Fix $K \subset\subset U_\lambda \cap W_k$ for testing and let $\Phi \in \widetilde{\mathcal{A}}_q(U_\lambda \cap W_k)$ for some $q \in \mathbb{N}$. There is $\varepsilon > 0$ such that $\operatorname{supp}\Phi(\varepsilon, p) \subseteq W_j$ for all $p \in K \cap \operatorname{supp}\chi_j$ and $\varepsilon < \varepsilon_0$, so $\pi_j(\Phi(\varepsilon, p)) = \Phi(\varepsilon, p)$ and the summand at such p is given by (we drop $\chi_j(p)$ from now on as it does not influence negligibility)

$$T_{\lambda(j)}(\Phi(\varepsilon, p))(p) - T_{\lambda(k)}(\Phi(\varepsilon, p))(p).$$

Using Lemma 4.3, negligibility of this expression immediately follows from negligibility of $T_{\lambda(j)} - T_{\lambda(k)}$

For $\hat{\mathcal{G}}^r_s$ to be a fine sheaf we need – supposing that $\{U_\lambda\}_\lambda$ is locally finite – a family of sheaf morphisms $\eta_\lambda \colon \hat{\mathcal{G}}^r_s \to \hat{\mathcal{G}}^r_s$ such that $\sum_\lambda \eta_\lambda = \mathrm{id}$ and that η_λ vanishes at $(\hat{\mathcal{G}}^r_s)_p$ (the stalk of $\hat{\mathcal{G}}^r_s$ at p) for all points p in a neighborhood of $U \setminus U_\lambda$. The needed sheaf morphisms are easily verified to be given on open subsets $V \subseteq U$ by

$$\eta_\mu|_V \colon \hat{\mathcal{G}}^r_s(V) \to \hat{\mathcal{G}}^r_s(V),$$
$$\eta_\mu R = \sum_{\{j|\lambda(j)=\mu\}} \chi_j \cdot (R|_{V\cap W_j} \circ \pi_j|_{\hat{\mathcal{A}}_0(V)}).$$

□

Chapter 5

Embedding of distributional tensor fields

Embedding distributional tensor fields amounts to a regularization procedure which we will first illustrate with a locally integrable tensor field. Unlike the scalar case we cannot simply multiply by an n-form with integral 1 and support around p and integrate – the values of the tensor field in different fibers first have to be related by a connection on the tangent bundle. On a Riemannian manifold there is a natural way to do this: locally (in convex neighborhoods) any two points are connected by a unique minimizing geodesic along which we can parallel transport tensor fields by means of the Levi-Civita connection.

In order to formalize this concept we employ the following definitions. For any two vector bundles $E \to M$ and $F \to N$ we define the vector bundle

$$\mathrm{TO}(E,F) := \bigcup_{(p,q)\in M\times N} \{(p,q)\} \times \mathrm{L}(E_p, F_q).$$

The fiber over (p,q) consists of the space of linear maps from E_p to F_q. A section of $\mathrm{TO}(E,F)$, called *transport operator*, is locally given by a smoothly parametrized matrix. The Lie derivative $\mathrm{L}_{X\times Y}A \in \Gamma(\mathrm{TO}(\mathrm{T}M,\mathrm{T}M))$ of a transport operator $A \in \Gamma(\mathrm{TO}(\mathrm{T}M,\mathrm{T}M))$ along a given pair of vector fields $X, Y \in \mathfrak{X}(M)$ is defined via the flow by

$$(\mathrm{L}_{X\times Y}A)(p,q) := \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} ((\mathrm{Fl}^X_\tau, \mathrm{Fl}^Y_\tau)^* A)(p,q) \tag{5.1}$$

which in turn rests on the pullback of A along a pair of diffeomorphisms (μ,ν), given by

$$((\mu,\nu)^*A)(p,q) := (\mathrm{T}_q\nu)^{-1} \cdot A(\mu(p),\nu(q)) \cdot \mathrm{T}_p\mu.$$

We abbreviate $\mathrm{L}_{X\times X}A$ by $\mathrm{L}_X A$. See [GKSV09, Appendix A] for further details about transport operators.

Following [Kli95, Definition 1.9.9] we call an open subset $U \subseteq M$ of a Riemannian manifold (M,g) *convex* if any two points p,q of U can be joined by a (not necessarily unique) geodesic of length $d(p,q)$ which lies entirely in U. We call U *strongly convex* if any two points $p,q \in U$ can be joined by a unique geodesic of length $d(p,q)$ which belongs entirely to U and if every ε-ball $B^g_\varepsilon(p) \subseteq U$ is convex. The *convexity radius*

$$c(p) := \sup\{r \in \mathbb{R} \cup \{\infty\} \mid B^g_r(p) \text{ is strongly convex}\} \tag{5.2}$$

then is a positive continuous function on M ([Kli95, Corollary 1.9.11]).

A transport operator on $V := \{(p,q) \in M \times M \mid d(p,q) < r(p)\}$ can be defined as follows: for $(p,q) \in V$ let $\sigma_{p,q}(t)\colon [0,1] \to M$ be the unique minimal geodesic from p to q. Denote by $\mathrm{P}_{\sigma_{p,q}}$ parallel transport along $\sigma_{p,q}$ with respect to the Levi-Civita connection. Then $\tilde{A}(p,q)\colon \mathrm{T}_pM \ni v_p \mapsto \mathrm{P}_{\alpha_{p,q}} v_p \in \mathrm{T}_qM$ (for $(p,q) \in V$) defines a transport operator $\tilde{A} \in \Gamma(V, \mathrm{TO}(\mathrm{T}M, \mathrm{T}M))$ which is smooth by standard results of ODE theory. For practical purposes we extend $\widetilde{A}$ to a global section: choose continuous functions r_1, r_2 on M such that $0 < r_1 < r_2 < c$ and a smooth cut-off function $\chi \in C^\infty(M \times M, \mathbb{R})$ satisfying $\chi(p,q) = 0$ for $d(p,q) \geq r_2(p)$ and $\chi(p,q) = 1$ for $d(p,q) \leq r_1(p)$. Then $A := \chi\tilde{A}$ is a global section of $\mathrm{TO}(\mathrm{T}M, \mathrm{T}M)$ which in the usual way extends to the tensor bundle of M, giving rise to a transport operator $A^r_s \in \Gamma(\mathrm{TO}(\mathrm{T}^r_s(M), \mathrm{T}^r_s(M)))$ for all (r,s). We call A resp. A^r_s the *canonical transport operator* obtained from the metric g.

Using the canonical transport operator we can approximate a locally integrable (r,s)-tensor field t at $p \in M$ by $t(p) \sim \int A^r_s(q,p)t(q)\omega(q)\,\mathrm{d}q$, where $\omega \in \hat{\mathcal{A}}_0(M)$ has support in a small ball around p. In order to get a distributional formula which we can use for the embedding we examine the action of t on a dual tensor field u:

$$\begin{aligned} t(p) \cdot u(p) &\sim \int (A^r_s(q,p)t(q) \cdot u(p))\omega(q)\,\mathrm{d}q \\ &= \int (t(q) \cdot A^s_r(p,q)u(p))\omega(q)\,\mathrm{d}q \\ &= \langle t(q), A^s_r(p,q)u(p) \otimes \omega(q)\rangle. \end{aligned}$$

The above considerations lead to the following definition of an embedding of $\mathcal{D}'^r_s(M)$ into $\hat{\mathcal{G}}^r_s(M)$.

Definition 5.1. The embedding $\iota^r_s\colon \mathcal{D}'^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$ is defined as

$$((\iota^r_s t)(\omega) \cdot v)(p) := \langle t, A^s_r(p,\cdot)v(p) \otimes \omega\rangle$$

where $t \in \mathcal{D}'^r_s(M)$, $\omega \in \hat{\mathcal{A}}_0(M)$, $v \in \mathcal{T}^s_r(M)$, and $p \in M$.

Remark 5.2. The (non-trivial) proof that $\iota^r_s(t)$ is smooth is to a large extent identical to the corresponding result in [GKSV09, Section 7], the necessary modifications being straightforward (we simply have one slot less to deal with).

We will now show that the embedding ι^r_s has the properties required for an embedding of distributions into Colombeau algebras, namely it has moderate values, for smooth tensor fields it reproduces σ^r_s, and it is injective.

Proposition 5.3. *The embeddings have the following properties.*

(i) $\iota_s^r(\mathcal{D}'^r_s(M)) \subseteq (\hat{\mathcal{E}}^r_s)_m(M)$.

(ii) $(\iota_s^r - \sigma_s^r)(\mathcal{T}^r_s(M)) \subseteq \hat{\mathcal{N}}^r_s(M)$.

(iii) For $v \in \mathcal{D}'^r_s(M)$, $\iota_s^r(v) \in \hat{\mathcal{N}}^r_s(M)$ *implies* $v = 0$.

Proof. (i) For testing we fix $K \subset\subset M$ and $l \in \mathbb{N}_0$. For any vector fields $X_1, \ldots, X_l \in \mathfrak{X}(M)$ and $\Phi \in \widetilde{\mathcal{A}}_0(M)$ by Theorem 4.8 (saturation) we need to calculate $\mathrm{L}_{X_1} \ldots \mathrm{L}_{X_l}(p \mapsto \langle t, A^s_r(p,\cdot)u(p) \otimes \Phi(\varepsilon, p)\rangle)$ on K for arbitrary $u \in \mathcal{T}^r_s(M)$. By the chain rule (for a detailed argument on why t commutes with the Lie derivative see the proof of [GKSV09, Proposition 6.8]) this is given by terms of the form

$$\langle t, v(p,\cdot) \otimes \mathrm{L}'_{Y_1} \ldots \mathrm{L}'_{Y_k} \Phi(\varepsilon, p)\rangle \tag{5.3}$$

for some $Y_i \in \mathfrak{X}(M)$ $(i = 1 \ldots k \in \mathbb{N})$ and $v \in \Gamma(\mathrm{pr}_2^*(\mathrm{T}^s_r(M)))$; the latter consists of Lie derivatives of u transported by Lie derivatives of A. By the definition of smoothing kernels, for ε small enough and p in a relatively compact neighborhood of K the support of $\Phi(\varepsilon, p)$ for $p \in K$ lies in a (bigger) relatively compact neighborhood L of K. Because t is continuous and linear and $\mathcal{T}^s_r(M) \otimes \Omega^n_c(M)$ carries the usual inductive limit topology (as in [GKSV09, Section 2]), the modulus of (5.3) can be estimated by a finite sum of seminorms of $\Gamma_{c,L}(\mathrm{T}^s_r(M) \otimes \Lambda^n \mathrm{T}^* M)$ applied to the argument of t in (5.3). These seminorms are given by $s \mapsto \sup_{x \in L} \left\| \mathrm{L}_{Z_1} \ldots \mathrm{L}_{Z_p} s(x) \right\|$ for some vector fields $Z_j \in \mathfrak{X}(M)$, $j = 1, \ldots, p \in \mathbb{N}$ (the norm is with respect to any Riemannian metric on M). It thus remains to estimate $\left\| \mathrm{L}_{Z_1} \ldots \mathrm{L}_{Z_p}(v(p,\cdot) \otimes \mathrm{L}'_{Y_1} \ldots \mathrm{L}'_{Y_k} \Phi(\varepsilon, p)) \right\|$. This in turn reduces to an estimate of L- and L′-derivatives of Φ, which immediately gives the desired moderateness estimate by definition of the space of smoothing kernels.

(ii) In order to show the claim we have to verify (using Theorem 4.8) that for arbitrary $u \in \mathcal{T}^s_r(M)$, $K \subset\subset M$ and $m \in \mathbb{N}_0$ there is some $k \in \mathbb{N}$ such that for all $\Phi \in \widetilde{\mathcal{A}}_k(M)$ we have the estimate

$$\sup_{p \in K} \left| \int_M (t \cdot (A^s_r(p,\cdot)u(p)))(q) \Phi(\varepsilon, p)(q) \,\mathrm{d}q - (t \cdot u)(p) \right| = O(\varepsilon^m). \tag{5.4}$$

Without loss of generality we may assume that K is contained in the domain of a chart (U, φ), by Lemma 4.3 we can then assume $\Phi \in \widetilde{\mathcal{A}}_k(U)$. Defining $f \in C^\infty(U \times U)$ by $f(p,q) := t(q) \cdot A^s_r(p,q)u(p)$ we can write (5.4) as $\sup_{p \in K} \left| \int_U \big(f(p,q) - f(p,p)\big) \Phi(\varepsilon, p)(q) \,\mathrm{d}q \right|$. Setting $\tilde{f} := f \circ (\varphi^{-1} \times \varphi^{-1})$ and $x := \varphi(p)$ the integral is given by

$$\int_{\varphi(U)} (\tilde{f}(x,y) - \tilde{f}(x,x)) \tilde{\phi}(\varepsilon, x)(y) \,\mathrm{d}y$$

where $\tilde{\phi} \in \widetilde{\mathcal{A}}_k(\varphi(U))$ is the local expression of Φ. By the remark after Definition 3.13 this is $O(\varepsilon^{k+1})$ uniformly for $x \in \varphi(K)$, so for $k+1 \geq m$ the required estimates are satisfied.

(iii) is shown in Corollary 5.5 below. □

Although we will not treat association in full detail, the following is a first step in this direction (cf. [GKSV09, Section 9] for the type of results that can be obtained). Let

$$\rho\colon \mathcal{T}^r_s(M) \to \mathcal{D}'^r_s(M)$$
$$\rho(t)(u \otimes \omega) := \int (t \cdot u)\, \omega$$

be the embedding of $\mathcal{T}^r_s(M)$ into $\mathcal{D}'^r_s(M)$. Given a tensor distribution $T \in \mathcal{D}'^r_s(M)$ and a smoothing kernel $\Phi \in \widetilde{\mathcal{A}}_0(M)$ we set

$$T_\varepsilon := [p \mapsto (\iota^r_s T)(\Phi(\varepsilon, p))(p)] \in \mathcal{T}^r_s(M).$$

T_ε can be seen as a regularization of T which gets more accurate for smaller ε. More precisely, we will now show that $\rho(T_\varepsilon)$ converges to T weakly in $\mathcal{D}'^r_s(M)$ for $\varepsilon \to 0$.

Fix $u \otimes \omega \in \mathcal{T}^s_r(M) \otimes_{C^\infty(M)} \Omega^n_c(M)$. We may assume that ω (and thus u) has support in a fixed compact set K contained in a chart (U, φ): using partitions of unity we can write $u \otimes \omega = \sum_i \chi_i u \otimes \chi_i \omega$ with $\operatorname{supp} \chi_i \subseteq U_i$. Then

$$\langle \rho(T_\varepsilon) - T, u \otimes \omega \rangle = \sum_i \langle \rho(T_\varepsilon) - T, \chi_i u \otimes \chi_i \omega \rangle$$

converges to 0 if the result holds for the case where K is contained in a chart (U, φ).

We abbreviate $\tilde{u}^{j_1 \dots j_s}_{i_1 \dots i_r}(p, q) := (A^s_r(p, q) u(p))^{j_1 \dots j_s}_{i_1 \dots i_r}$ and note that $u^{j_1 \dots j_s}_{i_1 \dots i_r}(p) = \tilde{u}^{j_1 \dots j_s}_{i_1 \dots i_r}(p, p)$. Given any neighborhood L of K which is relatively compact in U there is as in the proof of Lemma 4.3 some $\varepsilon_0 > 0$ and a smoothing kernel $\Phi_1 \in \widetilde{\mathcal{A}}_0(U)$ such that for all $p \in L$ and $\varepsilon < \varepsilon_0$ the support of $\Phi(\varepsilon, p)$ is contained in U and $\Phi(\varepsilon, p)|_U = \Phi_1(\varepsilon, p)$. Let Φ_1 have local expression $\tilde{\phi} := \hat{\lambda}_*(\varphi_* \Phi_1)$. Let $\psi \in \mathcal{D}(\varphi(U))$ be determined by $\varphi_* \omega = \psi \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$, i.e., $\psi = \lambda(\psi_* \omega)$. Then for $\varepsilon < \varepsilon_0$ (denoting the local expressions of $T^{i_1 \dots i_r}_{j_1 \dots j_s}$

and $\tilde{u}^{i_1\dots i_s}_{j_1\dots j_r}$ by the same letter)

$$
\begin{aligned}
\langle \rho(T_\varepsilon), u \otimes \omega\rangle &= \int_M \langle T(q), A^s_r(p,q)u(p) \otimes \Phi(\varepsilon,p)(q)\rangle\, \omega(p) \\
&= \int_M \langle T^{i_1\dots i_r}_{j_1\dots j_s}(q), (A^s_r(p,q)u(p))^{j_1\dots j_s}_{i_1\dots i_r} \cdot \Phi_1(\varepsilon,p)(q)\rangle\, \omega(p) \\
&= \int_{\varphi(U)} \langle T^{i_1\dots i_r}_{j_1\dots j_s}(y), \tilde{u}^{j_1\dots j_s}_{i_1\dots i_r}(x,y) \cdot \tilde{\phi}(\varepsilon,x)(y)\rangle \psi(x)\,\mathrm{d}^n x \\
&= \int_{\varphi(U)} \langle T^{i_1\dots i_r}_{j_1\dots j_s}(y), \tilde{u}^{j_1\dots j_s}_{i_1\dots i_r}(x,y) \cdot \psi(x) \cdot \tilde{\phi}(\varepsilon,x)(y)\,\mathrm{d}^n x \\
&= \langle T^{i_1\dots i_r}_{j_1\dots j_s}(y), \int_{\varphi(U)} \tilde{u}^{j_1\dots j_s}_{i_1\dots i_r}(x,y) \cdot \psi(x) \cdot \tilde{\phi}(\varepsilon,x)(y)\,\mathrm{d}^n x\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\langle T, u \otimes \omega\rangle &= \langle T^{i_1\dots i_r}_{j_1\dots j_s}(p), u^{j_1\dots j_s}_{i_1\dots i_r}(p) \cdot \omega(p)\rangle \\
&= \langle T^{i_1\dots i_r}_{j_1\dots j_s}(y), u^{j_1\dots j_s}_{i_1\dots i_r}(y) \cdot \psi(y)\rangle.
\end{aligned}
$$

Integration here commutes with the distributional action, as can be seen from writing the above as the tensor product of the distribution $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ with the distribution 1. Now for each choice of $j_1,\dots,j_s,i_1,\dots,i_r$ we abbreviate $f(x,y) := \tilde{u}^{j_1\dots j_s}_{i_1\dots i_r}(x,y) \cdot \psi(x)$ and note that $f(y,y) = u^{j_1\dots j_s}_{i_1\dots i_r}(y) \cdot \psi(y)$. Because $\int_{\varphi(U)} f(x,y)\tilde{\phi}(\varepsilon,x)(y)\,\mathrm{d}x - f(y,y)$ as a function in y has support in a compact set in $\varphi(U)$, for each component of $T_\varepsilon - T$ by [Tre76, Proposition 21.1] there exist $m > 0$ and $C > 0$ such that

$$
\langle (T_\varepsilon - T)^{i_1\dots i_r}_{j_1\dots j_s}, u^{j_1\dots j_s}_{i_1\dots i_r} \cdot \omega\rangle \le \sup_{|\alpha|\le m} \sup_{y\in\varphi(U)} \left\| \partial^\alpha \Big(\int_{\varphi(U)} f(x,y)\tilde{\phi}(\varepsilon,x)(y)\,\mathrm{d}x - f(y,y)\Big) \right\|
$$

which is $O(\varepsilon)$ by proposition 3.19. Summarizing, we have shown:

Proposition 5.4. *Given $T \in \mathcal{D}'^r_s(M)$ and $\Phi \in \widetilde{\mathcal{A}}_0(M)$ the regular distribution*

$$
p \mapsto (\iota^r_s T)(\Phi(\varepsilon,p))(p)
$$

converges weakly to T in $\mathcal{T}^s_r(M)$ for $\varepsilon \to 0$.

Corollary 5.5. *For $T \in \mathcal{D}'^r_s(M)$, $\iota^r_s(T) \in \hat{\mathcal{N}}^r_s(M)$ implies $T = 0$.*

Proof. For suitable $k \in \mathbb{N}$, $u \otimes \omega \in \mathcal{T}^s_r(M) \otimes_{C^\infty(M)} \Omega^n_c(M)$, and $\Phi \in \widetilde{\mathcal{A}}_k(M)$

$$
\begin{aligned}
|\langle T, u \otimes \omega\rangle| &= \left| \lim_{\varepsilon\to 0} \langle (\iota^r_s T)(\Phi(\varepsilon,p))(p), (u \otimes \omega)(p)\rangle \right| \\
&= \left| \lim_{\varepsilon\to 0} \int_M \langle T(q), A^s_r(p,q)u(p) \otimes \Phi(\varepsilon,p)(q)\rangle\, \omega(p) \right| \\
&\le \lim_{\varepsilon\to 0} \sup_{p\in\operatorname{supp}\omega} |\langle T(q), A^s_r(p,q)u(p) \otimes \Phi(\varepsilon,p)(q)\rangle| \cdot \left| \int_M \omega(p) \right|
\end{aligned}
$$

which is $O(\varepsilon^m)$ because of negligibility of T. □

Chapter 6

Pullback and Lie derivatives

In this section we will define pullback along a diffeomorphism and Lie derivatives of generalized tensor fields. For the pullback there is essentially only one sensible definition.

Definition 6.1. Let $\mu\colon M \to N$ be a diffeomorphism and $R \in \hat{\mathcal{E}}^r_s(N)$. Then the map $\mu^* R \in \hat{\mathcal{E}}^r_s(M)$ defined by $(\mu^* R)(\omega) := \mu^*(R(\mu_* \omega))$ for $\omega \in \hat{\mathcal{A}}_0(M)$ is called the *pullback* of R along μ.

Lemma 6.2. *The map $\mu^*\colon \hat{\mathcal{E}}^r_s(N) \to \hat{\mathcal{E}}^r_s(M)$ of Definition 6.1 preserves moderateness and negligibility, thus it defines a map $\mu^*\colon \hat{\mathcal{G}}^r_s(N) \to \hat{\mathcal{G}}^r_s(M)$.*

Proof. Given $R \in \hat{\mathcal{E}}^r_s(N)$ and $\Phi \in \widetilde{\mathcal{A}}_k(M)$ define $t \in \mathcal{T}^r_s(M)$ by $t(p) := (\mu^* R)(\Phi(\varepsilon, p))(p) = \mu^*(R(\mu_*(\Phi(\varepsilon,p))))(p)$. By Definition 4.1 moderateness and negligibility of $\mu^* R$ are established by evaluating Lie derivatives of t on a compact set $K \subset\subset M$. Given an arbitrary vector field $X \in \mathfrak{X}(M)$, $\mathrm{L}_X t$ is given by $\mu^*(\mathrm{L}_{\mu_* X} \mu_* t)$, where

$$(\mu_* t)(p) = R(\mu_*(\Phi(\varepsilon, \mu^{-1}(p))))(p) = R((\mu_* \Phi)(\varepsilon, p))(p).$$

By Proposition 3.9 $\mu_* \Phi$ is in $\widetilde{\mathcal{A}}_k(N)$, thus the growth conditions on $\mathrm{L}_X t$ (and similarly for any number of Lie derivatives) are obtained directly from those of R with help of Lemma 3.8 (ii). □

We can define the Lie derivative $\mathrm{L}_X R \in \hat{\mathcal{E}}^r_s(M)$ of $R \in \hat{\mathcal{E}}^r_s(M)$ along a complete vector field $X \in \mathfrak{X}(M)$ in a geometric manner via its flow, namely as $(\mathrm{L}_X R)(\omega) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}((\mathrm{Fl}^X_t)^* R)(\omega)$ for $\omega \in \hat{\mathcal{A}}_0(M)$. By the chain rule this is seen to be equal to $-\mathrm{d}R(\omega)(\mathrm{L}_X \omega) + \mathrm{L}_X(R(\omega))$ (see [GKSV09, Section 6] for the smoothness argument). Thus the Lie derivative is formally the same as for elements of $\hat{\mathcal{G}}(M)$ ([GKSV02, Definition 3.8]). For non-complete vector fields we use the formula obtained from the flow for defining the Lie derivative.

Definition 6.3. For $X \in \mathfrak{X}(M)$ we define the Lie derivative $\mathrm{L}_X \hat{R}$ of $\hat{R} = \mathrm{cl}[R] \in \hat{\mathcal{G}}^r_s(M)$ as

$$\mathrm{L}_X \hat{R} := \mathrm{cl}[\omega \mapsto -\mathrm{d}R(\omega)(\mathrm{L}_X \omega) + \mathrm{L}_X(R(\omega))] \in \hat{\mathcal{G}}^r_s(M).$$

We still need to show that this is well-defined. We use the following notation: let $\varphi\colon \hat{\mathcal{E}}^r_s(M) \to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \hat{\mathcal{E}}(M))$ be the isomorphism from Theorem 4.8; then for $R \in \hat{\mathcal{E}}^r_s(M)$ and $t \in \mathcal{T}^s_r(M)$ we write $R \cdot t$ instead of $\varphi(R)(t)$.

Lemma 6.4. *For full tensor contraction of $R \in \hat{\mathcal{E}}^r_s(M)$ and $t \in \mathcal{T}^s_r(M)$ the product rule holds: $\mathrm{L}_X(R \cdot t) = \mathrm{L}_X R \cdot t + R \cdot \mathrm{L}_X t$.*

Proof. Because contraction with t (i.e., the map $R \mapsto R \cdot t$ from $\hat{\mathcal{E}}^r_s(M)$ into $\hat{\mathcal{E}}(M)$) is linear and bounded it commutes with the differential and we obtain

$$\begin{aligned}
\mathrm{L}_X(R \cdot t)(\omega) &= -\mathrm{d}(R \cdot t)(\omega)(\mathrm{L}_X \omega) + \mathrm{L}_X((R \cdot t)(\omega)) \\
&= -\mathrm{d}R(\omega)(\mathrm{L}_X \omega) \cdot t + \mathrm{L}_X(R(\omega)) \cdot t + R(\omega) \cdot \mathrm{L}_X t \\
&= (\mathrm{L}_X R)(\omega) \cdot t + R(\omega) \cdot \mathrm{L}_X t \\
&= (\mathrm{L}_X R \cdot t)(\omega) + (R \cdot \mathrm{L}_X t)(\omega).
\end{aligned}$$

□

Corollary 6.5. *The mapping $\mathrm{L}_X\colon \hat{\mathcal{E}}^r_s(M) \to \hat{\mathcal{E}}^r_s(M)$ preserves moderateness and negligibility.*

Proof. By Theorem 4.8 $R \in \hat{\mathcal{E}}^r_s(M)$ is moderate resp. negligible if and only if $R \cdot t$ is moderate resp. negligible for all $t \in \mathcal{T}^s_r(M)$. By Lemma 6.4 $(\mathrm{L}_X R) \cdot t = \mathrm{L}_X(R \cdot t) - R \cdot \mathrm{L}_X t$, so the claim follows because $\mathrm{L}_X\colon \hat{\mathcal{E}}(M) \to \hat{\mathcal{E}}(M)$ preserves moderateness and negligibility ([GKSV02, Theorem 4.6]). □

Chapter 7

Commutation relations

Proposition 7.1. *The operations μ^* and L_X on $\hat{\mathcal{E}}^r_s(M)$ extend the usual pull-back and Lie derivative of smooth tensor fields:*

$$\mu^* \circ \sigma^r_s = \sigma^r_s \circ \mu^* \quad \textit{and} \quad \mathrm{L}_X \circ \sigma^r_s = \sigma^r_s \circ \mathrm{L}_X.$$

Proof. For $t \in \mathcal{T}^r_s(N)$ and $\omega \in \hat{\mathcal{A}}_0(M)$ we have

$$\mu^*(\sigma^r_s(t))(\omega) = \mu^*(\sigma^r_s(t)(\mu_*\omega)) = \mu^* t = \sigma^r_s(\mu^* t)(\omega)$$

and for $t \in \mathcal{T}^r_s(M)$, $X \in \mathfrak{X}(M)$, and $\omega \in \hat{\mathcal{A}}_0(M)$

$$\begin{aligned} \mathrm{L}_X(\sigma^r_s(t))(\omega) &= -\mathrm{d}(\sigma^r_s(t))(\omega)(\mathrm{L}_X\omega) + \mathrm{L}_X(\sigma^r_s(t)(\omega)) = \mathrm{L}_X t \\ &= \sigma^r_s(\mathrm{L}_X t)(\omega). \end{aligned}$$

□

As to commutation relations with ι^r_s, we first formulate the following lemma.

Lemma 7.2. *Let (M, g) and (N, h) be oriented Riemannian manifolds and $\mu\colon M \to N$ an isometry. Then $\iota^r_s \circ \mu^* - \mu^* \circ \iota^r_s$ has values in $\hat{\mathcal{N}}^r_s(M)$.*

Proof. Fix $K \subset\subset M$ for testing. Denoting by r_1 the function used in the construction of the canonical transport operator, let for each $p \in M$ r_p be a positive real number smaller than $c(p)$ (the convexity radius (5.2)) such that $U_p := B^g_{r_p}(p)$ is relatively compact in M. By compactness of K there are points $p_1, \dots, p_m \in M$ (for some number $m \in \mathbb{N}$) such that $K \subseteq \bigcup_{i=1}^m U_{p_i}$. Then with $K_i := K \cap \overline{U_{p_i}}$ we can write $K = \bigcup_{i=1}^m K_i$ and each K_i is compact and contained in $B^g_{c(p_i)}(p_i)$, the strongly convex open ball at p_i with radius $c(p_i)$.

Because of these considerations we may assume K itself to be contained in a strongly convex open ball $U_0 := B^g_{r_0}(p_0)$ for some $p_0 \in K$ and $0 < r_0 < r_1(p)$. Let L be a compact neighborhood of K in U_0. Given $\Phi \in \widetilde{\mathcal{A}}_0(M)$, there exists $\varepsilon_0 > 0$ such that $\Phi(\varepsilon, p)$ has support in U_0 for all $\varepsilon < \varepsilon_0$ and $p \in L$. Now let A and B denote the canonical transport operators of M and N, respectively.

We then claim that for all $p \in L$, $t \in \mathcal{D}'^r_s(N)$, $v \in \mathcal{T}^s_r(M)$, and $\omega \in \hat{\mathcal{A}}_0(M)$ with support in U_0 the expression

$$\begin{aligned}(\iota^r_s(\mu^* t)(\omega) \cdot v)(p) &= \langle \mu^* t, A^s_r(p, \cdot) v(p) \otimes \omega \rangle \\ &= \langle t, \mu_*(A^s_r(p, \cdot) v(p)) \otimes \mu_* \omega \rangle\end{aligned}$$

equals

$$\begin{aligned}(\mu^*(\iota^r_s t)(\omega) \cdot v)(p) &= (\mu^*((\iota^r_s t)(\mu_* \omega)) \cdot v)(p) \\ &= \mu^*((\iota^r_s t)(\mu_* \omega) \cdot \mu_* v)(p) \\ &= ((\iota^r_s t)(\mu_* \omega) \cdot \mu_* v)(\mu(p)) \\ &= \langle t, B^s_r(\mu(p), \cdot) \mu_* v(\mu(p)) \otimes \mu_* \omega \rangle.\end{aligned}$$

These expressions are equal if

$$\mu_*(A^s_r(p, \cdot) v(p))(\mu(q)) = B^s_r(\mu(p), \mu(q))(\mu_* v)(\mu(p))$$

for $q \in U_0$. But this is clear because μ is an isometry, thus it preserves minimizing geodesics, (strongly) convex sets, and parallel displacement ([KN63, Chapter IV Proposition 2.5]). □

This means that the embedding of distributional tensor fields commutes with pullback along isometries and consequently with Lie derivatives along Killing vector fields.

Lemma 7.2 allows to reformulate the question of whether pullback along an arbitrary (orientation preserving) diffeomorphism $\mu \colon M \to N$ commutes with ι^r_s, for if one endows M with the pullback metric $\mu^* h$ this question reduces to checking whether the embeddings $(\iota^g)^r_s$ and $(\iota^{\mu^* h})^r_s$ arising from the Riemannian metrics g and $\mu^* h$ are equal. We then have the following main result.

Theorem 7.3. *We have the following no-go result about commutation with the embedding.*

(i) Let g, h be Riemannian metrics on M with Levi-Civita connections ∇^g, ∇^h and corresponding embeddings $(\iota^g)^r_s$, $(\iota^h)^r_s$. Then

$$((\iota^g)^r_s - (\iota^h)^r_s)(\mathcal{D}'^r_s(M)) \subseteq \hat{\mathcal{N}}^r_s(M) \Longleftrightarrow \nabla^g = \nabla^h.$$

(ii) The embedding ι^r_s does not commute with arbitrary Lie derivatives.

The proof consists of several steps. First, the assumptions are written as conditions having the same form in both cases, namely negligibility of the generalized function $(\omega, p) \mapsto \langle T, Z(p, \cdot) \otimes \omega \rangle \in \mathcal{E}(M)$ for all $T \in \mathcal{D}'^r_s(M)$ and some $Z \in \Gamma(\mathrm{pr}_2^*(\mathrm{T}^s_r(M)))$. Then, choosing T appropriately we obtain that derivatives of Z in the second slot vanish. Finally, the derivatives of Z are calculated explicitly. This involves the derivatives of the transport operator, which are related to the connection.

Beginning with the first step, we show that both $(\iota^g)^r_s - (\iota^h)^r_s$ and $\iota^r_s \circ \mathrm{L}_X - \mathrm{L}_X \circ \iota^r_s$ give rise to expressions of the same form. Let A and B be the canonical transport operators obtained from g and h. In the first case, the equality $(\iota^g)^r_s = (\iota^h)^r_s$ in the quotient means that for all $T \in \mathcal{D}'^r_s(M)$ the generalized function $R := (\iota^g - \iota^h)T \in (\hat{\mathcal{E}}^r_s)_m(M)$ given by

$$(R(\omega) \cdot v)(p) = \langle T, (A^s_r(p, \cdot) - B^s_r(p, \cdot))v(p) \otimes \omega \rangle \tag{7.1}$$

for $v \in \mathcal{T}^s_r(M)$ and $\omega \in \Omega^n_c(M)$ is negligible. Note that the difference $(p, q) \mapsto (A^s_r(p, q) - B^s_r(p, q))v(p)$ is an element of $\Gamma(\mathrm{pr}_2^*(\mathrm{T}^s_r(M)))$ and vanishes on the diagonal in $M \times M$.

