

Abstract

Massless bosonic string-localized quantum fields are studied. After reviewing some facts about massive fields, the corresponding intertwiners for vector- and tensor potentials and for more general representations are constructed. These fields are fixed by the requirement that they are generalized potentials for the field strengths. Furthermore it is proven that they satisfy generalized versions of the Lorentz- and axial gauge and certain symmetry properties. It is also illustrated why they need additional indices. In the end their two-point function is analyzed and it is shown that their short-distance behavior is independent of the helicity.

Zusammenfassung

Masselose bosonische string-lokalisierte Quantenfelder werden untersucht. Nach einem kurzen Überblick über massive Felder werden die Intertwiner für Vektor- und Tensor-Potentiale und für allgemeinere Darstellungen konstruiert. Diese Felder sind eindeutig festgelegt durch die Forderung, dass sie verallgemeinerte Potentiale für die Feldstärken sind. Weiters wird bewiesen, dass sie verallgemeinerte Versionen der Lorentz- und der axialen Eichung erfüllen und gewisse symmetrie Eigenschaften besitzen. Es wird außerdem erläutert, warum sie zusätzliche Indizes benötigen. Zum Schluss wird noch deren Zwei-Punkt Funktion untersucht und es wird gezeigt, dass ihr Verhalten für kleine Abstände unabhängig von der Helizität ist.

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1 Introduction and Motivation

The principle of locality is one of the most important in modern quantum field theory. It states that observables are measurable in bounded space-time regions and that measurements of space-like separated observables are compatible. Typically one uses point-localized quantum fields (or more precisely quantum fields that are located in bounded space-time regions). To implement locality they should commute for space-like separated arguments. But in the modern formalism of local quantum field theory [8] only the observables have to be localized in bounded regions, the (usually unobservable) fields can have different localization properties.

One possibility to generalize the notion of localization are the recently studied string-localized fields [16]. The basis for their analysis is provided by the paper of Brunetti, Guido and Longo [4], where they establish a connection between irreducible positive energy representations of the Poincaré group and localization in space-like cones. This is achieved by using the formalism of modular localization, which is summarized in chapter 2. The cores of this space-like cones are then the localization regions of the string-localized fields, which are the subject of the present work.

While for representations of the Poincaré group for finite helicity the localization region can be tightened to a double cone (i.e. point-like localization), this is not possible for the so-called infinite spin representations (also called “continuous spin”) [32], where the sharpest possible localization regions are space-like cones with arbitrarily small opening angles. Nevertheless, string-like generating fields for massless particles and finite helicity can also be useful and are interesting to investigate for the following reasons:

- For massless particles it is well known, that in the classical point-like formalism the number of possible intertwiners between the canonical Wigner representation and a covariant representation is strongly limited [30, p.254]. By using string-like intertwiners one is able to recover the full spectrum of representations. Particularly interesting is the case of helicity ± 1 (Photons), where one has to give up either covariance or positivity of the Hilbert space norm in order to work with a vector potential [27]. Using string-localized intertwiners one can construct a covariant vector field acting on the photon Hilbert space, which is a potential for the (point-localized) field strength.
- The string-localized vector (or tensor) potentials offer a better short distance behavior. While the short distance dimension (sdd) of the point-localized field strengths

is increasing with the helicity, the sdd of the string-localized potentials is independent of the spin. This fact also guarantees that the p -dependence of the two-point function does not get worse with increasing spin.

- Although string-localized fields could not yet be implemented in perturbation theory, they are expected to admit a larger class of interactions, because of their nice short distance behavior [14].

Despite the interesting possibility to incorporate string-localized fields in perturbation theory, only non-interacting fields will be considered in this work.

The first time that string-like objects appeared was 1935, when P. Jordan [9] used exponential line-integrals over electromagnetic vector potentials to construct gauge invariant terms out of matter fields. Some decades later Mandelstam [11] used expressions involving integrals over local gauge fields to formulate quantum electrodynamics. In more recent times Buchholz and Fredenhagen [5] systematically analyzed the properties of semi-infinity string-like objects in the formalism of algebraic quantum field theory. At about the same time it was discovered that string-localized fields appear naturally (for massive representations of the Poincaré group) in space-time dimension $d = 1+2$, where non-half-integer spins are possible. These particles, which obey neither Bose nor Fermi statistics, but more general braid group statistics, are called anyons. String-localized anyon states were constructed and studied e.g. in [12] by J. Mund.

Due to the popularity of String Theory, the name “string-localized” fields can be a bit misleading. The string-localized fields considered in this paper have very little to do with the strings of String Theory. The construction of string-localized objects in this work fits perfectly into the well-known framework of quantum field theory. The only difference lies in the more general localization properties of the fields considered.

The content of the present work is the following: At first, after reviewing some basic facts about modular localization and Wigner particle theory, some results about massive string-fields from the paper by Mund, Schroer, Yngvason [16] are recapitulated. In their work they show a detailed construction of string-localized fields for massive representations and especially for the massless infinite spin representation. Then, string-localized intertwiners for massless representations of the Poincaré group to arbitrary (integer) helicity λ are constructed. In contrast to the massive “scalar” string fields in [16], the fields for mass zero need additional tensor (or spinor) indices. They also obey certain “gauge”-conditions, like

a generalization of the Lorentz-gauge and the axial gauge. However, the use of the word “gauge”-conditions is incorrect in the present framework, because they are not due to a gauge-freedom but are fixed by the desired properties of the fields.

The string-localized tensor potentials transform according to the $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$ representation of the Lorentz group, which would not be possible in the classical point-like formalism [30, sec. 5.9]. The requirement that they are potentials for the point-localized field strengths fixes them almost uniquely. Interestingly, there is also the possibility to write these fields as infinite line-integrals over the field-strengths.

After that, intertwiners for general representations of the Lorentz group $D^{[A,B]}$ are presented in the spinor formalism and their connection with the tensor fields is established. Also some properties of these fields are analyzed, e.g. their short distance behavior and their two-point functions.

The whole work is mainly based on the article by J. Mund, B. Schroer and J. Yngvason [16] (and other related articles by these authors [13, 14, 15, 23]), because these are almost the only rigorous treatments of the topic of constructing string-localized quantum fields.

2 Modular Localization

2.1 Motivation

In his famous paper from 1939 [31] E. Wigner established the general connection between relativistic particles and irreducible positive energy representations of the Poincaré group \mathcal{P} . This was one of the first steps toward a formulation of relativistic quantum theory without classical analogies. In this setting there are two different notions of localization: The *Born-Newton-Wigner localization* [17] and the modern approach called *modular localization* [4, 7].

The *Newton-Wigner localization* uses Born's quantum mechanical probability density in Wigner's representation theoretical setting. One introduces position operators and their spectral projectors, which are supposed to measure the probability to find a particle at a specific space-time point. For a fixed time one gets orthogonality of states with different spatial support. The problem is, that this is not consistent with relativistic covariance and causality (although approximately for distances larger than the Compton wave length). Moreover it's impossible to have *any* localization concept, where states localized in a given space-time region are orthogonal to those in the causal complement, if one wants to stay compatible with covariance and positivity of the energy [10, 19]. Despite this shortcomings of the Newton-Wigner localization, it is well-suited for scattering theory, because of its validity for large space-time separation.

A way to illustrate the Newton-Wigner localization [8] is to consider a wave function $\tilde{\Psi}(p)$ and Fourier transform it with the Lorentz invariant measure on the mass-shell $d\mu(p)$ ¹ to get a covariant wave function in x -space,

$$\Psi(x) = \frac{1}{(2\pi)^2} \int \tilde{\Psi}(p) e^{-ipx} d\mu(p) := \frac{1}{(2\pi)^2} \int \tilde{\Psi}(\epsilon_p, \vec{p}) e^{-i(\epsilon_p t - \vec{p}\vec{x})} \frac{d^3p}{2\epsilon_p}, \quad (2.1)$$

where ϵ_p is the relativistic energy $\epsilon_p^2 = m^2 + \vec{p}^2$. This is a covariant wave function, but using it to write down the scalar product, one gets

$$\langle \Psi', \Psi \rangle = \int \overline{\Psi'(x)} \Psi(y) K(x-y) d^4x d^4y, \quad (2.2)$$

with the non-local kernel $K(x-y)$, which is proportional to the Fourier transform of the energy factor $\frac{1}{2\epsilon_p}$.

Even if the wave functions $\Psi'(x), \Psi(x)$ have disjoint support, their scalar product doesn't

¹an explicit expression for this measure will be given later on

vanish, because of the non-local factor $K(x - y)$. To get a scalar product that takes the same form as in the Schrödinger theory, one can introduce the Newton-Wigner wave function

$$\tilde{\Psi}^{NW}(p) := \frac{1}{(2\epsilon_p)^{1/2}} \tilde{\Psi}(p), \quad (2.3)$$

and correspondingly the Newton-Wigner wave function in x -space as its Fourier transform. The scalar product then takes the usual form

$$\langle \Psi', \Psi \rangle = \int_{x^0=t} \overline{\Psi'^{NW}(x)} \Psi^{NW}(x) d^3x. \quad (2.4)$$

One could now regard $\Psi^{NW}(x)$ as the probability amplitude for finding the particle at time $x^0 = t$ at the position \vec{x} , but this definition of locality is dependent on the Lorentz-frame and not strictly compatible with causality [19, 10].

A more recent concept of localization, which is compatible with covariance and causality, is the *modular localization*. Here one uses the formalism of algebraic quantum field theory [8], where localization refers to local measurements of observables, not to particle positions. The observables in quantum field theory form a net of algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, where \mathcal{O} is a space-time region and $\mathcal{A}(\mathcal{O})$ is the associated von Neumann algebra, e.g. $\mathcal{A}(\mathcal{O})$ could be the algebra of smeared field operators $\mathcal{A}(\mathcal{O}) = \{\phi(f) \mid \text{supp}(f) \subset \mathcal{O}\}$, acting on the Fock space $\mathcal{F}(\mathcal{H}_1)$.

Important in this context is the famous Reeh-Schlieder property [20], the fact that applying the operators $\mathcal{A}(\mathcal{O})$ (for any open space time region \mathcal{O}) to the vacuum results in a dense set in the Hilbert space. Therefore the subspaces of the Hilbert space obtained in this way can never be orthogonal, even if the space-time regions are spacelike separated. That's why locality is implemented through commutativity of observables, localized in spacelike separated regions, and not through orthogonality of Hilbert space vectors with disjoint spatial support.

2.2 Construction of localization spaces

The concept of modular localization does not need any external assumptions, but is intrinsically defined within the representation theory of the Poincaré group \mathcal{P} . (The following treatment on modular localization is based on [4] and [16, sec. 2].)

Following Wigner [31] one starts with a unitary irreducible representation U_1 of \mathcal{P}_+ on

the one-particle Hilbert space \mathcal{H}_1 (For explicit expressions for \mathcal{H}_1 and U_1 see chapter 3). One then defines a standard wedge region

$$W_0 := \{x \in \mathbb{R}^4 \mid x^3 > |x^0|\}. \quad (2.5)$$

Other more general wedges W can be constructed by Poincaré-transforming the standard wedge W_0 . Moreover, because $\mathcal{P}_+^{\uparrow 2}$ acts transitively on the family of wedges, every wedge can be obtained by a suitable Poincaré-transformation of W_0 :

$$\text{for every wedge } W, \exists g \in \mathcal{P}_+^{\uparrow} : W = gW_0. \quad (2.6)$$

Now consider the family of Lorentz boosts $\Lambda_{W_0}(t)$, which leave W_0 invariant (they form a one-parameter subgroup $t \mapsto \Lambda_{W_0}(t)$) and the reflection $j_{W_0} \in \mathcal{P}_+$ across the edge of the wedge, where the edge of W_0 is the two-dimensional plane $\{x \in \mathbb{R}^4 : x^0 = x^3 = 0\}$. Their explicit representation as 4×4 matrices is,

$$\Lambda_{W_0}(t) := \begin{pmatrix} \cosh(t) & 0 & 0 & \sinh(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(t) & 0 & 0 & \cosh(t) \end{pmatrix}, \quad (2.7a)$$

$$j_{W_0} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.7b)$$

Since \mathcal{P}_+^{\uparrow} acts transitively on the family of wedges, one can assign to each wedge W such a one-parameter subgroup $\Lambda_W(t)$, which leaves W invariant, and a time reversing reflection j_W , mapping W into its causal complement W' . Using the Poincaré transformation $gW_0 = W$, they are defined according to

$$\Lambda_W := g \Lambda_{W_0} g^{-1} \quad (2.8a)$$

$$j_W := g j_{W_0} g^{-1} \quad , \text{ for } gW_0 = W \quad (2.8b)$$

² \mathcal{P}_+^{\uparrow} denotes the proper, orthochronous part of the Poincaré group. In this work only the part \mathcal{P}_+ will be of importance.

Using the (anti-)unitary representation U_1 of \mathcal{P}_+^3 one can then define the operators [16, sec. 2]

$$\Delta_W^{it} := U_1(\Lambda_W(-2\pi t)) \quad (2.9a)$$

$$J_W := U_1(j_W) \quad (2.9b)$$

$$S_W := J_W \Delta_W^{\frac{1}{2}} \quad (2.9c)$$

If Δ_W is written in exponential form $\Delta_W = e^{H_W}$ one gets the so-called *modular Hamiltonian* H_W [8]. For its connection to thermal states and the Unruh effect, the reader is referred to [8, chapter 5]. The operator S_W is the so-called *Tomita involution*. The connection between its polar decomposition $(J_W, \Delta_W^{\frac{1}{2}})$ and the geometric interpretation in terms of the Poincaré group is the application of the spatial *Bisognano-Wichmann theorem* [2] to the Wigner one-particle theory.

The operators (2.9a)-(2.9c) have the following properties: (the proofs can be found in [4])

- Δ_W is a densely defined, closed, positive and in general unbounded operator.
- J_W is anti-unitary with $J_W^2 = \mathbb{1}$.
- $J_W \Delta_W^{it} J_W^{-1} = \Delta_W^{it} \implies J_W \Delta_W J_W^{-1} = \Delta_W^{-1}$
- From these properties follows: S_W is a densely defined, closed, anti-linear and unbounded operator with $Ran(S_W) = Dom(S_W)$ and $S_W^2 \subset \mathbb{1}$

The Tomita involution S_W is uniquely defined by its eigenspace to the eigenvalue 1,

$$K(W) := \{\psi \in Dom(\Delta_W^{\frac{1}{2}}) \mid S_W \psi = \psi\}. \quad (2.10)$$

This is a closed, real-linear standard subspace of \mathcal{H}_1 in the sense of [21], where standardness means that,

$$\overline{K(W) + iK(W)}^4 = \mathcal{H}_1 \quad (2.11a)$$

$$K(W) \cap iK(W) = 0. \quad (2.11b)$$

Because of $U_1(g)S_W U_1(g)^{-1} = S_{gW}$, the representation U_1 acts covariantly on this family of standard subspaces, i.e.

$$U_1(g)K(W) = K(gW) \quad , g \in \mathcal{P}_+ \quad (2.12)$$

³For $m = 0$ and finite helicity, an irreducible representation of \mathcal{P}_+^\uparrow has to be doubled in order to allow for representing the anti-unitary reflection [30]

⁴The bar denotes closure in the Hilbert space norm.

Furthermore the subspaces $K(W)$ satisfy the important property [4, Thm. 2.5]

$$J_W K(W) = K(W') = K(W)^\perp \quad (2.13)$$

where \perp refers to the symplectic complement

$$K(W)^\perp := \{\psi \in \mathcal{H}_1 \mid \text{Im}(\psi, \phi) = 0, \forall \phi \in K(W)\}. \quad (2.14)$$

This is the spatial version of the so-called *Haag-duality*.

Conversely every real standard subspace K defines a unique involution S , according to $S(\psi + i\phi) = \psi - i\phi$, for $\psi, \phi \in K$.

It is important to note here that the Tomita operators S_W only differ in their domains K_W , while their action on Hilbert space vectors looks the same for every wedge W . The unboundedness of the operators S_W makes it possible to encode geometric localization properties into the domain of S_W .

Another crucial fact, which makes modular localization such a useful concept, is that positivity of the energy implies *isotony* [4, Thm. 3.4], where isotony means that

$$W_1 \subset W_2 \Rightarrow K(W_1) \subset K(W_2). \quad (2.15)$$

Until now only wedge regions have been considered, but a sharpening of the localization can be obtained for causally complete convex regions \mathcal{O} . For this purpose one simply intersects the localization spaces for all wedges that include the region \mathcal{O} [4]:

$$K(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} K(W) \quad (2.16)$$

Some important properties like *covariance*, *isotony*, *locality* and *Haag duality* also hold for these smaller subspaces (for a proof see [4]). But the important question is for which \mathcal{O} one can also prove standardness. In their paper [4] Brunetti, Guido and Longo showed, that standardness (among some other properties) holds, for every positive energy representation of the Poincaré group, if \mathcal{O} is a *space-like cone* \mathcal{C} , i.e. [16, sec. 2]

$$\mathcal{C} := a + \bigcup_{\lambda} \lambda D, \quad (2.17)$$

with D a double cone⁵ and a the apex of the space-like cone.

