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DISSERTATION

Weighted Jet Bundles and Differential Operators for Parabolic Geometries

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Abstract

A filtered manifold is a smooth manifold M, whose tangent bundle TM is endowed with a filtration by vector subbundles $TM = T^{-k}M \supset \ldots \supset T^{-1}M$, which is compatible with the Lie bracket of vector fields. Studying differential operators on filtered manifolds, it turns out that the notion of order of differential operators should be changed by adapting it to the filtration of the tangent bundle. This leads to a concept of weighted jet bundles for sections of vector bundles over filtered manifolds, which provides a convenient framework to investigate differential equations on filtered manifolds.

An interesting class of filtered manifolds are regular infinitesimal flag manifolds, which occur as underlying structures of parabolic geometries. In this thesis, we study differential operators on regular infinitesimal flag manifolds within the framework of weighted jet bundles.

In the first part of this thesis, we will deal with the problem of prolongation of differential equations on regular infinitesimal flag manifolds. First we will show that a linear system of differential equations of weighted finite type on a filtered manifold is always canonically equivalent to a certain linear system of weighted order one. This will imply that the solution space of such a system is always finite dimensional. Then we will show how one can construct for some class of semi-linear systems of differential equations on certain regular infinitesimal flag manifolds a linear connection ∇ on some vector bundle V over the regular infinitesimal flag manifold M and a bundle map $C : V \to T^*M \otimes V$ such that solutions of the studied system are in one to one correspondence with solutions of the system $\nabla\Sigma + C(\Sigma) = 0$. In particular, this will lead to sharp bounds for the dimension of the solution space for a wide class of linear systems of weighted finite type on certain regular infinitesimal flag manifolds.

In the second part, we will be concerned with the construction of invariant operators for parabolic geometries via curved Casimir operators. We will construct invariant operators for Lagrangean contact structures using curved Casimir operators.

Introduction

A filtered manifold is a smooth manifold M together with a filtration of the tangent bundle $TM = T^{-k}M \supset T^{-k+1}M \supset ... \supset T^{-1}M$ by vector subbundles, which is compatible with the Lie bracket of vector fields, meaning that for sections $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$. The associated graded vector bundle of a filtered manifold is given by forming the pairwise quotient of the filtration components of the tangent bundle $\operatorname{gr}(TM) = \bigoplus_{i=1}^{k} T^{-i}M/T^{-i+1}M$. The Lie bracket of vector fields induces a Lie bracket on each fiber $\operatorname{gr}(T_xM)$ over some point $x \in M$, which makes $\operatorname{gr}(T_xM)$ into a nilpotent graded Lie algebra, called the symbol algebra of the filtered manifold at the point $x \in M$. The symbol algebra $\operatorname{gr}(T_xM)$ should be viewed as the first order approximation to the filtered manifold at the point $x \in M$, playing the same role as the tangent space at some point for usual manifolds.

Studying differential equations on filtered manifolds it turns out that, in addition to replacing the usual tangent space at x by the graded nilpotent Lie algebra $\operatorname{gr}(T_xM)$, one should also change the notion of order of differential operators according to the filtration of the tangent bundle. One of the best studied examples of a filtered manifold structure is a contact structure $TM = T^{-2}M \supset T^{-1}M$ on a manifold M. In this special case, to adjust the notion of order according to the filtration of the tangent bundle means that a derivative in direction transversal to the contact subbundle $T^{-1}M$ should be considered as a differential operator of order two rather than one. Doing this leads to a notion of symbol for differential operators on M, which fits naturally together with the contact structure and which can be considered as the principal part of such operators. In the context of contact geometry the idea to study differential operators on M by means of their *weighted* symbol goes back to the 70's and 80's of the last century and is usually referred to as Heisenberg calculus, see [**1**] and [**41**].

Independently of these developments in contact geometry, Morimoto started in the 90's to study differential equations on general filtered manifolds and developed a formal theory, see [**30**], [**31**] and [**32**]. By adjusting the notion of order of differentiation to the filtration of a filtered manifold, he introduced a concept of weighted jet bundles, which provides a convenient framework to

study differential operators between sections of vector bundles over a filtered manifold. In particular, it leads to a notion of symbol, which can be naturally viewed as the principal part of differential operators between sections of vector bundles over a filtered manifold.
A geometric structure, which always gives rise to a filtered manifold structure, is a regular parabolic geometry. Parabolic geometries are special types of Cartan geometries, namely Cartan geometries of type (G, P), where G is a semisimple Lie group and $P \subset G$ is a parabolic subgroup. So a parabolic geometry of type (G, P) consists of a principal P-bundle $\mathcal{G} \to M$ and a one form ω on $\mathcal{G}$ with values in the Lie algebra of G, which is compatible with the principal P-action, trivialises the tangent bundle of $\mathcal{G}$ and reproduces fundamental vector fields. If the parabolic geometry is regular, it induces a regular infinitesimal flag structure on M, which consists of a filtration of the tangent bundle, making M into a filtered manifold, and a reduction of the structure group of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of the associated graded bundle $\mathrm{gr}(TM)$. With two exceptions, under a certain normalisation condition a regular parabolic geometry is always equivalent to its underlying regular infinitesimal flag structure. Interpreting these underlying regular infinitesimal flag structures in more conventional terms, one can see that parabolic geometries offer a uniform approach to a broad variety of geometric structures. Among these structures we have for instance conformal structures, partially integrable almost CR-structures, Lagrangean contact structures, quaternionic contact structures and certain types of generic distributions. In the last decades parabolic geometries and their underlying structures were intensively studied and formidable and profound results have been obtained. For an overview of the development of parabolic geometries see [**15**].
In this thesis, we will study differential operators between sections of natural vector bundles over regular infinitesimal flag manifolds within the framework of weighted jet bundles. The first part of this thesis is devoted to the problem of prolongation of differential equations on filtered manifolds and on regular infinitesimal flag manifolds. In the second part, we will be concerned with the construction of invariant operators for parabolic geometries via curved Casimir operators. We will show how to construct invariant operators for Lagrangean contact structures via curved Casimir operators.

Prolongation of systems of partial differential equations on filtered manifolds. Given some system of linear partial differential equations on a smooth manifold, one can ask the question whether this system can be rewritten as a first order system in closed form, meaning that all first order partial derivatives of the dependent variables are expressed in the dependent variables themselves. This actually demands to introduce new variables for certain unknown higher partial derivatives until all first order partial derivatives of all the variables can be obtained as differential consequences of the original system of equations. It can be rephrased as the need to construct a vector bundle and a linear connection such that its parallel sections correspond bijectively to solutions of the original system of equations. Having rewritten a system of linear differential equations in this way implies that the dimension of the solution space is bounded by the rank of the vector bundle and by looking at the curvature of the connection and its covariant derivatives one may derive obstructions to the existence of solutions. Hence rewriting a linear system of differential equation as a first order system in closed form leads to considerable information on this system.

In [**38**] Spencer studies a class of systems of differential equations, namely systems of so called finite type. For a system of differential equations of finite type it can be shown that a solution is already determined by a finite jet in a single point. Hence this is a class of systems of differential equations, for which one can expect such a rewriting procedure to exist. The difficulty is just how to really rewrite such a system. Even for easy equations it can become quite tricky to decide for which higher derivatives one introduces new variables and which differential consequences of the equation one should follow up, see [**4**] and [**20**].

On filtered manifolds, there exists a lot of examples of linear differential equations, for which a solution is already determined by finitely many partial derivatives in a single point, but which are not of finite type in the classical sense of Spencer. This indicates that differential equations on filtered manifolds should be better studied within the framework of weighted jet bundles and the notion of finite type should be adjusted to the weighted setting.

Using ideas of [**23**], we will show in chapter 1 that to a linear system of differential equations of weighted finite type on a filtered manifold one can always associate canonically a differential operator of weighted order one with injective weighted symbol whose kernel describes the solution of the original system. Moreover, we will see that rewriting a linear system of weighted finite type in this form, implies that a solution is already determined by a finite weighted jet in a single point, hence its solution space is always finite dimensional. In addition, we will obtain obstructions to the existence of solutions. These results have been already published in [**35**]. Although from a theoretical point of view this canonical weighted first order operator is as good as a linear connection on some vector bundle whose parallel sections correspond to solutions of the studied linear systems, this approach remains

in general too abstract to obtain concrete information.
In chapter 3, studying overdetermined systems on regular infinitesimal flag manifolds, we will therefore take up a different point of view. Suppose that $(\mathcal{G} \to M, \omega)$ is a regular parabolic geometry of some type (G, P) and let $\mathbb{V}$ be some irreducible representation of G. It was shown in [**13**] and the construction was improved in [**6**] that to the natural vector bundle V corresponding to $\mathbb{V}$ one can associate a sequence of invariant linear differential operators for the parabolic geometry $(\mathcal{G} \to M, \omega)$, which is called the BGG-sequence associated to V. In the case of the homogeneous model of parabolic geometries of type (G, P) this sequence is a complex and corresponds dually to the algebraic Bernstein-Gelfand-Gelfand resolution of the representation $\mathbb{V}$. It turns out that a huge class of invariant differential operators for $(\mathcal{G} \to M, \omega)$ actually occurs as differential operators in some BGG-sequence and that the first operator occurring in such a sequence gives always rise to an overdetermined system of partial differential equations. The BGG-operators can be also naturally viewed as differential operators between natural vector bundles over the underlying regular infinitesimal flag structure on M.
In chapter 3, we will consider semi-linear differential operators between sections of natural vector bundles over certain regular infinitesimal flag manifolds, which have the same weighted symbol as some first BGG-operator. Working in the setting of weighted jet bundles, we will show how the system of partial differential equations associated to such an operator can be rewritten as a system of partial differential equations of the form $\nabla\Sigma + C(\Sigma) = 0$, where ∇ is a linear connection on some vector bundle V over the regular infinitesimal flag manifold M and $C : V \to T^*M \otimes V$ is some bundle map. If the studied system is linear, we will see that it is of weighted finite type and that the rewriting procedure in this case produces a vector bundle map C. Hence $\nabla + C$ will be a linear connection, whose parallel sections are in bijection with solutions of the studied linear system. In particular, we will obtain in this way sharp bounds for a wide class of linear overdetermined systems on certain regular infinitesimal flag manifolds. This prolongation procedure will generalise the procedure, which was presented in [**4**], for overdetermined systems on regular infinitesimal flag manifolds corresponding to $|1|$-graded semisimple Lie algebras to regular infinitesimal flag manifolds corresponding to $|k|$-graded semisimple Lie algebras, where the center of its Levi subalgebra is one dimensional. In particular, it will apply to contact manifolds.
Let us remark that for the case of a first BGG-operator itself associated to some vector bundle V corresponding to an irreducible G representation it was recently shown, see [**25**], how to construct a linear connection on V,

whose parallel sections correspond bijectively to solutions of the linear system defined by the first BGG-operator. This approach has the feature that the connection on V is natural with respect to the parabolic geometry respectively to its underlying geometric structure. In contrast our method, although not natural, works as well for all the first BGG operators on any regular infinitesimal flag manifold, but also for all semi-linear differential operators on regular infinitesimal flag manifolds corresponding to $|k|$-graded semisimple Lie algebras, where the center of its Levi subalgebra is one dimensional, which have exactly the same weighted symbol as some first BGG-operator. In the latter case, we see that our approach is not only working for a larger class of differential operators, but also that it is very convenient for applications, since to apply our rewriting procedure one just has to check, if the operator one studies has the right weighted symbol, one doesn't need to know, if one is dealing with a BGG-operator.

Construction of invariant differential operators for parabolic geometries via Curved Casimir operators. We already mentioned that a large class of invariant operators for parabolic geometries occur as differential operators in some BGG-sequence. However, the construction of BGG-operators doesn't produce all the invariant differential operators. On the one hand one only obtains differential operators acting between natural vector bundles corresponding to P representations with regular infinitesimal character and on the other hand one can't construct so called non-standard invariant differential operators, like in the case of conformal geometries conformally invariant powers of Laplacians.
Given a parabolic geometry of some type (G, P) and a finite dimensional representation $\mathbb{W}$ of P, it was shown in **[16]** that there is always a basic invariant differential operator acting between sections of the natural vector bundle W corresponding to $\mathbb{W}$. It is called the curved Casimir operator on W. In **[10]** it was demonstrated how curved Casimir operators can be used to construct in a conceptual way invariant operators for parabolic geometries acting between sections of natural vector bundles corresponding to completely reducible representations of P. The only difficulty one has to face is that it is not apparent from the construction that the obtained operator is non-zero. In a forthcoming article Čap and Gover therefore developed in the case of $|1|$-graded parabolic geometries a method for computing the principal symbol of the constructed operator. In chapter 4, we will see that dealing with parabolic geometries corresponding to $|k|$-graded semisimple Lie algebras for $k > 1$, one should investigate the weighted symbol of the constructed operator rather than the usual principal symbol. Similarly as in the $|1|$-graded case one should be able to find a method to conceptually

compute the weighted symbol of the constructed operator. We will demonstrate this by constructing via curved Casimir operators invariant operators for Langrangean contact structures, which are related to the square of a Sub-Laplacian.

Short summary and structure of the following text. In the first chapter, we introduce the notion of a filtered manifold and expatiate on the concept of weighted jet bundles. Moreover, we will see that the solution space of a regular linear system of differential equations of weighted finite type on a filtered manifold is always finite dimensional by showing that it can be canonically rewritten as a certain linear system of weighted first order.
In the second chapter, we give a short introduction to parabolic geometries and collect some results, which will be crucial in the course of this thesis.
In the third chapter, we will study a large class of geometrically interesting semi-linear systems of differential equations on certain regular infinitesimal flag manifolds and present a systematical way to rewrite them as systems of the form $\nabla\Sigma + C(\Sigma) = 0$, where ∇ is a linear connection on some vector bundle V over the regular infinitesimal flag manifold M and $C : V \to T^*M \otimes V$ is a bundle map. In particular, this will lead to sharp bounds for the dimension of the solution space of a huge class of linear overdetermined systems on certain regular infinitesimal flag manifolds. Moreover, we will show how this prolongation procedure can be applied to contact manifolds.
In the fourth chapter, we are dealing with the construction of invariant operators for parabolic geometries via curved Casimir operators. We will show how to construct invariant operators for Lagrangean contact structures via curved Casimir operators, which are related to the square of a Sub-Laplacian.

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CHAPTER 1

Filtered Manifolds and Weighted Jet Bundles

In the first two sections of this chapter we introduce the notion of a filtered manifold and discuss the concept of weighted jet bundles of sections of vector bundles over filtered manifolds as it was introduced by Morimoto in order to study differential equations on filtered manifolds, see [**30**], [**31**] and [**32**]. In the last section, we will show that systems of linear partial differential equations of weighted finite type can be canonically rewritten as certain systems of weighted order one. This will immediately imply that the solution space of a system of weighted finite type is always finite dimensional and it will lead to obstructions to the existence of solutions.

1.1. Filtered manifolds

We start by collecting some basic facts about filtered manifolds and by considering some particularly interesting examples.

1.1.1. Definitions and notations.

Definition 1.1.

(1) A *filtered manifold* is a smooth manifold M together with a filtration of its tangent bundle TM by vector subbundles $\{T^iM\}_{i\in\mathbb{Z}}$ such that:
   - $T^iM \supseteq T^{i+1}M$
   - $T^0M = 0$ and there exists $\ell \in \mathbb{N}$ with $T^{-\ell}M = TM$
   - for sections $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$

(2) By the first two properties, the tangential filtration can be written as
$$TM = T^{-k}M \supsetneq T^{-k+1}M \supsetneq ... \supsetneq T^{-1}M$$
for some $k \in \mathbb{N}$ with $T^iM = TM$ for $i \leq -k$ and $T^iM = 0$ for $i \geq 0$. The number $k \in \mathbb{N}$ is called the *depth* of the filtered manifold.

(3) A *(local) isomorphism* between two filtered manifolds M and N is a (local) diffeomorphism $f : M \to N$, whose tangent map Tf satisfies $Tf(T^iM) = T^iN$.

Let M be a filtered manifold of depth k with tangential filtration $TM = T^kM \supset ... \supset T^{-1}M$. Then one can form the *associated graded vector bundle*

$\mathrm{gr}(TM)$ to the filtered vector bundle TM. It is obtained by taking the pairwise quotients of the filtration components of the tangent bundle

$$\mathrm{gr}(TM) = \bigoplus_{i=-k}^{-1} \mathrm{gr}_i(TM),$$

where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$. We denote by $q_i : T^iM \to \mathrm{gr}_i(TM)$ the natural projection.

Now consider the operator $\Gamma(T^iM) \times \Gamma(T^jM) \to \Gamma(\mathrm{gr}_{i+j}(TM))$ given by $(\xi, \eta) \mapsto q_{i+j}([\xi, \eta])$. Since $[f\xi, g\eta] = fg[\xi, \eta] + f(\xi \cdot g)\eta - g(\eta \cdot f)\xi$ for smooth functions $f, g \in C^\infty(M, \mathbb{R})$ and since for $i, j \leq -1$ the subbundles T^iM and T^jM are contained in $T^{i+j+1}M$, this operator is bilinear over smooth functions and therefore induced by a bilinear bundle map $T^iM \times T^jM \to \mathrm{gr}_{i+j}(TM)$. Moreover, it obviously factorises to a bundle map $\mathrm{gr}_i(TM) \times \mathrm{gr}_j(TM) \to \mathrm{gr}_{i+j}(TM)$, since for $\xi \in \Gamma(T^{i+1}M)$ or $\eta \in \Gamma(T^{j+1}M)$ we have $[\xi, \eta] \in \Gamma(T^{i+j+1}M)$. Hence we obtain a bilinear bundle map

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$$

on the associated graded bundle. Since $\mathcal{L}$ is induced from the Lie bracket of vector fields, $\mathcal{L}_x$ makes the fiber $\mathrm{gr}(T_xM)$ over $x \in M$ into a Lie algebra. Viewing $\mathrm{gr}(T_xM)$ as a graded vector space

$$\mathrm{gr}(T_xM) = \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}_i(T_xM)$$

where $\mathrm{gr}_i(T_xM) = 0$ for $i < -k$ or $i \geq 0$, we see that the Lie bracket $\mathcal{L}_x$ by construction satisfies

$$\mathcal{L}_x(\mathrm{gr}_i(T_xM), \mathrm{gr}_j(T_xM)) \subset \mathrm{gr}_{i+j}(T_xM).$$

Therefore $(\mathrm{gr}(T_xM), \mathcal{L}_x)$ is actually a graded nilpotent Lie algebra.

Definition 1.2. Let M be a filtered manifold.

(1) The tensorial bracket $\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$ induced from the Lie bracket of vector fields on $\mathrm{gr}(TM)$ is called the *Levi bracket.*

(2) The nilpotent graded Lie algebra $(\mathrm{gr}(T_xM), \mathcal{L}_x)$ is called the *symbol algebra* of the filtered manifold M at the point $x \in M$.