In the second case, from the proof of [GKSV09, Proposition 6.8] (in particular, equations (6.13) and (6.14) therein) we immediately obtain the identity

$$((\iota^r_s \circ \mathrm{L}_X - \mathrm{L}_X \circ \iota^r_s)(T)(\omega) \cdot v)(p) = \langle T, (\mathrm{L}_{X \times X} A)^s_r(p, \cdot)v(p) \otimes \omega \rangle \tag{7.2}$$

where the term on the right hand side is exactly the additional term of the Lie derivative of generalized tensor fields in [GKSV09] which makes it commute with the embedding already in the basic space there. As in our case pullback of generalized tensor fields cannot act on the transport operator this term does not cancel. Note that also $(p, q) \mapsto (\mathrm{L}_{X \times X} A)^s_r(p, \cdot)v(p)$ is an element of $\Gamma(\mathrm{pr}_2^*(\mathrm{T}^s_r(M)))$ and vanishes on the diagonal.

Thus in both cases (i) and (ii) for each $v \in \mathcal{T}^s_r(M)$ we have found some $Z \in \Gamma(\mathrm{pr}_2^*(\mathrm{T}^s_r(M)))$ such that for all $T \in \mathcal{D}'^r_s(M)$ the generalized function $R \cdot v \in \hat{\mathcal{E}}_m(M)$ defined by

$$\omega \mapsto [p \mapsto \langle T, Z(p, \cdot) \otimes \omega \rangle] \tag{7.3}$$

is negligible (i.e., an element of $\hat{\mathcal{N}}(M)$). The next proposition and the subsequent corollary allow us to get information about Z by the right choices of the distribution T.

The idea behind the following proof is the following: locally negligibility of (7.3) means that an expression like $\langle T, f(x, \cdot)\mathrm{T}_x \mathrm{S}_\varepsilon \varphi \rangle$ converges to 0. As a simple case consider $n = 1$, $x = 0$ and f depending on the second slot only with $f(0) = 0$. Then $\langle T, f \cdot S_\varepsilon \varphi \rangle \to 0$ one the one hand, but on the other hand we can write this as (neglecting the remainder of the Taylor expansion, which converges to zero anyways):

$$\langle T(y), (f(0) + f'(0) \cdot y + \ldots + f^{(k)}(0) \cdot y^k / k!) S_\varepsilon \varphi \rangle \to 0$$

As the support of $S_\varepsilon \varphi$ gets arbitrarily small we can only hope to get information about f at 0. It vanishes there, but we can determine its derivatives there by taking for T the principal value of $1/y$: this gives the terms

$$f(0) \cdot \langle 1/y, S_\varepsilon \varphi \rangle, \quad f'(0) \cdot \langle 1, S_\varepsilon \varphi \rangle, \quad \ldots \quad f^{(k)}(0) \langle y^{k-1}/k!, S_\varepsilon \varphi \rangle.$$

If φ now has vanishing moments of order $k-1$ and is even the only remaning term is $f'(0)$, so we can conclude $f'(0) = 0$.

In the general case the proof is slightly more involved. Note that in what follows $\mathcal{E}'(\Omega) \subseteq \mathcal{D}(\Omega)$ is the space of compactly supported distributions on Ω.

Proposition 7.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^\infty(\Omega \times \Omega)$. Then*

(i) For each $T \in \mathcal{D}'(\Omega)$ the mapping in $\mathcal{E}^C(\Omega)$ given by

$$(\varphi, x) \mapsto \langle T, f(x, \cdot)\varphi(. - x)\rangle \tag{7.4}$$

is moderate, i.e., an element of $\mathcal{E}^C_M(\Omega)$.

(ii) If for all compactly supported distributions $T \in \mathcal{E}'(\Omega)$ the mapping (7.4) is in $\mathcal{N}^C(\Omega)$ then all first order partial derivatives in the second slot of f vanish on the diagonal, i.e., $\partial_i(y \mapsto f(x, y))|_x = 0\ \forall x \in \Omega\ \forall i = 1 \ldots n$.

Proof. (i) resembles the statement that the embedding of distributions into $\mathcal{E}(\Omega)$ has moderate values; the proof is virtually the same (see [GFKS01, Theorem 7.4 (i)]), inserting $f(x, \cdot)$ at the appropriate places. This results in an application of the chain rule and the appearance of some extra constants (suprema of derivatives of f on compact sets), but leaves moderateness intact.

(ii) Let x be an arbitrary point of $\Omega \subseteq \mathbb{R}^n$. Choose some $\eta > 0$ with $\eta < \operatorname{dist}(x, \partial\Omega)$ and a smooth bump function $\chi \in \mathcal{D}(\mathbb{R})$ with $\chi = 1$ on $\overline{B_{\eta/2}}(0)$ and $\operatorname{supp}\chi \subseteq B_\eta(0)$.

Consider the distribution $t \mapsto \operatorname{sign} t \cdot |t|^{n-2}$. For $n > 1$ this is a locally integrable function, for $n = 1$ this means the principal value of $\frac{1}{t}$. This distributions thus is given for all $n \in \mathbb{N}$ by

$$\langle \operatorname{sign} t \cdot |t|^{n-2}, \omega\rangle = \lim_{\delta \to 0} \int_\delta^\infty t^{n-2}(\omega(t) - \omega(-t))\, \mathrm{d}t \qquad \forall \omega \in \mathcal{D}(\mathbb{R}). \tag{7.5}$$

We introduce the distribution

$$P := \delta \otimes \ldots \otimes \delta \otimes \chi(t) \operatorname{sign} t \cdot |t|^{n-2} \otimes \delta \otimes \ldots \otimes \delta \in \mathcal{D}'(\mathbb{R}^n)$$

or more explicitly

$$\langle P, \omega\rangle = \langle \operatorname{sign} t \cdot |t|^{n-2}, \chi(t)\omega(0, \ldots, t, \ldots, 0)\rangle \qquad \forall \omega \in \mathcal{D}(\mathbb{R}^n)$$

where $\chi(t) \operatorname{sign} t \cdot |t|^{n-2}$ resp. t appears at the kth position for an arbitrary $k \in \{1, \ldots, n\}$ which shall be fixed from now on.

$u := \mathrm{T}_x P = P(. - x)$ then is a compactly supported distribution on Ω: because $\operatorname{supp} P \subseteq \{0\} \times \ldots \times B_\eta(0) \times \ldots \times \{0\} \subseteq B_\eta(0)$ we have $\operatorname{supp} u \subseteq B_\eta(x) \subseteq \Omega$.

With $K = \{x\}$ and arbitrary $m \in \mathbb{N}$, by negligibility of (7.4) there is some $q \in \mathbb{N}$ (which can be chosen arbitrarily high) such that for any fixed $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ we have

$$\langle u, f(x,\cdot)\mathrm{T}_x\mathrm{S}_\varepsilon\varphi\rangle = O(\varepsilon^m) \qquad (\varepsilon \to 0). \tag{7.6}$$

Choose $\varphi_1 \in \mathcal{D}([0,\infty))$ which is constant in a neighborhood of 0 and satisfies

$$\int_0^\infty s^{j/n}\varphi_1(s)\,\mathrm{d}s = \begin{cases} \dfrac{n}{\omega_n} & j = 0 \\ 0 & j = 1,2,3,\ldots,q \end{cases}$$

where ω_n is the area of the $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

Such a function exists by a straightforward adaption of the proof of [GKOS01, Proposition 1.4.30], and we set $\varphi := \varphi_1 \circ \|\ \|^n \in \mathcal{D}(\mathbb{R}^n)$. Then φ is in $\mathcal{A}_q(\mathbb{R}^n)$, as we will show now. Denote by $x = \Phi_n(r,\phi,\theta_1,\ldots,\theta_{n-2})$ polar coordinates in $\mathbb{R}^n$ (as in [Wal95, 7.19.4]) and set $B_1 := [0,2\pi]\times[0,\pi]\times\ldots\times[0,\pi] \subseteq \mathbb{R}^{n-1}$. Noting that $\det\Phi_n'(r,\phi,\theta_1,\ldots,\theta_{n-2}) = r^{n-1}\det\Phi_n'(1,\phi,\theta_1,\ldots,\theta_{n-2}))$ we have for any multi-index $\alpha \in \mathbb{N}_0^n$

$$\begin{aligned}\int_{\mathbb{R}^n} & x^\alpha\varphi_1(\|x\|^n)\,\mathrm{d}x = \\ &= \int_0^\infty\int_{B_1}\underbrace{\Phi_n(r,\phi,\theta_1,\ldots,\theta_{n-2})^\alpha}_{=r^{|\alpha|}\Phi_n(1,\phi,\theta_1,\ldots,\theta_{n-2})}\varphi_1(r^n)\left|\det\Phi_n'\right|\,\mathrm{d}(\phi,\theta_1,\ldots,\theta_{n-2})\,\mathrm{d}r \\ &\qquad = M_\alpha\cdot\int_0^\infty r^{|\alpha|+n-1}\varphi_1(r^n)\,\mathrm{d}r = \frac{M_\alpha}{n}\cdot\int_0^\infty s^{|\alpha|/n}\varphi_1(s)\,\mathrm{d}s\end{aligned}$$

with constants M_α defined as

$$M_\alpha := \int_{B_1}\Phi_n(1,\phi,\theta_1,\ldots,\theta_{n-2})^\alpha\left|\det\Phi_n'(1,\phi,\theta_1,\ldots,\theta_{n-2})\right|\,\mathrm{d}(\phi,\theta_1,\ldots,\theta_{n-2}).$$

Each $M_\alpha > 0$ is a constant depending only on n and α; as $M_0 = \omega_n$, φ has integral 1. Furthermore, it has vanishing moments up to order q.

Choosing $r > 0$ such that $\operatorname{supp}\varphi \subseteq B_r(0)$, let $\varepsilon < \eta/(2r)$ from now on, which implies $\operatorname{supp}\mathrm{T}_x\mathrm{S}_\varepsilon\varphi \subseteq B_{\eta/2}(x) \subseteq \Omega$ and $\operatorname{supp}[t \mapsto \varphi_1(t^n/\varepsilon^n)] \subseteq B_{\eta/2}(0)$. By equation (7.5) the expression $\langle u, f(x,\cdot)\mathrm{T}_x\mathrm{S}_\varepsilon\rangle$ on the left-hand side of (7.6) is given by

$$\langle P, f(x,x+.)S_\varepsilon\varphi\rangle = \lim_{\delta\to 0}\int_\delta^{\eta/2}\chi(t)t^{n-2}(\tilde{f}(t)-\tilde{f}(-t))\varepsilon^{-n}\varphi_1((t/\varepsilon)^n)\,\mathrm{d}t \tag{7.7}$$

where $\tilde{f}(t) := f(x, x+t\cdot e_k)$ for $|t| < \eta/2$; e_k is the kth unit vector in $\mathbb{R}^n$. Note that $\chi(t) = 1$ on the range of integration. Let us now consider the Taylor expansion of $\tilde{f}$ at 0 of order q:

$$\tilde{f}(t) = \sum_{l=0}^q\frac{\tilde{f}^{(l)}(0)}{l!}t^l + \int_0^1\frac{(1-v)^q}{q!}\tilde{f}^{(q+1)}(vt)\cdot t^{q+1}\,\mathrm{d}v$$

for $|t| < \eta/2$. With this we can write (7.7) as

$$\lim_{\delta\to 0}\int_\delta^{\eta/2}\sum_{l=0}^{q} t^{n-2}\frac{\tilde f^{(l)}(0)}{l!}(t^l-(-t)^l)\varepsilon^{-n}\varphi_1(t^n/\varepsilon^n)\,\mathrm{d}t$$
$$+\lim_{\delta\to 0}\int_\delta^{\eta/2} t^{n-2}\int_0^1\frac{(1-v)^q}{q!}\big(\tilde f^{(q+1)}(vt)-(-1)^{q+1}\tilde f^{(q+1)}(-vt)\big)\,\mathrm{d}v \cdot t^{q+1}\varepsilon^{-n}\varphi_1(t^n/\varepsilon^n)\,\mathrm{d}t.$$

The terms for even l vanish, while for odd l they are given by

$$2\lim_{\delta\to 0}\int_\delta^{\eta/2} t^{n-2}\frac{\tilde f^{(l)}(0)}{l!}t^l\varepsilon^{-n}\varphi_1((\frac{t}{\varepsilon})^n)\,\mathrm{d}t.$$

Substituting $t=\varepsilon s^{1/n}$ the term for l is given by

$$\frac{2\tilde f^{(l)}(0)}{n\cdot l!}\lim_{\delta\to 0}\int_{(\delta/\varepsilon)^n}^{(\eta/2\varepsilon)^n} s^{(l-1)/n}\varepsilon^{l-1}\varphi_1(s)\,\mathrm{d}s.$$

By definition of φ the terms for l odd and ≥ 3 vanish and the term for $l=1$ gives exactly $2\tilde f'(0)/\omega_n$. Finally, the remainder term is

$$\int_0^{\eta/2}\int_0^1\frac{(1-v)^q}{q!}(\tilde f^{(q+1)}(vt)-(-1)^{q+1}\tilde f^{(q+1)}(-vt))t^{q+n-1}\varepsilon^{-n}\varphi_1(\frac{t^n}{\varepsilon^n})\,\mathrm{d}v\,\mathrm{d}t$$

and after substituting $t=\varepsilon s^{1/n}$ this is

$$\frac{\varepsilon^q}{n}\int_0^{(\eta/(2\varepsilon))^n}\int_0^1\frac{(1-v)^q}{q!}\big(\tilde f^{(q+1)}(\varepsilon v s^{1/n})- (-1)^{q+1}\tilde f^{(q+1)}(-\varepsilon v s^{1/n})\big)s^{q/n}\varphi_1(s)\,\mathrm{d}v\,\mathrm{d}s$$

and the integral is bounded by a finite constant independently of ε.

Concluding, from Taylor expansion on the one hand and the assumption on the other hand we have

$$\langle u, f(x,\cdot)\mathrm{T}_x\mathrm{S}_\varepsilon\varphi\rangle = 2\tilde f'(0)/\omega_n + O(\varepsilon^q)$$
$$\text{and } \langle u, f(x,\cdot)\mathrm{T}_x\mathrm{S}_\varepsilon\varphi\rangle = O(\varepsilon^m)$$

Together, this gives $\tilde f'(0) = O(\varepsilon^{\min(q,m)})$ where m and q can be chosen arbitrarily high. Thus $\tilde f'(0) = D_2 f(x,x)\cdot e_k = 0$, which concludes the proof because x and k were arbitrary. □

Now follows the corresponding result on a manifold.

Corollary 7.5. *Let* $Z \in \Gamma(\mathrm{pr}_2^*(\mathrm{T}^s_r(M)))$ *satisfy* $Z(p,p) = 0\ \forall p \in M$. *Then*

(i) For each $T \in \mathcal{D}'^r_s(M)$ *the mapping from* $\hat{\mathcal{A}}_0(M) \times M$ *into* $\mathbb{K}$ *defined by*

$$(\omega, p) \mapsto \langle T, Z(p, \cdot) \otimes \omega\rangle \tag{7.8}$$

is moderate, i.e., an element of $\hat{\mathcal{E}}_m(M)$.

(ii) If for all $T \in \mathcal{D}'^r_s(M)$ *the mapping* (7.8) *is negligible then* $\mathrm{L}_Y(Z(p, \cdot))(p)$ *vanishes for all* $Y \in \mathfrak{X}(M)$ *and* $p \in M$.

Proof. As in Proposition 7.4, (i) follows in the same way as moderateness of embedded distributions (see [GKSV02, Section 5]).

(ii) Let (U, ψ) be a chart on M and $\{b_\lambda\}_\lambda$ a basis of $\mathcal{T}^r_s(U)$ with dual basis $\{b^\lambda\}_\lambda$ of $\mathcal{T}^s_r(U)$. Denote the coordinates of Z on U by $Z_\lambda \in C^\infty(U \times U)$, i.e., $Z(p, q) = Z_\lambda(p, q)b^\lambda(q)$ for all $p, q \in U$.

We will show that for any compactly supported distribution $t_U \in \mathcal{E}'(\psi(U))$ the mapping defined by

$$(\varphi, x) \mapsto \langle t_U, Z_\lambda(\psi^{-1}(x), \cdot)\varphi(. - x)\rangle$$

is negligible, i.e., an element of $\mathcal{N}^C(\psi(U))$. For this purpose define $S \in \mathcal{D}'^r_s(U) \cong \mathcal{T}^r_s(U) \otimes_{C^\infty(M)} \mathcal{D}'(U)$ by $S := b_\lambda \otimes t$ (where $t \in \mathcal{D}'(U)$ corresponds to t_U as in Section 2.2), which has compact support and thus a trivial extension to a distributional tensor field $T \in \mathcal{D}'^r_s(M)$ with $T|_U = S$. By assumption the map $\hat{\mathcal{A}}_0(M) \times M \to \mathbb{K}$ given by

$$(\omega, p) \mapsto \langle T, Z(p, \cdot) \otimes \omega\rangle$$

is negligible, thus also its restriction to U which is the map $\hat{\mathcal{A}}_0(U) \times U \to \mathbb{K}$ given by

$$(\omega, p) \mapsto \langle T, Z(p, \cdot) \otimes \omega\rangle = \langle T|_U, Z(p, \cdot)|_U \otimes \omega\rangle = \langle t, Z_\lambda(p, \cdot)\omega\rangle.$$

This implies that the corresponding map $\mathcal{A}_0(\psi(U)) \times \psi(U) \to \mathbb{K}$ given by

$$\begin{aligned}(\varphi, x) \mapsto &\langle t, Z_\lambda(\psi^{-1}(x), \cdot)\psi^*(\varphi(. - x)\, \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^n)\rangle \\ &= \langle t, \psi^*(Z_\lambda(\psi^{-1}(x), \psi^{-1}(\cdot))\varphi(. - x)\, \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^n)\rangle \\ &= \langle t_U, (Z_\lambda \circ (\psi^{-1} \times \psi^{-1}))(x, \cdot)\varphi(. - x)\rangle\end{aligned}$$

is in $\mathcal{N}^C(\psi(U))$ for any choice of $t_U \in \mathcal{E}'(\psi(U))$. Proposition 7.4 now implies that $\partial_i(y \mapsto Z_\lambda(\psi^{-1}(x), \psi^{-1}(y)))|_x = 0$ for all x in $\psi(U)$ and all i. Noting that $Z(p, p) = 0$ by assumption, the local formula for $\mathrm{L}_Y(Z(p, \cdot))(p)$ evaluates to 0. □

Assuming (7.1) resp. (7.2) to be negligible for all choices of T, Corollary 7.5 implies in the case $(r,s) = (0,1)$ for all X, Y, $Z \in \mathfrak{X}(M)$ and $p \in M$ the identities

$$\begin{aligned}\text{(i) } &\mathrm{L}_Y(q \mapsto (A(p,q) - B(p,q))Z(p) = 0 \quad \text{and}\\ \text{(ii) } &\mathrm{L}_Y(q \mapsto (\mathrm{L}_{X\times X}A)(p,q)Z(p))(p) = 0.\end{aligned}$$

We will now calculate these expressions explicitly in a chart. Fix a chart (U,φ) containing p_0 for the remainder of this chapter. As this is the only chart we will use we will refrain from indexing local expressions of vector fields, flows etc. by U, e.g., for $X \in \mathfrak{X}(M)$ we will write $X \in C^\infty(U)$ for its local expression.

Given a vector field $X \in \mathfrak{X}(M)$, by its *local flow* on U we mean the map $\alpha \colon \mathscr{D}(X) \to \varphi(U)$ determined by the ordinary differential equation

$$\alpha(0,x) = x, \quad \alpha'(t,x) = X(\alpha(t,x)) \tag{7.9}$$

where $X \in C^\infty(\varphi(U), \mathbb{R}^n)$ is the local representation of X on U and $\mathscr{D}(X)$, the maximal domain of definition of α, is an open subset of $\mathbb{R} \times \varphi(U)$. For $p \in U$, its flow along X is given by $\mathrm{Fl}_t^X p = \mathrm{T}\varphi^{-1}(\alpha(t,\varphi(p)))$ for all t such that $(t,\varphi(p)) \in \mathscr{D}(X)$. Furthermore, α is smooth. By differentiating (7.9) one sees that for all $(t,x) \in \mathscr{D}(X)$ the local flow α satisfies

$$\begin{aligned}\alpha(t,x) &= x + \int_0^t X(\alpha(u,x))\,\mathrm{d}u\\ D_2\alpha(t,x) &= I + \int_0^t X'(\alpha(u,x))D_2\alpha(u,x)\,\mathrm{d}u\\ D_2^2\alpha(t,x) &= \int_0^t \big(X''(\alpha(u,x)) \circ (D_2\alpha(u,x) \times D_2\alpha(u,x))\\ &\qquad + X'(\alpha(u,x))D_2^2\alpha(u,x)\big)\,\mathrm{d}u\\ D_1\alpha(t,x) &= X(\alpha(t,x))\\ D_1D_2\alpha(t,x) &= X'(\alpha(t,x))D_2\alpha(t,x)\end{aligned}$$

In particular, we have

$$\begin{aligned}\alpha(0,x) &= x\\ D_2\alpha(0,x) &= I\\ D_2D_2\alpha(0,x) &= 0\\ D_1D_2\alpha(0,x) &= X'(x).\end{aligned} \tag{7.10}$$

In order to calculate the above expressions we need normal neighborhoods and smoothness of geodesics in starting and end points as well as the initial direction. As can be seen from standard results in differential geometry, for every $p \in M$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the following holds:

1. The open ball $W := B_\varepsilon(p)$ is contained in U.
2. Any two points $q, r \in W$ can be joined by a unique geodesic $(-2,2) \to W$, $t \mapsto \sigma(t,q,r)$ with $\sigma(0,q,r) = q$ and $\sigma(1,q,r) = r$. The map $\sigma\colon (-2,2) \times W \times W \to W$ is smooth.
3. The map $W \times W \to \mathrm{T}M$, $(q,r) \mapsto X_{qr} := \mathrm{d}/\mathrm{d}t|_{t=0}\,\sigma(t,q,r)$ is smooth; X_{qr} is the unique element of T_qM such that $\exp_q X_{qr} = r$.

We sketch the essential ideas for obtaining these results, following [Kli95].

Set $U' := \varphi(U)$. Geodesics are obtained locally by solving the ODE system

$$\begin{cases} u' = v & u(0) = x \in U' \subseteq \mathbb{R}^n \\ v' = -\Gamma(u)(v,v) & v(0) = w \in \mathbb{R}^n. \end{cases} \tag{7.11}$$

The initial conditions are the starting point x and the initial direction w of the geodesic u. There are open neighborhoods U_1', U_2' of $\varphi(p)$ in U' with $U_1' \subseteq U_2'$ and a constant $\eta > 0$ such that (7.11) has solutions $u(t,x,w)\colon (-2,2) \times U_1' \times B_\eta(0) \to U_2'$ resp. $v(t,x,w)\colon (-2,2) \times U_1' \times B_\eta(0) \to \mathbb{R}^n$ for all $t \in (-2,2)$ and (x,w) in $U_1' \times B_\eta(0) \subseteq \mathrm{T}U' = U' \times \mathbb{R}^n$. As Γ is smooth u and v are smooth in the independent variable as well as the initial conditions.

On the manifold this procedure gives an open neighborhood $\widetilde{\mathrm{T}}M$ of the submanifold $M \subseteq \mathrm{T}M$ such that for every $X \in \widetilde{\mathrm{T}}M$ the geodesic $c_X(t)$ with initial direction X starting at the footpoint of X is defined at least for $|t| < 2$ [Kli95, Lemma 1.6.7]. The exponential mapping $\exp\colon \widetilde{\mathrm{T}}M \to M$ then is defined as $\exp X := c_X(1)$.

In order to obtain geodesics joining two points one defines the mapping

$$\begin{aligned} F\colon \widetilde{\mathrm{T}}M &\to M \times M \\ X &\mapsto (\pi X, \exp X) \end{aligned}$$

where π is the projection of the tangent bundle. Note that we can always make U_1' and η smaller, so we can assume that the open set $W_1 := \mathrm{T}\varphi^{-1}(U_1' \times B_\eta(0))$ is contained in $\widetilde{\mathrm{T}}M$. The local expression of $F|_{W_1}$ is given by

$$\begin{aligned} F_U\colon U_1' \times B_\eta(0) &\to U_1' \times U_2' \\ (x,w) &\mapsto (x, u(1,x,w)). \end{aligned} \tag{7.12}$$

For each $x \in U_1'$ the Jacobian of F_U at $(x,0)$ is given by

$$DF_U(x,0) = \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} & \mathrm{id} \end{pmatrix}$$

which is regular, thus F_U is invertible at $(x,0)$. For $x = \varphi(p)$ this means that F is invertible at p, i.e, there is a neighborhood $\widetilde{W} \subseteq W_1 \subseteq \widetilde{\mathrm{T}}M$ of 0_p in $\mathrm{T}M$

and a neighborhood $W(p,p) \subseteq \varphi^{-1}(U_1' \times U_2')$ of $(p,p) \in M \times M$ such that $F|_{\widetilde{W}} \colon \widetilde{W} \to W(p,p)$ is a diffeomorphism. Choosing a neighborhood W of p in M such that $W \times W \subseteq W(p,p)$, for $q, r \in W$ we set $X_{qr} := (F|_{\widetilde{W}})^{-1}(q,r)$ (as in [Kli95, Theorem 1.6.12]).

Given a Riemannian metric inducing the Levi-Civita connection one can take for W a sufficiently small metric ball $B_\varepsilon(p)$ ([Kli95, Theorem 1.8.15]). Furthermore, for all ε small enough $B_\varepsilon(p)$ is strongly convex, i.e., the geodesics connecting points of W are unique and contained in W ([Kli95, Theorem 1.9.10]). Clearly ε can also be taken so small that $B_\varepsilon(p)$ is contained in U. Finally, we note that $\sigma(t,q,r) = \varphi^{-1}(u(t, \varphi(q), \mathrm{pr}_2 \circ \mathrm{T}\varphi(X_{qr})))$.

This enables us to calculate the derivatives of the transport operator.

Lemma 7.6. *Let (U, φ) be some chart on M. Then the local representation $a \in C^\infty(U \times U, \mathrm{L}(\mathbb{R}^n, \mathbb{R}^n))$ of the canonical transport operator A satisfies the following identities for all $x \in U$ and $\xi, \eta, \zeta \in \mathbb{R}^n$:*

(i) $a(x,x) = \mathrm{id}$

(ii) $(Da)(x,x)(\xi,\eta) \cdot \zeta = -\Gamma(x, \eta - \xi, \zeta)$

(iii)
$$2(D^2 a)(x,x)((\xi_1,\eta_1),(\xi_2,\eta_2))\zeta = -(\Gamma'(x) \cdot (\eta_1 + \xi_1))(\eta_2 - \xi_2, \zeta) - (\Gamma'(x) \cdot (\eta_2 + \xi_2))(\eta_1 - \xi_1, \zeta) + \Gamma(x, \eta_1 - \xi_1, \Gamma(x, \eta_2 - \xi_2, \zeta)) + \Gamma(x, \eta_2 - \xi_2, \Gamma(x, \eta_1 - \xi_1, \zeta))$$

Proof. Given $p \in U$ let $W = B_\varepsilon(p)$ be a neighborhood of p as above with $\varepsilon < r_1(p)$, with corresponding maps σ, X_{qr}, F, and u. We will use the following notation.

- x, y are points in $W' := \varphi(W)$. We set $p := \varphi^{-1}(x)$ and $q := \varphi^{-1}(y)$.
- σ has local expression $\sigma(t,x,y) := \varphi \circ \sigma(t,p,q)$ defined on $(-2,2) \times W' \times W'$.
- define $w \in C^\infty(W' \times W', \mathbb{R}^n)$ by $w(x,y) := \mathrm{pr}_2 \circ T\varphi(X_{pq})$ (the principal part of the local expression of X_{pq}). Note that $w(x,y) \in B_\eta(0)$ because $\widetilde{W} \subseteq W_1$.
- $u(t,x,w)$ and $v(t,x,w)$ are as above, defined on $(-2,2) \times U_1' \times B_\eta(0)$. Thus $\sigma(t,x,y)$ is given by $u(t,x,w(x,y))$.
- $F|_{\widetilde{W}}$ has local expression $F_U \colon (x,w) \mapsto (x, u(1,x,w))$, defined on $U_1' \times B_\eta(0)$.
- $a(x,y) \cdot \zeta = (\mathrm{pr}_2 \circ T\varphi)(A(p,q) \cdot T\varphi^{-1}(x,\zeta))$ for all $x, y \in W'$ and $\zeta \in \mathbb{R}^n$.
- Where convenient we write $\Gamma(u,v,w)$ in place of $\Gamma(u)(v,w)$ for any arguments u, v, w.

By definition a is given by $a(x,y)\cdot\zeta = \rho(1,x,y,\zeta)$, where $\rho(t) = \rho(t,x,y,\zeta)$ is the unique solution of the ODE

$$\begin{cases} \rho(0,x,y) = \zeta \\ \rho'(t,x,y) = -\Gamma(\sigma(t,x,y), \sigma'(t,x,y), \rho(t,x,y)). \end{cases} \tag{7.13}$$

This means that $\rho(1,x,y,\zeta)$ is the parallel transport of the vector ζ along the unique geodesic from x to y; ρ is a map $(-2,2)\times W'\times W'\times\mathbb{R}^n\to\mathbb{R}^n$.