Using this theorem one gets a resulting net $\mathcal{C} \rightarrow K(\mathcal{C})$ of standard subspaces $K(\mathcal{C})$ with the following properties (for a proof see [4]):

⁵A double cone can be equally defined as the intersection of a forward light-cone with a backward light-cone, or the causal completion of a 3-dimensional sphere.

- Covariance: $U_1(g)K(\mathcal{C}) = K(g\mathcal{C})$, $g \in \mathcal{P}_+$
- Locality: $\mathcal{C}_1 \subset \mathcal{C}'_2 \Rightarrow K(\mathcal{C}_1) \subset K'(\mathcal{C}_2)$
- Isotony: $\mathcal{C}_1 \subset \mathcal{C}_2 \Rightarrow K(\mathcal{C}_1) \subset K(\mathcal{C}_2)$
- Haag duality: $K(\mathcal{C}') = K'(\mathcal{C})$

It may seem at first sight, that locality follows trivially by isotony, but this is not the case, because the causal complement \mathcal{C}' of a spacelike cone is not a convex region.

After the construction of these standard subspaces, one can also define the corresponding Tomita operators according to

$$S_{\mathcal{C}}(\Psi + i\Phi) = \Psi - i\Phi, \quad \forall \Psi, \Phi \in K(\mathcal{C}). \quad (2.18)$$

The modular objects $\Delta_{\mathcal{C}}$ and $J_{\mathcal{C}}$ can then be obtained by forming the polar decomposition $S_{\mathcal{C}} = J_{\mathcal{C}}\Delta_{\mathcal{C}}^{\frac{1}{2}}$. In contrast to the operators for wedge regions though, there is in general no geometrical interpretation for these operators.

So far only the one-particle Hilbert space \mathcal{H}_1 and subspaces thereof have been considered, but for quantum field theory one needs the Fock space over \mathcal{H}_1 and operator algebras acting on it. In the next section it will be explained how to get von Neumann algebras from the above standard subspaces.

2.3 Construction of a net of von Neumann algebras⁶:

After having constructed such a family of real subspaces $K(\mathcal{O})$ one can use the CCR/Weyl second quantization functor to get a (bosonic) net of von Neumann algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ on the Fock space $\mathcal{H} = \mathcal{F}(\mathcal{H}_1)$ over \mathcal{H}_1 . The bosonic Fock space is defined according to

$$\mathcal{F}(\mathcal{H}_1) := \bigoplus_{n=0}^{\infty} \mathcal{H}_1^{\otimes_{s^n}}, \quad (2.19)$$

where $\mathcal{H}_1^{\otimes_{s^n}}$ denotes the symmetrized tensor product of n factors \mathcal{H}_1 and $\mathcal{H}_1^0 := \mathbb{C}$ by definition.

Given any open region \mathcal{O} of Minkowski space, one can then define the corresponding von Neumann algebra,

$$\mathcal{A}(\mathcal{O}) := \{Weyl(\psi) \mid \psi \in K(\mathcal{O})\}'' , \quad (2.20)$$

⁶This is only a brief outline. A more thorough treatment can be found in [8] or [7]

⁷ \mathcal{A}' denotes the commutant of \mathcal{A} in $\mathcal{B}(\mathcal{H})$

by introducing the Weyl operators [16]

$$Weyl(\psi) := \exp i(a^*(\psi) + a(\psi)) \in \mathcal{B}(\mathcal{H}). \quad (2.21)$$

These operators are elements of $\mathcal{B}(\mathcal{H})$, which denotes the algebra of all bounded operators acting on \mathcal{H} , and $a^*(\psi)/a(\psi)$ are the creation/annihilation operators on the Fock space. The double commutant theorem of von Neumann guarantees that the algebra $\mathcal{A}(\mathcal{O})$ is weakly closed.

Because the functorial relations (2.20) and (2.21) relate the orthocomplemented lattice of real subspaces to that of von Neumann algebras, the algebras $\mathcal{A}(\mathcal{O})$ satisfy the following properties [1]:

- $\mathcal{A}(\mathcal{O}) = \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W)$, i.e. the Weyl functor commutes with the sharpening of localization.
- Haag duality: $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$
- Covariance: $\mathcal{A}(g\mathcal{O}) = U(g)\mathcal{A}(\mathcal{O})U(g)^{-1}$, where $g \in \mathcal{P}_+$ and $U(g)$ is the second quantization of the operator $U_1(g)$.
- Isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$
- Locality: $\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$

The last three points, together with positivity of the energy, are often called the *Haag-Kastler axioms* of algebraic quantum field theory. These are the desired properties of a net of local observables in local quantum physics.

Using the Weyl functor one can also define the second quantization of the operators Δ_W, J_W and S_W acting on the Fock space \mathcal{H} [7, sec. 3]. Their adjoint action on the algebra $\mathcal{A}(W)$ is given by

$$J_W \mathcal{A}(W) J_W^{-1} = \mathcal{A}(W)', \quad (2.22a)$$

$$\Delta_W^{it} \mathcal{A}(W) \Delta_W^{-it} = \mathcal{A}(W). \quad (2.22b)$$

So the anti-unitary J_W maps the algebra on its commutant and the operators Δ_W^{it} define a family of automorphisms of the algebra.

The Bisognano-Wichmann theorem now states that the above definition of the operator

S_W coincides with the following one, which is possible for a large class of quantum field theories:

$$S_W \phi(f)\Omega = \phi(f)^*\Omega \quad , \text{if } \text{supp}(f) \subset W \quad (2.23)$$

Here $\phi(f)$ is the considered quantum field and Ω is the Poincaré invariant vacuum vector.

For the representations of the Poincaré group to mass m , spin s and to mass 0 and finite helicity the localization can be tightened to double cones by using the well known point-like fields in the Wightman framework [26]. The only fields where this isn't possible are the fields transforming according to the infinite spin representation. However, string-like generating fields are nevertheless interesting for $m = 0$ and finite helicity, for reasons that will be explained later in this work.

3 Wigner particle theory

3.1 Representations of the Lorentz Group

Transformations of Minkowski space, which leave the metric $ds^2 = dt^2 - d\vec{x}^2$ invariant, lead to the Poincaré-group [3]. It is the semi-direct product of the Lorentz group \mathcal{L} and the translation group \mathbb{R}^4

$$\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}. \quad (3.1)$$

The Lorentz group consists of real 4×4 matrices Λ , that satisfy the condition

$$\Lambda^\top \eta \Lambda = \eta, \quad (3.2)$$

where η is the Minkowski metric $\eta = \text{diag}(1, -1, -1, -1)$. Because of this product structure, elements of \mathcal{P} will be denoted by (a, Λ) , where $a \in \mathbb{R}^4$ and $\Lambda \in \mathcal{L}$. The product of two elements of the Poincaré group reads

$$(a_1, \Lambda_1) \circ (a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2), \quad (3.3)$$

which is why it is called a *semi*-direct product.

The Lorentz group is composed of four disconnected components

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_-^\downarrow, \quad (3.4)$$

and in this chapter mainly the proper orthochronous part \mathcal{L}_+^\uparrow will be considered. (For an introduction to relativistic symmetry in particle physics see [30, chapter 2 and 5] or [24].)

To determine the representations U of \mathcal{P}_+^\uparrow , it is comfortable and customary to consider infinitesimal transformations $U(a, \Lambda) \approx \text{id} - \frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} + i a_\mu P^\mu$, with a 4-vector a_μ and an anti-symmetric matrix $\omega_{\mu\nu}$. The generators of the translations P^μ and the generators of Lorentz transformations $J^{\mu\nu}$ are elements of the Lie-algebra of the Poincaré group. A representation of this Lie-algebra is provided by a 4-vector P^μ and a set of antisymmetric matrices $J_{\mu\nu}$, satisfying the commutation relations [24, p.280]

$$i[J_{\mu\nu}, J_{\rho\sigma}] = J_{\nu\sigma} \eta_{\mu\rho} + J_{\rho\nu} \eta_{\mu\sigma} - J_{\mu\sigma} \eta_{\nu\rho} - J_{\rho\mu} \eta_{\nu\sigma}, \quad (3.5a)$$

$$i[J_{\mu\nu}, P_\rho] = \eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu, \quad (3.5b)$$

$$i[P_\mu, P_\nu] = 0. \quad (3.5c)$$

The anti-symmetric generators $J_{\mu\nu}$ can be divided into an angular momentum part

$$J_1 = J_{23}, \quad J_2 = J_{31}, \quad J_3 = J_{12} \quad (3.6)$$

and a boost part

$$K_1 = J_{10}, \quad K_2 = J_{20}, \quad K_3 = J_{30}. \quad (3.7)$$

It is, however, more convenient to introduce a new basis in the (complexified) Lie-algebra by forming the linear combinations [30, p.230]

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) \quad (3.8a)$$

$$\mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}). \quad (3.8b)$$

These new generators now satisfy the simpler commutation relations

$$[A_i, A_j] = i\epsilon_{ijk}A_k \quad (3.9a)$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k \quad (3.9b)$$

$$[A_i, B_j] = 0. \quad (3.9c)$$

The Lie-algebra of the Lorentz group therefore decomposes as the direct sum of two Lie-algebras, which have the structure of the rotation algebra $SO(3)$ or its universal covering group $SU(2)$. Thus its representations can be classified as follows: The irreducible representations of \mathcal{L}_+^\uparrow are of the form

$$D^{[A,B]} := D^{(A)} \otimes D^{(B)}, \quad (3.10)$$

where $D^{(A)}, D^{(B)}$ are the irreducible representations of the covering of the rotation group $SU(2)$ to highest weight (which corresponds to the spin) A, B . Hence the product representation has dimension $(2A+1)(2B+1)$. This is a proper representation of \mathcal{L}_+^\uparrow , if the values of A and B are both integer or both half-integer. Otherwise one obtains a representation of $SL(2, \mathbb{C})$, the universal covering group of the Lorentz group. Representations of $SL(2, \mathbb{C})$ then lead to ray representations (representations up to a phase) of \mathcal{L}_+^\uparrow (see section 5.5).

A general Lorentz transformation $\Lambda(\vec{v}, \vec{\alpha})$ with boost vector \vec{v} and rotation axis $\vec{\alpha}$ is then represented by [24, p.232]

$$D^{[A,B]}(\Lambda(\vec{v}, \vec{\alpha})) = D^{(A)}(\vec{\alpha})D^{(A)}(-i\vec{u}) \otimes D^{(B)}(\vec{\alpha})D^{(B)}(i\vec{u}), \quad (3.11)$$

where the rapidity $\vec{u} := \frac{\vec{v}}{|\vec{v}|} \operatorname{artanh}(|\vec{v}|)$ has been introduced. More explicit expressions for $D^{(j)}(\vec{\alpha})$ will be given in chapter 3.2. and 5.5.

When restricted to the rotation subgroup, such a representation decomposes as [24, p.232]

$$D^{[A,B]}(L(0, \vec{\alpha})) = D^{(A+B)}(\vec{\alpha}) \oplus \dots \oplus D^{(|A-B|)}(\vec{\alpha}), \quad (3.12)$$

which is known as the Clebsch-Gordon decomposition.

In chapter 5 it will be shown how to construct string-localized fields that transform under general $[A, B]$ representations.

3.2 Single Particle States

In this section the classification of one-particle states according to their transformation under the Poincaré group is considered. (The summary given here is based on chapter 2 of Weinberg's book [30]). This classification dates back to the famous paper from Wigner [31], where he established the general connection between particles and irreducible ray representations of the Poincaré group. His work was a first step toward an intrinsic formulation of relativistic quantum physics without the use of classical analogies. In the present work mainly bosons will be considered, so only proper representations of the Poincaré group will be needed.

To obtain these representations U_1 one fixes the mass value m , which determines the energy-momentum spectrum of the corresponding particle. This is the joint spectrum of the generators of the translation subgroup. For positive mass m it has the form of a mass-hyperboloid

$$H_m^+ := \{p \in \mathbb{R}^4 : p^2 = m^2, p^0 > 0\}, \quad (3.13)$$

and for mass zero it is the forward light cone H_0^+ .

Then a point $\bar{p} \in H_m^+$ is fixed, the so-called *standard momentum*, which determines its stabilizer subgroup (also called “little group”) $G_{\bar{p}}$ of \mathcal{L}_+^\uparrow , i.e.

$$G_{\bar{p}} := \{\Lambda \in \mathcal{L}_+^\uparrow : \Lambda\bar{p} = \bar{p}\}. \quad (3.14)$$

For mass $m > 0$ the standard momentum is usually taken to be $\bar{p} = (0, 0, 0, m)$, which is the momentum of the particle in its rest frame. In this case the group $G_{\bar{p}}$ is isomorphic to the three-dimensional rotation group $SO(3)$ [24, p.291].

If massless particles are considered, there is no rest frame, so a possible and convenient choice for the standard momentum is e.g. $\bar{p} = (1, 0, 0, 1)$. The group $G_{\bar{p}}$ is then isomorphic to the Euclidean group in two dimensions $E(2)$ [24, p.294]. This group $E(2)$ consists of rotations $\mathcal{R}(\theta)$ and translations $S(\alpha, \beta)$ of the two-dimensional plane. In contrast to $SO(3)$ this group is not semi-simple, which causes interesting complications.

The massless case then leads to the so-called infinite spin representations if $G_{\bar{p}}$ is represented faithfully, and to a helicity representation if it is not faithfully but non-trivially

represented. The infinite spin representations have usually been discarded as “not used by nature” [30]. But although a localization in the sense of point-like fields is not possible for them [32], there is no reason why they shouldn’t be considered as string-like localized fields.

Besides the value for the mass m , one needs an additional data to fully characterize an irreducible positive energy representation of \mathcal{P}_+^\uparrow , namely a unitary irreducible representation D of the stabilizer group $G_{\bar{p}}$. This representation acts in the so-called “little Hilbert space” \mathfrak{h} , which determines the full representation space of U_1 ,

$$\mathcal{H}_1 := L^2(H_m^+, d\mu_m(p)) \otimes \mathfrak{h} = L^2(H_m^+, d\mu_m(p); \mathfrak{h}), \quad (3.15)$$

where $d\mu_m(p)$ is the Lorentz invariant measure on H_m^+ . It can be written as

$$d\mu_m(p) = \delta(p^2 - m^2)\Theta(p^0)d^4p.{}^8 \quad (3.16)$$

Now the representation U_1 acts on vectors $\Psi(p, \sigma) \in \mathcal{H}_1$, according to

$$(U_1(a, \Lambda)\Psi)(p, \sigma) = e^{iap} \sum_{\sigma'} D(R(\Lambda, p))_{\sigma\sigma'} \Psi(\Lambda^{-1}p, \sigma'). \quad (3.17)$$

The dependence on σ corresponds to the spin or the helicity of the particle respectively, and $R(\Lambda, p)$ denotes the Wigner-rotation ([30, p.65], but note that Weinberg uses the notation $W(\Lambda, p) = R(\Lambda, \Lambda p)$!)

$$R(\Lambda, p) := L_p^{-1} \Lambda L_{\Lambda^{-1}p}, \quad (3.18)$$

where L_p is a Lorentz-boost that maps \bar{p} to p , i.e. $L_p \bar{p} = p$. (For a detailed derivation of these equations see e.g. [30, sec. 2.5].) The representation U_1 of \mathcal{P}_+^\uparrow obtained in this way is said to be induced by the representation D of $G_{\bar{p}}$.

For $m > 0$ the irreducible unitary representations of $G_{\bar{p}} \cong SO(3)$ are given by the well-known spin j representations $D^{(j)}(R)$ of dimensionality $\dim \mathfrak{h} = 2j + 1$, with $j = 0, \frac{1}{2}, 1, \dots$. (Again, a proper representation of the rotation group is only obtained for $j = \text{integer}$. For $j = \text{half-integer}$ one gets a representation of $SU(2)$, which leads to a representation up to a phase of $SO(3)$ (see section 5.5).)