Suppose M and N are filtered manifold and let $f : M \to N$ be a local isomorphism of filtered manifolds. Then for each point $x \in M$ the tangent T_xf at x is an isomorphism of filtered vector spaces $T_xf(T_x^iM) = T_x^iN$ and hence induces an isomorphism of graded vector spaces between $\mathrm{gr}(T_xM)$ and $\mathrm{gr}(T_{f(x)}N)$. The compatibility of the pullback of vector fields with the Lie bracket easily implies that this actually is an isomorphism of graded Lie algebras. Therefore the symbol algebras are the basic invariants one can

associate to a filtered manifold. In fact, the symbol algebra at x should be seen as the first order linear approximation to the filtered manifold at the point x replacing the role of the tangent space for ordinary manifolds.
In general the symbol algebra of a filtered manifold may change from point to point. Hence the associated graded vector bundle need not to be locally trivial as a bundle of Lie algebras. If each fiber $\mathrm{gr}(T_xM)$ of the associated graded bundle is isomorphic to some fixed nilpotent graded Lie algebra $\mathfrak{n} = \mathfrak{n}_{-k} \oplus ... \oplus \mathfrak{n}_{-1}$, the filtered manifold is called *regular of type* $\mathfrak{n}$.

Example 1.1. Suppose $\mathfrak{n} = \mathfrak{n}_{-k} \oplus ... \oplus \mathfrak{n}_{-1}$ is a nilpotent graded Lie algebra and let N be a Lie group with Lie algebra $\mathfrak{n}$. Then N naturally admits the structure of a filtered manifold with symbol algebra in each point isomorphic to $\mathfrak{n}$. In fact, the tangential filtration is given as follows: Setting $\mathfrak{n}^\ell := \bigoplus_{i \geq \ell} \mathfrak{n}_i$, we obtain a filtration $\mathfrak{n} = \mathfrak{n}^{-k} \supset .... \supset \mathfrak{n}^{-1}$ of the vector space $\mathfrak{n}$, which makes $\mathfrak{n}$ into a filtered Lie algebra, meaning that $[\mathfrak{n}^i, \mathfrak{n}^j] \subset \mathfrak{n}^{i+j}$. Via the left trivialisation of the tangent bundle of a Lie group $TN \cong N \times \mathfrak{n}$, we see that $N \times \mathfrak{n}^\ell$ defines an left invariant subbundle $T^\ell N \subseteq TN$. Explicitly, $T^\ell N$ is spanned by the left invariant vector fields L_X generated by elements $X \in \mathfrak{n}^\ell$. Hence we obtain a filtration of the tangent bundle by vector subbundles $TN = T^{-k}N \supset ... \supset T^{-1}N$, which makes N into a filtered manifold. Since $[L_X, L_Y] = L_{[X,Y]}$, we conclude that the symbol algebra of N in each point is isomorphic to $\mathfrak{n}$. This provides the standard example of a regular filtered manifold of type $\mathfrak{n}$.

Given some regular filtered manifold of type $\mathfrak{n}$ one has a natural notion of a frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of the associated graded $\mathrm{gr}(TM)$. Denoting by $\mathcal{P}_x(\mathrm{gr}(TM))$ the space of all graded Lie algebra isomorphisms $\phi : \mathfrak{n} \to \mathrm{gr}(T_xM)$, the frame bundle is defined by the disjoint union

$$\mathcal{P}(\mathrm{gr}(TM)) := \sqcup_{x \in M} \mathcal{P}_x(\mathrm{gr}(TM)).$$

The bundle $\mathcal{P}(\mathrm{gr}(TM))$ is a principal bundle with structure group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n})$, the group of all Lie algebra automorphisms of $\mathfrak{n}$, which in addition preserve the grading. The right action of $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{n})$ on $\mathcal{P}(\mathrm{gr}(TM))$ is given by composition.

Remark 1.1. Any ordinary smooth manifold can be viewed as a trivial filtered manifold $TM = T^{-1}M$. The associated graded bundle is then just the tangent bundle, where the tangent space at each point is viewed as an abelian Lie algebra.

1.1.2. Regular Distributions. Let M be a smooth manifold. A *distribution* on M is a vector subbundle $H \subset TM$. By the theorem of Frobenius a distribution H is integrable if and only if it is involutive, i.e. for sections

$\xi, \eta \in \Gamma(H)$ also $[\xi, \eta] \in \Gamma(H)$.
If H is a non-integrable distribution, then it is expedient to consider the weak derived flag of subsheaves of the sheave of sections of the tangent bundle

$$\mathcal{H}^{-1} \subset \mathcal{H}^{-2} \subset ... \subset \Gamma(TM)$$

defined inductively by

$$\mathcal{H}^{-1} = \Gamma(H)$$

$$\mathcal{H}^{i-1} = \mathcal{H}^{i} + [\mathcal{H}^{i}, \mathcal{H}^{-1}].$$

The distribution H is called *regular*, if each $\mathcal{H}^i$ spans a subbundle $H^i \subset TM$ ($H^{-1} = H$). A regular distribution H endows M with the structure of a filtered manifold as follows: Since TM is of finite rank, there must be a $k \in \mathbb{N}$ with $H^i = H^{-k}$ for all $i \leq -k$. Let k be the smallest number with this property and set

$$T^iM = TM \text{ for } i \leq -k-1$$

$$T^iM = H^i \text{ for } -k \leq i \leq -1.$$

$$T^iM = 0 \text{ for } i \geq 0.$$

This is a filtration of the tangent bundle by subbundles, which by construction is compatible with the Lie bracket of vector fields. Hence M together with this filtration is a filtered manifold.
If $H^{-k} = TM$, then H is a *regular bracket generating* distribution. In other words, this means that H is a regular distribution such that the induced filtration of the tangent bundle satisfies that $\mathrm{gr}_{-1}(T_xM)$ generates $\mathrm{gr}(T_xM)$ as Lie algebra for all $x \in M$.
If not, H^{-k} is involutive and therefore integrable. So M is a manifold foliated by manifolds endowed with bracket generating distributions.
This shows that bracket generating distributions stand at the opposite end to integrable distributions.
The best studied examples of regular bracket generating distributions are contact distributions, which we will consider in detail in the next section.
Other well known examples of regular bracket generating distributions are for instance generic rank 2 distribution on manifolds of dimension 5, studied by É. Cartan in [**17**], generic rank 3 distribution on manifolds of dimension 6, considered by R. Bryant in [**5**] as well as quaternionic contact structures introduced by O. Biquard in [**3**].

Remark 1.2. A distribution H can be equivalently viewed as a differential system by duality. This is the point of view Tanaka has in [**40**]. His definition of a regular differential system coincides with the one of a regular distribution above.

1.1.3. Contact distributions. One of the best studied examples of filtered manifolds are contact manifolds.

Definition 1.3. A *contact structure* on a smooth manifold M of dimension $2n+1$ is a distribution $H \subset TM$ of rank $2n$ such that in each point $x \in M$ the Levi bracket $\mathcal{L}_x : H_x \times H_x \to T_xM/H_x$ is non-degenerate.
A *contact manifold* is a manifold of dimension $2n+1$ endowed with a contact structure.

By definition a contact manifold is a filtered manifold M of depth 2 such that the symbol algebra $\mathrm{gr}(T_xM) = TM/T_x^{-1}M \oplus T_x^{-1}M$ in each point is isomorphic to the nilpotent graded Lie algebra $\mathfrak{h}_{2n+1} := \mathfrak{h}_{-2} \oplus \mathfrak{h}_{-1} := \mathbb{R} \oplus \mathbb{R}^{2n}$, where the Lie bracket $[\,,\,] : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ is given by the standard symplectic form on $\mathbb{R}^{2n}$. The graded nilpotent Lie algebra $\mathfrak{h}_{2n+1}$ is called the *Heisenberg Lie algebra* of dimension $2n+1$. So contact manifolds are filtered manifold such that the symbol algebra in each point is isomorphic to a Heisenberg algebra.
The simply connected Lie group H_{2n+1} with Lie algebra $\mathfrak{h}_{2n+1}$ is called the *Heisenberg group*. The Heisenberg group H_{2n+1} is $\mathbb{R}^{2n+1}$ with the group law given by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}\sum_{i=1}^{n} x_i y_i - x'_i y'_i),$$

where $(x_1, .., x_n, y_1, .., y_n, z)$ are the coordinates of $\mathbb{R}^{2n+1}$. In example 1.1 we saw that H_{2n+1} is naturally endowed with the structure of a filtered a manifold such that the symbol algebra in each point is isomorphic to $\mathfrak{h}_{2n+1}$, hence with a canonical contact structure. Explicitly, the canonical contact structure is given as follows: The left invariant vector fields generated by the elements of the standard basis of $\mathbb{R}^{2n+1}$ are:

$$X_i = \frac{\partial}{\partial x_i} - \frac{1}{2} y_i \frac{\partial}{\partial z} \quad \text{for } i = 1, ..., n$$

$$Y_i = \frac{\partial}{\partial y_i} + \frac{1}{2} x_i \frac{\partial}{\partial z} \quad \text{for } i = 1, ..., n$$

$$Z = \frac{\partial}{\partial z}$$

The contact distribution is then spanned by the vector fields X_i, Y_i for $i = 1, ..., n$. One of the fundamental results in contact geometry says that locally any contact structure looks like the contact structure on a Heisenberg group.

Proposition 1.1. *Every contact manifold (M, H) of dimension $2n+1$ is locally isomorphic to an open subset of the Heisenberg group H_{2n+1} endowed with its canonical contact structure.*

1.2. Differential operators on filtered manifolds and weighted jet bundles

Studying analytic properties of differential operators on some manifold M, one may first look at the principal symbols of these operators. If M is a filtered manifold, it turns out that the usual symbol is not the appropriate object to consider and it should be replaced by a notion of symbol that reflects the geometric structure on M given by the filtration on the tangent bundle. To illustrate this let us consider an example.
Suppose M is the Heisenberg group $H_{2n+1} = \mathbb{R}^{2n+1}$ endowed with its canonical contact structure $T^{-1}M \subset TM$ as in section 1.1.3. Now consider the following differential operator acting on smooth functions

$$D : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$$

$$D = -\sum_{j=1}^{n}(X_j^2 + Y_j^2) + iaZ \text{ with } a \in \mathbb{C},$$

where $X_1, ..., X_n, Y_1, ..., Y_n, Z$ are the left invariant vector fields from section 1.1.3. It can be shown, see **[19]**, that the analytic properties of D, like for instance hypoellipticity, highly depend on the constant $a \in \mathbb{C}$. However, this can never be read off from the usual principal symbol, since the term iaZ is not part of it. This suggests that a derivative transversal to the contact distribution should have order two rather than one to obtain a notion of symbol that includes the term iaZ.
In the case of a general filtered manifold the situation is similar. Once one has replaced the role of the usual tangent space at some point $x \in M$ by the symbol algebra at that point, one should also adjust the notion of order of differentiation according to the filtration of the tangent bundle, in order to obtain a notion of principal symbol that can be seen as representing the principal parts of operators on M.

1.2.1. The algebra of differential operators on a filtered manifold.

Suppose that M is a manifold and denote by $\mathcal{D}(M)$ the complex vector space of linear differential operators $D : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ on M with coefficients in $\mathbb{C}$. The composition of two operators defines a multiplication on $\mathcal{D}(M)$, which makes $\mathcal{D}(M)$ into an associative unitial algebra over $\mathbb{C}$ and in particular into a module over $C^\infty(M, \mathbb{C})$. It is well known that differential operators are local and in fact the following holds:

Theorem 1.2. *Let $\{X_1, ..., X_n\}$ be a local frame of the tangent bundle TM or of the complexified tangent bundle $T_\mathbb{C}M$ defined on an open subset $U \subset M$. Then the differential operators given by the monomials*

$$X_1^{\alpha_1} ... X_n^{\alpha_n} \textit{ with } \alpha \in \mathbb{N}_0^n$$

form a basis of the $C^\infty(U, \mathbb{C})$-module $\mathcal{D}(U)$.
In particular, if $D \in \mathcal{D}(M)$ is a linear differential operator, then its restriction to U is given by

$$D|_U = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha X_1^{\alpha_1} ... X_n^{\alpha_n},$$

for unique smooth functions $a_\alpha \in C^\infty(U, \mathbb{C})$, where only finitely many of the functions a_α are non zero.

PROOF. See e.g. chapter 1 in **[42]**. □

Assume now that M is a filtered manifold. Then we adapt the concept of order for linear differential operators on M according to the filtration of the tangent bundle as follows:

Definition 1.4. Let M be a filtered manifold with filtration given by

$$TM = T^{-k}M \supset ... \supset T^{-1}M \supset T^0 M = \{0\}$$

(1) A local vector field ξ of M is of weighted order $\leq r$, if ξ is a local section of $T^{-r}M$. The smallest number $r \in \mathbb{N}_0$ such that this holds is called the *weighted order* $\mathrm{ord}(\xi)$ of ξ.
(2) A linear differential operator $D : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ is of weighted order $\leq r$, if for each point $x \in M$ there exists a local frame $\{X_1, ..., X_n\}$ of TM defined on an open neighbourhood $U \subset M$ of x such that

$$D|_U = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha X_1^{\alpha_1} ... X_n^{\alpha_n},$$

where for all non zero terms in this sum $\sum_{i=1}^n \alpha_i \mathrm{ord}(X_i) \leq r$. The smallest number $r \in \mathbb{N}_0$ such this holds is called the *weighted order* of D.

Remark 1.3. For a trivial filtered manifold $TM = T^{-1}M$ the notion of weighted order obviously coincides with the ordinary notion of order.

To deduce from a local description of a differential operator on M its weighted order, one needs to write it in terms of a local frame adapted to the filtration of TM.

Definition 1.5. Let M be a filtered manifold of depth k. An *adapted local frame* for TM (resp. $T_{\mathbb{C}}M$) is a local frame

$$\{X_{1,1}, ..., X_{1,i(1)}, ..., X_{k,1}, ..., X_{k,i(k)}\}$$

such that $\{X_{1,1}, ..., X_{1,i(1)}, ..., X_{\ell,1}, ..., X_{\ell,i(\ell)}\}$ is a local frame of $T^{-\ell}M$ (resp. $T_{\mathbb{C}}^{-\ell}M$) for all $\ell \leq k$.

Such an adapted local frame can always be constructed by choosing an open subset of M, over which all subbundles $T^{-\ell}M$ (resp. $T^{-\ell}_{\mathbb{C}}M$) trivialise. Note that, given an adapted local frame defined on an open subset $U \subset M$, for each point $x \in U$ the vectors $\{X_{\ell,1}(x), ..., X_{\ell,i(\ell)}(x)\}$ span a vector space compliment $V_x^{-\ell}$ of $T_x^{-\ell+1}M$ in $T_x^{-\ell}M$. Hence an adapted local frame induces an isomorphism $T_xM \cong \mathrm{gr}(T_xM)$ for all $x \in U$, where $V_x^{-\ell}$ is mapped onto $\mathrm{gr}_{-\ell}(T_xM)$.

Remark 1.4. Consider the Heisenberg group H_{2n+1} endowed with its canonical contact structure. We have seen that the generators of its contact structure are given by $X_i = \frac{\partial}{\partial x_i} - \frac{1}{2}y_i\frac{\partial}{\partial z}$ and $Y_i = \frac{\partial}{\partial y_i} + \frac{1}{2}x_i\frac{\partial}{\partial z}$ for $i = 1, ..., n$. The vector fields X_i and Y_i are of weighted order one and hence may be viewed as differential operators of weighted order one. However, in terms of coordinate vector fields they are the sum of vector fields of weighted order two. This shows that, conversely to the usual notion of order, it is not enough to look at the local description of a differential operator in terms of some local frame to read off the weighted order of a differential operator. The operator has to be expressed locally in terms of an adapted frame.

In fact, we obtain:

Proposition 1.3. *Let M be a filtered manifold of depth k with filtration $TM = T^{-k}M \supset ... \supset T^{-1}M$. A linear differential operator D is of weighted order r if and only if the following two conditions hold:*

(1) *for each point $x \in M$ there exists an adapted local frame of TM defined on some open neighbourhood U of x such that*
$$D|_U = \sum_{|\alpha| \leq r} a_\alpha X_{1,1}^{\alpha_{1,1}} ... X_{k,i(k)}^{\alpha_{k,i(k)}} \tag{1.1}$$
where $\alpha = (\alpha_{1,1}, ..., \alpha_{k,i(k)}) \in \mathbb{N}_0^n$ is a multi-index with $|\alpha| := \sum_{j=1}^{k}\sum_{\ell=1}^{i(j)} j\alpha_{j,\ell}$ and $a_\alpha \in C^\infty(U, \mathbb{C})$.

(2) *there exists at least one point $x_0 \in M$ such that in some adapted local frame defined on an open neighbourhood of x_0 the operator D is of the form (1.1) and $a_\alpha(x_0) \neq 0$ for some α with $|\alpha| = r$.*

Moreover, if D is a linear differential operator of weighted order r, then it is for any choice of local adapted frames of TM (resp. $T_{\mathbb{C}}M$) of this form.

Proof. Let D be a differential operator of weighted order r. Then for each point $x \in M$ there exists a local frame $\{Y_1, ..., Y_n\}$ of TM defined on an open neighbourhood U of x such that
$$D|_U = \sum_\beta b_\beta Y_1^{\beta_1} ... Y_n^{\beta_n}$$

for finitely many multi-indices $\beta \in \mathbb{N}_0^n$ satisfying $\sum_{j=1}^n \beta_j \mathrm{ord}(Y_j) \leq r$ and non-zero smooth functions $b_\beta \in C^\infty(U, \mathbb{C})$. Without loss of generality, we assume that all filtration components of TM trivialises over U and choose an adapted local frame $\{X_{1,1}, ..., X_{1,i(1)}, ..., X_{k,1}, ..., X_{k,i(k)}\}$ defined on U. Every vector field Y_p can be written as

$$Y_p = \sum_{j=1}^{ord(Y_p)} \sum_{\ell=1}^{i(j)} f_{j,\ell}^{Y_p} X_{j,\ell} \tag{1.2}$$

for unique functions $f_{j,\ell}^{Y_p} \in C^\infty(U, \mathbb{C})$.
Recall that a vector field ξ acts as derivation on the vector space of smooth functions $\xi \cdot (fg) = (\xi \cdot f)g + f(\xi \cdot g)$ and so we have that $\xi \cdot ((f\eta) \cdot g) = (\xi \cdot f)(\eta \cdot g) + f(\xi \cdot (\eta \cdot g))$ for another vector field η. Therefore inserting for all Y_p the expression (1.2) in the local formula of D, we obtain a description of $D|_U$ as sum of certain operators of the form $a_\alpha X_{1,1}^{\alpha_{1,1}} ... X_{k,i(k)}^{\alpha_{k,i(k)}}$ with $a_\alpha \in C^\infty(U, \mathbb{C})$. Since all the $X_{j,\ell}$ occurring in (1.2) are of weighted order $\leq ord(Y_p)$ and the filtration of TM is compatible with the Lie bracket of vector fields, we obtain that $D|_U = \sum_{|\alpha| \leq r} a_\alpha X_{1,1}^{\alpha_{1,1}} ... X_{k,i(k)}^{\alpha_{k,i(k)}}$ for smooth functions $a_\alpha \in C^\infty(U, \mathbb{C})$.
For (2) note that the definition of weighted order immediately implies that there exists a point $x_0 \in M$ and an adapted local frame $\{Y_1, ..., Y_n\}$ defined on an open neighbourhood U of x_0 such that

$$D|_U = \sum_\beta b_\beta Y_1^{\beta_1} ... Y_n^{\beta_n} \text{ with } \sum_{j=1}^n \beta_j \mathrm{ord}(Y_j) \leq r \text{ for all } \beta$$

and such that there is at least one term in this sum

$$b_\gamma Y_1^{\gamma_1} ... Y_n^{\gamma_n} \text{ with } b_\gamma(x_0) \neq 0 \text{ and } \sum_{j=1}^n \gamma_j \mathrm{ord}(Y_j) = r.$$

The second assertion then follows immediately by inserting (1.2) in the formula for $D|_U$.
Conversely, if D is a differential operator such that (1) and (2) are satisfied, then the weighted order of D has to be $\leq r$. The same argumentation as above now shows that D cannot have weighted order $< r$.
The last statement follows in the same way. □

We can define a filtration of the space $\mathcal{D}(M)$ of linear differential operators by subspaces

$$\mathcal{D}(M)^0 \subset \mathcal{D}(M)^1 \subset ... \subset \mathcal{D}(M)^i \subset ....,$$

where $\mathcal{D}^i(M) \subset \mathcal{D}(M)$ is the subspace of differential operators of weighted order $\leq i$. Since the composition of an operator of weighted order r and an operator of weighted order s, is obviously an operator of weighted order at most $r + s$, the filtration by the weighted order of differential operators makes $\mathcal{D}(M)$ into a filtered associative unitial algebra. The associated graded algebra to the filtered algebra $\mathcal{D}(M)$ is defined by

$$\mathrm{gr}(\mathcal{D}(M)) = \bigoplus_{i=0}^{\infty} \mathrm{gr}_i(\mathcal{D}(M)),$$

where $\mathrm{gr}_i(\mathcal{D}(M)) = \mathcal{D}(M)^i/\mathcal{D}(M)^{i-1}$. Elements of $\mathrm{gr}_i(\mathcal{D}(M))$ are equivalence classes of differential operators.