The claims of the Lemma are about the derivatives of ρ. For these we first need the derivatives of $\sigma(t,x,y)$. In what follows now, D denotes the differential with respect to the pair of variables (x,y) while differentiation with respect to t will be denoted by a prime, as in σ'; the latter is also used for other functions depending on only one variable, like the local expression of vector fields. We will mostly omit arguments of σ, u, and v for shorter notation. For the direction of differentiation we use arbitrary vectors $e = (\xi_1, \eta_1)$ and $f = (\xi_2, \eta_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Then σ and its derivatives are given by

$$\begin{aligned} \sigma(t,x,y) &= u(t,x,w(x,y)) \\ (D\sigma)(t,x,y)\cdot e &= (Du)(t,x,w(x,y)\cdot(\xi_1, Dw(x,y)\cdot e) \\ (D^2\sigma)(t,x,y)\cdot(e,f) &= (Du)(t,x,w(x,y))\cdot(0, D^2w(x,y)\cdot(e,f)) \\ &\quad + (D^2u)(t,x,w(x,y))\cdot((\xi_1, Dw(x,y)\cdot e),(\xi_2, Dw(x,y)\cdot f)) \end{aligned} \tag{7.14}$$

and similarly for σ' with v in place of u. The derivatives of u and v are determined by the following ODE systems obtained by differentiating (7.11), whose solutions exist on $(-2,2)\times U_1'\times B_\eta(0)$.

$$\begin{cases} (Du)'\cdot e = Dv\cdot e \\ (Dv)'\cdot e = (-\Gamma'(u)\cdot Du\cdot e)(v,v) - 2\Gamma(u, Dv\cdot e, v) \\ (Du)(0)\cdot e = \xi_1 \\ (Dv)(0)\cdot e = \eta_1 \end{cases}$$

$$\begin{cases} (D^2u)'\cdot(e,f) = (D^2v)\cdot(e,f) \\ (D^2v)'\cdot(e,f) = (-\Gamma''(u)\cdot(Du\cdot e, Du\cdot f))(v,v) \\ \qquad - (\Gamma'(u)\cdot D^2u\cdot(e,f))(v,v) - 2(\Gamma'(u)\cdot Du\cdot e)(Dv\cdot f, v) \\ \qquad - 2(\Gamma'(u)\cdot Du\cdot f)(Dv\cdot e, v) - 2\Gamma(u, D^2v\cdot(e,f), v) \\ \qquad - 2\Gamma(u, Dv\cdot e, Dv\cdot f) \\ (D^2u)(0) = 0 \\ (D^2v)(0) = 0 \end{cases}$$

For $w = 0$ we obtain the following solutions:

$$\begin{aligned}
v'(t,x,0) = 0 &\Rightarrow v(t,x,0) = w = 0 \\
u'(t,x,0) = 0 &\Rightarrow u(t,x,0) = x \\
(Dv')(t,x,0)\cdot e = 0 &\Rightarrow (Dv)(t,x,0)\cdot e = \eta_1 \\
&\Rightarrow (Du)(t,x,0)\cdot e = \xi_1 + t\eta_1 \\
(D^2v')(t,x,0)\cdot(e,f) & \\
= -2\Gamma(x,\eta_1,\eta_2) &\Rightarrow (D^2v)(t,x,0)\cdot(e,f) = -2t\Gamma(x,\eta_1,\eta_2) \\
&\Rightarrow (D^2u)(t,x,0)\cdot(e,f) = -t^2\Gamma(x,\eta_1,\eta_2)
\end{aligned}$$

For derivatives of w, first note that by (7.12) w is given by the second component of the inverse of $G := F_U|_{\widetilde{W}'}$ on $W' \times W'$ (where $\widetilde{W}' := \varphi(\widetilde{W})$). Writing $G = (G_1, G_2)$ with $G_1(x,w) = x$, $G_2(x,w) = u(1,x,w)$ we know from above that G is a diffeomorphism from $\widetilde{W}'$ onto $\varphi(W(p,p))$. The Jacobian of G^{-1} at (x,x) is given by

$$D(G^{-1})(x,x) = (DG(x,0))^{-1} = \begin{pmatrix} \mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id} \end{pmatrix}$$

and (as $w = \mathrm{pr}_2 \circ (G^{-1})|_{W'\times W'}$) $Dw(x,x)(\xi,\eta) = \eta - \xi$. Next, by the chain rule we see that $D^2(G^{-1}\circ G)(x,w) = 0$ implies

$$D^2G^{-1}(G(x,w)) \circ (DG(x,w) \times DG(x,w)) = -DG^{-1}(G(x,w))D^2G(x,w)$$

Furthermore, using the elementary fact that

$$D^2G(x,w)\cdot(e,f) = (D^2G_1(x,w)\cdot(e,f), D^2G_2(x,w)\cdot(e,f))$$

and the relations for derivatives of u from above we obtain

$$\begin{aligned}
&D^2G^{-1}(x,x)((\xi_1,\xi_1+\eta_1),(\xi_2,\xi_2+\eta_2)) \\
&\qquad\qquad = -\begin{pmatrix} \mathrm{id} & 0 \\ -\mathrm{id} & \mathrm{id} \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -\Gamma(x,\eta_1,\eta_2) \end{pmatrix} \\
&\qquad\qquad = (0,\Gamma(x,\eta_1,\eta_2))
\end{aligned}$$

and thus

$$\begin{aligned}
D^2w(x,x)((\xi_1,\eta_1),(\xi_2,\eta_2)) &= D^2(\mathrm{pr}_2 \circ G^{-1})(x)((\xi_1,\eta_1),(\xi_2,\eta_2)) \\
&= \mathrm{pr}_2\big(D^2G^{-1}(x)((\xi_1,\eta_1),(\xi_2,\eta_2))\big) \\
&= \Gamma(x,\eta_1-\xi_1,\eta_2-\xi_2).
\end{aligned}$$

Inserting into (7.14) we obtain the derivatives of σ:

$$\begin{aligned}
\sigma(t,x,x) &= x \\
(D\sigma)(t,x,x)(\xi,\eta) &= \xi + t(\eta - \xi) \\
(D^2\sigma)(t,x,x)(e,f) &= (t - t^2)(\Gamma(x,\eta_1 - \xi_1, \eta_2 - \xi_2) \\
\sigma'(t,x,x) &= 0 \\
(D\sigma')(t,x,x)(\xi,\eta) &= \eta - \xi \\
(D^2\sigma')(t,x,x)(e,f) &= (1-2t)\Gamma(x,\eta_1 - \xi_1, \eta_2 - \xi_2)
\end{aligned}$$

Now we calculate the derivatives of ρ by differentiating (7.13):

$$\begin{cases}
(D\rho)(0)\cdot e = 0 \\
\quad (D\rho)'\cdot e = -(\Gamma'(\sigma)\cdot D\sigma\cdot e)(\sigma',\rho) - \Gamma(\sigma, D\sigma'\cdot e, \rho) - \Gamma(\sigma,\sigma',D\rho\cdot e)
\end{cases}$$

$$\begin{cases}
(D^2\rho)(0)\cdot(e,f) = 0 \\
\qquad (D^2\rho)'(e,f) = -(\Gamma''(\sigma)(D\sigma\cdot e, D\sigma\cdot f))(\sigma',\rho) \\
\qquad\qquad - (\Gamma'(\sigma)\cdot D^2\sigma\cdot(e,f))(\sigma',\rho) \\
\qquad\qquad - (\Gamma'(\sigma)\cdot D\sigma\cdot e)(D\sigma'\cdot f,\rho) \\
\qquad\qquad - (\Gamma'(\sigma)\cdot D\sigma\cdot e)(\sigma', D\rho\cdot f) \\
\qquad\qquad - (\Gamma'(\sigma)\cdot D\sigma\cdot f)(D\sigma'\cdot e,\rho) \\
\qquad\qquad - \Gamma(\sigma, D^2\sigma'\cdot(e,f),\rho) \\
\qquad\qquad - \Gamma(\sigma, D\sigma'\cdot e, D\rho\cdot f) \\
\qquad\qquad - (\Gamma'(\sigma)\cdot D\sigma\cdot f)(\sigma', D\rho\cdot e) \\
\qquad\qquad - \Gamma(\sigma, D\sigma'\cdot f, D\rho\cdot e) \\
\qquad\qquad - \Gamma(\sigma,\sigma', D^2\rho\cdot(e,f))
\end{cases}$$

From this we finally obtain

$$\begin{aligned}
& & \rho'(t,x,x) &= 0 \\
&\Rightarrow & \rho(t,x,x) &= \zeta \\
& & (D\rho')(t,x,x)(\xi,\eta) &= -\Gamma(x,\eta-\xi,\zeta) \\
&\Rightarrow & (D\rho)(t,x,x)(\xi,\eta) &= -t\cdot\Gamma(x,\eta-\xi,\zeta)
\end{aligned}$$

and furthermore

$$
\begin{aligned}
(D^2\rho')(t,x,x)\cdot(e,f) &= -(\Gamma'(x)\cdot(\xi_1+t(\eta_1-\xi_1)))(\eta_2-\xi_2,\zeta)\\
&\quad -(\Gamma'(x)\cdot(\xi_2+t(\eta_2-\xi_2)))(\eta_1-\xi_1,\zeta)\\
&\quad -\Gamma(x,(1-2t)\Gamma(x,\eta_1-\xi_1,\eta_2-\xi_2),\zeta)\\
&\quad +\Gamma(x,\eta_1-\xi_1,t\cdot\Gamma(x,\eta_2-\xi_2,\zeta))\\
&\quad +\Gamma(x,\eta_2-\xi_2,t\cdot\Gamma(x,\eta_1-\xi_1,\zeta))\\
\Rightarrow\qquad (D^2\rho)(t,x,x)\cdot(e,f) &= -(\Gamma'(x)\cdot(t\xi_1+\frac{t^2(\eta_1-\xi_1)}{2}))(\eta_2-\xi_2,\zeta)\\
&\quad -(\Gamma'(x)\cdot(t\xi_2+\frac{t^2(\eta_2-\xi_2)}{2}))(\eta_1-\xi_1,\zeta)\\
&\quad -(t-t^2)\Gamma(x,\Gamma(x,\eta_1-\xi_1,\eta_2-\xi_2),\zeta)\\
&\quad +\frac{t^2}{2}\cdot\Gamma(x,\eta_1-\xi_1,\Gamma(x,\eta_2-\xi_2,\zeta))\\
&\quad +\frac{t^2}{2}\cdot\Gamma(x,\eta_2-\xi_2,\Gamma(x,\eta_1-\xi_1,\zeta)).
\end{aligned}
$$

As $a(x,x)\zeta=\rho(1,x,x,\zeta)=\zeta$ we are done. □

We now return to the proof of Theorem 7.3. For (i), $\mathrm{L}_X(q\mapsto A(p,q)Z(p))(p)$ is the derivative at $t=0$ of

$$\mathrm{T}\,\mathrm{Fl}^X_{-t}\,A(p,\mathrm{Fl}^X_t\,p)Z(p).$$

This means we have to differentiate the local expression

$$D\alpha(-t,\alpha(t,x))a(x,\alpha(t,x))Z(x)$$

which results in

$$
\begin{aligned}
&-D\alpha'(-t,\alpha(t,x))a(x,\alpha(t,x))Z(x)\\
&\qquad + D^2\alpha(-t,\alpha(t,x))X(\alpha(t,x))a(x,\alpha(t,x))Z(x)\\
&\qquad\qquad + D\alpha(-t,\alpha(t,x))D_2a(x,\alpha(t,x))X(\alpha(t,x))Z(x)
\end{aligned}
$$

which by (7.10) and Lemma 7.6 at $t=0$ evaluates to

$$-X'(x)Z(x)-\Gamma(x,X(x),Z(x))=-\nabla_Z X(x).$$

As we can choose X, Z, and x freely this immediately implies that both covariant derivatives are equal. This proves Theorem 7.3 (i).

Now to (ii). Higher derivatives of a map $F\colon U\times V\subseteq E_1\times E_2\to F$ like D_1D_2F are maps $D_1D_2F\colon U\times V\to \mathrm{L}^2(E\times E,F)$ and we write $D_1D_2F(x,y)(e_1,e_2)$ for $e_i\in E_i$, i.e., the order of the arguments is the same as the order of the derivatives.

By equation (5.1) $\mathrm{L}_Y(q \mapsto \mathrm{L}_{X\times X} A(p,q)Z(p))(p)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} T\,\mathrm{Fl}^Y_{-s} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{T}_{\mathrm{Fl}^X_t q} \mathrm{Fl}^X_{-t} \cdot A(\mathrm{Fl}^X_t p, \mathrm{Fl}^X_t \mathrm{Fl}^Y_s p) \cdot \mathrm{T}_p \mathrm{Fl}^X_t \cdot V(p). \quad (7.15)$$

We will first calculate the inner expression, which (setting $q := \mathrm{Fl}^Y_s p$) is given by

$$\mathrm{T}_{\mathrm{Fl}^X_t q} \mathrm{Fl}^X_{-t} \cdot A(\mathrm{Fl}^X_t p, \mathrm{Fl}^X_t q) \cdot \mathrm{T}_p \mathrm{Fl}^X_t \cdot V(p). \quad (7.16)$$

Note that for $p, q \in U$ and the modulus of s, t small enough the flows in (7.16) and (7.15) stay inside U, thus we have for (7.16) the local expression

$$F(t,x,y) := D\alpha(-t, \alpha(t,y)) a(\alpha(t,x), \alpha(t,y)).D\alpha(t,x)Z(x)$$

Here α (and below β) denotes the local flow of X (and Y, respectively). The derivative at $t = 0$ of this is

$$\begin{aligned} D_1F(t,x,y) = \big(&-D\alpha'(-t,\alpha(t,y))a(\alpha(t,x),\alpha(t,y))D\alpha(t,x) \\ &+ D^2\alpha(-t,\alpha(t,y))X(\alpha(t,y))a(\alpha(t,x),\alpha(t,y))D\alpha(t,x) \\ &+ D\alpha(-t,\alpha(t,y))Da(\alpha(t,x),\alpha(t,y))\big(X(\alpha(t,x)),X(\alpha(t,y))\big)D\alpha(t,x) \\ &+ D\alpha(-t,\alpha(t,y))a(\alpha(t,x),\alpha(t,y))D\alpha'(t,x)\big)Z(x). \end{aligned}$$

Evaluating at $t = 0$ we obtain by (7.10) that $F'(0,x,y)$ equals

$$\big(-X'(y)a(x,y) + Da(x,y)(X(x),X(y)) + a(x,y)X'(x)\big)Z(x).$$

Note that for $x = y$ this expression vanishes by Lemma 7.6. Now we set $y = \beta(t,x)$; then (7.15) is locally given by the derivative at $s = 0$ of

$$\begin{aligned} G(s,x) := D\beta(-s,\beta(s,x))\big(&-X'(\beta(s,x))a(x,\beta(s,x)) \\ &+ Da(x,\beta(s,x))(X(x),X(\beta(s,x)) + a(x,\beta(s,x))X'(x)\big)Z(x). \end{aligned}$$

The derivative of this with respect to s is

$$\begin{aligned} D_1G'(s,x,y) = &- D\beta'(-s,\beta(s,x))F'(0,x,\beta(s,x)) + \\ &+ D^2\beta(-s,\beta(s,x))Y(\beta(s,x))F'(0,x,\beta(s,x)) \\ &+ D\beta(-s,\beta(s,x)) \cdot \big(-X''(\beta(s,x))Y(\beta(s,x))a(x,\beta(s,x)) \\ &\quad - X'(\beta(s,x))D_2a(x,\beta(s,x))Y(\beta(s,x)) \\ &\quad + D^2a(x,\beta(s,x))\big((X(x),X(\beta(s,x))),(0,Y(\beta(s,x)))\big) \\ &\quad + Da(x,\beta(s,x))(0,X'(\beta(s,x))Y(\beta(s,x))) \\ &\quad + D_2a(x,\beta(s,x))Y(\beta(s,x))X'(x)\big)Z(x) \end{aligned}$$

and at $s = 0$ the first two terms vanish, while for the rest we obtain

$$D_1 G'(0,x) = (-X''Y - X'D_2aY + D^2a((X,Y),(0,Y)) \\ + D_2aX'Y + D_2aYX')Z$$

which by Lemma 7.6 equals

$$\begin{aligned} &- X''YZ + X'\Gamma(Y,Z) - 1/2\big(\Gamma' \cdot (X+Y)(Y,Z) + (\Gamma' \cdot Y)(Y-X,Z) \\ &\quad - \Gamma(Y-X,\Gamma(Y,Z)) - \Gamma(Y,\Gamma(Y-X,Z))\big) - \Gamma(X'Y,Z) - \Gamma(Y,X'Z) \\ &\qquad = -X''YZ + X'\Gamma(Y,Z) - (\Gamma' \cdot Y)(Y,Z) + \Gamma(Y,\Gamma(Y,Z)) \\ &\quad - 1/2\big((\Gamma' \cdot X)(Y,Z) - (\Gamma' \cdot Y)(X,Z) + \Gamma(X,\Gamma(Y,Z)) + \Gamma(Y,\Gamma(X,Z))\big) \\ &\qquad\qquad - \Gamma(X'Y,Z) - \Gamma(Y,X'Z). \end{aligned} \tag{7.17}$$

By assumption, this vanishes for all possible choices of X, Y, Z, and x. Setting $X = 0$ gives

$$(\Gamma' \cdot Y)(Y,Z) = \Gamma(Y,\Gamma(Y,Z))$$

and, applying this formula to $\Gamma' \cdot (X+Y)(X+Y,Z)$ for any X, Y, Z we obtain

$$(\Gamma' \cdot X)(Y,Z) + (\Gamma' \cdot Y)(X,Z) = \Gamma(X,\Gamma(Y,Z)) + \Gamma(Y,\Gamma(X,Z))$$

and thus, inserting this into (7.17)

$$-X''YZ + X'\Gamma(Y,Z) - (\Gamma' \cdot X)(Y,Z) - \Gamma(X'Y,Z) - \Gamma(Y,X'Z) = 0$$

for all choices of X, Y, Z, x.

Choosing X constant in a neighborhood of x gives $(\Gamma' \cdot X)(Y,Z) = 0$, thus $\Gamma' = 0$ and we can drop this term. Then, choosing X such that $X' = \mathrm{id}$ around x implies $\Gamma(Y,Z) = 0$. It remains that $X''YZ = 0$, which clearly cannot hold for arbitrary X, Y, Z. This proves the assertion that ι_s^r cannot commute with arbitrary Lie derivatives.

We thus established Theorem 7.3. $\square$

Part II

Topology and tensor products of section spaces

Chapter 8

Introduction to Part II

This part is devoted to introducing topologies on spaces of sections of vector bundles appropriate for defining distributions on manifolds. Furthermore, we endow their tensor product with a suitable topology such that the following become bornological isomorphisms:

$$\begin{aligned}\Gamma(E \otimes F) &\cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)\\ \Gamma_{c,K}(E \otimes F) &\cong \Gamma_{c,K}(E) \otimes_{C^\infty(M)} \Gamma(F)\\ \Gamma_c(E \otimes F) &\cong \Gamma_c(E) \otimes_{C^\infty(M)} \Gamma(F)\end{aligned}$$

In the beginning we will review inductive locally convex topologies and final convex bornologies defined by bilinear maps. Then the bornological and projective tensor product of locally convex resp. bounded modules are defined and their usual algebraic properties in the topological resp. bornological setting are established. We will then describe the natural Fréchet topology on spaces of sections and show that some usual algebraic isomorphisms for spaces of sections are homeomorphisms as well. Finally, we establish the above bornological isomorphisms and are able to obtain the bornological isomorphisms

$$\begin{aligned}\mathcal{D}'^r_s(M) &\cong (\mathcal{T}^s_r(M) \otimes_{C^\infty(M)} \Omega^n_c(M))'\\ &\cong \mathrm{L}^b_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M))\\ &\cong \mathcal{T}^r_s(M) \otimes_{C^\infty(M)} \mathcal{D}'(M).\end{aligned}$$

We will see that the bornological tensor product has to be preferred to the projective tensor product for our purposes: it has better algebraic properties (it commutes with direct limits), we can use the exponential law for spaces of bounded linear functions, and multiplication of distributions by smooth functions is jointly bounded but only separately continuous.

Chapter 9

Preliminaries

In this chapter we will lay down notation and give some background on the inductive locally convex topology and the final convex bornology on a vector space. Often these are defined with respect to linear maps only, but we need them for the canonical bilinear map $\otimes\colon E \times F \to E \otimes F$.

9.1 Notation

All locally convex spaces are over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$, and will be assumed to be Hausdorff. In the non-Hausdorff case we speak of a topological vector space with locally convex topology. We refer to [Jar81, Sch71, Tre76] for notions of topological vector spaces, to [HN77] for notions of bornological spaces, and to [Lan99] for notions of differential geometry.

We will use the following notation:

1. For any vector spaces $E_1, \ldots, E_n$, and F, $\mathrm{L}(E_1, \ldots, E_n; F)$ is the space of all n-multilinear mappings from $E_1 \times \ldots \times E_n$ to F. We write $L(E, F)$ instead of $L(E; F)$. $F^* = \mathrm{L}(F, \mathbb{K})$ denotes the algebraic dual of F.

2. For any locally convex spaces $E_1, \ldots, E_n$, and F, $\mathrm{L}^b(E_1, \ldots, E_n; F)$ is the space of *bounded* multilinear mappings as in [KM97, Section 5], equipped with the topology of uniform convergence on bounded sets ([Sch71, Chapter III §3]). $\mathrm{L}^c(E_1, \ldots, E_n; F)$ is the subspace of all *continuous* such mappings, equipped with the subspace topology.

3. $E' = \mathrm{L}^c(E, \mathbb{K})$ denotes the topological dual with the strong dual topology ([Tre76, Chapter 19]) (i.e., uniform convergence on bounded sets).

4. For any R-modules $M_1, \ldots, M_n$, and N, $\mathrm{L}_R(M_1, \ldots, M_n, N)$ is the space of R-multilinear mappings from $M_1 \times \ldots \times M_n$ to N.

5. For any locally convex R-modules $M_1, \ldots, M_n$, and N (as in Definition 10.3 below), the subspace $\mathrm{L}^b_R(M_1, \ldots, M_n, N) \subseteq \mathrm{L}^b(M_1, \ldots, M_n, N)$ is

the space of *bounded* R-multilinear mappings from $M_1 \times \ldots \times M_n$ to N, equipped with the subspace topology. We also equip the subspace $\mathrm{L}^c_R(E_1, \ldots, E_n; F) \subseteq \mathrm{L}^b_R(E_1, \ldots, E_n; F)$ of all *continuous* such mappings with the subspace topology. In all the above cases the subspace topology is again the topology of uniform convergence on bounded sets.

9.2 Inductive locally convex topologies

It is well known ([Jar81, Section 4.1]) that given a family $(E_j)_{j\in J}$ of topological vector spaces (where J is any index set), a vector space E, and linear maps $S_j \colon E_j \to E$ for each j, there is a finest linear topology on E such that all the S_j are continuous. A linear map T from E endowed with this topology into any topological vector space F is continuous if and only if all compositions $T \circ S_j$ are continuous.

Similarly, if the topologies of E_j, E, and F are locally convex the finest locally convex topology on E such that all S_j are continuous (called the inductive locally convex topology) has the property that a linear map $T \colon E \to F$ into any topological vector space F with locally convex topology is continuous if and only if all the $T \circ S_j$ are continuous ([Jar81, Section 6.6]).

Now let E and F be locally convex spaces. One prominent way to put a topology on the tensor product $E \otimes F$ is to take the finest locally convex topology such that the canonical bilinear map $\otimes \colon E \times F \to E \otimes F$ is continuous, in other words the *inductive* locally convex topology defined by this map. This is commonly called the *projective tensor topology.* $E \otimes F$ with this topology has the property that it linearizes continuous bilinear mappings [Jar81, 5.1 Theorem 2].

Now in [Jar81, Tre76] this topology is not treated satisfactorily for our purposes:

- In [Jar81, Section 15.1] $E \otimes F$ is endowed with the *finest topology* (not locally convex topology) which makes $\otimes$ continuous, and it is claimed that this topology is locally convex by referring to a Proposition about the *projective* topology, which does not apply here as we are instead dealing with the inductive locally convex topology defined by $\otimes$. As is well-known, the finest topology which makes $\otimes$ continuous need not be a linear topology ([Jar81, 5.7 G]) and the inductive linear topology need not be locally convex. Furthermore, for the inductive locally convex topology the universal property is only mentioned for linear mappings S_j: if a vector space E carries the inductive locally convex topology defined by *linear* mappings $S_j \colon E_j \to E$ from any topological vector spaces E_j with locally convex topology into E, a linear map $T \in \mathrm{L}(E, F)$ is continuous if and only if all $T \circ S_j$ are continuous. But for the projective tensor product we need the inductive topology with respect to the *bilinear* mapping $\otimes\colon E \times F \to E \otimes F$.

- While [Tre76] takes the finest locally convex topology on $E \otimes F$ such that $\otimes$ is continuous, [Tre76, Proposition 43.4] only shows that there is at most one topology on $E \otimes F$ such that for any locally convex space G there is an isomorphism $\mathrm{L}^c(E \times F, G) \cong \mathrm{L}^c(E \otimes F; G)$ – but not that the projective tensor topology on $E \otimes F$ has this property.

We will therefore treat this topology as well as its universal property in some more detail. We will use the fact that the preimage of a linear or locally convex topology is a topology of the same type.

As it will turn out, the projective tensor product will not be suited to our applications; in fact, one reason for this is that multiplication of distributions by smooth functions is bounded (Lemma 13.3) while it is not jointly continuous. The other reason is that the bornological tensor product has better algebraic properties (it has a right adjoint) than the topological tensor product, so Lemma 12.2 works only in the bornological setting. Thus we will also have to consider the bornological tensor product, which we will introduce from the topological and the bornological point of view.

Lemma 9.1. *(i) Let X be a set and $\mathscr{A}$ a family of subsets of X. Then the family of all finite intersections of elements of $\mathscr{A}$, together with $\emptyset$ and X, is a basis of the coarsest topology on E such that all sets in $\mathscr{A}$ are open. $\mathscr{A}$ is a subbasis of this topology, which we say to be generated by $\mathscr{A}$.*

(ii) Let E be a set, E_j a topological space with basis $\mathscr{U}_j$, and $T_j\colon E \to E_j$ a map for each j in some index set J. Then

$$\mathscr{A} := \bigcup_j \{T_j^{-1}(U_j) : U_j \in \mathscr{U}_j\}$$

generates the coarsest topology $\mathscr{T}_i$ on E such that all T_j are continuous. It suffices to take for each $\mathscr{U}_j$ a subbasis instead of a basis.

(iii) A map S from any topological space F into $(E, \mathscr{T}_i)$ is continuous if and only if all $T_j \circ S$ are.

(iv) If each E_j is a topological vector space, E is a vector space, and the T_j are linear, $\mathscr{T}_i$ is a linear topology.

(v) If the topology of each E_j is locally convex, E is a vector space, and the T_j are linear, $\mathscr{T}_i$ is locally convex.

$\mathscr{T}_i$ is called the projective topology defined by the family $(T_j)_j$.

Proof. (i) is well known. (ii): [Eng89, Proposition 1.4.8]. (iii): [Eng89, Proposition 1.4.9]. (iv): [Jar81, 2.4 Proposition 1]. (v): [Jar81, 6.6 Proposition 2]. □

Lemma 9.2. *Let $(E_j)_j$ be a family of topological vector spaces resp. topological vector spaces with locally convex topologies, E a vector space, and $S_j\colon E_j \to E$ any map for each j.*

(i) There is a finest linear resp. locally convex topology $\mathscr{T}_l$ on E such that all S_j are continuous.

(ii) A linear map T from E into any topological vector space resp. into any vector space with a locally convex topology is continuous if and only if all $T \circ S_j$ are so.

Proof. (i) $\mathscr{T}_l$ is obtained as the projective topology defined by the identities from E into all linear resp. locally convex topologies $\mathscr{T}$ on E such that the S_j are continuous w.r.t. $\mathscr{T}$. (ii) Given $(F, \mathscr{T})$ with $\mathscr{T}$ a linear resp. locally convex topology, $T\colon (E, T^{-1}(\mathscr{T})) \to (F, \mathscr{T})$ is continuous, all S_j are continuous into the linear resp. locally convex topology $T^{-1}(\mathscr{T})$ because $S_j^{-1}(T^{-1}(\mathscr{T})) = (T \circ S_j)^{-1}(\mathscr{T})$ is a family of open sets by assumption, thus $\mathscr{T}_l$ is finer than $T^{-1}(\mathscr{T})$ and $T\colon (E, \mathscr{T}_l) \to (E, T^{-1}(\mathscr{T})) \to (F, \mathscr{T})$ is continuous. □

$\mathscr{T}_l$ is called the inductive topology defined by the $(S_j)_j$.

9.3 Final convex bornologies

Our main results will be of a bornological nature which is why we will also mention the construction of final bornologies. The standard reference [HN77] for bornologies only defines final convex bornologies with respect to linear maps. The construction can easily be generalized to arbitrary maps; we will fill in some details along the route which were omitted in [HN77].

The proof of the following is straightforward from the definitions.

Lemma 9.3. *Let X be a set and $\mathscr{B}_0$ a family of subset of X. Then $\mathscr{B}_0$ is a base for a bornology on X if and only if*

(i) $\mathscr{B}_0$ covers X and

(ii) every finite union of elements of $\mathscr{B}_0$ is contained in a member of $\mathscr{B}_0$.

If X is a vector space, $\mathscr{B}_0$ is a base for a vector bornology on X if and only if additionally it satisfies *(iii) every finite sum of elements of $\mathscr{B}_0$ is contained in a member of $\mathscr{B}_0$,*

(iv) every homothetic image (scalar multiple) of an element of $\mathscr{B}_0$ is contained in a member of $\mathscr{B}_0$, and

(v) every circled hull of an element of $\mathscr{B}_0$ is contained in a member of $\mathscr{B}_0$.

and it is a base for a convex bornology on X if and only if it satisfies (i)-(v) as well as

(iv) every convex hull of elements of $\mathscr{B}_0$ is contained in a member of $\mathscr{B}_0$.

Lemma 9.4. *Let X be a set and $\mathscr{A}$ be any family of subsets of X. Define the family $\mathscr{D} := \mathscr{A} \cup \{ \{x\} \mid x \in X \}$. Then:*

(i) A base of the bornology generated by $\mathscr{A}$ is given by all finite unions of elements of $\mathscr{D}$.

(ii) If X is a vector space a base of the vector bornology generated by $\mathscr{A}$ is given by all subsets of X which can be obtained from elements of $\mathscr{D}$ by any finite combination of finite sums, finite unions, homothetic images, and circled hulls.

(iii) If X is a vector space a base of the convex bornology generated by $\mathscr{A}$ is given by all subsets of X which can be obtained from elements of $\mathscr{D}$ by any finite combination of finite sums, finite unions, homothetic images, circled hulls, and convex hulls.

Proof. Let $\mathscr{B}_0$ be the family of all subsets of X which can be obtained from elements of $\mathscr{D}$ by the respective operations in (i),(ii), and (iii). By Lemma 9.3 $\mathscr{B}_0$ is a base for a bornology (resp. vector bornology resp. convex bornology) on X. Any bornology (resp. vector bornology resp. convex bornology) $\mathscr{C}$ on X containing $\mathscr{A}$ has to contain $\mathscr{D}$ and because it is closed under the same operations which are applied to elements of $\mathscr{D}$ in order to construct $\mathscr{B}_0$, $\mathscr{B}_0$ is finer than $\mathscr{C}$. This means that $\mathscr{B}_0$ is a base of the bornology (resp. vector bornology resp. convex bornology) generated by $\mathscr{A}$ (i.e., of the finest bornology containing $\mathscr{A}$). □

Proposition 9.5. *Let X be a set and $(X_i, \mathscr{B}_i)$ bornological sets with mappings $v_i \colon X_i \to X$. Let $\mathscr{B}_f$ be the bornology on X generated by the family $\mathscr{A} = \bigcup_{i \in I} v_i(\mathscr{B}_i)$. Then $\mathscr{B}_f$ is the finest bornology on X such that all v_i are bounded. A mapping v from $(X, \mathscr{B}_f)$ into a bornological set $(Y, \mathscr{C})$ is bounded if and only if all compositions $v \circ v_i$ are bounded.*

The same holds analogously for the vector (resp. convex) bornology on a vector space X generated by $\mathscr{A}$ and a linear map v into a vector (resp. convex) bornological space $(Y, \mathscr{C})$.

Proof. Any bornology (resp. vector bornology resp. convex bornology) $\mathscr{C}$ on X such that the v_i are bounded has to contain $\bigcup_i v_i(\mathscr{B}_i)$. By definition $\mathscr{B}_f$ is the finest bornology (resp. vector bornology resp. convex bornology) containing this set so $\mathscr{B}_f$ is the finest bornology such that all v_i are bounded.

If v is bounded the $v \circ v_i$ trivially are so. Conversely, assume that all the $v \circ v_i$ are bounded into $(Y, \mathscr{C})$. Let $\mathscr{C}_f$ be the bornology (resp. vector bornology resp. convex bornology) on Y generated by $\bigcup_i v \circ v_i(\mathscr{B}_i)$. Because $\mathscr{C}_f$ is finer than $\mathscr{C}$ it suffices to show that v is bounded into $\mathscr{C}_f$. As v is linear it maps the base of $\mathscr{B}_f$ given by Lemma 9.4 to a base of $\mathscr{C}_f$, which implies that v is bounded into $\mathscr{C}_f$. □

We call $\mathscr{B}_f$ the final bornology (resp. vector resp. convex bornology) defined by the v_i.

Given any locally convex topology $\mathscr{T}$ we denote by ${}^{\mathrm{b}}\mathscr{T}$ its von Neumann bornology. Conversely, ${}^{\mathrm{t}}\mathscr{B}$ denotes the locally convex topology associated with a convex bornology $\mathscr{B}$ ([HN77, 4:1]). Whenever we talk of boundedness of a mapping from or into a topological vector space with locally convex topology this is meant with respect to its von Neumann bornology.

9.4 Relations between bornology and topology

Lemma 9.6. *Let $(E, \mathscr{T})$ be a topological vector space with locally convex topology and $f\colon E \to F$ an arbitrary map into a vector space F. Denote by $\mathscr{T}_f$ the finest locally convex topology on F such that f is bounded and by $\mathscr{B}_f$ the finest convex bornology on F which makes f bounded. Then $\mathscr{T}_f = {}^{\mathrm{t}}\mathscr{B}_f$.*

Proof. We show that f is bounded into ${}^{\mathrm{t}}\mathscr{B}_f$, which implies that $\mathscr{T}_f$ is finer than ${}^{\mathrm{t}}\mathscr{B}_f$. Given a bounded set B in $(E, \mathscr{T})$ its image $f(B)$ is bounded in $\mathscr{B}_f$ by assumption. As ${}^{\mathrm{t}}\mathscr{B}_f$ is the finest locally convex topology such that the identity $(F, \mathscr{B}_f) \to (F, {}^{\mathrm{t}}\mathscr{B}_f)$ is bounded ([HN77, 4:1]), $f(B)$ is bounded in ${}^{\mathrm{t}}\mathscr{B}_f$.

Conversely, the identity $(F, \mathscr{B}_f) \to (F, \mathscr{T}_f)$ is bounded if and only if the map $f\colon (E, \mathscr{T}) \to (F, \mathscr{T}_f)$ is bounded, which is the case by construction, thus $\mathscr{B}_f$ is finer than ${}^{b}\mathscr{T}_f$. By definition of the locally convex topology associated with a convex bornology ([HN77, 4:1'2]) ${}^{\mathrm{t}}\mathscr{B}_f$ is finer than $\mathscr{T}_f$. □

By [HN77, 4:1'5 Definition (2) and Lemma (2)] we obtain

Corollary 9.7. *In the situation of Lemma 9.6, $\mathscr{T}_f$ is bornological.*

We recall that a bornological vector space is separated if and only if the subspace $\{0\}$ is Mackey-closed ([HN77, 2:11 Proposition (1)]).

Lemma 9.8. *Let* $(E, \mathscr{T})$ *be a vector space with locally convex topology. If* $(E, \mathscr{T})$ *is Hausdorff then* $(E, {}^{\mathrm{b}}\mathscr{T})$ *is separated.*

Proof. We have to show that if the constant sequence 0 converges Mackey to x then $x = 0$. By [HN77, 1:4'2 Proposition (1)] this means that there exists a circled bounded subset B of E such that $x \in \varepsilon \cdot B$ for all $\varepsilon > 0$. As each circled 0-neighborhood U in $\mathscr{T}$ absorbs B there is some $\lambda > 0$ such that $x \in \varepsilon \cdot B \subseteq \varepsilon\lambda \cdot U$ for all ε, which implies $x \in U$ and hence $x = 0$ because $\mathscr{T}$ is Hausdorff. □

Chapter 10

Tensor product of locally convex modules

10.1 Bornological and projective tensor product of locally convex spaces

We will need the following definitions of the tensor product of locally convex spaces as in [KM97, 5.7] and [Tre76, Definition 43.2].

Definition 10.1. The *bornological* resp. *projective tensor product* of two locally convex spaces E and F is the algebraic tensor product $E \otimes F$ of vector spaces equipped with the finest locally convex topology such that the canonical map $(x, y) \to x \otimes y$ from $E \times F$ into $E \otimes F$ is bounded resp. continuous. The resulting space is denoted by $E \otimes_\beta F$ resp. $E \otimes_\pi F$.

By Corollary 9.7 $E \otimes_\beta F$ is bornological. $E \otimes_\pi F$ and thus $E \otimes_\beta F$ are Hausdorff ([Jar81, 15.1 Proposition 3]). For any locally convex space G there are bornological isomorphisms of locally convex spaces

$$\mathrm{L}^b(E \otimes_\beta F, G) \cong \mathrm{L}^b(E, F; G) \cong \mathrm{L}^b(E, \mathrm{L}^b(F, G)) \tag{10.1}$$

where the first isomorphism is given by the transpose of the canonical bilinear map $\otimes\colon E \times F \to E \otimes_\beta F$ and the second one by the exponential law ([KM97, 5.7]). Consequently, a bilinear map $E \times F \to G$ is bounded if and only if the associated linear map $E \otimes_\beta F \to G$ is bounded.

