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Semi-Riemannian Manifolds with Distributional Curvature

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0 Introduction

This work is concerned with the foundations of Semi-Riemannian geometry in case of low regularity. More precisely, using distributional connections and distributional Semi-Riemannian metrics respectively as a starting point, we will study to what extent classical results can be generalized to the distributional setting. As the limitations of such a general distributional approach are reached fairly quickly, we will subsequently narrow down the class of metrics and connections considered to those which belong to some appropriate local Sobolev spaces.

The motivation to look at a distributional formulation of Semi-Riemannian geometry comes from the theory of general relativity. More specifically, one is interested in gravitational sources concentrated in a region of space which can be 'approximated' by a line, a two- or a three-dimensional surface. Such sources model point particles, strings or shells of matter or radiation, with the idealization being physically admissible provided the appropriate internal structure of the source can be neglected. As it is natural to use delta distributions to describe such objects, a rigorous mathematical description has to incorporate the theory of distributions on manifolds. Moreover, distributional geometries are also useful in describing other physical scenarios in general relativity such as shock waves and junction conditions between matter and vacuum regions. This has led several authors to apply distributional geometry ([dR84, YCB78]) to general relativity, e.g. [GT87, Par79, Tau80, Lic79, CB93]. For a pedagogical account of distributional geometry see [GKOS01, Sec.3].

In linear field theories such as electrodynamics, the theory of distributions actually furnishes a consistent framework. More precisely, since Maxwell equations are linear in both sources and fields they can be formulated within the framework of distributions and thus allow for distributional as well as classical smooth solutions. Moreover, it is guaranteed that smooth sources sufficiently near to a distributional source in the sense of convergence of distributions produce fields which are close to the corresponding distributional fields. While the former is necessary for e.g. point charges to make sense mathematically, it is the latter property which renders them physically meaningful.

However, Einstein equations, which govern the behaviour of sources and fields in general relativity link the curvature of the space-time metric to the energy contents of the space-time in a non-linear way. Due to the impossibility to define a general product of distributions, distributional metrics have only limited use, since we cannot simply compute their curvature. However, imposing additional regularity on the metric, it is nevertheless possible to avoid ill-defined distributional products. The first work in this direction was conducted by Geroch and Traschen in [GT87], where they explored the lowest regularity requirements on a metric still allowing for Einstein equations to be formulated within the context of distributional geometry. More precisely, they introduced a class of 'regular' metrics, from now on referred to as gt-regular metrics, which apart from enabling the curvature to be computed also possessed certain stability properties. That is, the notion of convergence adopted for these metrics implies distributional convergence of the corresponding curvature tensors. As noted previously, it is precisely this latter property that makes it reasonable to use gt-regular metrics to model singular matter configurations in general relativity. However, Geroch and Traschen also showed that curvature tensor of a gt-regular metric can only be supported on a manifold of codimension at most one, which, while including thin-shells of matter and radiation, explicitly excludes other interesting physical scenarios such as point particles and cosmic strings.

The desire to model a wider class of non-smooth spacetimes has led several authors to introduce algebras of generalized functions, in the sense of J. F.Colombeau [Col84], into general relativity. These algebras are essentially based on regularization by convolution and the use of asymptotic estimates with respect to an appropriate regularization parameter. They contain both the space of smooth functions as a subalgebra as well as the space of distributions as a linear subspace and allow a product to be assigned to every pair of distributions. Moreover, via a procedure called association the result of calculations in the algebra of generalized functions can be compared to that obtained within the framework of classical distributions. This is important as it allows for these calculations to be interpreted physically. Algebras of generalized functions have been successfully applied to various problems in general relativity such as cosmic strings, Kerr-Schild geometries and impulsive pp-waves, see [SV06] and references therein.

Recently, there has been renewed interest in Semi-Riemannian metrics of low regularity. On the one hand, in [LM07] the formalism of [GT87] has been rederived in a coordinate-free manner and significantly extended. These results have been subsequently applied in [LR10] to the initial value problem for Einstein-Euler equations in Gowdy models. On the other hand there has also been interest in the question of compatibility of the classical distributional approach to calculating curvature due to [GT87] with the approach based on algebras of generalized functions. Both methods have been shown to be equivalent in [SV09]. For an extensive review of the use of generalized function and distributions in general relativity, again see [SV06].

In the following we describe the contents of this thesis in some detail. In order to make this work self-contained, we start by collecting all the necessary prerequisites. Essentially, these include distributional geometry, Sobolev spaces on manifolds and, in particular, the corresponding trace theorems. To this end we also recall some basic notions from smooth differential geometry, the theory of distributions and Sobolev spaces on open subsets of $\mathbb{R}^n$. A brief account of all these issues is given in section 1. The heart of this work is comprised of sections 2 and 3. In section 2 we review distributional Semi-Riemannian geometry. In particular, we deal with the limitations and the extent to which it is possible to incorporate the genuinely linear theory of distributions into the nonlinear theory of general relativity. We end section 2 by examining properties of gt-regular metrics introduced in [GT87] by Geroch and Traschen. In section 3, the results of section 2 are specialized in a coordinate free manner, to the situation, where a metric or a connection is 'smooth' off some hypersurface and suffers 'discontinuity' across the hypersurface in question. In particular, we discuss several 'jump formulas' for the respective curvature quantities.

While we have used standard references like [Ada75, O'N83, CP82] in the preparatory section 1, our main references for sections 2 and 3 have been the research articles [LM07, Ste08, GT87].

1 Preliminaries

1.1 Some function spaces

To begin with we recall some facts about function spaces on subsets of $\mathbb{R}^n$. For this purpose Ω will denote an open subset of $\mathbb{R}^n$, $\overline{\Omega}$ its closure and $\partial\Omega$ its boundary. In particular, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_n > 0\}$ with closure $\overline{\mathbb{R}}^n_+ := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ and boundary $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n = 0\}$. Here we have used $(x_1, ..., x_n) = x \in \mathbb{R}^n$. For a given $x \in \mathbb{R}^n$ we will write $x = (x', x_n)$ where x' denotes the first n-1 coordinates. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ we set

$$|\alpha| = \alpha_1 + ... + \alpha_n,$$

$$\alpha! = \alpha_1!...\alpha_n!$$

$$x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n} \text{ and } \partial^\alpha = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$$

where $x \in \mathbb{R}^n$ and $\partial_i = \partial/\partial_{x_i}$. Moreover, we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for $i = 1, ..., n$.

1.1.1. By $C^\infty(\Omega)$ we will denote the space of smooth complex-valued functions on Ω. Equipped with the topology induced by the family of semi-norms

$$\phi \longmapsto p_{m,K}(\phi) := \sup_{x \in K, 1 \leq |\alpha| \leq m} |\partial^\alpha \phi(x)|$$

where K is a compact subset of Ω and m is an integer, $C^\infty(\Omega)$ is a Fréchet space. For a given compact subset K of Ω, $\mathcal{D}_K(\Omega)$ will denote the closed subspace of $C^\infty(\Omega)$ containing all $\phi \in C^\infty(\Omega)$ that vanish on $\Omega \backslash K$. Finally, by

$$\mathcal{D}(\Omega) = \bigcup_K \mathcal{D}_K(\Omega)$$

we will denote the subspace of $C^\infty(\Omega)$ formed by all compactly supported functions on Ω. $\mathcal{D}(\Omega)$ will be equipped with the strict inductive limit topology with respect to all $\mathcal{D}_K(\Omega)$, that is the finest locally convex topology on $\mathcal{D}(\Omega)$ such that all injections $\mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$ are continuous, see for instance [Jar81, Ch.4]. In particular, a linear mapping from $\mathcal{D}(\Omega)$ into some topological vector space is continuous if and only if its composition with the injection $\mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega)$ is continuous for every K. Consequently the inclusion of $\mathcal{D}(\Omega)$ in $C^\infty(\Omega)$ is continuous.

1.1.2. The space $C^\infty(\overline{\Omega})$ is defined to be the set of all functions $\phi \in C^\infty(\Omega)$ for which $\partial^\alpha\phi$ is bounded and uniformly continuous on Ω for all multiindices α. Every $\phi \in C^\infty(\overline{\Omega})$ admits a unique, bounded and continuous extension of all $\partial^\alpha\phi$ to the whole of $\overline{\Omega}$. We equip $C^\infty(\overline{\Omega})$

with the family of seminorms

$$\phi \longmapsto p_{m,K}(\phi) := \sup_{x\in K, 1\leq|\alpha|\leq m}|\partial^\alpha \phi(x)|,$$

where K now runs over all compact subsets of $\overline{\Omega}$ and m is an integer. This turns $C^\infty(\overline{\Omega})$ into a Fréchet space. The inclusion of $C^\infty(\overline{\Omega})$ in $C^\infty(\Omega)$ is obviously continuous.
Similarly to 1.1.1, $\mathcal{D}_K(\overline{\Omega})$ denotes the closed subspace of $C^\infty(\overline{\Omega})$ formed by $\phi \in C^\infty(\overline{\Omega})$ which vanish outside some compact subset K of $\overline{\Omega}$, whereas $\mathcal{D}(\overline{\Omega}) = \bigcup_K \mathcal{D}_K(\overline{\Omega})$ denotes the set of compactly supported elements of $C^\infty(\overline{\Omega})$. As before, $\mathcal{D}(\overline{\Omega})$ will be equipped with the strict inductive limit topology with respect to all $\mathcal{D}_K(\overline{\Omega})$, which as in 1.1.1, implies that the inclusion of $\mathcal{D}(\overline{\Omega})$ in $C^\infty(\overline{\Omega})$ is continuous.

We now introduce the notion of regular open subsets of $\mathbb{R}^n$ which are a special case of manifolds with boundary as we shall see further below. Although we will only be interested in the special case $\mathbb{R}^n_+$, we include, for the sake of completness, some results for general regular open sets as well.

1.1.3. Definition. *An open set Ω is called regular open subset of $\mathbb{R}^n$ if for every $x \in \partial\Omega$ there exists an open neighbourhood U in $\mathbb{R}^n$ and a diffeomorphism $\psi : U \to \psi(U) \subseteq \mathbb{R}^n$ such that*

$$\psi(U \cap \Omega) = \{y \in \psi(U) \mid y_n > 0\}$$

$$\psi(U \cap \partial\Omega) = \{y \in \psi(U) \mid y_n = 0\}.$$

If Ω is assumed to be a regular open subset of $\mathbb{R}^n$, the space $C^\infty(\overline{\Omega})$ is the set of restrictions of $C^\infty(\mathbb{R}^n)$-functions to Ω. More precisely we have:

1.1.4. Proposition. *Let Ω be a regular open subset of $\mathbb{R}^n$. Every $\phi \in C^\infty(\overline{\Omega})$ can be extended to $C^\infty(\mathbb{R}^n)$, the extension operator ρ being linear and continuous. The restriction of ρ to $\mathcal{D}(\overline{\Omega})$ is a continuous, linear operator from $\mathcal{D}(\overline{\Omega})$ to $\mathcal{D}(\mathbb{R}^n)$.*

Sketch of proof. First, the case $\Omega = \mathbb{R}^n_+$ is proven using the Seeley extension. To be more precise, the extension of $\phi \in C^\infty(\overline{\mathbb{R}}^n_+)$ to $C^\infty(\mathbb{R}^n)$ is defined to be

$$\rho(\phi) = \begin{cases} \phi(x) & x_n \geq 0 \\ \sum_{k=1}^\infty \lambda_k \chi(-2^k x_n)\phi(x', -2^k x_n) & x_n < 0 \end{cases}$$

where $\chi \in \mathcal{D}(\mathbb{R})$ is a bump function which is equal to 1 in a neighbourhood of zero and $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ for which $\sum_{k=1}^\infty \lambda_k 2^{kj}$ converges absolutely to $(-1)^j$.

Using partitions of unity and the fact that Ω is locally diffeomorphic to $\mathbb{R}^n_+$, one subsequently proves the assertion for arbitrary regular open subsets Ω of $\mathbb{R}^n$.

For a complete proof see [CP82, Prop.9.2, p.64] □

All manifolds considered in this work will be assumed to be oriented, connected, paracompact, Hausdorff and smooth. They will generally be denoted by M or N. Observe that these assumptions on the manifold M imply the existence of C^∞-partitions of unity and of exhaustive sequences of compact sets i.e. the existence of sequences $K_n \subseteq M$ of compact sets such that $K_n \subseteq \overset{\circ}{K}_{n+1}$ and $\bigcup K_n = M$. The notation (U, ψ) will be used to denote a chart ψ of M with domain U.

1.1.5. Definition. *A C^∞-manifold with boundary is a set $\overline{X}$ together with an atlas of charts $\psi_U : U \to \psi_U(U)$ where $U \subseteq \overline{X}$ and $\psi_U(U)$ is open in $\overline{\mathbb{R}}^n_+$.*

Note that the compatibility of charts of a manifold with boundary means that the mappings $\psi_V \circ \psi_U^{-1} : \psi_U^{-1}(V \cap U) \to \psi_V(V \cap U)$ are diffeomorphisms of open subsets of $\overline{\mathbb{R}}^n_+$. Recall that a mapping between open subsets of $\overline{\mathbb{R}}^n_+$ is considered to be smooth if it possesses a smooth extension to open sets in $\mathbb{R}^n$.

In addition, both the boundary $\partial X = \{p \in \overline{X} \mid \exists \psi_U \text{ such that } \psi_U(p) \in \partial \mathbb{R}^n_+\}$ and the interior $X := \{p \in \overline{X} \mid \exists \psi_U \text{ such that } \psi_U(p) \in \mathbb{R}^n_+\}$ of $\overline{X}$ are manifolds without boundary embedded in $\overline{X}$ in the sense defined below. An atlas for ∂X is given by the restrictions $\psi_U|_{\partial X}$ and that for X by the restrictions $\psi_U|_X$ of the charts ψ_U of an $\overline{X}$-atlas.

Given two smooth manifolds M and N without boundary, a function $f : N \to M$ is called smooth if $\psi \circ f \circ \phi^{-1}$ is smooth for every chart ψ of M and ϕ of N. The space of smooth functions between N and M will be denoted by $C^\infty(N, M)$.

1.1.6. Definition. *Let N and M be smooth manifolds. $f \in C^\infty(N, M)$ is called an embedding if it is an injective immersion and $f : N \to f(N)$ is a homeomorphism with respect to the trace topology of $f(N)$ in M.*

Recall that $f \in C^\infty(N, M)$ is called an immersion if $T_x f : T_x N \to T_{f(x)} M$ is injectiv for every $x \in N$, where $T_x N$ and $T_{f(x)} M$ denote the respective tangent spaces.

1.1.7. $f \in C^\infty(N, M)$ is an embedding if and only if for every $x \in N$ there exists a chart ψ of M centered at $f(x)$ for which

$$\psi \circ f : f^{-1}(\mathrm{dom}\psi) \to \mathrm{im}\psi \cap \mathbb{R}^m \tag{1.1}$$

is a well-defined diffeomorphism, hence a chart of N at x, see [Kri, 21.11]. Here, m denotes the dimension of N. In particular, if $N \subseteq M$ and inclusion $\iota : N \hookrightarrow M$ is an embedding, N is called a (regular) submanifold of M. If, in addition, $\mathrm{dim}N = \mathrm{dim}M - 1$ then N is called a hypersurface of M.

On the other hand any subset N of the manifold M, such that for every $x \in N$ there exists a chart ψ of M centered at x for which

$$N \cap \mathrm{dom}\psi = \psi^{-1}(\mathrm{im}\psi \cap \mathbb{R}^m), \tag{1.2}$$

is itself an m-dimensional manifold with charts given by the restrictions of the ψ's. Moreover, according to (1.1), the inclusion $\iota : N \hookrightarrow M$ is an embedding.
Observe that in definition 1.1.6, M and/or N can be assumed to be manifolds with boundary. In particular, if N is a manifold with boundary both (1.1) and (1.2) hold provided $\mathbb{R}^m$ is replaced by $\overline{\mathbb{R}}^m_+$.

1.1.8. For a given smooth manifold M without boundary, $C^\infty(M)$ will denote the space of smooth functions on M, that is functions $f : M \to \mathbb{C}$ such that $\psi_* f \in C^\infty(\psi(U))$ for every chart (U, ψ) of M. Here we use the notation $\psi_* f = f \circ \psi^{-1}$. $C^\infty(M)$ will be equipped with the topology defined by the family of seminorms

$$q_{m,K,\psi}(f) := p_{m,K}(\psi_* f),$$

where (U, ψ) are charts of M and $p_{m,K}$ are the seminorms of 1.1.1. Using partitions of unity, one can show that seminorms associated with the charts of some atlas suffice to define the topology and that $C^\infty(M)$ is a Fréchet space.

For a fixed compact subset K of M, $\mathcal{D}_K(M)$ denotes the closed subspace of $C^\infty(M)$ consisting of all $f \in C^\infty(M)$ which vanish outside K, whereas $\mathcal{D}(M)$ denotes the inductive limit over all $\mathcal{D}_K(M)$.

If $\overline{X}$ is a manifold with boundary, $C^\infty(\overline{X})$ denotes the space of all $f \in C^\infty(X)$ such that $\psi_* f$ belong to $C^\infty(\overline{\psi(U)})$ for all charts (U, ψ) of $\overline{X}$. The spaces $\mathcal{D}_K(\overline{X})$ and $\mathcal{D}(\overline{X})$ are defined as in 1.1.2. The corresponding topological statements generalize without difficulty.

1.1.9. Proposition. *Let $\overline{X} \subseteq M$, where $\overline{X}$ and M are manifolds of the same dimension with, respectively without boundary. Moreover, assume the inclusion $\iota : \overline{X} \to M$ is an embedding. Then there exists a linear, continuous extension operator $\rho : C^\infty(\overline{X}) \to C^\infty(M)$ whose restriction to $\mathcal{D}(\overline{X})$ has values in $\mathcal{D}(M)$ and is linear and continuous.*

Sketch of proof. Use partitions of unity and proposition 1.1.4 for $\Omega = \mathbb{R}^n_+$ (which is a regular open subset of $\mathbb{R}^n$). □

1.1.10. Remark. Using yet another characterisation of embeddings, it can be shown that provided $f \in C^\infty(N, M)$ is an embedding and $f(N)$ is closed in M, for every $\varphi_0 \in C^\infty(N)$ there exists a $\varphi \in C^\infty(M)$ such that $\varphi \circ f = \varphi_0$.

In fact, $f \in C^\infty(N, M)$ is an embedding if and only if f has a local left inverse i.e. for every $x \in N$ there exists an open neighbourhood $V_{f(x)}$ of $f(x)$ in M and a function $h : V_{f(x)} \to N$ such that $h \circ f = \mathrm{id}$ on $f^{-1}(V_{f(x)})$, see [Kri, 21.11]. This implies that φ exists locally (set $\varphi = \varphi_0 \circ h$ on $V_{f(x)}$ and $\varphi \equiv 0$ on $M \backslash f(N)$). Gluing these local φ's by a partition of unity subordinated to the open covering of M consisting of $(V_{f(x)})_{x \in N}$ and $M \backslash f(N)$, we obtain $\varphi \in C^\infty(M)$ with the desired properties.

Note that under the same conditions on f and N an analogous statement holds for $\mathcal{D}(N)$ as well, i.e. for every $\varphi_0 \in \mathcal{D}(N)$ there exists a $\varphi \in \mathcal{D}(M)$ such that $\varphi \circ f = \varphi_0$. In fact, as shown above there exists a $\phi \in C^\infty(M)$ such that $\phi \circ f = \varphi_0$. Setting $\varphi := \chi\phi$, where χ is a bump function equal to 1 on $f(\mathrm{supp}(\varphi_0))$, the claim follows.

1.2 Sobolev spaces on $\mathbb{R}^n$

We first recall some basic facts from distribution theory, for details see for instance [FJ82, ch.1,2].

1.2.1. Definition. *The space of distributions $\mathcal{D}'(\Omega)$ is the topological dual of the space $\mathcal{D}(\Omega)$, i.e. the space of continuous linear forms on $\mathcal{D}(\Omega)$.*

The action of $u \in \mathcal{D}'(\Omega)$ on $\phi \in \mathcal{D}(\Omega)$ will be denoted $\langle u, \phi \rangle$. Note that a linear functional u on Ω belongs to $\mathcal{D}'(\Omega)$ if and only if for every compact subset K of Ω there exists a constant $C \geq 0$ and an integer m such that

$$|\langle u, \phi \rangle| \leq C p_{m,K}(\phi)$$

for every $\phi \in \mathcal{D}_K(\Omega)$.

$\mathcal{D}'(\Omega)$ will be equipped with the weak-*-convergence topology, i.e. a sequence u_n converges to u in $\mathcal{D}'(\Omega)$ if $\langle u_n, \phi \rangle$ converges to $\langle u, \phi \rangle$ in $\mathbb{C}$ for every $\phi \in \mathcal{D}(\Omega)$.

The space of locally integrable functions on Ω is embedded in $\mathcal{D}'(\Omega)$ via

$$j : L^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}'(\Omega)$$

$$\langle j(f), \phi \rangle = \int_\Omega f \phi dx \ , \ \phi \in \mathcal{D}(\Omega).$$

The image of $\mathcal{D}(\Omega)$ under j is dense in $\mathcal{D}'(\Omega)$.

For an open subset Ω' of Ω, the restriction $u|_{\Omega'}$ of $u \in \mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega')$ is defined to be the distribution $\langle u|_{\Omega'}, \phi \rangle = \langle u, \phi \rangle$ for all ϕ in $\mathcal{D}(\Omega')$.

If $u \in \mathcal{D}'(\Omega)$, the distributional derivative $\partial^\alpha u$ is defined by $\langle \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$ for all $\phi \in \mathcal{D}(\Omega)$ and multiindices α.

Next we collect some facts on Sobolev spaces. For details see [Ada75].

1.2.2. Definition. *For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ Sobolev spaces $W^{m,p}(\Omega)$ are defined to be the sets*

$$W^{m,p}(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \ \forall \alpha : 0 \leq |\alpha| \leq m\}.$$

Equipped with the norm

$$\|u\|_{m,p} = \begin{cases} (\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha u\|_p)^{1/p} & p < \infty \\ \max_{0 \leq |\alpha| \leq m} \|\partial^\alpha u\|_\infty & p = \infty \end{cases}$$

they are Banach spaces. Here $\| \ \|_p$ denotes the norm in $L^p(\Omega)$. If $p = 2$, then $W^{m,2}(\Omega)$ is even a Hilbert space. In this case we write H^m instead of $W^{m,2}$. Note that $W^{0,p}(\Omega)$ are just the usual Lebesque spaces $L^p(\Omega)$.