Definition 1.6.

(1) For a differential operator D of weighted order r, the image of D under the projection

$$\sigma_r : \mathcal{D}(M)^r \to \mathrm{gr}_r(\mathcal{D}(M))$$

is called the *weighted symbol* of D.

(2) The algebra $\mathrm{gr}(\mathcal{D}(M))$ with multiplication given by

$$\sigma_r(D)\sigma_s(E) = \sigma_{r+s}(DE)$$

for operators D and E of weighted order r respectively s is called the *algebra of weighted symbols* of differential operators on M.

Since a vector field ξ acts as a derivation on the space of smooth functions $\xi\cdot(fg) = f(\xi\cdot g)+(\xi\cdot f)g$, it follows from the local description of a differential operator of weighted order r in terms an adapted local frame [proposition 1.3] that

$$D(fg) = fD(g) + \text{terms of weighted order} \leq r-1 \text{ in } g. \tag{1.3}$$

Therefore elements in $\mathrm{gr}(\mathcal{D}(M))$ may be viewed as sections of a formal infinite dimensional vector bundle $\mathcal{U}$ over M. Explicitly, define $\mathcal{U}$ as the disjoint union $\mathcal{U} := \bigsqcup_{x\in M} \mathcal{U}(x)$ of the infinite dimensional graded vector spaces

$$\mathcal{U}(x) := \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(x),$$

where $\mathcal{U}_{-r}(x)$ is defined as the space of equivalence classes of differential operators of weighted order r, where two differential operators D and $\tilde{D}$ of weighted order r are equivalent $\sim_x$, if

$$D = \tilde{D} + \sum_i f_i D_i + \textit{differential operators of weighted order } \leq r-1,$$

for differential operators D_i of weighted order r and smooth functions f_i, which vanish at x.
An element $\sigma_r(D) \in \mathrm{gr}_r(D(M))$ can then be identified with the section of $\mathcal{U}$ given by $x \mapsto \sigma_r(D)(x)$, where $\sigma_r(D)(x) \in \mathcal{U}_{-r}(x)$ is the equivalence class of D in $\mathcal{U}(x)$. (Note that $\sigma_r(D) = \sigma_r(\tilde{D})$ if and only if $\sigma_r(D)(x) = \sigma_r(\tilde{D})(x)$ for all $x \in M$).

Suppose $D \sim_x \tilde{D}$ are equivalent operators of weighted order r and $E \sim_x \tilde{E}$ are equivalent operators of weighted order s with

$$E = \tilde{E} + \sum_j g_j E_j + \textit{differential operators of weighted order } \leq s-1$$

for differential operators E_j of weighted order s and smooth functions g_j vanishing at x, then (1.3) implies

$$DE = \tilde{D}\tilde{E} + \tilde{D}\sum_j g_j E_j + \sum_i f_i D_i \tilde{E} + \sum_{i,j} f_i D_i g_j E_j + \textit{terms of lower weighted order}$$

$$= \tilde{D}\tilde{E} + \sum_j g_j \tilde{D} E_j + \sum_i f_i D_i \tilde{E} + \sum_{i,j} f_i g_j D_i E_j + \textit{terms of lower weighted order.}$$

Therefore we have a well defined multiplication on $\mathcal{U}(x)$ given by

$$\sigma_r(D)(x)\sigma_s(E)(x) := \sigma_{r+s}(DE)(x),$$

which makes $\mathcal{U}(x) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(x)$ into an associative graded complex algebra.

We will see in section 1.2.5 that $\mathcal{U}$ can be in fact viewed as a formal infinite dimensional vector bundle, where the fiber $\mathcal{U}(x)$ can be identified with the universal enveloping algebra of the complexification of the symbol algebra $\mathrm{gr}(T_xM)$ of the filtered manifold.

1.2.2. Universal enveloping algebras. Let us first recall the notion of the universal enveloping algebra of a Lie algebra and collect some of its properties, which we will need in the sequel. Suppose $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and denote by $\mathcal{T}(\mathfrak{g})$ the tensor algebra of the vector space $\mathfrak{g}$. Recall that $\mathcal{T}(\mathfrak{g})$ is the unitial associative graded algebra given by

$$\mathcal{T}(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} T_i(\mathfrak{g}) \quad \text{with } T_i(\mathfrak{g}) = \otimes^i \mathfrak{g} \text{ and } T_0(\mathfrak{g}) = \mathbb{K},$$

where the multiplication is just the tensor product.
Let $\mathcal{I}$ be the two-sided ideal in $\mathcal{T}(\mathfrak{g})$ generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y] \text{ for } X, Y \in \mathfrak{g}.$$

The unitial associative algebra $\mathcal{U}(\mathfrak{g})$ defined as the quotient of the tensor algebra by this ideal

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I}$$

is called the *universal enveloping algebra* of $\mathfrak{g}$. Note that if $\mathfrak{g}$ is abelian, its universal enveloping algebra is just the symmetric algebra $\mathcal{S}(\mathfrak{g})$.
The canonical inclusion $\mathfrak{g} \hookrightarrow \mathcal{T}(\mathfrak{g})$ induces a homomorphism of Lie algebras

$$i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g}),$$

where the Lie algebra structure on $\mathcal{U}(\mathfrak{g})$ is given by the commutator $[u, v] = uv - vu$ for $v, u \in \mathcal{U}(\mathfrak{g})$. Obviously, $\mathcal{U}(\mathfrak{g})$ is generated by 1 and the elements in $i(\mathfrak{g})$. The pair $(\mathcal{U}(\mathfrak{g}), i)$ can be uniquely characterised by the following *universal property*, for a proof see e.g. chapter 2 of **[18]**.

Proposition 1.4. *If A is an unitial associative algebra over $\mathbb{K}$ and $\phi : \mathfrak{g} \to A$ a homomorphism of Lie algebras, where A is viewed as Lie algebra with the commutator as Lie bracket, then there exists a unique homomorphism of unitial associative algebras $\tilde{\phi} : \mathcal{U}(\mathfrak{g}) \to A$ such that $\tilde{\phi} \circ i = \phi$.*

If $X_1, ..., X_n$ is a linear basis of $\mathfrak{g}$, then it is clear that $\mathcal{U}(\mathfrak{g})$ is generated by 1 and all monomials $i(X_{j_1})...i(X_{j_s})$ with $j_i \in \{1, ..., n\}$ and $s \geq 1$. From

$$i(X_j)i(X_k) = i(X_k)i(X_j) + i([X_j, X_k]) \tag{1.4}$$

it follows that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is already generated by 1 and all monomials $i(X_{j_1})...i(X_{j_s})$ with $j_1 \leq ... \leq j_s$. The basic structure theorem of $\mathcal{U}(\mathfrak{g})$, called the Poincaré-Birkhoff-Witt theorem, says that 1 and these monomials are a basis of the vector space $\mathcal{U}(\mathfrak{g})$, see e.g. chapter 2 of **[18]**. In particular, this implies that $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective and hence one can identify elements of $\mathfrak{g}$ with its image in $\mathcal{U}(\mathfrak{g})$. Therefore the Poincaré-Birkhoff-Witt theorem can be formulated as follows:

Theorem 1.5. (Poincaré-Birkhoff-Witt theorem)
If $(X_1, .., X_n)$ is a linear basis of $\mathfrak{g}$, then the monomials $X_1^{\alpha_1} X_2^{\alpha_2} ... X_n^{\alpha_n}$ with $\alpha_1, ..., \alpha_s \in \mathbb{N}_0$ form a linear basis of the vector space $\mathcal{U}(\mathfrak{g})$.

Suppose $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. By proposition 1.4 the composition $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ uniquely extends to a homomorphism $\mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{g})$ of unitial associative algebras. Then theorem 1.5 directly implies that this homomorphism is injective and hence we have:

Corollary 1.6. *Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$.
Then $\mathcal{U}(\mathfrak{h})$ can be canonically identified with the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{h}$ and 1.*

Another immediate consequence of theorem 1.5 is:

Corollary 1.7. *Suppose $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}$ decomposes as vector space as $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Then the linear map*

$$\mathcal{U}(\mathfrak{h}_1) \otimes \mathcal{U}(\mathfrak{h}_2) \to \mathcal{U}(\mathfrak{g})$$

$$u_1 \otimes u_2 \mapsto u_1 u_2$$

*defines an isomorphism of vector spaces.
If, in addition, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ commute and hence $\mathfrak{g}$ equals the Lie algebra direct sum $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, the isomorphism is even an isomorphism of algebras.*

The Poincaré-Birkhoff-Witt theorem shows that there is a linear isomorphism from $\mathcal{U}(\mathfrak{g})$ to the symmetric algebra $\mathcal{S}(\mathfrak{g})$, defined by mapping a monomial to the same monomial in $\mathcal{S}(\mathfrak{g})$, but with product now taken in $\mathcal{S}(\mathfrak{g})$. However this isomorphism can also be established in a basis independent way. Let us sketch this. Since the ideal $\mathcal{I}$ is generated by non-homogeneous elements, the grading on the tensor algebra doesn't factorise to an algebra grading on the universal enveloping algebra. However one can consider the filtration associated to the grading on the tensor algebra defined by

$$T^0(\mathfrak{g}) \subset T^1(\mathfrak{g}) \subset .... \subset T^i(\mathfrak{g}) \subset ....$$

where $T^i(\mathfrak{g}) = \bigoplus_{j\leq i} T_j(\mathfrak{g})$, which makes $\mathcal{T}(\mathfrak{g})$ into a filtered algebra. The ideal $\mathcal{I}$ behaves well with respect to the filtration and so we obtain an algebra filtration on the universal enveloping algebra

$$U^0(\mathfrak{g}) \subset U^1(\mathfrak{g}) \subset .... \subset U^i(\mathfrak{g}) \subset ....$$

where $U^i(\mathfrak{g}) = T^i(\mathfrak{g})/(I \cap T^i(\mathfrak{g}))$. If $X_1, ..., X_n$ is a basis of $\mathfrak{g}$, then $U^i(\mathfrak{g})$ is of course exactly spanned by the monomials $X_1^{\alpha_1}...X_n^{\alpha_n}$ with $\alpha_1 + ... + \alpha_n \leq i$. Now consider the associated graded algebra of the filtered algebra $\mathcal{U}(\mathfrak{g})$ defined by

$$\mathrm{gr}(\mathcal{U}(\mathfrak{g})) = \bigoplus_i \mathrm{gr}_i(U(\mathfrak{g})),$$

where $\mathrm{gr}_i(U(\mathfrak{g})) = U^i(\mathfrak{g})/U^{i-1}(\mathfrak{g})$. In particular, one has $\mathrm{gr}_0(\mathcal{U}(\mathfrak{g})) = \mathbb{K}$ and $\mathrm{gr}_1(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}$.
The equation (1.4) implies that $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))$ is a commutative and therefore the canonical inclusion $\mathfrak{g} \hookrightarrow \mathrm{gr}(\mathcal{U}(\mathfrak{g}))$ extends uniquely to an algebra homomorphism

$$\phi : \mathcal{S}(\mathfrak{g}) \to \mathrm{gr}(\mathcal{U}(\mathfrak{g})).$$

Using the PBW-theorem [theorem 1.5], one deduces that this is in fact an isomorphism of algebras. Moreover, ϕ maps the subspace $S^i(\mathfrak{g}) \subset \mathcal{S}(\mathfrak{g})$ of symmetric tensors of degree i onto $\mathrm{gr}_i(\mathcal{U}(\mathfrak{g}))$. Hence ϕ is a isomorphism of graded algebras.
As vector space $\mathcal{U}(\mathfrak{g})$ is isomorphic to $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))$. To construct an isomorphism amounts to choose vector space compliments of $U^{i-1}(\mathfrak{g})$ in $U^i(\mathfrak{g})$ for all i. There is a very natural choice of such compliments available. In fact, we have the following commutative diagram

$$\begin{tikzcd} T_i(\mathfrak{g}) \arrow[r, "q_i"] \arrow[d, "p_i"] & U^i(\mathfrak{g}) \arrow[d, "\mathrm{gr}_i"] \\ S^i(\mathfrak{g}) \arrow[r, "\phi_i"] & \mathrm{gr}_i(U(\mathfrak{g})) \end{tikzcd}$$

where q_i is the restriction to $T_i(\mathfrak{g})$ of the quotient map $q : \mathcal{T}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, p_i is the restriction to $T_i(\mathfrak{g})$ of the quotient map $p : \mathcal{T}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g})$ and gr_i the quotient map $U^i(\mathfrak{g}) \to \mathrm{gr}_i(U(\mathfrak{g}))$ The space $S^i(\mathfrak{g})$ of symmetric tensors of degree i can not only viewed as a quotient of $T_i(\mathfrak{g})$, but also naturally as a subspace $T_i(\mathfrak{g})$. The restriction of the linear map $\phi_i \circ p_i$ to this subspace is an isomorphism and therefore, by the commutativity of the diagram above, also $\mathrm{gr}_i \circ q_i$ restricted to this subspace is an isomorphism. Hence q_i has to map this subspace to a linear compliment of $U^{i-1}(\mathfrak{g})$ in $U^i(\mathfrak{g})$, which we denote by W_i and we have $U^i(\mathfrak{g}) = W_i \oplus U^{i-1}(\mathfrak{g})$. In particular, denoting by $\mathfrak{S}_i$ the group of permutations of i elements, we obtain linear isomorphisms $\Phi_i : S^i(\mathfrak{g}) \to W_i$ defined by

$$\Phi_i(X_1...X_i) = \frac{1}{i!} \sum_{\theta \in \mathfrak{S}_i} X_{\theta(1)}...X_{\theta(i)} \tag{1.5}$$

where the product on the left side is taken in $S^i(\mathfrak{g})$ and on the right side in $\mathcal{U}(\mathfrak{g})$. Since as a vector space $\mathcal{U}(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} W_i$, the direct sum of all maps Φ_i defines a linear isomorphism

$$\Phi : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}),$$

sometimes called the symmetrisation.
Suppose $\psi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism between two Lie algebras. By proposition 1.4 the composition $\mathfrak{g} \to \mathfrak{h} \hookrightarrow \mathcal{U}(\mathfrak{h})$, extends uniquely to a homomorphism $\mathcal{U}(\psi) : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ of unitial associative algebras. Explicitly, it is obtained from the induced map $\mathcal{T}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ by passing to the quotient $\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I}$. In fact, it can be directly seen that:

Proposition 1.8. *$\mathcal{U}$ is a covariant functor from the category of Lie algebras over $\mathbb{K}$ to the category of unitial associative algebras over $\mathbb{K}$.*

It is well know that $\mathcal{S}$ can be viewed as a covariant functor from the category of vector spaces to the category of symmetric associative unitial algebras. For a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$, one can deduce easily from the explicit formula (1.5) of Φ that $\mathcal{U}(\psi) \circ \Phi = \Phi \circ \mathcal{S}(\psi)$.

Summing up, we have:

Theorem 1.9.

(1) *The map $\Phi : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is an isomorphism of filtered vector spaces, where $\mathcal{S}(\mathfrak{g})$ is endowed with the filtration associated to its canonical grading. Moreover, the linear map induced by Φ between the associated graded spaces $\mathcal{S}(\mathfrak{g}) = \bigoplus_i S^i(\mathfrak{g})$ and $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))$ equals ϕ.*

(2) *For a Lie algebra homomorphism* $\psi : \mathfrak{g} \to \mathfrak{h}$ *between two Lie algebras, we have*

$$\mathcal{U}(\psi) \circ \Phi = \Phi \circ \mathcal{S}(\psi).$$

Moreover, from theorem 1.5 and the construction of Φ one deduces immediately the following proposition. For proof see section 2.4. in [**18**].

Proposition 1.10. *Suppose* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *for vector subspaces* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *of* $\mathfrak{g}$. *Denote by* Φ_i *the restriction of* Φ *to* $\mathcal{S}(\mathfrak{g}_i)$ *for* $i = 1, 2$. *The map*

$$\mathcal{S}(\mathfrak{g}_1) \otimes \mathcal{S}(\mathfrak{g}_2) \to \mathcal{U}(\mathfrak{g})$$

$$x_1 \otimes x_2 \mapsto \Phi_1(x_1)\Phi_2(x_2)$$

defines an isomorphism of vector spaces.