These irreducible representations are generated by the standard matrices for infinitesimal rotations $J_k^{(j)} \in \mathfrak{so}(3)$ [30, p.68],

$$(J_1^{(j)} \pm J_2^{(j)})_{\sigma'\sigma} = \delta_{\sigma', \sigma \pm 1} \sqrt{(j \pm \sigma)(j \pm \sigma + 1)}, \quad (3.19a)$$

$$(J_3^{(j)})_{\sigma'\sigma} = \sigma \delta_{\sigma'\sigma}, \quad (3.19b)$$

⁸For mass $m = 0$ the subscript 0 will be omitted for convenience.

where σ can take the values $j, j-1, \dots, -j$. Hence a rotation about the axis $\vec{\alpha}$ can be written as

$$D(R(\vec{\alpha})) = e^{-i\vec{\alpha}\vec{J}}. \quad (3.20)$$

For $m = 0$ a general element of $G_{\bar{p}} \cong E(2)$ can be written as [30, p.69-73]

$$W(\alpha, \beta, \theta) = S(\alpha, \beta)\mathcal{R}(\theta), \quad (3.21)$$

where $S(\alpha, \beta)$ is a combined rotation and boost in the x - y plane and $\mathcal{R}(\theta)$ is a rotation around the z -axis by the angle θ :

$$S(\alpha, \beta) = \begin{pmatrix} 1+\gamma & \alpha & \beta & -\gamma \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \gamma & \alpha & \beta & 1-\gamma \end{pmatrix}, \quad \gamma = \frac{\alpha^2 + \beta^2}{2} \quad (3.22a)$$

$$\mathcal{R}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.22b)$$

For infinitesimal transformations one gets

$$W(\alpha, \beta, \theta) = 1 - i\alpha M - i\beta N - i\theta J_3, \quad (3.23)$$

where the generators M, N are defined according to

$$M = K_1 + J_2, \quad (3.24a)$$

$$N = K_2 - J_1. \quad (3.24b)$$

From (3.5a) their commutation relations can be read off:

$$[J_3, M] = iN, \quad (3.25a)$$

$$[J_3, N] = -iM, \quad (3.25b)$$

$$[M, N] = 0. \quad (3.25c)$$

Due to this commutation relations one can see that both the S - and the \mathcal{R} -part are Abelian subgroups, but only the S -part is a normal subgroup, so there are two different kinds of representations. On the one hand faithful representations, which lead to so-called infinite spin particles [32], and on the other hand the helicity representations, where the part $S(\alpha, \beta)$ is represented trivially. In the second case the general element $W(\alpha, \beta, \theta)$ is then represented by the one-dimensional matrix [30, p.72]

$$D^{(\sigma)}(W(\alpha, \beta, \theta)) = e^{i\sigma\theta}. \quad (3.26)$$

This is the representation of $E(2)$ that will be important in this work.

Because the helicity is invariant under \mathcal{L}_+^\uparrow , one could think of massless particles of each different helicity as different types of particles and only work in the Hilbert space for one specific helicity. However, particles of opposite helicity are related by space or time inversions. Thus, assuming the PCT-Theorem holds, one has to extend a representation U_1 of \mathcal{P}_+^\uparrow to a representation of \mathcal{P}_+ , the proper Poincaré group (see [16, sec.3]).

In the relevant case of $m = 0$ and finite helicity, however, the Hilbert space has to be doubled. One takes the direct sum of the Hilbert spaces, $\mathcal{H}_1^{(\lambda)}$, $\mathcal{H}_1^{(-\lambda)}$ for helicities $\sigma = +\lambda$ and $\sigma = -\lambda$:

$$\mathcal{H}_1 := \mathcal{H}_1^{(\lambda)} \oplus \mathcal{H}_1^{(-\lambda)} \quad (3.27)$$

On this Hilbert space the extended representation of \mathcal{P}_+ acts as follows. Taking the reflection j_0 at the edge of the wedge W_0 from chapter 2, one can form an anti-unitary involution $D(j_0)$, which extends D to a representation of \mathcal{L}_+ . (For explicit expressions and a proof of the representation property see [16, Lemma B1].) Now an anti-unitary involution $U_1(j_0)$ can be defined by

$$(U_1(j_0)\Psi)(p, \sigma) = D(j_0)\Psi(-j_0p, -\sigma). \quad (3.28)$$

This $U_1(j_0)$ extends U_1 to a representation of \mathcal{P}_+ .

3.3 Quantum Fields

After having constructed the single particle space \mathcal{H}_1 and the representation U_1 , one can go over to the corresponding quantum field $\Phi(x)$ by introducing the creation and annihilation operators $a^*(\Psi)/a(\Psi)$ on the Fock space $\mathcal{F}(\mathcal{H}_1)$ over \mathcal{H}_1 . Taking a homogenous element of the Fock space $\phi_1 \otimes_s \dots \otimes_s \phi_n \in \mathcal{H}_1^{\otimes_s n} \subset \mathcal{F}(\mathcal{H}_1)$, they act on it in the following way:

$$a^*(\psi)(\phi_1 \otimes_s \dots \otimes_s \phi_n) = \sqrt{n+1}(\psi \otimes_s \phi_1 \otimes_s \dots \otimes_s \phi_n), \quad (3.29a)$$

$$a(\psi)(\phi_1 \otimes_s \dots \otimes_s \phi_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \langle \psi, \phi_k \rangle (\phi_1 \otimes_s \dots \hat{\phi}_k \dots \otimes_s \phi_n), \quad (3.29b)$$

where the hat means that $\hat{\phi}_k$ is omitted and of course $a(\psi)\Omega = 0$, where Ω is the Fock vacuum. These creation and annihilation operators satisfy the commutation relations

$$[a(\psi), a(\phi)] = [a^*(\psi), a^*(\phi)] = 0, \quad (3.30a)$$

$$[a(\psi), a^*(\phi)] = \langle \psi, \phi \rangle \text{id}_{\mathcal{F}}. \quad (3.30b)$$

Then, to define operators $a^*(p, \sigma)$, $a(p, \sigma)$, one can write symbolically [16, sec. 3.1]

$$a^*(\Psi) =: \sum_{\sigma} \int d\mu(p) \Psi(p, \sigma) a^*(p, \sigma), \quad (3.31a)$$

$$a(\Psi) =: \sum_{\sigma} \int d\mu(p) \overline{\Psi(p, \sigma)} a(p, \sigma). \quad (3.31b)$$

Under the representation U they transform according to [30, sec. 5.1]:

$$U(\Lambda) a^*(p, \sigma) U(\Lambda^{-1}) = \sum_{\sigma'} a^*(\Lambda p, \sigma') D(R(\Lambda, \Lambda p))_{\sigma' \sigma}. \quad (3.32)$$

This can be seen by using the transformation behavior (3.17) of the single particle state $\Psi(p, \sigma)$ in the expression

$$U(\Lambda) a^*(\Psi) U(\Lambda)^{-1} = a^*(U(\Lambda)\Psi). \quad (3.33)$$

Now, if the quantum field $\Phi(x)$ is written as a Fourier-transform,

$$\Phi(x) = \sum_{\sigma} \int d\mu(p) [e^{ipx} u(p, \sigma) a^*(p, \sigma) + e^{-ipx} \overline{u(p, \sigma)} a(p, \sigma)], \quad (3.34)$$

with the “wave function” $e^{ipx} u(p, \sigma)$, one gets a non-local transformation behavior in x , because of the p -dependent transformation matrix $D(R(\Lambda, p))$ in (3.32).

To overcome this, the “Wigner bases” are replaced by “covariant bases”. This is achieved by demanding that the wave functions $u(p, \sigma)$ in (3.34) obey the intertwiner equation [30, p. 195]

$$\sum_{\sigma'} D(R(\Lambda, p))_{\sigma \sigma'} u(\Lambda^{-1} p, \sigma') = u(p, \sigma) D'(\Lambda). \quad (3.35)$$

The representation $D'(\Lambda)$ then specifies the transformation behavior of the field $\Phi(x)$. A similar intertwiner relation also holds in the string-localized case for string-like intertwiners $u(p, e, \sigma)$ (see section 5.1). With the intertwiner function $u(p, \sigma)$, satisfying (3.35), the quantum field $\Phi(x)$ transforms according to

$$U(a, \Lambda) \Phi(x) U(a, \Lambda)^{-1} = \Phi(\Lambda x + a) D'(\Lambda). \quad (3.36)$$

All of these relations will be calculated explicitly in the string-localized case in chapter 5. The main idea for the construction of such intertwiners is to first specify them for the standard momentum \bar{p} and then apply an appropriate boost to obtain the intertwiner for arbitrary momentum.

An important result (which can be found in [30, sec. 5.7]) is, that these intertwiner functions $u(p, \sigma)$ exist for massive particles with spin j for every representation $D^{[A,B]}$, satisfying the relation

$$|A - B| \leq j \leq A + B \quad (\text{cf. equation (3.12)}). \quad (3.37)$$

For massless particles, however, the possible representations are more restricted. For helicity σ , intertwiners can only be constructed for representations satisfying [30, sec. 5.9]

$$\sigma = B - A. \quad (3.38)$$

Because photons have helicity $\sigma = \pm 1$, vector potentials, transforming according to $D^{[\frac{1}{2}, \frac{1}{2}]}$, are in principle excluded from this formalism. This is why one usually has to use the formalism of quantum gauge theory [22], including unphysical ghost degrees of freedom, to incorporate vector potentials for photons into quantum field theory. Another possibility to overcome the restriction (3.38) is to use string-localized intertwiners for massless particles, which will be constructed in chapter 5. That way one can recover the full spectrum of possible intertwiners (3.37).

4 String-localized Fields

4.1 Definition and General Construction

In this chapter the exact definition of a string-localized quantum field will be given and some of the results of [16] will be recapitulated.

The term “string” in string-localized fields denotes a ray, which starts at a point x of Minkowski space and extends to infinity in a space-like direction e . The vector e is an element of the three-dimensional manifold of space-like directions, denoted by H^3 :

$$H^3 := \{e \in \mathbb{R}^4 : e \cdot e = -1\}^9 \quad (4.1)$$

H^3 is a submanifold of Minkowski space and can be seen as a three-dimensional deSitter space. The string can be written as $S_{x,e} = x + \mathbb{R}^+ e$ and can be envisaged as the core of a space-like cone. These space-like cones can then be seen as the localization regions of chapter 2.

To properly define what is meant by a string-localized quantum field, one considers the Fock space $\mathcal{H} = \mathcal{F}(\mathcal{H}_1)$ over the one-particle Hilbert space \mathcal{H}_1 with Poincaré invariant vacuum vector Ω . Furthermore one has a unitary representation of the Poincaré group U_1 on \mathcal{H}_1 and its second quantization on the Fock space.

The following definition is a slight generalization of the definition given in [13, Definition 1]. The only difference is, that the fields considered here are allowed to have additional indices (subsumed in the index r) and thus are not scalar, but transform according to a certain representation D of the Lorentz group.

Definition 1. *A free string-localized quantum field is an operator valued distribution $\Phi_r(x, e)$ over $\mathbb{R}^4 \times H^3$ acting on $\mathcal{H} = \mathcal{F}(\mathcal{H}_1)$, satisfying the following properties:*

- i) String-locality: If the strings $x_1 + \mathbb{R}^+ e'_1$ and $x_2 + \mathbb{R}^+ e_2$ are space-like separated for all e'_1 in an open neighborhood of e_1 , then the fields $\Phi_r(x_1, e_1), \Phi_{r'}(x_2, e_2)$ commute, i.e.*

$$[\Phi_r(x_1, e_1), \Phi_{r'}(x_2, e_2)] = 0. \quad (4.2)$$

- ii) Covariance: Given $(x, e) \in \mathbb{R}^4 \times H^3$ and an element of the Poincaré group $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, then the field has the following transformation behavior for a certain representation*

⁹Here and in the following only four-dimensional Minkowski space will be considered.

of the Lorentz group $D(\Lambda)$:

$$U(a, \Lambda)\Phi_r(x, e)U(a, \Lambda)^{-1} = \sum_{r'} D(\Lambda^{-1})_{rr'}\Phi_{r'}(\Lambda x + a, \Lambda e) \quad (4.3)$$

iii) Positivity of the energy: *The restriction of the representation U to the translation subgroup satisfies the spectrum condition. This means that the spectrum of the generators of the translation group lies in the forward light cone.*

iv) Free fields: *The field creates only single particle states when acting on the vacuum vector Ω , i.e.*

$$\Phi_r(f, g)\Omega \in \mathcal{H}_1, \quad (4.4)$$

where f, g are functions over \mathbb{R}^4, H^3 respectively.

The definition of string-locality given in *i*) is equivalent to demanding that the strings $x_1 + \mathbb{R}^+e_1$ and $x_2 + \mathbb{R}^+e_2$ are space-like separated *and* that the directions e_1 and e_2 are space-like separated, cf. [16, Lemma A1].

Using the Jost-Schroer theorem for string-fields [25], one can reduce the problem of constructing such fields to the construction of single particle vectors $\Psi_r(f, h) = \Phi_r(f, h)\Omega \in \mathcal{H}_1$ [13]. However, it has to be noted that in [25] the Jost-Schroer theorem has only been proven for fields with a fixed direction e . Work in proving it for string-localized fields, where e can vary over all space-like directions (in the sense of Definition 1), is currently in progress. The requirement of string-locality (4.2) for the field $\Phi_r(f, h)$ can then be translated to the following one-particle version: Given a space-like cone \mathcal{C} , then [13, Definition 2]

$$\Psi_r(f, h) = \Phi_r(f, h)\Omega \in \text{Dom}(S_{\mathcal{C}}) \quad , \quad \text{whenever } \text{supp}(f) + \mathbb{R}^+\text{supp}(h) \subset \mathcal{C}, \quad (4.5)$$

where $S_{\mathcal{C}}$ is the Tomita operator defined in (2.18). This means that for $\text{supp}(f)$ and $\text{supp}(h)$ bounded, the wave function $\Psi_r(f, h)$ is localized in the so-called *truncated space-like cone* $\text{supp}(f) + \mathbb{R}^+\text{supp}(h)$.

Due to the concept of modular localization, this property is intrinsic to the representation U_1 of the Poincaré group and it does not need any reference to the field $\Phi_r(x, e)$. Such a covariant and local \mathcal{H}_1 valued distribution $\Psi_r(x, e)$ is called a *string-localized wave function* for U_1 [13, Definition 2].

This facts suggest to reverse the strategy, namely to construct such a $\Psi_r(x, e)$ first and

then obtain the field $\Phi_r(x, e)$ via second quantization. A detailed construction of scalar one-particle vectors $\Psi(x, e)$ and a mathematically rigorous treatment of their properties can be found in [16, sec. 3]. The difference to the case of massless, finite helicity particles lies mainly in just writing more indices.

To get a covariant transformation behavior of $\Phi_r(x, e)$ after second quantization, one uses so-called intertwiner functions $u_\sigma(p, e)_r$ (see section 5.1), which satisfy the intertwiner relation

$$\sum_{\sigma'} D(R(\Lambda, p))_{\sigma\sigma'} u_{\sigma'}(\Lambda^{-1}p, e)_r = \sum_{r'} D^{[A,B]}(\Lambda^{-1})_{rr'} u_\sigma(p, \Lambda e)_{r'}, \quad (4.6)$$

where D is the Wigner representation of the little group and $R(\Lambda, p)$ is the Wigner rotation. These intertwiners $u_\sigma(p, e)_r$ have to obey certain bounds and analyticity properties (see [16, Definition 3.1]) in order to guarantee the desired localization properties of $\Psi_r(f, h)$. For $f \in \mathcal{S}(\mathbb{R}^4)$ and $h \in \mathcal{D}(H^3)$ one can then define the single-particle wave functions $\Psi_r(f, h, \sigma)$ by [16, sec. 3]

$$\Psi_r(f, h, \sigma)(p) := (Ef)(p)u_\sigma(p, h)_r, \quad (4.7)$$

where Ef denotes the restriction to the mass shell of the Fourier transform of f :

$$(Ef)(p) := \frac{1}{(2\pi)^2} \int d^4x e^{ipx} f(x) \Big|_{p \in H_0^+} \quad (4.8)$$

The smeared $u_\sigma(p, h)_r$ is defined according to

$$u_\sigma(p, h)_r := \int d\sigma(e) h(e) u_\sigma(p, e)_r, \quad (4.9)$$

where $d\sigma(e)$ is the Lorentz invariant measure on H^3 : $d\sigma(e) = \delta(e^2 + 1)d^4e$.