1.2.3. Definition. *Let Ω be $\mathbb{R}^n_+$, $\mathbb{R}^n$ or a bounded regular open subset of $\mathbb{R}^n$. For $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $1 \leq p \leq \infty$, the space $W^{s,p}(\Omega)$ is defined as the set of all $u \in W^{\lfloor s \rfloor,p}(\Omega)$ for which the norm:*

$$\|u\|_{s,p} = \begin{cases} \|u\|_{\lfloor s \rfloor,p} + \sum\limits_{|\alpha| = \lfloor s \rfloor} (\int_\Omega \int_\Omega \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+(s-\lfloor s \rfloor)p}} dx dy)^{1/p} & p < \infty \\ \|u\|_{s,\infty} = \max(\|u\|_{\lfloor s \rfloor,\infty}, \max\limits_{|\alpha| = \lfloor s \rfloor} \operatorname*{esssup}\limits_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^{s-\lfloor s \rfloor}}) & p = \infty \end{cases}$$

is finite, where $\lfloor s \rfloor$ is the largest integer smaller than s.

With this norm topology, the spaces $W^{s,p}(\Omega)$ are Banach spaces. For the definition of $W^{s,p}(\Omega)$ for more general Ω see [Ada75, 7.35] (compare this also with [Ada75, 7.48]), however we will not need these spaces in the following.

1.2.4. Definition. *For $s \in \mathbb{R}_+$ and $1 \leq p \leq \infty$ the Sobolev spaces $W^{s,p}_{\mathrm{loc}}(\Omega)$ are defined to be the sets*

$$W^{s,p}_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid \varphi u \in W^{s,p}(\mathbb{R}^n) \ \forall \varphi \in \mathcal{D}(\Omega)\}.$$

The spaces $W^{s,p}_{\mathrm{loc}}(\Omega)$ are Fréchet spaces, their topology being defined by the family of seminorms

$$p_{s,p,\varphi}(u) := \|\varphi u\|_{s,p}$$

for $\varphi \in \mathcal{D}(\Omega)$.

If m is a non-negative integer, u belongs to $W^{m,p}_{\mathrm{loc}}(\Omega)$ if and only if u belongs to $W^{m,p}(\Omega')$ for all open and relatively compact $\Omega' \subseteq \Omega$.

1.2.5. Remark. $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$ for any $s \geq 0$ and $1 \leq p < \infty$, see [Ada75, 7.38]. For arbitrary Ω the situation is more subtle, the above statement being false in general. For our purposes it suffices to know that for $s \geq 0$ and $p < \infty$ the restrictions of $\mathcal{D}(\mathbb{R}^n)$ to $\mathbb{R}^n_+$ are dense in $W^{s,p}(\mathbb{R}^n_+)$. This implies that $\mathcal{D}(\overline{\mathbb{R}}^n_+)$ is dense in $W^{s,p}(\mathbb{R}^n_+)$, since $\mathcal{D}(\mathbb{R}^n)|_{\mathbb{R}^n_+} = \mathcal{D}(\overline{\mathbb{R}}^n_+)$ as we already know from proposition 1.1.4. Density also holds if Ω is a bounded regular open set i.e. in this case $\mathcal{D}(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$, see [Tri83, 3.2.4].

Luckily the situation is far more simple for $W^{s,p}_{\mathrm{loc}}(\Omega)$, since $\mathcal{D}(\Omega)$ is indeed dense in $W^{s,p}_{\mathrm{loc}}(\Omega)$ for every open Ω and $p < \infty$. To see this just note that after choosing an exhaustive sequence for Ω and corresponding plateau functions φ_n we clearly have $\varphi_n u \to u$ in $W^{s,p}_{\mathrm{loc}}(\Omega)$ for every $u \in W^{s,p}_{\mathrm{loc}}(\Omega)$. Since every $\varphi_n u$ can be approximated in the respective $W^{s,p}(\Omega)$-norm through mollification by compactly supported smooth functions the claim follows, see [Ada75, 3.15].

The Sobolev Imbedding Theorem

Imbedding properties of Sobolev spaces, known as Sobolev imbedding theorem, assert the existence of imbeddings of $W^{m,p}(\Omega)$ into various Banach spaces. Here, we will restrict ourselves to two cases, namely the imbedding of $W^{m,p}(\Omega)$ into $L^q(\Omega)$ and $C^j_B(\Omega)$. For a more complete account on the subject we refer to [AF03, Ch.4].

1.2.6. Theorem. *Let Ω be $\mathbb{R}^n_+$, $\mathbb{R}^n$ or a bounded regular open subset of $\mathbb{R}^n$. Furthermore let $1 \leq p < \infty$, and let $m \geq 1$ and $j \geq 0$ be integers. If either $mp > n$ or $m = n$ and $p = 1$, then*

$$W^{j+m,p}(\Omega) \subseteq C^j_B(\Omega)$$

continuously. Here $C^j_B(\Omega)$ denotes the space of functions on Ω whose derivatives are bounded and continuous up to order j. Moreover

$$W^{m,p}(\Omega) \subseteq L^q(\Omega)$$

continuously, if

(1.) $mp > n$ *for* $p \leq q \leq \infty$,

(2.) $mp = n$ *for* $p \leq q < \infty$,

(3.) $mp < n$ *for* $p \leq q \leq np/(n - mp)$.

Proof. [AF03, 4.12] □

Note that the assumptions on Ω can be considerably relaxed; for a discussion of the geometric conditions needed for Ω see [AF03, Ch.4].

Trace Theorems

The next topic we are going to disscus are trace theorems. Heuristically they give conditions under which a function from some $W^{m,p}(\Omega)$ can be 'restricted' to the boundary of Ω. Technically, the trace of a function will be realized as a value of the continuous extension of the restriction operator:

$$\begin{aligned} \gamma : \mathcal{D}(\overline{\Omega}) &\to \mathcal{D}(\partial\Omega) \\ \gamma U &:= U|_{\partial\Omega} \end{aligned} \tag{1.3}$$

to

$$\gamma : W^{m,p}(\Omega) \to W^{k(m,p),p}(\partial\Omega)$$

where $k(m,p) \in \mathbb{R}_+$. The corresponding theorems state that for certain Ω such an extension exists and is onto. The proofs for such theorems are usually based on the case $\Omega = \mathbb{R}^n$, where in (1.3) the boundary $\partial\Omega$ is replaced by the hyperplane $\{x \in \mathbb{R}^n \mid x_n = 0\}$. In what follows, the hyperplane $\{x \in \mathbb{R}^n \mid x_n = 0\}$ will be identified with $\mathbb{R}^{n-1}$ via $\mathbb{R}^{n-1} \ni x' = (x', 0) \in \mathbb{R}^n$.

1.2.7. Theorem. *Let* $1 \leq p < \infty$ *and* $m > r + 1/p$ *for some* $r \in \mathbb{N}$ *then the mapping*

$$\begin{aligned} \gamma : \mathcal{D}(\mathbb{R}^n) &\to \prod_{j=0}^{r} \mathcal{D}(\mathbb{R}^{n-1}) \\ \gamma U(x') &:= (U(x', 0), \partial_n U(x', 0), ..., \partial_n^r U(x', 0)) \end{aligned} \tag{1.4}$$

admits a unique, linear and continuous extension to a mapping (again denoted γ*)*

$$\gamma : W^{m,p}(\mathbb{R}^n) \to \prod_{j=0}^{r} W^{m-j-1/p,p}(\mathbb{R}^{n-1}).$$

This extension is onto.

Proof. [Tri83, 3.3.3], for the case $r = 0$ see also [AF03, 7.39]. □

To generalize this theorem to more general open subsets of $\mathbb{R}^n$ we will make use of extension theorems. In fact, if Ω denotes an open subset of $\mathbb{R}^n$, we call a continuous linear mapping

$$E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$$

an extension operator if

$$Eu|_\Omega = u.$$

We will say that $U \in W^{s,p}(\mathbb{R}^n)$ is an extension of $u \in W^{s,p}(\Omega)$ if there exists a continuous linear extension operator E such that $Eu = U$.

1.2.8. Proposition. *Let Ω be $\mathbb{R}^n_+$ or a bounded regular open subset of $\mathbb{R}^n$. For every non-negative integer m and every $1 \leq p < \infty$ there exists a continuous, linear operator extending $u \in W^{m,p}(\Omega)$ to $U \in W^{m,p}(\mathbb{R}^n)$. In particular,*

$$\|U\|_{m,p} \leq C\|u\|_{m,p} \tag{1.5}$$

for some constant C depending only on m and p.

Sketch of proof. First, as in proposition 1.1.4, the Seeley extension is used to construct the extension of $u \in W^{s,p}(\mathbb{R}^n_+)$ to $W^{s,p}(\mathbb{R}^n)$. Partitions of unity and the special case $\Omega = \mathbb{R}^n_+$ are subsequently used to prove the assertion for bounded regular open subsets. For proofs see [Tri83] and [Ada75, 4.26]. □

1.2.9. Corollary. *Let Ω be a bounded, regular open subset of $\mathbb{R}^n$ or the halfspace $\mathbb{R}^n_+$. For $1 \leq p < \infty$ and $r, m \in \mathbb{N}$ with $m > r + 1/p$ the mapping*

$$\begin{aligned} \gamma : \mathcal{D}(\overline{\Omega}) &\to \prod_{j=0}^{r} \mathcal{D}(\partial\Omega) \\ \gamma U :&= (U|_{\partial\Omega}, \partial_\nu U|_{\partial\Omega}, ..., \partial_\nu^r U|_{\partial\Omega}) \end{aligned} \tag{1.6}$$

where ν is the outward normal to the boundary $\partial\Omega$, can be uniquely, continuously extended to a mapping (again denoted γ)

$$\gamma : W^{m,p}(\Omega) \to \prod_{j=0}^{r} W^{m-j-1/p,p}(\partial\Omega)$$

This extension is onto.

Proof. We only prove the case $\Omega = \mathbb{R}^n_+$; the proof of the general case is reduced to the halfspace case by using partitions of unity. The complete proof can be found in [Tri83, 3.3.3].

Throughout this proof, to avoid confusion, γ in the statement of the corollary will be denoted $\overline{\gamma}$, whereas γ will be used to denote the trace operator of theorem 1.2.7. For arbitrary $U \in W^{m,p}(\mathbb{R}^n_+)$ define

$$\overline{\gamma}U := \gamma\widetilde{U} \tag{1.7}$$

where $\widetilde{U}$ is any extension of U to $W^{m,p}(\mathbb{R}^n)$ according to proposition 1.2.8. Since γ is continuous and the inequality (1.5) holds, one concludes that

$$\sum_{j=0}^{r} \|\partial_{x_n}^j \widetilde{U}(x',0)\|_{m-j-1/p,p} \leq K\|U\|_{m,p}$$

for some constant K not depending on U or its particular extension $\widetilde{U}$. This implies that $\overline{\gamma}$ is well defined since taking $U = 0$ implies $\gamma\widetilde{U} = 0$. Moreover it also implies continuity of $\overline{\gamma}$.

If $U \in \mathcal{D}(\overline{\mathbb{R}}^n_+)$ then definition (1.7) of $\overline{\gamma}$ coincides with the one given in the statement of the corollary i.e.

$$\overline{\gamma}U(x) = (U(x,0), ..., \partial_n^r U(x,0))$$

since such U possess smooth extension to the boundary. The density of $\mathcal{D}(\overline{\mathbb{R}}^n_+)$ in $W^{s,p}(\mathbb{R}^n_+)$ implies the uniqueness of $\overline{\gamma}$.

To prove that $\overline{\gamma}$ is onto, consult theorem 1.2.7 and the definition of $\overline{\gamma}$ given at the beginning of this proof. □

1.2.10. Remark. If $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$ and $u \in W^{m,p}(\Omega)$, then for m large enough

$$\partial_{x_i}(\gamma u) = \gamma(\partial_{x_i} u) \text{ for } i = 1, ..., n-1$$

by an easy convergence argument. Indeed u may be approximated by a sequence $u_n \in \mathcal{D}(\Omega)$ for which $\gamma(\partial_i u_n) = \partial_i(\gamma u_n)$ obviously holds. The claim now follows from convergence in $W^{m-1-1/p,p}(\partial\Omega)$.

The following remark is actually not relevant for the understanding of the rest of the text and it is rather technical, nevertheless of independent interest.

1.2.11. Remark. In [Tri83], the spaces $W^{s,p}(\mathbb{R}^n)$ as defined above, are called Slobodeckij spaces. Since their exact form will not actually be needed in the rest of the text, one could ask why not using the much more natural generalization of Sobolev spaces, the Bessel potential spaces, defined by means of Fourier transform: $u \in H^{s,p}(\mathbb{R}^n)$ iff $(1+|x|^2)^{s/2}\widehat{u}(x) \in L^p(\mathbb{R}^n)$ ($\widehat{u}$ denoting the Fourier transform of u). Both families of spaces agree with $W^{m,p}(\mathbb{R}^n)$ when m is an integer and $1 < p < \infty$ up to the equivalence of norms. The reason for taking the less intuitive Slobodeckij spaces lies in the fact that $H^{s,p}$ do not allow for a trace theorem as formulated above, except in the case p=2! In fact, it is shown in [Tri83, p.132] that the restriction operator $\gamma : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^{n-1})$ extends to a surjective, continuous, linear operator

$$F^s_{p,q}(\mathbb{R}^n) \twoheadrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \tag{1.8}$$

for certain families of spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, where s, p and q are some non-negative real parameters. These spaces obey the following relations (which can be found in [Tri83, p.47]):

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \subseteq F^s_{p,q}(\mathbb{R}^n) \subseteq B^s_{p,\max(p,q)}(\mathbb{R}^n)$$

$$F^{s_1}_{p_1,q_1}(\mathbb{R}^n) = B^{s_2}_{p_2,q_2}(\mathbb{R}^n) \Leftrightarrow s_1 = s_2,\ p_1 = p_2 = q_1 = q_2 < \infty.$$

Since $F^s_{p,2}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$ for every $s \in \mathbb{R}$ and $1 < p < \infty$, see [Tri83, p.88], for $p > 2$ equation (1.8) implies that

$$H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n) \twoheadrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \supsetneqq F^{s-1/p}_{p,2}(\mathbb{R}^{n-1}) = H^{s-1/p,p}(\mathbb{R}^{n-1})$$

which in turn implies that there are some elements from $H^{s,p}(\mathbb{R}^n)$ whose image does not lie in $H^{s-1/p,p}(\mathbb{R}^{n-1})$. On the other hand, for $p < 2$

$$H^{s,p}(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n) \twoheadrightarrow B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \subsetneqq F^{s-1/p}_{p,2}(\mathbb{R}^{n-1}) = H^{s-1/p,p}(\mathbb{R}^{n-1})$$

implying that the restriction, again, fails to be onto.

1.3 Distributions on manifolds

We start by introducing some more notation from differential geometry. We will denote the tangent bundle of M by TM and the cotangent bundle by T^*M, the respective fibers over $x \in M$ being denoted T_xM and T^*_xM. Both TM and T^*M will be endowed with the usual canonical C^∞-manifold structure.

$\mathcal{T}^r_s(M)$ will denote the space of smooth (r,s)-tensorfields, that is the space of smooth sections of the (r,s)-tensor bundle $T^r_s(M)$. The tensorbundle chart (Ψ, U) over the chart (ψ, U) in M is a mapping

$$\Psi : \pi^{-1}(U) \to \psi(U) \times \mathbb{R}^{n^{r+s}}$$
$$z \mapsto (\psi(p), \widetilde{\psi}(z)),$$

where π denotes the base mapping of the tensor bundle $T^r_s(M)$ and $\pi(z) = p$. Recall that $\mathcal{T}^r_s(M)$ is a $C^\infty(M)$-module and as such isomorphic to the space of $C^\infty(M)$-multilinear maps from $\mathcal{T}^0_1(M)^r \times \mathcal{T}^1_0(M)^s$ to $C^\infty(M)$, i.e. to the $C^\infty(M)$-module
$L_{C^\infty}(\mathcal{T}^0_1(M)^r, \mathcal{T}^1_0(M)^s; C^\infty(M))$.

Locally, in a chart (ψ, U) of M, the components $t(dx^{i_1}, ..., dx^{i_r}, \partial^{j_1}, ..., \partial^{j_s})$ of $t \in \mathcal{T}^r_s(M)$ will be denoted $t^{i_1,...,i_r}_{j_1,...,j_s}$. For a given compact subset K of M, $\mathcal{D}_K\mathcal{T}^r_s(M)$ will denote the subspace of $\mathcal{T}^r_s(M)$ formed by all $t \in \mathcal{T}^r_s(M)$ which vanish outside K whereas $\mathcal{D}\mathcal{T}^r_s(M)$ will denote the space of all compactly supported (r,s)-tensorfields. Note that all of the previous assertions can be adapted to fit manifolds with boundary in much the same way it was done for manifold charts and smooth functions in section 1.1.

We will denote the space of k-forms on M by $\Omega^k(M)$ and the corresponding subspace of compactly supported k-forms by $\Omega^k_c(M)$. Furthermore we will use $E \otimes_{C^\infty(M)} F$ to denote the $C^\infty(M)$-balanced tensor product of the $C^\infty(M)$-modules E and F.

1.3.1. In what follows we will define the space of distributional tensorfields as the topological dual of some appropriate space of compactly supported testfields, consequently we first need to construct a topology for $\mathcal{D}\mathcal{T}^r_s(M)$. There are several equivalent approaches yielding the same topology on $\mathcal{D}\mathcal{T}^r_s(M)$, for a detailed discussion we refer to [GKOS01, Sec.3.1] and we choose to follow [YCB78, Sec.VII.8] where the topology is constructed in an explicitely covariant way. Observe that this construction is only possible for tensor-fields and not in the more general case of sections in vector bundles.

To begin with we choose a smooth Riemannian metric **h** on M and denote by ∇ its Levi-Civita connection as well as its extension to all tensor-bundles; in particular for $t \in \mathcal{T}^r_s(M)$ we will denote by ∇t its covariant derivative. For basic facts on (Semi-)Riemannian geometry we refer to [O'N83] and section 2.1. We define the pointwise norm $|t(x)|$ of the tensorfield t by

$$|t(x)| = |t^{i_1,\dots,i'_n}(x)t_{i_1,\dots,i'_n}(x)|^{1/2}$$

where we have used the abstract index notation, see [PR84], and indices were lowered and raised by the metric, see section 2.1. Equipped with the family of seminorms

$$p_{m,K}(t) := \sum_{k\leq m} \sup_{x\in K} |(\nabla^k t)(x)|$$

where m is a non-negative integer and K is a compact subset, the space $\mathcal{T}^r_s(M)$ becomes a Fréchet space. Note that this topology does not depend on the particular choice of the Riemannian metric h. Finally we endow $\mathcal{D}\mathcal{T}^r_s(M)$ with the strict inductive limit topology with respect to closed subspaces $\mathcal{D}_K\mathcal{T}^r_s(M)$ of $\mathcal{D}\mathcal{T}^r_s(M)$.

Before actually stating our formal approach to introducing $\mathcal{D}'(M)$, i.e. the space of scalar distributions on M we comment on the issue of distributions versus distributional densities; for a detailed discussion see [GKOS01, Sec.3.1]. We will define $\mathcal{D}'(M)$ as the dual space of $\Omega^n_c(M)$, hence put the burden of integration onto the test object side. As a consequence one obtains a natural embedding of smooth or more generally locally integrable functions into $\mathcal{D}'(M)$ via the assignment

$$\tau \to \int_M f\tau$$

where $\tau \in \Omega^n_c(M)$. The alternative would be to consider distributional densities as members of the dual space of $\mathcal{D}(M)$. In this case one could naturally embedd smooth resp. locally integrable n-forms. However since we assume the manifold M to be oriented, both approaches lead to isomorphic spaces - an isomorphism being induced by each non-vanishing n-form. More precisely, $\mathcal{D}(M)$ is isomorphic to $\Omega^n_c(M)$ via the mapping $f \to f\theta$ where θ is an arbitrary orientation inducing n-form. We henceforth identify both of these spaces to obtain a topology on $\Omega^n_c(M)$. Moreover, the notation $\mathcal{D}'(M) \cong (\mathcal{D}(M))'$ is justified.

The space of scalar distributions $\mathcal{D}'(M)$

1.3.2. Definition. *The space of distributions $\mathcal{D}'(M)$ on a manifold M is the space of continuous linear functionals on the space of compactly supported smooth n-forms $\Omega^n_c(M)$, i.e.*

$$\mathcal{D}'(M) \cong (\Omega^n_c(M))'.$$

Analogous to the $\mathbb{R}^n$-case, the action of $u \in \mathcal{D}'(M)$ on some $\tau \in \Omega^n_c(M)$ will be denoted by

$\langle u, \tau \rangle$. $\mathcal{D}'(M)$ will be equipped with the weak-*-topology, that is

$$u_m \to u \text{ in } \mathcal{D}'(M) \;:\Leftrightarrow\; \langle u_m, \tau \rangle \to \langle u_m, \tau \rangle \text{ in } \mathbb{C},\ \forall \tau \in \Omega^n_c(M).$$

If U is an open subset of M, the restriction $u|_U$ of a distribution u to U is defined to be the element of $\mathcal{D}'(U)$ given by $\langle u|_U, \tau \rangle = \langle u, \tau \rangle$ for all $\tau \in \Omega^n_c(U)$, where on the right hand side we have trivially extended τ to all of M.

1.3.3. $\mathcal{D}'(M)$ is a fine sheaf of $C^\infty(M)$-modules, that is, given an open covering $(U_\alpha)_{\alpha \in A}$ of M the following holds:

(i) For every $u \in \mathcal{D}'(M)$ local vanishing i.e. $u|_{U_\alpha} = 0$ for all $\alpha \in A$ implies global vanishing i.e. $u = 0$.

(ii) For any family of distributions $u_\alpha \in \mathcal{D}'(U_\alpha)$ such that $u_\alpha|_{U_\alpha \cap U_\beta} = u_\beta|_{U_\alpha \cap U_\beta}$ for every $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ there exists a unique distribution $u \in \mathcal{D}'(M)$ such that $u|_{U_\alpha} = u_\alpha$.

1.3.4. Remark. The sheaf property of $\mathcal{D}'(M)$ implies that, given an atlas (U_α, ψ_α) of M, every distribution $u \in \mathcal{D}'(M)$ can be identified with the family of distributions

$$u_\alpha := \psi_{\alpha *}(u|_{U_\alpha}) \in \mathcal{D}'(\psi_\alpha(U_\alpha))$$

that satisfy the transformation law $u_\alpha = (\psi_\alpha \circ \psi_\beta^{-1})_* u_\beta$ on $\psi_\alpha(U_\alpha \cap U_\beta)$. Here, we presuppose the definition of the distributional pushforward $\psi_{\alpha *}$ by the chart ψ_α given in 1.3.6 below. This local description is sometimes taken to be the definition of $\mathcal{D}'(M)$, for example in [Hör90, 6.3.3].

1.3.5. A function $f : M \to \mathbb{C}$ is called locally integrable if $\psi_* f$ is locally integrable for every chart (U, ψ) of M. For later use we note that if f is locally integrable, then

$$\int f\tau = 0 \;\; \forall \tau \in \Omega^n_c(M) \tag{1.9}$$

implies $f = 0$. The space of locally integrable functions $L^1_{\mathrm{loc}}(M)$ is injectively embedded in $\mathcal{D}'(M)$ via

$$\begin{aligned} j &: L^1_{\mathrm{loc}}(M) \to \mathcal{D}'(M) \\ \langle j(f), \tau \rangle &= \int_M f\tau, \end{aligned}$$

where $\tau \in \Omega^n_c(M)$. The image of $C^\infty(M)$ under j is dense in $\mathcal{D}'(M)$, i.e. $\overline{j(C^\infty(M))} = \mathcal{D}'(M)$.