For an algebra A we denote by $\bar{A}$ the opposite algebra, i.e. A with the multiplication given by $(a, b) \mapsto -ab$, the negative of the multiplication of A. Note that we obviously have a linear injection $\tau : \mathfrak{g} \hookrightarrow \overline{\mathcal{U}(\bar{\mathfrak{g}})}$. One verifies directly that τ is a homomorphism of Lie algebras and hence by proposition 1.4 we obtain an algebra homomorphism $\tilde{\tau} : \mathcal{U}(\mathfrak{g}) \to \overline{\mathcal{U}(\bar{\mathfrak{g}})}$, which extends the identity on $\mathfrak{g}$. The PBW- theorem [theorem 1.5] implies that $\tilde{\tau}$ maps a linear basis of $\mathcal{U}(\mathfrak{g})$ to a linear basis of $\overline{\mathcal{U}(\bar{\mathfrak{g}})}$ and therefore $\tilde{\tau}$ is a isomorphism of unitial associative algebras. So we can identify the algebras $\overline{\mathcal{U}(\bar{\mathfrak{g}})}$ and $\mathcal{U}(\mathfrak{g})$. By proposition 1.8 the Lie algebra isomorphism $X \mapsto -X$ from $\mathfrak{g}$ to $\bar{\mathfrak{g}}$ extends to an isomorphism of associative algebras $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\bar{\mathfrak{g}})$. Composing this isomorphism with the anti-isomorphism $\mathcal{U}(\bar{\mathfrak{g}}) \to \overline{\mathcal{U}(\bar{\mathfrak{g}})}$ and identifying $\overline{\mathcal{U}(\bar{\mathfrak{g}})}$ with $\mathcal{U}(\mathfrak{g})$, one obtains the following proposition:

Proposition 1.11. *There exists a unique anti-automorphism* $u \mapsto u^\top$ *of the universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ *such that* $X^\top = -X$ *for all* $X \in \mathfrak{g}$. *Moreover, for elements* $X_1, ..., X_n \in \mathfrak{g}$ *we have*

$$(X_1 X_2 ... X_n)^\top = (-1)^n X_n X_{n-1} ... X_1.$$

The map $u \mapsto u^\top$ *is called the principal anti-automorphism of* $\mathcal{U}(\mathfrak{g})$.

1.2.3. Universal enveloping algebras and their relation to invariant differential operators. Suppose G is a real Lie group with Lie algebra $\mathfrak{g}$ and denote by $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ the complexification of $\mathfrak{g}$. Then it is well known that the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ can be identified with the algebra $\mathcal{D}_G(G)$ of linear left invariant differential operators on G.

Definition 1.7. A linear differential operator $D : C^\infty(G, \mathbb{C}) \to C^\infty(G, \mathbb{C})$ is called *left invariant*, if $\ell_g D(f) = D(\ell_g f)$ for all $g \in G$, where ℓ_g is the left translation by g defined by $\ell_g(f)(h) = f(g^{-1}h)$.

The space $\mathcal{D}_G(G)$ of all linear left invariant differential operators is obviously a subalgebra of the algebra of all linear differential operators $\mathcal{D}(G)$. For $X \in \mathfrak{g}$ we denote by L_X the left invariant vector field on G generated by X and for $Y = Y_1 + iY_2$ contained in $\mathfrak{g}^{\mathbb{C}}$ we set $L_Y = L_{Y_1} + iL_{Y_2}$.
Now suppose $X_1, ..., X_n$ is a basis of $\mathfrak{g}^{\mathbb{C}}$, then $L_{X_1}, ..., L_{X_n}$ is a global frame of the complexified tangent bundle $T_{\mathbb{C}}G$ of G. Hence by theorem 1.2 the left invariant operators $L_X^\alpha := L_{X_1}^{\alpha_1}...L_{X_n}^{\alpha_n}$ with $\alpha \in \mathbb{N}_0^n$ some multi-index, form a basis of the $C^\infty(G, \mathbb{C})$-module $\mathcal{D}(G)$. Therefore any linear differential operator $D \in \mathcal{D}(G)$ is of the form

$$D = \sum_\alpha a_\alpha L_X^\alpha \quad \text{ with } a_\alpha \in C^\infty(G, \mathbb{C}).$$

If $D \in \mathcal{D}_G(G)$, denoting by $e \in G$ the neutral element of G, we have

$$D(f)(g) = D(\ell_{g^{-1}} f)(e) = \sum a_\alpha(e)(L_X^\alpha(\ell_{g^{-1}} f))(e) = \sum a_\alpha(e)(L_X^\alpha f)(g)$$

and so the operators L_X^α form a basis of the complex vector space $\mathcal{D}_G(G)$. Moreover, since $L_{[X,Y]} = [L_X, L_Y]$, the map

$$\mathfrak{g}^{\mathbb{C}} \to \mathcal{D}_G(G)$$

$$X \mapsto L_X$$

is a Lie algebra homomorphism and hence by the universal property of the universal enveloping algebra [proposition 1.4] lifts to a homomorphisms of associative algebras

$$\mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \to \mathcal{D}_G(G).$$

This homomorphism is an isomorphism, since by the PBW-theorem [theorem 1.5] the monomials $X^\alpha = X_1^{\alpha_1}...X_n^{\alpha_n}$ form a basis of $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$. Therefore we have:

Theorem 1.12. *The Lie algebra homomorphism $X \mapsto L_X$ from $\mathfrak{g}^{\mathbb{C}}$ to $\mathcal{D}_G(G)$ induces an isomorphism of unitial associative algebras $\mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \cong \mathcal{D}_G(G)$.*

Remark 1.5. The map $X \mapsto L_X$ from $\mathfrak{g}$ to the algebra $\mathcal{D}_G(G, \mathbb{R})$ of left invariant operators with coefficients in $\mathbb{R}$ extends to an isomorphism between the real algebra $\mathcal{U}(\mathfrak{g})$ and $\mathcal{D}_G(G, \mathbb{R})$. The algebra $\mathcal{D}_G(G, \mathbb{R})$ is of course a real subalgebra of $\mathcal{D}_G(G)$. In accordance with this, applying proposition 1.4 to the canonical map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g} \otimes \mathbb{C})$, we obtain an injective homomorphism of real associative algebras $\mathcal{U}(\mathfrak{g}) \hookrightarrow \mathcal{U}(\mathfrak{g} \otimes \mathbb{C})$. In addition, this map induces an isomorphism of complex algebras $\mathcal{U}(\mathfrak{g}) \otimes \mathbb{C} \cong \mathcal{U}(\mathfrak{g} \otimes \mathbb{C})$.

1.2.4. Universal enveloping algebras of nilpotent graded Lie algebras. Let $\mathfrak{n} = \mathfrak{n}_{-1} \oplus ... \oplus \mathfrak{n}_{-k}$ be some real nilpotent graded Lie algebra and put $\mathfrak{n}_i = 0$ for $i < -k$ or $i \geq 0$. Suppose N is a Lie group with Lie algebra $\mathfrak{n}$. In example 1.1 we saw that N is naturally a filtered manifold with a filtration of the form $TN = T^{-k}N \supset ... \supset T^{-1}N$ and symbol algebra in each point isomorphic to $\mathfrak{n}$. Consider the algebra of differential operators $\mathcal{D}(N)$ filtered by the weighted order of differential operators. Filtering the algebra $\mathcal{D}_N(N)$ of left-invariant operators as well by the weighted order of differential operators

$$\mathcal{D}_N(N)^0 \subset \mathcal{D}_N(N)^1 \subset .... \subset \mathcal{D}_N(N)^i \subset ...,$$

the space $\mathcal{D}_N(N)$ becomes a filtered subalgebra of $\mathcal{D}(N)$. In accordance with this filtration we can introduce a *weighted* filtration on $\mathcal{U}(\mathfrak{n}^{\mathbb{C}})$ such that the isomorphism $\mathcal{U}(\mathfrak{n}^{\mathbb{C}}) \cong \mathcal{D}_N(N)$ is, after flipping signs, filtration preserving for these filtrations. In fact, we may even define an algebra grading on $\mathcal{U}(\mathfrak{n}^{\mathbb{C}})$ inducing this filtration.
For this purpose observe that the grading of $\mathfrak{n}$ induces a weighted algebra grading on the tensor algebra given by

$$\mathcal{T}(\mathfrak{n}) = \bigoplus_{i=0}^{\infty} \mathcal{T}_{-i}(\mathfrak{n}), \tag{1.6}$$

where

$$\mathcal{T}_{-i}(\mathfrak{n}) = \{\sum_j X_{j_1} \otimes ... \otimes X_{j_{s(j)}} : X_{j_\ell} \in \mathfrak{n}_{j_\ell} \text{ and } \sum_{\ell=1}^{s(j)} j_\ell = -i \quad \forall j\}.$$

Since $\mathfrak{n}$ is a graded Lie algebra $[\mathfrak{n}_\ell, \mathfrak{n}_m] \subset \mathfrak{n}_{\ell+m}$, the ideal $\mathcal{I}$ is homogeneous for this grading. Therefore the grading passes to an algebra grading on the universal enveloping algebra

$$\mathcal{U}(\mathfrak{n}) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(\mathfrak{n}), \tag{1.7}$$

where

$$\mathcal{U}_{-i}(\mathfrak{n}) = \mathcal{T}_{-i}(\mathfrak{n})/\mathcal{I} \cap \mathcal{T}_{-i}(\mathfrak{n}) = \{\sum_j X_{j_1} ... X_{j_{s(j)}} : X_{j_\ell} \in \mathfrak{n}_{j_\ell} \text{ and } \sum_{\ell=1}^{s(j)} j_\ell = -i \quad \forall j\}.$$

We will denote by upper indices the associated filtration

$$\mathcal{U}^0(\mathfrak{n}) \subset \mathcal{U}^{-1}(\mathfrak{n}) \subset ... \subset \mathcal{U}^{-i}(\mathfrak{n}) \subset ... \quad \text{with } \mathcal{U}^{-i}(\mathfrak{n}) = \bigoplus_{j=0}^{i} \mathcal{U}_{-j}(\mathfrak{n}).$$

The complexification $\mathfrak{n}^{\mathbb{C}}$ is also naturally a graded nilpotent Lie algebra and therefore we have as well a grading on $\mathcal{U}(\mathfrak{n}^{\mathbb{C}})$. The canonical isomorphism

$\mathcal{U}(\mathfrak{n}) \otimes \mathbb{C} \cong \mathcal{U}(\mathfrak{n}^{\mathbb{C}})$ thereby restricts to a linear isomorphism

$$\mathcal{U}_{-i}(\mathfrak{n}) \otimes \mathbb{C} \cong \mathcal{U}_{-i}(\mathfrak{n}^{\mathbb{C}}).$$

Obviously, we obtain:

Proposition 1.13. *Let $\mathfrak{n} = \mathfrak{n}_{-1} \oplus ... \oplus \mathfrak{n}_{-k}$ be a nilpotent graded Lie algebra and N a Lie group with Lie algebra $\mathfrak{n}$ endowed with its canonical tangential filtration $TN = T^{-k}N \supset ... \supset T^{-1}N$ making N into a regular filtered manifold of type $\mathfrak{n}$.*
Then the algebra isomorphism $\mathcal{U}(\mathfrak{n}^{\mathbb{C}}) \cong \mathcal{D}_N(N)$ of theorem 1.12 maps $\mathcal{U}^{-i}(\mathfrak{n}^{\mathbb{C}})$ onto $\mathcal{D}_N(N)^i$ and hence becomes an isomorphism of filtered algebras after flipping signs. Moreover, the unitial associative algebra $\mathcal{D}_N(N)$ is naturally a graded algebra

$$\mathcal{D}_N(N) = \bigoplus_{i=0}^{\infty} \mathcal{D}_N(N)_i,$$

where $\mathcal{D}_N(N)_i \subset \mathcal{D}_N(N)$ is the image of $\mathcal{U}_{-i}(\mathfrak{n}^{\mathbb{C}})$ under the isomorphism $\mathcal{U}(\mathfrak{n}^{\mathbb{C}})$ to $\mathcal{D}_N(N)$.

Remark 1.6. The reason for using negative indices for the weighted filtration on $\mathcal{U}(\mathfrak{n}^{\mathbb{C}})$ rather than positive, which would make the isomorphism $\mathcal{U}(\mathfrak{n}^{\mathbb{C}}) \cong \mathcal{D}_N(N)$ to an isomorphism of filtered algebras, will become comprehensible in chapter 3.

1.2.5. The algebra gr($\mathcal{D}(M)$) of weighted symbols of differential operators on a filtered manifold. Let M be a filtered manifold of depth k with filtration given by $TM = T^{-k}M \supset ... \supset T^{-1}M$. We know from section 1.1.1 that the Levi bracket $\mathcal{L}_x$ makes

$$\mathrm{gr}(T_xM) = \mathrm{gr}_{-1}(T_xM) \oplus ... \oplus \mathrm{gr}_{-k}(T_xM)$$

into a nilpotent graded Lie algebra, called the symbol algebra of M at $x \in M$. Therefore the universal enveloping algebra of the symbol algebra $\mathrm{gr}(T_xM)$ can be endowed with the grading defined in the previous section

$$\mathcal{U}(\mathrm{gr}(T_xM)) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(\mathrm{gr}(T_xM))$$

$$\mathcal{U}(\mathrm{gr}(T_xM)^{\mathbb{C}}) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(\mathrm{gr}(T_xM)^{\mathbb{C}}) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(\mathrm{gr}(T_xM)) \otimes \mathbb{C} = \mathcal{U}(\mathrm{gr}(T_xM)) \otimes \mathbb{C}.$$

Proposition 1.14. *For $r \in \mathbb{N}_0$ the disjoint union*

$$\mathcal{U}_{-r}(\mathrm{gr}(TM)) := \bigsqcup_{x \in M} \mathcal{U}_{-r}(\mathrm{gr}(T_xM))$$

is a vector bundle over M. In particular, we can view the direct sum

$$\mathcal{U}(\mathrm{gr}(TM)) := \bigoplus_{i=0}^{\infty} \mathcal{U}_{-r}(\mathrm{gr}(TM))$$

as a formal infinite dimensional vector bundle over M.

Proof. For a general filtered manifold, the Levi bracket $\mathcal{L}_x$ may change from point to point and hence symbol algebras need not to be isomorphic. Recall that $\mathcal{U}_{-r}(\mathrm{gr}(T_xM)) = \mathcal{T}_{-r}(\mathrm{gr}(T_xM))/(\mathcal{I} \cap \mathcal{T}_{-r}(\mathrm{gr}(T_xM)))$, where $\mathcal{I} \in \mathcal{U}(\mathrm{gr}(T_xM))$ is the ideal generated by elements of the form $X \otimes Y - Y \otimes X - \mathcal{L}_x(X,Y)$. So at the first glance this vector space depends on $\mathcal{L}_x$. However, by proposition 1.10 the space $\mathcal{U}_{-r}(\mathrm{gr}(T_xM))$ is always isomorphic to

$$\mathcal{S}_{-r}(\mathrm{gr}(T_xM)) := \bigoplus_{1i_1+\ldots+ki_k=r} S^{i_1}(\mathrm{gr}_{-1}(T_xM)) \otimes \ldots \otimes S^{i_k}(\mathrm{gr}_{-k}(T_xM))$$

and hence is actually independent of $\mathcal{L}_x$. Therefore $\mathcal{U}_{-r}(\mathrm{gr}(TM))$ can be given the structure of a vector bundle over M. □

In section 1.2.1 we considered the algebra of weighted symbols of linear differential operators on M defined as the associated graded algebra

$$\mathrm{gr}(\mathcal{D}(M)) = \bigoplus_{i=0}^{\infty} \mathrm{gr}_i(\mathcal{D}(M))$$

of the filtered algebra $\mathcal{D}(M)$ (filtered by the weighted order of operators). We observed that an element $\sigma_r(D) \in \mathrm{gr}_r(\mathcal{D}(M))$ can be considered as a map $x \mapsto \sigma_r(D)(x)$ from M to $\mathcal{U}_{-r}(x)$, where $\mathcal{U}_{-r}(x)$ is the $-r$-th grading component of the graded unitial associative algebra $\mathcal{U}(x) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(x)$ defined in section 1.2.1.

Recall that $\mathcal{U}_{-r}(x)$ consists of equivalence classes of linear differential operators of weighted order r, where two operators D and $\tilde{D}$ of weighted order r are equivalent, if

$$D - \tilde{D} = \sum_i f_i D_i + \textit{differential operators of weighted order } \leq r-1,$$

for differential operators D_i of weighted order r and smooth functions f_i, which vanish at x.

We have the following proposition, see also **[19]** in the case of contact manifolds:

Proposition 1.15. *For each $x \in M$ the graded unitial associative algebra $\mathcal{U}(x)$ is a isomorphic to the graded unitial associative algebra $\mathcal{U}(\mathrm{gr}(T_xM)^{\mathbb{C}})$. Moreover, the disjoint union $\mathcal{U} := \bigsqcup_{x \in M} \mathcal{U}(x)$ can be seen as a formal infinite dimensional vector bundle and sections of $\mathcal{U}$ can be identified with elements of $\mathrm{gr}(\mathcal{D}(M))$.*

PROOF. Let $U \subset M$ be an open subset over which all filtration components of the tangent bundle trivialise and choose an adapted local frame of $T_{\mathbb{C}}M$ defined on U given by

$$\{X_{1,1}, ..., X_{1,i(1)}, ..., X_{k,1}, ..., X_{k,i(k)}\}.$$

Recall that an adapted local frame induces for each point $x \in U$ an isomorphism $T_xM^{\mathbb{C}} \cong \mathrm{gr}(T_xM)^{\mathbb{C}}$.
Sending an element $X_x \in \mathrm{gr}_{-i}(T_xM)^{\mathbb{C}}$ to $\sigma_i(\xi)(x)$, where $\xi \in \Gamma(T^{-i}_{\mathbb{C}}M)$ such that $q_{-i}(\xi(x)) = X_x$ with $q_{-i} : T^{-i}_{\mathbb{C}}M \to \mathrm{gr}_{-i}(T_{\mathbb{C}}M)$ the natural projection, defines a linear map

$$\Omega : \mathrm{gr}(T_xM)^{\mathbb{C}} \to \mathcal{U}(x).$$

Since for $X_x \in \mathrm{gr}_{-i}(T_xM)^{\mathbb{C}}$ and $Y_x \in \mathrm{gr}_{-j}(T_xM)^{\mathbb{C}}$ the Levi bracket is given by

$$\mathcal{L}_x(X_x, Y_x) = q_{-(i+j)}([\xi, \eta](x))$$

with $\eta \in \Gamma(T^{-j}_{\mathbb{C}}M)$ such that $q_{-j}(\eta(x)) = Y_x$, we obtain that

$$\Omega(\mathcal{L}_x(X_x, Y_x)) = \sigma_{i+j}([\xi, \eta])(x),$$

which equals

$$\sigma_i(\xi)(x)\sigma_j(\eta)(x) - \sigma_j(\eta)(x)\sigma_i(\xi)(x)$$

by definition of the multiplication in $\mathcal{U}(x)$. Hence by the universal property of the universal enveloping algebra [proposition 1.4] the map Ω extends to a homomorphism of unitial associative algebras

$$\mathcal{U}(\mathrm{gr}(T_xM)^{\mathbb{C}}) \to \mathcal{U}(x).$$

By the PBW- theorem [theorem 1.5], the monomials $X^{\alpha_{1,1}}_{1,1}(x)...X^{\alpha_{k,i(k)}}_{k,i(k)}(x)$ are a basis of $\mathcal{U}(\mathrm{gr}(T_xM)^{\mathbb{C}})$ and such an element is mapped by Ω to the equivalence class in $\mathcal{U}_{-|\alpha|}(x)$ of the differential operator $X^{\alpha_{1,1}}_{1,1}...X^{\alpha_{k,i(k)}}_{k,i(k)}$, which can be seen as differential operator defined on M by trivial extension of the local vector fields $X_{\ell,j}$ to M. This implies immediately that Ω is injective.
Conversely, assume that D is a differential operator on M, whose restriction to U is of weighted order r and given by

$$D|U = \sum_{|\alpha| \leq r} a_\alpha X^{\alpha_{1,1}}_{1,1}...X^{\alpha_{k,i(k)}}_{k,i(k)},$$

for smooth functions $a_\alpha \in C^\infty(U, \mathbb{C})$. The element

$$\sum_{|\alpha|=r} a_\alpha(x) X^{\alpha_{1,1}}_{1,1}(x)...X^{\alpha_{k,i(k)}}_{k,i(k)}(x) \in \mathcal{U}_{-|\alpha|}(\mathrm{gr}(T_xM)^{\mathbb{C}})$$

is then mapped by Ω to the equivalence class in $\mathcal{U}_{-r}(x)$ of the differential operator

$$\tilde{D} = \sum_{|\alpha|=r} a_\alpha(x) X^{\alpha_{1,1}}_{1,1}...X^{\alpha_{k,i(k)}}_{k,i(k)},$$

where $a_\alpha(x)$ is viewed as the constant function $y \mapsto a_\alpha(x)$ on M. Since

$$D|_U - \tilde{D}|_U = \sum_{|\alpha|=r} (a_\alpha - a_\alpha(x)) X_{1,1}^{\alpha_{1,1}} ... X_{k,i(k)}^{\alpha_{k,i(k)}} +, \sum_{|\alpha|\leq r-1} a_\alpha X_{1,1}^{\alpha_{1,1}} ... X_{k,i(k)}^{\alpha_{k,i(k)}},$$

the operators D and $\tilde{D}$ define the same equivalence class in $\mathcal{U}_{-r}(x)$ and we conclude that Ω is an isomorphism of algebras. Since Ω obviously respects the grading, its an isomorphism of graded algebras. The rest now follows from proposition 1.14. □

Let $G(x)$ be the simply connected Lie group with Lie algebra $\mathrm{gr}(T_xM)$. We have seen in theorem 1.12 that elements in $\mathcal{U}(\mathrm{gr}(T_xM)^{\mathbb{C}})$ can be identified with left-invariant operators on the Lie group $Gr(x)$. Suppose M is a regular filtered manifold with symbol algebra $\mathfrak{n}$. Starting with a linear differential operator D of weighted order r on M, its weighted symbol $\sigma_r(D)(x)$ can be seen a smooth family of homogeneous left invariant differential operators of weighted order r on the simply connected Lie group N with Lie algebra $\mathfrak{n}$. Studying analytic properties of a linear differential operator D on M, the family of left-invariant operators $\sigma_r(D)(x)$ on N plays an important role, see **[19]**.