The main part of this work will therefore be dedicated to the problem of constructing such intertwiner functions for different possible massless representations of the Poincaré group. After their construction, the fields $\Phi_r(x, e)$ can be defined as follows: Let $a^*(\Psi)/a(\Psi)$ denote the creation and annihilation operators on the Fock space. Then $a^*(p, \sigma)/a(p, \sigma)$ can be defined implicitly by [16, sec. 3]

$$a^*(\Psi) =: \sum_\sigma \int d\mu(p) \Psi(p, \sigma) a^*(p, \sigma) \quad (4.10a)$$

$$a(\Psi) =: \sum_\sigma \int d\mu(p) \overline{\Psi(p, \sigma)} a(p, \sigma). \quad (4.10b)$$

The field $\Phi_r(x, e)$ can then be written as

$$\Phi_r(x, e) = \sum_{\sigma} \int d\mu(p) \left(e^{ipx} u_{\sigma}(p, e)_r a^*(p, \sigma) + e^{-ipx} \overline{u_{\sigma}(p, e)_r} a(p, \sigma) \right). \quad (4.11)$$

This field satisfies the requirement of locality, it is covariant and it satisfies the Reeh-Schlieder property [16, Theorem 3.3].

4.2 Necessity of Additional Tensor (Spinor-) Indices

In contrast to [16, 13] the massless fields $\Phi_r(x, e)$ in this work need additional indices (except for helicity $\lambda = 0$) to guarantee a covariant transformation behavior. All possible (spinor or tensor) indices will be subsumed in the index r here for reasons of simplicity. Therefore these fields are not “scalar” in the sense of [16], but transform under a certain representation of the Lorentz group. This is one of the main differences to most of the fields considered in [16] and it is necessitated by the different stabilizer groups of massive and massless particles (see chapter 3).

The reason that one needs additional indices in the case of massless particles is the following. In section 3.2, the authors of [16] describe the general idea of constructing scalar intertwiner functions for faithful representations of the Poincaré group (i.e. the massive and the massless infinite spin case). They consider the pullback representations of the stabilizer subgroups G on specific G -orbits Γ ,

$$\Gamma = \{q \in H_0^+ : q \cdot \bar{p} = 1\}, \quad (4.12)$$

where \bar{p} is the standard momentum, satisfying $R\bar{p} = \bar{p}, \forall R \in G$. In the case of massive fields, i.e. $\bar{p} = (0, 0, 0, m)$, this G -orbit Γ is isomorphic to the two-dimensional sphere S_2 , and in the case of massless fields, i.e. $\bar{p} = (1, 0, 0, 1)$, it is isomorphic to the euclidean plane \mathbb{R}^2 .

Considering functions $v(q)$ on $L^2(\Gamma, d\nu)$, where $d\nu$ is the G -invariant measure on Γ , one gets a unitary representation \tilde{D} of G ,

$$(\tilde{D}(R)v)(q) := v(R^{-1}q), \quad R \in G. \quad (4.13)$$

Now they exploit the fact that \tilde{D} decomposes into the direct sum (or direct integral for $m = 0$) of all faithful representations of G . Hence for every faithful representation D of G

there is a partial isometry V , which intertwines \tilde{D} and D ,

$$D(R)V = V\tilde{D}(R), \quad R \in G. \quad (4.14)$$

Introducing the function $F(w) = w^\alpha$, for suitable $\alpha \in \mathbb{C}$, one can define intertwiner functions $u(p, e)$

$$u(p, e) := VF(q \cdot L_p^{-1}e), \quad (4.15)$$

where L_p denotes the Lorentz boost which maps \bar{p} to p . It is then easy to see, that this $u(p, e)$ satisfies the desired intertwiner relation

$$D(R(\Lambda, p))u(\Lambda^{-1}p, e) = u(p, \Lambda e). \quad (4.16)$$

The massless finite helicity representations, however, are not faithful representations of the stabilizer group $G = E(2)$, because the translations in $E(2)$ are represented trivially there. Therefore the above procedure for constructing intertwiners cannot work. That's why it is not possible to have "scalar" intertwiners for $m = 0$, but needs to introduce additional indices and a more general transformation behavior and thus gets a slightly more complicated intertwiner relation.

4.3 Previous Results on String-Localized Fields

In [16] Mund, Schroer and Yngvason proved a lot of important properties, which the scalar versions of the above intertwiners and the corresponding massive and massless infinite spin fields obey. The following is a brief summary of their major results.

- Free covariant string-localized fields can be constructed in dimension $d = 4$ and $d = 3$ for all irreducible representations of the Lorentz group with faithful or trivial representation of the little group. This is true especially for the massless infinite spin representations, which do not allow for a description in the sense of point-localized fields.

Apart from covariance and string-locality these fields obey the Reeh-Schlieder and the Bisognano-Wichmann properties [16, Theorem 3.3, i)].

- Every string-localized field is (up to unitary equivalence) of the form

$$\int d\mu(p) [e^{ipx} u(p, e) \cdot a^*(p) + h.c.]^{10}. \quad (4.17)$$

and the intertwiner functions $u(p, e)$ are unique up to multiplication with a function $F(e \cdot p)$ [16, Theorem 3.3, ii) and iii)], which has to be meromorphic in the upper half of the complex plane. This means, if $u(p, e)$ and $\hat{u}(p, e)$ are both intertwiners, then there is a function $F(e \cdot p)$, such that $u(p, e) = F(e \cdot p)\hat{u}(p, e)$.

- The string-localized fields for massive representations can be written as infinite line integrals over point-like tensor fields [16, Theorem 4.4],

$$\phi(x, e) = \int_0^\infty f(t) \sum_r \Phi_r(x + te) w(e)_r, \quad (4.18)$$

with a function $f(t)$, supported in the interval $[0, \infty)$. The last factor $w(e)_r$ is a tensor in e , which satisfies the condition

$$w(e)_r = D'(\Lambda^{-1})_{rr'} w(\Lambda e)_{r'}, \quad (4.19)$$

where $D'(\Lambda)$ is the representation of the Lorentz group according to which the tensor field $\Phi_r(x)$ transforms. For the infinite spin representation this is not possible, because there exist no point-localized fields to integrate over.

- The massive string-localized fields have a better UV-behavior than the point-like realization of these fields. More precisely the distributional character of the free fields is less singular than that of the point-like fields, especially in the direction of the string e .

For large p the intertwiner functions $u(p, e)$ are bounded in p , independent of the spin s , whereas the intertwiners for point-like fields go at least like $|p|^s$. [16, sec. 4.3].

- Photons can also be described by a string-localized field, but it needs an additional 4-vector index, $A_\mu(x, e)$. It can be defined as an infinite line integral over the field strength tensor,

$$A_\mu(x, e) = \int_0^\infty dt f(t) \mathcal{F}_{\mu\nu}(x + te) e^\nu. \quad (4.20)$$

¹⁰Here and in the following, h.c. stands for *hermitian conjugate*

The function $f(t)$ turns out to be the Heaviside function $\Theta(t)$ and it is essentially fixed by the requirement that $A_\mu(x, e)$ is a vector potential for $\mathcal{F}_{\mu\nu}(x)$ [16, sec. 5], i.e. the expression

$$\partial_\mu A_\nu(x, e) - \partial_\nu A_\mu(x, e) = \mathcal{F}_{\mu\nu}(x) \quad (4.21)$$

has to be independent of the string direction e .

Furthermore, the photon field satisfies the Lorentz ($\partial^\mu A_\mu(x, e) = 0$) and the axial gauge condition ($e^\mu A_\mu(x, e) = 0$).

The last result was the motivation and the starting point for the present work, where string-localized tensor fields for higher helicities and different representations of the Poincaré group will be studied and the above results for the photon field will be generalized to higher helicities.

5 Massless String-Localized Fields

5.1 String-Localized Intertwiners

In chapter 3 point-localized fields were constructed for different representations of the Lorentz-group, by introducing intertwiner functions $u(p, \sigma)$, which intertwine the Wigner bases with covariant bases. The same procedure is possible for string-localized fields, introducing string-localized intertwiner functions $u(p, e, \sigma)$.

One starts with string-like one particle states [16, sec. 3.1]

$$\Psi(f, h)(p, \sigma) : \mathcal{S}(\mathbb{R}^4) \times \mathcal{D}(H^3) \rightarrow \mathcal{H}_1 = L^2(H_m^+, d\mu; \mathfrak{h}) \quad (5.1)$$

and their corresponding creation and annihilation operators on Fock space $a^*(\Psi), a(\Psi)$, where $a^*(p, \sigma)$ is again implicitly defined by

$$a^*(\Psi) = \sum_{\sigma} \int d\mu(p) \Psi(p, \sigma) a^*(p, \sigma). \quad (5.2)$$

Inserting the transformation behavior of the state into this equation one gets:

$$\begin{aligned} U(\Lambda) a^*(\Psi) U(\Lambda)^{-1} &= \sum_{\sigma} \sum_{\sigma'} \int d\mu(p) D_{\sigma\sigma'}(R(\Lambda, p)) \Psi(\Lambda^{-1}p, \sigma') a^*(p, \sigma) \\ &= \sum_{\sigma'} \int d\mu(p) \Psi(\Lambda^{-1}p, \sigma') \sum_{\sigma} a^*(p, \sigma) D_{\sigma\sigma'}(R(\Lambda, p)) \\ &= \sum_{\sigma} \int d\mu(p) \Psi(p, \sigma) \sum_{\sigma'} a^*(\Lambda p, \sigma') D_{\sigma'\sigma}(R(\Lambda, \Lambda p)) \end{aligned} \quad (5.3)$$

From this the transformation behavior of $a^*(p, \sigma)$ can be read off:

$$U(\Lambda) a^*(p, \sigma) U(\Lambda)^{-1} = \sum_{\sigma'} a^*(\Lambda p, \sigma') D_{\sigma'\sigma}(R(\Lambda, \Lambda p)) \quad (5.4)$$

Using this creation and annihilation operators and certain string-like intertwiner functions $u_{\sigma}(p, e)$, one can define the string-localized quantum field by [16, Thm. 3.3]

$$\Phi(x, e)_r = \sum_{\sigma} \int d\mu(p) \left(e^{ipx} u_{\sigma}(p, e)_r a^*(p, \sigma) + e^{-ipx} \overline{u_{\sigma}(p, e)_r} a(p, \sigma) \right). \quad (5.5)$$

This field should transform covariantly under a certain representation of the Lorentz group $D'(\Lambda)$

$$U(a, \Lambda) \Phi(x, e)_r U(a, \Lambda)^{-1} = \sum_{r'} D'_{rr'}(\Lambda^{-1}) \Phi(\Lambda x + a, \Lambda e)_{r'}. \quad (5.6)$$

Using equation (5.4) one can derive the necessary intertwiner relation for $u_\sigma(p, e)_r$, that guarantees that the field $\Phi(x, e)_r$ transforms as in (5.6).

$$\begin{aligned}
U(\Lambda)\Phi(x, e)_rU(\Lambda)^{-1} &= \sum_\sigma \int d\mu(p)e^{ipx}u_\sigma(p, e)_r \sum_{\sigma'} a^*(\Lambda p, \sigma')D_{\sigma'\sigma}(R(\Lambda, \Lambda p)) + h.c. \\
&= \sum_{\sigma'} \int d\mu(p)e^{ipx} \sum_\sigma D_{\sigma'\sigma}(R(\Lambda, \Lambda p))u_\sigma(p, e)_r a^*(\Lambda p, \sigma') + h.c. \\
&= \sum_\sigma \int d\mu(p)e^{ip\cdot(\Lambda x)} \sum_{\sigma'} D_{\sigma\sigma'}(R(\Lambda, p))u_{\sigma'}(\Lambda^{-1}p, e)_r a^*(p, \sigma) + h.c. \\
&\stackrel{!}{=} \sum_\sigma \int d\mu(p)e^{ip\cdot(\Lambda x)} D'_{rr'}(\Lambda^{-1})u_\sigma(p, \Lambda e)_{r'} a^*(p, \sigma) + h.c.
\end{aligned}$$

Thus the intertwiner relation that $u_\sigma(p, e)_r$ must fulfill reads

$$\sum_{\sigma'} D_{\sigma\sigma'}(R(\Lambda, p))u_{\sigma'}(\Lambda^{-1}p, e)_r = \sum_{r'} D'_{rr'}(\Lambda^{-1})u_\sigma(p, \Lambda e)_{r'}. \quad (5.7)$$

This is a generalization of the point-like intertwiner relation (3.35).

The strategy to find such $u_\sigma(p, e)_r$, satisfying (5.7), is to first consider the intertwiner $u_\sigma(\bar{p}, e)_r$ for the standard momentum \bar{p} (cf. [30, sec. 5.1]). Inserting $\Lambda = L_p^{-1}$ and $p = \bar{p}$ into (5.7) one sees that the intertwiner for arbitrary momentum p is fixed by the intertwiner for standard momentum \bar{p} by the equation

$$u_\sigma(p, e)_r = \sum_{r'} D_{rr'}(L_p)u_\sigma(\bar{p}, L_p^{-1}e)_{r'}. \quad (5.8)$$

The intertwiner for standard momentum \bar{p} then has to satisfy the relation

$$\sum_{\sigma'} D_{\sigma\sigma'}(W)u_{\sigma'}(\bar{p}, e)_r = \sum_{r'} D_{rr'}(W^{-1})u_\sigma(\bar{p}, We)_{r'}, \quad (5.9)$$

for every little group element $W \in G_{\bar{p}}$. This equation is obtained by inserting $p = \bar{p}$ and $\Lambda = W \in G_{\bar{p}}$ into (5.7), and using the fact that $R(W, p) = W$ for $W \in G_{\bar{p}}$.

So in general one can first try to find intertwiners for the standard momentum \bar{p} , satisfying the simpler relation (5.9), and then obtain the intertwiner for arbitrary momentum p by using the equation (5.8).

Because only massless particles will be concerned here, the representation matrix $D_{\sigma\sigma'}(R(\Lambda, p))$ will always be just $e^{i\sigma\theta(\Lambda, p)}\delta_{\sigma\sigma'}$, where $\theta(\Lambda, p)$ is defined according to the decomposition $R(\Lambda, p) = S(\alpha, \beta)\mathcal{R}(\theta)$, with the rotation around the z -axis \mathcal{R} [30, sec. 2.5]. Also, the standard momentum \bar{p} is always taken as $\bar{p} = k = (1, 0, 0, 1)$, and in the following will be denoted by k for convenience.

5.2 Intertwiners for $\lambda = 1$

Before deriving the expressions for the intertwiners for general representations of \mathcal{L}_+^\uparrow , they are constructed for helicity $\sigma = \pm 1$, which describes photons. More explicitly, intertwiners for the $D^{[\frac{1}{2}, \frac{1}{2}]}$ representation will be given, because this allows one to describe photons as a vector field, which is not possible in the usual point-like approach [30, p. 249-251].

To get these intertwiners, one first tries to find the $u(k, e, \sigma)_\mu$ for standard momentum k , satisfying

$$e^{\pm i\theta} u_\pm(k, e)_\mu = (\Lambda^{-1})_\mu{}^\nu u_\pm(k, \Lambda e)_\nu, \quad (5.10)$$

where $u_\pm(k, e)_\mu$ is the intertwiner for helicity $\sigma = \pm 1$. They are constructed using the polarization vectors

$$\hat{e}_\pm := \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \quad (5.11)$$

Restricting the transformation $W(\alpha, \beta, \theta)$ to rotations around the z -axis $\mathcal{R}(\theta)$ they satisfy the relation [30, p.249]

$$\mathcal{R}(\theta)^{-1} \hat{e}_\pm = e^{\pm i\theta} \hat{e}_\pm, \quad (5.12)$$

but for a general little group element $W(\alpha, \beta, \theta) = S(\alpha, \beta) \mathcal{R}(\theta)$ one gets [30, p.250]

$$\mathcal{R}(\theta)^{-1} S(\alpha, \beta)^{-1} \hat{e}_\pm = e^{\pm i\theta} (\hat{e}_\pm + f_\pm(\alpha, \beta) k), \quad (5.13)$$

with a certain function $f(\alpha, \beta)$, whose detailed form is not important here. To cancel the second term one forms the antisymmetric combination [30, p. 251]

$$\tilde{u}_\pm(k)_{\mu\nu} := (\hat{e}_{\pm\mu} k_\nu - k_\mu \hat{e}_{\pm\nu}) = \hat{e}_{\pm[\mu} k_{\nu]}, \quad (5.14)$$

which is the usual intertwiner for the $D^{[1,0]}$ or $D^{[0,1]}$ representation respectively and transforms according to

$$(W(\alpha, \beta, \theta)^{-1})_\mu{}^\rho (W(\alpha, \beta, \theta)^{-1})_\nu{}^\sigma \tilde{u}_\pm(k)_{\rho\sigma} = e^{\pm i\theta} \tilde{u}_\pm(k)_{\mu\nu}. \quad (5.15)$$

To get rid of the second tensor index and arrive at the desired vector potential, one now goes over to a string-like intertwiner $u_\pm(k, e)_\mu$, by contracting (5.14) with e^ν and defining [16, sec. 5]

$$u_\pm(k, e)_\mu = F(e \cdot p) \tilde{u}_\pm(k)_{\mu\nu} e^\nu = F(e \cdot p) [\hat{e}_{\pm\mu}(k \cdot e) - k_\mu(\hat{e}_\pm \cdot e)], \quad (5.16)$$

with an appropriate function $F(e \cdot p)$, which is not determined by (5.9). It will be shown below that it has to be taken as $F(e \cdot p) = \frac{1}{(e \cdot p)}$ to make the field a vector potential for the field strength tensor.