1.3.6. Let $f : M \to N$ be a smooth mapping. The pullback $f^* : \Omega^k(N) \to \Omega^k(M)$ by f is defined by

$$(f^*\tau)(\xi_1, .., \xi_k) = \tau(Tf\xi_1, ..., Tf\xi_k),$$

where ξ_i belong to $\mathcal{T}^1_0(M)$ and Tf is the tangential mapping of f.

Recall that the pushforward $f_* : \Omega^k(M) \to \Omega^k(N)$ is not defined for general smooth $f : M \to N$. However, it is certainly defined if f is assumed to be a diffeomorphism. In this case $f_* = (f^{-1})^*$ i.e.,

$$(f_*\omega)(\eta_1, .., \eta_k) = \omega(Tf^{-1}\eta_1, ..., Tf^{-1}\eta_k)$$

where η_i belong to $\mathcal{T}^1_0(N)$.

To extend the notion of pullback respectively pushforward to scalar distributions we use transposition. For that purpose let $f : M \to N$ now denote an orientation preserving diffeomorphism. We then define the *distributional pullback* by f as the mapping $f^* : \mathcal{D}'(N) \to \mathcal{D}'(M)$ given by

$$\langle f^*u, \tau\rangle = \langle u, f_*\tau\rangle$$

where $\tau \in \Omega^n_c(M)$. It is the unique continuous extension of the classical pullback $f^* : C^\infty(N) \to C^\infty(M)$. Note that locally, for charts ψ of M and ϕ of N, f^* is given by the formula

$$\langle f^*u, \tau\rangle = \langle u, \det\frac{\partial(y^j \circ f^{-1})}{\partial x^i}(\theta \circ f^{-1})\mathrm{d}^n x\rangle$$

where $\tau = \theta\mathrm{d}^n x \in \Omega^n_c(U_\alpha)$, $x^i = \mathrm{pr}^i \circ \phi^{-1}$ and $y^j = \mathrm{pr}^j \circ \psi^{-1}$. The *distributional pushforward* $f_* : \mathcal{D}'(M) \to \mathcal{D}'(N)$ is defined analogously.

1.3.7. The *product* of a smooth function $f \in C^\infty(M)$ with a distribution $u \in \mathcal{D}'(M)$ is defined by transposition, i.e.

$$\langle fu, \tau\rangle := \langle u, f\tau\rangle$$

for all $\tau \in \Omega^n_c(M)$.

1.3.8. The distributional Lie derivative is constructed as the unique continuous extension of $L_\xi : C^\infty(M) \to C^\infty(M)$ to distributions. Before proceeding with the actual definition of the distributional Lie derivative we briefly recall the classical Lie derivative. Indeed, for a smooth vectorfield ξ on M, $L_\xi : \Omega^k(M) \to \Omega^k(M)$ is given by

$$L_\xi\omega(\xi_1, ..., \xi_k) = \xi(\omega(\xi_1, ..., \xi_k)) - \sum_{i=1}^{k} \omega(\xi_1, ..., [\xi, \xi_i], ..., \xi_k)$$

where $\xi_i \in \mathcal{T}^1_0(M)$ and by brackets we denote the Lie bracket of vectorfields. In particular, $L_\xi(f) = \xi(f)$ for $f \in C^\infty(M)$. We are now ready to define the distributional Lie derivative $L_\xi : \mathcal{D}'(M) \to \mathcal{D}'(M)$ by transposition, i.e. by

$$\langle L_\xi u, \tau\rangle := -\langle u, L_\xi\tau\rangle$$

for $\tau \in \Omega^n_c(M)$. As in case of $\mathcal{D}'(\Omega)$ the minus sign ensures compatibility with the classical Lie derivative $L_\xi : C^\infty(M) \to C^\infty(M)$ via integration by parts. In this way, the distributional Lie derivative inherits many properties of the classical Lie derivative, see e.g.[GKOS01, 3.1.24]. Here we just note the Leibnitz rule, i.e. if $f \in C^\infty(M)$ and $u \in \mathcal{D}'(M)$, then

$$L_\xi(fu) = (L_\xi f)u + fL_\xi u,$$

and that as a derivative on the space of distributions, L_ξ is localizes appropriately, i.e. for any open set $V \subseteq M$, $\xi \in \mathcal{T}^1_0(M)$ and $u \in \mathcal{D}'(M)$:

$$(L_\xi u)|_V = L_{\xi|_V} u|_V$$

We will often abbreviate the action of the distributional Lie derivative along $\xi \in \mathcal{T}^1_0(M)$ on $u \in \mathcal{D}'(M)$ by $\xi(u)$.

The space of distributional tensorfields $\mathcal{D}'\mathcal{T}^r_s(M)$

1.3.9. Definition. *The space $\mathcal{D}'\mathcal{T}^r_s(M)$ of distributional tensorfields is the topological dual of the space $\mathcal{D}\mathcal{T}^s_r(M) \otimes \Omega^n_c(M)$ where r and s are two non-negative integers which do not vanish at the same time.*

For different representation of the space of test objects used above and on extensive discussion of the various approaches to tensor distributions, see [Gro08].

As in the scalar case we will denote the action of $T \in \mathcal{D}'\mathcal{T}^r_s(M)$ on some $\mu \in \mathcal{D}\mathcal{T}^s_r(M) \otimes \Omega^n_c(M)$ by $\langle T, \mu\rangle$. Furthermore if U is an open subset of M, the restriction $T|_U \in \mathcal{D}'\mathcal{T}^r_s(U)$ is defined by $\langle T|_U, \mu\rangle = \langle T, \mu\rangle$ for all $\mu \in \mathcal{D}\mathcal{T}^s_r(U) \otimes \Omega^n_c(U)$ where on the right hand side we have trivially extended μ to all of M.

$\mathcal{D}'\mathcal{T}^r_s(M)$ can be characterised in two very useful ways. First the following local isomorphism of C^∞-modules holds

$$\mathcal{D}'\mathcal{T}^r_s(U_\alpha) \cong \mathcal{D}'(U_\alpha) \otimes_{C^\infty(U_\alpha)} \mathcal{T}^r_s(U_\alpha), \tag{1.10}$$

where U_α are the domains of the respective tensor bundle charts; for a proof see [GKOS01, 3.1.11]. This implies that $T|_{U_\alpha}$ can be written as:

$$T|_{U_\alpha} = (T^\alpha)^{i_1,\dots,i_r}_{j_1,\dots,j_s} \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

with $(T^\alpha)^{i_1,\dots,i_r}_{j_1,\dots,j_s} \in \mathcal{D}'(U_\alpha)$. This will henceforth be the preferred, local description for elements of $\mathcal{D}'\mathcal{T}^r_s(M)$. The local isomorphism (1.10) also holds globally, since both $\mathcal{D}'(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M)$ and $\mathcal{D}'\mathcal{T}^r_s(M)$ are sheaves of $C^\infty(M)$-modules, see [GKOS01, 3.1.7].

Finally by a purely algebraic argument, see for instance [Bou74, ChII, 4.2], one can show that

$$\mathcal{D}'(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \cong L_{C^\infty(M)}(\mathcal{T}^0_1(M)^r, \mathcal{T}^1_0(M)^s; \mathcal{D}'(M))$$

holds. To summarize:

1.3.10. Theorem. *The following isomorphisms of $C^\infty(M)$-modules hold*

$$\mathcal{D}'\mathcal{T}^r_s(M) \cong \mathcal{D}'(M) \otimes_{C^\infty(M)} \mathcal{T}^r_s(M) \cong L_{C^\infty(M)}(\mathcal{T}^0_1(M)^r, \mathcal{T}^1_0(M)^s; \mathcal{D}'(M))$$

Distributional tensorfields can therefore be viewed either as tensorfields with distributional coefficients or C^∞-multilinear maps with values in $\mathcal{D}'(M)$. We will prefer the latter when describing distributional tensorfields globally.

1.3.11. When we consider the topology of $\mathcal{D}'\mathcal{T}^r_s(M)$ we will always identify $\mathcal{D}'\mathcal{T}^r_s(M)$ with $L_{C^\infty(M)}(\mathcal{T}^0_1(M)^r, \mathcal{T}^1_0(M)^s; \mathcal{D}'(M))$ equipped with the topology of pointwise convergence, that is

$$T_m \to T \;:\Leftrightarrow T_m(\omega_1, ..., \omega_r, \xi_1, ..., \xi_s) \to T(\omega_1, ..., \omega_r, \xi_1, ..., \xi_s) \text{ in } \mathcal{D}'(M),$$
$$\forall \omega_i \in \mathcal{T}^0_1(M),\ \forall \xi_j \in \mathcal{T}^1_0(M).$$

Analogous to the scalar case, locally integrable tensorfields $L^1_{\text{loc}}\mathcal{T}^r_s(M)$ which we here identify with $L_{C^\infty(M)}(\mathcal{T}^0_1(M)^r, \mathcal{T}^1_0(M)^s; L^1_{\text{loc}}(M))$, can be injectively embedded in $\mathcal{D}'\mathcal{T}^r_s(M)$ via

$$G^r_s : L^1_{\text{loc}}\mathcal{T}^r_s(M) \to L_{C^\infty(M)}(\mathcal{T}^0_1(M)^r, \mathcal{T}^1_0(M)^s; \mathcal{D}'(M))$$

$$G^r_s(t)(\omega_1, ...\omega_r, \xi_1, .., \xi_s) := j(t(\omega_1, ...\omega_r, \xi_1, .., \xi_s)),$$

where j was defined in paragraph 1.3.5. Moreover $\overline{G^r_s(\mathcal{T}^r_s(M))} = \mathcal{D}'\mathcal{T}^r_s(M)$.

1.3.12. The *product* of $f \in C^\infty(M)$ with a tensor distribution $T \in \mathcal{D}'\mathcal{T}^r_s(M)$ or of a distribution $f \in \mathcal{D}'(M)$ with a smooth tensorfield $T \in \mathcal{T}^r_s(M)$ is defined as an element of $\mathcal{D}'\mathcal{T}^r_s(M)$ given by

$$(fT)(\xi_1, ..., \xi_r, \omega_1, ..., \omega_s) := fT(\xi_1, ..., \xi_r, \omega_1, ..., \omega_s).$$

Here the product on the right hand side being the product of a scalar distribution with a smooth function, see 1.3.7. Any given $t \in \mathcal{T}^r_s(M)$ can be extended to a $C^\infty(M)$-multilinear mapping

$$T : \mathcal{D}'\mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M)^{s-1} \times \mathcal{T}^0_1(M)^r \to \mathcal{D}'(M)$$

via

$$T(\tau, \xi_1, ..., \xi_{s-1}, \omega_1, ..., \omega_r) := \tau((\xi \longmapsto t(\xi, \xi_1, ..., \xi_{s-1}, \omega_1, ..., \omega_r)))$$

for $\xi,\ \xi_i \in \mathcal{T}^1_0(M)$, $\omega_j \in \mathcal{T}^0_1(M)$ and $\tau \in \mathcal{D}'\mathcal{T}^1_0(M)$. Likewise we may extend $t \in \mathcal{T}^r_s(M)$ to a $C^\infty(M)$-multilinear mapping

$$T : \mathcal{T}^1_0(M)^s \times \mathcal{D}'\mathcal{T}^0_1(M) \times \mathcal{T}^0_1(M)^{r-1} \to \mathcal{D}'(M)$$

and likewise for any other slot.

1.3.13. To end this section we extend the *contraction* to act on distributional tensorfields. The definition is completely analogous to the case of smooth tensorfields, see [O'N83, p.40]. The (1,1)- distributional contraction is the $C^\infty(M)$-linear mapping

$$C : \mathcal{D}'\mathcal{T}^1_1(M) \to \mathcal{D}'(M)$$

locally defined via

$$C(T) := \sum_i T(dx^i, \partial_i) = \sum_i (T^\alpha)^i_i$$

For general tensorfields we obtain a family of mappings contracting the i-th contravariant slot with the j-th covariant slot:

$$C^i_j : \mathcal{D}'\mathcal{T}^r_s(M) \to \mathcal{D}'\mathcal{T}^{r-1}_{s-1}(M)$$

defined via

$$C^i_j(T)(\omega_1, ..., \omega_{r-1}, \xi_1, ..., \xi_{s-1}) :=$$
$$C((\omega, \xi) \to T(\omega_1, ..., \omega_{i-1}, \omega, \omega_i, ..., \xi_{j-1}, \xi, \xi_j, ..., \xi_{s-1})).$$

1.4 Local Sobolev spaces on manifolds

In this section we generalize the local Sobolev spaces introduced in section 1.2 to manifolds and prove the corresponding trace theorems. Throughout this chapter, M will denote an n-dimensional manifold without boundary whereas $\overline{X}$ will denote an n-dimensional manifold with boundary unless explicitely stated otherwise.

1.4.1. Definition. *Let $s \in \mathbb{R}_+$ and $1 \leq p \leq \infty$. The local Sobolev spaces on a manifold M are defined to be the sets*

$$W^{s,p}_{\mathrm{loc}}(M) = \{u \in \mathcal{D}'(M) \mid \psi_*(u|_U) \in W^{s,p}_{\mathrm{loc}}(\psi(U)) \textit{ for all charts } (U, \psi) \textit{ of M}\}.$$

$W^{s,p}_{\mathrm{loc}}(M)$ will be equipped with the topology defined by the family of seminorms

$$u \to p_{\varphi,\psi}(u) := \|\varphi\psi_* u\|_{s,p}$$

for charts (U, ψ) of M and $\varphi \in \mathcal{D}(\psi(U))$. Using partitions of unity one can show that the seminorms associated with the charts of some atlas suffice to define the topology and that $W^{s,p}_{\mathrm{loc}}(M)$ is a Fréchet space. The injection of $W^{s,p}_{\mathrm{loc}}(M)$ in $\mathcal{D}'(M)$ is continuous.

Observe that $W^{s,p}_{\mathrm{loc}}(M)$ is a subspace of $L^1_{\mathrm{loc}}(M)$ for all $s \in \mathbb{R}_+$ and $p \geq 1$. More precisely, for every $u \in W^{s,p}_{\mathrm{loc}}(M)$ there exists a locally integrable function f such that $j(f) = u$. Here, j denotes the mapping from 1.3.5.

1.4.2. Remark. For $s \geq 1$, the distributional Lie derivative defined in 1.3.8 has values in $W^{s-1,p}_{\mathrm{loc}}(M)$ when restricted to $W^{s,p}_{\mathrm{loc}}(M)$. Furthermore it is continuous in the respective topologies on local Sobolev spaces.

1.4.3. Lemma. *$\mathcal{D}(M)$ is dense in $W^{s,p}_{\mathrm{loc}}(M)$ with continuous injection for $s \in \mathbb{R}_+$ and $1 \leq p < \infty$.*

Proof. Let u be in $W^{s,p}_{\mathrm{loc}}(M)$. Furthermore let $(U_j, \psi_j)_{j\in J}$ be an atlas of M and $(f_j)_{j\in J}$ a partition of unity subordinate to U_j. By virtue of 1.2.5, for every $j \in J$ there exists a sequence $\varphi^j_n \in \mathcal{D}(\psi_j(U_j))$ such that $\varphi^j_n \to \psi_{j*}u$ in $W^{s,p}_{\mathrm{loc}}(\psi_j(U_j))$. Then $\varphi_n := \sum_j f_j \psi^*_j \varphi^j_n \in C^\infty(M)$ since the φ_n's are locally finite sums of smooth functions and $\varphi_n \to u$ in $W^{s,p}_{\mathrm{loc}}(M)$. Now choose an exhaustive sequence of compact sets K_n of M and corresponding plateau functions ϕ_n. Then $\phi_n\varphi_n \in \mathcal{D}(M)$ and $(\phi_n\varphi_n)_n$ converges to u in $W^{s,p}_{\mathrm{loc}}(M)$. □

1.4.4. Theorem. *Let N be a hypersurface in M, and let m and r be non-negative integers such that $m > r + 1/p$ and $1 \leq p < \infty$. Moreover let $\xi_i, ..., \xi_r$ be vectorfields on M such that $\xi_i(x) \in T_xN^\perp$ for $i = 1, .., r$ and all $x \in N$. Here, $T_xN^\perp$ denotes the orthogonal complement of T_xN in T_xM with respect to any Riemannian metric on M. Then the restriction operator*

$$\gamma : \mathcal{D}(M) \to \prod_{j=0}^{r} \mathcal{D}(N)$$

$$\gamma u := (u|_N, (L_{\xi_1} u)|_N, ..., (L_{\xi_r} ... L_{\xi_1} u)|_N) \tag{1.11}$$

uniquely extends to a continuous linear operator (again denoted γ)

$$\gamma : W^{m,p}_{\text{loc}}(M) \to \prod_{j=0}^{r} W^{m-j-1/p,p}_{\text{loc}}(N).$$

Moreover, γ is onto.

Sketch of proof. Throughout this proof, γ defined in equation (1.11) will be denoted by $\overline{\gamma}$, whereas γ will be used to denote the trace operator from theorem 1.2.7.

If fact, to prove the assertion of the theorem, we only need to show continuity of the restriction operator

$$\overline{\gamma} : \mathcal{D}(M) \to \prod_{j=0}^{r} \mathcal{D}(N),$$

when considered with the topology inherited from the respective Sobolev spaces. Indeed, since $\mathcal{D}(M)$ is dense in $W^{m,p}_{\text{loc}}(M)$, the continuity of $\overline{\gamma}$ implies that there exists a continuous linear extension of $\overline{\gamma}$ to arbitrary u in $W^{m,p}_{\text{loc}}(M)$. In fact, the extension is given by

$$\overline{\gamma} u := \lim_{n \to \infty} \overline{\gamma} u_n,$$

where $(u_n)_{n \in \mathbb{N}}$ is an arbitrary sequence in $\mathcal{D}(M)$ converging to u in $W^{m,p}_{\text{loc}}(M)$.

To show continuity of $\overline{\gamma}$, it suffices to show continuity of the components

$$u \to (\overline{\gamma} u)_j,$$

$$\mathcal{D}(M) \subseteq W^{m,p}_{\text{loc}}(M) \to \mathcal{D}(N) \subseteq W^{m-j-1/p,p}_{\text{loc}}(N),$$

which we will prove using induction on r, the number of vectorfields. The subscript j will be used to denote the j-th component of γ as well.

Since N is assumed to be embedded in M, for every $y \in N$ we can choose a chart $(U_\circ, \psi_\circ)$ of y in N such that a chart (U, ψ) of y in M exists for which $U_\circ = U \cap N$, $\psi_\circ = \psi_{|U_\circ}$ and $\psi(U_\circ) = \{x \in \psi(U) | x_n = 0\}$, see 1.1.7. Furthermore, by remark 1.1.10 for a given $\varphi_\circ \in \mathcal{D}(\psi(U_\circ))$ we can choose a $\varphi \in \mathcal{D}(\psi(U))$ such that $\varphi|_{\psi(U_\circ)} = \varphi_\circ$.

If $r = 0$, then clearly $\varphi_\circ \psi_{\circ *}(\overline{\gamma} u) = \gamma(\varphi \psi_* u)$ for every $u \in \mathcal{D}(M)$. From the trace theorem 1.2.7 one obtains

$$\|\varphi_\circ \psi_{\circ *}(\overline{\gamma} u)\|_{m-1/p,p} \leq C \|\varphi \psi_* u\|_{m,p}$$

where $C > 0$ depends only on m, p.

Now assume $r-1$ vectorfields as in the statement of the theorem are given and for every $j \le r-1$ there exists some constant $C > 0$ independent of u such that

$$\|\varphi_\circ \psi_{\circ *}(\overline{\gamma} u)_j\|_{m-j-1/p,p} \le C\|\varphi\psi_* u\|_{m,p}.$$

Now, let r vectorfields $(\xi_i)_{i=1,\dots,r}$ as in the statement of the theorem be given. We have to estimate $L_{\xi_r}...L_{\xi_1}u$ for $u \in \mathcal{D}(M)$. To this end, using Leibniz rule, we obtain

$$\gamma(L_{\psi_*\xi_r}...L_{\psi_*\xi_1}(\varphi\psi_* u)) = F(\varphi_\circ, \psi_\circ, u) + \varphi_\circ\psi_{\circ *}(\underbrace{L_{\xi_r}...L_{\xi_1}u|_N}_{=(\overline{\gamma}u)_r}) \tag{1.12}$$

where F includes all the terms which include at least one derivative of $\varphi_\circ$. By induction hypothesis and Leibniz rule there exists a constant $K > 0$ independent of u such that

$$\|F\|_{m-r-1/p,p} \le K\|\varphi\psi_* u\|_{m,p}. \tag{1.13}$$

On the other hand, from theorem 1.2.7 one can conclude that there exists a positive constant D independant on u such that

$$\|\varphi_\circ\psi_{\circ *}(\overline{\gamma}u)_r\|_{m-r-1/p,p} \le D\|\varphi\psi_* u\|_{m,p} \tag{1.14}$$

Putting (1.12), (1.13) and (1.14) together, the claim follows. □

Local Sobolev spaces on manifolds with boundary

1.4.5. Definition. *Let $s \in \mathbb{R}_+$ and $1 \le p \le \infty$. The local Sobolev spaces $W^{s,p}_{\text{loc}}(\overline{X})$ consist of all u belonging to $W^{s,p}_{\text{loc}}(X)$ for which $\varphi(\psi_* u)$ belongs to $W^{s,p}_{\text{loc}}(\mathbb{R}^n_+)$ for all charts (U,ψ) at the boundary of $\overline{X}$ and every $\varphi \in \mathcal{D}(\psi(U))$.*

As in the in case of Sobolev spaces for manifolds without boundary one can conclude that $W^{s,p}_{\text{loc}}(\overline{X})$ is a Fréchet space, equipped with the topology induced by the family of seminorms

$$p_{\varphi,\psi}(u) := \|\varphi(\psi_* u)\|_{s,p}$$

for charts (U,ψ) and $\varphi \in \mathcal{D}(\psi(U))$. As before one can show that charts of some atlas suffice to define the topology.

1.4.6. Proposition. *Every manifold with boundary $\overline{X}$ can be "extended" to a manifold M without boundary of the same dimension, i.e. so that the inclusion $\iota : \overline{X} \to M$ is an embedding.*

Sketch of proof. $\overline{X}$ can be embedded in $M := \overline{X}\backslash\partial X$ via the flow of an 'inner' vectorfield. For a sketch of the proof see [Kri, 47.10]. □

Our next goal is to generalize theorem 1.4.4 to manifolds with boundary. In analogy to the $\Omega \subseteq \mathbb{R}^n$-case we will make use of an extension theorem. We call a continuous linear mapping

$$E : W^{s,p}_{\text{loc}}(\overline{X}) \to W^{s,p}_{\text{loc}}(M)$$

an extension operator if

$$Eu|_{\overline{X}} = u,$$

where we assume that $\overline{X}$ is embedded in M. As before, $U \in W^{s,p}_{\text{loc}}(M)$ is considered to be an extension of $u \in W^{s,p}_{\text{loc}}(\overline{X})$ if there exists an extension operator E such that $Eu = U$.