1.2.6. Weighted jet bundles. Suppose $\pi : E \to M$ is a complex or real vector bundle over some filtered manifold M. Following **[32]** we have a natural notion of weighted jet spaces of sections of E. The notion of weighted jet spaces presented here coincides with the one in **[32]** in the case of E being a trivially filtered vector bundle over M.

Definition 1.8. Let $\Gamma_x(E)$ be the space of germs of smooth sections of E at the point $x \in M$.
For $r \in \mathbb{N}_0$ two sections $s, s' \in \Gamma_x(E)$ are *r-equivalent* $\sim_r$, if

$$D(\langle \lambda, s - s' \rangle)(x) = 0$$

for all linear differential operators D on M of weighted order $\leq r$ and all sections λ of the dual bundle E^*, where $\langle\ ,\ \rangle : \Gamma(E^*) \times \Gamma(E) \to C^\infty(M, \mathbb{C})$ is the evaluation.

The relation $\sim_r$ obviously defines an equivalence relation on $\Gamma_x(E)$ and one can consider the quotient $\Gamma_x(E)/\sim_r$.

Definition 1.9. The quotient space of $\Gamma_x(E)$ by the relation $\sim_r$

$$\mathcal{J}_x^r(E) := \Gamma_x(E)/\sim_r$$

is called *the space of jets of weighted order r with source $x \in M$*.
For $s \in \Gamma_x(E)$ we denote by $j_x^r s$ the class of s in $\mathcal{J}_x^r(E)$.

Since for $s < r$ the relation $\sim_s$ is coarser as the relation $\sim_r$, we have linear projections from

$$\pi^r_s : \mathcal{J}^r_x(E) \to \mathcal{J}^s_x(E) \text{ for } s < r.$$

Proposition 1.16. *For $x \in M$ and $r \in \mathbb{N}$ we have an exact sequence of vector spaces*

$$0 \longrightarrow \mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x \xrightarrow{\iota} \mathcal{J}^r_x(E) \xrightarrow{\pi^r_{r-1}} \mathcal{J}^{r-1}_x(E) \longrightarrow 0.$$

Moreover, for any $r \in \mathbb{N}_0$ the vector space $\mathcal{J}^r_x(E)$ is finite dimensional and isomorphic to $\bigoplus_{i=0}^r \mathcal{U}_{-i}(\mathrm{gr}(T_xM))^* \otimes E_x$.

Proof. Assume E is a vector bundle with standard fiber $\mathbb{R}^m$ and fix a point $x \in M$.
Suppose s is a local section defined around x with $j^{r-1}_x s = 0$. Choosing some local trivialisation of E over an open neighbourhood U of x, we can view s as a smooth function $(s_1, ..., s_m) : U \subseteq M \to \mathbb{R}^m$.
For vector fields $\xi_1, .., \xi_\ell \in \Gamma(TM)$ with $\sum_i \mathrm{ord}(\xi_i) = r$ the value

$$(\xi_1 \cdot ... \cdot \xi_\ell \cdot s)(x) \in \mathbb{R}^m$$

depends only on the values of the vector fields at the point x, since $j^{r-1}_x s = 0$ (see (1.3) of section 1.2.1). By the same reason, it actually just depends on the elements $q_{-ord(\xi_i)}(\xi_i(x)) \in \mathrm{gr}_{-ord(\xi_i)}(T_xM)$. Therefore we obtain a well defined linear map

$$\mathcal{T}_{-r}(\mathrm{gr}(T_xM)) \to \mathbb{R}^m.$$

Additionally we have the symmetries of differentiation, like for example

$$\xi_1 \cdot \xi_2 \cdot ... \cdot \xi_\ell \cdot s - \xi_2 \cdot \xi_1 \cdot ... \cdot \xi_\ell \cdot s = [\xi_1, \xi_2] \cdot ... \cdot \xi_\ell \cdot s$$

and since

$$q_{-(ord(\xi_1)+ord(\xi_2))}([\xi_1, \xi_2](x)) = \mathcal{L}_x(\xi_1(x), \xi_2(x)),$$

the map above factorises to a to a linear map

$$\mathcal{U}_{-r}(\mathrm{gr}(T_xM)) \to \mathbb{R}^m.$$

Hence any element in the kernel of the projection $\mathcal{J}^r_x(E) \to \mathcal{J}^{r-1}_x(E)$ defines an element in $\mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x$ and so we have a linear map $\tau : \ker(\pi^r_{r-1}) \to \mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x$, which obviously is injective.
To see that it is even surjective we construct an inverse map. Let U be an open neighbourhood of x over which all filtration components of the tangent bundle and E trivialise. Choose an adapted local frame

$$\{X_{1,1}, ..., X_{1,i(1)}, ..., X_{k,1}, ..., X_{k,i(k)}\}$$

of TM defined on U. Recall that such an adapted local frame defines an isomorphism $T_yM \cong \mathrm{gr}(T_yM)$ for all $y \in U$, where the vector space spanned by $\{X_{j,1}(y), ..., X_{j,i(j)}(y)\}$ is mapped onto $\mathrm{gr}_{-j}(T_yM)$. Now suppose $\{f_{1,1}, ..., f_{1,i(1)}, ..., f_{k,1}, ..., f_{k,i(k)}\}$ are smooth functions such that

$$f_{\ell,q}(x) = 0 \quad \text{for all } \ell \text{ and } q \tag{1.8}$$

$$(X_{j,p} \cdot f_{\ell,q})(x) = 1 \quad \text{for } j = \ell \text{ and } p = q \tag{1.9}$$

$$(X_{j,p} \cdot f_{\ell,q})(x) = 0 \quad \text{otherwise.} \tag{1.10}$$

Recall that the monomials $X_{1,1}^{\alpha_{1,1}}(x)...X_{k,i(k)}^{\alpha_{k,i(k)}}(x)$ with $|\alpha| = r$ form a basis of $\mathcal{U}_{-r}(\mathrm{gr}(T_xM))$. For each multi-index α with $|\alpha| = r$ define $\phi_\alpha \in \mathcal{U}_{-r}(\mathrm{gr}(T_xM))^*$ as the linear functional given by

$$\phi_\alpha(X_{1,1}^{\alpha_{1,1}}(x)...X_{k,i(k)}^{\alpha_{k,i(k)}}(x)) = X_{1,1}^{\alpha_{1,1}} \cdot ... \cdot X_{k,i(k)}^{\alpha_{k,i(k)}}(f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}})(x)$$

$$\phi_\alpha((X_{1,1}^{\beta_{1,1}}(x)...X_{k,i(k)}^{\beta_{k,i(k)}}(x))) = 0 \quad \text{for } \beta \neq \alpha.$$

By its construction the functionals $\{\phi_\alpha : |\alpha| = r\}$ form a basis of $\mathcal{U}_{-r}(\mathrm{gr}(T_xM))^*$ and we define a linear map

$$\iota : \mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x \to \mathcal{J}_x^r(E)$$

by

$$\phi_\alpha \otimes e \mapsto j_x^r(f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}}s),$$

where s is some section of E with $s(x) = e \in E_x$. This is well defined, since whenever one of the functions $f_{\ell,q}$ occurring in $f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}}s$ is not differentiated, the resulting expression evaluated at x vanishes by (1.8) and hence $j_x^r((f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}}s)$ doesn't depend on the choice of s.
Moreover, by (1.8) - (1.10) the expression

$$X_{1,1}^{\beta_{1,1}} \cdot ... \cdot X_{k,i(k)}^{\beta_{k,i(k)}}(f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}}s)(x)$$

is zero for all multi-indices β with $|\beta| \leq r-1$ and so ι has values in $\ker(\pi_{r-1}^r)$. In addition, by (1.8) - (1.10) for any multi-index β with $|\beta| = r$ we have that the expression

$$X_{1,1}^{\beta_{1,1}} \cdot ... \cdot X_{k,i(k)}^{\beta_{k,i(k)}}(f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}}s)(x)$$

is zero for $\beta \neq \alpha$ and if $\beta = \alpha$, then

$$X_{1,1}^{\alpha_{1,1}} \cdot ... \cdot X_{k,i(k)}^{\alpha_{k,i(k)}}(f_{1,1}^{\alpha_{1,1}}...f_{k,i(k)}^{\alpha_{k,i(k)}}s)(x) =$$
$$= ((X_{1,1} \cdot f_{1,1})(x))^{\alpha_{1,1}}...((X_{k,i(k)} \cdot f_{k,i(k)})(x))^{\alpha_{k,i(k)}}s(x) = s(x) = e.$$

Therefore ι is injective and induces an isomorphism $\mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x \cong \ker(\pi_{r-1}^r)$, which is inverse to τ. Hence we have an exact sequence of vector spaces as claimed

$$0 \longrightarrow \mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x \xrightarrow{\iota} \mathcal{J}_x^r(E) \xrightarrow{\pi_{r-1}^r} \mathcal{J}_x^{r-1}(E) \longrightarrow 0.$$

Observing that $\mathcal{J}^0_x(E) = E_x$, the last statement follows by induction on r from this exact sequence. □

The space of jets of weighted order r is given as the disjoint union of all the vector spaces $\mathcal{J}^r_x(E)$.

Definition 1.10. For $r \in \mathbb{N}_0$ the disjoint union over all x of $\mathcal{J}^r_x(E)$

$$\mathcal{J}^r(E) := \bigsqcup_{x \in M} \mathcal{J}^r_x(E)$$

is called the *space of jets of weighted order* r. We denote by $\pi^r : \mathcal{J}^r(E) \to M$ the natural projection.

The weighted jet space $\mathcal{J}^r(E)$ can be endowed with the structure of a smooth manifold such that $\pi^r : \mathcal{J}^r(E) \to M$ is a vector bundle.

Theorem 1.17. *Let M be a filtered manifold and $\pi : E \to M$ a vector bundle over M.*

(1) *For $r \in \mathbb{N}_0$ the natural projection $\pi^r : \mathcal{J}^r(E) \to M$ is a vector bundle with fiber $\mathcal{J}^r_x(E)$ isomorphic to $\bigoplus_{i=0}^r \mathcal{U}_{-i}(\mathrm{gr}(T_xM))^* \otimes E_x$.*
(2) *For $r > s$ the projections*

$$\pi^r_s : \mathcal{J}^r(E) \to \mathcal{J}^s(E)$$

are vector bundle homomorphisms and for $r \in \mathbb{N}$ we have an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \xrightarrow{\iota} \mathcal{J}^r(E) \xrightarrow{\pi^r_{r-1}} \mathcal{J}^{r-1}(E) \longrightarrow 0.$$

Proof.

(1) Suppose E is a vector bundle with standard fiber $\mathbb{R}^m$. The associated graded bundle $\mathrm{gr}(TM)$ can be seen as a vector bundle modeled on a graded vector space $V = V_{-1} \oplus ... \oplus V_{-k}$. For a point $x \in M$ choose an open neighbourhood $U_x \subset M$ of x, over which E and all filtration components of TM trivialise. Denote by $\phi_x : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^m$ the local trivialisation of E over U_x and choose an adapted local frame $X_{1,1}, ..., X_{k,i(k)}$ of TM defined on U_x. Recall that the choice of an adapted frame induces a linear isomorphism $T_xM \cong \mathrm{gr}(T_xM) \cong V$ for all $x \in U$.
Now consider the map

$$\Phi^r_x : \mathcal{J}^r(E|U) \to U_x \times \bigoplus_{\ell=0}^r \mathcal{S}_{-\ell}(V)^* \otimes \mathbb{R}^m$$

$$\Phi^r_x : j^r_y s \mapsto (x, \sum_{\ell=0}^r \phi^{\ell,r}_x(j^r_y s)),$$

where $\mathcal{S}_{-\ell}(V)$ is defined as in the proof of proposition 1.14 and

$$\phi_x^{\ell,r}(j_y^r s) : \mathcal{S}_{-\ell}(V) \to \mathbb{R}^m$$

is defined by mapping a monomial in $\mathcal{S}_{-\ell}(V)$ of the form

$$X_{1,1}^{\alpha_{1,1}}(y)...X_{1,i(1)}^{\alpha_{1,i(1)}}(y) \otimes ... \otimes X_{k,1}^{\alpha_{k,1}}(y)...X_{k,i(k)}^{\alpha_{k,i(k)}}(y)$$

to

$$(X_{1,1}^{\alpha_{1,1}} \cdot ... \cdot X_{k,i(k)}^{\alpha_{k,i(k)}} \cdot \phi(s))(y).$$

By proposition 1.14 and proposition 1.16 the map Φ_x^r is a bijection satisfying that the restriction of Φ_x^r to $\mathcal{J}_y^r(E|U)$ defines a linear isomorphism between $\mathcal{J}_y^r(E|U) \cong \bigoplus_{\ell=0}^r \mathcal{S}_{-\ell}(V)^* \otimes \mathbb{R}^m$ for all $y \in U$. Obviously, we also have $\pi^r|_{\mathcal{J}^r(E|U)} = pr_1 \circ \Phi_x^r$, where pr_1 is the projection onto U_x.
We can endow $\mathcal{J}^r(E)$ with the unique manifold structure such that the maps Φ_x^r are diffeomorphisms. Then $\pi^r : \mathcal{J}^r(E) \to M$ is a vector bundle with a vector bundle chart defined around $x \in M$ given by Φ_x^r.
Finally, note that for U_x and U_y with $U_{x,y} = U_x \cap U_y \neq \emptyset$, the chart change $\Phi_x^r \circ (\Phi_y^r)^{-1}$ is given by

$$U_{x,y} \times \bigoplus_{\ell=0}^{r} \mathcal{S}_{-\ell}(V)^* \otimes \mathbb{R}^m \to U_{x,y} \times \bigoplus_{\ell=0}^{r} \mathcal{S}_{-\ell}(V)^* \otimes \mathbb{R}^m$$
$$\Phi_x^r \circ (\Phi_y^r)^{-1} : (z, u) \mapsto (z, \Phi_{x,y}^r(z)(u)),$$

where the isomorphism

$$\Phi_{x,y}^r(z) : \bigoplus_{\ell=0}^{r} \mathcal{S}_{-\ell}(V)^* \otimes \mathbb{R}^m \cong \bigoplus_{\ell=0}^{r} \mathcal{S}_{-\ell}(V)^* \otimes \mathbb{R}^m$$

is induced from the graded vector space isomorphism $V \cong V$ corresponding to change of local adapted frames and the linear isomorphism $\phi_{x,y}(z) : \mathbb{R}^m \cong \mathbb{R}^m$ defined by $\phi_x \circ \phi_y^{-1}(z, u) = (z, \phi_{x,y}(z)(u))$.

(2) In terms of vector bundle charts the projections π_s^r are given by the natural projections

$$\Phi_x^s \circ \pi_s^r \circ (\Phi_x^r)^{-1}$$
$$U_x \times \bigoplus_{i=0}^{r} \mathcal{S}_{-i}(V)^* \otimes \mathbb{R}^m \to U_x \times \bigoplus_{i=0}^{s} \mathcal{S}_{-i}(V)^* \otimes \mathbb{R}^m$$

and therefore they are vector bundle homomorphisms.
The kernel of π_{r-1}^r is the vector bundle $\mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E$ with standard fiber $\mathcal{S}_{-r}(V)^* \otimes \mathbb{R}^m$ and so we have an injective vector bundle map $\iota : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \hookrightarrow \mathcal{J}^r(E)$ corresponding to the inclusion $\mathcal{S}_{-r}(V)^* \otimes \mathbb{R}^m \hookrightarrow \bigoplus_{i=0}^r \mathcal{S}_{-i}(V)^* \otimes \mathbb{R}^m$. The map of course coincides with the one defined in proposition 1.16.