This string-like vector-intertwiner now satisfies the correct intertwiner relation

$$(W(\alpha, \beta, \theta)^{-1})_{\mu}^{\nu} u_{\pm}(k, e)_{\nu} = e^{\pm i\theta} u_{\pm}(k, W(\alpha, \beta, \theta)e)_{\mu}, \quad (5.17)$$

because the second factor $W(\alpha, \beta, \theta)^{-1}$ in (5.15) gets canceled by trading it for an additional transformation of the vector e .

To go over to intertwiners for arbitrary momentum, one needs the polarization vectors for momentum p ,

$$\hat{e}_{\pm}(p) := L_p \hat{e}_{\pm} \quad , \quad \text{with } L_p \bar{p} = p. \quad (5.18)$$

Using them, the general intertwiners for $\sigma = \pm 1$ and the representation $D^{[\frac{1}{2}, \frac{1}{2}]}$ can be defined as

$$u_{\pm}(p, e)_{\mu} = \frac{1}{(e \cdot p)} [(e \cdot p) \hat{e}_{\pm}(p)_{\mu} - (e \cdot \hat{e}_{\pm}(p)) p_{\mu}]. \quad (5.19)$$

They now satisfy the full intertwiner relation [16, sec. 5]

$$e^{\pm i\theta(\Lambda, p)} u_{\pm}(\Lambda^{-1} p, e)_{\mu} = (\Lambda^{-1})_{\mu}^{\nu} u_{\pm}(p, \Lambda e)_{\nu}, \quad (5.20)$$

which guarantees a covariant transformation law for the corresponding field.

With these intertwiners the string-localized vector field for photons can now be written as

$$A(x, e)_{\mu} = \sum_{\sigma=\pm 1} \int d\mu(p) \left[e^{ipx} u_{\sigma}(p, e)_{\mu} a^{*}(p, \sigma) + e^{-ipx} \overline{u_{\sigma}(p, e)_{\mu}} a(p, \sigma) \right]. \quad (5.21)$$

Due to the intertwiner relation (5.20) this field transforms under the Poincaré-group according to

$$U(a, \Lambda) A(x, e)_{\mu} U(a, \Lambda)^{-1} = (\Lambda^{-1})_{\mu}^{\nu} A(\Lambda x + a, \Lambda e)_{\nu}. \quad (5.22)$$

Now this vector field should be a potential for the point-localized electromagnetic field strength,

$$\mathcal{F}(x)_{\mu\nu} = \sum_{\sigma=\pm 1} \int d\mu(p) \left[e^{ipx} \tilde{u}_{\sigma}(p)_{\mu\nu} a^{*}(p, \sigma) + e^{-ipx} \overline{\tilde{u}_{\sigma}(p)_{\mu\nu}} a(p, \sigma) \right]. \quad (5.23)$$

This means that the expression

$$\partial_{\mu} A(x, e)_{\nu} - \partial_{\nu} A(x, e)_{\mu} = \mathcal{F}(x)_{\mu\nu} \quad (5.24)$$

has to be independent of the direction e . In p -space, this condition translates to e -independence of the expression [16, Proposition 5.1]

$$p_\mu u_\pm(p, e)_\nu - p_\nu u_\pm(p, e)_\mu = F(e \cdot p)(e \cdot p) [\hat{e}_\pm(p)_\nu p_\mu - \hat{e}_\pm(p)_\mu p_\nu]. \quad (5.25)$$

This is independent of e if and only if the function $F(e \cdot p)$ equals $\frac{c}{(e \cdot p)}$, with an unspecified constant c , which can be adjusted for correct normalization. This now leads to the intertwiner (5.19).

The vector potential (5.21) satisfies the following conditions [16, Proposition 5.1]:

- Lorentz “gauge”: $\partial^\mu A(x, e)_\mu = 0$
- Axial “gauge”: $e^\mu A(x, e)_\mu = 0$

The proof of this properties is very simple, because in p -space they translate to $p^\mu u_\pm(p, e)_\mu = 0$ and $e^\mu u_\pm(p, e)_\mu = 0$, which can be checked easily by using the facts that $p^2 = 0$, $\hat{e}_\pm(p)^2 = 0$ and $p \cdot \hat{e}_\pm(p) = k \cdot \hat{e}_\pm = 0$.

The word “gauge” is a bit of a misnomer here, because these are not actual gauge conditions, but are satisfied by every free vector field $A(x, e)_\mu$, transforming as in (5.22) and acting in the physical Hilbert space. The proof of this statement and the uniqueness of the field (5.21) can be found in [16, Proposition 5.1].

Such axial gauge fields have already been discussed in the literature before, but the direction e has always been considered fixed. The transformation behavior then becomes non-covariant and one gets additional gauge terms [16, sec. 5],

$$U(\Lambda)A(x, e)_\mu U(\Lambda)^{-1} = (\Lambda^{-1})_\mu{}^\nu A(\Lambda x, e)_\nu + \text{gauge term}. \quad (5.26)$$

Another disadvantage of these fields is that they create divergences at momenta orthogonal to e ($\frac{1}{(e \cdot p)} \rightarrow \infty$, for $e \cdot p \rightarrow 0$), when used in perturbative calculations. The string-localized fields $A(x, e)_\mu$ studied in the present setting overcome these difficulties, because they are considered as a distribution in e . To emphasize this fact, one could write the intertwiners (5.19) as [14]

$$u_\pm(p, e)_\mu = \hat{e}_\pm(p)_\mu - \lim_{\epsilon \rightarrow 0} \frac{\hat{e}_\pm(p) \cdot e}{e \cdot p + i\epsilon} p_\mu, \quad (5.27)$$

but the ϵ -part will be omitted here for brevity.

5.3 Intertwiners for Higher Helicities

5.3.1 General Intertwiners

The method from the previous section for constructing intertwiners for the string-localized photon field can now be generalized to higher helicities $\sigma = \pm\lambda$. The corresponding intertwiners will be denoted $u_{\pm}^{(\lambda)}(p, e)_{\mu_1 \dots \mu_\lambda}$. They can be generated by simply forming tensor products of the fundamental $\lambda = 1$ intertwiners $u_{\pm}(p, e)_{\mu}$, which leads to the definition

$$\begin{aligned} u_{\pm}^{(\lambda)}(p, e)_{\mu_1 \dots \mu_\lambda} &:= F^{(\lambda)}(e \cdot p) \left[\hat{e}_{\pm}(p)_{[\mu_1 p_{\nu_1}]} \dots \hat{e}_{\pm}(p)_{[\mu_\lambda p_{\nu_\lambda}]} \right] e^{\nu_1} \dots e^{\nu_\lambda} \\ &= F^{(\lambda)}(e \cdot p) \tilde{u}_{\pm}^{(\lambda)}(p)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} e^{\nu_1} \dots e^{\nu_\lambda}. \end{aligned} \quad (5.28)$$

The function $F^{(\lambda)}(e \cdot p)$ will be determined below and $\tilde{u}_{\pm}^{(\lambda)}(p)$ denotes the point-like intertwiner for the generalized field strength $\mathcal{F}^{(\lambda)}(x)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda}$ to helicity λ and representation $D^{[\lambda, 0]}$, $D^{[0, \lambda]}$ respectively.

It can easily be seen that the intertwiner (5.28) satisfies the desired relation

$$e^{\pm i\lambda\theta(\Lambda, p)} u_{\pm}^{(\lambda)}(\Lambda^{-1}p, e)_{\mu_1 \dots \mu_\lambda} = (\Lambda^{-1})_{\mu_1}^{\rho_1} \dots (\Lambda^{-1})_{\mu_\lambda}^{\rho_\lambda} u_{\pm}^{(\lambda)}(p, \Lambda e)_{\rho_1 \dots \rho_\lambda}. \quad (5.29)$$

To convince oneself that this is true, one first notes that the $\tilde{u}_{\pm}^{(\lambda)}(p)$ satisfy the relation

$$\begin{aligned} e^{\pm i\lambda\theta(\Lambda, p)} \tilde{u}_{\pm}^{(\lambda)}(\Lambda^{-1}p)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} &= \\ (\Lambda^{-1})_{\mu_1}^{\rho_1} (\Lambda^{-1})_{\nu_1}^{\sigma_1} \dots (\Lambda^{-1})_{\mu_\lambda}^{\rho_\lambda} (\Lambda^{-1})_{\nu_\lambda}^{\sigma_\lambda} \tilde{u}_{\pm}^{(\lambda)}(p)_{\rho_1 \sigma_1 \dots \rho_\lambda \sigma_\lambda}. \end{aligned} \quad (5.30)$$

The additional factors $e^{\nu_1} \dots e^{\nu_\lambda}$ in (5.28) then cancel λ factors of Λ^{-1} , by trading them for a transformation $e \mapsto \Lambda e$ in (5.29). With the yet undefined function $F^{(\lambda)}(e \cdot p)$ equation (5.28) defines the general intertwiners for helicity λ and the representation $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$ of the Lorentz-group.

With these intertwiners one can now define the string-localized quantum fields for general helicity λ :

$$A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda} = \sum_{\sigma} \int d\mu(p) \left[e^{ipx} u_{\sigma}^{(\lambda)}(p, e)_{\mu_1 \dots \mu_\lambda} a^{*}(p, \sigma) + e^{-ipx} \overline{u_{\sigma}^{(\lambda)}(p, e)_{\mu_1 \dots \mu_\lambda}} a(p, \sigma) \right] \quad (5.31)$$

Due to the general intertwiner relation (5.29) these fields transform like a covariant tensor of degree λ ,

$$U(a, \Lambda) A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda} U(a, \Lambda)^{-1} = (\Lambda^{-1})_{\mu_1}^{\nu_1} \dots (\Lambda^{-1})_{\mu_\lambda}^{\nu_\lambda} A^{(\lambda)}(\Lambda x + a, \Lambda e)_{\nu_1 \dots \nu_\lambda}. \quad (5.32)$$

Like the photon field they also have certain properties:

- Total symmetry: $A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_r \dots \mu_k \dots \mu_\lambda} = A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_k \dots \mu_r \dots \mu_\lambda} \quad \forall k, r$
- Generalized Lorentz condition: $\partial^{\mu_1} A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda} = 0$
- Axial gauge condition: $e^{\mu_1} A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda} = 0$
- $A^{(\lambda)}(x, e)$ is “trace free”: $A^{(\lambda)}(x, e)^{\mu_1}_{\mu_1 \dots \mu_\lambda} = 0$

Just as in the previous section, the last three can easily be verified by just translating them to the corresponding properties for the $u_{\pm}^{(\lambda)}(p, e)$ in p -space, inserting the definition (5.28) and then using that $p^2 = 0$, $\hat{e}_{\pm}(p)^2 = 0$ and $\hat{e}_{\pm}(p) \cdot p = 0$. Total symmetry obviously follows by definition.

Of course there is also a point-like field strength for helicity λ ,

$$\mathcal{F}^{(\lambda)}(x)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} = \sum_{\sigma} \int d\mu(p) \left[e^{ipx} \tilde{u}_{\sigma}^{(\lambda)}(p)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} a^*(p, \sigma) + h.c. \right], \quad (5.33)$$

transforming according to the $D^{[\lambda, 0]}$, $D^{[0, \lambda]}$ representation respectively. Again, one wants the field $A^{(\lambda)}(x, e)$ to be a generalized potential for the field strength (5.33). More explicitly, this means that if one takes the expression

$$\partial_{\mu_1} \dots \partial_{\mu_\lambda} A^{(\lambda)}(x, e)_{\nu_1 \dots \nu_\lambda} \quad (5.34)$$

and antisymmetrizes it in every index pair μ_k, ν_k (denoted by ”AntiSym[...]” below), the result should be independent of the direction e and should yield the field strength $\mathcal{F}^{(\lambda)}(x)$.

In p -space this again amounts to e -independence of the expression

$$p_{\mu_1} \dots p_{\mu_\lambda} u_{\pm}^{(\lambda)}(p, e)_{\nu_1 \dots \nu_\lambda} \quad (5.35)$$

when antisymmetrized in all the index pairs μ_k, ν_k . Using (5.28) and recalling the definition of $\tilde{u}_{\pm}^{(1)}(p)_{\mu}$ one can calculate this antisymmetrized product in the following way:

$$\begin{aligned} \text{AntiSym} \left[p_{\mu_1} \dots p_{\mu_\lambda} u_{\pm}^{(\lambda)}(p, e)_{\nu_1 \dots \nu_\lambda} \right] &= \\ &= F^{(\lambda)}(e \cdot p) \left\{ p_{[\mu_1} \tilde{u}_{\pm}^{(1)}(p)_{\nu_1] \sigma_1} \dots p_{[\mu_\lambda} \tilde{u}_{\pm}^{(1)}(p)_{\nu_\lambda] \sigma_\lambda} \right\} e^{\sigma_1} \dots e^{\sigma_\lambda} \\ &= F^{(\lambda)}(e \cdot p) (e \cdot p)^\lambda \tilde{u}_{\pm}^{(\lambda)}(p)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} \end{aligned} \quad (5.36)$$

For the choice

$$F^{(\lambda)}(e \cdot p) = \frac{1}{(e \cdot p)^\lambda} \quad (5.37)$$

this is clearly independent of e and the desired relation

$$\text{AntiSym} \left[\partial_{\mu_1} \dots \partial_{\mu_\lambda} A^{(\lambda)}(x, e)_{\nu_1 \dots \nu_\lambda} \right] = \mathcal{F}^{(\lambda)}(x)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} \quad (5.38)$$

holds. Therefore the full expression for the intertwiner (5.28) reads

$$u_{\pm}^{(\lambda)}(p, e)_{\mu_1 \dots \mu_\lambda} = \frac{1}{(e \cdot p)^\lambda} \left[\hat{e}_{\pm}(p)_{[\mu_1 p \nu_1]} \dots \hat{e}_{\pm}(p)_{[\mu_\lambda p \nu_\lambda]} \right] e^{\nu_1} \dots e^{\nu_\lambda} . \quad (5.39)$$

Again, this has to be seen as a distribution in e , so that the divergence for $e \cdot p = 0$ gets regularized.

5.3.2 Example: $\lambda = 2$, Gravitons

Another interesting example, besides the photon field, is the string-localized field for helicity $\sigma = \pm 2$, describing hypothetical gravitons. The field $h(x, e)_{\mu\nu}$ describes the perturbation of the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and is a potential for the linearized (point-localized) Riemann tensor $R(x)_{\mu\nu\rho\sigma}$ (see e.g. [29, sec. 4.4]). This means that the classical relation between $R_{\mu\nu\rho\sigma}$ and the field $h_{\mu\nu}$ holds, i.e.

$$R(x)_{\mu\nu\rho\sigma} = \frac{1}{2} [\partial_\mu \partial_\rho h(x, e)_{\nu\sigma} + \partial_\nu \partial_\sigma h(x, e)_{\mu\rho} - \partial_\nu \partial_\rho h(x, e)_{\mu\sigma} - \partial_\mu \partial_\sigma h(x, e)_{\nu\rho}] , \quad (5.40)$$

which is a special case of the general relation (5.38).

The intertwiner for the case $\sigma = \pm 2$ can be written as

$$u_{\pm}(p, e)_{\mu\nu} = \hat{e}_{\pm}(p)_\mu \hat{e}_{\pm}(p)_\nu - \frac{e \cdot \hat{e}_{\pm}(p)}{e \cdot p} \left(\hat{e}_{\pm}(p)_\mu p_\nu + p_\mu \hat{e}_{\pm}(p)_\nu \right) + \frac{(e \cdot \hat{e}_{\pm}(p))^2}{(e \cdot p)^2} p_\mu p_\nu . \quad (5.41)$$

It satisfies the intertwiner relation

$$e^{\pm 2i\theta(\Lambda, p)} u_{\pm}(\Lambda^{-1}p, e)_{\mu\nu} = (\Lambda^{-1})_\mu^\rho (\Lambda^{-1})_\nu^\sigma u_{\pm}(p, \Lambda e)_{\rho\sigma} , \quad (5.42)$$

so that the field

$$h(x, e)_{\mu\nu} = \sum_{\sigma} \int d\mu(p) \left(e^{ipx} u_{\sigma}(p, e)_{\mu\nu} a^*(p, \sigma) + \text{h.c.} \right) \quad (5.43)$$

transforms like a second rank covariant tensor field, i.e.

$$U(a, \Lambda) h(x, e)_{\mu\nu} U(a, \Lambda)^{-1} = (\Lambda^{-1})_\mu^\rho (\Lambda^{-1})_\nu^\sigma h(\Lambda x + a, \Lambda e)_{\rho\sigma} . \quad (5.44)$$

The field (5.43) has all the desired properties, one demands a quantum field describing (linearized) gravity to have. Namely it is *symmetric*, it satisfies the *axial gauge condition*

$e^\mu h(x, e)_{\mu\nu} = 0$ and the remaining properties $\partial^\mu h(x, e)_{\mu\nu} = 0$ and $h(x, e)^\mu{}_\mu = 0$ are usually called *harmonic gauge*.