1.4.7. Proposition. *Let $\overline{X}$ and M be manifolds of dimension n with resp. without boundary such that $\overline{X} \subseteq M$. Moreover, let the inclusion $\iota : \overline{X} \to M$ be an embedding. Then, for every non-negative integer m and $1 \leq p < \infty$, there exists an extension operator*

$$E : W^{m,p}_{\text{loc}}(\overline{X}) \to W^{m,p}_{\text{loc}}(M).$$

Sketch of proof. Use partitions of unity and proposition 1.2.8. □

1.4.8. Lemma. *$\mathcal{D}(\overline{X})$ is dense in $W^{m,p}_{\text{loc}}(\overline{X})$ with continuous injection for every $m \in \mathbb{N}$ and $1 \leq p < \infty$.*

Proof. Following proposition 1.4.7 every $u \in W^{m,p}_{\text{loc}}(\overline{X})$ can be extended to some $\widetilde{u} \in W^{m,p}_{\text{loc}}(M)$ where M is chosen as in proposition 1.4.6. According to lemma 1.4.3, $\widetilde{u}$ can be approximated in $W^{m,p}_{\text{loc}}(M)$ by a sequence $\varphi_n \in \mathcal{D}(M)$. Since the restriction $r : W^{m,p}_{\text{loc}}(M) \to W^{m,p}_{\text{loc}}(\overline{X})$ is continuous and $r(\mathcal{D}(M)) \subseteq \mathcal{D}(\overline{X})$ the claim follows. □

1.4.9. Theorem. *Let $m > r + 1/p$ for $m, r \in \mathbb{N}$ and $1 \leq p < \infty$. Moreover let $\xi_i \in T\overline{X}$ be such that $\xi_i(x) \in T_x\partial X^{\perp}$ for $i = 0, ..., r$ and $x \in \partial X$. Then the restriction operator*

$$\gamma : \mathcal{D}(\overline{X}) \to \prod_{j=0}^{r} \mathcal{D}(\partial X)$$

$$\gamma u := (u|_{\partial X}, (L_{\xi_1} u)|_{\partial X}, ..., (L_{\xi_r} ... L_{\xi_1} u)|_{\partial X})$$

can be uniquely extended to a continuos linear operator

$$\gamma : W^{m,p}_{\text{loc}}(\overline{X}) \to \prod_{j=0}^{r} W^{m-j-1/p,p}_{\text{loc}}(\partial X).$$

Moreover, γ is onto.

Proof. The proof is analogous to the proof of the corollary 1.2.9, using 1.4.7, 1.4.6 and 1.4.8 as well as the trace theorem 1.4.4. □

1.4.10. Remark. (i) Assume N is a hypersurface of M. If we take $r = 0$ or equivalently consider only the first component of γ in the above trace theorems, one can conclude, applying an easy convergence argument, that γ is $C^\infty(M)$-linear, i.e.

$$\gamma(fu) = f|_N \gamma(u)$$

for $f \in C^\infty(M)$ and $u \in W^{m,p}_{\mathrm{loc}}(M)$. Moreover, as in the $\mathbb{R}^n$-case, cf. remark 1.2.10, γ commutes with certain derivatives. More precisely if $u \in W^{m,p}_{\mathrm{loc}}(M)$ then:

$$L_{\xi|_N}(\gamma u) = \gamma(L_\xi u)$$

for all $\xi \in \mathcal{T}^1_0(M)$ such that $\xi(x) \in T_xN$ for all $x \in N$. For $r > 0$ more care is needed since Lie derivatives do not commute in general.

(ii) If $u \in W^{m,p}_{\mathrm{loc}}(M) \cap C(M)$ then $u|_N = \gamma(u)$. This is a local question, but locally there exists a sequence of smooth functions converging simultaneously uniformly on compact sets and in the respective Sobolev topology to u, see [Ada75, 2.18, 3.15], therefore the claim follows since γ is continuous.

(iii) The statements (i) and (ii) still hold if M is replaced by a manifold with boundary $\overline{X}$ and N is replaced by the boundary ∂X of $\overline{X}$.

Tensorfields with coefficients in $W^{s,p}_{\mathrm{loc}}$

1.4.11. Definition. *Let $s \in \mathbb{R}_+$ and $1 \le p \le \infty$. We define the space of $W^{s,p}_{\mathrm{loc}}(M)$-tensorfields by*

$$W^{s,p}_{\mathrm{loc}}\mathcal{T}^k_l(M) := \{t \in \mathcal{D}'\mathcal{T}^k_l(M) \,|\Psi_* t \in W^{s,p}_{\mathrm{loc}}(\psi(U))^{n^{k+l}} \forall (U,\Psi)\}.$$

Here Ψ_* denotes the pushforward by the tensor bundle chart (Ψ, U).

$W^{s,p}_{\mathrm{loc}}\mathcal{T}^k_l(M)$ is a Fréchet space equipped with the family of seminorms $t \to p(\Psi_* t)$, where p runs through all the seminorms in the product topology of $W^{s,p}_{\mathrm{loc}}(\psi(U))^{n^{k+l}}$.

Furthermore, the isomorphisms of theorem 1.3.10 clearly restrict to $W^{s,p}_{\mathrm{loc}}(M)$-spaces, more precisely, the following C^∞-module isomorphisms hold

$$\begin{aligned} W^{s,p}_{\mathrm{loc}}\mathcal{T}^k_l(M) &\cong W^{s,p}_{\mathrm{loc}}(M) \otimes_{C^\infty(M)} \mathcal{T}^k_l(M) \\ &\cong L_{C^\infty(M)}(\mathcal{T}^1_0(M)^k, \mathcal{T}^0_1(M)^l; W^{s,p}_{\mathrm{loc}}(M)), \end{aligned} \tag{1.15}$$

where the last space denotes the space of all $C^\infty(M)$-multilinear maps with values in $W^{s,p}_{\mathrm{loc}}(M)$.

1.4.12. Definition. *Let $\overline{X}$ be a manifold with boundary, $s \in \mathbb{R}_+$ and $1 \le p \le \infty$. The space of $W^{s,p}_{\mathrm{loc}}(\overline{X})$-tensorfields is defined as the set of all t in $W^{s,p}_{\mathrm{loc}}\mathcal{T}^k_l(X)$ for which $\Psi_* t \in W^{s,p}_{\mathrm{loc}}(\mathbb{R}^n_+)^{n^{k+l}}$ for all tensor bundle charts (U, Ψ) at the boundary of $\overline{X}$.*

For manifolds with boundary, the isomorphisms (1.15) hold as well, in other words

$$\begin{aligned} W^{s,p}_{\mathrm{loc}}\mathcal{T}^k_l(\overline{X}) &\cong W^{s,p}_{\mathrm{loc}}(\overline{X}) \otimes_{C^\infty(\overline{X})} \mathcal{T}^k_l(\overline{X}) \\ &\cong L_{C^\infty(\overline{X})}(\mathcal{T}^1_0(\overline{X})^k, \mathcal{T}^0_1(\overline{X})^l; W^{s,p}_{\mathrm{loc}}(\overline{X})) \end{aligned}$$

where the spaces $\mathcal{T}^k_l(\overline{X})$ are defined analogous to $C^\infty(\overline{X})$ in 1.1.8 with tensor bundle charts replacing the corresponding manifold charts.

We end this section by stating, without proof, a trace theorem for tensorfields which have Sobolev space regularity, in analogy with scalar trace theorems 1.4.4 and 1.4.9.

More precisely, since it can be shown, in analogy to lemma 1.4.8, that $\mathcal{DT}_l^k(M)$ is dense in $W_{\mathrm{loc}}^{m,p}\mathcal{T}_l^k(M)$, a generalization of theorem 1.4.4 to $W_{\mathrm{loc}}^{m,p}(M)$-tensorfields is straightforward, i.e.

1.4.13. Theorem. *Let N be a hypersurface in M and let $m > 1/p$ be a non-negative integer and $1 \leq p < \infty$. The restriction operator*

$$\gamma : \mathcal{DT}_l^k(M) \to \mathcal{DT}_l^k(N)|_N$$

can be uniquely, continuously extended to an operator (again denoted γ)

$$\gamma : W_{\mathrm{loc}}^{m,p}\mathcal{T}_l^k(M) \to W_{\mathrm{loc}}^{m-1/p,p}\mathcal{T}_l^k(M)|_N.$$

Moreover, γ is onto. Here $\mathcal{T}_l^k(M)|_N$ denotes the space of smooth sections of the bundle $T_s^r(M)|_N$.

1.4.14. Remark. (i) A restriction of $t \in \mathcal{T}_l^k(M)$ to N does not necessarily yield an element of $\mathcal{T}_l^k(N)$. Therefore $\mathcal{T}_l^k(M)|_N$ has to be used in the statement of theorem 1.4.13 and not $\mathcal{T}_l^k(N)$.

(ii) Concerning traces in $W_{\mathrm{loc}}^{m,p}\mathcal{T}_l^k(\overline{X})$, the assertions of 1.4.7 and 1.4.8 can easily be generalized to $W_{\mathrm{loc}}^{m,p}(\overline{X})$-tensorfields. Hence, theorem 1.4.13 still holds when M is replaced by $\overline{X}$ and N is replaced by the boundary ∂X of $\overline{X}$.

(iii) Theorem 1.4.13 can also be formulated including (Lie) derivatives in analogy to 1.4.4 and 1.4.9, however we will not need it here.

2 Distributional Semi-Riemannian geometry

In this chapter we review some basic notions in distributional Semi-Riemannian geometry. The history of distributional geometry and global analysis using distributions goes back to L. Schwartz [Sch66] and G. De Rham [dR84]. The topic was later on pursued by A. Lichnerowicz [] and J. Marsden. Prominent papers using distributional geometry in relativity are [Par79], [Tau80] and the classical paper by R. Geroch and J. Trachen [GT87] in which they investigate the limits of the use of the (genuine linear) theory of distributions in the (nonlinear) theory of relativity. Recently the topic has been taken up by [LM07]; for an overview see [SV06].

More specifically, we concentrate on the following issues: first we discuss the notions of metrics and connections in the distributional setting and explore the possibilities for associating a Levi-Civita connection to metrics of low regularity, i.e. metrics with coefficients in some appropriate Sobolev space. We then pass on to discuss, within the framework of distributional tensorfields, the definition of the curvature arising from either the connection or the metric.

For the convenience of the reader we start by recalling, some notions in (smooth) Semi-Riamannian geometry and then proceed to discuss the corresponding generalizations in distributional geometry. For the classical results we generally use [O'N83, Ch. 2-4] as our main reference. To begin with we fix our baisc notation: We denote the action of the metric $\mathbf{g}$ on vectorfields ξ and η by $\mathbf{g}(\xi, \eta)$. The distributiona action will be denoted using angular brackets, e.g. $\langle u, \tau \rangle$ stands for the action of a scalar distribution u on some compactly supported n-form τ. Furthermore we will use the summation convention.

2.1 Metrics and connections in smooth Semi-Riemannian geometry

2.1.1. Definition. *A (smooth) Semi-Riemannian metric $\mathbf{g}$ on M is a symmetric, non-degenerate element of $\mathcal{T}^0_2(M)$ whoose index is constant.*

In other words to each point p of M, $\mathbf{g} \in \mathcal{T}^0_2(M)$ smoothly assigns a symmetric, non-degenerate bilinear form $\mathbf{g}_p$ on the tangent space T_pM, i.e.

$$\mathbf{g}_p(v, w) = \mathbf{g}_p(w, v) \quad \forall v, w \in T_pM,$$

and

$$\mathbf{g}_p(v, w) = 0 \ \ \forall w \in T_pM \ \Rightarrow \ v = 0 \ \text{in} T_pM \tag{2.1}$$

Moreover, recall that the index of a symmetric bilinear form b on a vector space V is defined to be the largest integer which equals the dimension of a subspace $W \subseteq V$ such that $b|_W$ is negative definite, i.e.

$$b(v,v) < 0 \quad \forall v \in W, v \neq 0.$$

If the index is equal to 0 we call $\mathbf{g}$ a Riemannian metric, however we will be mostly concerned with the Lorentzian case, i.e. the index being 1.

2.1.2. At every point $p \in M$ the metric $\mathbf{g}$ induces an isomorphism $\flat : T_pM \to T_p^*M$. Indeed fixing $v \in T_pM$, the map $T_pM \ni w \mapsto \mathbf{g}_p(v,w)$ is a linear form on T_pM and we may define

$$\flat : T_pM \to T_p^*M$$

$$v \mapsto v^\flat : v^\flat(w) := \mathbf{g}_p(v,w) \quad \forall w \in T_pM.$$

By non-degeneracy of $\mathbf{g}$ the map $\flat$ is injective ($v^\flat = 0 \;\Rightarrow\; v^\flat(w) = 0 \;\forall w \in T_pM \;\Rightarrow\; v = 0$), hence surjective since $\dim T_pM = \dim T_p^*M$.

The map $\flat$ extends to a C^∞-linear isomorphism of the respective section spaces which we again denote $\flat$. More precisely we have

$$\flat : \mathcal{T}_0^1(M) \to \mathcal{T}_1^0(M)$$

$$\xi \to \xi^\flat : \xi^\flat(\eta) = \mathbf{g}(\xi,\eta)$$

for all $\eta \in TM$. In local coordinates, $\flat$ corresponds to the classical operation of lowering an index. Indeed, choose a chart (U,ψ) of M and denote by $\mathbf{g}_{ij}$ the components of $\mathbf{g}$ with respect to this chart, i.e. $\mathbf{g}_{ij} = \mathbf{g}(\partial_i, \partial_j)$ on U. Obviously $\mathbf{g}_{ij}(p)$ determine the components of a non-singular matrix for all $p \in U$. If the components of the vectorfield ξ are denoted by ξ^i, i.e. $\xi = \xi^i \partial_i$, we obtain

$$(\xi^\flat)_i = \mathbf{g}_{ij}\xi^j$$

for the components of the one-form $\xi^\flat$. We will use the notation ξ_i instead of $(\xi^\flat)_i$; in other words we have $\xi^\flat = \xi_i dx^i$.

The inverse of $\flat$ is the mapping

$$\sharp : \mathcal{T}_1^0(M) \to \mathcal{T}_0^1(M)$$

defined via

$$\omega \to \omega^\sharp : \quad \omega^\sharp(\nu) = \mathbf{g}^{-1}(\omega,\nu), \;\forall \nu \in \mathcal{T}_1^0(M),$$

where $\mathbf{g}^{-1}$ is the inverse metric of $\mathbf{g}$. For every one-form ω, we therefore locally have

$$(\omega^\sharp)^i \equiv \omega^i = \mathbf{g}^{ij}\omega_j$$

where $\mathbf{g}^{ij}$ are the components of $\mathbf{g}^{-1}$. This, of course, corresponds to the classical operation of raising an index.

Both $\flat$ and $\sharp$ can be extended to C^∞-linear mappings acting on general tensorfields. More precisely for fix $1 \leq k \leq r$ and $1 \leq l \leq s$ we have

$$\downarrow_k^l \colon \mathcal{T}_s^r(M) \to \mathcal{T}_{s+1}^{r-1}(M)$$

$$(\downarrow_k^l T)(\xi_1, ...\xi_{s+1}, \omega_1, ..., \omega_{r-1}) = T(\xi_1, ...\xi_{k-1}, \xi_{k+1}, ...\omega_1, ...\omega_{l-1}, \xi_k^\flat, \omega_l...\omega_{r-1})$$

for vectorfields ξ_i and one-forms ω_j. Locally, in a coordinate chart, we have

$$(\downarrow_k^l T)^{i_1...i_{r-1}}_{j_1...j_{s+1}} \equiv T^{i_1...i_{r-1}}_{j_1...j_{s+1}} = g_{j_k m} T^{i_1...i_{l-1} m ... i_{r-1}}_{j_1...j_{k-1} j_{k+1}...j_{s+1}}.$$

thus $\downarrow_k^l$ is actually a composition of a suitable contraction C and the tensor product of T with $\mathbf{g}$, i.e. $\downarrow_k^l T = C(\mathbf{g} \otimes T)$. $\downarrow_k^l$ is an isomorphism, its inverse given by the mapping

$$\uparrow_k^l \colon \mathcal{T}_{s+1}^{r-1}(M) \to \mathcal{T}_s^r(M)$$

which extracts the l-th one-form ω_l and inserts $\omega_l^\sharp$ into the k-th vectorfield slot. As before, from the local formula for $\uparrow_k^l T$ one can conclude that $\uparrow_k^l$ is a composition of the tensor product of T with $\mathbf{g}^{-1}$ and a contraction.

A family of $\mathbb{R}$-linear mappings $D \equiv D_s^r : \mathcal{T}_s^r(M) \to \mathcal{T}_s^r(M)$ is called a tensor derivation on M if

$$D(T_1 \otimes T_2) = DT_1 \otimes T_2 + T_1 \otimes DT_2$$

$$D(CT_1) = C(DT_1)$$

for any two tensorfields T_1 and T_2 and any contraction C. In particular for $f \in C^\infty(M)$, we have $D(fT) = fDT + D(f)T$ for all $T \in \mathcal{T}_s^r(M)$. A tensor derivation is consequently not C^∞-linear in general, hence the value of DT at a point $p \in M$ depends only on the values of T on some arbitrarily small neighbourhood of p. Tensor derivations can therefore be restricted to act on open subsets of M or more precisely, if U is open in M there exists a unique tensor derivation D_U on U, called the restriction of D to U, such that

$$D_U(T|_U) = D(T)|_U$$

for every $T \in \mathcal{T}_s^r(M)$. In fact for $p \in U$, now omiting the subscript U, D is defined as $(DT)_p := D(fT)_p$ where $T \in \mathcal{T}_s^r(U)$ and f is some bump function around p.

A tensor derivation can be reconstructed from its values on smooth functions and vectorfields:

2.1.3. Theorem. *Given $\eta \in \mathcal{T}_0^1(M)$ and $\delta : \mathcal{T}_0^1(M) \to \mathcal{T}_0^1(M)$, an $\mathbb{R}$-linear mapping such that for all smooth functions f and vectorfields η*

$$\delta(f\xi) = \eta(f)\xi + f\delta(\xi), \tag{2.2}$$

then there exists a unique tensor derivation D on M such that $D^0_0 = L_\eta$ and $D^1_0 = \delta$.

Sketch of proof. Two tensor derivations coincide if they agree on $C^\infty(M)$ and $\mathcal{T}^1_0(M)$, which proves the uniqueness assertion of the theorem. As for the existence assertion, define $D : \mathcal{T}^0_1(M) \to \mathcal{T}^0_1(M)$ as

$$(D\omega)(\xi) := \eta\big(\omega(\xi)\big) - \omega(\delta(\xi)) \tag{2.3}$$

where $\eta, \xi \in \mathcal{T}^1_0(M)$, whereas for general (r,s)- tensorfields with $r + s \geq 2$, take $D : \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M)$ to be

$$\begin{aligned}(DT)(\xi_1, ...\xi_s, \omega_1, ..., \omega_r) := &\ \eta(T(\xi_1, ...\xi_s, \omega_1, ..., \omega_r)) \\ &- \sum_{i=1}^{s} T(\xi_1, ..., \xi_{i-1}, \delta(\xi_i), .., \xi_s, \omega_1, ..., \omega_r) \\ &- \sum_{j=1}^{r} T(\xi_1, ..., \xi_s, \omega_1, ..., \omega_{j-1}, D\omega_j, ..., \omega_r)\end{aligned} \tag{2.4}$$

for all $\xi_i \in \mathcal{T}^1_0(M)$, $\omega_j \in \mathcal{T}^0_1(M)$. Observe that analogs of equations (2.3)-(2.4) actually hold for all tensor derivations, which can be seen by applying the tensor product and contraction rules. □

2.1.4. Definition. *A (smooth) connection ∇ on M is a mapping*

$$\nabla : \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \to \mathcal{T}^1_0(M)$$

satisfying

(∇_1) $\nabla_{f\eta+\eta'}\xi = f\nabla_\eta\xi + \nabla_{\eta'}\xi$

(∇_2) $\nabla_\eta(\xi + \alpha\xi') = \nabla_\eta\xi + \alpha\nabla_\eta\xi'$

(∇_3) $\nabla_\eta(f\xi) = f\nabla_\eta\xi + \eta(f)\xi$

for all $\eta, \eta', \xi, \xi' \in \mathcal{T}^1_0(M)$, $f \in C^\infty(M)$ and $\alpha \in \mathbb{R}$. $\nabla_\eta\xi$ is called covariant derivative of ξ w.r.t. η.

2.1.5. By condition (∇_1), $\nabla_\eta\xi$ is a tensor in η, hence for any $v \in T_pM$ there exists a well defined tangent vector $\nabla_v\xi \in T_pM$, namely $(\nabla_\eta\xi)_p$ where η is any vectorfield satisfying $\eta_p = v$. On the other hand $\nabla_\eta\xi$ is not a tensor in ξ by (∇_3), however by $(\nabla_2) - (\nabla_3)$ the mapping

$$\xi \to \nabla_\eta\xi$$

satisifes the requirements of the map δ in theorem 2.1.3 for every fixed vectorfield η. Consequently there exists a unique tensor derivation $D_\eta : \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M)$ such that $D_\eta(f) = \eta(f)$ and $D_\eta(\xi) = \nabla_\eta\xi$ for all vectorfields ξ and smooth functions f. Clearly the D_η's induce a mapping

$$\nabla : \mathcal{T}^1_0(M) \times \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M)$$

such that

$$\nabla_\eta f := \eta(f) \tag{2.5}$$

for all $f \in C^\infty(M)$ and $\eta \in \mathcal{T}^1_0(M)$. Next we want to obtain explicit formulae for ∇: using equations (2.3)-(2.4) we have

$$(\nabla_\eta \omega)(\xi) := \eta\big(\omega(\xi)\big) - \omega(\nabla_\eta \xi) \tag{2.6}$$

for $\omega \in \mathcal{T}^1_0(M)$ where $\eta, \xi \in \mathcal{T}^1_0(M)$, whereas for $T \in \mathcal{T}^r_s(M)$ with $r + s \geq 2$ we have

$$\begin{aligned}(\nabla_\eta T)(\xi_1, ...\xi_s, \omega_1, ..., \omega_r) : = \ & \eta(T(\xi_1, ...\xi_s, \omega_1, ..., \omega_r)) \\ & - \sum_{i=1}^{s} T(\xi_1, ..., \xi_{i-1}, \nabla_\eta \xi_i, .., \xi_s, \omega_1, ..., \omega_r) \\ & - \sum_{j=1}^{r} T(\xi_1, ..., \xi_s, \omega_1, ..., \omega_{j-1}, \nabla_\eta \omega_j, ..., \omega_r)\end{aligned} \tag{2.7}$$

for all $\xi_i \in \mathcal{T}^1_0(M)$, $\omega_j \in \mathcal{T}^0_1(M)$.