□

Suppose E and F are vector bundles over M and let $\phi : E \to F$ be a vector bundle map. Then we can lift ϕ to a vector bundle map $\mathcal{J}^r(\phi) : \mathcal{J}^r(E) \to \mathcal{J}^r(F)$ defined by

$$\mathcal{J}^r(\phi)(j^r_x s) = j^r_x(\phi(s)).$$

This is well defined, since the right-hand side just depends on the r-jet of s at $x \in M$. It is the unique vector bundle map such that the diagram

$$\begin{tikzcd} \Gamma(\mathcal{J}^r(E)) \arrow[r, "\mathcal{J}^r(\phi)"] & \Gamma(\mathcal{J}^r(F)) \\ \Gamma(E) \arrow[u, "j^r"] \arrow[r, "\phi"] & \Gamma(F) \arrow[u, "j^r"'] \end{tikzcd}$$

commutes. We have $\mathcal{J}^r(id_E) = id_{\mathcal{J}^r(E)}$ and for vector bundle maps $\phi : E \to F$ and $\psi : F \to G$ we obtain $\mathcal{J}^r(\psi \circ \phi) = \mathcal{J}^r(\psi) \circ \mathcal{J}^r(\phi)$. Therefore we obtain:

Theorem 1.18. *For any $r \geq 1$, $\mathcal{J}^r$ is a covariant functor acting on the category of vector bundles.*

In particular, this implies:

Corollary 1.19. *If E is a vector bundle endowed with a complex structure $I : E \to E$, then $\mathcal{J}^r(E)$ admits as well a complex structure given by $\mathcal{J}^r(I) : \mathcal{J}^r(E) \to \mathcal{J}^r(E)$. In addition, the projections $\pi^r_s : \mathcal{J}^r(E) \to \mathcal{J}^s(E)$ are complex vector bundle homomorphism.*

Now we can define the weighted order of a differential operator $D : \Gamma(E) \to \Gamma(F)$ between sections of vector bundles E and F as follows:

Definition 1.11. Suppose E and F are vector bundles over a filtered manifold M.
A differential operator $D : \Gamma(E) \to \Gamma(F)$ is of *weighted order* $\leq r$, if for any point $x \in M$ and any two section $s, t \in \Gamma(E)$ the equation $j^r_x s = j^r_x t$ implies that $D(s)(x) = D(t)(x)$.
The smallest number $r \in \mathbb{N}_0$ such that this holds, is called the *weighted order* of D.

Given a differential operator $D : \Gamma(E) \to \Gamma(F)$ of weighted order r, we obtain a bundle map $\phi : \mathcal{J}^r(E) \to F$ defined by $\phi(j^r_x s) = D(s)(x)$.
Conversely, if $\phi : \mathcal{J}^r(E) \to F$ is a bundle map, then $D = \phi \circ j^r$ defines a differential operator of weighted order r, where $j^r : \Gamma(E) \to \Gamma(\mathcal{J}^r(E))$ is the *universal differential operator of weighted order* r given by $s \mapsto (x \mapsto j^r_x s)$.
Therefore we can equivalently view a differential operator $D : \Gamma(E) \to \Gamma(F)$

of weighted order r as a bundle map from $\mathcal{J}^r(E) \to F$.
Note that a differential operator $D : \Gamma(E) \to \Gamma(F)$ of weighted order r is linear if and only if the associated bundle map $\phi : \mathcal{J}^r(E) \to F$ is a vector bundle map.

Definition 1.12. Let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator of weighted order r with associated bundle map $\phi : \mathcal{J}^r(E) \to F$.
The *weighted symbol* $\sigma_r(\phi)$ of D is the composition of ϕ with the canonical inclusion $\iota : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \hookrightarrow \mathcal{J}^r(E)$ (see theorem 1.17). So we have:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \arrow[r, "\iota"] \arrow[dr, "\sigma_r(\phi)"'] & \mathcal{J}^r(E) \arrow[r, "\pi^r_{r-1}"] \arrow[d, "\phi"] & \mathcal{J}^{r-1}(E) \arrow[r] & 0 \\
& & F & &
\end{tikzcd}$$

We sometimes will also just write $\sigma(\phi)$ for the weighted symbol.

Note that by its definition the weighted symbol of a differential operator D describes exactly the part of highest weighted order of D.

Remark 1.7. Let $E = M \times \mathbb{C}$ be the trivial complex line bundle over a filtered manifold M. Suppose $\phi : \mathcal{J}^r(E) \to E$ is a vector bundle map and let $D = \phi \circ j^r : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ be the corresponding differential operator of weighted order r. The weighted symbol is a bundle map

$$\sigma_r(\phi) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes \mathbb{C} \to \mathbb{C}.$$

Hence $\sigma_r(\phi)$ can be viewed as a section of $\mathcal{U}_{-r}(\mathrm{gr}(TM)) \otimes \mathbb{C}$. Note that this coincides with our definition of the weighted symbol $\sigma_r(D)$ of D in section 1.2.1.

Remark 1.8.

(1) If M is a trivial filtered $TM = T^{-1}M$, then the weighted jet bundle $\mathcal{J}^r(E)$ of a vector bundle E coincides with the usual vector bundle $J^r(E)$ of jets of order r. The symbol algebra of M at some point $x \in M$ is just the tangent space T_xM viewed as abelian Lie algebra and the weighted symbol $\sigma_r(\phi)$ of a differential operator $\phi : \mathcal{J}^r(E) \to F$ is the usual principal symbol $\sigma_r(\phi) : S^r(T_xM)^* \otimes E \to F$.

(2) Suppose that M is a filtered manifold and let E be a vector bundle over M. Note that there are always projections $J^r(E) \to \mathcal{J}^r(E)$ from the usual jet bundle of order r to the jet bundle of weighted order r, since the equivalence relation defining $J^r(E)$ is finer than the relation $\sim_r$ of definition 1.8.
Conversely, there are projections $\mathcal{J}^\ell(E) \to J^r(E)$ for all $\ell \geq d \geq r$, where d is a number depending on the depth of the filtered manifold.

1.3. Systems of partial differential equation of weighted finite type

In [**38**] Spencer investigates an interesting class of linear systems of partial differential equations, which he calls linear systems of finite type. For a linear system of differential equations of finite type it can be shown that a solution is already determined by a finite jet in a single point and hence the dimension of its solution space is always finite dimensional.
On filtered manifolds the phenomenon occurs that there are many examples of differential equations for which a solution is already determined by finitely many partial derivatives in a single point, but which are not of finite type in the classical sense of Spencer. This indicates again that differential equations on filtered manifold should be better studied within the framework of weighted jet bundles and the notion of finite type should be adapted to the weighted setting.
In this section we introduce the notion of a linear system of differential equations of weighted finite type on a filtered manifold and show, using ideas of [**23**], that to such a system one may always associate canonically a differential operator of weighted order one with injective weighted symbol whose kernel describes the solution of the original system. We will see that rewriting a linear system of weighted finite type in this form, implies that a solution is already determined by a finite weighted jet in a single point, hence its solution space is always finite dimensional. In addition, we will obtain obstruction to the existence of solutions.

1.3.1. The formal theory. Throughout this section we will assume that M is a filtered manifold of depth k such that the symbol algebra $\mathrm{gr}(T_xM) = \mathrm{gr}_{-k}(T_xM) \oplus ... \oplus \mathrm{gr}_{-1}(T_xM)$ in each point $x \in M$ is generated by $\mathrm{gr}_{-1}(T_xM)$. Comparing with section 1.1.2, this means that $T^{-1}M \subset TM$ is a regular bracket generating distribution. Further let E and F be vector bundles over M and suppose $\phi : \mathcal{J}^r(E) \to F$ is a vector bundle map.

Definition 1.13. For all $\ell \geq 0$ the *ℓ-th prolongation* $p_\ell(\phi) : \mathcal{J}^{r+\ell}(E) \to \mathcal{J}^\ell(F)$ of ϕ is defined by

$$p_\ell(\phi)(j_x^{r+\ell}s) = j_x^\ell(\phi(j^r s)).$$

This is well defined, since the righthand side just depends on the weighted $r+\ell$ jet of s at the point x.
The ℓ-th prolongation $p_\ell(\phi)$ can be characterised as the unique vector bundle

map such that the diagram

$$\begin{tikzcd}
\Gamma(\mathcal{J}^{r+\ell}(E)) \arrow[r, "p_\ell(\phi)"] & \Gamma(\mathcal{J}^{\ell}(F)) \\
\Gamma(E) \arrow[u, "j^{r+\ell}"] \arrow[r, "\phi\circ j^r"] & \Gamma(F) \arrow[u, "j^\ell"']
\end{tikzcd}$$

commutes.

Remark 1.9. The notion of the ℓ-th prolongation of a linear differential operator $\phi : \mathcal{J}^r(E) \to F$ doesn't require of course that the filtered base manifold M satisfies that $\mathrm{gr}_{-1}(T_xM)$ generates $\mathrm{gr}(T_xM)$. Moreover, let us remark that the notion of prolongation here coincides with the one in **[32]** in the case, where E and F are trivially filtered vector bundles over M.

In particular, we have the vector bundle map

$$p_\ell(id_r) : \mathcal{J}^{r+\ell}(E) \to \mathcal{J}^{\ell}(\mathcal{J}^r(E)),$$

where id_r is the identity map on $\mathcal{J}^r(E)$. Any derivative in direction transversal to $T^{-1}M$ can be expressed by iterated derivatives in direction of the subbundle $T^{-1}M$, since we assumed that $\mathrm{gr}_{-1}(T_xM)$ generates $\mathrm{gr}(T_xM)$ as Lie algebra. Therefore $p_\ell(id_r)$ is injective.
By definition we have $\mathcal{J}^\ell(\phi) \circ p_\ell(id_r) = p_\ell(\phi)$, where $\mathcal{J}^\ell(\phi) : \mathcal{J}^\ell(\mathcal{J}^r(E)) \to \mathcal{J}^\ell(F)$ is the bundle map induced by ϕ on the ℓ-jet spaces, see proposition 1.18.

Since we have the inclusion $p_1(id_r) : \mathcal{J}^{r+1}(E) \hookrightarrow \mathcal{J}^1(\mathcal{J}^r(E))$, we can consider the operator δ^r of weighted order one defined by the projection

$$\mathcal{J}^1(\mathcal{J}^r(E)) \to \mathcal{J}^1(\mathcal{J}^r(E))/\mathcal{J}^{r+1}(E).$$

This operator can now be characterised, analogously as in **[23]** for usual jet bundles:

We have the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{U}_{-(r+1)}(\mathrm{gr}(TM))^* \otimes E \arrow[r] \arrow[d, "\iota"'] & \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^r(E) \arrow[r] \arrow[d, "\iota"'] & W^r \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{J}^{r+1}(E) \arrow[r, "p_1(id_r)"] \arrow[d, "\pi_r^{r+1}"'] & \mathcal{J}^1(\mathcal{J}^r(E)) \arrow[r, "\delta^r"] \arrow[d, "\pi_0^1"'] & \mathcal{J}^1(\mathcal{J}^r(E))/\mathcal{J}^{r+1}(E) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}^r(E) \arrow[r] \arrow[d] & \mathcal{J}^r(E) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the inclusion of $\mathcal{U}_{-(r+1)}(\mathrm{gr}(TM))^* \otimes E$ into $\mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^r(E)$ is obtained by the commutativity of the next two rows and the space W^r is defined by the diagram. Moreover, this diagram induces an isomorphism of vector bundles between W^r and $\mathcal{J}^1(\mathcal{J}^r(E))/\mathcal{J}^{r+1}(E)$. Therefore we can view δ^r as an operator from $\mathcal{J}^1(\mathcal{J}^r(E))$ to W^r. Hence we have the following proposition:

Proposition 1.20. *There exists a unique differential operator*

$$\delta^r : \mathcal{J}^1(\mathcal{J}^r(E)) \to W^r$$

of weighted order one such that
- *the kernel of* δ^r *is* $\mathcal{J}^{r+1}(E)$
- *the weighted symbol* $\sigma(\delta^r) : \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^r(E) \to W^r$ *is the projection.*

Proof. The uniqueness follows from the exactness of the diagram above. ☐

By a similar reasoning as in **[23]** we can now deduce the existence of a first order operator $S : \mathcal{J}^1(\mathcal{J}^r(E)) \to \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)$. We will call S the *weighted Spencer operator.*

Proposition 1.21. *There exists a unique differential operator*

$$S : \mathcal{J}^1(\mathcal{J}^r(E)) \to \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)$$

of weighted order one such that
- $\mathcal{J}^{r+1}(E) \subseteq \ker(S)$
- *the weighted symbol* $\sigma(S) : \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^r(E) \to \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)$ *is* $id \otimes \pi^r_{r-1}$.

Moreover, we have the following exact sequence of sheaves:

$$0 \longrightarrow \Gamma(E) \xrightarrow{j^r} \Gamma(\mathcal{J}^r(E)) \xrightarrow{S \circ j^1} \Gamma(\mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)) \longrightarrow 0$$

Proof. If such an operator exists, it must factorise over δ^r by proposition 1.20, since $\mathcal{J}^{r+1}(E) \subseteq \ker(S)$. This means that it has to be of the form $S = \psi \circ \delta^r$ for some bundle map $\psi : W^r \to \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)$. By the second property ψ has to satisfy that $\sigma(S) = \psi \circ \sigma(\delta^r)$ equals the projection $id \otimes \pi^r_{r-1}$. To see that such a map ψ exists, we have to show that $id \otimes \pi^r_{r-1}$ factorises over $\sigma(\delta^r)$.
We already know that $\ker(\sigma(\delta^r)) = \ker(\pi^{r+1}_r)$, which is mapped under the inclusion $p_1(id_r) : \mathcal{J}^{r+1}(E) \hookrightarrow \mathcal{J}^1(\mathcal{J}^r(E))$ to $\mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^r(E)$. Since the map $\mathcal{J}^1(\pi^r_{r-1}) : \mathcal{J}^1(\mathcal{J}^r(E)) \to \mathcal{J}^1(\mathcal{J}^{r-1}(E))$ has symbol $\iota \circ id \otimes \pi^r_{r-1}$ and we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{J}^{r+1}(E) & \xrightarrow{p_1(id_r)} & \mathcal{J}^1(\mathcal{J}^r(E)) \\ {\scriptstyle \pi^{r+1}_r}\downarrow & & \downarrow{\scriptstyle \mathcal{J}^1(\pi^r_{r-1})} \\ \mathcal{J}^r(E) & \xrightarrow{p_1(id_{r-1})} & \mathcal{J}^1(\mathcal{J}^{r-1}(E)) \end{array}$$

we conclude that $\ker(\pi^{r+1}_r)$ is mapped under the inclusion $p_1(id_r)$ to the kernel of $id \otimes \pi^r_{r-1}$. Hence $id \otimes \pi^r_{r-1}$ factorises over $\sigma(\delta^r)$. So there exists a unique bundle map $\psi : W^r \to \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)$ with $\psi \circ \sigma(\delta^r) = id \otimes \pi^r_{r-1}$ and we can define $S = \psi \circ \delta^r$.
To show the exactness of the sequence above let us describe S in another way. Consider the bundle map

$$\mathcal{J}^1(\pi^r_{r-1}) - p_1(id_{r-1}) \circ \pi^1_0 : \mathcal{J}^1(\mathcal{J}^r(E)) \to \mathcal{J}^1(\mathcal{J}^{r-1}(E)).$$

Since $\pi^1_0 \circ \mathcal{J}^1(\pi^r_{r-1}) = \pi^1_0 \circ p_1(id_{r-1}) \circ \pi^1_0$, this operator actually has values in $\mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E)$. Moreover, $\mathcal{J}^{r+1}(E)$ lies in its kernel by the commutative diagram above and the symbol is given by the symbol of $\mathcal{J}^1(\pi^r_{r-1})$ which equals $\iota \circ (id \otimes \pi^r_{r-1})$. Hence viewing S as an operator from $\mathcal{J}^1(\mathcal{J}^r(E))$ to $\mathcal{J}^1(\mathcal{J}^{r-1}(E))$ by means of the inclusion $\iota : \mathrm{gr}_{-1}(TM)^* \otimes \mathcal{J}^{r-1}(E) \hookrightarrow \mathcal{J}^1(\mathcal{J}^{r-1}(E))$, we must have

$$S = \mathcal{J}^1(\pi^r_{r-1}) - p_1(id_{r-1}) \circ \pi^1_0 : \mathcal{J}^1(\mathcal{J}^r(E)) \to \mathcal{J}^1(\mathcal{J}^{r-1}(E)).$$

Suppose now we have a section of $\mathcal{J}^r(E)$ which can be written as $j^r s$ for some $s \in \Gamma(E)$. Then it lies in the kernel of $S \circ j^1$, since $\mathcal{J}^1(\pi^r_{r-1}) \circ j^1(j^r s) = j^1(\pi^r_{r-1}(j^r s)) = j^1(j^{r-1} s) = p_1(id_{r-1})(j^r s)$.
To show the converse one can proceed by induction on r.
If $r = 1$, then for $s \in \Gamma(\mathcal{J}^1(E))$ to be in the kernel of $S \circ j^1$ means $j^1(\pi^1_0(s)) = p_1(id_0)s = s$. Now suppose the assertion holds for r. If $s \in \Gamma(\mathcal{J}^{r+1}(E))$ satisfies $j^1(\pi^{r+1}_r(s)) = p_1(id_r)(s)$, then

$$\mathcal{J}^1(\pi^r_{r-1})(j^1(\pi^{r+1}_r(s))) = \mathcal{J}^1(\pi^r_{r-1})(p_1(id_r)(s)).$$

From the commutative diagram above we know that the right side coincides with $p_1(id_{r-1})(\pi^{r+1}_r(s))$. By the induction hypothesis $\pi^{r+1}_r(s) = j^r(u)$ for some $u \in \Gamma(E)$. Now s must equal $j^{r+1}u$, since $p_1(id_r)(s) = j^1(\pi_r^{r+1}(s)) = j^1(j^r u)$ and $p_1(id_r)$ is injective. □

1.3.2. Universal prolongation of systems of weighted finite type. Suppose that E and F are vector bundles over a filtered manifold M and let $\phi : \mathcal{J}^r(E) \to F$ be a vector bundle map of constant rank, then the vector subbundle of $\mathcal{J}^r(E)$ defined by its kernel

$$Q^r := \ker(\phi)$$

is called the *linear system of differential equations associated to the differential operator* ϕ. A *solution* of Q^r is a section s of E satisfying $\phi(j^r s) = 0$.

The *symbol* of $Q^r \subset \mathcal{J}^r(E)$ is the family of vector spaces $K = \{K_x\}_{x\in M}$ over M given by the kernel of the projection

$$\pi^r_{r-1}|_{Q^r_x} : Q^r_x \to \mathcal{J}^{r-1}_x(E).$$

Note that K_x is exactly the kernel of the weighted symbol

$$\sigma(\phi) : \mathcal{U}_{-r}(\mathrm{gr}(T_xM))^* \otimes E_x \to F_x.$$

Definition 1.14. For $\ell \geq 0$ the *ℓ-th prolongation* $Q^{r+\ell}$ of Q^r is the kernel of the ℓ-th prolongation $p_\ell(\phi) : \mathcal{J}^{r+\ell}(E) \to \mathcal{J}^\ell(F)$ of ϕ.

Since the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
 & & \mathcal{J}^{r+\ell}(E) \arrow[r, "p_\ell(\phi)"] \arrow[d, "p_\ell(id_r)"'] & \mathcal{J}^\ell(F) \arrow[d, "id"] \\
0 \arrow[r] & \mathcal{J}^\ell(Q^r) \arrow[r] & \mathcal{J}^\ell(\mathcal{J}^r(E)) \arrow[r, "\mathcal{J}^\ell(\phi)"] & \mathcal{J}^\ell(F)
\end{tikzcd}$$

commutes, we have

$$Q^{r+\ell} = \mathcal{J}^\ell(Q^r) \cap \mathcal{J}^{r+\ell}(E).$$

In general, the vector bundle map $p_\ell(\phi)$ is not of constant rank and $Q^{r+\ell}$ need not to be a vector bundle.

Definition 1.15. We call a vector bundle map $\phi : \mathcal{J}^r(E) \to F$ respectively the corresponding linear differential operator *regular*, if $p_\ell(\phi)$ is of constant rank for all $\ell \geq 0$.