For the projective tensor product however the algebraic isomorphism of vector spaces ([Tre76, Proposition 43.4])

$$\mathrm{L}^c(E \otimes_\pi F, G) \cong \mathrm{L}^c(E, F; G) \tag{10.2}$$

given by the transpose of the canonical map $\otimes\colon E \times F \to E \otimes_\pi F$ in general is *not* continuous and $\mathrm{L}^c(E, F; G)$ is not isomorphic to $\mathrm{L}^c(E, \mathrm{L}^c(F, G))$. $E \otimes_\pi F$ has the universal property for continuous bilinear mappings, i.e., a bilinear map $E \times F \to G$ is continuous if and only if the associated linear map $E \otimes_\pi F \to G$ is continuous.

10.2 Vector space structures on rings and modules

Let R be a nonzero ring, $\mathbb{K}$ a field, and $\iota\colon \mathbb{K} \to R$ any mapping. Define the action of $\mathbb{K}$ on R (scalar multiplication) by the map $\mathbb{K}\times R \to R$, $(\lambda, r) \mapsto \iota(\lambda)\cdot r$. It is easily seen that this action turns R into a vector space over $\mathbb{K}$ if and only if ι is a ring homomorphism. By [Bou70, I §9.1 Theorem 2] the subring $\iota(\mathbb{K})$ of R then is a field and ι is an isomorphism of $\mathbb{K}$ onto $\iota(\mathbb{K})$. Consequently, R is a unital algebra over $\mathbb{K}$ which is associative if $\mathbb{K}$ is commutative.

Definition 10.2. We call a locally convex space A over $\mathbb{K}$ with a bilinear multiplication map $A \times A \to A$ a *bounded algebra* resp. a *locally convex algebra* if this multiplication is bounded resp. continuous.

We will only be concerned with the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, thus by an *algebra* from now on we will always mean an associative and unital algebra over $\mathbb{K}$. Any such algebra A contains $\mathbb{K}$ injectively via the ring homomorphism $\iota\colon \mathbb{K} \to A$, $\lambda \mapsto \lambda \cdot 1$. This turns every module M over A into a module over the subring $\iota(\mathbb{K})$ of A and hence into a vector space over $\mathbb{K}$.

Definition 10.3. Let A be a bounded (resp. locally convex) algebra over $\mathbb{K}$. A left A-module M carrying a topology $\mathscr{T}$ which is locally convex with respect to the vector space structure induced by the subring $\mathbb{K} \subseteq A$ is called a *bounded left module* resp. a *locally convex left module* if module multiplication $A \times M \to M$ is bounded (resp. continuous) with respect to $\mathscr{T}$. The definition for right modules is analogous.

Remark 10.4. It is equivalent to define a bounded (resp. locally convex) left module M over A as a topological vector space M with locally convex topology together with a $\mathbb{Z}$-bilinear bounded (resp. continuous) mapping $A \times M \to M$, $(a, m) \mapsto a \cdot m$ such that $a \cdot (b \cdot m) = (ab) \cdot m$ and $1 \cdot m = m$.

10.3 Bornological and projective tensor product of locally convex modules

We will from now on assume that the algebra A contains $\mathbb{K}$ in its center. This is necessary for the tensor product $M \otimes_A N$ of modules over A and the quotient $M \otimes_{\mathbb{K}} N/J_0$ with J_0 as below to be a vector space.

Let A be a bounded algebra over $\mathbb{K}$, M a right bounded A-module, and N a left bounded N-module. Define J_0 as the sub-$\mathbb{Z}$-module of $M \otimes_{\mathbb{K}} N$ generated by all elements of the form $ma \otimes n - m \otimes an$ with $a \in A$, $m \in M$, and $n \in N$. The vector spaces $M \otimes_A N$ and $(M \otimes_{\mathbb{K}} N)/J_0$ are isomorphic [Cap96, Theorem I.5.1], but in order to obtain a Hausdorff space we need to take the quotient with respect to the closure J of J_0 in $M \otimes_\beta N$. Because the vector space operations are continuous J is a sub-$\mathbb{Z}$-module of $M \otimes_\beta N$ and we define the $\mathbb{Z}$-module quotient

$$M \otimes_A^\beta N := (M \otimes_\beta N)/J$$

which is a vector space because $\mathbb{K}$ is contained in the center of A. We endow it with the quotient topology, which is locally convex and Hausdorff. Denoting by q the quotient map we have a canonical bilinear map

$$\otimes_A^\beta := q \circ \otimes \colon M \times N \to M \otimes_A^\beta N.$$

Similarly, if A, M, N are taken to be locally convex instead of bounded, we denote the resulting space by $M \otimes_A^\pi N$ with corresponding map $\otimes_A^\pi$:

$$M \otimes_A^\pi N := (M \otimes_\pi N)/J$$
$$\otimes_A^\pi := q \circ \otimes \colon M \times N \to M \otimes_A^\pi N.$$

Definition 10.5. We call $M \otimes_A^\beta N$ resp. $M \otimes_A^\pi N$ the *bornological* resp. *projective tensor product* of M and N over A.

By [Jar81, 13.5 Prop. 1 (b)] $M \otimes_A^\beta N$ is bornological. These spaces have the following universal properties.

Proposition 10.6. *Let M be a right module over an algebra A, N a left module over A, and E any locally convex space. If M, N, and A are locally convex then:*

(i) Given a continuous $\mathbb{K}$-linear mapping $g\colon M \otimes_A^\pi N \to E$, the mapping $f := g \circ \otimes_A^\pi$ is continuous, $\mathbb{K}$-bilinear and A-balanced.

(ii) Given a continuous A-balanced $\mathbb{K}$-bilinear mapping $f\colon M \times N \to E$ there exists a unique continuous $\mathbb{K}$-linear mapping $g\colon M \otimes_A^\pi N \to E$ such that $f = g \circ \otimes_A^\pi$.

This gives a vector space isomorphism

$$\mathrm{L}^{A,c}(M, N; E) \cong \mathrm{L}^c(M \otimes_A^\pi N, E). \tag{10.3}$$

If M, N, and A are bounded then:

(iii) Given any bounded $\mathbb{K}$-linear mapping $g\colon M \otimes_A^\pi N \to E$, the mapping $f := g \circ \otimes_A^\pi$ is bounded, $\mathbb{K}$-bilinear, and A-balanced.

(iv) Given a bounded A-balanced $\mathbb{K}$-bilinear mapping $f\colon M \times N \to E$ there exists a unique bounded $\mathbb{K}$-linear mapping $g\colon M \otimes_A^\pi N \to E$ such that $f = g \circ \otimes_A^\pi$.

This gives a vector space isomorphism

$$\mathrm{L}^{A,b}(M,N;E) \cong \mathrm{L}^b(M \otimes_A^\pi N, E). \tag{10.4}$$

Proof. (i) and (iii) are trivial.

For (ii) and (iv) we obtain from (10.2) resp. (10.1) a unique mapping $\tilde{f}$ in $\mathrm{L}^c(M \otimes_\pi N, G)$ resp. $\mathrm{L}^b(M \otimes_\beta N, G)$ such that $f = \tilde{f} \circ \otimes$. Noting that $M \otimes_\beta N$ is bornological, $\tilde{f}$ is continuous in both cases and thus vanishes on J, whence there exists a unique linear mapping g from $M \otimes_A^\pi N$ resp. $M \otimes_A^\beta N$ into E such that $f = g \circ q \circ \otimes$ which equals $g \circ \otimes_A^\pi$ resp. $f = g \circ \otimes_A^\beta$, where q is the projection onto the quotient. Clearly g is continuous resp. bounded by definition.

It is furthermore easily verified that the correspondence $f \leftrightsquigarrow g$ is a vector space isomorphism. □

In order to show that the isomorphism (10.4) in Proposition 10.6 is bornological we need the following Lemma.

Lemma 10.7. *Let E be a bornological locally convex space, N a closed subspace of E, and F an arbitrary locally convex space. Then there is a bornological isomorphism*

$$\mathrm{L}^b(E/N, F) \cong \{T \in \mathrm{L}^b(E,F) : N \subseteq \ker T\}$$

where the latter space is equipped with the subspace topology.

Proof. Denote by $p\colon E \to E/N$ the canonical projection. As to the algebraic part, for $\tilde{T} \in \mathrm{L}^b(E/N, F)$ the map $T := \tilde{T} \circ p$ is in $\mathrm{L}^b(E,F)$ and vanishes on N; conversely, given such T there exists a unique linear map $\tilde{T}$ such that $T = \tilde{T} \circ p$. Now T is continuous (equivalently bounded) if and only if $\tilde{T}$ is ([Tre76, Proposition 4.6]). The correspondences $T \leftrightsquigarrow \tilde{T}$ are inverse to each other and linear because the transpose p^* of p is linear.

For boundedness of p^* let $\tilde{B} \subseteq L^b(E/N, F)$ be bounded and set $B := p^*(\tilde{B})$. Let $D \subseteq E$ be bounded and V a 0-neighborhood in F. Then $\tilde{D} := p(D) \subseteq E/N$ is bounded so there exists $\lambda > 0$ such that

$$\tilde{B} \subseteq \lambda \cdot \{\, \tilde{T} \in \mathrm{L}^b(E/N, F) : \tilde{T}(\tilde{D}) \subseteq V \,\}$$

and thus

$$\begin{aligned} B &\subseteq \lambda \cdot \{\, p^*(\tilde{T}) : \tilde{T} \in \mathrm{L}^b(E/N,F), \tilde{T}(\tilde{D}) \subseteq V \,\} \\ &\subseteq \lambda \cdot \{\, T \in \mathrm{L}^b(E/N,F) : N \subseteq \ker T, T(D) \subseteq V \,\}. \end{aligned}$$

which implies that B is bounded. Conversely, let $B \subseteq \{T \in \mathrm{L}^b(E,F) : N \subseteq \ker T\}$ be bounded and set $\tilde{B} := (p^*)^{-1}(B) \subseteq \mathrm{L}^b(E/N,F)$. Let $\tilde{D} \subseteq E/N$

be bounded and V a 0-neighborhood in F. Because the images of bounded subsets of E form a basis of the bornology of E/N ([HN77, 2:7]) there exists a bounded set $D \subseteq E$ such that $\tilde{D} \subseteq p(D)$. By assumption there is $\lambda > 0$ such that

$$B \subseteq \lambda \cdot \{T \in \mathrm{L}^b(E, F) : N \subseteq \ker T, T(D) \subseteq V\}$$

and thus

$$\begin{aligned} \tilde{B} &\subseteq \lambda \cdot \{\, (p^*)^{-1}(T) : T \in \mathrm{L}^b(E, F), N \subseteq \ker T, T(D) \subseteq V \,\} \\ &\subseteq \lambda \cdot \{\, \tilde{T} \in \mathrm{L}^b(E/N, F) : \tilde{T}(\tilde{D}) \subseteq V \,\}. \end{aligned}$$ □

Corollary 10.8. *Let M be a right bounded module and N a left bounded module over a bounded algebra A, and let E be any locally convex space. Then the isomorphism $\mathrm{L}^{A,b}(M, N; E) \cong \mathrm{L}^b(M \otimes_A^\beta N, E)$ of Proposition 10.6 is a bornological isomorphism. These spaces furthermore are bornologically isomorphic to $\mathrm{L}_A^b(M, \mathrm{L}^b(N, E))$.*

Proof. The first isomorphism of (10.1) restricts to a bornological isomorphism

$$\mathrm{L}^{A,b}(M, N; E) \cong \{\, T \in \mathrm{L}^b(M \otimes_\beta N, E) : J \subseteq \ker T \,\}.$$

Together with Lemma 10.7 this gives the first result. For the second claim we note that $\mathrm{L}^b(N, E)$ has a canonical right A-module structure with respect to which the exponential law for spaces of linear bounded maps [KM97, 5.7] gives an isomorphism $\mathrm{L}^{A,b}(M, N; E) \cong \mathrm{L}_A^b(M, \mathrm{L}^b(N, E))$ which obviously is bounded in both directions. □

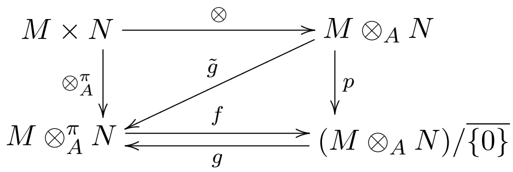
The tensor product can also be constructed in a different way. Remember that as $\mathbb{K}$ is in the center of A, $E \otimes_A F$ has a canonical vector space structure [Bou70, II §3.6 Remark (2)]). In the following Lemma, the separated vector bornology associated with a vector bornology is defined as the quotient bornology with respect to the Mackey closure $\overline{\{0\}}^{\mathrm{b}}$ of $\{0\}$ ([HN77, 2:12 Definition (2)]).

Lemma 10.9. *Let M and N be A-modules. Then*

(i) The Hausdorff space associated with the algebraic tensor product $M \otimes_A N$ endowed with the finest locally convex topology such that the canonical map $\otimes \colon M \times N \to M \otimes_A N$ is continuous is homeomorphic to $M \otimes_A^\pi N$.

(ii) The separated bornological vector space associated with the algebraic tensor product $M \otimes_A N$ endowed with the finest convex bornology such that the canonical map $\otimes$ is bounded is bornologically isomorphic to $M \otimes_A^\beta N$.

Proof. (i) Let $p\colon M \otimes_A N \to (M \otimes_A N)/\overline{\{0\}}$ denote the canonical projection onto the quotient space, which is Hausdorff.



Let f be the continuous linear map induced by the continuous bilinear map $p \circ \otimes$. $\otimes_A^\pi$ induces a continuous linear map $\tilde{g}$, which is continuous (and thus its kernel contains the closure of $\{0\}$); hence there exists a linear continuous map g with $g \circ p = \tilde{g}$. In order to see that f and g are inverse to each other, we note that as p is surjective and the images of $\otimes$ resp. $\otimes_A^\pi$ generate $M \otimes_A N$ resp. $M \otimes_A^\pi N$ it suffices to have the identities

$$f \circ g \circ p \circ \otimes = f \circ \otimes_A^\pi = p \circ \otimes$$
$$g \circ f \circ \otimes_A^\pi = g \circ p \circ \otimes = \otimes_A^\pi$$

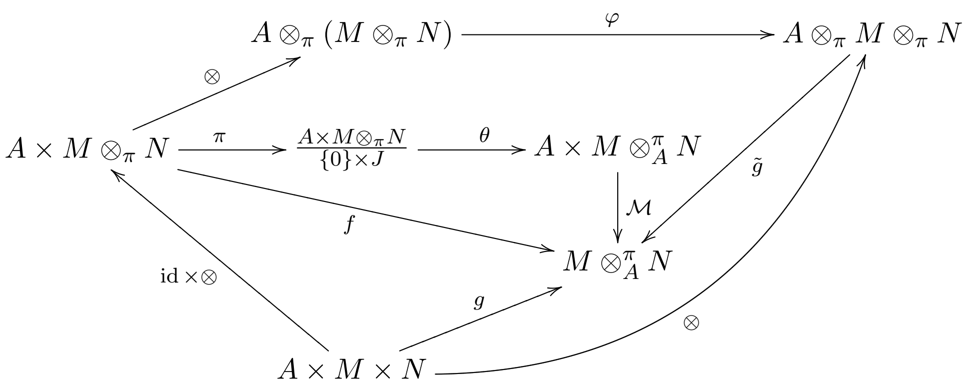
thus we are done.

(ii) Replace $\overline{\{0\}}$ by $\overline{\{0\}}^{\mathrm{b}}$, $\otimes_A^\pi$ by $\otimes_A^\beta$ and "continuous" by "bounded" in (i). Apply Lemma 9.8 to see that $M \otimes_A^\beta N$ is a separated bornological space, and use [HN77, 2:12 Proposition (2)] for obtaining g. $\square$

If A is commutative $M \otimes_A^\beta N$ resp. $M \otimes_A^\pi N$ has a canonical structure of an A-module with the action given by $a \cdot (m \otimes_A^\pi n) := (ma) \otimes_A^\pi n$.

Proposition 10.10. *If A is commutative then $M \otimes_A^\beta N$ resp. $M \otimes_A^\pi N$ is a bounded resp. locally convex A-module.*

Proof. For the bounded case see [KM97, 5.21]. For the continuous case, following the proof of [Cap96, Proposition II.2.2] we have the following diagram.



Multiplication on $M \otimes_A^\pi N$ is defined by the map $\mathcal{M}(a, m \otimes_A^\pi n) := am \otimes_A^\pi n$. It is easily seen that there is a isomorphism of locally convex spaces $\theta\colon (A \times M \otimes_\pi N)/(\{0\} \times J) \to A \times M \otimes_A^\pi N$. The map $g(a, m, n) := am \otimes_A^\pi n$ is continuous and trilinear, thus it induces a continuous map $\tilde{g}$ such that $\tilde{g}(a \otimes m \otimes n) = am \otimes_A^\pi n$. Define f as the continuous map $\tilde{g} \circ \varphi \circ \otimes$, where φ is the canonical isomorphism as in the diagram. It is easily verified that $f = \mathcal{M} \circ \theta \circ \pi$ on the image of $A \times M \times N$ under the continuous map $\mathrm{id} \times \otimes$, which generates $A \times M \otimes_\pi N$, thus $f = \mathcal{M} \otimes \theta \circ \pi$ on the whole space. As π is the quotient map, $\mathcal{M}$ is continuous because f is. □

Corollary 10.11. *If A is commutative then the isomorphism* (10.3) *resp.* (10.4) *induces, for any bounded resp. locally convex A-modules M, N, and P, a bornological isomorphism*

$$\mathrm{L}_A^b(M, N; P) \cong \mathrm{L}_A^b(M \otimes_A^\beta N, P)$$

and an algebraic isomorphism

$$\mathrm{L}_A^c(M, N; P) \cong \mathrm{L}_A^c(M \otimes_A^\pi N, P).$$

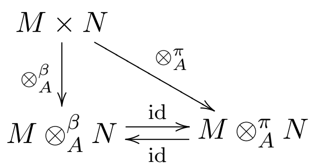
Proposition 10.12. *Let $f\colon M \to M'$ and $g\colon N \to N'$ be bounded (resp. continuous) A-linear maps between bounded (resp. locally convex) A-modules. Then $f \otimes g$ is bounded (resp. continuous).*

Proof. As the mapping $(m, n) \mapsto f(m) \otimes g(n)$ from $M \times N$ into $M' \otimes_A N'$ is A-bilinear and bounded (resp. continuous) the corresponding A-linear map $f \otimes g$ from $M \otimes_A^\beta N$ to $M' \otimes_A^\beta N'$ (resp. from $M \otimes_A^\pi N$ to $M' \otimes_A^\pi N'$) such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ is bounded (resp. continuous). □

The following is an analogue of [KM97, Proposition 5.8].

Lemma 10.13. *If every bounded bilinear mapping on $M \times N$ into an arbitrary locally convex space is continuous then $M \otimes_A^\pi N = M \otimes_A^\beta N$.*

Proof. By construction, the topology of $M \otimes_A^\beta N$ is finer than the topology of $M \otimes_A^\pi N$: the identity $M \otimes_A^\beta N \to M \otimes_A^\pi N$ is continuous if and only if it is bounded (as $M \otimes_A^\beta N$ is bornological), which is the case if and only if $\mathrm{id} \circ \otimes_A^\beta = \otimes_A^\pi$ is bounded, but this map is even continuous.



Conversely, the identity $M \otimes_A^\pi N \to M \otimes_A^\beta N$ is continuous if and only if $\mathrm{id} \circ \otimes_A^\pi = \otimes_A^\beta$ is continuous, which is the case by assumption because it is bounded and bilinear. $\square$

By [KM97, Proposition 5.8] the assumption of Lemma 10.13 is satisfied if M and N are metrizable, or if M and N are bornological and every separately continuous bilinear mapping on $E \times F$ is continuous.

Chapter 11

Topology on section spaces

We will now define a suitable topology on the space of sections of a finite dimensional vector bundle.

All manifolds M are supposed to be finite dimensional, second countable, and Hausdorff. For any open subset Ω of $\mathbb{R}^n$ or M a sequence of sets K_i such that

1. each K_i is compact,
2. K_i is contained in the interior of K_{i+1}, and
3. $\Omega = \bigcup_{i=1}^{\infty} K_i$

is called a *compact exhaustion* of Ω.

Let $\Omega \subseteq \mathbb{R}^n$ be open and $(\mathbb{E}, \| \ \|)$ a Banach space. The space $C^\infty(\Omega, \mathbb{E})$ of all smooth functions from Ω to $\mathbb{E}$ has the usual Fréchet structure ([Tre76, Chapter 40]). Defining the seminorms $\mathfrak{p}_{K,k}$ (for $K \subseteq \Omega$ compact and $k \in \mathbb{N}_0$) on $C^\infty(\Omega, \mathbb{E})$ by

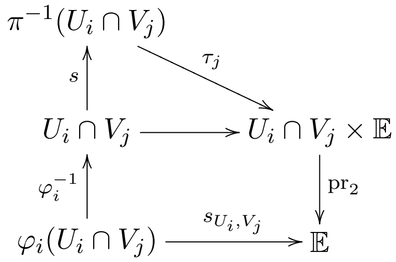
$$\mathfrak{p}_{K,k} := \max_{|\alpha| \leq k, x \in K} \|\partial^\alpha f(x)\|$$

the topology of $C^\infty(\Omega, \mathbb{E})$ has as basis of continuous seminorms the family $\{ \mathfrak{p}_{K_n,k} \mid n \in \mathbb{N}, k \in \mathbb{N}_0 \}$ where $(K_n)_n$ is a fixed compact exhaustion of Ω. This topology evidently does not depend on the choice of the compact exhaustion.

Let M be an n-dimensional manifold with atlas $\{(U_i, \varphi_i)\}_i$ and $\pi \colon E \to M$ a vector bundle whose typical fiber is the m-dimensional Banach space $\mathbb{E}$. Let $\{(V_j, \tau_j)\}_j$ be a trivializing covering of E (for the terminology used here see [Lan99, Chapter III]). Denote by $\Gamma(E)$ the space of sections of E. For any i and j a section $s \in \Gamma(E)$ has local representation

$$s_{U_i,V_j} := \mathrm{pr}_2 \circ \tau_j \circ s|_{U_i \cap V_j} \circ (\varphi_i|_{U_i \cap V_j})^{-1} \in C^\infty(\varphi_i(U_i \cap V_j), \mathbb{E}).$$

This is illustrated by the following diagram.



$\Gamma(E)$ is endowed with the projective topology $\mathscr{T}_E$ with respect to the linear mappings

$$\Gamma(E) \ni s \mapsto s_{U_i,V_j} \in C^\infty(\varphi_i(U_i \cap V_j), \mathbb{E})$$

(for all i, j) which turns it into a complete locally convex topological vector space by [Sch71, II 5.3]. For a description by seminorms we set $\mathfrak{p}_{U_i,V_j,K,k}(s) := \mathfrak{p}_{\varphi_i(K),k}(s_{U_i,V_j})$ for $s \in \Gamma(E)$. The topology $\mathscr{T}_E$ has as basis of continuous seminorms the family $\mathfrak{P}_E$ given by all $\mathfrak{p}_{U_i,V_j,K_n,k}$ for $k \in \mathbb{N}_0$, $(K_n)_n$ a compact exhaustion of $U_i \cap V_j$, and all i and j. As for each $s \in \Gamma(E) \setminus \{0\}$ there is some $\mathfrak{p} \in \mathfrak{P}_E$ such that $\mathfrak{p}(s) > 0$, $\mathscr{T}_E$ is Hausdorff by [Jar81, Section 2.7 Proposition 1].

Proposition 11.1. *$\mathscr{T}_E$ is independent of the atlas, the trivializing covering, and the compact exhaustions.*

Proof. Let M have atlases $\{(U_i, \varphi_i)\}_i$ and $\{(\tilde{U}_k, \tilde{\varphi}_k)\}_k$ and let E have trivializing coverings $\{(V_j, \tau_j)\}_j$ and $\{(\tilde{V}_l, \tilde{\tau}_l)\}_l$. This gives rise to topologies $\mathscr{T}_E$ resp. $\tilde{\mathscr{T}}_E$ on $\Gamma(E)$. For continuity of the identity map $(\Gamma(E), \mathscr{T}_E) \to (\Gamma(E), \tilde{\mathscr{T}}_E)$ it suffices to show that for all k, l and compact exhaustions $(\tilde{K}_m)_m$ of $\tilde{U}_k \cap \tilde{V}_l$, and all m, p there is a continuous seminorm $\mathfrak{p}$ of $(\Gamma(E), \mathscr{T}_E)$ such that

$$\mathfrak{p}_{\tilde{U}_k,\tilde{V}_l,\tilde{K}_m,p}(s) \leq \mathfrak{p}(s). \tag{11.1}$$

First, we show that we can assume that $\tilde{K}_m$ is contained in some $U_i \cap V_j$. As the open sets $U_i \cap V_j$ form an open cover of M we can write $\tilde{K}_m$ as the disjoint union of finitely many $\tilde{K}_m^{a,b} \subset\subset U_{i(a)} \cap V_{j(b)} \cap \tilde{U}_k \cap \tilde{V}_l$. Assuming that (11.1) holds in this case there are continuous seminorms $\mathfrak{p}_{a,b}$ of $\mathscr{T}_E$ such that

$$\mathfrak{p}_{\tilde{U}_k,\tilde{V}_l,\tilde{K}_m^{a,b},p}(s) \leq \mathfrak{p}_{a,b}(s)$$

for all a, b. We take the maximum over all a, b on both sides and obtain $\mathfrak{p}_{\tilde{U}_k,\tilde{V}_l,\tilde{K}_m,p}$ on the left side and a continuous seminorm $\mathfrak{p}$ on the right side. Thus we may assume that $K := \tilde{K}_m \subset\subset U_i \cap V_j \cap \tilde{U}_k \cap \tilde{V}_l$ for some i, j, k, l.

The left side of (11.1) is then given by

$$\sup_{\substack{|\alpha|\le p\\ x\in\tilde\varphi_k(K)}} \left\|\partial^\alpha s_{\tilde U_k,\tilde V_l}(x)\right\|.$$

For $x\in\tilde\varphi_k(K)$ we then write

$$\begin{aligned} s_{\tilde U_k,\tilde V_l}(x) &= \mathrm{pr}_2\circ\tilde\tau_l\circ\tau_j^{-1}(\tilde\varphi_k^{-1}(x), s_{U_i,V_j}\circ\varphi_i\circ\tilde\varphi_k^{-1}(x))\\ &= (\tilde\tau_l\circ\tau_j^{-1})_{\tilde\varphi_k^{-1}(x)}(s_{U_i,V_j}\circ\varphi_i\circ\tilde\varphi_k^{-1}(x)) \end{aligned}$$

where $(\tilde\tau_l\circ\tau_j^{-1})_{\varphi_k^{-1}(x)}$ is a transition map and the function $x\mapsto(\tilde\tau_l\circ\tau_j^{-1})_{\tilde\varphi_k^{-1}(x)}$ is smooth from $\tilde\varphi_k(\tilde U_k)$ to $L(\mathbb{E},\mathbb{E})$. By the product rule we obtain for $\partial^\alpha s_{\tilde U_k,\tilde V_l}(x)$ terms of the form

$$\partial^\beta[x\mapsto(\tilde\tau_l\circ\tau_j^{-1})_{\tilde\varphi_k^{-1}(x)}]\cdot\partial^\gamma[x\mapsto s_{U_i,V_j}(\varphi_i\circ\tilde\varphi_k^{-1}(x))]$$

for some multi-indices β,γ. Taking the supremum over $x\in\tilde\varphi_k(K)$, the first factor gives a constant and the second factor gives a sum of terms of the form

$$\sup_{x\in\varphi_i(K)}\left\|\partial^{\gamma'}s_{U_i,V_j}(x)\right\|\le\mathfrak{p}_{U_i,V_j,K,|\gamma'|}(s)$$

for some multi-indices γ'. Altogether, these terms give a continuous seminorm of $\mathscr{T}_E$, so the identity map from $(\Gamma(E),\tilde{\mathscr{T}}_E)\to(\Gamma(E),\mathscr{T}_E)$ is continuous. By symmetry we have a homeomorphism. □

As the trivializing covering of E and the atlas of M can be assumed to be countable ([BC70, 1.4.8]) $\mathscr{T}_E$ is determined by a countable family of seminorms whence $(\Gamma(E),\mathscr{T}_E)$ as well as its closed subspace $\Gamma_{c,L}(E)$ (the subspace of sections with support in the compact set $L\subseteq M$) with the subspace topology are Fréchet spaces.

In order to turn $\Gamma_c(E)$ (the space of all sections with compact support) into a complete topological space we have to endow it with the strict inductive limit topology of a suitable sequence of Fréchet subspaces, which by [Sch71, II 6.6] is complete. As M is σ-compact we obtain an (LF)-space $\Gamma_c(E)=\varinjlim\Gamma_{c,L}(E)$, where L ranges through a compact exhaustion of M.

For the particular case $C^\infty(M)$ we abbreviate $\mathfrak{p}_{i,K,k}:=\mathfrak{p}_{U_i,U_i,K,k}$. Then we obtain a basis of continuous seminorms

$$\mathfrak{P}_M:=\{\,\mathfrak{p}_{i,K_n^i,k}\mid k\in\mathbb{N}_0,k\in\mathbb{N},i\,\}$$

where $(K_n^i)_n$ is a fixed compact exhaustion of $\varphi_i(U_i)$.

We now state simple lemmata about continuity of bilinear maps with respect to seminorms. For any seminorm p and $\lambda>0$ set $p_{\le\lambda}:=p^{-1}([0,\lambda])$ and $p_{<\lambda}:=p^{-1}([0,\lambda)]$.

Lemma 11.2. *Let E, F, and G be topological vector spaces with locally convex topology. A bilinear map $f\colon E \times F \to G$ is continuous if and only if for each continuous seminorm r on G there are continuous seminorms p on E and q on F such that for all $x \in E$ and $y \in F$,*

$$r(f(x, y)) \leq p(x)q(x). \tag{11.2}$$

Proof. Suppose f is continuous. By [Tre76, Proposition 7.2]) there are barrels and 0-neighborhoods $U \subseteq E$ and $V \subseteq F$ such that $f(U \times V) \subseteq W := r_{\leq 1}$. U and V are the closed unit balls of continuous seminorms p on E and q on F [Tre76, Proposition 7.5]. For any $\varepsilon > 0$, $x \in E$, and $y \in F$ we see that $(p(x) + \varepsilon)^{-1}x \in U$ and $(q(y) + \varepsilon)^{-1}y \in V$, thus

$$f(\frac{x}{p(x) + \varepsilon}, \frac{y}{q(y) + \varepsilon}) = \frac{f(x, y)}{(p(x) + \varepsilon)(q(y) + \varepsilon)} \in W$$

and consequently $f(x, y) \in (p(x) + \varepsilon)(q(y) + \varepsilon) \cdot W \ \forall \varepsilon > 0$, i.e., $r(f(x, y)) \leq (p(x) + \varepsilon) \cdot (q(y) + \varepsilon) \ \forall \varepsilon > 0$. Because this holds for all $\varepsilon > 0$ it implies (11.2).

For the converse, by [Jar81, Section 5.1 Proposition 3] we only have to check continuity at $(0, 0)$. Let W be a neighborhood of 0 in G. Then there is a continuous seminorm r on G such that $r_{\leq 1} \subseteq W$. By assumption there are continuous seminorms p on E and q on F such that $r(f(x, y)) \leq p(x)q(x)$ for all $(x, y) \in E \times F$, thus for $x \in U := p_{\leq 1}$ and $y \in V := q_{\leq 1}$ we have $f(x, y) \in r_{\leq 1}$, i.e., $f(U \times V) \subseteq W$. As $U \times V$ is a 0-neighborhood in $E \times F$, f is continuous. □

In the following, the notions of a base of continuous seminorms and a family of seminorms defining the topology is as in [Tre76, Chapter 7].

Corollary 11.3. *Let E, F, G be topological vector spaces with locally convex topology. Let $\mathcal{P}_E$ resp. $\mathcal{P}_F$ be bases of continuous seminorms on E resp. F and $\mathcal{S}_G$ a family of seminorms on G defining the topology of G. Then a bilinear map $f\colon E \times F \to G$ is continuous if and only if for each $r \in \mathcal{S}_G$ there are seminorms $p \in \mathcal{P}_E$ and $q \in \mathcal{P}_F$ and a constant $C > 0$ such that $r(f(x, y)) \leq Cp(x)q(x)$ for all $x \in E$, $y \in F$.*

Lemma 11.4. *(i) $C^\infty(M)$ is a locally convex unital commutative associative algebra.*

(ii) For any vector bundle E, the space of sections $\Gamma(E)$ is a Hausdorff locally convex module over $C^\infty(M)$.

Proof. We will only check continuity of the respective multiplication maps, the rest being immediately clear from the definitions. Let $\{(U_i, \varphi_i)\}_i$ be an atlas of M and $\{(U_i, \tau_i)\}_i$ a trivializing covering of $\Gamma(E)$ – by Proposition 11.1 we can always intersect the domains of the atlas and the trivializing covering in order to have them in this form. By the product rule for differentiation we obtain

$$\mathfrak{p}_{i,K,k}(fg) \leq C\mathfrak{p}_{i,K,k}(f) \cdot \mathfrak{p}_{i,K,k}(g) \text{ and}$$
$$\mathfrak{p}_{i,K,k}(fs) \leq C\mathfrak{p}_{i,K,k}(f) \cdot \mathfrak{p}_{U_i,U_i,K,k}(s)$$

for all $K \subset\subset U_i$, $k \in \mathbb{N}_0$, $f, g \in C^\infty(M)$, $s \in \Gamma(E)$, and some constant $C > 0$. □

Lemma 11.5. *Given a trivial vector bundle E and a basis $\{b_1, \dots, b_n\}$ of $\Gamma(E)$, the elements of the corresponding dual basis $\{b_1^*, \dots, b_n^*\}$ are continuous, i.e., elements of $\mathrm{L}^c_{C^\infty(M)}(\Gamma(E), C^\infty(M))$.*

Proof. Let $\tau\colon E \to M \times \mathbb{R}^n$ be trivializing. For the basis $\alpha_i(x) := \tau^{-1}(x, e_i)$ where $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$ the result is clear, as the dual basis is then given by $\alpha_i^*(s)(x) = \mathrm{pr}_i \circ \mathrm{pr}_2 \circ \tau \circ s$. For an arbitrary basis $\{b_1, \dots, b_n\}$ we know that $b_i^* = a_i^j \alpha_j^*$ for some $a_i^j \in C^\infty(M)$. As for $f \in C^\infty(M)$ the map $s \mapsto (f\alpha_j^*)(s) = f \cdot \alpha_j(s)$ is the composition of α_j and multiplication with f, both continuous, b_i^* is the sum of continuous maps and thus continuous. □

We recall the following basic facts about products and direct sums of topological vector spaces. Let $(M_i)_i$ be a family of topological vector spaces. The product $\prod_i M_i$ carries the projective topology w.r.t. the canonical projections π_i and the external direct sum $\bigoplus_i M_i$ the inductive linear topology with respect to the canonical injections, which makes them topological vector spaces. If all M_i are locally convex A-modules $\prod_i M_i$ is a locally convex A-module: denoting the multiplication maps by $m\colon A \times \prod_i M_i \to \prod_i M_i$ resp. $m_i\colon A \times M_i \to M_i$, m is continuous because $\pi_i \circ m = m_i \circ (\mathrm{id} \times \pi_i)$ is continuous for each i. For finitely many factors $\bigoplus_i M_i = \prod_i M_i$ topologically.