In the formalism of point-like localization such a field $h(x)_{\mu\nu}$ does not exist in a Hilbert space representation with positive energy [28]. In [28] it is shown that under the general assumptions (1) existence of an invariant vacuum, (2) the fields transform as tensors and (3) the two-point function is analytic in the forward tube, the Einstein equations have no solutions apart from $R_{\mu\nu\rho\sigma} = 0$.

A possible solution to this problem is to give up covariance and use a field that does not transform as a tensor,

$$U(\Lambda)h(x)_{\mu\nu}U(\Lambda)^{-1} = (\Lambda^{-1})_\mu{}^\rho(\Lambda^{-1})_\nu{}^\sigma h(\Lambda x)_{\rho\sigma} + \text{gauge terms}. \quad (5.45)$$

Another solution could be to use string-localized fields $h(x, e)_{\mu\nu}$, where all the desired conditions can be met. The problem is, that it is still unclear how to realize interacting string-localized fields and how to use such fields in perturbation theory [14].

5.4 Description by Line-Integral

Interestingly it is also possible to describe the fields (5.31) as an infinite line integral along the string direction e . In [16, Thm. 4.4] it is shown that in the massive case, every string-localized field can be written as a line integral over a point-localized tensor field

$$\Phi(x, e) = \int_0^\infty dt f(t) \sum_r \Phi_r(x + te) w(e)_r, \quad (5.46)$$

where $f(t)$ has support in the interval $[0, \infty)$ and $w(e)_r$ is a tensor in e which is subject to the condition

$$w(e)_r = \sum_{r'} D'(\Lambda^{-1})_{rr'} w(\Lambda e)_{r'}. \quad (5.47)$$

Such a description is also possible for the above tensor potentials. It has also been shown in [16], that the string-localized photon field $A(x, e)_\mu$ can be written as a line integral over the point-localized field strength $\mathcal{F}(x)_{\mu\nu}$, according to

$$A(x, e)_\mu = \int_0^\infty dt f(t) \mathcal{F}(x + te)_{\mu\nu} e^\nu, \quad (5.48)$$

where $f(t)$ again is supported in $[0, \infty)$. Clearly, e^ν is the simplest tensor satisfying (5.47). This could also be taken as the definition of $A(x, e)_\mu$.

The properties $\partial^\mu A(x, e)_\mu = 0$ and $e^\mu A(x, e)_\mu = 0$ then follow by the vacuum Maxwell equations $\partial^\mu \mathcal{F}(x)_{\mu\nu} = 0$ and the antisymmetry of $\mathcal{F}(x)_{\mu\nu}$.

By inserting the definition of $\mathcal{F}(x)_{\mu\nu}$ into equation (5.48), one can check the equivalence with the prior definition.

$$\begin{aligned}
A(x, e)_\mu &= \int_0^\infty dt f(t) \mathcal{F}(x + te)_{\mu\nu} e^\nu \\
&= \int_0^\infty dt f(t) \sum_\sigma \int d\mu(p) \left(e^{ip(x+te)} \tilde{u}_\sigma(p)_{\mu\nu} a^*(p, \sigma) + \text{h.c.} \right) e^\nu \\
&= \sum_\sigma \int d\mu(p) \left[e^{ipx} \left(\int_0^\infty dt f(t) e^{itp \cdot e} \right) \tilde{u}_\sigma(p)_{\mu\nu} e^\nu a^*(p, \sigma) + \text{h.c.} \right] \\
&= \sum_\sigma \int d\mu(p) \left[e^{ipx} u_\sigma(p, e)_\mu a^*(p, \sigma) + \text{h.c.} \right],
\end{aligned} \tag{5.49}$$

where $u_\pm(p, e)_\mu = F(e \cdot p) \tilde{u}_\pm(p)_{\mu\nu} e^\nu$ and the function $F(e \cdot p) = \mathfrak{F}(f)(e \cdot p)$ is obviously proportional to the Fourier-transform of the function $f(t)$. To get the usual choice $F(e \cdot p) = \frac{1}{e \cdot p}$, one has to take $f(t)$ as the Heaviside function $\Theta(t)$ (cf. [16, Proposition 5.1]). With this choice for $f(t)$, $A(x, e)_\mu$ is again a potential for $\mathcal{F}(x)_{\mu\nu}$ in the sense that $\partial_\mu A(x, e)_\nu - \partial_\nu A(x, e)_\mu = \mathcal{F}(x)_{\mu\nu}$.

This can even be checked without first inserting the definition of $\mathcal{F}(x)_{\mu\nu}$, by using the homogeneous Maxwell equation

$$\partial_\mu \mathcal{F}(x)_{\nu\rho} + \partial_\nu \mathcal{F}(x)_{\rho\mu} + \partial_\rho \mathcal{F}(x)_{\mu\nu} = 0, \tag{5.50}$$

and the antisymmetry of $\mathcal{F}(x)_{\mu\nu}$. The idea is as follows:

$$\begin{aligned}
\partial_\mu A(x, e)_\nu - \partial_\nu A(x, e)_\mu &= \int_0^\infty dt \left[\partial_\mu \mathcal{F}(x + te)_{\nu\rho} e^\rho - \partial_\nu \mathcal{F}(x + te)_{\mu\rho} e^\rho \right] \\
&= - \int_0^\infty dt e^\rho \partial_\rho \mathcal{F}(x + te)_{\mu\nu} \\
&= - \left(\mathcal{F}(x + te)_{\mu\nu} \Big|_{t=\infty} - \mathcal{F}(x + te)_{\mu\nu} \Big|_{t=0} \right) \\
&= \mathcal{F}(x)_{\mu\nu},
\end{aligned} \tag{5.51}$$

if one assumes that the field strength $\mathcal{F}(x)$ is zero at spatial infinity.

In the case of the gravitational field, $h(x, e)_{\mu\nu}$ can also be defined according to

$$h(x, e)_{\mu\nu} = \int dt f^{(2)}(t) R(x + te)_{\mu\nu\rho\sigma} e^\mu e^\nu e^\rho e^\sigma, \tag{5.52}$$

with $f^{(2)}(t) = \Theta(t) \cdot t$. The proof that this $h(x, e)$ is a potential for the linearized Riemann tensor is a bit tedious and uses the Bianchi identity.

For higher helicities λ , the definition by a line-integral like (5.48) of the fields $A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda}$ can be slightly generalized to

$$A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda} = \int dt f^{(\lambda)}(t) \mathcal{F}^{(\lambda)}(x)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda} e^{\nu_1} \dots e^{\nu_\lambda}, \quad (5.53)$$

where $f^{(\lambda)}(t)$ is proportional to the inverse Fourier transform of $F^{(\lambda)}(e \cdot p) = \frac{1}{(e \cdot p)^\lambda}$ for $t > 0$, when interpreted as the distribution $F^{(\lambda)}(\omega) = \lim_{\epsilon \rightarrow 0} \frac{1}{(\omega + i\epsilon)^\lambda}$.

The function $f^{(\lambda)}(t)$ then results in

$$f^{(\lambda)}(t) = \begin{cases} t^{\lambda-1}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0. \end{cases} \quad (5.54)$$

With this choice for $f^{(\lambda)}(t)$, the definition of $A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda}$ by a line integral is again equivalent to (5.31) and it is a potential for $\mathcal{F}^{(\lambda)}(x)_{\mu_1 \nu_1 \dots \mu_\lambda \nu_\lambda}$. The proof of the equivalence is essentially the same as in the case of the photon field, and amounts simply to keeping track of more indices.

By using this infinitely extended line-integrals, one can get string-localized potentials, transforming according to the $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$ -representation, from point-localized field strength, transforming according to $D^{[\lambda, 0]}, D^{[0, \lambda]}$ respectively. In contrast to the massive case this is not possible for other representations of the Lorentz-group $D^{[A, B]}$. This is because the only potential massless fields for helicity λ are the $D^{[\lambda+b, b]}$ or $D^{[b, \lambda+b]}$ fields, with a half-integer $b \geq 0$, because the relation $\lambda = |B - A|$ has to be satisfied. However, it can be shown that these fields are just linear combinations of the $2b$ th derivatives of fields of type $D^{[\lambda, 0]}$ or $D^{[0, \lambda]}$, and so they don't offer any new alternatives [30, sec. 5.9].

5.5 General Representations

5.5.1 Spinor-Representations

To construct general representations of the Lorentz group it is convenient to work in the *spinor formalism* [18, 24]. Thus one considers the universal covering group of \mathcal{L}_+^\uparrow , the group $SL(2, \mathbb{C})$, because every representation of $SL(2, \mathbb{C})$ is related to a multivalued representation of \mathcal{L}_+^\uparrow [24]. This simply connected group consists of two-dimensional complex matrices A with $\det A = 1$.

The relationship between $SL(2, \mathbb{C})$ and \mathcal{L}_+^\uparrow is given by a 2:1 homomorphism

$$SL(2, \mathbb{C}) \longrightarrow \mathcal{L}_+^\uparrow, \quad (5.55)$$

i.e. two elements of $SL(2, \mathbb{C})$ (more precisely the elements A and $-A$) are mapped to the same Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$.

The fundamental representations of $SL(2, \mathbb{C})$ are $D^{[\frac{1}{2}, 0]}$ and $D^{[0, \frac{1}{2}]}$, which are generated by trace-free 2×2 matrices, so the representation matrix $D^{[\frac{1}{2}, 0]}(\vec{\alpha}, \vec{v})$ of a rotation around $\vec{\alpha}$ and a boost in direction \vec{v} can be written as [24, sec. 8.2]

$$D^{[\frac{1}{2}, 0]}(\vec{\alpha}, \vec{v}) = e^{-i\vec{\alpha}\vec{\sigma}/2} e^{-\vec{v}\vec{\sigma}/2} \equiv A(\vec{\alpha}, \vec{v}), \quad (5.56)$$

where again $\vec{u} = (\vec{v}/v)\text{artanh}(v)$. This matrix is hermitian for $\vec{\alpha} = 0$ and anti-hermitian for $\vec{v} = 0$.

Again, it has to be noted here, that the rotational part $e^{-i\vec{\alpha}\vec{\sigma}/2}$ does not yield the identity for a rotation around 2π , but minus the identity matrix. Thus it is not a proper representation of $SO(3)$ but only of $SU(2)$. Equation (5.56) is just a special case of the general representation matrix (3.11), that has been discussed in section 3.1.

The representation $D^{[\frac{1}{2}, 0]}$ acts on so called spinors $u \in \mathcal{S} = \mathbb{C}^2$ (see [24, sec. 8.3] for an introduction to the topic of spinor algebra), according to

$$u' = Au, \quad (5.57)$$

This can also be written using the index notation

$$u'^J = A^J_K u^K, \quad (5.58)$$

where capital letters now denote spinor indices, which can have the values 1 or 2.

One can get spinors of higher degree by forming tensor products of these fundamental spinors. They transform according to

$$u'^{JK\dots} = A^J_M A^K_N \dots u^{MN\dots}. \quad (5.59)$$

To get irreducible representations, these spinors have to be totally symmetrized

$$u^{(K_1 \dots K_n)} = \frac{1}{n!} \sum_{\pi \in S_n} u^{K_{\pi(1)} \dots K_{\pi(n)}}, \quad (5.60)$$

where the sum is taken over all permutations of $(1, \dots, n)$. This spinor then transforms according to the irreducible representation $D^{[\frac{n}{2}, 0]}$.

An analogous formalism exists on the conjugate spinor space \bar{S} , where the complex conjugate representation $D^{[0, \frac{1}{2}]}$ acts. The complex conjugate spinors then transform according to

$$\bar{u}' = \bar{A} \bar{u} = \bar{u} A^\dagger. \quad (5.61)$$

Elements of the representation space of $D^{[0, \frac{1}{2}]}$ are also often written with dotted indices, so using the index notation again equation (5.61) reads

$$\bar{u}'^{\dot{J}} = \bar{A}^{\dot{J}}_{\dot{K}} \bar{u}^{\dot{K}}. \quad (5.62)$$

A general irreducible representation $D[A, B]$ then acts on spinors with A undotted and B dotted indices, which are totally symmetric in the undotted and dotted parts respectively,

$$u'^{(AB\dots)(\dot{J}\dot{K}\dots)} = A^A_M A^B_N \dots \bar{A}^{\dot{J}}_{\dot{X}} \bar{A}^{\dot{K}}_{\dot{Y}} u^{(MN\dots)(\dot{X}\dot{Y}\dots)}. \quad (5.63)$$

The intertwiners for general $D^{[A, B]}$ representations will be constructed as such dotted and undotted spinors.

5.5.2 Relation between Spinors and Tensors

To illustrate the homomorphism $SL(2, \mathbb{C}) \rightarrow \mathcal{L}_+^\uparrow$ and the relation between spinors and tensors, one usually considers a 4-vector v^μ and forms the following hermitian 2×2 matrix from it,

$$\tilde{v} := \sum_k v^k \sigma_k := \mathbb{1} v^0 + \vec{v} \vec{\sigma} = \begin{pmatrix} v^0 + v^3 & v^1 - i v^2 \\ v^1 + i v^2 & v^0 - v^3 \end{pmatrix}, \quad (5.64)$$

where $\sigma_k = \{\mathbb{1}, \vec{\sigma}\}$. (For a more detailed treatment of the relation between spinors and tensors see [24, sec. 8.4].) Conversely, every hermitian 2×2 matrix can be written like that and one can get back to the 4-vector by the following equation,

$$v^\mu = \frac{1}{2} \text{Tr}(\tilde{v} \sigma_\mu). \quad (5.65)$$

From the definition (5.64) it can easily be seen that the relation

$$\det \tilde{v} = v_\mu v^\mu \quad (5.66)$$

holds. Now the important fact is, that this determinant is invariant when \tilde{v} is being transformed, using an arbitrary complex unimodular 2×2 matrix, according to

$$\tilde{v}' = A \tilde{v} A^\dagger, \quad (5.67)$$

$$v'_\mu v'^\mu = \det \tilde{v}' = \det \tilde{v} = v_\mu v^\mu. \quad (5.68)$$

Because of (5.68), equation (5.67) defines a Lorentz transformation, which is given by [24, p.236]

$$\Lambda^\mu{}_\nu = \frac{1}{2} \text{Tr}(A\sigma_\nu A^\dagger \sigma_\mu). \quad (5.69)$$

This is the desired homomorphism $SL(2, \mathbb{C}) \rightarrow \mathcal{L}_+^\uparrow$, which is only 2:1 because A and $-A$ clearly lead to the same Lorentz transformation.

The above scheme of passing from a 4-vector v^μ to a spinor \tilde{v} can also be generalized to tensors and spinors of higher degree [24, sec. 8.4]. If the value of $A + B$ is integer, the representation $D^{[A,B]}$ is single valued, i.e. it is a proper representation of \mathcal{L}_+^\uparrow . Then there exists a relation between the tensor and the spinor representations, and as a matter of fact, every 4-tensor can be constructed from a certain spinor.

In general it is also possible to introduce spinors with lower indices, also called *covariant spinors*,

$$u_J := \epsilon_{JK} u^K, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (5.70)$$

where the two-dimensional epsilon tensor ϵ has been introduced. They transform according to

$$u'_J = A_J{}^K u_K. \quad (5.71)$$

The simplest case, which has already been defined above, is the relation between a 4-vector and a 2×2 matrix, transforming according to the $D^{[\frac{1}{2}, \frac{1}{2}]}$ representation

$$\tilde{v}_{J\dot{K}} = v^\mu (\sigma_\mu)_{J\dot{K}}. \quad (5.72)$$

Another interesting example is the $D^{[1,0]}$ representation, whose representation space contains the anti-symmetric field strength tensor $F_{\mu\nu}$. In the spinor formalism it is carried by symmetric spinors u^{JK} . Introducing the self-dual extension of $F_{\mu\nu}$,

$$f_{\mu\nu} := F_{\mu\nu} - \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad (5.73)$$

one can write down the relation between the tensor $f_{\mu\nu}$ and the spinor u^{JK} [24, p.248],

$$f_{\mu\nu} = \frac{1}{2} u^{JK} \epsilon^{\dot{M}\dot{N}} (\sigma_\mu)_{J\dot{M}} (\sigma_\nu)_{K\dot{N}}. \quad (5.74)$$

Of course these two examples can be generalized to higher spinors and tensors. To each

4-tensor T of degree p , one can construct an equivalent spinor t [24, p.249],

$$t_{J\dot{M}K\dot{N}\dots} = 2^{-p/2} T^{\mu\nu\dots} (\sigma_\mu)_{J\dot{M}} (\sigma_\nu)_{K\dot{N}} \dots, \quad (5.75)$$

and conversely there is a 4-tensor for every spinor with equal numbers of dotted and undotted indices $2A = 2B = p$ [24, p.249],

$$T_{\mu\nu\dots} = 2^{-p/2} t^{J\dot{M}K\dot{N}\dots} (\sigma_\mu)_{J\dot{M}} (\sigma_\nu)_{K\dot{N}} \dots \quad (5.76)$$

The tensors formed that way transform according to the $D^{[\frac{p}{2}, \frac{p}{2}]}$ representation and are totally symmetric and trace free, which serves to characterize them as irreducible.