The mapping $\nabla : \mathcal{T}^1_0(M) \times \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M)$ is C^∞-linear in the first variable and therefore induces a mapping, again denoted ∇,

$$\nabla : \mathcal{T}^r_s(M) \to \mathcal{T}^r_{s+1}(M)$$

$$(\nabla T)(\xi_1, ..., \xi_{s+1}, \omega_1, ..., \omega_r) = (\nabla_{\xi_{s+1}} T)(\xi_1, ..., \xi_s, \omega_1, ..., \omega_r)$$

for all $\xi_i \in \mathcal{T}^1_0(M)$ and $\omega_j \in \mathcal{T}^0_1(M)$. ∇T is called the covariant differential of T.

As a tensor derivation in the second variable and a C^∞-linear mapping in the first, ∇ is local and therefore can be restricted to act on open subsets of M. In particular, in some coordinate chart it is given by

$$\nabla_{\partial_i}(\sum_k \xi^k \partial_k) = \sum_k \big(\frac{\partial \xi^k}{\partial x^i} + \sum_j \Gamma^k_{ij} \xi^j\big)\partial_k \tag{2.8}$$

where Γ^k_{ij} are the Christoffel symbol, defined as $\nabla_{\partial_i} \partial^j = \sum_k \Gamma^k_{ij} \partial_k$.

We end this section with one of fundamental results of (Semi-)Riemmanian geometry which states that a unique metric and torsion-free connection can be assigned to every smooth metric. More precisely:

2.1.6. Proposition. *To any metric* $\boldsymbol{g}$ *one can associate a unique smooth connection* ∇ *which satisfies*

(∇_4) $\nabla \boldsymbol{g} = 0$ ($\Leftrightarrow$ $\eta(\boldsymbol{g}(\xi, \zeta)) = \boldsymbol{g}(\nabla_\eta \xi, \zeta) + \boldsymbol{g}(\xi, \nabla_\eta \zeta)$)

(∇_5) $T(\eta, \xi) := \nabla_\eta \xi - \nabla_\xi \eta - [\eta, \xi] = 0$

for all $\xi, \eta, \zeta \in \mathcal{T}^1_0(M)$. We call ∇ Levi-Civita connection of $\boldsymbol{g}$ and say ∇ is metric and torsion-free. The Levi-Civita connection of $\boldsymbol{g}$ is implicitely given by the Koszul formula

$$\begin{aligned} 2(\nabla_\eta \xi)^\flat(\zeta) = 2\boldsymbol{g}(\nabla_\eta \xi, \zeta) =& \eta(\boldsymbol{g}(\xi,\zeta)) + \xi(\boldsymbol{g}(\zeta,\eta)) - \zeta(\boldsymbol{g}(\eta,\xi)) \\ &- \boldsymbol{g}(\eta,[\xi,\zeta]) + \boldsymbol{g}(\xi,[\zeta,\eta]) + \boldsymbol{g}(\zeta,[\eta,\xi]). \end{aligned} \tag{2.9}$$

Sketch of proof. For later use, we first abbreviate the right hand side of the Koszul formula by

$$\begin{aligned} F(\xi,\eta,\zeta) := \frac{1}{2}\big(&\eta(\mathbf{g}(\xi,\zeta)) + \xi(\mathbf{g}(\zeta,\eta)) - \zeta(\mathbf{g}(\eta,\xi)) \\ &- \mathbf{g}(\eta,[\xi,\zeta]) + \mathbf{g}(\xi,[\zeta,\eta]) + \mathbf{g}(\zeta,[\eta,\xi])\big) \end{aligned} \tag{2.10}$$

for $\xi, \eta, \zeta \in \mathcal{T}^1_0(M)$.

To show uniqueness note that any connection satisfying $(\nabla_4) - (\nabla_5)$ for the metric $\mathbf{g}$, also satsifies equation (2.9). Since $\flat$ is an isomorphism and the right hand side of (2.9) depends only on the metric the claim follows.

As for the existence, observe that for fixed ξ and η the mapping $\zeta \to F(\xi,\eta,\zeta)$ is $C^\infty(M)$-linear, hence a one-form. Consequently, there exists a unique vectorfield denoted $\nabla_\eta \xi$ such that $\nabla_\eta \xi = (\zeta \to F(\xi,\eta,\zeta))^\sharp$. By inserting the appropriate combinations of vectorfields in $F(\xi,\eta,\zeta)$ it follows that ∇ is a smooth connection satisfying required properties $(\nabla_1) - (\nabla_5)$.

For the complete proof see for instance [O'N83, p.61]. □

Finally we recall that Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_m \mathbf{g}^{km}\{\partial_i \mathbf{g}_{jm} + \partial_j \mathbf{g}_{im} - \partial_m \mathbf{g}_{ij}\}.$$

2.2 Distributional metrics and connections

This section is devoted to the study of distributional metrics and connections. Distributional metrics and connections have first been defined in [Mar67] and later on in [Par79]. The notion of non-degeneracy used for the metrics in these two papers differs significantly, so we start by discussing appropriate non-degeneracy conditions, see also [SV09]. We then turn to defining distributional connections and discuss extensions of their action on distributional vectorfields. We end this section by discussing the difficulties which arise when attempting to associate a distributional Levi-Civita connection to a distributional metric, mainly following the presentation in [LM07].

Distributional metrics

2.2.1. Definition. *A distributional metric $\boldsymbol{g}$ is a nondegenerate, symmetric element of $\mathcal{T}^0_2(M)'$.*

Obviously $\mathbf{g} \in \mathcal{D}'\mathcal{T}^0_2(M)$ is called symmetric if

$$\mathbf{g}(\xi, \eta) = \mathbf{g}(\eta, \xi)$$

for all $\eta, \xi \in \mathcal{T}^1_0(M)$. It is called nondegenerate if it satisfies

$$g(\xi, \eta) = 0 \quad \forall \eta \in \mathcal{T}^1_0(M) \Rightarrow \xi = 0 \text{ in } \mathcal{T}^1_0(M) \tag{2.11}$$

and it is in addition pointwise non-degenerate in the usual sense (see equation (2.1)) away from its singular support. Recall that the singular support of $\mathbf{g} \in \mathcal{D}'\mathcal{T}^0_2(M)$ is defined to be the set

$$\text{singsupp}(\mathbf{g}) = \{x \in M \mid \exists U \text{ ngbh. of } x : \mathbf{g}|_U \text{ smooth}\}^c.$$

2.2.2. Remark. The notion of non-degeneracy in the definition of the distributional metrics is not consistent throughout the literature. According to [Mar67] non-degeneracy is defined solely via equation (2.11), whereas [Par79] demandes only pointwise non-degeneracy away from the singular support in his definition of the distributional metric. Both of these conditions suffer some serious drawbacks when considered on their own: the former classifies some classically singular line elements such as $ds^2 = x^2 dx^2$ as non-degenerate, whereas the latter imposes no control whatsoever on the behavior of the metric on its singular support. Therefore following [Ste08], we have decided to combine both of them, thereby preserving the usual notion of non-degeneracy for smooth metrics.

A distributional metric $\mathbf{g}$ does not induce an isomorphism from $\mathcal{D}'\mathcal{T}^1_0(M)$ to $\mathcal{D}'\mathcal{T}^0_1(M)$ analogous to the mapping $\flat$ in the smooth case. Actually, the mere definition of $\flat$ as a mapping from $\mathcal{D}'\mathcal{T}^1_0(M)$ into $\mathcal{D}'\mathcal{T}^0_1(M)$ is already problematic, since for a distributional vectorfield ξ and metric $\mathbf{g}$, the definition of $\mathbf{g}(\xi, \,.\,)$ requires multiplication of distributions. However, restricting the domain of $\flat$ to $\mathcal{T}^1_0(M)$ avoids this problem and we can therefore define

$$\flat : \mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^0_1(M)$$
$$\xi \to \xi^\flat : \xi^\flat(\eta) = \mathbf{g}(\xi, \eta)$$

for all $\eta \in \mathcal{T}^1_0(M)$. The non-degeneracy condition given in equation (2.11) renders $\flat$ injective. Surjectivity is obviously not to be expected in general.

Observe however that there are metrics with distributional coefficients which are invertible: consider for instance the line element

$$ds^2 = \delta(u) du^2 + 2 du dv$$

on $\mathbb{R}^2$ which is a two-dimensional model of an impulsive pp-wave in Brinkmann form, see [Pen68]. Here δ is a one-dimensional Dirac measure. That is we have the metric

$$\mathbf{g}_{ij}(u, v) = \begin{pmatrix} \delta(u) & 1 \\ 1 & 0 \end{pmatrix}$$

whose inverse metric is obviously given by

$$\mathbf{g}^{ij}(u,v) = \begin{pmatrix} -\delta(u) & 1 \\ 1 & 0 \end{pmatrix}.$$

Distributional connections

2.2.3. Definition. *A distributional connection ∇ on M is a mapping*

$$\nabla : \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^1_0(M)$$

satisfying properties $(\nabla_1) - (\nabla_3)$ of definition 2.1.4, i.e.

(∇_1) $\nabla_{f\eta+\eta'}\xi = f\nabla_\eta\xi + \nabla_{\eta'}\xi$

(∇_2) $\nabla_\eta(\xi+\xi') = \nabla_\eta\xi + \nabla_\eta\xi'$

(∇_3) $\nabla_\eta f\xi = f\nabla_\eta\xi + (\eta(f))\xi$

for all $\eta, \eta', \xi, \xi' \in \mathcal{T}^1_0(M)$ and $f \in C^\infty(M)$.

As in the smooth case a distributional connection ∇ is local, i.e. if $\xi|_U \equiv 0$ on some open subset U of M then $(\nabla_\eta\xi)|_U \equiv 0$ for all $\eta \in \mathcal{T}^1_0(M)$. This can be seen by applying (∇_3) to $f\xi$ where f is some compactly supported smooth function whose support is contained in U. Since ∇ is C^∞-linear in the first variable the analogous statement is also true for η. This implies ∇ can be restricted to a unique distributional connection on U such that $\nabla_{\eta|_U}(\xi|_U) = (\nabla_\eta\xi)|_U$. In particular, the usual coordinate formula (2.8) employing Christoffel symbols holds, where clearly the Christoffel symbols are now distributions. A distributional connection can be extended to act on the full smooth tensor algebra in the usual way, i.e. it can be extended to a mapping

$$\nabla : \mathcal{T}^1_0(M) \times \mathcal{T}^r_s(M) \to \mathcal{D}'\mathcal{T}^r_s(M) \tag{2.12}$$

via equations (2.5)-(2.7). ∇ will always be considered with its extension to the full tensor algebra.

2.2.4. Remark. The definition of a distributional connection as given here first appeared in [Mar67]. However, the very definition contains a typo and literally defines ∇ to be a mapping

$$\nabla : \mathcal{T}^1_0(M) \times \mathcal{D}'\mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^1_0(M) \tag{2.13}$$

(in the notation used here) which satisfies $(\nabla_1) - (\nabla_3)$ as defined in 2.2.3. Let us assume for a moment that we do have such a mapping. Then strictly speaking condition (∇_3) reads

(∇_3') $\nabla_\eta f\xi = f\nabla_\eta\xi + (\eta(f))\xi \ , \ \forall f \in \mathcal{D}'(M), \forall \eta, \xi \in \mathcal{T}^1_0(M).$

Using this equation to derive the usual local expressions (2.8) for the Christoffel symbols gives

$$\nabla_{\partial_i}(\sum_j \xi^j\partial_j) = \sum_j \big(\frac{\partial\xi^j}{\partial x^i} + \sum_k \Gamma^j_{ik}\xi^k\big)\partial_j$$

where both ξ_i and Γ^k_{ij} are distributions. This leads to multiplication of distributions and therefore (2.13) does not let us pass to usual coordinate formulas.

There is another argument which casts doubt on defining the connection according to (2.13). Interpreting the product $\xi\Gamma^k_{ij}$ in the sense of a suitable irregular intrinsic product of distributions [Obe92, Chap. II] we would have

$$\forall v \in \mathcal{D}'(M),\ \exists\, u \cdot v \ \Rightarrow\ u \in C^\infty(M),$$

which leads us to conjecture that (2.13) forces Γ^k_{ij} to be smooth, hence ∇ to be classical!

The dual Levi-Civita connection

Having fixed the notions of a distributional metric and connection we examine if, in analogy to the smooth case, it is possible to associate a distributional connection to a distributional metric **g**, i.e. we discuss possible analogs to proposition 2.1.6. We follow the discussion of [LM07] respectively [Ste08].

The first obstacle we meet is the impossibility to formulate condition (∇_4) i.e. $\nabla \mathbf{g} = 0$ or for that matter $\mathbf{g}(\nabla_\xi \eta, \zeta)$ if both ∇ and **g** are distributional.

An attempt to, nevertheless, define a distributional connection via the Koszul formula (2.9) fails, since $\flat$ is only injective and not surjectve for a general distributional metric **g**, as already discussed. More precisely, for fixed $\xi, \eta \in \mathcal{T}^1_0(M)$, the mapping

$$\mathcal{T}^1_0(M) \ni \zeta \longmapsto F(\xi, \eta, \zeta) \in \mathcal{D}'(M),$$

where F is the right hand side of the Koszul formula (cf.(2.10)), is $C^\infty(M)$-linear, which can be seen on inspection of (2.10). Hence it defines an element in $\mathcal{D}'\mathcal{T}^1_0(M)$. However lacking an isomorphism $\mathcal{D}'\mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^0_1(M)$ induced by **g**, we cannot mimic the final step in the proof of 2.1.6, i.e. the definition $\nabla_\eta \xi := (\zeta \longmapsto F(\xi, \eta, \zeta))^\sharp$.

The above discussion implies that even though it is in general not possible to associate a distributional connection in the sense of definition 2.2.3 to some general distributional metric **g**, we can obtain “dual“ Levi-Civita connection as described in [LM07]:

2.2.5. Definition. *The dual Levi-Civita connection associated with the distributional metric* ***g*** *is a mapping* $\nabla^\flat : \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^0_1(M)$ *defined via:*

$$\nabla^\flat_\eta \xi(\zeta) := F(\xi, \eta, \zeta) \tag{2.14}$$

for $\eta, \xi, \zeta \in \mathcal{T}^1_0(M)$.

This connection is obviously not a distributional connection in the sense of the definition 2.2.3 . However $\nabla^\flat$ does satisfy dual versions of $(\nabla_4) - (\nabla_5)$, i.e.

2.2.6. Proposition. *The dual Levi-Civita connection $\nabla^\flat$ of the metric $\boldsymbol{g}$ satisfies:*

$$\begin{aligned} &(\nabla_4') \quad \eta(\boldsymbol{g}(\xi,\zeta)) = \nabla^\flat_\eta \xi(\zeta) + \nabla^\flat_\eta \zeta(\xi) \\ &(\nabla_5') \quad \nabla^\flat_\eta \xi - \nabla^\flat_\xi \eta - [\eta,\xi]^\flat = 0. \end{aligned} \tag{2.15}$$

Sketch of proof. Indeed, using (2.14) equations (2.15) are obviously, for all $\zeta \in \mathcal{T}^1_0(M)$, equivalent to

$$\eta(\mathbf{g}(\xi,\zeta)) = F(\xi,\eta,\zeta) + F(\zeta,\xi,\eta)$$

$$\mathbf{g}([\eta,\xi],\zeta) = F(\xi,\eta,\zeta) - F(\eta,\xi,\zeta).$$

By writing F out according to (2.10) and using the symmetry of $\mathbf{g}$ most of the terms cancel in pairs and the claim follows. □

2.3 Curvature in smooth Semi-Riemannian geometry

In the following we recall some basic facts on the notion of curvature on Semi-Riemannian manifolds, for details see eg. [O'N83, p.74].

2.3.1. For a smooth connection ∇ on M (cf. definition 2.1.4) the Riemann curvature tensor **Riem** is the (1,3)-tensorfield

$$\mathbf{Riem} : \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \times \mathcal{T}^1_0(M) \to \mathcal{T}^1_0(M)$$

defined by the formula

$$\mathbf{Riem}(\eta,\xi)\zeta = \nabla_\eta \nabla_\xi \zeta - \nabla_\xi \nabla_\eta \zeta - \nabla_{[\eta,\xi]}\zeta \tag{2.16}$$

for all $\xi, \eta, \zeta \in \mathcal{T}^1_0(M)$. Locally we have $\mathbf{Riem}(\partial_k,\partial_l)\partial_j = \mathbf{Riem}^i{}_{jkl}\partial_i$ where classical indexing has been used. In terms of Christoffel symbols, the components $\mathbf{Riem}^i{}_{jkl}$ of the Riemann tensor are given by

$$\mathbf{Riem}^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}. \tag{2.17}$$

Recall that a (semi-)Riemannian manifold is called flat if its Riemann tensor vanishes.

2.3.2. The Ricci curvature tensor is a symmetric (0,2)-tensorfield obtained by contracting the Riemann tensor i.e.

$$\mathbf{Ric}(\eta,\xi) := (C^1_3\mathbf{Riem})(\eta,\xi) \tag{2.18}$$

For the notation regarding the contraction operator see paragraph 1.3.13. In particular, the components of the Ricci tensor relativ to some coordinate system are given by $\mathbf{Ric}_{ij} := \sum_m \mathbf{Riem}^m{}_{ijm}$.

Another local description of the Ricci tensor can be obtained in a given local frame $(E_{(\alpha)})_{\alpha=1,\dots,n}$, i.e. in a system of n(= dim(M)) linearly independent smooth vectorfields, namely

$$\mathbf{Ric}(\eta,\xi) := E^{(\alpha)}\big(\mathbf{Riem}(\eta,E_{(\alpha)})\xi\big) \tag{2.19}$$

where $(E^{(\alpha)})_{\alpha=1,\dots,n}$ is the dual frame of $E_{(\alpha)}$ defined via

$$E^{(\alpha)}(E_{(\beta)}) := \delta^\alpha_\beta = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases}$$

In case of a Semi-Riemannian manifold we always use the Riemann and Ricci tensors induced by the Levi-Civita connection of the corresponding SR-metric **g**. Observe that in this case, symmetries of the Riemann tensor imply that ±**Ric** are the only non-vanishing tensorfields which can be obtained by contracting **Riem**. In addition, if the frame $(E_{(\alpha)})_{\alpha=1,\dots,n}$ is chosen to be orthonormal with respect to **g**, i.e.

$$\mathbf{g}(E_{(\alpha)}, E_{(\beta)}) = \epsilon_\alpha \delta^\alpha_\beta$$

where $\epsilon_\alpha = \pm 1$, we also have the following description of the Ricci tensor:

$$\mathbf{Ric}(\eta, \xi) := \epsilon_\alpha E^{(\alpha)}\big(\mathbf{Riem}(\eta, E_{(\alpha)})\xi\big) \tag{2.20}$$

where the dual frame is obtained by raising the indices of $E_{(\alpha)}$ by the metric, that is $E^{(\alpha)} := E^\flat_{(\alpha)}$ or $E^{(\alpha)}(E_{(\beta)}) = \epsilon_\alpha \delta^\alpha_\beta$. Even though this definition of the frame is common, it involves the metric and hence cannot be generalized to the level of distributional geometry, i.e. when **g** is a distributional metric. We will therefore always, unless specifically stated otherwise, use the former definition of the frame.

A semi-Riemannian manifold M is called Ricci flat, if its Ricci tensor vanishes. Obviously any flat manifold is Ricci flat, whereas the converse does not hold in general.

2.3.3. The scalar curvature R is defined as the contraction of the Ricci curvature associated with the Levi-Civita connection of the metric **g**, that is

$$R = \mathrm{C}(\uparrow^1_1 \mathbf{Ric})$$

or locally

$$R = \mathbf{g}^{mk}\mathbf{Ric}_{mk} \tag{2.21}$$

where $\mathbf{g}^{\alpha\beta}$ denote the components of the inverse metric $\mathbf{g}^{-1}$.

2.4 Distributional Curvature

In this section we will discuss the perspective of defining curvature for a distributional connection. As already clearly stated by Marsden, [Mar67] (cf. 2.2.4) for a general distributional connection it is not possible to define the curvature due to the impossibility of a general product for distributions. This can be seen most explicitely from formula (2.16) which may be used to define **Riem** in the smooth setting. Indeed the terms involving second covariant derivatives, e.g. $\nabla_\eta \nabla_\xi \zeta$, are not defined since ∇_ξ maps $\zeta \in \mathcal{T}^1_0(M)$ into $\mathcal{D}'\mathcal{T}^1_0(M)$ and therefore ∇_η cannot be applied to $\nabla_\xi \zeta$. Another aspect of the same obstruction can be seen from the coordinate formula (2.17) which involves products of the

Christoffel symbols. Since in our case these are distributional (cf. discussion following definition 2.2.3) we directly see that **Riem** cannot be defined in this way.

Following an idea developed in [LM07] we try to define the curvature for distributional connections ∇ that have additional regularity. In other words we look for a class of connections with range in some appropriate subspace $\mathcal{A} \subseteq \mathcal{D}'(M)$ such that ∇ can be extended to a map acting on $\mathcal{A}$-objects in its second slot. More precisely we assume that ∇ has values in $\mathcal{A}$, i.e.

$$\nabla : \mathcal{T}_0^1(M) \times \mathcal{T}_0^1(M) \to \mathcal{A}\mathcal{T}_0^1(M)$$

where the choice of $\mathcal{A}$ has to permit an extension of ∇ to a mapping

$$\nabla : \mathcal{T}_0^1(M) \times \mathcal{A}\mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^1_0(M).$$

In fact, if we take

$$(\nabla_\eta \xi)\big(\omega\big) := \eta\big(\xi(\omega)\big) - \xi(\nabla_\eta \omega), \tag{2.22}$$

we obtain such an extension of ∇, provided the right hand side of (2.22) is defined for all ξ in $\mathcal{A}\mathcal{T}^1_0(M)$. Note that the problem lies in the term $\xi(\nabla_\eta \omega)$. For it to be defined, it suffices to require that multiplication of two function in $\mathcal{A}$ results in a well defined distribution and that the extension of ∇ to smooth one-forms has values in $\mathcal{A}$.

As noted by [LM07], one possible choice for $\mathcal{A}$ is $L^2_{\text{loc}}(M)$, since it satisfies all the requirements outlined above. Indeed multiplication of two $L^2_{\text{loc}}(M)$-functions yields a function in $L^1_{\text{loc}}(M) \subseteq \mathcal{D}'(M)$. Moreover, the usual extension of an L^2_{loc}-connection ∇ to the smooth tensor algebra, defined via equations (2.5)-(2.7), has values in $L^2_{\text{loc}}\mathcal{T}^r_s(M)$. This enables us to define:

2.4.1. Definition. *A distributional connection ∇ is called an L^2_{loc}-connection if $\nabla_\eta \xi$ belongs to $L^2_{\text{loc}}\mathcal{T}^1_0(M)$ for all $\eta, \xi \in \mathcal{T}^1_0(M)$. With other words ∇ is a mapping*

$$\nabla : \mathcal{T}_0^1(M) \times \mathcal{T}_0^1(M) \to L^2_{\text{loc}}\mathcal{T}_0^1(M)$$

which satisfies $(\nabla_1) - (\nabla_3)$.