For $\ell \geq 1$ the symbol of the prolonged equation $Q^{r+\ell}$ is the family of vector spaces $g^{r+\ell} := \{g_x^{r+\ell}\}_{x\in M}$ over M, where $g_x^{r+\ell}$ is the kernel of the linear map $Q_x^{r+\ell} \to Q_x^{r+\ell-1} \subset \mathcal{J}_x^{r+\ell-1}(E)$ given by the restriction of the

projection $\pi^{r+\ell}_{r+\ell-1}$ to $Q^{r+\ell}_x$.
For all $\ell \geq 1$ we have a vector bundle map

$$\overline{\sigma}_\ell(\phi) : \mathcal{U}_{-(r+\ell)}(\mathrm{gr}(TM))^* \otimes E \to \mathcal{U}_{-\ell}(\mathrm{gr}(TM))^* \otimes F,$$

which we call the ℓ-th *symbol mapping*. It is defined by the following (fiber-wise) commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & g^{r+\ell} \arrow[r] \arrow[d] & \mathcal{U}_{-(r+\ell)}(\mathrm{gr}(TM))^* \otimes E \arrow[r, "\overline{\sigma}_\ell(\phi)"] \arrow[d] & \mathcal{U}_{-\ell}(\mathrm{gr}(TM))^* \otimes F \arrow[d] \\
0 \arrow[r] & Q^{r+\ell} \arrow[r] \arrow[d] & \mathcal{J}^{r+\ell}(E) \arrow[r, "p_\ell(\phi)"] \arrow[d] & \mathcal{J}^{\ell}(F) \arrow[d] \\
0 \arrow[r] & Q^{r+\ell-1} \arrow[r] & \mathcal{J}^{r+\ell-1}(E) \arrow[r, "p_{\ell-1}(\phi)"] & \mathcal{J}^{\ell-1}(F)
\end{tikzcd}$$

Remark 1.10. The ℓ-th symbol mapping composed with the inclusion $\mathcal{U}_{-\ell}(\mathrm{gr}(TM))^* \otimes F \hookrightarrow \mathcal{J}^\ell(F)$ coincides of course just with the weighted symbol of the ℓ-th prolongation $p_\ell(\phi)$ of ϕ.

By definition the kernel of the ℓ-th symbol mapping $\overline{\sigma}_\ell(\phi)$ is $g^{r+\ell}$ viewed as a subset of $\mathcal{U}_{-(r+\ell)}(\mathrm{gr}(TM))^* \otimes E$.
Since $Q^{r+\ell} = \mathcal{J}^\ell(Q^r) \cap \mathcal{J}^{r+\ell}(E)$ and the diagram

$$\begin{tikzcd}
\mathcal{J}^{r+\ell}(E) \arrow[r, "p_\ell(id_r)"] \arrow[d, "\pi^{r+\ell}_{r+\ell-1}"'] & \mathcal{J}^\ell(\mathcal{J}^r(E)) \arrow[d, "\pi^\ell_{\ell-1}"] \\
\mathcal{J}^{r+\ell-1}(E) \arrow[r, "p_{\ell-1}(id_r)"] & \mathcal{J}^{\ell-1}(\mathcal{J}^r(E))
\end{tikzcd}$$

commutes, we conclude that

$$g^{r+\ell}_x = \mathcal{U}_{-(r+\ell)}(\mathrm{gr}(T_xM))^* \otimes E_x \cap \mathcal{U}_{-\ell}(\mathrm{gr}(T_xM))^* \otimes K_x$$

where $K = \{K_x\}_{x\in M}$ is the kernel of the weighted symbol $\sigma(\phi)$ of ϕ.

Definition 1.16. A system of linear differential equations $Q^r \subset \mathcal{J}^r(E)$ is called of *weighted finite type*, if there exists $m \in \mathbb{N}$ such that $g^{r+\ell}_x = 0$ for all $x \in M$ and $\ell \geq m$.

For equations of weighted finite type we can prove the following theorem:

Theorem 1.22. *Suppose M is a filtered manifold such that $\mathrm{gr}_{-1}(T_xM)$ generates $\mathrm{gr}(T_xM)$ for all $x \in M$ and suppose E and F are vector bundles over M.*
Let $D : \Gamma(E) \to \Gamma(F)$ be a regular differential operator of weighted order r

defining a system of linear differential equations of weighted finite type. Then for some $\ell_0 \in \mathbb{N}$ *there exists a differential operator*

$$D' : \Gamma(Q^{r+\ell_0}) \to \Gamma(W^{r+\ell_0}) \quad \textit{of weighted order one with injective symbol}$$

such that $s \mapsto j^{r+\ell_0}s$ *induces a bijection:*

$$\{s \in \Gamma(E) : D(s) = 0\} \leftrightarrow \{s' \in \Gamma(Q^{r+\ell_0}) : D'(s') = 0\}.$$

Proof. Let us denote by $\phi : \mathcal{J}^r(E) \to F$ the bundle map associated to D and by $Q^r \subset \mathcal{J}^r(E)$ the differential equation given by the kernel of ϕ. For all $\ell \geq 0$ we can consider the operator $D^{r+\ell} : \Gamma(Q^{r+\ell}) \to \Gamma(W^{r+\ell})$ of weighted order one given by the restriction of $\delta^{r+\ell} \circ j^1$ to $\Gamma(Q^{r+\ell})$.
If $s \in \Gamma(E)$ is a solution $Ds = 0$, then $j^{r+\ell}s \in \Gamma(Q^{r+\ell})$ and since $j^1(j^{r+\ell}s)$ is a section of $\mathcal{J}^{r+\ell+1}(E) \subset \mathcal{J}^1(\mathcal{J}^{r+\ell}(E))$ we also have $D^{r+\ell}(j^{r+\ell}s) = 0$.
And conversely, if s' is a section of $Q^{r+\ell}$ such that $D^{r+\ell}(s') = 0$, then j^1s' is a section of $\mathcal{J}^{r+\ell+1}(E)$. Since $\mathcal{J}^{r+\ell+1}(E)$ is contained in the kernel of the weighted Spencer operator $\mathcal{J}^1(\mathcal{J}^{r+\ell}(E)) \to \mathcal{J}^1(\mathcal{J}^{r+\ell-1}(E))$, the section s' equals $j^{r+\ell}s$ for some section $s \in \Gamma(E)$. Obviously $\pi_0^{r+\ell}(j^{r+\ell}s) = s$ then satisfies $Ds = 0$.
This shows that for all $\ell \geq 0$ the map $j^{r+\ell}$ induces a bijection between solutions of D and solutions of $D^{r+\ell}$. So it remains to prove that there exists some ℓ_0 such that $D^{r+\ell_0}$ has injective symbol.
The symbol of $D^{r+\ell}$ is a bundle map $\mathcal{U}_{-1}(\mathrm{gr}(TM))^* \otimes Q^{r+\ell} \to W^{r+\ell}$. We know that the kernel of $\sigma(\delta^{-(r+\ell)})$ is $\mathcal{U}_{r+\ell+1}(\mathrm{gr}(TM))^* \otimes E$. Since $D^{r+\ell}$ is just the restriction of $\delta^{r+\ell} \circ j^1$ to $\Gamma(Q^{r+\ell})$, we obtain that

$$\ker(\sigma(D^{r+\ell}))_x = \mathcal{U}_{-(r+\ell+1)}(\mathrm{gr}(T_xM))^* \otimes E_x \cap \mathcal{U}_{-1}(\mathrm{gr}(T_xM))^* \otimes g_x^{r+\ell}.$$

But $g_x^{r+\ell} = \mathcal{U}_{-(r+\ell)}(\mathrm{gr}(T_xM))^* \otimes E_x \cap \mathcal{U}_{-\ell}(\mathrm{gr}(T_xM))^* \otimes K_x$ where K is the kernel of the symbol of D. Therefore we get

$$\ker(\sigma(D^{r+\ell}))_x = \mathcal{U}_{-(r+\ell+1)}(\mathrm{gr}(T_xM))^* \otimes E_x \cap \mathcal{U}_{-(\ell+1)}(\mathrm{gr}(T_xM))^* \otimes K_x$$

which coincides with $g_x^{r+\ell+1}$.
Since the equation Q^r is of finite type, there exists ℓ_0 such that $g^{r+\ell_0+1} = 0$ and hence $D^{r+\ell_0} : \Gamma(Q^{r+\ell_0}) \to \Gamma(W^{r+\ell_0})$ is a differential operator of weighted order one with injective symbol, whose solutions are in bijective correspondence with solutions of the original equation Q^r. □

As a consequence of this theorem, we obtain:

Corollary 1.23. *Let* M *be a connected filtered manifold such that for all* $x \in M$ *the symbol algebra* $\mathrm{gr}(T_xM)$ *is generated by* $\mathrm{gr}_{-1}(T_xM)$ *. Suppose* $D : \Gamma(E) \to \Gamma(F)$ *is a regular linear differential operator of weighted order* r *defining a system of linear differential equations of weighted finite type.*

Then any solution $s \in \Gamma(E)$ is already determined by a finite weighted jet in a single point. In particular, the solution space is finite dimensional.

PROOF. By theorem 1.22, we know that solutions of $D(s) = 0$ are in bijective correspondence with solutions of $D^{r+\ell_0}(s') = 0$, where $D^{r+\ell_0}$ is the operator of weighted order one with injective symbol of the theorem. Denote by $\phi^{r+\ell_0}$ the bundle map corresponding to $D^{r+\ell_0}$ and consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{gr}_{-1}(TM)^* \otimes Q^{r+\ell_0} \arrow[r] \arrow[dr, "\sigma(\phi^{r+\ell_0})"'] & \mathcal{J}^1(Q^{r+\ell_0}) \arrow[r, "\pi^1_0"] \arrow[d, "\phi^{r+\ell_0}"'] & Q^{r+\ell_0} \arrow[r] & 0 \\ & & W^{r+\ell_0} & & \end{tikzcd}$$

Now choose a splitting $\mathcal{J}^1(Q^{r+\ell_0}) \to \mathrm{gr}_{-1}(TM)^* \otimes Q^{r+\ell_0}$ of the short exact sequence. Viewing this splitting as a partial connection $\nabla : \Gamma(Q^{r+\ell_0}) \to \Gamma(\mathrm{gr}_{-1}(TM)^* \otimes Q^{r+\ell_0})$, we see from the diagram that

$$D^{r+\ell_0} = \sigma(\phi^{r+\ell_0}) \circ \nabla + a, \tag{1.11}$$

where $a : \Gamma(Q^{r+\ell_0}) \to \Gamma(W^{r+\ell_0})$ is induced by some vector bundle map. Suppose $c : I \to M$ is a smooth curve satisfying $c'(t) \in T^{-1}_{c(t)}M$ for all $t \in I$ and consider the equation for a section μ of the pullback bundle $c^*Q^{r+\ell_0}$

$$\sigma(\phi^{r+\ell_0})(\nabla_{c'(t)}\mu(t)) + a(\mu(t)) = 0. \tag{1.12}$$

Since the $\sigma(\phi^{r+\ell_0})$ is injective, this defines an ordinary differential equation of order one and therefore a solution is determined by its value in one point. Suppose $s' \in \Gamma(E)$ is a solution of (1.11), then $t \mapsto s'(c(t))$ must be a solution of the equation (1.12). Since $T^{-1}M$ is bracket generating and M is connected, any two points can be connected by curve c with $c'(t) \in T^{-1}_{c(t)}M$ for all $t \in I$ due to the theorem of Chow, see e.g. chapter 2 in **[29]**. Therefore a solution s' is already determined by its value in one point. Hence the original equation is determined by its weighted $r + \ell_0$-jet in one point. □

Moreover, since $D^{r+\ell_0}$ is of weighted order one with injective symbol, it induces a vector bundle map $\rho : Q^{r+\ell_0} \to W^{r+\ell_0}/(\mathrm{gr}_{-1}(TM)^* \otimes Q^{r+\ell_0})$. Any solution s' of $D^{r+\ell_0}$ must clearly also satisfy $\rho(s') = 0$, which leads to obstructions for the existence of solutions.

Remark 1.11. The fact that a differential equation of weighted finite type has finite dimensional solution space was (by other means) already observed earlier by Morimoto, **[33]**.

Summing up, we have seen in theorem 1.22 that there is always a canonical bijection between solutions of a system of linear differential equations of weighted finite type and solutions of a certain prolonged system of weighted order one. Considering this prolonged system leads to bounds for the dimension of the solution space of the original system and even to obstructions to the existence of solutions. However, as it stands, this approach is quite abstract. In chapter 3 we will therefore return to this problem and will show that for a large class of system of weighted finite type on filtered manifolds, admitting an additional geometric structure, the bundle $Q^{r+\ell_0}$ can be explicitly computed and present a conceptual way to construct a linear connection on $Q^{r+\ell_0}$, whose parallel sections correspond to solutions of the studied weighted finite type system. Finally, let us remark that Eastwood and Gover in **[21]**, recently developed a prolongation theory on contact manifolds using higher order connections, which is closely related to the results of this section.

CHAPTER 2

Parabolic Geometries

A geometric structure on a manifold M, which always induces a filtration of the tangent bundle TM, is a parabolic geometry. Parabolic geometries are special types of Cartan geometries. Under certain conditions a parabolic geometry on some manifold M is equivalent to some underlying geometric structure, which consists of a filtration of the tangent bundle TM, making M into a filtered manifold, and a reduction of the structure group of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of the associated graded bundle $\mathrm{gr}(TM)$. In this chapter we will give a short introduction to parabolic geometries and their underlying structures. In our presentation we mainly follow [**15**] and [**7**].

2.1. Basic definitions and notations

2.1.1. Cartan Geometries. In order to relate geometry in the sense of Felix Klein's Erlangen program and differential geometry, Élie Cartan introduced at the beginning of the last century a common generalisation of both concepts under the name *éspaces généralisés*. In the Erlangen program a geometry is given by a manifold together with a transitive left action of a Lie group G. Hence up to a base point by a homogeneous space G/H where G is acting from the left by multiplication. The constitutive idea of Cartan now was to endow the homogeneous space G/H with a geometric structure such that the automorphisms of this structure become the left multiplications by elements of G. The ingredient to recognise left multiplications among all diffeomorphisms of G/H is provided by the Maurer-Cartan form of G, the canonical one form on G with values in the Lie algebra of G, which encodes the left trivialisation of the tangent bundle of G. Let us explain this more explicitly.

We fix a Lie group G and a closed subgroup $H \subseteq G$ and denote by $\mathfrak{g}$ the Lie algebra of G. Then the homogeneous space G/H can be endowed with the geometric structure consisting of the following data:

(1) the principal H-bundle given by the projection $p : G \to G/H$
(2) the Maurer-Cartan form $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$ on G defined by

$$\omega_{MC}(\xi_g) = T_g\lambda_{g^{-1}}\xi_g,$$

where $\lambda_{g^{-1}} : G \to G$ denotes the left multiplication by g^{-1}.

Considering G/H endowed with this geometric structure, the automorphisms of G/H are defined to be the principal bundle isomorphisms $\phi : G \to G$, which in addition preserve the Maurer-Cartan form $\phi^*\omega_{MC} = \omega_{MC}$. It is not hard to see, that if G/H is connected, the automorphisms are exactly the left multiplications by elements of G.
A Cartan geometry of type (G, H) is a generalisation of the situation above. It is given by a principal H-bundle over a manifold of the same dimension as G/H and a $\mathfrak{g}$-valued one form on the total space of this bundle, which has all the basic properties of the Maurer-Cartan form that still make sense.

Definition 2.1. Suppose G is a Lie group and $H \subseteq G$ a closed subgroup and denote by $\mathfrak{g}$ and $\mathfrak{h}$ its Lie algebras.
A *Cartan Geometry* of type (G, H) on a manifold M is given by

(1) a principal fiber bundle $p : \mathcal{G} \to M$ with structure group H
(2) a *Cartan connection*: a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with values in $\mathfrak{g}$ such that:
   - ω is H-equivariant : $(r^h)^*\omega = Ad(h^{-1}) \circ \omega$ for all $h \in H$
   - ω reproduces the generators of fundamental vector fields: $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{h}$
   - ω trivialises $T\mathcal{G}$: $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$

where $r : \mathcal{G} \times H \to \mathcal{G}$ denote the principal right action of H on $\mathcal{G}$ and $Ad : G \to GL(\mathfrak{g})$ the adjoint representation.

Note that the definition implies that the dimension of M equals the one of G/H.
Since the Maurer-Cartan form satisfies the defining properties of a Cartan connection, the canonical bundle $p : G \to G/H$ endowed with the Maurer-Cartan form defines a Cartan geometry on G/H, which is called the *homogeneous model* for Cartan geometries of type (G, H).

Definition 2.2. A *morphism* between Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\mathcal{G}' \to M', \omega')$ of type (G, H) is a principal bundle morphism $\phi : \mathcal{G} \to \mathcal{G}'$ satisfying $\phi^*\omega' = \omega$.

Note that the condition $\phi^*\omega' = \omega$ implies that the tangent map of ϕ in each point is a linear isomorphism. Hence ϕ and its base map are local diffeomorphisms.
For a Cartan geometry $(\mathcal{G} \to M, \omega)$ one has the notion of its curvature.

Definition 2.3. The *curvature* of a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, H) is the $\mathfrak{g}$-valued two from $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

The fact that the Cartan connection ω by definition trivialises the tangent bundle $T\mathcal{G}$ implies that any differential form on $\mathcal{G}$ is already determined by its values on the constant vector fields $\omega^{-1}(X)$ for $X \in \mathfrak{g}$. Therefore, the curvature of a Cartan geometry can be equivalently encoded in the *curvature function*

$$\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$$

defined by

$$\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(X)(u)).$$

From the first two properties of a Cartan connection follows immediately that the curvature is horizontal and H-equivariant. Thus, we have:

Lemma 2.1. *The curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ is horizontal and hence the curvature function can be viewed as a function $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$. In addition, the equivariancy of the Cartan connection implies*

$$(r^h)^*K = Ad(h^{-1}) \circ K$$

$$\kappa \circ r^h = \lambda(h^{-1}) \circ \kappa,$$

where λ is the representation of H defined as the tensor product of the H-representations $\Lambda^2(\mathfrak{g}/\mathfrak{h})^*$ and $\mathfrak{g}$, which are induced by the adjoint action $Ad : G \to GL(\mathfrak{g})$.

Note that the Maurer-Cartan form always satisfies

$$d\omega_{MC}(\xi, \eta) + [\omega_{MC}(\xi), \omega_{MC}(\eta)] = 0.$$

Hence the curvature of the homogeneous model vanishes identically. Conversely, one can show that vanishing of the curvature is already a full obstruction to local flatness of the geometry:

Proposition 2.2. *The curvature of a Cartan geometry $(\mathcal{G} \to M, \omega)$ vanishes identically if and only if $(\mathcal{G} \to M, \omega)$ is locally isomorphic to the homogeneous model.*

This proposition implies that the curvature of a Cartan geometry of type (G, H) measures to which amount the geometry differs from its homogeneous model. In that sense a general Cartan geometry of type (G, H) can be viewed as a curved analog of the homogeneous space G/H. For a proof of the proposition see for instance section 1.5.2. in **[15]**.
Against the background of the intention of Cartan to reconcile different concepts of geometry, one of the most motivating examples of a geometric structure, which can be also considered as a Cartan geometry, was the one of a Riemannian structure.