We will now establish some preliminaries we will need for the isomorphism $\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.

Proposition 11.6. *Given vector bundles $E_1, \dots, E_n$ the canonical isomorphism of $C^\infty(M)$-modules*

$$\Gamma(\bigoplus_{j=1\dots n} E_j) \cong \bigoplus_{j=1\dots n} \Gamma(E_j)$$

is a homeomorphism.

Proof. For each $x \in M$ let $\iota_j\colon E_{jx} \to \bigoplus_{i=1\dots n} E_{ix}$ denote the canonical injection of the fiber E_{jx} and $\pi_j\colon \bigoplus_{i=1\dots n} E_{ix} \to E_{jx}$ the canonical projection onto it. Define injections resp. projections

$$\tilde{\iota}_j\colon \Gamma(E_j) \to \Gamma(\bigoplus_{i=1\dots n} E_i), \quad (\tilde{\iota}_j s_j)(x) := \iota_j(s_j(x)) \quad \text{for } s_j \in \Gamma(E_j),$$

$$\tilde{\pi}_j\colon \Gamma(\bigoplus_{i=1\dots n} E_i) \to \Gamma(E_j), \quad (\tilde{\pi}_j s)(x) := \pi_j(s(x)) \quad \text{for } s \in \Gamma(\bigoplus_{i=1\dots n} E_i).$$

We have to verify that the images of $\tilde{\iota}_j$ and $\tilde{\pi}_j$ are indeed smooth sections. Let $\{U_l, \varphi_l\}_l$ be an atlas of M and $\{(V^j_{k_j}, \tau^j_{k_j})\}_{k_j}$ trivializing coverings of E_j, then $\bigoplus_{i=1\dots n} E_i$ has trivializing covering

$$\{(\bigcap_{j=1\dots n} V^j_{k_j}, \sigma_{k_1,\dots,k_n})\}_{k_1,\dots,k_n}$$

where $(\sigma_{k_1,\dots,k_n})_x(t) := (x, (\mathrm{pr}_2\, \tau^1_{k_1}\pi_1 t, \dots, \mathrm{pr}_2\, \tau^n_{k_n}\pi_n t))$ for $t \in \bigoplus_{j=1\dots n} E_{jx}$ and $x \in \bigcap_{j=1\dots n} V^j_{k_j}$. First, let $s_j \in \Gamma(E_j)$; then on each chart domain $U_l \cap V^1_{k_1} \cap \dots \cap V^n_{k_n}$, $\mathrm{pr}_2 \circ \sigma_{k_1,\dots,k_n} \circ \tilde{\iota}_j(s_j) \circ \varphi_l^{-1}$ is smooth because its only nonzero component is $\mathrm{pr}_2 \circ \tau^j_{k_j} \circ s_j \circ \varphi_l^{-1}$ which is smooth by assumption. Conversely, let $s \in \Gamma(\bigoplus_{i=1\dots n} E_i)$. Then on each chart domain as above $\mathrm{pr}_2 \circ \tau^j_{k_j} \circ \tilde{\pi}_j(s) \circ \varphi_l^{-1} = \mathrm{pr}_2 \circ \tau^j_{k_j} \circ \pi_j \circ s \circ \varphi_l^{-1} = \mathrm{pr}_j \circ \mathrm{pr}_2 \circ \sigma_{k_1,\dots,k_n} \circ s \circ \varphi_l^{-1}$ is smooth. Finally, $\tilde{\pi}_k \circ \tilde{\iota}_j = \mathrm{id}$ for $k = j$ and 0 otherwise; as $\sum_j \tilde{\iota}_j \circ \tilde{\pi}_j(s) = s$, $\Gamma(\bigoplus_{j=1\dots n} E_j)$ is a direct product for the family of $C^\infty(M)$-modules $(\Gamma(E_j))_j$ ([Bly77, Theorem 6.7]) and algebraically isomorphic to $\bigoplus_{j=1\dots n} \Gamma(E_j)$. The isomorphism $\psi\colon \Gamma(\bigoplus_{j=1\dots n} E_j) \to \bigoplus_{j=1\dots n} \Gamma(E_j)$ is given by

$$\psi(s) = (\tilde{\pi}_1(s), \dots, \tilde{\pi}_n(s)) \text{ and}$$
$$\psi^{-1}(s_1, \dots, s_n) = \tilde{\iota}_1(s_1) + \dots + \tilde{\iota}_n(s_n).$$

Continuity of $\tilde{\pi}_j$ and $\tilde{\iota}_j$ is easily seen from the respective seminorms, which implies continuity of ψ and ψ^{-1}. □

Proposition 11.7. *For vector bundles $E_1, \dots, E_n$ and $F_1, \dots, F_m$ over M we have a canonical vector bundle isomorphism*

$$(\bigoplus_{i=1\dots n} E_i) \otimes (\bigoplus_{j=1\dots m} F_j) \cong \bigoplus_{\substack{i=1\dots n\\ j=1\dots m}} (E_i \otimes F_j)$$

Proof. Evidently the fiberwise defined map

$$(v_1, \dots, v_n) \otimes (w_1, \dots, w_m) \mapsto (v_1 \otimes w_1, \dots, v_n \otimes w_m)$$

(where $v_i \in E_{ix}$ and $w_j \in F_{jx}$ for all i, j and fixed x) is a strong vector bundle isomorphism. Its inverse is induced by the maps

$$e_i \otimes f_j \mapsto \iota_i e_i \otimes \iota_j f_j \qquad (e_i \in E_{ix}, f_j \in F_{jx})$$

for all i, j, where ι_i,ι_j are the canonical injections $E_{ix} \to \bigoplus_i E_{ix}$ and $F_{jx} \to \bigoplus_j F_{jx}$. □

Lemma 11.8. *For isomorphic vector bundles $E \cong F$ the canonical $C^\infty(M)$-module isomorphism $\Gamma(E) \cong \Gamma(F)$ is a homeomorphism.*

Proof. If (f, f_0) is the vector bundle isomorphism from E to F the isomorphism $\Gamma(E) \to \Gamma(F)$ is given by $s \mapsto f \circ s \circ f_0^{-1}$. It is readily verified by using the respective seminorms that this assignment and its inverse are continuous. □

Lemma 11.9. *Let A be a locally convex algebra, M_i $(i = 1, \ldots, n)$ locally convex right A-modules, and N_j $(j = 1, \ldots, m)$ locally convex left A-modules. Then the canonical vector space isomorphism*

$$(\bigoplus_{i=1\ldots n} M_i) \otimes (\bigoplus_{j=1\ldots m} N_j) \cong \bigoplus_{\substack{i=1\ldots n\\ j=1\ldots m}} (M_i \otimes N_j)$$

induces isomorphisms of locally convex spaces

$$(\bigoplus_{i=1\ldots n} M_i) \otimes_\pi (\bigoplus_{j=1\ldots m} N_j) \cong \bigoplus_{\substack{i=1\ldots n\\ j=1\ldots m}} (M_i \otimes_\pi N_j)$$

$$(\bigoplus_{i=1\ldots n} M_i) \otimes_A^\pi (\bigoplus_{j=1\ldots m} N_j) \cong \bigoplus_{\substack{i=1\ldots n\\ j=1\ldots m}} (M_i \otimes_A^\pi N_j).$$

If A is commutative these are isomorphisms of A-modules.

Proof. By [Bou70, II §3.7 Proposition 7] the mapping

$$g\colon (\bigoplus_{i=1\ldots n} M_i) \otimes (\bigoplus_{j=1\ldots m} N_j) \to \bigoplus_{\substack{i=1\ldots n\\ j=1\ldots m}} (M_i \otimes N_j)$$

$$(m_i)_i \otimes (n_j)_j \mapsto (m_i \otimes n_j)_{i,j}$$

is a vector space isomorphism. Its inverse h is induced by the maps $h_{ij} := \iota_i \otimes \iota_j$, where $\iota_i\colon M_i \to \bigoplus M_i$ and $\iota_j\colon N_j \to \bigoplus N_j$ are the canonical injections. This means that h is given by $\sum_{ij} h_{ij} \circ \mathrm{pr}_{ij}$ where pr_{ij} is the canonical projection $\bigoplus_{ij}(M_i \otimes N_j) \to M_i \otimes N_j$.

Define J_0 as the sub-$\mathbb{Z}$-module of $(\bigoplus M_i) \otimes (\bigoplus N_h)$ generated by all elements of the form $(m_i)_i a \otimes (n_j)_j - (m_i)_i \otimes a(n_j)_j$, and J_{ij} as the sub-$\mathbb{Z}$-module of $M_i \otimes N_j$ generated by all elements of the form $m_i a \otimes n_j - m_i \otimes a n_j$. As $\mathbb{K}$ is in the center of A these are vector subspaces. By [Bou70, II §1.6] there is a canonical isomorphism of vector spaces

$$f\colon \bigoplus_{i,j} \frac{M_i \otimes N_j}{\overline{J_{ij}}} \to \frac{\bigoplus_{i,j}(M_i \otimes N_j)}{\bigoplus_{i,j} \overline{J_{ij}}}$$

induced by the maps $f_{ij}(m_i \otimes n_j + \overline{J_{ij}}) := \iota(m_i) \otimes \iota(n_j) + \bigoplus_{i,j} \overline{J_{ij}}$. Thus we obtain the following commutative diagram.

$$
\begin{array}{ccccc}
(\bigoplus_i M_i) \otimes (\bigoplus_j N_j) & \underset{h}{\overset{g}{\rightleftarrows}} & \bigoplus_{i,j}(M_i \otimes N_j) & \xrightarrow{(p_{ij})_{i,j}} & \\
\downarrow q & & \downarrow r & & \searrow \\
\dfrac{(\bigoplus_i M_i) \otimes (\bigoplus_j N_j)}{\overline{J_0}} & \overset{\lambda}{\dashrightarrow} & \dfrac{\bigoplus_{i,j}(M_i \otimes N_j)}{\bigoplus_{i,j} \overline{J_{ij}}} & \overset{f}{\longleftarrow} & \bigoplus_{i,j} \dfrac{M_i \otimes N_j}{\overline{J_{ij}}}
\end{array}
$$

Here q, r, and p_{ij} are the projections onto the respective quotient.

It is now easily seen that $g(J_0) = \bigoplus_{i,j} J_{ij}$, and if g and h are continuous, $g(\overline{J_0}) = \bigoplus_{i,j} \overline{J_{ij}}$, which immediately implies that there exists a vector space isomorphism λ as in the diagram. Now endow the tensor products with the projective tensor product topology. The claims then follow if we show f, f^{-1}, g and h to be continuous.

First, g is induced by the $C^\infty(M)$-bilinear map

$$
\begin{aligned}
\tilde{g}\colon (\bigoplus_i M_i) \times (\bigoplus_j N_j) &\to \bigoplus_{i,j}(M_i \otimes N_j) \\
((m_i)_i, (n_j)_j) &\mapsto (m_i \otimes n_j)_{i,j}
\end{aligned}
$$

and g is continuous if and only if $\tilde{g}$ is. Because the target space has only finitely many summands continuity can be tested by composition with the projections π_{ij} onto $M_i \otimes N_j$. As $\pi_{ij} \circ \tilde{g} = \otimes \circ (\pi_i \times \pi_j)$ is continuous g is continuous.

Second, by definition of the inductive topology h is continuous if and only if all h_{ij} are, which is the case because they are the tensor product of continuous mappings. Similarly, f is continuous because $f \circ \iota_{ij} \circ p_{ij} = r \circ (\iota_i \circ \iota_j)$ is continuous, where $\iota_{ij} : (M_i \otimes_\pi N_j)/\overline{J_{ij}} \to \bigoplus_{i,j}(M_i \otimes_p i N_j)/\overline{J_{ij}}$ is the canonical inclusion.

Finally, f^{-1} is continuous if and only if $f^{-1} \circ r = (p_{ij})_{i,j}$ is, which is the case because all p_{ij} are continuous and we can test continuity into the finite direct sum by composition with the projections on each factor. □

Note that for infinitely many summands the previous lemma is false, in general ([Jar81, 15.5, 1. Example]).

Chapter 12

Tensor product of section spaces

Theorem 12.1. *For any vector bundles E and F on M the canonical $C^\infty(M)$-module isomorphism $\Gamma(E)\otimes_{C^\infty(M)}\Gamma(F) \cong \Gamma(E\otimes F)$ induces a homeomorphism $\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) \cong \Gamma(E \otimes F)$.*

Proof. Suppose first that E and F are trivial, then there are finite bases $\{\alpha_i\}_i$ and $\{\beta_j\}_j$ of $\Gamma(E)$ and $\Gamma(F)$, respectively. Clearly $E \otimes F$ then also is trivial and $\Gamma(E \otimes F)$ has a finite basis $\{\gamma_{ij}\}_{i,j}$. Explicitly these bases can be given as follows: suppose we have trivializing maps $\tau\colon E \to M \times \mathbb{E}$, $\sigma\colon F \to M \times \mathbb{F}$ and $\mu\colon E \otimes F \to M \times (\mathbb{E} \otimes \mathbb{F})$, with $\mu_x(v \otimes w) = (x, \mathrm{pr}_2 \circ \tau_x(v) \otimes \mathrm{pr}_2 \circ \sigma_x(w))$. Let $\{e_i\}_i$, $\{f_j\}_j$ be bases of $\mathbb{E}$ resp. $\mathbb{F}$, which gives a basis $\{e_i \otimes f_j\}_{i,j}$ of $E \otimes F$. Then we set

$$\begin{aligned}
\alpha_i(x) &:= \tau^{-1}(x, e_i),\\
\beta_j(x) &:= \sigma^{-1}(x, f_j), \text{ and}\\
\gamma_{ij}(x) &:= \mu^{-1}(x, e_i \otimes f_j) = \alpha_i(x) \otimes \beta_j(x).
\end{aligned}$$

Now $\{(\alpha_i, \beta_j)\}_{i,j}$ is a basis of $\Gamma(E) \times \Gamma(F)$. There is a unique $C^\infty(M)$-bilinear mapping

$$\tilde{g}\colon \Gamma(E) \times \Gamma(F) \to \Gamma(E \otimes F)$$

such that $\tilde{g}(\alpha_i, \beta_j) = \gamma_{ij}$ $\forall i, j$. Writing

$$\tilde{g} = \sum_{i,j} m \circ (\mathrm{id} \times m(\cdot, \gamma_{ij})) \circ (\alpha_i^* \times \beta_j^*)$$

where $m\colon C^\infty(M) \times \Gamma(E\otimes F) \to \Gamma(E\otimes F)$ is module multiplication on $\Gamma(E\otimes F)$ and α_i^*, β_j^* are elements of the bases dual to $\{\alpha_i\}_i$ and $\{\beta_j\}_j$ (which are continuous by Lemma 11.5) one sees that $\tilde{g}$ is continuous. Note that $g(t \otimes s)(x) = t(x) \otimes s(x)$ for $t \in \Gamma(E)$, $s \in \Gamma(E)$, and $x \in M$. By Corollary 10.11 $\tilde{g}$ induces a unique continuous $C^\infty(M)$-linear mapping $g\colon \Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) \to$

$\Gamma(E \otimes F)$ such that $\tilde{g} = g \circ \otimes^{\pi}_{C^\infty(M)}$.

$$\begin{array}{ccc} \Gamma(E) \times \Gamma(F) & \xrightarrow{\tilde{g}} & \Gamma(E \otimes F) \\ \downarrow{\scriptstyle \otimes^{\pi}_{C^\infty(M)}} & \overset{g}{\nearrow} \ \underset{h}{\swarrow} & \\ \Gamma(E) \otimes^{\pi}_{C^\infty(M)} \Gamma(F) & & \end{array}$$

For the inverse we define $h\colon \Gamma(E \otimes F) \to \Gamma(E) \otimes^{\pi}_{C^\infty(M)} \Gamma(F)$ by $h(\gamma_{ij}) = \alpha_i \otimes^{\pi}_{C^\infty(M)} \beta_j$, i.e., $h(s) = \sum_{i,j} \gamma^*_{ij}(s) \alpha_i \otimes^{\pi}_{C^\infty(M)} \beta_j$ for $s \in \Gamma(E \otimes F)$, which is continuous and $C^\infty(M)$-linear. Now it suffices to note that g and h are inverse to each other:

$$\begin{aligned} h(g(t \otimes^{\pi}_{C^\infty(M)} u)) &= h(\tilde{g}(t^i\alpha_i, u^j\beta_j)) = h(t^i u^j \gamma_{ij}) = t^i u^j \alpha_i \otimes^{\pi}_{C^\infty(M)} \beta_j \\ &= t^i\alpha_i \otimes^{\pi}_{C^\infty(M)} u^j\beta_j = t \otimes^{\pi}_{C^\infty(M)} u \text{ and} \\ g(h(s)) &= g(s^{ij}\alpha_i \otimes^{\pi}_{C^\infty(M)} \beta_j) = s^{ij}\tilde{g}(\alpha_i, \beta_j) = s^{ij}\gamma_{ij} = s. \end{aligned}$$

Thus for trivial bundles we have established the $C^\infty(M)$-module isomorphism and homeomorphism $\varphi_{E,F} := h$,

$$\varphi_{E,F}\colon \Gamma(E \otimes F) \to \Gamma(E) \otimes^{\pi}_{C^\infty(M)} \Gamma(F).$$

Now suppose that E and F are arbitrary non-trivial vector bundles. Then by [GHV72, 2.23] there exist vector bundles E' and F' over M such that $E \oplus E'$ and $F \oplus F'$ are trivial, giving an isomorphism $\varphi := \varphi_{E \oplus E', F \oplus F'}$ as above:

$$\Gamma((E \oplus E') \otimes (F \oplus F')) \cong \Gamma(E \oplus E') \otimes^{\pi}_{C^\infty(M)} \Gamma(F \oplus F'). \tag{12.1}$$

We now distribute the direct sums on both sides and write down all isomorphisms involved. First, by Proposition 11.6 we have an isomorphism of $C^\infty(M)$-modules and homeomorphism $\psi_{E,E'} : \Gamma(E \oplus E') \to \Gamma(E) \oplus \Gamma(E')$ given by

$$\begin{aligned} \psi_{E,E'}(s) &= [x \mapsto (\mathrm{pr}_1 \circ s(x), \mathrm{pr}_2 \circ s(x))] = (\mathrm{pr}_1 \circ s, \mathrm{pr}_2 \circ s) \\ \psi^{-1}_{E,E'}(s_1, s_2) &= [x \mapsto (s_1(x), s_2(x))]. \end{aligned}$$

As both $\psi := \psi_{E,E'} \otimes^{\pi}_{C^\infty(M)} \psi_{F,F'}$ and its inverse $\psi^{-1}_{E,E'} \otimes^{\pi}_{C^\infty(M)} \psi^{-1}_{F,F'}$ are continuous (Proposition 10.12) we obtain an isomorphism of $C^\infty(M)$-modules

$$\psi : \Gamma(E \oplus E') \otimes^{\pi}_{C^\infty(M)} \Gamma(F \oplus F') \to (\Gamma(E) \oplus \Gamma(E')) \otimes^{\pi}_{C^\infty(M)} (\Gamma(F) \oplus \Gamma(F'))$$

which also is a homeomorphism. For the left hand side of (12.1) we use the vector bundle isomorphism of Proposition 11.7 given on each fiber by

$$\kappa\colon (e, e') \otimes (f, f') \mapsto (e \otimes f, e \otimes f', e' \otimes f, e' \otimes f')$$

which by Lemma 11.8 gives a $C^\infty(M)$-module isomorphism and homeomorphism $\lambda\colon s \mapsto \kappa \circ s$.

Let ρ be the isomorphism from Lemma 11.9 (denoted by g in its proof). In our case it is explicitly given by the $C^\infty(M)$-linear mapping

$$\rho\colon (s,s') \otimes^\pi_{C^\infty(M)} (t,t') \mapsto (s \otimes^\pi_{C^\infty(M)} t, s \otimes^\pi_{C^\infty(M)} t', s' \otimes^\pi_{C^\infty(M)} t, s' \otimes^\pi_{C^\infty(M)} t').$$

Its inverse ρ^{-1} is induced by the following maps, all having image in the space $(\Gamma(E) \oplus \Gamma(E')) \otimes^\pi_{C^\infty(M)} (\Gamma(F) \oplus \Gamma(F'))$:

$$\begin{aligned}
\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) \ni s_1 \otimes^\pi_{C^\infty(M)} t_1 &\mapsto (s_1, 0) \otimes^\pi_{C^\infty(M)} (t_1, 0),\\
\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F') \ni s_2 \otimes^\pi_{C^\infty(M)} t'_1 &\mapsto (s_2, 0) \otimes^\pi_{C^\infty(M)} (0, t'_1),\\
\Gamma(E') \otimes^\pi_{C^\infty(M)} \Gamma(F) \ni s'_1 \otimes^\pi_{C^\infty(M)} t_2 &\mapsto (0, s'_1) \otimes^\pi_{C^\infty(M)} (t_2, 0), \text{ and}\\
\Gamma(E') \otimes^\pi_{C^\infty(M)} \Gamma(F') \ni s'_2 \otimes^\pi_{C^\infty(M)} t'_2 &\mapsto (0, s'_2) \otimes^\pi_{C^\infty(M)} (0, t'_2).
\end{aligned}$$

This means that $\rho^{-1}(s_1 \otimes^\pi_{C^\infty(M)} t_1, s_2 \otimes^\pi_{C^\infty(M)} t'_1, s'_1 \otimes^\pi_{C^\infty(M)} t_2, s'_2 \otimes^\pi_{C^\infty(M)} t'_2)$ is given by

$$\begin{aligned}
(s_1, 0) \otimes^\pi_{C^\infty(M)} (t_1, 0) &+ (s_2, 0) \otimes^\pi_{C^\infty(M)} (0, t'_1)\\
&+ (0, s'_1) \otimes^\pi_{C^\infty(M)} (t_2, 0) + (0, s'_2) \otimes^\pi_{C^\infty(M)} (0, t'_2).
\end{aligned}$$

The isomorphism $\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) \cong \Gamma(E \otimes F)$ we are looking for will now be obtained as a component of $f := \lambda \circ \varphi^{-1} \circ \psi^{-1} \circ \rho^{-1}$. Note that f is an isomorphism of $C^\infty(M)$-modules and a homeomorphism by what was said so far. The composition f is depicted in the following diagram.

$$\begin{array}{c}
(\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F)) \oplus (\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F'))\oplus\\
(\Gamma(E') \otimes^\pi_{C^\infty(M)} \Gamma(F)) \oplus (\Gamma(E') \otimes^\pi_{C^\infty(M)} \Gamma(F'))\\
\Big\downarrow \rho^{-1}\\
(\Gamma(E) \oplus \Gamma(E')) \otimes^\pi_{C^\infty(M)} (\Gamma(F) \oplus \Gamma(F'))\\
\Big\downarrow \psi^{-1}\\
\Gamma(E \oplus E') \otimes^\pi_{C^\infty(M)} \Gamma(F \oplus F')\\
\Big\downarrow \varphi^{-1}\\
\Gamma((E \oplus E') \otimes (F \oplus F'))\\
\Big\downarrow \lambda\\
\Gamma(E \otimes F) \oplus \Gamma(E \otimes F') \oplus \Gamma(E' \otimes F) \oplus \Gamma(E' \otimes F')
\end{array}$$

From this we obtain

$$\begin{aligned}
(\lambda \circ \varphi^{-1} \circ \psi^{-1} \circ \rho^{-1})\,&(s_1 \otimes^\pi_{C^\infty(M)} t_1, s_2 \otimes^\pi_{C^\infty(M)} t_1', \\
&\qquad s_1' \otimes^\pi_{C^\infty(M)} t_2, s_2' \otimes^\pi_{C^\infty(M)} t_2') \\
&= (\lambda \circ \varphi^{-1} \circ \psi^{-1})\,((s_1,0) \otimes^\pi_{C^\infty(M)} (t_1,0) + (s_2,0) \otimes^\pi_{C^\infty(M)} (0,t_1') \\
&\qquad + (0,s_1') \otimes^\pi_{C^\infty(M)} (t_2,0) + (0,s_2') \otimes^\pi_{C^\infty(M)} (0,t_2')) \\
&= (\lambda \circ \varphi^{-1})\,([x \mapsto (s_1(x),0)] \otimes^\pi_{C^\infty(M)} [x \mapsto (t_1(x),0)] \\
&\qquad + [x \mapsto (s_2(x),0)] \otimes^\pi_{C^\infty(M)} [x \mapsto (0,t_1'(x))] \\
&\qquad + [x \mapsto (0,s_1'(x))] \otimes^\pi_{C^\infty(M)} [x \mapsto (t_2(x),0)] \\
&\qquad + [x \mapsto (0,s_2'(x))] \otimes^\pi_{C^\infty(M)} [x \mapsto (0,t_2'(x))]) \\
&= \lambda\,([x \mapsto (s_1(x),0) \otimes (t_1(x),0)] + [x \mapsto (s_2(x),0) \otimes (0,t_1'(x))] \\
&\qquad + [x \mapsto (0,s_1'(x)) \otimes (t_2(x),0)] + [x \mapsto (0,s_2'(x)) \otimes (0,t_2'(x))]) \\
&= ([x \mapsto s_1(x) \otimes t_1(x)], [x \mapsto s_2(x) \otimes t_1'(x)], \\
&\qquad [x \mapsto s_1'(x) \otimes t_2(x)], [x \mapsto s_2'(x) \otimes t_2'(x)]).
\end{aligned}$$

This means we can write $f = (f_1, f_2, f_3, f_4)$ with $f_1 \colon \Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) \to \Gamma(E \otimes F)$ and analogously for the other components. Because f is bijective all f_i have to be ([Bou70, Chapter II §1.6 Corollary 1 to Proposition 7]). As f is a homeomorphism it follows immediately that all f_i are homeomorphisms. □

We now see that $\Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F)$ is a Fréchet space.

The above isomorphism induces a homeomorphism for spaces of sections supported in a fixed compact set $K \subset\subset M$. By Lemma 10.13 we have

$$\Gamma(E \otimes F) \cong \Gamma(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) = \Gamma(E) \otimes^\beta_{C^\infty(M)} \Gamma(F)$$

and

$$\Gamma_{c,K}(E \otimes F) \cong \Gamma_{c,K}(E) \otimes^\pi_{C^\infty(M)} \Gamma(F) = \Gamma_{c,K}(E) \otimes^\beta_{C^\infty(M)} \Gamma(F).$$

Recall that $\Gamma_c(E)$ is the strict inductive limit of the spaces $\Gamma_{c,K}(E)$ for K running through a compact exhaustion of M.

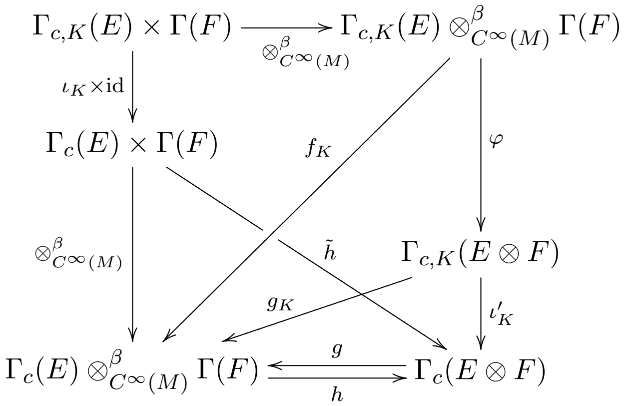
Lemma 12.2. *Let a locally convex space E be the strict inductive limit of a sequence of subspaces E_n with embeddings $\iota_n \colon E_n \to E$ and let F and G be arbitrary locally convex spaces. Then a bilinear mapping $f \colon E \times F \to G$ is bounded if and only if $f \circ (\iota_n \times \mathrm{id}) \colon E_n \times F \to G$ is bounded for all n.*

Proof. Necessity is clear. For sufficiency, let $B \subseteq E \times F$ be bounded. As the canonical projections π_1 onto E and π_2 onto F are bounded $B_1 := \pi_1(B)$ and $B_2 := \pi_2(B)$ are bounded B is contained in the bounded set $B_1 \times B_2$. Because B_1 is bounded it is contained in some E_n, thus by assumption $f(B) \subseteq f(B_1 \times B_2) = f(\iota_n(B_1) \times B_2) = f \circ (\iota_n \times \mathrm{id})(B_1 \times B_2)$ is bounded. □

Theorem 12.3. *There is a a bornological $C^\infty(M)$-module isomorphism*

$$\Gamma_c(E) \otimes^\beta_{C^\infty(M)} \Gamma(E) \cong \Gamma_c(E \otimes F).$$

Proof. Consider the following diagram.



Here $\iota_K : \Gamma_{c,K}(E) \to \Gamma_c(E)$ and $\iota'_K : \Gamma_{c,K}(E \otimes F) \to \Gamma_c(E \otimes F)$ are the inclusion maps. For $K \subset\subset M$ the $C^\infty(M)$-bilinear bounded map $\otimes^\beta_{C^\infty(M)} \circ (\iota_K \times \mathrm{id})$ by Corollary 10.11 induces a bounded (and thus continuous) linear map f_K. Because φ is a homeomorphism there is a corresponding linear continuous map $g_K := \varphi^{-1} \circ f_K$. Because $\Gamma_c(E \otimes F)$ is the strict inductive limit of the spaces $\Gamma_{c,K}(E \otimes F)$ and for different K the maps g_K are compatible with each other there is a unique continuous linear map g such that $g \circ \iota'_K = g_K$.

By Lemma 12.2 the bilinear map $\tilde{h}$ defined by $\tilde{h}(s,t)(x) := s(x) \otimes t(x)$ is bounded because all $\tilde{h} \circ (\iota_K \times \mathrm{id}) = \iota'_K \circ \varphi \circ \otimes^\beta_{C^\infty(M)}$ are bounded, thus a unique bounded linear map h completing the diagram exists. It is easily verified that g and h are inverse to each other, which completes the proof. ☐

Remark 12.4. Similarly one can obtain

$$\Gamma(E) \otimes_{C^\infty(M)} \Gamma_c(F) \cong \Gamma_c(E) \otimes_{C^\infty(M)} \Gamma_c(F) \cong \Gamma_c(E \otimes F).$$

Note that Lemma 12.2 and thus Theorem 12.3 only work in the bornological setting but not in the topological one.

Chapter 13

Distributions on manifolds

In this chapter we will finally define the space of tensor distributions and give bornologically isomorphic representations. For additional information we refer to [GKOS01, Section 3.1].

Definition 13.1. The space of distributions on an orientable manifold M is defined as

$$\mathcal{D}'(M) := [\Omega^n_c(M)]'$$

and the space of tensor distributions of rank (r,s) on M as

$$\mathcal{D}'^r_s(M) := [\Gamma_c(M, \mathrm{T}^s_r(M) \otimes \Lambda^n \mathrm{T}^* M)]'.$$

Here $\Gamma_c(M, \mathrm{T}^s_r(M) \otimes \Lambda^n \mathrm{T}^* M)$ (where $\Lambda^n \mathrm{T}^* M$ is the n-fold exterior bundle) and $\Omega^n_c(M)$ (the space of compactly supported n-forms on M) are equipped with the (LF)-topology discussed in Chapter 11. Because this topology is bornological these are exactly the bounded linear functionals. $\mathcal{D}'(M)$ and $\mathcal{D}'^r_s(M)$ carry the strong dual topology ([Tre76, Chapter 19]).

13.1 Isomorphic representations of distributions

Theorem 13.2. *We have the following bornological $C^\infty(M)$-module isomorphisms*

$$\mathcal{D}'^r_s(M) \cong (\mathcal{T}^s_r(M) \otimes^\beta_{C^\infty(M)} \Omega^n_c(M))' \tag{13.1}$$

$$\cong \mathrm{L}^b_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)) \tag{13.2}$$

$$\cong \mathcal{T}^r_s(M) \otimes^\beta_{C^\infty(M)} \mathcal{D}'(M). \tag{13.3}$$

Proof. (13.1) is clear from the bornological isomorphism of $C^\infty(M)$-modules

$$\Gamma_c(M, \mathrm{T}^s_r(M) \otimes \Lambda^n \mathrm{T}^* M) \cong \mathcal{T}^s_r(M) \otimes^\beta_{C^\infty(M)} \Omega^n_c(M)$$

given by Theorem 12.3. As both spaces are bornological it is also an isomorphism of topological vector spaces, thus the duals are homeomorphic ([Tre76, Chapter 23]).

(13.1) $\leftrightsquigarrow$ (13.2) is clear from Corollary 10.8.