We now state and prove that an $L^2_{\text{loc}}(M)$-connection indeed allows an extension to $L^2_{\text{loc}}\mathcal{T}^1_0(M)$ in the second slot.

2.4.2. Proposition. *Every L^2_{loc}-connection extends to an operator $\nabla : \mathcal{T}^1_0(M) \times L^2_{\text{loc}}\mathcal{T}^1_0(M) \to \mathcal{D}'\mathcal{T}^1_0(M)$ defined via*

$$(\nabla_\eta \xi)\big(\omega\big) = \eta\big(\xi(\omega)\big) - \xi(\nabla_\eta \omega) \tag{2.23}$$

for $\eta \in \mathcal{T}^1_0(M)$, $\omega \in \mathcal{T}^0_1(M)$ and $\xi \in L^2_{\text{loc}}\mathcal{T}^1_0(M)$.

Proof. We have to show that both terms on the r.h.s of equation (2.23) make sense distributionally. As to the first one, $\xi(\omega)$ belongs to $L^2_{\text{loc}}(M)$ if $\xi \in L^2_{\text{loc}}\mathcal{T}^1_0(M)$, cf. equation (1.15). Interpreting it as a distribution we may apply Lie derivatives along smooth vectorfield to it. As for the second term, both $\nabla_\eta \omega$ (defined by equation (2.6)) and ξ have L^2_{loc}-coefficients, which implies that this term belongs to $L^1_{\text{loc}}(M)$. Consequently, the right hand side of

(2.23) is a well defined distribution. Furthermore it is $C^\infty(M)$-linear in ω, implying $\nabla_\eta \xi$ belongs to $\mathcal{D}'\mathcal{T}^1_0(M)$. In fact from (2.6) it follows

$$\begin{aligned}(\nabla_\eta \xi)(f\omega) &= \eta(f)\xi(\omega) + f\eta(\xi(\omega)) - \eta(f)\xi(\omega) - f\xi(\nabla_\eta \omega) \\ &= f(\eta(\xi(\omega)) - \xi(\nabla_\eta \omega)).\end{aligned}$$

for all $f \in C^\infty(M)$. □

Proposition 2.4.2 allows us to define second order covariant derivatives for L^2_{loc}-connections. Indeed for $\eta, \zeta, \xi \in \mathcal{T}^1_0(M)$ we have $\nabla_\eta \zeta \in L^2_{\mathrm{loc}}\mathcal{T}^1_0(M)$ hence $\nabla_\xi \nabla_\eta \zeta \in \mathcal{D}'\mathcal{T}^1_0(M)$. We therefore may define distributional Riemann and Ricci curvature of an L^2_{loc}-connection as usual. Note, however that in this case third order covariant derivatives are not defined as follows from an iteration of the discussion preceding the definition of L^2_{loc}-connection.

2.4.3. Definition. *The distributional Riemann curvature of an L^2_{loc}-connection ∇ is an element of $\mathcal{D}'\mathcal{T}^1_3(M)$ defined via*

$$\boldsymbol{Riem}(\eta, \xi)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta \tag{2.24}$$

for all $\eta, \xi, \zeta \in \mathcal{T}^1_0(M)$.

Inserting (2.23) in (2.24), one obtains an alternative description of the distributional Riemann curvature which has been used as definition of distributional Riemann tensor in [LM07]:

$$\begin{aligned}(\mathbf{Riem}(\eta, \xi)\zeta)(\omega) =& \xi(\nabla_\eta \zeta(\omega)) - \nabla_\eta \zeta(\nabla_\xi \omega) \\ &- \eta(\nabla_\xi \zeta(\omega)) + \nabla_\xi \zeta(\nabla_\eta \omega) + \nabla_{[\eta,\xi]}\zeta(\omega)\end{aligned} \tag{2.25}$$

where $\omega \in \mathcal{T}^0_1(M)$ and $\xi, \eta, \zeta \in \mathcal{T}^1_0(M)$. Moreover since Christoffel symbols of an L^2_{loc}-connection obviously belong to $L^2_{\mathrm{loc}}(M)$, we obtain the usual coordinate formula employing Christoffel symbols as well:

$$\mathbf{Riem}^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}. \tag{2.26}$$

Having successfully defined Riemannian curvature, the definition of the Ricci curvature is straghtforward:

2.4.4. Definition. *The distributional Ricci curvature **Ric** of an L^2_{loc}-connection ∇ is an element of $\mathcal{D}'\mathcal{T}^0_2(M)$ defined via*

$$\boldsymbol{Ric}(\eta, \xi) := (C^1_3 \boldsymbol{Riem})(\eta, \xi)$$

for every $\eta, \xi \in \mathcal{T}^1_0(M)$.

Since we may use the local formulas we obtain exactly as in the smooth case:

2.4.5. Lemma. *In a local frame $(E_{(\alpha)})_{\alpha=1,\dots,n}$, the distributional Ricci tensor is given by*

$$\boldsymbol{Ric}(\eta,\xi) = E^{(\alpha)}\big(\boldsymbol{Riem}(\eta, E_{(\alpha)})\xi\big) \tag{2.27}$$

for all $\eta, \xi \in \mathcal{T}^1_0(M)$, where $E^{(\alpha)}$ denotes the dual frame of $E_{(\alpha)}$ defined via $E^{(\alpha)}(E_{(\beta)}) = \delta^\alpha_\beta$.

Remark In order to define distributional scalar curvature more care is needed, since it involves the Levi-Civita connection which is, in the present case, the Levi-Civita connection of a distributional metric $\mathbf{g}$ - a concept not entirely without problem as we have seen in discussion preceding definition 2.2.5. Moreover the inverse of the metric is involved as well. Observe that the mere existence of the inverse metric or for that matter the associated Levi-Civita connection is not sufficient to guarantee the existence of the scalar curvature since "multiplication" of possibly only distributional Ricci tensor and $\mathbf{g}^{-1}$ must be defined as well. The class of metrics introduced in the next section is regular enough to overcome these difficulties.

2.5 Geroch-Traschen class of metrics

In this section a special class of metrics, called gt-regular metrics, is defined. First introduced in [GT87], their purpose lies in the fact that all of the results of the classical geometry discussed so far hold for this metrics, i.e. it is possible to associate an L^2_{loc}-connection to such metrics which satisfies requirements $(\nabla_4) - (\nabla_5)$ (as opposed to $(\nabla'_4) - (\nabla'_5)$) and allows Riemann and Ricci curvature tensors to be defined. Moreover such metrics are invertible and their product with the Ricci tensor is defined, hence the distributional scalar curvature can be defined as well.

The regularity of gt-regular metrics will be $H^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$, so we start by examining this space.

2.5.1. Lemma. *$H^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$ is an algebra.*

Proof. We have to show that $H^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$ is closed with respect to the (pointwise) product. Since this is a local issue, it suffices to consider $f, g \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$.

Now clearly, $fg \in L^\infty_{\mathrm{loc}}(\Omega) \subseteq L^2_{\mathrm{loc}}(\Omega)$ and we only have to deal with the derivative, i.e. we have to show that $\partial_i(fg) \in L^2_{\mathrm{loc}}(\Omega)$. By the lemma below, the product rule applies and we have $\partial_i(fg) = \partial_i f g + f \partial_i g$. The right hand side is a sum of products of the form $L^2_{\mathrm{loc}}(\Omega) \cdot L^\infty_{\mathrm{loc}}(\Omega)$ and therefore belongs to $L^2_{\mathrm{loc}}(\Omega)$. □

2.5.2. Lemma. *The Leibnitz rule holds in $H^1_{\mathrm{loc}}(\Omega)$. More precisely for $fg \in H^1_{\mathrm{loc}}(\Omega)$ we have*

$$\partial_i(fg) = \partial_i f g + f \partial_i g \quad 1 \le i \le n$$

in $L^1_{\mathrm{loc}}(\Omega)$.

Proof. Let f and g be elements of $H^1_{\text{loc}}(\Omega)$ and f_ϵ respectively g_ϵ nets obtained by convoluting f resp. g with a mollifier. By standard results on smoothing (cf. e.g. [AF03]), f_ϵ and g_ϵ are smooth and converge to f resp. g. We then clearly have

$$\partial_i(f_\epsilon g_\epsilon) = (\partial_i f_\epsilon)g_\epsilon + (\partial_i g_\epsilon)f_\epsilon.$$

By continuity of the product $L^2_{\text{loc}}(\Omega) \times L^2_{\text{loc}}(\Omega) \to L^1_{\text{loc}}(\Omega)$ the terms on the right hand side converge to $(\partial_i f)g + (\partial_i g)f$ in $L^1_{\text{loc}}(\Omega)$ and the term on the right hand side to $\partial_i(fg)$ in $\mathcal{D}'(\Omega)$, where $\partial_i(fg)$ initially exists only in $\mathcal{D}'(\Omega)$. Putting both limits together, we obtain that $\partial_i(fg)$ also belongs to $L^1_{\text{loc}}(\Omega)$. □

Finally we discuss invertibility of functions in $L^\infty_{\text{loc}}(M) \cap H^1_{\text{loc}}(M)$ with respect to multiplication:

2.5.3. Lemma. *$f \in H^1_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M)$ is invertible with respect to multiplication if and only if it is locally uniformly bounded from below, i.e. for every compact subset K of M there exists a constant $C > 0$ such that*

$$|f(x)| \geq C \quad \textit{a.e on K.} \tag{2.28}$$

Proof. As above it is sufficient to consider open subsets of $\mathbb{R}^n$.

($\Rightarrow$) If f is invertible in $L^\infty_{\text{loc}}(\Omega)$, its inverse $1/f$ belongs to $L^\infty_{\text{loc}}(\Omega)$ by definition. Consequently, for every compact subset K of Ω there exists a constant $C > 0$ such that $|1/f(x)| \leq C$ almost everywhere on K.

($\Leftarrow$) Conversely let f be uniformly bounded from below. Then $f(x) \neq 0$ for almost every $x \in \Omega$, which implies that $1/f$ is defined a.e. on Ω. Moreover, $1/f$ belongs to $L^\infty_{\text{loc}}(\Omega) \subseteq L^2_{\text{loc}}(\Omega)$ by (2.28). It remains to show $\partial_i(1/f) \in L^2_{\text{loc}}(\Omega)$ for $1 \leq i \leq n$. In fact, since Leibnitz rule holds we also have

$$\partial_i(1/f) = -\partial_i f/f^2 \in L^2_{\text{loc}}(\Omega) \cdot L^\infty_{\text{loc}}(\Omega) \subseteq L^2_{\text{loc}}(\Omega)$$

by lemma 2.5.2, hence the claim holds.

□

We are now ready to give the main definition of this section:

2.5.4. Definition. *A distributional metric $\boldsymbol{g}$ is said to be gt-regular if $\boldsymbol{g} \in (H^1_{\text{loc}} \cap L^\infty_{\text{loc}})\mathcal{T}^0_2(M)$. A gt-regular metric is called non-degenerate if its determinant is locally uniformly bounded from below i.e.*

$$\forall K \textit{ compact } \exists C \ : \ |\det(\boldsymbol{g}(x))| \geq C > 0 \textit{ for a.e. } x \in K \tag{2.29}$$

By the non-degeneracy condition (2.29) and lemma 2.5.3, gt-regular metrics are invertible, with the inverse metric being gt-regular as well. More precisely:

2.5.5. Lemma. *A non-degenerate gt-regular metric $\boldsymbol{g}$ is invertible. Furthermore its inverse $\boldsymbol{g}^{-1}$ is an element of $(H^1_{\text{loc}} \cap L^\infty_{\text{loc}})\mathcal{T}^2_0(M)$ whose determinant is locally uniformly bounded from below.*

Proof. Since $(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})(M)$ is an algebra, the determinant det(**g**) of a gt-regular metric **g** belongs to $(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})(M)$. By lemma 2.5.3, the non-degeneracy condition (2.29) imposed on the determinant of **g** is equivalent to the assertion that $1/\det(\mathbf{g}) \in (H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})(M)$ a.e. on M. Employing the cofactor formula, this implies that **g** is invertible a.e. with an inverse in $(H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})\mathcal{T}^2_0(M)$, whose determinant is locally uniformly bounded from below. □

The existence of the inverse metric enables us to invert the dual Levi-Civita connection (cf. definition 2.2.5) as in the smooth case, thereby obtaining the fundamental lemma of Semi-Riemannian geometry for gt-regular metrics.

2.5.6. Theorem. *Let **g** is be a non-degenerate gt-regular metric on M. The dual Levi-Civita connection of **g** has values in $L^2_{\mathrm{loc}}\mathcal{T}^0_1(M)$. Consequently there exists a unique L^2_{loc}-connection on M that is torsion free and metric in the usual sense, i.e. this connection satisfies:*

$$(\nabla_4) \qquad \nabla \boldsymbol{g} = 0$$
$$(\nabla_5) \qquad T = 0.$$

*As usual it is called the Levi-Civita connection associated with the metric **g**.*

Proof. The dual Levi-Civita connection $\nabla^\flat$ defined in 2.2.5 by the Koszul formula has values in L^2_{loc} since

$$\begin{aligned} 2(\nabla^\flat_\eta \xi)(\zeta) = & \underbrace{\eta(\mathbf{g}(\xi,\zeta)) + \xi(\mathbf{g}(\zeta,\eta)) - \zeta(\mathbf{g}(\eta,\xi))}_{L^2_{\mathrm{loc}}} \\ & - \underbrace{\mathbf{g}(\eta,[\xi,\zeta]) + \mathbf{g}(\xi,[\zeta,\eta]) + \mathbf{g}(\zeta,[\eta,\xi])}_{H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}}. \end{aligned}$$

This and $\mathbf{g}^{-1} \in (H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})\mathcal{T}^2_0(M)$ imply that $\nabla_\eta \xi := \mathbf{g}^{-1}(\nabla^\flat_\eta \xi, \ .\ )$ belongs to $L^2_{\mathrm{loc}}\mathcal{T}^1_0(M)$. To show that ∇ is the unique connection satisfying $(\nabla_1) - (\nabla_5)$ one proceeds exactly as in the smooth case, by plugging appropriate combinations of vectorfields in the Koszul formula, cf. propositions 2.1.6 and 2.2.6. □

By 2.4.3 and 2.4.4, the existence of the Levi-Civita L^2_{loc}-connection implies the existence of the corresponding Riemann and Ricci curvature tensors, but as already mentioned at the beginning of this section the scalar curvature is defined as well. In order to define scalar curvature we will generalize equation (2.21) to distributional setting, which requires some care since it is not a priori clear what the product of $\mathbf{g}^{-1}$ with distribution **Ric** should mean. In what follows we first discuss the definition of multiplication of $\mathbf{g}^{-1}$ with **Ric** in remark 2.5.7 and then collect all the results concerning curvature of gt-regular metrics in theorem 2.5.8.

2.5.7. Remark. To define the product of $\mathbf{g}^{-1}$ with Ricci tensor **Ric** observe that in the smooth case

$$\begin{aligned}
&\mathbf{Ric}(E_{(\alpha)}, E_{(\beta)}) \\
&\quad = E^{(\sigma)}\big(\mathbf{Riem}(E_{(\alpha)}, E_{(\sigma)})E_{(\beta)}\big) \\
&\quad = E_{(\alpha)}\big(E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\alpha)})\big) - E_{(\sigma)}\big(E^{(\sigma)}(\nabla_{E_{(\alpha)}}E_{(\beta)})\big) \\
&\quad - \nabla_{E_{(\alpha)}}E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\beta)}) + \nabla_{E_{(\sigma)}}E^{(\sigma)}(\nabla_{E_{(\alpha)}}E_{(\beta)}) - E^{(\sigma)}(\nabla_{[E_{(\alpha)},E_{(\sigma)}]}E_{(\beta)})
\end{aligned}$$

Now in our case the product of $\mathbf{g}^{\alpha\beta}$ with the last three terms is well defined since all of these terms belong to L^1_{loc} and $\mathbf{g}^{\alpha\beta} \in H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ according to lemma 2.5.5. However, the first two terms are in general only distributions so the product has to be defined by the following trick: we mimic the Leibnitz rule to define

$$\begin{aligned}
&g^{\alpha\beta}E_{(\alpha)}\big(E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\beta)})\big) := \\
&\qquad = E_{(\alpha)}\big(g^{\alpha\beta}E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\beta)})\big) - (E_{(\alpha)}(g^{\alpha\beta}))E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\beta)})
\end{aligned}$$

and analogously for the second term. Observe that $g^{\alpha\beta}E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\beta)})$ is in $H^1_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ hence its derivative is a distribution, while $E_{(\alpha)}(g^{\alpha\beta})$ is an L^2_{loc}-function which is multiplied with another L^2_{loc}-function. The above expression is thus well-defined as a distribution. If the terms of the form $E^{(\sigma)}(\nabla_{E_{(\sigma)}}E_{(\alpha)})$ and the components of $\mathbf{g}^{-1}$ are sufficiently regular, this definition coincides with the usual pointwise product of measurable functions.

2.5.8. Theorem. *Let **g** be a non-degenerate gt-regular metric on M. Then:*

1. *The Riemann and Ricci curvature associated with the Levi-Civita connection of the metric **g** are well defined as distributions on M.*
2. *The scalar curvature is a well-defined distribution on M given locally by*

$$R := g^{\alpha\beta}\boldsymbol{Ric}(E_{(\alpha)}, E_{(\beta)}) \tag{2.30}$$

where $(E_{(\alpha)})_{\alpha=1,\dots,n}$ is a local frame on M.

Proof. 1. Having shown that ∇ is an L^2_{loc}-connection, proposition 2.4.2 implies that the distributional Riemman and Ricci curvature can be defined as in 2.4.3 and 2.4.4.

2. It follows from remark 2.5.7, that scalar curvature defined in (2.30) really is a distribution. □

Even tough the class of non-degenerate gt-regular metrics enables a formulation of curvature tensors as distributional tensorfields, it is too limited to describe arbitrary concentration of gravitating sources on M.

More precisely, a gt-regular metric **g** can only be assigned a stress-energy tensor supported on a submanifold of codimension at most 1, as was first shown in [GT87]. In particular gt-regular metrics cannot be used to describe point particles or strings (which are supported on 1- respectively 2-dimensional submanifolds of M). As we shall see in a moment, this is a consequence of the form of the Riemannian curvature tensor associated with the metric **g**, which is a sum of a locally integrable tensorfield and a derivation of the locally square-integrable tensorfield, cf. 2.5.6 resp. equation (2.25).

2.5.9. Theorem. *Let X be a submanifold of the n-dimensional manifold M with dim$(X) = d \leq n$. Let $0 \neq t \in \mathcal{D}'\mathcal{T}^r_s(M)$ be supported in X and of the form*

$$t = t_1 + L_\xi t_2$$

where $t_1 \in L^1_{\text{loc}}\mathcal{T}^r_s(M)$, $t_2 \in L^2_{\text{loc}}\mathcal{T}^r_s(M)$ and $\xi \in \mathcal{T}^1_0(M)$. Then $d = n - 1$.

Proof. We assume w.l.o.g that t is a scalar distribution. Moreover, since we have to evaluate $\langle t, \tau\rangle$ for some $\tau \in \Omega^n_c(M)$, we assume w.l.o.g. that the supp(τ) is contained in the domain of some chart. We therefore may work entirely in some open subset $U \subseteq \mathbb{R}^n$ centered around the origin.

So let $u = u_1 + \partial(u_2)$, where $u_1 \in L^1_{\text{loc}}(U)$, $u_2 \in L^2_{\text{loc}}(U)$ and ∂ denoting some partial derivative. Let V be the intersection of U with some subspace of $\mathbb{R}^n$ of dimension d. Denote by V_ϵ an ϵ-neighbourhood of V in U, i.e. $V_\epsilon = \{p \in U \mid \inf_{x\in V}\|p - x\| < \epsilon\}$. Now choose a smooth function f_ϵ that is identically equal to 1 on $U\backslash V_\epsilon$ but vanishes on a smaller neighborhood of V. Moreover we demand $|\text{grad}(f_\epsilon) \leq c/\epsilon|$ for some constant $c > 0$.

Observe that by integration by parts, we have

$$\langle u, \varphi\rangle = \int_U u_1\varphi + u_2\partial\varphi.$$

for all $\varphi \in \Omega^n_c(U)$, so by the properties of f_ϵ it follows

$$\begin{aligned} |\int_U (u_1\varphi - u_2\partial(\varphi))f_\epsilon| &= |\langle u, f_\epsilon\varphi\rangle + \int_U f\partial(f_\epsilon)\varphi| = |\int_{V_\epsilon} f\partial(f_\epsilon)\varphi| \\ &\leq \Big(\int_{V_\epsilon} |\varphi|\;|u_2|^2\Big)^{1/2}\Big(\int_{V_\epsilon} |\varphi|\|\text{grad} f_\epsilon\|^2\Big)^{1/2} \\ &\leq C\Big(\int_{V_\epsilon} |\varphi|\;|u_2|^2\Big)^{1/2}\Big(\frac{1}{\epsilon^2}\int_{V_\epsilon\cap K} dx\Big)^{1/2} \\ &\leq C'\epsilon^{(n-d-1)/2}\Big(\int_{V_\epsilon} |\varphi|\;|u_2|^2\Big)^{1/2} \qquad \textbf{(*)} \end{aligned}$$

where in the first step $\langle u, f_\epsilon\varphi\rangle$ vanishes since u is supported on V. The first inequality is a consequence of the Cauchy-Schwarz inequality, whereas the last one follows from the fact that the integral over $V_\epsilon \cap K$ can easily be seen to amount to a multiple of ϵ^{n-d}.

Now the left hand side of (*) converges to $\langle u, \varphi\rangle$ while the integral on the right hand side approaches zero as $\epsilon \to 0$. Therefore $\epsilon^{(n-d-1)/2}$ has to be unbounded which is only the case if $d = n - 1$. ☐

3 Jump formulas for distributional curvature

As we have seen in the previous section the curvature tensor arising from a gt-regular metric or more generally from an L^2_{loc}-connection can have its support concentrated only on hypersurfaces. Of course the support of a general curvature tensor does not have to be concentrated on any submanifold at all, however in this section we will show, following [LM07], that under certain additional assumptions on the metric resp. on the connection the curvature tensor can be written as a sum of a 'regular' part belonging to some Lebesgue space and a 'singular' part supported on a hypersurface.

Space-times containing such singularities in the curvature have been used to model impulsive gravitational waves. For instance, in an empty space-time a curvature discontinuity can be interpreted as a gravitational schock wave, whereas a delta function in the curvature may be interpreted as the impulsive gravitational wave front, cf. [Pen72].

We remark that jump formulas along hypersurfaces have already been discussed among others by Lichnerovicz, see e.g. [Lic79], Choquet-Bruhat [CB93] and Mars and Senovilla [MS93].

3.1 Preliminaries

First we fix some notation and assumptions on the manifold M and the connection ∇ which will hold throughout chapter 3. Unless explicitely stated otherwise, we will always assume these conditions whenever referring to either the manifold or the connection.