Considering spinors with an uneven number of dotted and undotted indices, carrying the representation $D^{[A,B]}$, but still satisfying $A + B = \text{integer}$, one can also generalize equation (5.74) [24, p.249]. If a spinor $u^{JK\dots\dot{M}\dot{N}}$ has more indices of one type, one just fills up the number of indices of the other kind by multiplying it with an appropriate number of factors ϵ^{AB} or $\epsilon^{\dot{X}\dot{Y}}$ and then uses equation (5.76).

For example consider the representation $D^{[2,0]}$, generated by symmetric spinors u^{JKLM} . To get the corresponding 4-tensor, one simply forms the expression

$$T_{\mu\nu\rho\eta} = \frac{1}{4} u^{JKLM} \epsilon^{\dot{A}\dot{B}} \epsilon^{\dot{C}\dot{D}} (\sigma_\mu)_{J\dot{A}} (\sigma_\nu)_{K\dot{B}} (\sigma_\rho)_{L\dot{C}} (\sigma_\eta)_{M\dot{D}}. \quad (5.77)$$

This tensor is antisymmetric in $\mu\nu$ and $\rho\eta$ and symmetric under the substitution $\mu\nu \leftrightarrow \rho\eta$, just like the Riemann-tensor $R_{\mu\nu\rho\eta}$.

In the next section string-localized intertwiners for general representations of \mathcal{L}_+^\uparrow will be constructed in the spinor formalism and the above formulas can then be used to get back to a tensor representation.

5.5.3 String-localized Intertwiners for General Representations

In this section string-localized intertwiners for general representations of the Lorentz group, respectively the group $SL(2, \mathbb{C})$, will be constructed, i.e. intertwiners $u_\pm(p, e)$ will be given that satisfy the relation

$$e^{\pm i\lambda\theta} u_\pm(\Lambda^{-1}p, e) = D^{[A,B]}(\Lambda^{-1}) u_\pm(p, \Lambda e). \quad (5.78)$$

To achieve this, at first intertwiners for the fundamental representations $D^{[\frac{1}{2}, 0]}$ and $D^{[0, \frac{1}{2}]}$, for helicity $\sigma = \pm\frac{1}{2}$, have to be found and by taking tensor products of them, intertwiners

for other representations can then be constructed. Of course these fundamental intertwiners no longer define bosonic fields, because of the half-integer helicity. But it will turn out that there is not really a difference between the construction of bosonic and fermionic intertwiners in the spinor formalism.

Again, one first tries to find intertwiners for the standard momentum $k = (1, 0, 0, 1)$, because they fix the intertwiners for arbitrary momenta. These standard intertwiners $u_{\pm}(k, e) =: u_{\pm}(e)$ have to satisfy the simpler relation

$$e^{\pm \frac{i}{2}\theta} u_{\pm}(e) = A(c, \theta)^{-1} u_{\pm}(\Lambda e), \quad (5.79)$$

where $A(c, \theta)$ is an element of the (covering of the) little group $G_k \cong \tilde{E}(2)$. The intertwiner for the complex conjugate representation $D^{[0, \frac{1}{2}]}$ is then given by the complex conjugate $\overline{u_{\pm}(e)}$.

These $u_{\pm}(e)$ are maps between the standard Wigner particle space and the spinor space $\mathcal{S} = \mathbb{C}^2$. Because of their spinorial nature their explicit dependence will not be on the 4-vector e , but on the 2×2 matrix \tilde{e} , which can be formed from it using equation (5.64).

$$\tilde{e} = \begin{pmatrix} e^0 + e^3 & e^1 - ie^2 \\ e^1 + ie^2 & e^0 - e^3 \end{pmatrix}. \quad (5.80)$$

Thus the intertwiner relation reads

$$e^{\pm \frac{i}{2}\theta} u_{\pm}(\tilde{e}) = A(c, \theta)^{-1} u_{\pm}(A \tilde{e} A^{\dagger}). \quad (5.81)$$

The general form of the matrix $A(c, \theta) \in \tilde{E}(2)$ is given by [24, p.294]

$$A(c, \theta) = \begin{pmatrix} e^{i\frac{\theta}{2}} & c e^{-i\frac{\theta}{2}} \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}, \quad (5.82)$$

where $0 \leq \theta < 4\pi$ is again the angle of the rotation around the z -axis and the complex number c parametrizes the “translational” part of the Euclidean group. It can easily be seen that it is really an element of the stabilizer group G_k , because

$$A(c, \theta) \tilde{k} A(c, \theta)^{\dagger} = \tilde{k}, \quad \text{for } \tilde{k} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.83)$$

This $A(c, \theta)$ is the $SL(2, \mathbb{C})$ version of the Lorentz transformation $W(\alpha, \beta, \theta) = S(\alpha, \beta) \mathcal{R}(\theta)$ from chapter 3.

The trick to find such intertwiners is similar to the construction of the tensor intertwiners in section 5.3. There one used the well known point-like intertwiners for the representations $D^{[\lambda,0]}$, $\tilde{u}_\pm(p)$, and contracted them with an appropriate number of factors e to get string-like intertwiners for the representation $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$. In simple terms, these factors e cancel out half of the transformation matrices Λ^{-1} and trade them for a transformation $e \rightarrow \Lambda e$. A similar procedure can be applied here.

First define the point-like intertwiners u_- and u_+ , where u_- intertwines the $D^{[\frac{1}{2},0]}$ representation with the $\sigma = -\frac{1}{2}$ Wigner representation and u_+ the $D^{[0,\frac{1}{2}]}$ with the $\sigma = +\frac{1}{2}$ representation. This is in accordance with the rule, that there can only be point-like intertwiners for $\sigma = B - A$. These standard intertwiners are given by

$$u_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (5.84)$$

where possible factors of $1/\sqrt{2}$ are omitted for simplicity. With the matrices

$$A(c, \theta)^{-1} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & -c e^{-i\frac{\theta}{2}} \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}, \quad \overline{A(c, \theta)^{-1}} = \begin{pmatrix} e^{i\frac{\theta}{2}} & -\bar{c} e^{i\frac{\theta}{2}} \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \quad (5.85)$$

one can immediately see that the u_\pm satisfy the relations

$$A(c, \theta)^{-1} u_- = e^{-i\frac{\theta}{2}} u_-, \quad (5.86a)$$

$$\overline{A(c, \theta)^{-1}} u_+ = e^{+i\frac{\theta}{2}} u_+. \quad (5.86b)$$

Now, to get an intertwiner for $D^{[\frac{1}{2},0]}$ and $\sigma = +\frac{1}{2}$ as well, simply multiply the spinor u_+ with a factor \tilde{e} , to get the string-like intertwiner $u_+(\tilde{e})$. The final intertwiners (for standard momentum k) for the fundamental representation $D^{[\frac{1}{2},0]}$ are then given by

$$u_+(\tilde{e}) := \tilde{e} u_+ = \begin{pmatrix} e^1 - i e^2 \\ e^0 - e^3 \end{pmatrix}, \quad (5.87a)$$

$$u_-(\tilde{e}) := u_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (5.87b)$$

It is straightforward to check that they satisfy the desired relation

$$e^{\pm i\frac{\theta}{2}} u_\pm(\tilde{e}) = A(c, \theta)^{-1} u_\pm(A \tilde{e} A^\dagger). \quad (5.88)$$

Indeed the u_- part is trivial and the u_+ part can be proven by inserting the definitions:

$$A^{-1} u_+(A \tilde{e} A^\dagger) = A^{-1} A \tilde{e} A^\dagger u_+ = \tilde{e} A^\dagger u_+ = e^{i\frac{\theta}{2}} \tilde{e} u_+ = e^{i\frac{\theta}{2}} u_+(\tilde{e}), \quad (5.89)$$

because $A^\dagger u_+ = e^{i\frac{\theta}{2}} u_+$.

The intertwiners for the $D^{[0, \frac{1}{2}]}$ representation are then just the complex conjugates of these, and they satisfy the complex conjugate intertwiner relation,

$$e^{\mp i\frac{\theta}{2}} \overline{u_\pm(\tilde{e})} = \overline{A(c, \theta)^{-1}} u_\pm(\overline{A} \tilde{e} \overline{A}^\dagger). \quad (5.90)$$

The proof can be seen immediately by just conjugating equation (5.89).

After having constructed these standard intertwiners, one can get the intertwiners for arbitrary momentum by the spinorial version of the general formula (5.8),

$$u_\pm(p, \tilde{e}) := A_p u_\pm(A_p^{-1} \tilde{e} (A_p^\dagger)^{-1}), \quad (5.91)$$

where A_p is an element of $SL(2, \mathbb{C})$, which maps the 2×2 matrix of the standard momentum k to that of the momentum p :

$$A_p \tilde{k} A_p^\dagger = \tilde{p} \quad (5.92)$$

A possible choice for this A_p is

$$A_p = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p^0 + p^3} & \frac{\sqrt{p^0 + p^3}}{p^1 + ip^2} \\ \frac{p^1 + ip^2}{\sqrt{p^0 + p^3}} & \frac{2}{\sqrt{p^0 + p^3}} \end{pmatrix}. \quad (5.93)$$

Finally, one has found spinors $u_\pm(p, \tilde{e})$, that intertwine the fundamental representation $D^{[\frac{1}{2}, 0]}$ with the canonical Wigner representation for helicity $\sigma = \pm \frac{1}{2}$:

$$e^{\pm i\frac{\theta}{2}\theta(p, A)} u_\pm(\Lambda^{-1}p, \tilde{e}) = A^{-1} u_\pm(p, A \tilde{e} A^\dagger), \quad (5.94)$$

where A now is an arbitrary element of $SL(2, \mathbb{C})$.

With the above intertwiners $u_\pm(p, \tilde{e})$ one could now construct a string-localized massless fermionic spinor field,

$$\phi(x, e) = \sum_\sigma \int d\mu(p) [e^{ipx} u_\sigma(p, \tilde{e}) a^*(p, \sigma) + \text{h.c.}], \quad (5.95)$$

for helicity $\lambda = \frac{1}{2}$. Of course its creation and annihilation operators have to obey anti-commutation relations instead of commutation relations. This field then transforms according to

$$U(a, A)\phi(x, e)U(a, A)^{-1} = A^{-1} \phi(\Lambda(A)x + a, \Lambda(A)e). \quad (5.96)$$

However, the main subject here are only bosonic fields, so this will not be pursued any further.

What's important now, is that one can find intertwiners for more general representations and higher helicities by just forming tensor products of these fundamental intertwiners. For example, the intertwiners $u_{\pm}^{(1)}(p, \tilde{e}) \in \mathbb{C}^4$ for helicity $\sigma = \pm 1$ and representation $D^{[\frac{1}{2}, \frac{1}{2}]}$ would be

$$u_{\pm}^{(1)}(p, \tilde{e}) = u_{\mp}(p, \tilde{e}) \otimes \overline{u_{\pm}(p, \tilde{e})}, \quad (5.97)$$

or in index notation

$$u_{\pm}^{(1)}(p, \tilde{e})_{J\dot{K}} = u_{\mp}(p, \tilde{e})_J \overline{u_{\pm}(p, \tilde{e})_{\dot{K}}}. \quad (5.98)$$

Because of the product structure, it satisfies the correct intertwiner relation

$$(A^{-1})_J^M (\overline{A^{-1}})_{\dot{K}}^{\dot{N}} u_{\pm}^{(1)}(\Lambda^{-1}p, \tilde{e})_{M\dot{N}} = e^{\pm i\theta} u_{\pm}^{(1)}(p, A\tilde{e}A^{\dagger})_{J\dot{K}}. \quad (5.99)$$

For the representation $D^{[1,0]}$, one gets on the one hand the usual point-like intertwiner for $\sigma = -1$

$$u_{-}^{(1)}(p, \tilde{e}) = u_{-}(p, \tilde{e}) \otimes u_{-}(p, \tilde{e}) = u_{-}(p) \otimes u_{-}(p) = u_{-}^{(1)}(p) \quad (5.100a)$$

$$(A^{-1} \otimes A^{-1}) u_{-}^{(1)}(p) = e^{-i\theta} u_{-}(\Lambda^{-1}p) \quad (5.100b)$$

which is independent of e , and on the other hand an additional string-like intertwiner for $\sigma = +1$

$$u_{+}^{(1)}(p, \tilde{e}) = u_{+}(p, \tilde{e}) \otimes u_{+}(p, \tilde{e}) \quad (5.101a)$$

$$(A^{-1} \otimes A^{-1}) u_{+}^{(1)}(p, A\tilde{e}A^{\dagger}) = e^{+i\theta} u_{+}(\Lambda^{-1}p, \tilde{e}). \quad (5.101b)$$

Using the tensor product notation, one can check immediately that they in fact satisfy the correct intertwiner relations.

To get intertwiners for more general representations $D^{[A,B]}$ and helicity $\pm\lambda$, one has to combine the fundamental intertwiners $u_{\pm}(p, \tilde{e})$ and their complex conjugates in such a way, that all the $A+B$ coefficients $e^{\pm i\frac{\theta}{2}}$ in each tensor factor add up to $e^{\pm i\lambda\theta}$. This means that in the expression

$$\begin{aligned} & \left[\underbrace{(A^{-1} \otimes \dots \otimes A^{-1})}_{A\text{-times}} \otimes \underbrace{(\overline{A^{-1}} \otimes \dots \otimes \overline{A^{-1}})}_{B\text{-times}} \right] \left[\underbrace{(u_{\pm} \otimes \dots \otimes u_{\pm})}_{A\text{-times}} \otimes \underbrace{(\overline{u_{\pm}} \otimes \dots \otimes \overline{u_{\pm}})}_{B\text{-times}} \right] \\ & = e^{\pm i\lambda\theta} \underbrace{(u_{\pm} \otimes \dots \otimes u_{\pm})}_{A\text{-times}} \otimes \underbrace{(\overline{u_{\pm}} \otimes \dots \otimes \overline{u_{\pm}})}_{B\text{-times}}, \end{aligned} \quad (5.102)$$

one has to adjust all the u_7 in such a way, that the correct factor $e^{\pm i\lambda\theta}$ comes out on the right side. Of course, this is only possible if $\lambda < A + B$, which is also the case for the point-like intertwiners for massive particles.

If the intertwiner, assembled in that way, has an integer number of indices (i.e. $A + B = \text{integer}$), one can go back to a tensor intertwiner, using the scheme described after equation (5.76). First, the intertwiner has to be multiplied with an appropriate number of factors ϵ^{JK} or $\epsilon^{\dot{M}\dot{N}}$ to adapt the number of dotted and undotted indices. Then the tensor intertwiner can be obtained according to

$$u_{\pm}(p, e)_{\mu\nu\dots} = 2^{-p/2} u_{\pm}(p, \tilde{e})^{JK\dots\dot{M}\dot{N}\dots} (\sigma_{\mu})_{J\dot{M}} (\sigma_{\nu})_{K\dot{N}} \dots, \quad (5.103)$$

where p is again the number of indices and the spinor $u_{\pm}(p, \tilde{e})^{JK\dots\dot{M}\dot{N}\dots}$ already contains the additional factors of ϵ .

After knowing how to construct these general intertwiners in the spinor formalism, one has to check if they lead to the same results as the 4-tensor intertwiners for the $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$ representations. That this is actually true will be shown in the special case of the photon intertwiners $u_{\pm}^{\mu}(p, e)$.