For (13.2) $\leftrightsquigarrow$ (13.3) consider the map

$$\theta_{\mathcal{T}^s_r(M)} \colon \mathcal{T}^s_r(M)^* \otimes_{C^\infty(M)} \mathcal{D}'(M) \to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M))$$

induced by the bilinear map

$$\begin{aligned} \mathcal{T}^s_r(M)^* \times \mathcal{D}'(M) &\to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)) \\ (u^*, v) &\mapsto [u \mapsto u^*(u) \cdot v]. \end{aligned} \tag{13.4}$$

Because $\mathcal{T}^s_r(M)$ is finitely generated and projective it is a direct summand of a free finitely generated $C^\infty(M)$-module F with injection ι and projection π. By [GHV72, 2.23] there exists a vector bundle $C \to M$ such that $\mathrm{T}^s_r(M) \oplus C$ is trivial, thus we can take $F = \mathcal{T}^s_r(M) \oplus \Gamma(C)$. Note that duals of F and $\mathcal{T}^s_r(M)$ here are always meant with respect to the $C^\infty(M)$-module structure. By standard methods (cf. the proof of [Bly77, Theorem 14.10]) one obtains the commutative diagram

$$\begin{array}{ccccc} F^* \otimes_{C^\infty(M)} \mathcal{D}'(M) & \xrightarrow{\iota^* \otimes \mathrm{id}} & \mathcal{T}^s_r(M)^* \otimes_{C^\infty(M)} \mathcal{D}'(M) & \xrightarrow{\pi^* \otimes \mathrm{id}} & F^* \otimes_{C^\infty(M)} \mathcal{D}'(M) \\ \downarrow{\scriptstyle \theta_F} & & \downarrow{\scriptstyle \theta_{\mathcal{T}^s_r(M)}} & & \downarrow{\scriptstyle \theta_F} \\ \mathrm{L}_{C^\infty(M)}(F, \mathcal{D}'(M)) & \xrightarrow[\iota^{\mathrm{t}}]{} & \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)) & \xrightarrow[\pi^{\mathrm{t}}]{} & \mathrm{L}_{C^\infty(M)}(F, \mathcal{D}'(M)) \end{array}$$

with mappings

$$\begin{aligned} \iota^* &\colon F^* \to \mathcal{T}^s_r(M)^*,\ u^* \mapsto u^* \circ \iota \\ \pi^* &\colon \mathcal{T}^s_r(M)^* \to F^*,\ u^* \mapsto u^* \circ \pi \\ \iota^{\mathrm{t}} &\colon \mathrm{L}_{C^\infty(M)}(F, \mathcal{D}'(M)) \to \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)),\ \ell \mapsto \ell \circ \iota \\ \pi^{\mathrm{t}} &\colon \mathrm{L}_{C^\infty(M)}(\mathcal{T}^s_r(M), \mathcal{D}'(M)) \to \mathrm{L}_{C^\infty(M)}(F, \mathcal{D}'(M)),\ \ell \mapsto \ell \circ \pi \end{aligned}$$

where $\iota^* \otimes \mathrm{id}$ and ι^{t} are surjective while $\pi^* \otimes \mathrm{id}$ and π^{t} are injective.

The inverse of θ_F can be given explicitly because F is free and finitely generated. Let $\{b_1, \dots, b_n\}$ be a basis of F and $\{b_1^*, \dots, b_n^*\}$ the corresponding dual basis of F^*. For $\ell \in \mathrm{L}_{C^\infty(M)}(F, \mathcal{D}'(M))$ we have

$$\theta_F^{-1}(\ell) = \sum_{i=1,\dots,n} b_i^* \otimes \ell(b_i) \in F^* \otimes_{C^\infty(M)} \mathcal{D}'(M).$$

This implies that also $\theta_{\mathcal{T}^s_r(M)}$ is an isomorphism, its inverse is given by the composition $(\iota^* \otimes \mathrm{id}) \circ \theta_F^{-1} \circ \pi^{\mathrm{t}}$.

As (13.4) is bounded from $\mathcal{T}_r^s(M)' \times \mathcal{D}'(M)$ into $\mathrm{L}^b_{C^\infty(M)}(\mathcal{T}_r^s(M), \mathcal{D}'(M))$ the induced map $\theta_{\mathcal{T}_r^s(M)}\colon \mathcal{T}_r^s(M)' \otimes^\beta_{C^\infty(M)} \mathcal{D}'(M) \to \mathrm{L}^b_{C^\infty(M)}(\mathcal{T}_r^s(M), \mathcal{D}'(M))$ is bounded and linear. Because ι and π obviously are continuous all maps in the following diagram are bounded.

$$\begin{array}{ccccc} F' \otimes^\beta_{C^\infty(M)} \mathcal{D}'(M) & \xrightarrow{\iota^* \otimes \mathrm{id}} & \mathcal{T}_r^s(M)' \otimes^\beta_{C^\infty(M)} \mathcal{D}'(M) & \xrightarrow{\pi^* \otimes \mathrm{id}} & F' \otimes^\beta_{C^\infty(M)} \mathcal{D}'(M) \\ \downarrow{\scriptstyle \theta_F} & & \downarrow{\scriptstyle \theta_{\mathcal{T}_r^s(M)}} & & \downarrow{\scriptstyle \theta_F} \\ \mathrm{L}^b_{C^\infty(M)}(F, \mathcal{D}'(M)) & \xrightarrow[\iota^{\mathrm{t}}]{} & \mathrm{L}^b_{C^\infty(M)}(\mathcal{T}_r^s(M), \mathcal{D}'(M)) & \xrightarrow[\pi^{\mathrm{t}}]{} & \mathrm{L}^b_{C^\infty(M)}(F, \mathcal{D}'(M)) \end{array}$$

Concluding, $\theta_F^{-1}\colon \ell \mapsto \sum_i b_i^* \otimes^\beta_{C^\infty(M)} \ell(b_i)$ is bounded into $F' \otimes^\beta_{C^\infty(M)} \mathcal{D}'(M)$ whence $\theta^{-1}_{\mathcal{T}_r^s(M)} = (\iota^* \otimes \mathrm{id}) \circ \theta_F^{-1} \circ \pi^{\mathrm{t}}$ also is bounded. □

Lemma 13.3. *Multiplication* $C^\infty(M) \times \mathcal{D}'(M) \to \mathcal{D}'(M)$, $(f, T) \mapsto f \cdot T = [\omega \mapsto \langle T, f \cdot \omega \rangle]$ *is bounded.*

Proof. As the bornology of $\mathcal{D}'(M)$ consists of all weakly bounded sets we only have to verify that for $B_1 \subseteq C^\infty(M)$ and $B_2 \subseteq \mathcal{D}'(M)$ both bounded $\{\, \langle T, f \cdot \omega \rangle \mid f \in B_1, T \in B_2 \,\}$ is bounded for each $\omega \in \Omega^n_c(M)$, which follows because $\{\, f \cdot \omega \mid f \in B_1 \,\}$ is bounded in $\Omega^n_c(M)$ and B_2 is uniformly bounded on bounded sets. □

Note that multiplication of distributions is not jointly continuous ([KM81]), thus the proof of Theorem 13.2 does not work in the topological setting. For $T \in \mathcal{D}'^r_s(M)$ we will denote its image in both spaces $(\mathcal{T}_r^s(M) \otimes \Omega^n_c(M))'$ and $\mathrm{L}_{C^\infty(M)}(\mathcal{T}_r^s(M), \mathcal{D}'(M))$ by the same letter T, as it is always clear from the arguments what is meant. Thus for $u \in \mathcal{T}_r^s(M)$, $\omega \in \Omega^n_c M$, and $\xi = [x \mapsto u(x) \otimes \omega(x)] \in \Gamma_c(M, T^s_r M \otimes \Lambda^n T^* M)$ we write

$$\langle T, \xi \rangle = \langle T, u \otimes \omega \rangle = \langle T(u), \omega \rangle.$$

13.2 Coordinates of distributions

Using isomorphism (13.2) and the fact that $\mathcal{D}'^r_s(M)$ is a sheaf ([GKOS01, Theorem 3.1.7]) we can now define coordinates of distributions. Let $(b_\lambda)_\lambda$ be a basis of $\mathcal{T}^r_s(U)$ with dual basis $(b^\lambda)_\lambda$ of $\mathcal{T}_r^s(U)$. Then for $T \in \mathcal{D}'^r_s(U)$, $u \in \mathcal{T}_r^s(U)$ and $\omega \in \Omega^n_c(U)$ we can write

$$\begin{aligned} \langle T, u \otimes \omega \rangle &= \langle T, u_\lambda b^\lambda \otimes \omega \rangle = \langle T(u_\lambda b^\lambda), \omega \rangle \\ &= \langle T(b^\lambda), u_\lambda \omega \rangle = \langle T^\lambda, u_\lambda \omega \rangle \end{aligned}$$

where $T^\lambda := T(b^\lambda) \in \mathcal{D}'(U)$ is called the λ-coordinate of T and $u_\lambda = b_\lambda(u)$ is the λ-coordinate of u.

Part III

Point values in full Colombeau algebras

Chapter 14

Introduction to Part III

Colombeau algebras ([Col85]) are spaces of generalized functions which serve to extend the theory of Schwartz distributions such that these can be multiplied, circumventing the well-known impossibility result by Schwartz [Sch54]. These commutative and associative differential algebras provide an embedding of the space of distributions as a linear subspace and the space of smooth functions as a faithful subalgebra.

For Schwartz distributions a concept of point values was introduced by [Łoj57], but an arbitrary distribution need not have a point value in this sense at every point. Furthermore, it is not possible to characterize distributions by their point values. Colombeau-type algebras of generalized functions are usually constructed as nets of smooth functions, which means that a given point can be inserted into each component of the net in order to give a generalized point value. This is not sufficient for uniquely characterizing a generalized function, though: there exist nonzero generalized functions that evaluate to zero at every classical point. However, with the introduction of generalized points one can obtain a point value characterization theorem. Note that for holomorphic generalized functions a stronger results holds, which states that such a function is zero already if its zero set has positive measure ([KS06]). Point values for Colombeau generalized functions were first introduced for $\mathcal{G}^s(\Omega)$, the special Colombeau algebra on an open set $\Omega \subseteq \mathbb{R}^n$, in [KO99] and later on also for the special Colombeau algebra on a manifold in [KS02a]. In the context of p-adic Colombeau-Egorov type generalized functions it was first claimed that classical points suffice to characterize a function ([AKS05]), but this claim was shown to be invalid later on and a characterization using generalized points was given in [May07].

The aim of this part is to introduce generalized points, numbers, and point values for the elementary full algebra $\mathcal{G}^e(\Omega)$ of [Col85] and the diffeomorphism invariant full algebra $\mathcal{G}^d(\Omega)$ of [GFKS01]. Both algebras are presented in a unifying framework in [GKOS01]. Our main result is a point value characterization theorem for each algebra (Theorems 17.6 and 18.8) which states that two generalized functions are equal if and only if they have the same generalized point value at all generalized points.

Let us mention some applications generalized numbers and point values have found so far. First, when one does Lie group analysis of differential equations in generalized function spaces, point values allow to transfer the classical procedure for computing symmetries to the generalized case ([KO00]). Second, consider mappings from the space of generalized points into the space of generalized numbers. For such mappings a discontinuous differential calculus was constructed, featuring a fundamental theorem of calculus, notions of sub-linear, holomorphic, and analytic mappings, generalized manifolds, and related results ([AFJ05]). Using point values, elements of $\mathcal{G}^s$ can be regarded as such mappings and their local properties can be analyzed from this viewpoint ([OPS03]). Moreover, point values have repeatedly turned out to be indispensable tools for doing analysis in algebras of generalized functions (cf., e.g., [Gar05b, Gar05a, PSV06, Ver09]).

Chapter 15

Preliminaries

15.1 Notation

The number $n \in \mathbb{N}$ will always denote the dimension of the underlying space $\mathbb{R}^n$. $\partial\Omega$ denotes the topological boundary of a set Ω. For $A \subseteq \mathbb{R}^n$ we write $K \subset\subset A$ if K is a compact subset of A°, the interior of A. Nets (here with parameter ε) are written in the form $(u_\varepsilon)_\varepsilon$. The class with respect to any equivalence relation is denoted by square brackets $[\dots]$. A family of objects x_i indexed by $i \in I$ is written as $\{x_i\}_{i\in I}$ or simply $\{x_i\}_i$ when the index set is clear from the context. We use Landau notation: for expressions $f(\varepsilon)$ and $g(\varepsilon)$ depending on and defined for small ε we write $f(\varepsilon) = O(g(\varepsilon))$ (always for $\varepsilon \to 0$) if and only if $\exists C > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0$: $|f(\varepsilon)| < C\,|g(\varepsilon)|$. $B_\eta(x)$ resp. $B_\eta(K)$ denotes the metric ball of radius η around $x \in \mathbb{R}^n$ resp. a set K, dist denotes the Euclidean distance function on $\mathbb{R}^n$. For a function $f(\varphi, x)$ of a variable φ and an n-dimensional real variable $x = (x_1, \dots, x_n)$, $\mathrm{d}_2 f$ denotes the total differential of f with respect to x and $\partial_i f$ its partial differential with respect to x_i. For the derivative of a function γ depending on $t \in \mathbb{R}$ we will write γ'. An n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a multi-index; we use the notation $|\alpha| = \alpha_1 + \dots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. A strictly decreasing sequence $(x_n)_{n\in\mathbb{N}}$ converging to x_0 is denoted by $x_n \searrow x_0$. A function between finite dimensional real vector spaces is said to be smooth if it is infinitely differentiable. The action of a distribution $u \in \mathcal{D}'(\Omega)$ on a test function $\varphi \in \mathcal{D}(\Omega)$ is written as $\langle u, \varphi \rangle$.

15.2 Calculus on convenient vector spaces

The construction of the diffeomorphism invariant full algebra $\mathcal{G}^d(\Omega)$ as defined below requires calculus on infinite-dimensional locally convex spaces as an indispensable prerequisite. The theoretical framework chosen for this by Grosser et al. [GFKS01, GKOS01] is smooth calculus on convenient vector spaces, which is presented by Kriegl and Michor in [KM97] using functional analysis and by Frölicher and Kriegl in [FK88] using category theory. For a detailed exposition of what is needed for the diffeomorphism invariant full algebra we refer to [GFKS01, Section 4]. Whenever we encounter smoothness on a subset of a locally convex space (or an affine subspace thereof) we endow it it with the initial smooth structure.

We will use that a sesquilinear form on a complex locally convex space is smooth if and only if it is bounded; this easily results from an adaptation of [KM97, Section 5] to antilinear maps.

Although the differential is at first only defined for mappings having as domain open subsets of locally convex spaces with respect to a certain topology ([KM97, Theorem 3.18]) this definition can be easily extended to maps defined on affine subspaces, as is remarked in the proof of Proposition 18.5. Properties like the chain rule and the symmetry of higher derivatives remain intact.

15.3 Colombeau algebras

We will now give the definitions of the special algebra $\mathcal{G}^s(\Omega)$ and the full algebras $\mathcal{G}^e(\Omega)$ and $\mathcal{G}^d(\Omega)$ on an arbitrary open subset $\Omega \subseteq \mathbb{R}^n$.

The special Colombeau algebra $\mathcal{G}^s(\Omega)$ ([GKOS01, Section 1.2]) consists of nets of smooth functions on Ω indexed by $I := (0,1]$. Such a net $(u_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ is said to be *moderate* if $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}$ such that $\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})$, or *negligible* if $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \forall m \in \mathbb{N}$: $\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m)$. $\mathcal{G}^s(\Omega)$ then is the quotient of $\mathcal{E}^s_M(\Omega)$ (the set of moderate nets) modulo $\mathcal{N}^s(\Omega)$ (the set of negligible nets).

The full algebras $\mathcal{G}^e(\Omega)$ and $\mathcal{G}^d(\Omega)$ require some auxiliary definitions. For $q \in \mathbb{N}_0$ let $\mathcal{A}_q(\Omega)$ be the set of all test functions $\varphi \in \mathcal{D}(\Omega)$ having integral 1, if $q \geq 1$ additionally satisfying $\int x^\alpha \varphi(x)\,\mathrm{d}x = 0$ for all multi-indices α with $1 \leq |\alpha| \leq q$. Let $\mathcal{A}_{0q}(\Omega)$ be defined in the same way but with integral 0. For any subset $M \subseteq \Omega$ define $\mathcal{A}_{0,M}(\Omega)$ as the set of those elements of $\mathcal{A}_0(\Omega)$ with support in M. $\mathcal{A}_q(\Omega)$ and $\mathcal{A}_{0q}(\Omega)$ are endowed with the initial topology and the initial smooth structure with respect to the embedding in $\mathcal{D}(\Omega)$ or $\mathcal{D}(\mathbb{R}^n)$. Let $U(\Omega)$ be the set of all pairs $(\varphi, x) \in \mathcal{A}_0(\mathbb{R}^n) \times \Omega$ satisfying $\operatorname{supp} \varphi + x \subseteq \Omega$. Furthermore, let $C^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$ be the space of those mappings which are smooth from $I \times \Omega$ into $\mathcal{A}_0(\mathbb{R}^n)$ such that for any compact set $K \subset\subset \Omega$ and any $\alpha \in \mathbb{N}_0^n$ the set $\{\, \partial^\alpha \phi(\varepsilon, x) \mid \varepsilon \in I,\ x \in K \,\}$ is bounded in $\mathcal{D}(\mathbb{R}^n)$. For $\varepsilon \in \mathbb{R}_+$ let $\mathrm{S}_\varepsilon \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ be the mapping given by $(\mathrm{S}_\varepsilon \varphi)(y) := \varepsilon^{-n} \varphi(y/\varepsilon)$ and set $\mathrm{S}^{(\varepsilon)}(\varphi, x) := (\mathrm{S}_\varepsilon \varphi, x)$ for $(\varphi, x) \in \mathcal{D}(\mathbb{R}^n) \times \mathbb{R}^n$. For $x \in \mathbb{R}^n$ denote by $\mathrm{T}_x \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ the mapping given by $(\mathrm{T}_x \varphi)(y) := \varphi(y - x)$ and define $\mathrm{T} \colon \mathcal{D}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{D}(\mathbb{R}^n) \times \mathbb{R}^n$ by $\mathrm{T}(\varphi, x) := (\mathrm{T}_x \varphi, x)$. For a map R we will frequently write R_ε instead of $R \circ \mathrm{S}^{(\varepsilon)}$.

For $\mathcal{G}^e(\Omega)$ ([GKOS01, Section 1.4]), the base space $\mathcal{E}^e(\Omega)$ is the set of all functions $R \colon U(\Omega) \to \mathbb{C}$ which are smooth in the second variable. R is called *moderate* if $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)$: $\sup_{x \in K} |\partial^\alpha R(\mathrm{S}_\varepsilon \varphi, x)| =$

$O(\varepsilon^{-N})$ and *negligible* if $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \forall m \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)$: $\sup_{x\in K} |\partial^\alpha R(\mathrm{S}_\varepsilon\varphi, x)| = O(\varepsilon^m)$. The corresponding sets $\mathcal{E}^e_M(\Omega)$ of moderate and $\mathcal{N}^e(\Omega)$ of negligible functions give rise to the differential algebra $\mathcal{G}^e(\Omega) := \mathcal{E}^e_M(\Omega)/\mathcal{N}^e(\Omega)$. Distributions $u \in \mathcal{D}'(\Omega)$ are embedded via the linear injective mapping $\iota\colon \mathcal{D}'(\Omega) \to \mathcal{E}^e_M(\Omega)$ given by $\iota(u)(\varphi, x) := \langle u, \mathrm{T}_x\varphi\rangle$ for $(\varphi, x) \in U(\Omega)$. The derivations of $\mathcal{G}^e(\Omega)$ which extend the distributional ones are given by $(D_i R)(\varphi, x) := (\partial_i R)(\varphi, x)$ for $R \in \mathcal{E}^e_M(\Omega)$ and $i = 1, \ldots, n$.

For $\mathcal{G}^d(\Omega)$ ([GKOS01, Chapter 2] or [GFKS01]) the base space is $\mathcal{E}^d(\Omega) := C^\infty(U(\Omega))$. A map $R \in \mathcal{E}^d(\Omega)$ is called *moderate* if $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}\ \forall \phi \in C^\infty_b(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n))$: $\sup_{x\in K} |\partial^\alpha R(\mathrm{S}_\varepsilon\phi(\varepsilon, x), x)| = O(\varepsilon^{-N})$ and *negligible* if it is moderate and $\forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \forall m \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall \phi \in C^\infty_b(I \times \Omega, \mathcal{A}_q(\mathbb{R}^n))$: $\sup_{x\in K} |\partial^\alpha R(\mathrm{S}_\varepsilon\phi(\varepsilon, x), x)| = O(\varepsilon^m)$. The corresponding sets $\mathcal{E}^d_M(\Omega)$ of moderate and $\mathcal{N}^d(\Omega)$ of negligible functions give rise to the differential algebra $\mathcal{G}^d(\Omega) := \mathcal{E}^d_M(\Omega)/\mathcal{N}^d(\Omega)$. The embedding (denoted by ι as well) of distributions $u \in \mathcal{D}'(\Omega)$ is given by $\iota(u)(\varphi, x) := \langle u, \mathrm{T}_x\varphi\rangle$ for $(\varphi, x) \in U(\Omega)$. The derivations which extend the distributional ones are given by $(D_i R)(\varphi, x) := (\partial_i R)(\varphi, x)$.

A *constant* in one of the preceding differential algebras (as in any differential ring) is defind as an element whose derivations are all zero ([Kol73, Chapter I Section 1]).

Remark 15.1. For later use we note the following.

(i) In all definitions of moderateness and negligibility above and below, when expanding the Landau symbol in expressions of the form $f(\varepsilon) = O(\varepsilon^{-N})$ into $\exists C > 0\ \exists \eta > 0\ \forall \varepsilon < \eta$: $|f(\varepsilon)| < C\varepsilon^{-N}$ (resp. ε^m for negligibility) one can always have $C = 1$.

(ii) In the definitions of negligibility one can disregard the derivatives and only consider $\alpha = 0$ if one presupposes the tested element to be moderate ([GKOS01, Theorems 1.2.3, 1.4.8, and 2.5.4]).

Chapter 16

Previous results in the special algebra $\mathcal{G}^s(\Omega)$

We first recall the definition of generalized points, numbers, and point values for $\mathcal{G}^s(\Omega)$. Two results justify these definitions: first, the ring of constants in $\mathcal{G}^s(\Omega)$ equals the space of generalized numbers. Second, two generalized functions are equal if and only if they have the same point values.

Definition 16.1 ([GKOS01, Definition 1.2.31])**.** Generalized numbers in the $\mathcal{G}^s$-setting are defined by

$$\begin{aligned} \mathbb{C}_M &:= \{ (r_\varepsilon)_\varepsilon \in \mathbb{C}^I \mid \exists N \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^{-N}) \}, \\ \mathbb{C}_N &:= \{ (r_\varepsilon)_\varepsilon \in \mathbb{C}^I \mid \forall m \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^{m}) \}, \\ \widetilde{\mathbb{C}} &:= \mathbb{C}_M / \mathbb{C}_N. \end{aligned}$$

Definition 16.2 ([GKOS01, Definition 1.2.44])**.** Generalized points in the $\mathcal{G}^s$-setting are defined by

$$\begin{aligned} \Omega_M &:= \{ (x_\varepsilon)_\varepsilon \in \Omega^I \mid \exists N \in \mathbb{N} : |x_\varepsilon| = O(\varepsilon^{-N}) \}, \\ (x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon &:\Leftrightarrow \forall m \in \mathbb{N} : |x_\varepsilon - y_\varepsilon| = O(\varepsilon^{m}), \\ \widetilde{\Omega} &:= \Omega_M / \sim, \\ \widetilde{\Omega}_c &:= \{ \tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega} \mid \exists K \subset\subset \Omega \ \exists \eta > 0 \ \forall \varepsilon < \eta : x_\varepsilon \in K \}. \end{aligned}$$

Clearly $\widetilde{\mathbb{C}}$ can be seen as a subset of $\mathcal{G}^s(\Omega)$.

Proposition 16.3 ([GKOS01, Proposition 1.2.35])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be connected and $\tilde{u} \in \mathcal{G}^s(\Omega)$. Then $D\tilde{u} = 0$ if and only if $\tilde{u} \in \widetilde{\mathbb{C}}$.*

Definition 16.4. Let $\tilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^s(\Omega)$ and $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c$. Then the generalized point value of $\tilde{u}$ at $\tilde{x}$ defined by $\tilde{u}(\tilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well-defined element of $\widetilde{\mathbb{C}}$.

Theorem 16.5 ([GKOS01, Theorem 1.2.64])**.** *Let $\tilde{u} \in \mathcal{G}^s(\Omega)$. Then $\tilde{u} = 0$ in $\mathcal{G}^s(\Omega)$ if and only if $\tilde{u}(\tilde{x}) = 0$ in $\widetilde{\mathbb{C}}$ for all $\tilde{x} \in \widetilde{\Omega}_c$.*

Chapter 17

Point values in $\mathcal{G}^e(\Omega)$

It was asserted by Grosser et al. ([GKOS01, Section 1.4.2]) that results concerning point values obtained in the special algebra can be recovered in the full algebra $\mathcal{G}^e(\Omega)$. This section explicitly states these results and their proofs for $\mathcal{G}^e(\Omega)$, which should not be regarded as a mere technical exercise but as an essential building step if one aims to obtain the corresponding results in $\mathcal{G}^d(\Omega)$, where in addition one needs to incorporate smoothness into the proofs presented here.

After recalling the definition of generalized numbers in the $\mathcal{G}^e$-setting we will define a suitable space of generalized points.

Definition 17.1 ([GKOS01, Definition 1.4.19])**.** Generalized numbers in the $\mathcal{G}^e$-setting are defined by

$$\begin{aligned}
\mathbb{C}_M(n) &:= \{\, r\colon \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C} \mid \exists N \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : |r(\mathrm{S}_\varepsilon\varphi)| = O(\varepsilon^{-N}) \,\},\\
\mathbb{C}_N(n) &:= \{\, r\colon \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C} \mid \forall m \in \mathbb{N}\ \exists q \in \mathbb{N}\\
&\qquad\qquad \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n) : |r(\mathrm{S}_\varepsilon\varphi)| = O(\varepsilon^{m}) \,\},\\
\widetilde{\mathbb{C}}(n) &:= \mathbb{C}_M(n)/\mathbb{C}_N(n).
\end{aligned}$$

Definition 17.2. Generalized points in the $\mathcal{G}^e$-setting are defined by

$$\begin{aligned}
\Omega_M(n) &:= \{\, X\colon \mathcal{A}_0(\mathbb{R}^n) \to \Omega \mid \forall \varphi \in \mathcal{A}_0(\mathbb{R}^n)\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0 :\\
&\qquad (\mathrm{S}_\varepsilon\varphi, X(\mathrm{S}_\varepsilon\varphi)) \in U(\Omega) \text{ and}\\
&\qquad \exists N \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n) : |X(\mathrm{S}_\varepsilon\varphi)| = O(\varepsilon^{-N}) \,\},\\
\Omega_N(n) &:= \{\, X \in \Omega_M(n) \mid \forall m \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n) : |X(\mathrm{S}_\varepsilon\varphi)| = O(\varepsilon^{m}) \,\},\\
\widetilde{\Omega}(n) &:= \Omega_M(n)/\Omega_N(n),\\
\widetilde{\Omega}_c(n) &:= \{\, \widetilde{X} \in \widetilde{\Omega}(n) \mid \text{for one (thus any) representative } X \text{ of } \widetilde{X}\\
&\qquad \exists K \subset\subset \Omega\ \exists N \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)\ \exists \eta > 0\ \forall \varepsilon < \eta : X(\mathrm{S}_\varepsilon\varphi) \in K \,\}.
\end{aligned}$$

We write $X \sim Y$ if $X - Y \in \Omega_N(n)$. Any $X \in \Omega_M(n)$ satisfying the condition in the definition of $\widetilde{\Omega}_c(n)$ is called *compactly supported* (in K). If one replaces $\mathbb{C}$ by $\mathbb{R}$ in Definition 17.1 the resulting space is denoted by $\widetilde{\mathbb{R}}(n)$.

Proposition 17.3. *Let $X, Y \in \Omega_M(n)$ be compactly supported generalized points and $R \in \mathcal{E}^e_M(\Omega)$. Define $R(X)\colon \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C}$ by*

$$R(X)(\varphi) := \begin{cases} R(\varphi, X(\varphi)) & (\varphi, X(\varphi)) \in U(\Omega) \\ 0 & \text{otherwise.} \end{cases}$$

Then $R(X)$ is in $\mathbb{C}_M(n)$, $R \in \mathcal{N}^e(\Omega)$ implies $R(X) \in \mathbb{C}_N(n)$, and $X \sim Y$ implies $R(X) - R(Y) \in \mathbb{C}_N(n)$.

Proof. Let X be compactly supported in $K \subset\subset \Omega$, which means that $\exists N \in \mathbb{N}$ $\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)$: $X(\mathrm{S}_\varepsilon\varphi) \in K$ for small ε. Given any $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$, for small ε we have $X(\mathrm{S}_\varepsilon\varphi) \in K$, $R(X)(\mathrm{S}_\varepsilon\varphi) = R(\mathrm{S}_\varepsilon\varphi, X(\mathrm{S}_\varepsilon\varphi))$, and thus $|R(X)(\mathrm{S}_\varepsilon\varphi)| \le \sup_{x\in K} |R(\mathrm{S}_\varepsilon\varphi, x)|$ whence $R(X)$ inherits moderateness respectively negligibility from R. For the last claim, choose some $m \in \mathbb{N}$ for the test for membership in $\mathbb{C}_N(n)$. Then we use the following ingredients.

(i) As $X \sim Y$, $\exists q_0 \in \mathbb{N}$ $\forall \varphi \in \mathcal{A}_{q_0}(\mathbb{R}^n)$: $|X(\mathrm{S}_\varepsilon\varphi) - Y(\mathrm{S}_\varepsilon\varphi)| < \varepsilon^m$ for small ε.

(ii) $\exists \eta > 0$: $\overline{B_\eta}(K) \subseteq \Omega$. Set $V := B_\eta(K)$.

(iii) As derivatives of R are moderate, there exists $N' \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_{N'}(\mathbb{R}^n)$ we have $\sup_{x\in\overline{V}} |\mathrm{d}_2 R(\mathrm{S}_\varepsilon\varphi, x)| \le \varepsilon^{-N'}$ for small ε.

(iv) From (i) we know in particular that given $\varphi \in \mathcal{A}_{\max(q_0,N)}(\mathbb{R}^n)$, $g(t) := (X + t(Y-X))(\mathrm{S}_\varepsilon\varphi)$ lies in V for small ε and all $t \in [0,1]$.

(v) $\forall \varphi \in \mathcal{A}_0(\mathbb{R}^n)$: $\operatorname{supp} \mathrm{S}_\varepsilon\varphi + V \subseteq \Omega$ for small ε.

Next let $\varphi \in \mathcal{A}_{\max(q_0,N,N')}(\mathbb{R}^n)$ and ε small enough. Then by (iv), $X(\mathrm{S}_\varepsilon\varphi)$ and $Y(\mathrm{S}_\varepsilon\varphi)$ are in V, $(R(X) - R(Y))(\mathrm{S}_\varepsilon\varphi) = R(\mathrm{S}_\varepsilon\varphi, X(\mathrm{S}_\varepsilon\varphi)) - R(\mathrm{S}_\varepsilon\varphi, Y(\mathrm{S}_\varepsilon\varphi))$, and the domain of $R(\mathrm{S}_\varepsilon\varphi, \cdot)$ contains V. Set $F(t) := R(\mathrm{S}_\varepsilon\varphi, g(t))$ for $t \in [0,1]$. Then F is smooth on $[0,1]$ and

$$\begin{aligned} |R(X)(\mathrm{S}_\varepsilon\varphi) - R(Y)(\mathrm{S}_\varepsilon\varphi)| = |F(1) - F(0)| &= \left| \int_0^1 F'(t)\,\mathrm{d}t \right| = \\ &= \left| \int_0^1 \mathrm{d}_2 R(\mathrm{S}_\varepsilon\varphi, g(t)) \cdot (X(\mathrm{S}_\varepsilon\varphi) - Y(\mathrm{S}_\varepsilon\varphi))\,\mathrm{d}t \right| \\ &\le |(X-Y)(\mathrm{S}_\varepsilon\varphi)| \cdot \sup_{x\in\overline{V}} |(\mathrm{d}_2 R)(\mathrm{S}_\varepsilon\varphi, x)| \le \varepsilon^m \varepsilon^{-N'}. \end{aligned}$$

As m was arbitrary this concludes the proof. □

The following lemma will be used to construct generalized points and numbers taking prescribed values.

Lemma 17.4. *Given $\varphi_q \in \mathcal{A}_q(\mathbb{R}^n)$, $\varepsilon_{q,k} \in (0,\infty)$ and x_0, $x_{q,k}$ in any set A for all $q,k \in \mathbb{N}$, there exists a mapping $X\colon \mathcal{A}_0(\mathbb{R}^n) \to A$ and strictly increasing sequences $(q_l)_{l\in\mathbb{N}}$ and $(a_l)_{l\in\mathbb{N}}$ of natural numbers such that $X(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}) = x_{q_l,k}$ $\forall k,l \in \mathbb{N}$, $X(\varphi) = x_0$ for all φ not equal to some $\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}$, and $\varphi_{q_l} \in \mathcal{A}_{a_l}(\mathbb{R}^n) \setminus \mathcal{A}_{a_l+1}(\mathbb{R}^n)$.*

Proof. Set $q_1 := 1$, a_1 such that $\varphi_{q_1} \in \mathcal{A}_{a_1}(\mathbb{R}^n) \setminus \mathcal{A}_{a_1+1}(\mathbb{R}^n)$ and inductively choose $q_{l+1} := a_l + 1$ and a_{l+1} appropriately. This is possible because for q increasing more and more moments of φ_q have to vanish. Then define $X\colon \mathcal{A}_0(\mathbb{R}^n) \to A$ as follows: given $\psi \in \mathcal{A}_0(\mathbb{R}^n)$, if $\psi = \mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}$ for some k,l then set $X(\psi) := x_{q_l,k}$, otherwise set $X(\psi) := x_0$. $\square$

Definition 17.5. For $\widetilde{R} = [R] \in \mathcal{G}^e(\Omega)$ and $\widetilde{X} = [X] \in \widetilde{\Omega}_c(n)$ we define the point value $\widetilde{R}(\widetilde{X})$ of $\widetilde{R}$ at $\widetilde{X}$ as the class in $\widetilde{\mathbb{C}}(n)$ of $R(X)$ as defined in Proposition 17.3.