3.1.1. Assumptions on M. Henceforth we assume that we are given an n-dimensional manifold M which is a union of two manifolds with boundary M^+ and M^-. Both M^+ and M^- are of the same dimension as M and are assumed to be embedded in M in the sense of 1.1.7. In addition, the boundaries of M^+ and M^- are pointwise identified with the resulting hypersurface denoted by X. The orientation on X is taken with respect to M^-. In other words:

$$M = M^+ \cup M^-$$

$$M^+ \cap M^- = X$$

$$\partial M^+ = \partial M^- = X.$$

For a given distributional tensorfield A on M, we will denote by $A^\pm$ restrictions of A to the interiors int($M^\pm$) of $M^\pm$. Note that int($M^\pm$) are open in M. If A is sufficiently regular for traces to be defined, we will denote the difference of traces of A from M^+ and M^- to X

by $[A]_X$. In other words

$$[A]_X := \gamma(A^+) - \gamma(A^-),$$

where γ is the trace operator defined in 1.4.9 resp. 1.4.13 (cf. remark 1.4.14(ii)). $[A]_X$ will be called the jump of A across X. On the other hand, we will call

$$A^{\mathrm{reg}} := \begin{cases} A^+ & \text{in} \quad \mathrm{int}(M^+) \\ A^- & \text{in} \quad \mathrm{int}(M^-) \end{cases}$$

the 'regular' part of A. If $[A]_X$ does not vanish, A is said to suffer a jump discontinuity across X.

We will be mainly interested in fields $A \in L^1_{\mathrm{loc}}\mathcal{T}^r_s(M)$, with the additional regularity requirement $A^\pm \in W^{1,p}_{\mathrm{loc}}\mathcal{T}^r_s(M^\pm)$ for some $p \geq 1$. By virtue of the trace theorem 1.4.13, the latter implies that both $\gamma(A^+)$ and $\gamma(A^-)$ belong to $W^{1,1-1/p}_{\mathrm{loc}}\mathcal{T}^r_s(X)$. We therefore have

$$[A]_X \in W^{1,1-1/p}_{\mathrm{loc}}\mathcal{T}^r_s(X)$$

$$A^{\mathrm{reg}} \in L^1_{\mathrm{loc}}\mathcal{T}^r_s(M)$$

where values of A^{reg} on X do not need to be specified. In order to simplify notation we will be, most of the time, omitting both the superscripts $^\pm$ and γ when denoting restrictions to $M^\pm$ and X respectively.

3.1.2. Assumptions on ∇**.** We will consider distributional connections on M that are more regular ("smooth") when restricted to $M^\pm$ but "discontinuous" across X. More precisely, we will assume that M is endowed with an L^2_{loc}-connection ∇ for which

$$(\nabla_\eta \xi)^\pm \in W^{1,p}_{\mathrm{loc}}\mathcal{T}^1_0(M^\pm) \quad \forall \xi, \eta \in \mathcal{T}^1_0(M)$$

for some $p \geq 1$ holds. In particular,

$$\gamma((\nabla_\eta \xi)^\pm) \in W^{1,1-1/p}_{\mathrm{loc}}\mathcal{T}^1_0(X)$$

and so we may define $[\nabla_\eta \xi]_X$ which we will assume not to be identically vanishing. In other words, ∇ will be assumed to possess a jump discontinuity across X.

In what follows we will abbreviate these regularity condition on ∇ by saying that ∇ is an $L^2_{\mathrm{loc}}(M) \cap W^{1,p}_{\mathrm{loc}}(M^\pm)$-connection. Note that these assumptions on ∇ imply that both M^+ and M^- are endowed with a well-defined, unique $L^2_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$-connection $\nabla^\pm$ which satisfies

$$\nabla^\pm_{\eta^\pm} \xi^\pm = (\nabla_\eta \xi)^\pm \quad \forall \xi^\pm, \eta^\pm \in \mathcal{T}^1_0(M^\pm) \tag{3.1}$$

for arbitrary extensions ξ and η of $\xi^\pm$ resp. $\eta^\pm$ to M.

Having fixed the assumptions on the connection we now address the question of curvature. Applying the results of section 2.4 to the connections ∇ and $\nabla^\pm$ we obtain the distributional Riemann and Ricci curvature **Riem** and **Ric** on M, as well as **Riem**$^\pm$ and

$\mathbf{Ric}^{\pm}$ on $M^{\pm}$ for which obviously

$$\mathbf{Riem}^{\pm}(\xi^{\pm},\eta^{\pm})\zeta^{\pm} = (\mathbf{Riem}(\xi,\eta)\zeta)^{\pm}$$

$$\mathbf{Ric}^{\pm}(\xi^{\pm},\eta^{\pm}) = (\mathbf{Ric}(\xi,\eta))^{\pm}.$$

holds, if ξ,η and ζ denote respective arbitrary extensions of $\xi^{\pm},\eta^{\pm}$ and $\zeta^{\pm}$ to M (cf. equation (3.1)). Comparing the assumptions on $\nabla^{\pm}$ with the assumptions needed to obtain distributional curvature tensors of a given distributional connection (cf. section 2.4), it is immediately clear that $\nabla^{\pm}$, as $L^2_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$-connections, possess some additional regularity. This additional regularity can be expected to imply some additional regularity of $\mathbf{Riem}^{\pm}$ and $\mathbf{Ric}^{\pm}$ as well. Indeed:

3.1.3. Lemma. *Let $p \geq n/2$. Then the product of two functions in $W^{1,p}_{\mathrm{loc}}(M)$ belongs to $L^p_{\mathrm{loc}}(M)$, i.e.*

$$(W^{1,p}_{\mathrm{loc}}(M))^2 \subseteq L^p_{\mathrm{loc}}(M).$$

Proof. Since this is a local question, it suffices to prove the statement for open subsets of $\mathbb{R}^n$. To simplify the arguments we will notationally supress the subsets altogether. In fact, by the Sobolev embedding theorem 1.2.6 we have

$$W^{1,p}_{\mathrm{loc}} \subseteq L^q_{\mathrm{loc}} \text{ for } p \leq q < \infty$$

for $p \geq n$, while for $p < n$ we have

$$W^{1,p}_{\mathrm{loc}} \subseteq L^q_{\mathrm{loc}} \text{ for } p \leq q \leq np/(n-p) =: q^*.$$

Moreover, by Hölder's inequality we have $(L^q_{\mathrm{loc}})^2 \subseteq L^{q/2}_{\mathrm{loc}}$, which for $p < n$ implies

$$(L^{q^*}_{\mathrm{loc}})^2 \subseteq L^{q^*/2}_{\mathrm{loc}} = L^{np/2(n-p)}_{\mathrm{loc}} \subseteq L^p_{\mathrm{loc}},$$

if

$$np/2(n-p) \geq p \Leftrightarrow n \geq 2(n-p) \Leftrightarrow p \geq n/2.$$

For $p \geq n$ just take $q = 2p$ and use Hölder inequality. Summing up we obtain $(W^{1,p}_{\mathrm{loc}})^2 \subseteq L^p_{\mathrm{loc}}$ provided $p \geq n/2$. $\square$

3.1.4. Lemma. *Let $\nabla^{\pm}$ be $L^2_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$-connections on $M^{\pm}$ for some $p \geq 1$. Then the Riemann and Ricci curvature tensors **Riem**$^{\pm}$ and **Ric**$^{\pm}$ belong to $L^1_{\mathrm{loc}}(M^{\pm})$. If, in addition, $p \geq n/2$, Riemann and Ricci curvature tensors even belong to $L^p_{\mathrm{loc}}(M^{\pm})$.*

Proof. Let $\eta,\zeta,\xi \in \mathcal{T}^1_0(M^{\pm})$. The Riemann curvature on $M^{\pm}$ is given by

$$\begin{aligned}(\mathbf{Riem}^{\pm}(\eta,\xi)\zeta)(\omega) = &\underbrace{\xi\big(\nabla^{\pm}_{\eta}\zeta(\omega)\big) - \eta\big(\nabla^{\pm}_{\xi}\zeta(\omega)\big)}_{\in L^p_{\mathrm{loc}} \subseteq L^1_{\mathrm{loc}}} \\ &\underbrace{-\nabla^{\pm}_{\eta}\zeta\big(\nabla^{\pm}_{\xi}\omega\big) + \nabla^{\pm}_{\xi}\zeta\big(\nabla^{\pm}_{\eta}\omega\big)}_{\in L^2_{\mathrm{loc}} \times L^2_{\mathrm{loc}} \subseteq L^1_{\mathrm{loc}}} + \nabla^{\pm}_{[\eta,\xi]}\zeta(\omega),\end{aligned}$$

see equation (2.25). The last term in the above expression has the same regularity as $\nabla^{\pm}$, so obviously **Riem**$^{\pm}$ belongs to $L^1_{\text{loc}}\mathcal{T}^1_3(M^{\pm})$. The same holds for **Ric**$^{\pm}$ since it is just a contraction of the Riemann tensor **Riem**$^{\pm}$, cf. 2.4.4.

To show the second claim, it is sufficient to show that the product of two functions in $L^2_{\text{loc}}(M^{\pm}) \cap W^{1,p}_{\text{loc}}(M^{\pm})$ actually belongs to $L^p_{\text{loc}}(M^{\pm})$ provided $p \geq n/2$, however by lemma 3.1.3 we even have $(W^{1,p}_{\text{loc}}(M^{\pm}))^2 \subseteq L^p_{\text{loc}}(M^{\pm})$. □

3.2 Jump formulas: the case of a singular connection

In the situation where the connection ∇ suffers a jump discontinuity across X, it can be expected that the curvature tensors, which involve an additional derivative, will be singular on X, i.e. will contain a delta distribution. To describe this behaviour and to make the notion of delta distribution more precise we introduce the following 1-form distribution δ which has its support concentrated on the hypersurface X:

3.2.1. Definition. *The delta distribution δ_X supported on the hypersurface X of M is the distributional 1-form on M defined as*

$$\delta_X(\xi)(\omega) := \int_X i_\xi \omega|_X$$

for $\xi \in \mathcal{T}^1_0(M)$ and $\omega \in \Omega^n_c(M)$.

Since $i_\xi\omega|_X = i_{\xi|_X}\omega|_X$, δ depends only on the values of ξ on X. Furthermore if $\xi|_X \in TX$ then $\delta_X(\xi)$ vanishes, since $i_\xi\omega|_X$ vanishes as an (n-1)-form on X.

To derive the expression for the curvature adapted to the form of the connection we need to study terms of the form $\nabla_\xi\nabla_\eta\zeta$, which in turn implies we need to study the behaviour of the connection on $L^2_{\text{loc}} \cap W^{1,p}_{\text{loc}}$-vectorfields suffering a jump discontinuity on X itself. To do this we prove the following preparatory lemma:

3.2.2. Lemma. *Let M' be an open subset of M with smooth boundary $\partial M'$ and let $p < \infty$. Then for all $u \in W^{m,p}_{\text{loc}}(M)$, $\theta \in \Omega^n_c(M)$ and $\xi \in \mathcal{T}^1_0(M)$ we have*

$$\int_{M'} \xi(u)\theta = \int_{\partial M'} \gamma(u) i_\xi\theta - \int_{M'} u\xi(\theta). \tag{3.2}$$

Proof. To begin with observe that by the Stoke's formula equation (3.2) holds if $u \in C^\infty(M)$. Indeed, for $u \in C^\infty(M)$ and θ, ξ as above, we have

$$\xi(u)\theta = \xi(u\theta) - u\xi(\theta) = d(i_\xi(u\theta)) - u\xi(\theta),$$

where we have used $\xi = di_\xi + i_\xi d$ and $d\theta = 0$. We thus obtain

$$\int_{\partial M'} u i_\xi\theta = \int_{M'} d(i_\xi(u\theta)) = \int_{M'} u\xi(\theta) + \int_{M'} \xi(u)\theta. \qquad (*)$$

To deal with the case $u \in W^{m,p}_{\mathrm{loc}}(M)$ recall that by lemma 1.4.3, u can be approximated by some sequence $f_n \in \mathcal{D}(M)$ in $W^{m,p}_{\mathrm{loc}}(M)$. By continuity, $(*)$ then holds for u as well, where, according to theorem 1.4.4, we have replaced u by $\gamma(u)$ in the first integral. $\square$

In what follows we will use the following shorthand notation for the restrictions of the connection and the curvature to $M^{\pm}$:

$$\nabla^{\pm}_{\eta}\xi := \nabla^{\pm}_{\eta^{\pm}}\xi^{\pm}$$

$$\mathbf{Riem}^{\pm}(\xi,\eta)\zeta := \mathbf{Riem}^{\pm}(\xi^{\pm},\eta^{\pm})\zeta^{\pm}$$

$$\mathbf{Ric}^{\pm}(\xi,\eta) := \mathbf{Ric}^{\pm}(\xi^{\pm},\eta^{\pm})$$

where ξ,η and ζ belong to $\mathcal{T}^1_0(M)$ and $\xi^{\pm},\eta^{\pm},\zeta^{\pm}$ denote respective restrictions to $M^{\pm}$. A similar notation will be used for all tensorfields on M.

3.2.3. Proposition. *Let ∇ be an $L^2_{\mathrm{loc}}(M) \cap W^{1,p}_{\mathrm{loc}}(M^{\pm})$-connection with $p \geq 1$. Then the extension of ∇ to*

$$\nabla : \mathcal{T}^1_0(M) \times L^2_{\mathrm{loc}}\mathcal{T}^1_0(M) \to \mathcal{D}'(M),$$

defined via (2.23), has the form

$$\nabla_\xi V := (\nabla_\xi V)^{\mathrm{reg}} + [V]_X \delta_X(\xi) \qquad \forall \xi \in \mathcal{T}^1_0(M) \tag{3.3}$$

for $V \in L^2_{\mathrm{loc}}\mathcal{T}^1_0(M)$ whose restrictions $V^{\pm}$ to $M^{\pm}$ belong to $W^{m,p}_{\mathrm{loc}}\mathcal{T}^1_0(M^{\pm})$.

3.2.4. Remark. The action of the distributional vectorfield $[V]_X\delta_X(\xi)$ on $\omega \in \mathcal{T}^0_1(M)$ is defined as

$$\big([V]_X\delta_X(\xi)\big)\big(\omega\big) := \Big(\theta \longmapsto \int_X [V]_X\big(\omega|_X\big)\ i_\xi\theta\Big). \tag{3.4}$$

To simplify notation we will sometimes write ω instead of $\omega|_X$ in the above integral. Recall that under the current assumptions $[V]_X$ belongs to $W^{1-1/p,p}_{\mathrm{loc}}\mathcal{T}^1_0(X)$, see theorem 1.4.13. Note that for any ξ, the distributional one-form $[V]_X\delta_X(\xi)$ vanishes iff $[V]_X$ vanishes.

Observe that for every fixed ω this vectorfield distribution induces a scalar distribution $[V(\omega)]_X\delta_X(\xi)$ which is given by

$$\langle [V(\omega)]_X\delta_X(\xi), \theta\rangle = \int_X [V]_X(\omega) i_\xi\theta = \int_X [V(\omega)]_X i_\xi\theta.$$

Clearly, $[V(\omega)]_X$ belongs to $W^{1-1/p,p}_{\mathrm{loc}}(X)$.

Proof of the proposition. By prop. 2.4.2, the extension of ∇ to $V \in L^2_{\mathrm{loc}}\mathcal{T}^1_0(M)$ is given by (2.23) i.e.

$$(\nabla_\xi V)\big(\omega\big) = \xi(V\big(\omega\big)) - V(\nabla_\xi\omega) \tag{3.5}$$

where ω is any smooth one-form. The first term is the distributional Lie derivative along ξ

which applied to $\theta \in \Omega_c^n(M)$ reads:

$$\begin{aligned}
&\langle \xi(V(\omega)), \theta\rangle = \\
&= -\langle V(\omega), \xi(\theta)\rangle = -\int_M V(\omega)\xi(\theta) \\
&= -\int_{M^+} \underbrace{V^+(\omega)}_{W^{1,p}_{\text{loc}}(M^+)} \xi(\theta) - \int_{M^-} \underbrace{V^-(\omega)}_{W^{1,p}_{\text{loc}}(M^-)} \xi(\theta) \\
&= \int_{M^+} \xi(V^+(\omega))\theta + \int_X V^+(\omega) i_X\theta + \int_{M^-} \xi(V^-(\omega))\theta - \int_X V^-(\omega) i_X\theta
\end{aligned}$$

where in the last step lemma 3.2.2 was used. Inserting the above expression for $\xi(V(\omega))$ in (3.5) and observing

$$V(\nabla_\xi \omega) = V^+(\nabla^+_\xi \omega) + V^-(\nabla^-_\xi \omega)$$

implies

$$\begin{aligned}
&\langle \nabla_\xi V(\omega), \theta\rangle = \\
&= \int_{M^+} \underbrace{(\ \xi(V^+(\omega)) - V^+(\nabla^+_\xi \omega)\ )}_{=(\nabla_\xi V)^+(\omega)} \theta \\
&+ \int_{M^-} \underbrace{(\ \xi(V^-(\omega)) - V^+(\nabla^-_\xi \omega)\ )}_{=(\nabla_\xi V)^-(\omega)} \theta + \int_X [V]_X(\omega) i_X\theta \\
&= \int_{M^+} (\nabla_\xi V)^+(\omega)\theta + \int_{M^-} (\nabla_\xi V)^-(\omega)\theta + \int_X \langle [V]_X(\omega) i_X\theta \\
&= \langle (\nabla_\xi V)^{\text{reg}}(\omega), \theta\rangle + \langle ([V]_X \delta_X(\xi))(\omega), \theta\rangle,
\end{aligned}$$

hence the claim holds. □

We call a frame $(E_{(\alpha)})_{\alpha=1,...,n}$ of M adapted to the hypersurface X if, for every point x in X, $(E_{(\alpha)}(x))_{\alpha=1,...,n-1}$ is a base of T_xX. We employ the following notation: A sum involving only the first $n-1$ frame fields will be indicated by a latin index e.g. $\nabla_{E_{(i)}} E_{(i)}$ means summation over $i = 1, ..., n-1$.

Inserting the expression for ∇ derived in proposition 3.2.3 into equations (2.24) and (2.27) we obtain a decomposition of the curvature tensors in a regular part and a singular part involving delta distributions. More precisely:

3.2.5. Theorem. *Let ∇ be an $L^2_{\text{loc}}(M) \cap W^{1,p}_{\text{loc}}(M^{\pm})$-connection with $p \geq 1$.*

(i) The distributional Riemann curvature defined in 2.4.3 takes the form

$$\boldsymbol{Riem}(\xi, \eta)\zeta = (\boldsymbol{Riem}(\xi, \eta)\zeta)^{\text{reg}} + [\nabla_\eta \zeta]_X \delta_X(\xi) - [\nabla_\xi \zeta]_X \delta_X(\eta)$$

for every $\eta, \xi, \zeta \in \mathcal{T}^1_0(M)$.

(ii) The distributional Ricci curvature defined in 2.4.4 takes the form

$$\begin{aligned}\boldsymbol{Ric}(\xi,\eta) =& (\boldsymbol{Ric}(\xi,\eta))^{\mathrm{reg}}+\\ &+[E^{(\alpha)}(\nabla_{E_{(\alpha)}}\eta)]_X\delta_X(\xi)-[E^{(n)}(\nabla_\xi\eta)]_X\delta_X(E_{(n)})\end{aligned}$$

for every $\xi,\eta\in\mathcal{T}_0^1(M)$ and every frame $(E_{(\alpha)})_{\alpha=1,\dots,n}$ of M adapted to the hypersurface X.

Here the regular parts $(\boldsymbol{Riem}(\xi,\eta)\zeta)^{\mathrm{reg}}$ and $(\boldsymbol{Ric}(\xi,\eta))^{\mathrm{reg}}$ of the Riemann resp. Ricci curvature tensor belong to $L^1_{\mathrm{loc}}(M)$, while the coefficients of the singular parts, e.g. the jumps $[\nabla_\eta\zeta]_X$, $[\nabla_\xi\zeta]_X$ of the Riemann tensor, belong to $W^{1-1/p,p}_{\mathrm{loc}}(X)$. Provided $p\geq n/2$, the regular parts of the curvature tensors belong to $L^p_{\mathrm{loc}}(M)$.

Proof. (i) According to definition 2.4.3, the distributional Riemann curvature is given by the formula

$$\mathbf{Riem}(\xi,\eta)\zeta=\nabla_\xi\nabla_\eta\zeta-\nabla_\eta\nabla_\xi\zeta-\nabla_{[\xi,\eta]}\zeta.$$

By the assumptions on ∇, the first order covariant derivatives $\nabla_\eta\zeta$ and $\nabla_\xi\zeta$ belong to $L^2_{\mathrm{loc}}\mathcal{T}_0^1(M)$ hence we may apply proposition 3.2.3 (note that at this point [LM07] is imprecise) to obtain the respective expressions of the second order covariant derivatives. Hence we obtain

$$\begin{aligned}&\mathbf{Riem}(\xi,\eta)\zeta=\\ &=(\nabla_\xi\nabla_\eta\zeta)^{\mathrm{reg}}+[\nabla_\eta\zeta]_X\delta_X(\xi)-(\nabla_\eta\nabla_\xi\zeta)^{\mathrm{reg}}-[\nabla_\xi\zeta]_X\delta_X(\eta)-\nabla_{[\xi,\eta]}\zeta\\ &=(\mathbf{Riem}(\xi,\eta)\zeta)^{\mathrm{reg}}+[\nabla_\eta\zeta]_X\delta_X(\xi)-[\nabla_\xi\zeta]_X\delta_X(\eta)\end{aligned}$$

for all $\eta,\xi\in\mathcal{T}_0^1(M)$. This proves (i).

(ii) By lemma 2.4.5 we may insert the above form of the Riemann curvature into

$$\mathbf{Ric}(\xi,\eta):=E^{(\alpha)}(\mathbf{Riem}(\xi,E_{(\alpha)})\eta),$$

to obtain

$$\begin{aligned}&\mathbf{Ric}(\xi,\eta)=\\ &=E^{(\alpha)}((\mathbf{Riem}(\xi,E_{(\alpha)})\eta)^{\mathrm{reg}})+E^{(\alpha)}([\nabla_{E_{(\alpha)}}\eta]_X\delta_X(\xi))-E^{(\alpha)}([\nabla_\xi\eta]_X\delta_X(E_{(\alpha)}))\\ &=(\mathbf{Ric}(\xi,\eta))^{\mathrm{reg}}+[(\nabla_{E_{(\alpha)}}\eta)\big(E^{(\alpha)}\big)]_X\delta_X(\xi)-[(\nabla_\xi\eta)\big(E^{(\alpha)}\big)]_X\delta_X(E_{(\alpha)}).\end{aligned}$$

However, $\delta_X(E_{(\alpha)})=0$ for $\alpha=1,..,n-1$ since $(E_{(\alpha)})_{\alpha=1,..,n-1}$ are all tangent to the hypersurface X, which gives the result.