For the standard momentum $k = (1, 0, 0, 1)$ they read

$$u_{\pm}^{\mu}(k, e) = \frac{1}{(e \cdot k)} [\hat{e}_{\pm}^{\mu}(e \cdot k) - k^{\mu}(e \cdot \hat{e}_{\pm})]. \quad (5.104)$$

To compare them with the spinor intertwiners for $\sigma = \pm 1$ and $D^{[\frac{1}{2}, \frac{1}{2}]}$, one translates them into the corresponding 2×2 matrix $\tilde{u}_{\pm}(k, e)$:

$$\begin{aligned} \tilde{u}_{\pm}(k, e) &= \sum_{\mu} u_{\pm}^{\mu}(k, e) \sigma_{\mu} = u_{\pm}^0(k, e) \mathbb{1} + \vec{u}_{\pm}(k, e) \vec{\sigma} \\ &= \frac{1}{(e \cdot k)} [(-\sigma_1 \mp i\sigma_2)(e^0 - e^3) - (\sigma_0 + \sigma_3)(e^1 \pm ie^2)] \end{aligned} \quad (5.105)$$

$$\begin{aligned} \Rightarrow \tilde{u}_{+}(k, e) &= \frac{1}{(e \cdot k)} \left[- \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} (e^0 - e^3) - \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} (e^1 + ie^2) \right] \\ &= -\frac{2}{(e \cdot k)} \begin{pmatrix} e^1 + ie^2 & e^0 - e^3 \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (5.106a)$$

$$\begin{aligned}
\widetilde{u}_-(k, e) &= \frac{1}{(e \cdot k)} \left[- \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} (e^0 - e^3) - \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} (e^1 - ie^2) \right] \\
&= -\frac{2}{(e \cdot k)} \begin{pmatrix} e^1 - ie^2 & 0 \\ e^0 - e^3 & 0 \end{pmatrix}
\end{aligned} \tag{5.106b}$$

The explicit matrices of the spinor intertwiners are:

$$\begin{aligned}
u_+^{(1)}(\tilde{e}) &= u_+(\tilde{e}) \otimes \overline{u_-(\tilde{e})} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} e^1 + ie^2 \\ e^0 - e^3 \end{pmatrix} = \begin{pmatrix} e^1 + ie^2 & e^0 - e^3 \\ 0 & 0 \end{pmatrix} \\
&= -\frac{(e \cdot k)}{2} \widetilde{u}_+(k, e)
\end{aligned} \tag{5.107a}$$

$$\begin{aligned}
u_-^{(1)}(\tilde{e}) &= u_-(\tilde{e}) \otimes \overline{u_+(\tilde{e})} = \begin{pmatrix} e^1 - ie^2 \\ e^0 - e^3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^1 - ie^2 & 0 \\ e^0 - e^3 & 0 \end{pmatrix} \\
&= -\frac{(e \cdot k)}{2} \widetilde{u}_-(k, e)
\end{aligned} \tag{5.107b}$$

It can be seen that the two versions of the intertwiners only differ in the factor $-(e \cdot k)/2$, which is acceptable, because in general the intertwiners are only unique up to multiplication with a function $F(e \cdot p)$.

To get the same intertwiners in both cases, one can redefine the spinorial intertwiners according to

$$u_{\pm}^{(1)}(p, \tilde{e}) := -\frac{2}{(e \cdot p)} [u_{\mp}(p, \tilde{e}) \otimes \overline{u_{\pm}(p, \tilde{e})}], \tag{5.108}$$

where $u_{\pm}(p, \tilde{e})$ are the same intertwiners as in equation (5.91). The factor $1/(e \cdot p)$ again serves to improve the short distance behavior and guarantees that the corresponding field is a potential for the field strength.

5.6 Two-Point Function

For potential applications for these string-localized quantum fields, one needs their two-point functions and the corresponding Feynman propagators. The simplest case is the photon field, where the two-point function takes the form [14]

$$\langle \Omega, A(x, e)_{\mu} A(x', e')_{\nu} \Omega \rangle = \int d\mu(p) e^{ip(x-x')} M(p, e, e')_{\mu\nu}, \tag{5.109}$$

where the tensor $M(p, e, e')$ is defined according to

$$M(p, e, e')_{\mu\nu} := \pi(p)_{\mu\nu} - \pi(p)_{\rho\mu} \frac{e'^{\rho} p_{\nu}}{(e' \cdot p)} - \pi(p)_{\rho\nu} \frac{e^{\rho} p_{\mu}}{(e \cdot p)} + \pi(p)_{\rho\eta} \frac{e^{\rho} e'^{\eta} p_{\mu} p_{\nu}}{(e \cdot p)(e' \cdot p)}, \quad (5.110a)$$

$$\pi(p)_{\mu\nu} := \sum_{\sigma} \overline{\hat{e}_{\sigma}(p)_{\mu}} \hat{e}_{\sigma}(p)_{\nu}. \quad (5.110b)$$

The sum over the polarization vectors $\pi(p)_{\mu\nu}$ has the form [30, p.354]

$$\pi(p)_{ik} = \delta_{ik} - \frac{p_i p_k}{|p|^2}, \quad (5.111)$$

$$\pi(p)_{0i} = \pi(p)_{i0} = \pi(p)_{00} = 0.$$

The two-point function (5.109) with $e = e'$ has the same form as in the ordinary point-like formalism when using the axial gauge, with the difference that there e is a *fixed* direction. However, the usual axial gauge field has two disadvantages compared with the string-like field $A(x, e)_{\mu}$. Since the direction e is fixed, the field and also the two-point function are not covariant. The other problem are the singularities $(e \cdot p)^{-1}$, which have to be regularized in a certain way, whereas in the string-like approach the fields are also distributions in the string direction e and the factors $(e \cdot p + i\epsilon)^{-1}$ are regular after smearing with a test function (cf. [14]).

The two-point function of the general fields can be calculated the following way:

$$\begin{aligned} & \left\langle \Omega, A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_{\lambda}} A^{(\lambda)}(x', e')_{\nu_1 \dots \nu_{\lambda}} \Omega \right\rangle = \\ & = \int d\mu(p) e^{ip(x-x')} \sum_{\sigma=\pm} \overline{u_{\sigma}^{(\lambda)}(p, e)_{\mu_1 \dots \mu_{\lambda}}} u_{\sigma}^{(\lambda)}(p, e')_{\nu_1 \dots \nu_{\lambda}} \\ & = \int d\mu(p) e^{ip(x-x')} \frac{e^{\rho_1} \dots e^{\rho_{\lambda}}}{(e \cdot p)^{\lambda}} \frac{e'^{\eta_1} \dots e'^{\eta_{\lambda}}}{(e' \cdot p)^{\lambda}} \sum_{\sigma=\pm} \overline{\tilde{u}_{\sigma}^{(\lambda)}(p)_{\mu_1 \rho_1 \dots \mu_{\lambda} \rho_{\lambda}}} \tilde{u}_{\sigma}^{(\lambda)}(p)_{\nu_1 \eta_1 \dots \nu_{\lambda} \eta_{\lambda}} \quad (5.112) \\ & \equiv \int d\mu(p) e^{ip(x-x')} \frac{1}{(e \cdot p)^{\lambda} (e' \cdot p)^{\lambda}} \left(\prod_{k=1}^s e^{\rho_k} e'^{\eta_k} \right) \widetilde{M}^{(\lambda)}(p)_{\mu_1 \rho_1 \dots \mu_{\lambda} \rho_{\lambda} \nu_1 \eta_1 \dots \nu_{\lambda} \eta_{\lambda}} \\ & \equiv \int d\mu(p) e^{ip(x-x')} M^{(\lambda)}(p, e, e')_{\mu_1 \dots \mu_{\lambda} \nu_1 \dots \nu_{\lambda}} \end{aligned}$$

In the last two lines the “spin sum” of the point-like intertwiners, $\widetilde{M}^{(\lambda)}(p)$, and of the string-like intertwiners, $M^{(\lambda)}(p, e, e')$, have been defined.

The first one, $\widetilde{M}^{(\lambda)}(p)$, is a polynomial in p of degree 2λ , because the intertwiners $\tilde{u}_{\pm}^{(\lambda)}(p)$ are polynomials of degree λ . The second one, $M^{(\lambda)}(p, e, e')$, includes the factor $\frac{1}{(e \cdot p)^{\lambda} (e' \cdot p)^{\lambda}}$ which is proportional to $\frac{1}{|p|^{2\lambda}}$, so $M^{(\lambda)}(p, e, e')$ is bounded for large p . This suggests that the Fourier transform of the Feynman propagator goes like $|p|^{-2}$ for large $|p|$, independent

of the helicity. This is a significant improvement over the point-like field strengths, whose large p behavior of the two-point function gets worse with increasing spin.

5.7 Short Distance Behavior

In this section it will be shown, that the string-localized fields $A^{(\lambda)}(x, e)$ have a nice short distance behavior, namely that their short distance dimension (sdd) is always $\text{sdd} = 1$ and it does not get worse with increasing spin.

By “short distance behavior” the behavior of the field under dilations is meant (cf. [16, sec. 5]). Therefore one extends the representation $U(a, \Lambda)$ of the Poincaré group to the dilations d_α , $\alpha > 0$. The operator $U_1(d_\alpha)$, representing d_α on the one particle space \mathcal{H}_1 , is defined according to

$$(U_1(d_\alpha)\Psi)(p) := \alpha\Psi(\alpha p) \equiv \Psi_\alpha(p), \quad (5.113)$$

Using the fact that

$$d\mu(\alpha p) = \Theta(\alpha p^0)\delta(\alpha^2 p^2)\alpha^4 d^4 p = \alpha^2 d\mu(p) \quad (5.114)$$

one can calculate the norm of the state $\Psi_\alpha(p)$,

$$\int d\mu(p)\alpha^2|\Psi(\alpha p)|^2 = \int d\mu(\alpha^{-1}p)\alpha^2|\Psi(p)|^2 = \int d\mu(p)|\Psi(p)|^2, \quad (5.115)$$

which shows that $U_1(d_\alpha)$ really is a unitary operator.

The definition (5.113) then determines the transformation behavior of the creation operators $a^*(\Psi)$ and $a^*(p)$:

$$\begin{aligned} U(d_\alpha)a^*(\Psi)U(d_\alpha)^{-1} &= a^*(\Psi_\alpha) = \int d\mu(p)\Psi_\alpha(p)a^*(p) \\ &= \int d\mu(p)\alpha\Psi(\alpha p)a^*(p) = \int d\mu(\alpha^{-1}p)\alpha\Psi(p)a^*(\alpha^{-1}p) \\ &= \int d\mu(p)\Psi(p)\alpha^{-1}a^*(\alpha^{-1}p) = \int d\mu(p)\Psi(p)U(d_\alpha)a^*(p)U(d_\alpha)^{-1} \end{aligned} \quad (5.116)$$

So under dilations the creation operators $a^*(p)$ behave according to

$$U(d_\alpha)a^*(p)U(d_\alpha)^{-1} = \alpha^{-1}a^*(\alpha^{-1}p). \quad (5.117)$$

Knowing this one can now calculate the behavior of the fields $A^{(\lambda)}(x, e)_{\mu_1 \dots \mu_\lambda}$ under dila-

tions (the cumbersome indices will be omitted in this calculation):

$$\begin{aligned}
U(d_\alpha)A^{(\lambda)}(x, e)U(d_\alpha^{-1}) &= \int d\mu(p) \left[e^{ipx} u^{(\lambda)}(p, e) U(d_\alpha)a^*(p)U(d_\alpha)^{-1} + \text{h.c.} \right] \\
&= \int d\mu(p) \left[e^{ipx} u^{(\lambda)}(p, e) \alpha^{-1}a^*(\alpha^{-1}p) + \text{h.c.} \right] \\
&= \int d\mu(\alpha p)\alpha^{-1} \left[e^{i(\alpha p)x} u^{(\lambda)}(\alpha p, e)a^*(p) + \text{h.c.} \right] \\
&= \int d\mu(p)\alpha \left[e^{ip(\alpha x)} u^{(\lambda)}(p, e) a^*(p) + \text{h.c.} \right] \\
&= \alpha A^{(\lambda)}(\alpha x, e)
\end{aligned} \tag{5.118}$$

This is because the intertwiners (5.28) satisfy the relation

$$u_{\pm}^{(\lambda)}(\alpha p, e) = u_{\pm}^{(\lambda)}(p, e), \tag{5.119}$$

which can be seen right away by making the substitution $p \mapsto \alpha p$ in the definition (5.28).

Therefore under dilations the fields behave according to

$$U(d_\alpha)A^{(\lambda)}(x, e)U(d_\alpha^{-1}) = \alpha A^{(\lambda)}(\alpha x, e), \tag{5.120}$$

and this is what is meant by “the field has short distance dimension $\text{sdd} = 1$ ”.

In contrast to this nice behavior of the potentials $A^{(\lambda)}(x, e)$, the fields strengths $\mathcal{F}^{(\lambda)}(x)$ have a short distance behavior that gets worse with increasing spin, i.e.

$$U(d_\alpha)\mathcal{F}^{(\lambda)}(x)U(d_\alpha^{-1}) = \alpha^{\lambda+1}\mathcal{F}^{(\lambda)}(\alpha x). \tag{5.121}$$

The worse short distance dimension stems from the fact that the fields $\mathcal{F}^{(\lambda)}(x)$ have point-like intertwiners $\tilde{u}_{\pm}^{(\lambda)}(p)$ that satisfy the relation

$$\tilde{u}_{\pm}^{(\lambda)}(\alpha p) = \alpha^\lambda \tilde{u}_{\pm}^{(\lambda)}(p), \tag{5.122}$$

because of the missing factor $\frac{1}{(e \cdot p)^\lambda}$, which reduces the short distance dimension to one in case of the potentials $A^{(\lambda)}(x, e)$.

6 Conclusion

After reviewing some previous results on massive string-localized quantum fields, string-like intertwiners and the corresponding quantum fields for massless bosonic particles have been constructed. It has been shown that, in contrast to the massive string-localized fields, they cannot be introduced as scalar fields, but need additional tensor- or spinor indices. After the calculation of the intertwiner relations they have to obey, two important examples have been considered. These are the string-localized vector potential for the photon field and the tensor potential for the gravitational field. They are potentials for the electromagnetic field strength and the Riemann tensor in the usual sense. It has also been shown that, apart from the axial gauge condition, they satisfy the Lorentz and the harmonic gauge conditions respectively.

After that, the intertwiners for general helicities λ for the representations $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$ have been given as tensor products of the fundamental photon intertwiner, and it has been shown, that they satisfy generalized Lorentz-, axial gauge- and symmetry conditions. Furthermore it has been proven that the string-localized potentials, transforming according to these representations, can all be written as infinite line integrals over the corresponding point-localized tensor field strengths, just like it is possible in the massive case.

Using the more general spinor formalism to obtain representations of $SL(2, \mathbb{C})$, intertwiners for arbitrary representations $D^{[A, B]}$ could also be constructed for general helicities, which are subject to the restriction $|A - B| < \lambda < A + B$. This construction is also possible for massless fermions with half-integer helicity. The difference lies mainly in using anti-commuting annihilation/creation operators instead of commuting ones.

In the end, other important properties of the string-localized $D^{[\frac{\lambda}{2}, \frac{\lambda}{2}]}$ fields have been discussed, like the short distance behavior and the two-point function of the fields. It turned out that the tensor potentials all have the same short distance dimension, $\text{sdd} = 1$, which also makes sure the propagator of these fields behaves like $|p|^{-2}$ for large $|p|$, independent of the helicity.

The two-point functions of the fields can be written as the Fourier transform of the point-like spin sum $\widetilde{M}^{(\lambda)}(p)$ times a factor $(e \cdot p)^{-\lambda}(e' \cdot p)^{-\lambda}$, which causes the nice large p behavior. Although this expression looks quite singular for $e \cdot p \rightarrow 0$, it becomes regular after smearing with a test function, which is an advantage over the usual axial gauge fields.

Until now only free fields have been considered, because they can be easily constructed by second-quantizing free single particle states. What would be interesting in the next step, is to incorporate these fields into a perturbative scheme, like the causal construction of Epstein and Glaser [6]. There are some difficulties, which prevent this approach from being generalized straightforward to string-localized fields. For example the time-ordering prescription of products of fields must take the strings $S_{x,e}$ into account and one has to assure that string-locality stays valid in every order of the perturbation series. Some ideas in this direction have been proposed by Mund in [14], but there are currently no rigorous results about the possibility of using string-localized fields to describe interactions.

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Curriculum Vitae

Persönliche Daten:

Name: Matthias Plaschke
Geburtstag: 03. September 1985
Geburtsort: Bregenz, Vorarlberg
Nationalität: Österreich

Ausbildung:

1992 - 1996 Volkschule Hörbranz, Vorarlberg
1996 - 2004 BG Gallusstraße, Bregenz, Vorarlberg
June, 2004 Matura am BG Gallustraße
09. 2004 - 04. 2005: Wehrdienst in Salzburg
seit 2005 Diplomstudium Physik an der Universtität Wien