Having defined suitable spaces of generalized points and numbers as well as a corresponding notion of point evaluation we can now state the point value characterization theorem for $\mathcal{G}^e$.

Theorem 17.6. *Let $\widetilde{R} = [R] \in \mathcal{G}^e(\Omega)$. Then $\widetilde{R} = 0$ if and only if $\widetilde{R}(\widetilde{X}) = 0$ in $\widetilde{\mathbb{C}}(n)$ for all $\widetilde{X} \in \widetilde{\Omega}_c(n)$.*

Proof. Necessity was already shown in Proposition 17.3. For sufficiency assume that $R \notin \mathcal{N}^e(\Omega)$; then by Remark 15.1 (ii) there exist $K \subset\subset \Omega$ and $m_0 \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there is some $\varphi_q \in \mathcal{A}_q(\mathbb{R}^n)$, a sequence $(\varepsilon_{q,k})_{k\in\mathbb{N}} \searrow 0$ and a sequence $(x_{q,k})_{k\in\mathbb{N}}$ in K such that $\left|R(\mathrm{S}_{\varepsilon_{q,k}}\varphi_q, x_{q,k})\right| \geq \varepsilon_{q,k}^{m_0}$.

Let $X\colon \mathcal{A}_0(\mathbb{R}^n) \to K$, $(q_l)_{l\in\mathbb{N}}$ and $(a_l)_{l\in\mathbb{N}}$ be as obtained from Lemma 17.4 with arbitrary $x_0 \in K$. Then clearly X is compactly supported, $[X] \in \widetilde{\Omega}_c$ and $R(X) \notin \mathbb{C}_N(n)$: for any $q \in \mathbb{N}$ there is some $l \in \mathbb{N}$ such that $a_l \geq q$, so $\varphi_{q_l} \in \mathcal{A}_q(\mathbb{R}^n)$. By construction,

$$\left|R(X)(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l})\right| = \left|R(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}, X(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}))\right| = \left|R(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}, x_{q_l,k})\right| > \varepsilon_{q_l,k}^{m_0}$$

for all large enough $k \in \mathbb{N}$, which ensures that the negligibility test for $R(X)$ fails. $\square$

The proof of the following proposition is evident.

Proposition 17.7. *The map $\rho\colon \mathbb{C}_M(n) \to \mathcal{E}^e(\Omega)$ given by $\rho(r)(\varphi,x) := r(\varphi)$ for $(\varphi,x) \in U(\Omega)$ is a ring homomorphism preserving moderateness and negligibility and thus induces an embedding $\tilde{\rho}\colon \widetilde{\mathbb{C}}(n) \to \mathcal{G}^e(\Omega)$.*

Lemma 17.8. *Let $\Omega \subseteq \mathbb{R}^n$ be connected and $K \subset\subset \Omega$. Then there exist a set $M \subset\subset \Omega$ containing K and a real number $L > 0$ such that any two points in K can be connected by a continuous curve $\gamma\colon [0,1] \to \Omega$ with image in M having length $\int_0^1 |\gamma'(t)|\,\mathrm{d}t \leq L$.*

Proof. Cover K by finitely many closed balls of some radius $\varepsilon > 0$ which are contained in Ω. As Ω is (pathwise) connected these can be joined by finitely many continuous curves in Ω. Taking as M the union of these ε-balls and the images of these curves, the existence of L as desired is obvious. □

In the differential algebra $\mathcal{G}^e(\Omega)$ the constant elements are by definition exactly those whose derivatives are zero. With the availability of point values one can also call a generalized function constant if it has the same generalized value at every generalized point. The following proposition shows that these properties in fact are equivalent.

Proposition 17.9. *If $\widetilde{R} \in \mathcal{G}^e(\Omega)$ has the property $\widetilde{R}(\widetilde{X}) = \widetilde{R}(\widetilde{Y})\ \forall \widetilde{X}, \widetilde{Y} \in \widetilde{\Omega}_c(n)$ then $D_i\widetilde{R} = 0$ for $i = 1, \ldots, n$; if Ω is connected the converse also holds.*

Proof. Given any $\widetilde{X} \in \widetilde{\Omega}_c(n)$ one easily sees that for all $\widetilde{Y} \in \widetilde{\Omega}_c(n)$ we have $\tilde{\rho}(\widetilde{R}(\widetilde{X}))(\widetilde{Y}) = \widetilde{R}(\widetilde{X})$ on the one hand and $\widetilde{R}(\widetilde{Y}) = \widetilde{R}(\widetilde{X})$ on the other hand by assumption. By Theorem 17.6 then $\tilde{\rho}(\widetilde{R}(\widetilde{X})) = \widetilde{R}$, whence $D_i\widetilde{R} = D_i\tilde{\rho}(\widetilde{R}(\widetilde{X})) = 0$ follows at once from the definitions.

For the converse we show that in case Ω is connected $D_i\widetilde{R} = 0$ (for $i = 1, \ldots, n$) in $\mathcal{G}^e(\Omega)$ implies $\widetilde{R} = \tilde{\rho}(\widetilde{R}(\widetilde{X}))$ for arbitrary $\widetilde{X} = [X] \in \widetilde{\Omega}_c(n)$. Fix $K_1 \subset\subset \Omega$ and $m \in \mathbb{N}$ for testing and let X be compactly supported in $K_2 \subset\subset \Omega$. Let M and L be as obtained from Lemma 17.8 applied to $K = K_1 \cup K_2$. By assumption,

(i) $\exists q \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0$: $\sup_{x \in M} |\mathrm{d}_2 R(\mathrm{S}_\varepsilon\varphi, x)| \leq \varepsilon^m$.

(ii) $\exists N \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)\ \exists \eta > 0\ \forall \varepsilon < \eta$: $X(\mathrm{S}_\varepsilon\varphi) \in K_2$.

Now let $\varphi \in \mathcal{A}_{\max(q,N)}(\mathbb{R}^n)$ and $\varepsilon < \min(\varepsilon_0, \eta)$. Then for every $y \in K_1$ there exists a continuous curve $\gamma\colon [0,1] \to \Omega$ with image in M connecting y and $X(\mathrm{S}_\varepsilon\varphi)$ and having length $\leq L$. Thus we can estimate

$$\begin{aligned}|R(\mathrm{S}_\varepsilon\varphi, y) - R(\mathrm{S}_\varepsilon\varphi, X(\mathrm{S}_\varepsilon\varphi))| &= \left|\int_0^1 \mathrm{d}_2 R(\mathrm{S}_\varepsilon\varphi, \gamma(t))\gamma'(t)\,\mathrm{d}t\right| \\ &\leq \sup_{x \in M} |\mathrm{d}_2 R(\mathrm{S}_\varepsilon\varphi, x)| \cdot \int_0^1 \left|\gamma'(t)\right|\,\mathrm{d}t \leq L\varepsilon^m\end{aligned}$$

which gives the claimed result. □

Definition 17.10. For $\tilde{r}, \tilde{s} \in \widetilde{\mathbb{R}}(n)$ we write $\tilde{r} \le \tilde{s}$ if there are representatives r, s such that $r(\varphi) \le s(\varphi)$ for all $\varphi \in \mathcal{A}_0(\Omega)$.

Proposition 17.11. *$(\widetilde{\mathbb{R}}(n), \le)$ is a partially ordered ring.*

Proof. Reflexivity is clear. For antisymmetry, $\tilde{r} \le \tilde{s}$ and $\tilde{s} \le \tilde{r}$ imply $r_1 \le s_1$ and $s_2 \le r_2$ for some representatives r_1, r_2 of $\tilde{r}$ and s_1, s_2 of $\tilde{s}$. Writing $s_1 = s_2 + n$ and $r_2 = r_1 + m$ with $n, m \in \mathcal{N}^e(\Omega)$ gives $r_1 \le s_2 + n$ and $s_2 \le r_1 + m$, thus $r_1 - s_2 \le n$ and $s_2 - r_1 \le m$, implying $|r_1 - s_2| \le \max(n, m)$ and finally $r_1 - s_2 \in \mathcal{N}^e(\Omega)$. For transitivity assume $\tilde{r} \le \tilde{s} \le \tilde{t}$. Then with representatives r, s_1, s_2 and t we have $s_1 = s_2 + n$ with $n \in \mathcal{N}^e(\Omega)$ and thus $r \le s_1 = s_2 + n \le t + n$, which is $\tilde{r} \le \tilde{t}$. Finally, $\tilde{r} \le \tilde{s}$ clearly implies $\tilde{r} + \tilde{t} \le \tilde{s} + \tilde{t}$ and $0 \le \tilde{r}$, $0 \le \tilde{s}$ reads $n \le r$, $m \le s$ in representatives which implies $nm \le rs$ or $0 \le \tilde{r}\tilde{s}$. ☐

We call a generalized number $\tilde{r} \in \widetilde{\mathbb{C}}(n)$ strictly nonzero if it has a representative $r \in \mathbb{C}_M(n)$ such that

$$\exists q \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)\ \exists C > 0\ \exists \eta > 0\ \forall \varepsilon < \eta : |r(\mathrm{S}_\varepsilon \varphi)| > C\varepsilon^q. \tag{17.1}$$

Note that Remark 15.1 (i) applies here and we can always have $C = 1$. We come to the following characterization of invertibility in $\widetilde{\mathbb{C}}(n)$.

Proposition 17.12. *An element of $\widetilde{\mathbb{C}}(n)$ is invertible if and only if it is strictly nonzero.*

Proof. Given $\tilde{r} = [r], \tilde{s} = [s] \in \widetilde{\mathbb{C}}(n)$ with $\tilde{r}\tilde{s} = 1$, there exists $t \in \mathbb{C}_N(n)$ such that $rs = 1 + t$. By the definition of negligibility $\exists q \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)$ $\exists \eta > 0\ \forall \varepsilon < \eta$: $|t(\mathrm{S}_\varepsilon \varphi)| < 1/2$, and thus also $s(\mathrm{S}_\varepsilon \varphi) \ne 0$. By moderateness of s $\exists N \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)\ \exists \eta' > 0$ such that for all $\varepsilon < \eta'$ we have $|s(\mathrm{S}_\varepsilon \varphi)| < \varepsilon^{-N}$. Thus for $q' := \max(q, N)$, $\varphi \in \mathcal{A}_{q'}(\mathbb{R}^n)$, and $\varepsilon < \min(\eta, \eta')$ we obtain

$$|r(\mathrm{S}_\varepsilon \varphi)| = \left| \frac{1 + t(\mathrm{S}_\varepsilon \varphi)}{s(\mathrm{S}_\varepsilon \varphi)} \right| > \frac{\varepsilon^N}{2} \ge \frac{\varepsilon^{q'}}{2}.$$

Conversely, given $r \in \mathbb{C}_M(n)$ satisfying (17.1) set $s(\varphi) := 1/r(\varphi)$ where defined and 0 elsewhere. Then $s \in \mathbb{C}_M(n)$ by definition and obviously $rs - 1 \in \mathcal{N}^e(n)$ because for $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ with q of (17.1) and small ε, $s(\mathrm{S}_\varepsilon \varphi) = 1/r(\mathrm{S}_\varepsilon \varphi)$, thus $rs - 1 = 0$ and the negligibility test succeeds trivially. ☐

Proposition 17.13. *For $\tilde{r} \in \widetilde{\mathbb{C}}(n)$ the following assertions are equivalent.*

(i) *$\tilde{r}$ is not invertible.*

(ii) *$\tilde{r}$ has a representative r such that for all $q \in \mathbb{N}$ there is some $\varphi_q \in \mathcal{A}_q(\mathbb{R}^n)$ and a sequence $(\varepsilon_{q,k})_{k \in \mathbb{N}} \searrow 0$ such that $r(\mathrm{S}_{\varepsilon_{q,k}} \varphi_q) = 0$ for all k.*

(iii) $\tilde{r}$ *is a zero divisor.*

Proof. (i) $\Rightarrow$ (ii): $\tilde{r}$ fails to be strictly nonzero, thus any representative r satisfies

$$\forall q \in \mathbb{N}\ \exists \varphi_q \in \mathcal{A}_q(\mathbb{R}^n)\ \exists (\varepsilon_{q,k})_{k\in\mathbb{N}} \searrow 0 : \left| r(\mathrm{S}_{\varepsilon_{q,k}}\varphi_q) \right| \leq \varepsilon_{q,k}^q.$$

With $x_{q,k} := r(\mathrm{S}_{\varepsilon_{q,k}}\varphi_q)$ for all $q, k \in \mathbb{N}$ and $x_0 := 0$ let $s\colon \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C}$, $(q_l)_l$, and $(a_l)_l$ be as obtained from Lemma 17.4. This map satisfies $s(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}) = x_{q_l,k}\ \forall k, l \in \mathbb{N}$. Then s is negligible: let $m \in \mathbb{N}$ be given and choose $l_0 \in \mathbb{N}$ such that $q_{l_0} > m$. Let $\varphi \in \mathcal{A}_{a_l}(\mathbb{R}^n)$. Then $s(\mathrm{S}_\varepsilon\varphi)$ can only be nonzero if $\varphi = \mathrm{S}_\eta\varphi_{q_l}$ for some $\eta > 0$ and $l \geq l_0$ and this requires that $\mathrm{S}_\varepsilon\varphi = \mathrm{S}_\varepsilon\mathrm{S}_\eta\varphi_{q_l} = \mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}$ for some $k \in \mathbb{N}$, that is $\varepsilon\eta = \varepsilon_{q_l,k}$. In this case

$$|s(\mathrm{S}_\varepsilon\varphi)| = \left| r(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}) \right| \leq \varepsilon_{q_l,k}^{q_l} = \eta^{q_l}\varepsilon^{q_l} < \eta^{q_l}\varepsilon^m$$

for all $\varepsilon = \varepsilon_{q_l,k}/\eta$ which are < 1. Finally $r - s$ has the desired property: given $q \in \mathbb{N}$, there is some l such that $q_l \geq q$ and for $\varphi_{q_l} \in \mathcal{A}_{q_l}(\mathbb{R}^n) \subseteq \mathcal{A}_q(\mathbb{R}^n)$ we have $(r - s)(\mathrm{S}_{\varepsilon_{q_l,k}}\varphi_{q_l}) = 0$.

(ii) $\Rightarrow$ (iii): Define $s\colon \mathcal{A}_0(\mathbb{R}^n) \to \mathbb{C}$ by $s(\varphi) := 1$ if $r(\varphi) = 0$ and $s(\varphi) := 0$ otherwise. Then $s \in \mathbb{C}_M(n)$ and $rs = 0$ but it is easily verified that $s \notin \mathbb{C}_N(n)$.

(iii) $\Rightarrow$ (i) is trivial. $\square$

The following is a characterization of non-degeneracy of matrices over $\widetilde{\mathbb{C}}(n)$.

Proposition 17.14. *Let $A \in \widetilde{\mathbb{C}}(n)^{m^2}$ be an $m \times m$ square matrix with entries from $\widetilde{\mathbb{C}}(n)$. The following are equivalent:*

(i) *A is non-degenerate, i.e., if $\xi, \eta \in \widetilde{\mathbb{C}}(n)^m$ then $\xi^t A \eta = 0\ \forall \eta$ implies $\xi = 0$.*

(ii) *$A\colon \widetilde{\mathbb{C}}(n)^m \to \widetilde{\mathbb{C}}(n)^m$ is injective.*

(iii) *$A\colon \widetilde{\mathbb{C}}(n)^m \to \widetilde{\mathbb{C}}(n)^m$ is bijective.*

(iv) $\det(A)$ *is invertible.*

Proof. The proof is purely algebraical and hence is entirely equivalent to the version for $\mathcal{G}^s(\Omega)$ ([GKOS01, Lemma 1.4.41]). More explicitly, (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) is dealt with by [Bou70, Chapter III §8 Proposition 3 and Theorem 1]. (i) $\Rightarrow$ (ii) follows by showing that (i) is equivalent to A^t being injective, after which (ii) $\Rightarrow$ (iv) can be applied to $\det(A) = \det(A^t)$. $\square$

The next theorem is a characterization of invertibility of generalized functions in $\mathcal{G}^e(\Omega)$.

Theorem 17.15. *For $\widetilde{R} \in \mathcal{G}^e(\Omega)$ the following are equivalent:*

(i) $\widetilde{R}$ is invertible.

(ii) For each representative R of $\widetilde{R}$ the following holds:

$$\forall K \subset\subset \Omega\ \exists m \in \mathbb{N}\ \exists q \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)\ \exists C > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0 : \sup_{x \in K} |R(\mathrm{S}_\varepsilon \varphi, x)| > C\varepsilon^m.$$

Remark 15.1 (i) applies here; furthermore, we can always have $m = q$.

Proof. Assuming (i) there exist $S \in \mathcal{E}^e_M(\Omega)$ and $Q \in \mathcal{E}^e_N(\Omega)$ such that $RS = 1 + Q$. Fix $K \subset\subset \Omega$. Then $\exists p \in \mathbb{N}\ \forall \varphi \in \mathcal{A}_p(\mathbb{R}^n)\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0$: $\sup_{x \in K} |Q(\mathrm{S}_\varepsilon \varphi, x)| < \frac{1}{2}$ and thus $S(\mathrm{S}_\varepsilon \varphi, x) > 0$. Furthermore, $\exists N \in \mathbb{N}$ $\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)\ \exists \varepsilon_1 > 0\ \forall \varepsilon < \varepsilon_1$: $\sup_{x \in K} |S(\mathrm{S}_\varepsilon \varphi, x)| < \varepsilon^{-N}$. Then for $q := \max(p, N)$, $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$, $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$ and $x \in K$ we obtain

$$|R(\mathrm{S}_\varepsilon \varphi, x)| = \left| \frac{1 + Q(\mathrm{S}_\varepsilon \varphi, x)}{S(\mathrm{S}_\varepsilon \varphi, x)} \right| \geq \frac{1 - |Q(\mathrm{S}_\varepsilon \varphi, x)|}{S(\mathrm{S}_\varepsilon \varphi, x)} > \frac{\varepsilon^N}{2}.$$

Conversely, given R satisfying (ii) set $S(\varphi) := 1/R(\varphi)$ where defined and 0 elsewhere. Then $S \in \mathcal{E}^e_M(\Omega)$ by definition and obviously $RS - 1 \in \mathcal{N}^e(\Omega)(n)$. ☐

The following proposition establishes a relation between invertibility and point values.

Proposition 17.16. *$\widetilde{R} \in \mathcal{G}^e(\Omega)$ is invertible if and only if $\widetilde{R}(\widetilde{X})$ is invertible in $\widetilde{\mathbb{C}}(n)$ for each $\widetilde{X} \in \widetilde{\Omega}_c$.*

Proof. Necessity holds because point evaluation at a fixed generalized point evidently is a ring homomorphism from $\mathcal{G}^e(\Omega)$ into $\widetilde{\mathbb{C}}(n)$, thus $\widetilde{R}\widetilde{S} = 1$ in $\mathcal{G}^e(\Omega)$ implies $\widetilde{R}(\widetilde{X})\widetilde{S}(\widetilde{X}) = 1$ in $\widetilde{\mathbb{C}}(n)$. For sufficiency suppose that $\widetilde{R}$ is not invertible. Then by Theorem 17.15 $\exists K \subset\subset \Omega\ \forall q \in \mathbb{N}\ \exists \varphi_q \in \mathcal{A}_q(\mathbb{R}^n)$ $\exists (\varepsilon_{q,k})_{k \in \mathbb{N}} \searrow 0\ \exists (x_{q,k})_{k \in \mathbb{N}} \in K^{\mathbb{N}}$ such that $\left|R(\mathrm{S}_{\varepsilon_{q,k}} \varphi_q, x_{q,k})\right| \leq \varepsilon^q_{q,k}$. Let $X \colon \mathcal{A}_0(\mathbb{R}^n) \to K$ and $(q_l)_{l \in \mathbb{N}}$ be as obtained from Lemma 17.4 with arbitrary $x_0 \in K$. Then clearly X is compactly supported and the class of $R(X)$ is not strictly nonzero and thus not invertible, because for arbitrary q we can choose any l such that $q_l \geq q$ and for large enough $k \in \mathbb{N}$ we obtain

$$\left|R(X)(\mathrm{S}_{\varepsilon_{q_l,k}} \varphi_{q_l})\right| = \left|R(\mathrm{S}_{\varepsilon_{q_l,k}} \varphi_{q_l}, x_{q_l,k})\right| \leq \varepsilon^{q_l}_{q_l,k} \leq \varepsilon^q_{q_l,k}.$$ ☐

Proposition 17.12 also follows directly from the following Lemma, whose validity is clear because for $\tilde{r} \in \widetilde{\mathbb{C}}(n)$ and $\widetilde{X} \in \widetilde{\Omega}_c(n)$ we have $\tilde{\rho}(\tilde{r})(\widetilde{X}) = \tilde{r}$.

Lemma 17.17. *$\tilde{r} \in \widetilde{\mathbb{C}}(n)$ is invertible if and only if $\tilde{\rho}(\tilde{r}) \in \mathcal{G}^e(\Omega)$ is.*

Chapter 18

Point values in $\mathcal{G}^d(\Omega)$

While in $\mathcal{G}^s$ and $\mathcal{G}^e$ one can essentially leave away the x-slot in order to obtain generalized numbers we have to be more careful when introducing generalized numbers in the diffeomorphism invariant setting. First, smoothness of the involved objects is a crucial factor requiring considerable technical machinery (cf. [GKOS01, Chapter 2]). Second, there are two equivalent formalisms for describing the algebra $\mathcal{G}^d$: one stems from the original construction by J. F. Colombeau [Col85], the other is used by J. Jelínek [Jel99] and is essential if one aims to construct a corresponding algebra intrinsically on a manifold. It is a sensible requirement that the translation mechanism between the C-formalism and the J-formalism ([GKOS01, Section 2.3.2]) remains intact in order to translate results related to point values.

As we are dealing with differential algebras we can define generalized numbers as constant generalized functions, which means those functions R satisfying $D_i R = 0\ \forall i$. For connected Ω this is a natural definition of a space of numbers, generalized points simply are vectors of such numbers. Now as D_i only acts on the x-slot one would be tempted to simply leave it away as we did in the $\mathcal{G}^e$-setting with the hope to get simpler objects. We refrain from doing so, however, because retaining the space of generalized numbers as a subspace of the space of generalized functions has two significant advantages: first, the existing technical background regarding smoothness which lies at the basis of $\mathcal{G}^d$ can be used. Second, the translation mechanism given by the map T works straightforward.

Instead of requiring $D_i R = 0$ one can equivalently demand that the function does not depend on the second slot. We thus come to the following definition of generalized points.

Definition 18.1. Let $V \subseteq \mathbb{R}^p$ be open for some $p \in \mathbb{N}$. Then generalized points of V in the $\mathcal{G}^d$-setting are defined by

$$\begin{aligned}
V_M(\Omega) := \{\, & X \in C^\infty(U(\Omega), V) \mid \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N} \\
& \forall \phi \in C_b^\infty(I \times \Omega, \mathcal{A}_0(\mathbb{R}^n)) : \sup_{x \in K} |\partial^\alpha X(\mathrm{S}_\varepsilon \phi(\varepsilon, x), x)| = O(\varepsilon^{-N}) \\
& \text{and } \forall (\varphi, x), (\varphi, y) \in U(\Omega) : X(\varphi, x) = X(\varphi, y) \,\}, \\
V_N(\Omega) := \{\, & X \in V_M(\Omega) \mid \forall K \subset\subset \Omega\ \forall m \in \mathbb{N}\ \exists q \in \mathbb{N} \\
& \forall \phi \in C_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^n)) : \sup_{x \in K} |X(\mathrm{S}_\varepsilon \phi(\varepsilon, x), x)| = O(\varepsilon^m) \,\}, \\
\widetilde{V}(\Omega) := {} & V_M(\Omega)/V_N(\Omega).
\end{aligned}$$

In order to obtain moderateness estimates of generalized point values one needs to introduce the concept of compactly supported generalized points, as is exemplified in the special algebra resp. elementary full algebra by the estimates

$$|(u(x))_\varepsilon| = |u_\varepsilon(x_\varepsilon)| \le \sup_{x \in K} |u_\varepsilon(x)|$$

resp.

$$|R(X)(\mathrm{S}_\varepsilon \varphi)| = |R(\mathrm{S}_\varepsilon \varphi, X(\mathrm{S}_\varepsilon \varphi))| \le \sup_{x \in K} |R(\mathrm{S}_\varepsilon \varphi, x)|$$

where $x_\varepsilon \in K$ for small ε resp. $X(\mathrm{S}_\varepsilon \varphi) \in K$ for all φ with sufficiently many vanishing moments and small ε. In order to find an analogous condition for $\mathcal{G}^d$ one could start with a representative $X \in V_M(\Omega)$ of a generalized point satisfying $X(\varphi, x) \in L$ for all $(\varphi, x) \in U(\Omega)$ and some compact set $L \subset\subset \Omega$. However, this condition is not preserved under change of representative: if one adds an element Y of $V_N(\Omega)$ to X one can only retain

$$\begin{gathered}
\forall K \subset\subset \Omega\ \exists q \in \mathbb{N}\ \forall \phi \in C_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^n))\ \exists \varepsilon_0 > 0 \\
\forall \varepsilon < \varepsilon_0\ \forall x \in K : (X + Y)(\mathrm{S}_\varepsilon \phi(\varepsilon, x), x) \in L'
\end{gathered}$$

where L' is an arbitrarily small compact neighborhood of L. The reason for this is that negligibility of $Y \in V_N(\Omega)$ gives uniformly small values of $Y(\mathrm{S}_\varepsilon \phi(\varepsilon, x), x)$ (for $x \in K$ and ε small) only if ϕ is an element of $C_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^n))$ for some certain q. This means that if ϕ has less than q vanishing moments $Y(\mathrm{S}_\varepsilon \phi(\varepsilon, x), x)$ may grow in any moderate way, leaving no hope of staying near L or even in any compact subset of V, in general.

The easiest remedy to this problem is to simply define a generalized point $\widetilde{X} \in \widetilde{V}(\Omega)$ as being compactly supported if it has at least one representative X whose image is contained in some compact set and only use such a suitable representative for the definition of point evaluation.

A different approach which is not pursued here but has to be mentioned is to use an equivalent description of $\mathcal{G}^d(\Omega)$ where tests for moderateness and negligibility are performed using test objects having asymptotically vanishing moments. Such an algebra, called $\mathcal{G}^2(\Omega)$, exists and is diffeomorphism invariant ([GFKS01, Section 17]). It was demonstrated by J. Jelínek in [Jel99] that this algebra actually is the same as $\mathcal{G}^d(\Omega)$. Using the moderateness and negligibility conditions of $\mathcal{G}^2(\Omega)$ it would be possible to redefine the spaces used here in order to have a definition of compact support which is stable under change of representatives. In order to be consistent with our formalism of $\mathcal{G}^d$, however, we chose not to take this route here, as it has no effect on the validity of the point value characterization theorem below and because there is no straightforward interface between $\mathcal{G}^2(\Omega)$ and $\mathcal{G}^d(\Omega)$.

Definition 18.2. A generalized point $\widetilde{X} \in \widetilde{V}(\Omega)$ is called compactly supported in $L \subset\subset V$ if it has a representative $X \in V_M(\Omega)$ such that $\forall(\varphi, x) \in U(\Omega)$: $X(\varphi, x) \in L$. Denote by $\widetilde{V_c}(\Omega)$ the subset of all compactly supported generalized points of $\widetilde{V}(\Omega)$.

As usual, elements of $V_M(\Omega)$ resp. $V_N(\Omega)$ are called moderate resp. negligible and we write $X \sim Y$ for $X - Y \in V_N(\Omega)$.

Setting $V = \mathbb{C}$ gives the space $\widetilde{\mathbb{C}}(\Omega)$ of generalized complex numbers over Ω. As $X \in C^\infty(U(\Omega), V)$ is moderate resp. negligible if and only if each component $\mathrm{pr}_i \circ X$ is, [GKOS01, Theorems 2.5.3 and 2.5.4] immediately give a characterization of moderateness resp. negligibility of X in terms of differentials of $X_\varepsilon := X \circ \mathrm{S}^{(\varepsilon)}$: $X \in C^\infty(U(\Omega), V)$ is moderate if and only if $\forall K \subset\subset \Omega$ $\forall \alpha \in \mathbb{N}_0^n$ $\forall k \in \mathbb{N}_0$ $\exists N \in \mathbb{N}$ $\forall B \subseteq \mathcal{D}(\mathbb{R}^n)$ bounded it holds that

$$\left\| \partial^\alpha \mathrm{d}_1^k X_\varepsilon(\varphi, x)(\psi_1, \ldots, \psi_k) \right\| = O(\varepsilon^{-N}) \qquad (\varepsilon \to 0)$$

resp. $X \in V_M(\Omega)$ is negligible if and only if $\forall K \subset\subset \Omega$ $\forall m \in \mathbb{N}$ $\exists q \in \mathbb{N}$ $\forall B \subseteq \mathcal{D}(\mathbb{R}^n)$ bounded it holds that

$$\|X_\varepsilon(\varphi, x)\| = O(\varepsilon^m) \qquad (\varepsilon \to 0),$$

where the estimate has to hold uniformly for $x \in K$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$ resp. $B \cap \mathcal{A}_q(\mathbb{R}^n)$, and $\psi_1, \ldots, \psi_k \in B \cap \mathcal{A}_{00}(\mathbb{R}^n)$.

In the C-setting the point value is obtained as in $\mathcal{G}^s$ and $\mathcal{G}^e$ by inserting the (generalized) point into the x-slot. The corresponding formula for the J-setting is obtained by using the translation mechanism provided by the map T^*. We fix the following abbreviations for the natural definitions of point evaluation in the J- and the C-setting, noting that no confusion can arise from using the expression $R(X)$ in both cases.

1. $R(X)(\varphi, x) := R(\mathrm{T}_{X(\varphi,x)-x}\varphi, X(\varphi, x))$ for $R \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega)$ and $X \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega, \Omega)$, and
2. $R(X)(\varphi, x) := R(\varphi, X(\varphi, x))$ for $R \in C^\infty(U(\Omega))$ and $X \in C^\infty(U(\Omega), \Omega)$.

Because $R(X)$ is not defined on the whole of $\mathcal{A}_0(\Omega) \times \Omega$ resp. $U(\Omega)$ one has to implement a smooth cut-off procedure as in the following proposition. We will do so first in the J-setting because there the smoothness issues are more perspicuous – the topology on $U(\Omega)$ is induced by the mapping T, so questions of smoothness on $U(\Omega)$ are most easily handled by transferring them to $\mathcal{A}_0(\Omega) \times \Omega$.