The assertion regarding the regularity of the regular parts follow from lemma 3.1.4, the one of the jumps from remark 3.2.4. □

3.2.6. Remark. Note that by the properties of the delta distribution and the trace itself, the singular part of the Riemann curvature actually takes the form

$$(\mathbf{Riem}(\xi,\eta)\zeta)^{\text{sing}} = [\nabla_{\eta_1}\zeta]_X\delta_X(\xi_2) - [\nabla_{\xi_1}\zeta]_X\delta_X(\eta_2) \tag{3.6}$$

where we have locally decomposed ξ and η as sums of two vectorfields; one tangential and one normal to the hypersurface X. More precisely, $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$ where ξ_1 and η_1 are tangential, whereas ξ_2 and η_2 are normal to X. This is locally always possible by the existence of local frames adapted to the hypersurface X. In fact, we have

$$\begin{aligned}[\nabla_\xi\zeta]_X\delta_X(\eta) &= \underbrace{[\nabla_\xi\zeta]_X\delta_X(\eta_1)}_{=0} + [\nabla_\xi\zeta]_X\delta_X(\eta_2)\\ &= [\nabla_{\xi_1}\zeta]_X\delta_X(\eta_2) + [\nabla_{\xi_2}\zeta]_X\delta_X(\eta_2).\end{aligned}$$

In some local frame adapted to the hypersurface, ξ_2 and η_2 can be written as $\xi_2 = fE_{(n)}$ resp. $\eta_2 = gE_{(n)}$ for some smooth functions f and g and therefore by remark 1.4.10(i) the second term on the right hand side can be written as

$$[\nabla_{\xi_2}\zeta]_X\delta_X(\eta_2) = fg[\nabla_{E_{(n)}}\zeta]_X\delta_X(E_{(n)}),$$

which is obviously symmetric in η and ξ. Hence

$$[\nabla_\eta\zeta]_X\delta_X(\xi) - [\nabla_\xi\zeta]_X\delta_X(\eta) = [\nabla_{\eta_1}\zeta]_X\delta_X(\xi_2) - [\nabla_{\xi_1}\zeta]_X\delta_X(\eta_2)$$

which proves (3.6).

We end this section by studying the conditions which make the singular part of the curvature tensors vanish, thereby setting straight a lapse in [LM07].

3.2.7. Corollary.

(i) The singular part of the Riemann curvature vanishes if and only if $[\nabla_\xi\zeta]_X = 0$ *for all* $\xi,\zeta \in \mathcal{T}^1_0(M)$ *such that* $\xi|_X \in TX$.

(ii) The singular part of the Ricci curvature vanishes if and only if $[(\nabla_{E_{(i)}}\eta)(E^{(i)})]_X = 0$ *and* $[(\nabla_\xi\eta)(E^{(n)})]_X = 0$ *for all* $\xi,\eta \in \mathcal{T}^1_0(M)$ *such that* $\xi|_X \in TX$.

Proof. (i) Since the space of distributional vectorfields forms a sheaf, cf. paragraph 1.3.3, it suffices to show that the singular part of the Riemann tensor vanishes locally. By remark 3.2.6, this is equivalent to showing

$$[\nabla_{\eta_1}\zeta]_X\delta_X(\xi_2) - [\nabla_{\xi_1}\zeta]_X\delta_X(\eta_2) = 0 \tag{3.7}$$

for all $\xi,\eta,\zeta \in \mathcal{T}^1_0(M)$, where we locally have $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$. Here ξ_1 and η_1 denote the tangential part of ξ resp. η, whereas ξ_2 and η_2 denote the respective normal parts.

Now let us first assume (3.7) holds. Then for every $\xi \in \mathcal{T}^1_0(M)$ tangential to the hypersurface, i.e. $\xi|_X \in TX$, the first term in (3.7) vanishes. Hence we obtain

$$[\nabla_\xi \zeta]_X = 0$$

for all $\zeta \in \mathcal{T}^1_0(M)$ as desired.

Conversely, assume $[\nabla_\xi \zeta]_X = 0$ for all $\xi, \zeta \in \mathcal{T}^1_0(M)$ such that $\xi|_X \in TX$. Then the second term in equation (3.7) vanishes, whereas the first one vanishes since ξ is tangential to X i.e. $\xi_2 = 0$.

(ii) First assume that for every $\xi, \eta \in \mathcal{T}^1_0(M)$ the singular part of the Ricci curvature vanishes i.e.

$$\underbrace{[(\nabla_{E_{(i)}}\eta)(E^{(i)})]_X \delta_X(\xi)}_{(*)} + \underbrace{[(\nabla_{E_{(n)}}\eta)(E^{(n)})]_X \delta_X(\xi) - [(\nabla_\xi \eta)(E^{(n)})]_X \delta_X(E_{(n)})}_{(**)} = 0 \qquad (3.8)$$

For $\xi \in \mathcal{T}^1_0(M)$ such that $\xi|_X \in TX$ the first two terms in (3.8) vanish, implying

$$[(\nabla_\xi \eta)(E^{(n)})]_X = 0$$

for all $\eta \in \mathcal{T}^1_0(M)$. On the other hand, setting $\xi = E_{(n)}$ implies $(**) = 0$, which in turn implies

$$[(\nabla_{E_{(i)}}\eta)(E^{(i)})]_X = 0 \quad \forall \eta \in \mathcal{T}^1_0(M).$$

Assume conversely $[E^{(i)}(\nabla_{E_{(i)}}\eta)]_X = 0$ and $[E^{(n)}(\nabla_\xi \eta)]_X = 0$ for every $\eta, \xi \in \mathcal{T}^1_0(M)$ such that $\xi|_X \in TX$. The first assumption implies the vanishing of (*), whereas the second one implies vanishing of (**) by an argument similar to the one in the proof of (i). $\square$

3.3 Jump formulas: the case of a singular metric

In this section, we start with a metric, that is 'smooth' on $M^\pm$ but behaves badly across X. We define the corresponding Levi-Civita connection and curvature tensor, and discuss jump formulas in terms of the derivatives of the metric.

3.3.1. Assumptions on g. Throughout this section we assume **g** is a continuous pointwise non-degenerate metric on M, which is of class $W^{2,p}_{\mathrm{loc}}(M^\pm)$ when restricted to $M^\pm$ for some $p \geq n/2$. Moreover, we assume that the derivatives of **g** suffer a jump discontinuity across X.

We will abbreviate these assumptions on **g**, by saying that **g** is a $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$-metric.

3.3.2. Lemma. *Let $p \geq n/2$. If **g** is a $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$-metric, then $\mathbf{g}^{-1}$ belongs to $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$ as well.*

Proof. By pointwise non-degeneracy we have $\mathbf{g}^{-1} \in \mathcal{C}\mathcal{T}^2_0(M)$ and also in $L^\infty_{\mathrm{loc}}\mathcal{T}^2_0(M)$. Hence we only need to show that $\mathbf{g}^{-1} \in W^{2,p}_{\mathrm{loc}}\mathcal{T}^2_0(M^\pm)$.

As usual we work locally. Again by pointwise non-degeneracy (since for continuous metrics, pointwise non-degeneracy implies local uniform boundedness from below) and lemma below we have $1/\det(\mathbf{g}), \operatorname{cof}(\mathbf{g})_{ij} \in W^{2,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(M^\pm)$ and hence we have

$$\mathbf{g}^{ij} = \frac{\operatorname{cof}(\mathbf{g})_{ji}}{\det(\mathbf{g})} \in W^{2,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(M^\pm)$$

as well. □

3.3.3. Lemma. *Let $p \geq n/2$. $W^{2,p}_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$ is an algebra and $f \in W^{2,p}_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$ is invertible iff f is locally uniformly bounded from below.*

Proof. As usual we may argue locally. So let $f, g \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Then clearly $f, g \in L^\infty_{\mathrm{loc}}(\Omega)$ and since the Leibnitz rule applies (by the same reasoning as in lemma 2.5.2) we have

$$\partial_i(fg) = \partial_i f g + f \partial_i g \in W^{1,p}_{\mathrm{loc}} \cdot L^\infty_{\mathrm{loc}} \subseteq L^p_{\mathrm{loc}}$$

and

$$\partial_{ij}(fg) = \partial_{ij}(f)g + \partial_i(f)\partial_j(g) + \partial_j(f)\partial_i(g) + f\partial_{ij}(g) \in L^p_{\mathrm{loc}} \cdot L^\infty_{\mathrm{loc}} + W^{1,p}_{\mathrm{loc}} \cdot W^{1,p}_{\mathrm{loc}} \subseteq L^p_{\mathrm{loc}}$$

by lemma 3.1.3.

If f is invertible in $W^{2,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ then we see as in the proof of lemma 2.5.3 that f is locally uniformly bounded from below.

Conversely if f us locally uniformly bounded from below then $1/f \in L^\infty_{\mathrm{loc}} \subseteq L^p_{\mathrm{loc}}$ and by the Leibnitz rule

$$\partial_i(1/f) = -\partial_i f/f^2 \in L^\infty_{\mathrm{loc}} \cdot W^{1,p}_{\mathrm{loc}} \subseteq L^p_{\mathrm{loc}}.$$

Finally we have

$$\partial_{ij}(1/f) = \frac{2f\partial_i f \partial_j f - f^2 \partial_{ij} f}{f^4} \in L^\infty_{\mathrm{loc}} \cdot W^{1,p}_{\mathrm{loc}} \cdot W^{1,p}_{\mathrm{loc}} + L^p_{\mathrm{loc}} \cdot L^\infty_{\mathrm{loc}} \subseteq L^p_{\mathrm{loc}},$$

again by lemma 3.1.3.

□

As a first step towards the definition of the Levi-Civita connection associated with $\mathbf{g}$, we check that a $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$-metric indeed satisfies the assumptions made in section 2.5, i.e. it is a non-degenerate gt-regular metric.

3.3.4. Lemma. *If $\mathbf{g}$ is a $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$-metric and $p \geq n/2$, then $\mathbf{g}$ also belongs to*

$$H^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M).$$

Proof. The $L^\infty_{\mathrm{loc}}(M)$-property is clear. To obtain $\mathbf{g} \in H^1_{\mathrm{loc}}(M)$, observe that we actually need to show $\mathbf{g} \in H^1(K)$ for any compact $K \subseteq M$. Since each K can be written as a union of a compact set in M^+ and a compact set in M^-, i.e. as $K = (K \cap M^+) \cup (K \cap M^-)$, we may take full advantage of the hypothesis $\mathbf{g} \in W^{2,p}_{\mathrm{loc}}(M^\pm)$. Hence, the result follows if we can show that

$$W^{2,p}_{\mathrm{loc}}(\Omega) \subseteq H^1_{\mathrm{loc}}(\Omega)$$

for $p \geq n/2$ and all open $\Omega \subseteq \mathbb{R}^n$.

In case $p \geq 2$ everything is clear, so we only need to consider the cases $2 > p \geq n/2$, i.e. $n = 1, 2, 3$.

In case $n = 1$, we have $1 \leq p < 2$ and we may use the Sobolev imbedding theorem 1.2.6 (1),(2) for $m = 1$ to obtain $W^{1,p}_{\mathrm{loc}}(\Omega) \subseteq L^2_{\mathrm{loc}}(\Omega)$.

In case $n = 2$, we have to consider $1 \leq p < 2$ again, but now we use 1.2.6(3) since $mp = p < 2 = n$. Indeed, $W^{1,p}_{\mathrm{loc}}(\Omega) \subseteq L^2_{\mathrm{loc}}(\Omega)$ since

$$\frac{np}{n - mp} = \frac{2p}{2 - p} \geq 2.$$

Finally, in case $n = 3$ we have to consider $3/2 \leq p < 2$ and we again may use 1.2.6(3) since $mp = p < 2 < n = 3$. From there we again obtain $W^{1,p}_{\mathrm{loc}}(\Omega) \subseteq L^2_{\mathrm{loc}}(\Omega)$ since

$$\frac{np}{n - mp} = \frac{3p}{3 - p} \geq 2.$$

for $p \geq 6/5$ and therefore, in particular for $p \geq 3/2$. $\square$

Since for continuous metrics, the notions of pointwise non-degeneracy and uniform non-degeneracy on compact subsets of M are equivalent, the metric **g** is, by the previous lemma, a non-degenerate gt-regular metric and hence the results of section 2.5 apply, i.e. by theorem 2.5.6, **g** induces an L^2_{loc}-Levi-Civita connection, which in turn enables the definition of the distributional Riemann, Ricci and scalar curvature. However, the Levi-Civita connection of a $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$-metric actually satisifies the assumptions of 3.1.2. More precisely:

3.3.5. Theorem. *Let **g** be a $\mathcal{C}(M) \cap W^{2,p}_{\mathrm{loc}}(M^\pm)$-metric with $p \geq n/2$. Then, the Levi-Civita connection ∇ of **g** is an L^2_{loc}-connection on M, whose restrictions to $M^\pm$ belong to $W^{1,p}_{\mathrm{loc}}(M^\pm)$.*

Proof. By lemma 3.3.4, **g** is gt-regular, hence by theorem 2.5.6 its Levi-Civita connection ∇ is defined and of class $L^2_{\mathrm{loc}}(M)$.

To show $\nabla^\pm \in W^{1,p}_{\mathrm{loc}}(M^\pm)$ we consider the right hand side of the Koszul formula (2.9). Indeed, since the restrictions of **g** to $M^\pm$ belong to $W^{2,p}_{\mathrm{loc}}(M^\pm)$, we have

$$(\nabla^\flat_\xi \eta)^\pm \in W^{1,p}_{\mathrm{loc}}\mathcal{T}^0_1(M^\pm) \quad \forall \xi, \eta \in \mathcal{T}^1_0(M).$$

Since ∇ is obtained by inverting $\nabla^\flat$ via the inverse metric, i.e.

$$\nabla_\xi \eta = \mathbf{g}^{-1}(\nabla^\flat_\xi \eta, \ . \ )$$

the claim follows from the lemma below. Indeed, by lemma 3.3.2, the inverse metric $\mathbf{g}^{-1}$ belongs to $W^{2,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}\mathcal{T}^2_0(M)$ hence its 'product' with $\nabla^\flat \in W^{1,p}_{\mathrm{loc}}\mathcal{T}^0_1(M^\pm)$ yields an element of $W^{1,p}_{\mathrm{loc}}\mathcal{T}^1_0(M^\pm)$. $\square$

3.3.6. Lemma. *Let $p \geq n/2$ and $\Omega \subseteq \mathbb{R}^n$ open. Then $f \in W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and $g \in W^{1,p}_{\text{loc}}(\Omega)$ imply $fg \in W^{1,p}_{\text{loc}}(\Omega)$.*

Proof. First observe that $fg \in L^\infty_{\text{loc}}(\Omega) \cdot W^{1,p}_{\text{loc}}(\Omega) \subseteq L^p_{\text{loc}}(\Omega)$. By the same reasoning as in lemma 2.5.2 one can show that the Leibnitz rule applies and therefore we may write

$$\partial_j(fg) = \partial_j(f)g + f\partial_j(g) \in W^{1,p}_{\text{loc}}(\Omega) \cdot W^{1,p}_{\text{loc}}(\Omega) + L^\infty_{\text{loc}}(\Omega) \cdot L^p_{\text{loc}}(\Omega).$$

which is, by lemma 3.1.3, in $L^p_{\text{loc}}(\Omega)$. □

3.3.7. Theorem. *Let $\boldsymbol{g}$ be a $\mathcal{C}(M) \cap W^{2,p}_{\text{loc}}(M^\pm)$-metric with $p \geq n/2$.*

(i) The Riemann and Ricci curvature associated with ∇ take the form derived in theorem 3.2.5(i)-(ii).

(ii) The scalar curvature associated with ∇, as defined in theorem 2.5.6, takes the form

$$R = R^{\text{reg}} + [(\boldsymbol{g}^{n\beta}E^{(j)} - \boldsymbol{g}^{j\beta}E^m)\big(\nabla_{E_{(j)}}E_{(\beta)}\big)]_X \delta_X(E_{(n)})$$

Moreover, the regular parts of the Riemann, Ricci and scalar curvature belong to $L^p_{\text{loc}}(M)$ whereas the jumps belong to $W^{1-1/p,p}_{\text{loc}}(X)$.

Proof. (i) By theorem 3.3.5, the Levi-Civita connection of **g** satisfies the assumptions of paragraph 3.1.2, hence theorem 3.2.5 applies.

(ii) The existence of the scalar curvature follows from theorem 2.5.6, while its form is obtained by inserting the expression for the Ricci tensor derived in theorem 3.2.5 in equation (2.30).

The regularity assertions for the regular resp. singular parts of the Riemann and Ricci curvature follow from theorem 3.2.5. As for the scalar curvature, its regular part belongs to $L^p_{\text{loc}}(M)$, by remark 2.5.7 and lemma 3.1.4, whereas the jump of R belongs to $W^{1-1/p,p}_{\text{loc}}(X)$ by the trace theorem 1.4.13 (cf.remark 1.4.14(ii)). □

Based on corollary 3.2.7, we now determine the conditions for the singular part of the Riemann curvature to vanish, in terms of the metric and its derivatives. In doing so we will make use of the following remark:

3.3.8. Remark. Observe that for any $\eta, \zeta, \xi \in \mathcal{T}^1_0(M)$ such that $\xi|_X \in TX$

$$[\xi(\mathbf{g}(\eta, \zeta))]_X = 0 \tag{3.9}$$

In fact $[\xi(\mathbf{g}(\eta, \zeta))]_X = \xi([\mathbf{g}(\eta, \zeta)]_X)$ according to remark 1.4.10(i). Since **g** is continuous, $[\mathbf{g}(\eta, \zeta)]_X$ vanishes (cf. remark 1.4.10(ii)), hence the claim holds.

3.3.9. Corollary. *The singular part of the Riemman curvature vanishes if and only if*

$$[\zeta(\boldsymbol{g}(\xi, \eta))]_X = 0 \tag{3.10}$$

for all $\eta, \zeta, \xi \in \mathcal{T}^1_0(M)$ such that $\xi|_X$ and $\eta|_X$ belong to $\mathcal{T}^1_0(X)$.

Proof. According to corollary 3.2.7, the singular part of the Riemann tensor vanishes if and only if $[\nabla_\xi \eta]_X$ vanishes for all smooth vectorfields ξ and η such that $\xi|_X \in \mathcal{T}^1_0(M)$. Since

$$\nabla_\xi \eta = \mathbf{g}^{-1}(\nabla^\flat_\xi \eta,\ .\ )$$

and both $\mathbf{g}$ and $\mathbf{g}^{-1}$ are continuous, by remark 1.4.10(ii) we have from the Koszul formula

$$\begin{aligned}
[\nabla_\xi \eta]_X = 0 \ &\Leftrightarrow\ [\nabla^\flat_\xi \eta]_X = 0 \\
&\Leftrightarrow\ [\xi(\mathbf{g}(\eta,\zeta))]_X + [\eta(\mathbf{g}(\xi,\zeta))]_X - [\zeta(\mathbf{g}(\xi,\eta))]_X = 0,\ \forall \zeta \in \mathcal{T}^1_0(M).
\end{aligned}$$

Assuming first $[\nabla_\xi \eta]_X = 0$ for all smooth vectorfields ξ and η such that $\xi|_X \in \mathcal{T}^1_0(X)$, equation (3.10) follows from the second equivalence above and applying equation (3.9).

Conversely, let (3.10) hold and choose $\eta, \zeta, \xi \in \mathcal{T}^1_0(M)$ such that $\xi|_X, \eta|_X \in \mathcal{T}^1_0(X)$. By remark (3.3.8), the claim follows, again using the equivalence above. □

3.3.10. Remark. Note that in general it is not possible to formulate vanishing of the singular part of Ricci curvature in terms of the 'continuity' of metric derivatives. However, assuming that the hypersurface is nowhere null we can locally choose an orthonormal frame $(E_{(\alpha)})_{\alpha=1,\dots,n}$ adapted to X which has the same regularity as $\mathbf{g}$. Recall that its dual frame $E^{(\alpha)}$, obtained by raising the indices by the metric, satisfies $E^{(\alpha)}(E_{(\beta)}) := \epsilon_\alpha \delta^\alpha_\beta$ where $\epsilon_\alpha = \mathbf{g}(E_{(\alpha)}, E_{(\alpha)})$.

It is easy to see that formula (2.27) for the Ricci tensor can be extended to such a frame. By the same reasoning as in corollary 3.3.9, the singular part of the Ricci tensor vanishes iff

$$[E_{(n)}(\mathbf{g}(\xi,\eta))]_X = 0$$

for all η and ξ tangential to X. By remark 3.3.8, this implies that the singular part of the Ricci tensor vanishes if and only if the singular part of the Riemann tensor vanishes. For null hypersurface this is not the case.

Abstract

This thesis is concerned with a low-regularity formulation of Semi-Riemannian geometry. More precisely, we deal with connections and Semi-Riemannian metrics which belong to some appropriate local Sobolev space.

The interest in geometries of low-regularity is motivated by applications in general relativity, where they are used to describe space-times with energy-matter concentration on some lower dimensional region. Such space-times are frequently used to model thin shells of matter and radiation, cosmic strings and impulsive gravitational waves.

After collecting all the necessary prerequisites (distributions and Sobolev spaces on manifolds), we discuss the basic notions of Semi-Riemannian geometry within the framework of distributional geometry. In particular, we study what regularity assumptions have to be imposed on a distributional metric, for its Levi-Civita connection resp. curvature to be defined. Finally, we specialize to the case of connections resp. metrics which are 'smooth' everywhere except on some hypersurface, across which, they suffer a 'jump discontinuity'. In this context we discuss several 'jump formulas' for the respective curvature quantities.

Zusammenfassung

Die vorliegende Magisterarbeit befasst sich mit den Grundlagen der Semi-Riemannschen Geometrie im Fall niedriger Regularität. Genauer gesagt behandeln wir Zusammenhänge bzw. Semi-Riemannsche Metriken die in einem geeigneten lokalen Sobolev Raum liegen.

Das Interesse an solchen Geometrien niedriger Regularität wird durch Anwedungen in der Allgemeinen Relativität motiviert, wo sie zur Beschreibung von Raumzeiten, deren Energie-Materie Inhalt auf einem niedrieger-dimensionalen Bereich konzentriert ist, eingesetzt werden. Solche Raumzeiten werden ihrerseits zur Modellierung dünner Schalen von Materie oder Strahlung, kosmischer Strings und impulsiver Gravitationswellen verwendet.

Nachdem wir die nötigen Grundkenntnise (i.e. Distributionen und Sobolev Räume auf Mannigfaltigkeiten) wiederholt haben, diskutieren wir die grundlegenden Begriffe der Semi-Riemannschen Geometrie im Rahmen der distributionellen Geometrie. Wir befassen uns insbesondere mit der Frage nach der minimalen Regularität distributioneller Metriken, welche es erlaubt, den Levi-Civita Zusammenhang bzw. die Krümmung zu definieren. Zum Abschluss, betrachten wir den Spezialfall von Metriken die außerhalb einer Hyperfläche 'glatt' sind, aber 'Sprünge' längs dieser Hyperfläche aufweisen. In diesem Kontext prsentieren wir einige 'Sprungformeln' für die entsprechenden Krümmungsgrößen.

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Curriculum Vitae

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