Proposition 18.3. *Given $R \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega)$ and $X \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega, \Omega)$ satisfying*

$$\exists L \subset\subset \Omega \ \forall (\varphi, x) \in \mathcal{A}_0(\Omega) \times \Omega : X(\varphi, x) \in L \tag{18.1}$$

there exists a map $J_{R,X} \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega)$ such that for any $K \subset\subset \Omega$ and any $B \subseteq \mathcal{D}(\mathbb{R}^n)$ satisfying $\exists \beta > 0\ \forall \omega \in B$: $\operatorname{supp} \omega \subseteq \overline{B_\beta}(0)$ there is a relatively compact open neighborhood U of K in Ω and $\varepsilon_0 > 0$ such that for all $x \in U$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$, and $\varepsilon < \varepsilon_0$ the expression $R(X)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x)$ is defined and

$$J_{R,X}(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x) = R(X)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x).$$

Proof. Let $z \in \Omega$ remain fixed for the following construction. For some $\delta_z > 0$ smaller than $\frac{1}{3}\operatorname{dist}(L, \partial\Omega)$ and $\frac{1}{2}\operatorname{dist}(z, \partial\Omega)$ we set $A_z := B_{\delta_z}(z) \subseteq \Omega$ and $B_z := B_{\delta_z}(A_z) = B_{2\delta_z}(z)$. Both sets are relatively compact in Ω. For all $x \in \overline{A_z}$ and $\varphi \in \mathcal{A}_{0,\overline{B_z}}(\Omega)$ we consequently obtain

$$\begin{aligned} \operatorname{supp} \mathrm{T}_{X(\varphi,x)-x}\varphi &= X(\varphi, x) - x + \operatorname{supp} \varphi \\ &\subseteq L - x + \overline{B_{2\delta_z}}(z) \subseteq L + \overline{B_{3\delta_z}}(0) \subseteq \Omega \end{aligned}$$

which means that $R(X)(\varphi, x) = R(\mathrm{T}_{X(\varphi,x)-x}\varphi, X(\varphi, x))$ is defined on the set $\mathcal{A}_{0,\overline{B_z}}(\Omega) \times A_z$. Furthermore $g_z := R(X)|_{\mathcal{A}_{0,\overline{B_z}}(\Omega) \times A_z} \in C^\infty(\mathcal{A}_{0,\overline{B_z}}(\Omega) \times A_z)$: this follows easily by writing down all maps and spaces involved, after which g_z is seen to be a composition of smooth functions. Set $D_z := B_{\delta_z/2}(A_z)$ and choose a smooth function $\rho_z \in C^\infty(\Omega, \mathbb{R})$ with support in $\overline{B_z}$ and $\rho_z \equiv 1$ on $\overline{D_z}$. Fixing an arbitrary $\varphi_z \in \mathcal{A}_{0,\overline{B_z}}(\Omega)$ define the projection

$$\pi_z(\varphi) := \varphi \cdot \rho_z + (1 - \int \varphi \cdot \rho_z) \cdot \varphi_z \qquad \forall \varphi \in \mathcal{A}_0(\Omega),$$

then clearly $\pi_z \in C^\infty(\mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n))$ and thus $\pi_z \in C^\infty(\mathcal{A}_0(\Omega), \mathcal{A}_{0,\overline{B_z}}(\Omega))$: the restriction to a set carrying the initial smooth structure with respect to the inclusion evidently is smooth, and as π_z has values in $\mathcal{A}_{0,\overline{B_z}}(\Omega)$ and this set also carries the initial smooth structure, π_z is smooth into this set. For $\operatorname{supp} \varphi \subseteq \overline{D_z}$ we have $\pi_z(\varphi) = \varphi$. There exists a smooth partition of unity $\{\chi_z\}_z$ subordinate to $\{A_z\}_z$, that is a collection of maps $\chi_z \in C^\infty(\Omega, [0, 1])$ with $\operatorname{supp} \chi_z \subseteq A_z$ such that set of supports $\{\operatorname{supp} \chi_z\}_z$ is locally finite and $\sum \chi_z(x) = 1\ \forall x \in \Omega$. Define a map f_z on $\mathcal{A}_0(\Omega) \times \Omega$ by

$$f_z(\varphi, x) := \begin{cases} g_z(\pi_z(\varphi), x)\chi_z(x) & \text{if } x \in A_z \\ 0 & \text{otherwise.} \end{cases}$$

We now see that $f_z \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega)$: given a smooth curve $c = (c_1, c_2)$ in $C^\infty(\mathbb{R}, \mathcal{A}_0(\Omega) \times \Omega)$, $c^* f_z$ is smooth because any $t_0 \in \mathbb{R}$ has a neighborhood whose image under c_2 lies either in A_z or in the complement of $\operatorname{supp} \chi_z$, which are open sets covering Ω. In the first case, $g_z(\pi_z(c_1(t)), c_2(t))\chi_z(c_2(t)) = g_z(\pi_z(c_1(t)), \tilde{c}_2(t))\chi_z(\tilde{c}_2(t))$ in a neighborhood of t_0 on which c_2 is equal to some curve $\tilde{c}_2 \in C^\infty(\mathbb{R}, A_z)$, thus one can employ smoothness of $(\varphi, x) \mapsto g_z(\pi_z(\varphi), x)\chi_z(x)$ on $\mathcal{A}_0(\Omega) \times A_z$. In the second case the function is zero on an open neighborhood of $c_2(t_0)$, thus smooth trivially. Now we can define $J_{R,X} \colon \mathcal{A}_0(\Omega) \times \Omega \to \mathbb{C}$ as $J_{R,X}(\varphi, x) := \sum_z f_z(\varphi, x)$, which also is easily seen to be smooth as the sum is locally finite in x. Now let K and B be given as stated in the proposition. K has an open neighborhood U which meets only finitely many supports of the χ_z, which means that there are $z_1, \dots, z_m \in \Omega$ for some $m \in \mathbb{N}$ such that $K \subseteq U \subseteq \bigcup_{i=1\dots m} \operatorname{supp} \chi_{z_i} \subseteq \bigcup_{i=1\dots m} A_{z_i}$, so on $\mathcal{A}_0(\Omega) \times U$ $J_{R,X}$ is given by $\sum_{i=1\dots m} f_{z_i}$. For $\varepsilon < \min_i \delta_{z_i}/(2\beta)$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$ and $x \in A_{z_i}$, $\operatorname{supp} \mathrm{T}_x \mathrm{S}_\varepsilon \varphi \subseteq \overline{B_{\varepsilon\beta}}(x) \subseteq \overline{D_{z_i}}$ and thus $\pi_{z_i}(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi) = \mathrm{T}_x \mathrm{S}_\varepsilon \varphi$; now $x \in \operatorname{supp} \chi_{z_i} \subseteq A_{z_i}$ implies $g_{z_i}(\pi_{z_i}(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi), x) = R(X)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x)$ and thus for $x \in U$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$, and ε as above we finally obtain the conclusion

$$\begin{aligned} J_{R,X}(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x) &= \sum_{i=1\dots m} g_{z_i}(\pi_{z_i}(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi), x)\chi_{z_i}(x) \\ &= R(X)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x) \cdot \sum_{i=1\dots m} \chi_{z_i}(x) = R(X)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x). \qquad \square \end{aligned}$$

Corollary 18.4. *Given $R \in C^\infty(U(\Omega))$ and $X \in C^\infty(U(\Omega), \Omega)$ satisfying*

$$\exists L \subset\subset \Omega \ \forall (\varphi, x) \in U(\Omega) : X(\varphi, x) \in L \tag{18.2}$$

there exists $S_{R,X} \in C^\infty(U(\Omega))$ such that for any $K \subset\subset \Omega$ and $B \subseteq \mathcal{D}(\mathbb{R}^n)$ satisfying $\exists \beta > 0\ \forall \omega \in B$: $\operatorname{supp} \omega \subseteq \overline{B_\beta}(0)$ there is a relatively compact open neighborhood U of K in Ω and $\varepsilon_0 > 0$ such that for all $x \in U$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$, and $\varepsilon < \varepsilon_0$, the expression $R(X)(\mathrm{S}_\varepsilon \varphi, x)$ is defined and

$$S_{R,X}(\mathrm{S}_\varepsilon \varphi, x) = R(X)(\mathrm{S}_\varepsilon \varphi, x).$$

Proof. Set

$$\begin{aligned} R^J &:= (\mathrm{T}^{-1})^* R \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega) \\ X^J &:= (\mathrm{T}^{-1})^* X \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega, \Omega). \end{aligned}$$

Then X^J satisfies (18.1), giving $J_{R^J,X^J} \in C^\infty(\mathcal{A}_0(\Omega) \times \Omega)$. Now by Proposition 18.3 there exists a relatively compact open neighborhood U of K in Ω and $\varepsilon_0 > 0$ such that $\forall x \in U$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$, and $\varepsilon < \varepsilon_0$ we know that $R^J(X^J)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x)$ is defined and $J_{R^J,X^J}(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x) = R^J(X^J)(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x)$. Thus because $\mathrm{T}^*(R^J(X^J)) = R(X)$ we obtain the result by setting $S_{R,X} := \mathrm{T}^* J_{R^J,X^J}$. $\square$

The following proposition establishes that the construction of $S_{R,X}$ defines a unique element of $\widetilde{\mathbb{C}}(\Omega)$ and enables us to use it for the definition of point values in $\mathcal{G}^d(\Omega)$.

Proposition 18.5. *Given $R \in \mathcal{E}^d_M(\Omega)$ and $X, Y \in \Omega_M(\Omega)$ satisfying* (18.2) *$S_{R,X}$ is in $\mathbb{C}_M(\Omega)$; if R is negligible $S_{R,X}$ is, and $X \sim Y$ implies $S_{R,X} \sim S_{R,Y}$.*

Proof. Fix $K \subset\subset \Omega$, $\alpha \in \mathbb{N}_0^n$, and $k \in \mathbb{N}_0$ for testing and let $B \subseteq \mathcal{D}(\mathbb{R}^n)$ be bounded for testing in terms of differentials. Moderateness of $S_{R,X}$ is tested by estimating

$$\left|\partial^\alpha \mathrm{d}_1^k (S_{R,X})_\varepsilon(\varphi, x)(\psi_1, \ldots, \psi_k)\right|$$

where $x \in K$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$, and $\psi_1, \ldots, \psi_k \in B \cap \mathcal{A}_{00}(\mathbb{R}^n)$. Let $J \subseteq \mathbb{R}$ be a bounded neighborhood of 0. Then $B + J\psi_1 + \cdots + J\psi_k$ is bounded in $\mathcal{D}(\mathbb{R}^n)$. Corollary 18.4 gives an open neighborhood U of K in Ω and $\varepsilon_0 > 0$ such that for $x \in U$, $\varphi \in B' \cap \mathcal{A}_0(\mathbb{R}^n)$, and $\varepsilon < \varepsilon_0$ the equation

$$(S_{R,X})_\varepsilon(\varphi, x) = (R(X))_\varepsilon(\varphi, x)$$

holds. Given $\varphi, \psi_1, \ldots, \psi_k$ as above we obtain for the kth differential

$$\begin{aligned}
&\mathrm{d}_1^k (S_{R,X})_\varepsilon(\varphi, x)(\psi_1, \ldots, \psi_k) \\
&\qquad = \left.\frac{\partial}{\partial t_1}\right|_0 \cdots \left.\frac{\partial}{\partial t_k}\right|_0 (S_{R,X})_\varepsilon(\varphi + t_1\psi_1 + \ldots + t_k\psi_k, x) \\
&\qquad = \left.\frac{\partial}{\partial t_1}\right|_0 \cdots \left.\frac{\partial}{\partial t_k}\right|_0 (R(X))_\varepsilon(\varphi + t_1\psi_1 + \ldots + t_k\psi_k, x) \\
&\qquad = \mathrm{d}_1^k (R(X))_\varepsilon(\varphi, x)(\psi_1, \ldots, \psi_k).
\end{aligned}$$

Note that this seemingly trivial equality and the following application of the chain rule rest on two hidden details. First, because in the first slot the mappings $S_{R,X}$ and $R(X)$ are defined on subsets of the affine subspace $\mathcal{A}_0(\Omega)$, their differentials have to be calculated by considering the corresponding maps on the linear subspace $\mathcal{A}_{00}(\Omega)$ which are obtained by pullback along an affine bibounded isomorphism $\mathcal{A}_{00}(\Omega) \to \mathcal{A}_0(\Omega)$. Second, these maps obtained actually have to be restricted to suitable subsets of $\mathcal{A}_0(\mathbb{R}^n) \times \Omega$ in order to give meaning to their differentials (cf. [GKOS01, Section 2.3.3] for a detailed discussion).

As $(R(X))_\varepsilon(\varphi, x) = R_\varepsilon(\varphi, X_\varepsilon(\varphi, x))$, by the chain rule ([GKOS01, Appendix A]) $\mathrm{d}_1^k (R(X))_\varepsilon(\varphi, x)(\psi_1, \ldots, \psi_k)$ consists of terms of the form

$$\begin{aligned}
(\mathrm{d}_2^l \mathrm{d}_1^m R_\varepsilon(\varphi, X_\varepsilon(\varphi, x))(\psi_{i_1}, \ldots, \psi_{i_m},& \\
(\mathrm{d}_1^{a_1} X_\varepsilon)(\varphi, x)(\psi_{A_1}), \ldots, &(\mathrm{d}_1^{a_l} X_\varepsilon(\varphi, x)(\psi_{A_l}))
\end{aligned}$$

where $m, l \in \mathbb{N}_0$, $i_1, \dots, i_m \in \{1 \dots k\}$, $a_1, \dots, a_l \in \mathbb{N}$, and $\psi_{A_1}, \dots, \psi_{A_l}$ are appropriate tuples of elements from $\{\psi_1, \dots, \psi_k\}$. Consequently, the expression $\partial^\alpha \mathrm{d}_1^k (R(X))_\varepsilon(\varphi, x)(\psi_1, \dots, \psi_k)$ consists of terms of the form

$$(\mathrm{d}_2^l \mathrm{d}_1^m \partial^\gamma R_\varepsilon(\varphi, X_\varepsilon(\varphi, x))(\psi_{i_1}, \dots, \psi_{i_m}, \\ (\partial^{\beta_1} \mathrm{d}_1^{a_1} X_\varepsilon(\varphi, x)(\psi_{A_1}), \dots, (\partial^{\beta_l} \mathrm{d}_1^{a_l} X_\varepsilon(\varphi, x)(\psi_{A_l}))$$

where $\gamma, \beta_1, \dots, \beta_l$ are some multi-indices. The norm of the last expression can be estimated by

$$\left\| (\mathrm{d}_2^l \mathrm{d}_1^m \partial^\gamma R_\varepsilon(\varphi, X_\varepsilon(\varphi, x))(\psi_{i_1}, \dots, \psi_{i_m}) \right\| \cdot \\ \cdot \left\| (\partial^{\beta_1} \mathrm{d}_1^{a_1} X_\varepsilon(\varphi, x)(\psi_{A_1}) \right\| \cdots \left\| (\partial^{\beta_l} \mathrm{d}_1^{a_l} X_\varepsilon(\varphi, x)(\psi_{A_l}) \right\|$$

whence the first two claims of the proposition follow immediately from moderateness and negligibility of R, respectively, and moderateness of the compactly supported X.

For the last claim, fix $K \subset\subset \Omega$ and $m \in \mathbb{N}$ for testing and let $B \subseteq \mathcal{D}(\mathbb{R}^n)$ be bounded. Let Y take values in $L \subset\subset \Omega$. We need to estimate the expression $|(S_{R,X} - S_{R,Y})_\varepsilon(\varphi, x)|$ for $x \in K$ and $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$. By Corollary 18.4 there exists an open neighborhood U of K in Ω such that for $x \in U$, $\varphi \in B \cap \mathcal{A}_0(\mathbb{R}^n)$, and small ε we have both $(S_{R,X})_\varepsilon(\varphi, x) = (R(X))_\varepsilon(\varphi, x)$ and $(S_{R,Y})_\varepsilon(\varphi, x) = (R(Y))_\varepsilon(\varphi, x)$, so we have to estimate $|(R(X) - R(Y))_\varepsilon(\varphi, x)|$. Setting $F(t) := R_\varepsilon(\varphi, (Y + t(X - Y))_\varepsilon(\varphi, x))$ the last expression can be written as $|F(1) - F(0)|$. As $X \sim Y$ there exists $q \in \mathbb{N}$ such that for $x \in K$, $\varphi \in B \cap \mathcal{A}_q(\mathbb{R}^n)$, and small ε we have $|(X - Y)_\varepsilon(\varphi, x)| < \varepsilon$, so $F(t)$ is defined and smooth on $[0, 1]$ and we can write

$$|F(1) - F(0)| = \left| \int_0^1 F'(t)\, \mathrm{d}t \right| = \left| \int_0^1 \mathrm{d}_2 R_\varepsilon(\varphi, g(t)) \cdot (X - Y)_\varepsilon(\varphi, x)\, \mathrm{d}t \right|$$

whence the claim follows directly from moderateness of R and negligibility of $X - Y$. □

Definition 18.6. For $\widetilde{R} \in \mathcal{G}^d(\Omega)$ and $X \in \widetilde{\Omega}_c(\Omega)$ we define the generalized point value of $\widetilde{R}$ at $\widetilde{X}$ as $\widetilde{R}(\widetilde{X}) := [S_{R,X}]$ where R is any representative of $\widetilde{R}$ and X is a representative of $\widetilde{X}$ satisfying (18.2).

Lemma 18.7. *Let K be a compact set. Given for each $q \in \mathbb{N}$ a sequence $(x_{q,k})_{k \in \mathbb{N}}$ in K it holds that*

$$\exists x_0 \in K\ \forall \delta > 0\ \forall q_0 \in \mathbb{N}\ \exists q = q(\delta, q_0) \geq q_0 \\ \forall k_0 \in \mathbb{N}\ \exists k = k(\delta, q_0, k_0) \geq k_0 : x_{q,k} \in B_\delta(x_0).$$

This means that x_0 is an accumulation point of infinitely many of the sequences $(x_{q,k})_k$.

Proof. Assuming the converse we would have $\forall x_0 \in K\ \exists \delta = \delta(x_0) > 0$ $\exists q_0 = q_0(x_0) \in \mathbb{N}\ \forall q \geq q_0\ \exists k_0 = k_0(x_0, q)\ \forall k \geq k_0$: $x_{q,k} \notin B_{\delta(x_0)}(x_0)$. As $K \subseteq \bigcup_{x \in K} B_{\delta(x)}(x)$ we can choose $x_1, \ldots, x_m$ ($m \in \mathbb{N}$) such that $K \subseteq \bigcup_{i=1,\ldots,m} B_{\delta(x_i)}(x_i)$. Then for $q \geq \max_i q_0(x_i)$ and $k \geq \max_i k_0(x_i, q)$ we obtain the contradiction $x_{q,k} \notin \bigcup_{i=1,\ldots,m} B_{\delta(x_i)}(x_i) \supseteq K$. □

After these preparations we are finally able to establish the point value characterization theorem for $\mathcal{G}^d(\Omega)$.

Theorem 18.8. *$\widetilde{R} \in \mathcal{G}^d(\Omega)$ is 0 if and only if $\widetilde{R}(\widetilde{X}) = 0$ in $\widetilde{\mathbb{C}}(\Omega)$ for all $\widetilde{X} \in \widetilde{\Omega}_c(\Omega)$.*

Proof. Let R be a representative of $\widetilde{R}$. We have already shown in Proposition 18.5 that $R \in \mathcal{N}^d(\Omega)$ implies $R(X) \in \mathbb{C}_{\mathcal{N}}(\Omega)$ for all $X \in \Omega_M(\Omega)$. For the converse we assume $R \notin \mathcal{N}^d(\Omega)$ and construct a generalized point X such that $R(X) \notin \mathbb{C}_{\mathcal{N}}(\Omega)$. By this assumption there exists $K \subset\subset \Omega$ and $m \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there is some $\phi_q \in C_b^\infty(I \times \Omega, \mathcal{A}_q(\mathbb{R}^n))$ such that $\forall k \in \mathbb{N}$ $\exists \varepsilon_{q,k} < \frac{1}{k}\ \exists x_{q,k} \in K$ such that with $\varphi_{q,k} := \mathrm{S}_{\varepsilon_{q,k}} \phi_q(\varepsilon_{q,k}, x_{q,k})$ we have

$$|R(\varphi_{q,k}, x_{q,k})| \geq \varepsilon_{q,k}^m.$$

For the negligibility test of $R(X)$ to fail it suffices to construct X such that for each of infinitely many q the equation $X(\varphi_{q,k}, x_{q,k}) = x_{q,k}$ holds for infinitely many k. Choose positive real numbers δ and η_1 both smaller than $\mathrm{dist}(x_0, \partial\Omega)$. Lemma 18.7 gives

$$\exists x_0 \in K\ \forall q_0 \in \mathbb{N}\ \exists q = q(\delta, q_0) \geq q_0\ \forall k_0 \in \mathbb{N}$$
$$\exists k = k(\delta, q_0, k_0) \geq k_0 : x_{q,k} \in B_\delta(x_0). \quad (18.3)$$

Furthermore, we know that for all $q \in \mathbb{N}$ there exists an index $k_1(q) \in \mathbb{N}$ such that $\mathrm{supp}\, \mathrm{S}_{\varepsilon_{q,k}} \phi_q(\varepsilon_{q,k}, x_{q,k}) \subseteq B_{\eta_1}(0)$ for all $k \geq k_1(q)$. Combining this with (18.3), there exists a strictly increasing sequence $(q_l)_{l \in \mathbb{N}}$ and for each $l \in \mathbb{N}$ a sequence $(k_{l,r})_{r \in \mathbb{N}}$ with $k_{l,r} \geq k_1(q_l)$ and $x_{q_l, k_{l,r}} \in B_\delta(x_0)$ for all $r \in \mathbb{N}$. Choose $\eta_2 > 0$ arbitrary and set $U := \{\varphi \in \mathcal{D}(\mathbb{R}^n) \mid \|\varphi\|_\infty < \eta_2\}$.

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{N}$ in which each natural number appears infinitely often. Set $\varphi_1 := \varphi_{q_{c_1}, k_{c_1,1}}$ and $x_1 := x_{q_{c_1}, k_{c_1,1}}$. Inductively, given φ_n choose r large enough such that $\left\|\varphi_{q_{c_{n+1}}, k_{c_{n+1},r}}\right\|_\infty > \|\varphi_n\|_\infty + 2\eta_2$ and set $\varphi_{n+1} := \varphi_{q_{c_{n+1}}, k_{c_{n+1},r}}$ and $x_{n+1} := x_{q_{c_{n+1}}, k_{c_{n+1},r}}$.

The sequences $(\varphi_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ then have the following properties:

1. $x_n \in B_\delta(x_0)\ \forall n \in \mathbb{N}$.
2. For each of infinitely many $q \in \mathbb{N}$ there are infinitely many $k \in \mathbb{N}$ such that $\varphi_{q,k}$ resp. $x_{q,k}$ appears in the sequence $(\varphi_n)_n$ resp. $(x_n)_n$.

3. $\operatorname{supp}\varphi_n \subseteq B_{\eta_1}(0)$ for all $n \in \mathbb{N}$.
4. All sets $\overline{U} + \mathrm{T}_{-x_n}\varphi_n$ for $n \in \mathbb{N}$ are pairwise disjoint, as $\|\varphi_n\|_\infty = \|\mathrm{T}_{-x_n}\varphi_n\|_\infty$.

Choose η_3 such that $0 < \eta_3 < \eta_2$. Set $U' := \{\varphi \in \mathcal{D}(\mathbb{R}^n) \mid \|\varphi\|_\infty < \eta_3\}$, $\mathbb{E} := \mathcal{D}_{\overline{B_{\eta_1}(0)}}(\mathbb{R}^n)$ and $U_1' := U' \cap \mathbb{E}$. Construct a smooth bump function $\chi_1 \in C^\infty(\mathbb{E}, \mathbb{R})$ with $\operatorname{supp}\chi_1 \subseteq \overline{U_1'}$ and $\chi_1(0) = 1$ as follows:

Let $g \in C^\infty(\mathbb{R}, \mathbb{R})$ be nonnegative such that $g(x) = 1$ for $x \le 0$ and $g(x) = 0$ for $x \ge 1$. As $\mathbb{E}$ is a nuclear locally convex space, there exist a convex, circled 0-neighborhood $V \subseteq U_1'$ and a positive semi-definite sesquilinear form σ on $\mathbb{E}$ such that $p\colon x \mapsto \sqrt{\sigma(x,x)}$ is the gauge function of V and a continuous seminorm on $\mathbb{E}$ ([Sch71, Chapter III 7.3]). From the Cauchy-Schwartz inequality we infer $|\sigma(x,y)| \le p(x)p(y)$, which means that σ is bounded and thus smooth. Consequently the associated hermitian form $h\colon x \mapsto \sigma(x,x)$ also is smooth. The differentials of h are given by

$$\begin{aligned} \mathrm{d}h(x)(v) &= 2\Re\sigma(x,v), \\ \mathrm{d}^2h(x)(v,w) &= 2\Re\sigma(v,w), \text{ and} \\ \mathrm{d}^3h &= 0 \end{aligned}$$

where $\Re$ denotes the real part. Now $\chi_1 := g \circ h$ is in $C^\infty(\mathbb{E}, \mathbb{R})$ with $\chi_1(0) = 1$ and $\operatorname{supp}\chi_1 \subseteq \overline{V} \subseteq \overline{U_1'} \subseteq U \cap \mathbb{E}$ because $g(h(x)) > 0$ implies $h(x) < 1$ and thus $x \in V$.

Then by [KM97, Lemma 16.6] and an obvious adaptation of the proof of [KM97, Proposition 16.7] there exists a function $\chi \in C^\infty(\mathcal{D}(\mathbb{R}^n), \mathbb{R})$ such that $\chi|_{\mathbb{E}} = \chi_1$, $\chi(0) = 1$ and $\operatorname{supp}\chi \subseteq \overline{U}$.

Set $\chi_m(\varphi) := \chi(\varphi - \mathrm{T}_{-x_m}\varphi_m)$ for $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We define a map $Y\colon \mathcal{D}(\mathbb{R}^n) \times \mathbb{R}^n \to \Omega$ by

$$Y(\varphi, x) := \sum_{m \in \mathbb{N}} (x_0 + \chi_m(\mathrm{T}_{-x}\varphi)(x_m - x_0)) \in B_\delta(x_0).$$

Because the supports of χ_m are disjoint Y has at most one summand near any given φ; it clearly is smooth and as $\mathcal{A}_0(\Omega) \times \Omega$ carries the initial smooth structure with respect to the inclusion its restriction to $\mathcal{A}_0(\Omega) \times \Omega$ also is smooth. Our prospective generalized point is defined as

$$X := \mathrm{T}^*(Y|_{\mathcal{A}_0(\Omega)\times\Omega}) \in C^\infty(U(\Omega), \Omega),$$

and satisfies $X(\varphi_n, x_n) = x_n$. X is compactly supported in $\overline{B_\delta}(x_0)$. In order to show moderateness of X we test in terms of differentials. Fix $K \subset\subset \Omega$, $\alpha \in \mathbb{N}_0^n$, $k \in \mathbb{N}_0$, and $B \subseteq \mathcal{D}(\mathbb{R}^n)$ bounded for testing. We then need to estimate the expression

$$\partial^\alpha \mathrm{d}_1^k X_\varepsilon(\varphi, x)(\psi_1, \ldots, \psi_k)$$

where $x \in K$, $\varphi \in B \cap \mathcal{A}_0(\Omega)$, and $\psi_1, \dots, \psi_k \in B \cap \mathcal{A}_{00}(\Omega)$. We first look at the function whose derivatives we need:

$$X_\varepsilon(\varphi, x) = Y(\mathrm{T}_x \mathrm{S}_\varepsilon \varphi, x) = \sum_m (x_0 + \chi_m(\mathrm{S}_\varepsilon \varphi)(x_m - x_0)).$$

As we see from the right hand side this expression does not depend on x so we only need to consider the case $\alpha = 0$. If the kth differential at φ in directions $\psi_1, \dots, \psi_k$ is nonzero it is given by only one term of the right hand side, so for each φ there exists an index $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\mathrm{d}_1^k X_\varepsilon(\varphi,x)(\psi_1, \dots, \psi_k) \\ &\quad = \mathrm{d}^k\big(\varphi \mapsto (x_0 + \chi_{m_0}(\mathrm{S}_\varepsilon\varphi)(x_{m_0} - x_0))\big)(\varphi)(\psi_1, \dots, \psi_k) \\ &\quad = \mathrm{d}^k\big(\varphi \mapsto (x_0 + \chi(\mathrm{S}_\varepsilon\varphi - \mathrm{T}_{-x_{m_0}}\varphi_{m_0})(x_{m_0} - x_0))\big)(\varphi)(\psi_1, \dots, \psi_k) \end{aligned}$$

In order to use that $\chi|_{\mathbb{E}} = \chi_1$ we need that the support of the argument of χ in the previous expression is contained in $\overline{B_{\eta_1}}(0)$. By construction this is the case for all φ_n and if ε is small enough it is also satisfied for $\mathrm{S}_\varepsilon\varphi$ for all $\varphi \in B$ uniformly. As $\chi_1 = g \circ h$ we need to obtain the differentials

$$\mathrm{d}^k\big(\varphi \mapsto g(h(\mathrm{S}_\varepsilon\varphi - \mathrm{T}_{-x_{m_0}}\varphi_{m_0}))\big)(\varphi, x)(\psi_1, \dots, \psi_k). \tag{18.4}$$

Abbreviate $f(\varphi) := \mathrm{S}_\varepsilon\varphi - \mathrm{T}_{-x_{m_0}}\varphi_{m_0}$. We can assume that $h(f(\varphi)) < 1$ holds, as otherwise expression (18.4) vanishes. By the chain rule we see that the kth differential is given by the product of derivatives of g (which are globally bounded) and terms of the form $\mathrm{d}^k(h \circ f)(\varphi)(\psi_1, \dots, \psi_k)$ for some $k \in \mathbb{N}$ which again by the chain rule are given by terms of the form

$$(\mathrm{d}^k h)(f(\varphi))(\mathrm{d}^{l_1} f(\varphi)(\psi_{A_1}), \dots, \mathrm{d}^{l_k} f(\varphi)(\psi_{A_k})) \tag{18.5}$$

for some $l_1, \dots, l_k \in \mathbb{N}$ and appropriate subsets $\psi_{A_1}, \dots, \psi_{A_k} \subseteq \{\psi_1, \dots, \psi_k\}$. Here only $k = 0, 1, 2$ are relevant as higher derivatives of h vanish. We obtain from (18.5) the three terms

$$\begin{aligned} h(f(\varphi)) &= \sigma(f(\varphi), f(\varphi)) \\ \mathrm{d}h(f(\varphi))(\mathrm{d}f(\varphi)(\psi_1)) &= 2\Re\sigma(f(\varphi), \mathrm{d}f(\varphi)(\psi_1)) \\ \mathrm{d}^2 h(f(\varphi))(\mathrm{d}f(\varphi)(\psi_1), \mathrm{d}f(\varphi)(\psi_2)) &= 2\Re\sigma(\mathrm{d}f(\varphi)(\psi_1), \mathrm{d}f(\varphi)(\psi_2)) \end{aligned}$$

The function f is differentiated at most once because its higher order derivatives vanish. Noting that $\mathrm{d}f(\varphi)(\psi) = \mathrm{S}_\varepsilon\psi$ we estimate these terms by the Cauchy-Schwartz inequality. We obtain products of $\sqrt{h(f(\varphi))}$ (which has been assumed to be smaller than 1) and $\sqrt{h(\mathrm{S}_\varepsilon\psi)} = p(\mathrm{S}_\varepsilon\psi)$ (where ψ is ψ_1 or ψ_2). Being a continuous seminorm, p is majorized by finitely many of the usual seminorms q_α of $\mathbb{E}$ given by $q_\alpha(\varphi) = \sup_{x \in \mathbb{R}^n} |\partial^\alpha \varphi(x)|$ for all $\alpha \in \mathbb{N}_0^n$.

We thus end up with the expression

$$q_\alpha(\mathrm{S}_\varepsilon\psi) = \sup_{x\in\mathbb{R}^n} |\partial^\alpha(\mathrm{S}_\varepsilon\psi)(x)| = \sup_{x\in\mathbb{R}^n} \left|\partial^\alpha(\varepsilon^{-n}\psi(x/\varepsilon))\right|$$
$$= \sup_{x\in\mathbb{R}^n} \left|\varepsilon^{-n-|\alpha|}(\partial^\alpha\psi)(x/\varepsilon)\right| = \varepsilon^{-n-|\alpha|}\,\|\psi\|_\infty$$

and as ψ is from the bounded set B we finally obtain the desired growth estimates independently of m_0 and conclude that X is moderate. By construction $R(X)$ is not negligible and the point value characterization theorem is established. □

Appendices

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Kurzfassung

Diese Dissertation behandelt drei verwandte Themenbereiche im Gebiet der vollen diffeomorphismeninvarianten Colombeau'schen Algebren.

Teil I umfasst eine Erweiterung der Theorie der vollen diffeomorphismeninvarianten Colombeau'schen Algebren ([GKSV02]) auf den Fall von Tensorfeldern auf Riemannschen Mannigfaltigkeiten. Eine wesentliche Rolle spielt dabei der Levi-Civita-Zusammenhang mittels welchem distributionelle Tensorfelder regularisiert und somit auf eine kanonische Art und Weise in einen Raum nichtlinearer verallgemeinerter Tensorfelder eingebettet werden können. Dies steht im Gegensatz zu einer verwandten Konstruktion ([GKSV09]) in der an Stelle des Zusammenhanges auf der Mannigfaltigkeit ein zusätzlicher Regularisierungsparameter für verallgemeinerte Tensorfelder eingeführt wurde, was im Vergleich zur vorliegenden Variante technisch aufwändiger ist.

Die wesentliche Frage zum konstruierten Raum verallgemeinerter Tensorfelder ist, ob die Einbettung von distributionellen Tensorfeldern mit Pullback entlang von Diffeomorphismen und Lie-Ableitungen kommutiert. Im Allgemeinen ist dies nicht der Fall, was ein Hauptresultat dieser Arbeit darstellt; jedoch erhält man ein positives Ergebnis für solche Operationen, welche die zugrunde liegende Struktur der Riemannschen Mannigfaltigkeit respektieren, das heißt für Pullback entlang von Isometrien beziehungsweise Lie-Ableitungen entlang von Killing-Vektorfeldern.

Teil II gibt eine detaillierte Beschreibung der Topologie auf Tensorprodukten von Schnitträumen endlichdimensionaler Vektorbündel, die für die Beschreibung von distributionellen Tensorfeldern nützlich ist. Man erhält dadurch bornologisch isomorphe Darstellungen letzterer als Ergänzung zur vorhandenen Literatur ([Gro08, GKOS01]).

Teil III schließlich gibt eine Punktwertecharakterisierung für verallgemeinerte Funktionen in der lokalen diffeomorphismeninvarianten Theorie, welche zuvor nur in einfacheren Fällen verfügbar war ([KO99]).

Abstract

This thesis presents three related topics in the field of full diffeomorphism-invariant Colombeau algebras.

Part I consists of an extension of the theory of full diffeomorphism-invariant Colombeau algebras ([GKSV02]) to the setting of generalized tensor fields on Riemannian manifolds. The Levi-Civita connection is used as a key element to regularize distributional tensor fields and thus embed them in a canonical way into a space of nonlinear generalized tensor fields. This stands in contrast to a related construction ([GKSV09]) in which instead of a connection on the manifold an additional regularization parameter of generalized tensor fields was used, which is technically more involved.

The central question about the constructed space of generalized tensor fields is whether the embedding of distributional tensor fields commutes with pullback along diffeomorphisms and Lie derivatives. In general this is not the case, which is a main result of this work. One gets however a positive answer for operations respecting the structure of the Riemannian manifold, i.e., for pullbacks along isometries and Lie-derivatives along Killing vector fields.

Part II gives a detailed description of the topology on tensor products of spaces of sections of finite dimensional vector bundles which is used for the description of distributional tensor fields. One obtains bornologically isomorphic representations of the latter, which complements the existing literature ([Gro08, GKOS01]).

Part III finally gives a point value characterization for generalized functions in the local full diffeomorphism-invariant theory. Previously, such a characterization has been available only in simpler cases ([KO99]).

Curriculum Vitæ

Eduard Nigsch

Date and Place of Birth: March 10, 1983 in Feldkirch, Austria

Academic Degree: Dipl.-Ing.

Current Position: Ph.D. student, scientific coworker

Children: Valentin and Florentin

Education:

Since Dec 2006: Doctoral studies of Mathematics, University of Vienna

2001 – 2006: Studies of Technical Mathematics, Vienna University of Technology. Diploma thesis "Colombeau generalized functions on manifolds", Dipl.-Ing. (comparable to "Master of Science") with distinction

1994 – 2001: Grammar school in Feldkirch, final examination passed with distinction

Academic Employment:

Dec 2006 - Dec 2010 (with 13 months of parental leave): Ph.D. scholar in the doctoral college "Differential Geometry and Lie Groups" of the University of Vienna; research assistant, START-project Y237 of the Austrian Science Foundation (FWF)

Dec 2006 - Jan 2008: Research assistant, project P16742 of the Austrian Science Foundation (FWF)