Example 2.1. Let G be the group of euclidean motion on $\mathbb{R}^n$ and $H = O(n) \subset G$ the subgroup of orthogonal transformations. Then G/H is the euclidean space $\mathbb{R}^n$. Suppose $(\mathcal{G} \to M, \omega)$ is a Cartan geometry of type (G, H). Since the Lie algebra $\mathfrak{g}$ of G is isomorphic as H-module to $\mathfrak{h} \oplus \mathbb{R}^n$, the Cartan connection decomposes into a $\mathfrak{h}$-valued one form γ and a $\mathbb{R}^n$-valued one form θ. It can be easily seen that θ defines a reduction of the linear frame bundle of M to H, which is the same as a Riemannian metric and γ a principal connection on $\mathcal{G}$ which is equivalent to a metric connection. If γ is torsion free, it must be the Levi-Civita connection. Conversely, given a Riemannian manifold of dimension n, one can construct a torsion free Cartan geometry by setting $\mathcal{G}$ the orthonormal frame bundle and defining ω to be the sum of the Levi-Civita-connection and the soldering form. In this way, one gets an equivalence of categories between torsion free Cartan geometries of type (G, H) and Riemannian manifolds of dimension n.

Knowing that a geometric structure can be equivalently described as a Cartan geometry has immediately strong consequences. For example, it can be shown that the automorphism group of a Cartan geometry of type (G, H) is a finite dimensional Lie group whose dimension is at most the dimension of G, see section 1.5.11. in [**15**]. The Lie algebra of the automorphism group can be completely described in terms of the Lie algebra of G and the curvature of the geometry. In certain cases the description can be even improved, which leads to interesting results about possible dimensions of such automorphism groups, see [**11**]. Also it is not hard to see that every local automorphism of the homogeneous model extends uniquely to a global one, see section 1.5.11. in [**15**] .
In the last decades, a special class of Cartan geometries, namely so called parabolic geometries, on which will focus in the next section, were studied intensively and a lot of tools were developed to study these geometries. Under certain conditions, parabolic geometries are always determined by some underlying geometric structure. These underlying structures cover a large variety of geometric structures and to consider them as parabolic geometries leads to powerful results.

2.1.2. Parabolic Geometries. A parabolic geometry is a Cartan geometry of type (G, P), where G is a semisimple Lie group and P a parabolic subgroup. Parabolic subgroups are defined via parabolic subalgebras. There are several ways to define parabolic subalgebras in semisimple Lie algebras. The most convenient way for our purposes is to view them as subalgebras which determine $|k|$-gradings on semisimple Lie algebras. In particular, this will enable us to deal with the real and complex case simultaneously.

Definition 2.4. Let $\mathfrak{g}$ be a complex or real semisimple Lie algebra and $k > 1$ an integer. A $|k|$-*grading* on $\mathfrak{g}$ is a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$$

such that

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, where we set $\mathfrak{g}_i = \{0\}$ for $|i| > k$
- the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_{-1}$ is generated as Lie algebra by $\mathfrak{g}_{-1}$
- $\mathfrak{g}_{\pm k} \neq \{0\}$

By the grading property $\mathfrak{g}_0$ is a subalgebra and each $\mathfrak{g}_i$ is a $\mathfrak{g}_0$-module. Given a $|k|$-graded semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$, we have the associated filtration of the vector space $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset ... \supset \mathfrak{g}^0 \supset ... \supset \mathfrak{g}^k \text{ where } \mathfrak{g}^i = \bigoplus_{j=i}^{k} \mathfrak{g}_j$$

which makes $\mathfrak{g}$ into a filtered Lie algebra. By the grading property, $\mathfrak{g}^0$ is subalgebra and each $\mathfrak{g}^i$ is $\mathfrak{g}^0$-modules.
For later use we fix some notation. We set $\mathfrak{p} := \mathfrak{g}^0$ and $\mathfrak{p}_+ := \mathfrak{g}^1$. Obviously $\mathfrak{p}_+$ is a subalgebra of $\mathfrak{g}$ and a nilpotent ideal in the algebra $\mathfrak{p}$.
Now we collect some basic properties about $|k|$-graded semisimple Lie algebras, for a proof see **[43]** and section 3.1. of **[15]**:

Proposition 2.3.

(1) *There exists a unique element $e \in \mathfrak{z}(\mathfrak{g}_0)$, called the grading element, which satisfies $[e, X] = jX$ for $X \in \mathfrak{g}_j$ for $j = -k, ..., k$.*
(2) *The isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ induced by the Killing form is compatible with the grading and the filtration. In particular, we have a duality between the $\mathfrak{g}_0$-modules $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ and a duality of $\mathfrak{p}$-modules between $\mathfrak{g}^i$ and $\mathfrak{g}/\mathfrak{g}^{-i+1}$. In particular, we have an isomorphism of $\mathfrak{g}_0$-modules $\mathfrak{p}_+ \cong \mathfrak{g}_-^*$ and an isomorphism of $\mathfrak{p}$-modules $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$.*
(3) *the algebras $\mathfrak{g}_0$ and $\mathfrak{p}$ can be characterised by*

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : ad(X)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = -k, ..., k\}$$
$$\mathfrak{p} = \{X \in \mathfrak{g} : ad(X)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for } i = -k, ..., k\}$$

Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a graded semisimple Lie algebra and G be a Lie group with Lie algebra $\mathfrak{g}$. Then the subgroup

$$P := \{g \in G : Ad(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \text{ for } i = -k, ..., k\}$$

is closed, since it is the intersection of the normalisers $N_G(\mathfrak{g}^i)$ of $\mathfrak{g}^i$ in G, and has Lie algebra $\mathfrak{p}$. Hence any other closed subgroup of G with Lie algebra $\mathfrak{p}$ lies between $P = \bigcap_{i=-k}^{k} N_G(\mathfrak{g}^i)$ and its connected component of the identity.

A closed subgroup of G with Lie algebra $\mathfrak{p}$ is called a *parabolic subgroup* (corresponding to the given $|k|$-grading). We will see in the next section that this definition of parabolic subgroups coincide with the one used in representation theory. Having fixed a parabolic subgroup P corresponding to a given $|k|$-grading, the closed subgroup

$$G_0 := \{g \in P : Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = -k, ..., k\}$$

is called the *Levi subgroup* of P. Its Lie algebra is $\mathfrak{g}_0$.
Moreover, we have the following theorem about the subgroups $G_0 \subset P \subset G$, for a proof see the section 3.1.3. in **[15]**:

Theorem 2.4. *The map $(g_0, Z) \mapsto g_0 exp(Z)$ defines a diffeomorphism $G_0 \times \mathfrak{p}_+ \to P$. In particular, $P_+ := exp(\mathfrak{p}_+)$ is a closed nilpotent subgroup of G.*

Note that, since for $g \in P$ and $Z \in \mathfrak{p}_+$ we have $Ad(g)(Z) \in \mathfrak{p}_+$, the group P_+ is a normal subgroup of P and $P/P_+ \cong G_0$.

Definition 2.5. A *parabolic geometry* is a Cartan geometry of type (G, P), where G is a semisimple Lie group whose Lie algebra is endowed with a $|k|$-grading and P is a closed subgroup with Lie algebra $\mathfrak{p}$.

The homogeneous space G/P is a so called *generalised flag manifold.* If G is connected, G/P is always compact in the complex case and in the real case provided that the center of G is finite, see e.g. section 3.2.6. and section 3.2.11. in **[15]**.

2.1.3. $|k|$-gradings and their relation to parabolic subalgebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is then defined to be *parabolic*, if $\mathfrak{p}$ contains a Borel subalgebra.
Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and denote by Δ the set of roots associated to $\mathfrak{h}$. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha \in \Delta$. Choosing a simple subsystem of roots $\Delta^0 \subset \Delta$, any root can be written as a linear combination of simple roots, where all coefficients are integers either all ≤ 0 or ≥ 0. Denoting by $\Delta^+ \subset \Delta$ the subset of positive roots, i.e. the subset of those roots which can be written as a linear combination of simple roots with positive coefficients, the set of roots Δ is actually given by the disjoint union $\Delta = \Delta^+ \sqcup -\Delta^+$, where $-\Delta^+ \subset \Delta$ is called the subset of negative roots.
The subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

is a maximal solvable subalgebra of $\mathfrak{g}$, called the *standard Borel subalgebra* associated to $\mathfrak{h}$ and Δ^0. A *standard parabolic subalgebra* is a subalgebra which contains $\mathfrak{b}$.

It is well known that standard parabolic subalgebras can be classified by subsets $\Sigma \subset \Delta^0$ of simple roots. In fact, the map

$$\mathfrak{p} \mapsto \Sigma_{\mathfrak{p}} = \{\alpha \in \Delta^0 : \mathfrak{g}_{-\alpha} \nsubseteq \mathfrak{p}\}$$

defines a bijection between standard parabolic subalgebras and subsets of simple roots, where the inverse is given by assigning to a subset Σ of simple roots the algebra $\mathfrak{p}_\Sigma$, which is the direct sum of $\mathfrak{b}$ and all root spaces corresponding to negative roots, which can be written as a linear combination of elements of $\Delta^0 \setminus \Sigma$.

The fact that Cartan subalgebras and the choice of a simple subsystem of roots are unique up to conjugation can as well be formulated as the fact that any two Borel subalgebras are conjugate by an inner automorphism of $\mathfrak{g}$. Therefore, having fixed a Cartan subalgebra and a simple subsystem of roots, every parabolic subalgebra is conjugate to a standard one. Hence up to conjugation a parabolic subalgebra can be uniquely described by a subset Σ of simple roots and we may denote a parabolic subalgebra by replacing in the Dynkin diagram of $\mathfrak{g}$ all dots corresponding to roots in Σ by crosses.

Let $\mathfrak{p}$ be a standard parabolic subalgebra and denote by $ht_{\Sigma_p}(\alpha)$ the $\Sigma_{\mathfrak{p}}$-height of α, i.e. the sum of all coefficients of elements in $\Sigma_{\mathfrak{p}}$ in the representation of α as linear combination of simple roots.

Then $\mathfrak{p}$ determines a $|k|$-grading on $\mathfrak{g}$ as follows

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{ht_{\Sigma_{\mathfrak{p}}}(\alpha)=0} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_i = \bigoplus_{ht_{\Sigma_{\mathfrak{p}}}(\alpha)=i} \mathfrak{g}_\alpha$$

with $\mathfrak{g}^0 = \mathfrak{p}$.

Conversely, given a $|k|$-grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$, one can show that there is a choice of $\mathfrak{h}$ and Δ^0 such that $\mathfrak{g}^0$ contains the standard Borel subalgebra. Hence $\mathfrak{g}^0$ is parabolic. Moreover, one can show that the grading is given by $\Sigma_{\mathfrak{g}^0}$-height. Therefore we have:

Proposition 2.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra with corresponding root system Δ and $\Delta^0 \subset \Delta$ a simple subsystem. Then the map, which maps a standard parabolic subalgebra $\mathfrak{p}$ to the $|k|$-grading of $\mathfrak{g}$ given by $\Sigma_{\mathfrak{p}}$-height , defines a one to one correspondence between conjugation classes of parabolic subalgebras of $\mathfrak{g}$ and isomorphism classes of $|k|$-gradings of $\mathfrak{g}$.*

Moreover, using the description of $|k|$-gradings in terms of weights, one can prove:

Proposition 2.6. *Suppose $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a complex semisimple $|k|$-graded Lie algebra. Then the following holds:*

(1) *The subalgebra $\mathfrak{g}_0$ is reductive. It is called the Levi subalgebra of the parabolic subalgebra $\mathfrak{p} := \mathfrak{g}^0$.*

(2) *If we choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a simple subsystem of roots Δ^0 such that $\mathfrak{p}$ becomes a standard parabolic subalgebra and denote by $\{H_\alpha\}_{\alpha\in\Delta^0}$ the basis of $\mathfrak{h}$, where $H_\alpha \in \mathfrak{h}$ corresponds under the isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ induced by the Killing form to $\alpha \in \Delta^0$, then*
$$\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}_0 \oplus \bigoplus_{ht_{\Sigma_\mathfrak{p}}(\alpha)=0} \mathfrak{g}_\alpha,$$
where $\mathfrak{z}(\mathfrak{g}_0)$ is the center of $\mathfrak{g}_0$ and $\mathfrak{h}_0$ is the linear span of all the H_α with $\alpha \in \Delta^0 \backslash \Sigma_\mathfrak{p}$. Hence the dimension of $\mathfrak{z}(\mathfrak{g}_0)$ equals the number of elements in $\Sigma_\mathfrak{p}$. Moreover, the subalgebra $\mathfrak{h}_0$ is a Cartan subalgebra of the semisimple part $\mathfrak{g}_0^{ss}$ of $\mathfrak{g}_0$ whose corresponding root decomposition is exactly
$$\mathfrak{g}_0^{ss} = \mathfrak{h}_0 \oplus \bigoplus_{ht_{\Sigma_\mathfrak{p}}(\alpha)=0} \mathfrak{g}_\alpha.$$

This proposition implies that the Dynkin diagram of the semisimple part $\mathfrak{g}_0^{ss}$ of the Levi subalgebra of a parabolic subalgebra $\mathfrak{p}$ is obtained by removing all crossed nodes and all edges connecting to them from the crossed Dynkin diagram of the parabolic subalgebra $\mathfrak{p}$.

In the real case one can proceed similar. Suppose $\mathfrak{g}$ is a real semisimple Lie algebra. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic*, if its complexification $\mathfrak{p}_\mathbb{C}$ is a parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$.
Choosing a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$, we can decompose $\mathfrak{g}$ into eigenspaces for θ
$$\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{k},$$
where $\mathfrak{q}$ is the eigenspace to the eigenvalue -1 and $\mathfrak{k}$ the one to the eigenvalue 1. Recall that then $\mathfrak{k}$ is a subalgebra, $[\mathfrak{k}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{k}$ and that $B_\theta(X, Y) := -B(X, \theta(Y))$ defines a positive definite bilinear form on $\mathfrak{g}$, where B is the Killing form of $\mathfrak{g}$.
Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{q}$ has maximal possible dimension under all θ-stable Cartan subalgebras. Such a Cartan subalgebra can always be constructed by first choosing a maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{q}$ and then choosing a maximal abelian subalgebra $\mathfrak{t}$ in the centraliser $Z_\mathfrak{k}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$ and setting $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$.

Having choosen θ and $\mathfrak{h}$, we denote by Δ the root system of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. It is easy to see that with respect to B_θ one always has $ad(\theta(X))^t = -ad(X)$. This implies that $ad(A)$ for $A \in \mathfrak{a}$ is a family of commuting symmetric linear maps and therefore simultaneously diagonalisable, where the eigenvalues are linear functionals $\lambda : \mathfrak{a} \to \mathbb{R}$. We denote by Δ_r the set of non-zero eigenfunctionals, called the *restricted roots* of $\mathfrak{g}$ (with respect to $\mathfrak{a}$). Then $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta_r} \mathfrak{g}_\lambda.$$

By construction elements of Δ_r are restrictions to $\mathfrak{a}$ of elements of Δ and a restricted root space $\mathfrak{g}_\lambda$ is just the intersection of $\mathfrak{g}$ with the direct sum of those root spaces $(\mathfrak{g}_{\mathbb{C}})_\alpha$ with $\alpha|_{\mathfrak{a}} = \lambda$.
Let $\Delta^+ \subset \Delta$ be an admissible positive subsystem, i.e. for $\alpha \in \Delta^+$ we have either $\sigma^*\alpha = -\alpha$ or $\sigma^*\alpha \in \Delta^+$, where $\sigma^* : \Delta \to \Delta$ is the involutive automorphism defined by $\sigma^*\alpha(H) := \overline{\alpha(\sigma(H))}$ with σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$. Then the image of Δ^+ under the restriction map $\Delta \to \Delta_r$ is a positive subsystem Δ_r^+ of the root system Δ_r and we denote by Δ_r^0 the corresponding subset of simple restricted roots.
A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is then called a *standard parabolic* subalgebra, if $\mathfrak{p}_{\mathbb{C}}$ is a standard parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$with respect to $\mathfrak{h}_{\mathbb{C}}$ and Δ^+. It can be shown that standard parabolic subalgebras are exactly those subalgebras of $\mathfrak{g}$, which contain the subalgebra $Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta_r^+} \mathfrak{g}_\lambda$. From this one can deduce that the map

$$\mathfrak{p} \mapsto \Sigma_{\mathfrak{p}} = \{\lambda \in \Delta_r^0 : \mathfrak{g}_{-\lambda} \nsubseteq \mathfrak{p}\}$$

defines a bijection between standard parabolic subalgebras and subsets of restricted simple roots.
Lets us briefly recall how the *Satake diagram* of a real semisimple Lie algebra $\mathfrak{g}$ is defined. Choosing θ, $\mathfrak{h}$ and Δ^+ as above, we may consider the subset of *compact roots* given by $\Delta_c := \{\alpha \in \Delta : \sigma^*(\alpha) = -\alpha\} \subset \Delta$. The fact that $ad(\theta(X))^t = -ad(X)$ with respect to B_θ implies that all roots are real valued on $\mathfrak{a} \oplus i\mathfrak{t}$. Therefore, restricting a root α to $\mathfrak{h} \subset \mathfrak{h}_{\mathbb{C}}$, we see that $\sigma^*(\alpha) = -\alpha$ if and only if $\alpha|_{\mathfrak{a}} = 0$. So we have $\Delta_c = \{\alpha \in \Delta : \alpha|_{\mathfrak{a}} = 0\}$. It can be shown that $\Delta_c^0 := \Delta^0 \cap \Delta_c$ is a simple subsystem of Δ_c and that for any simple non-compact root α there exists a unique simple non-compact root α' such that $\sigma^*(\alpha) - \alpha'$ can be written as a linear combination of compact roots. By mapping α to α' one obtains an involutive automorphism on $\Delta^0 \setminus \Delta_c^0$. The Satake diagram of $\mathfrak{g}$ is then defined as the Dynkin diagram of Δ^0, where elements in Δ_c^0 are denoted by black dots $\bullet$ and all others by white dots $\circ$ and where for $\alpha \in \Delta^0 \setminus \Delta_c^0$ with $\alpha \neq \alpha'$ the two roots are connected by an

arrow.
Since Δ^+ is admissible, the image Δ_r^+ of Δ^+ under the surjective restriction map $\Delta \to \Delta_r$ is a positive subsystem of Δ_r and this easily implies that the corresponding subset of simple restricted roots Δ_r^0 is the quotient of $\Delta^0 \setminus \Delta_c^0$ constructed by identifying α with α'.
This shows that subsets of simple restricted roots can be identified with subsets of $\Delta^0 \setminus \Delta_c^0$ satisfying that if α is contained in the subset then α' is contained as well. Therefore we may denote a standard parabolic subalgebra of $\mathfrak{g}$ by replacing white dots in the Satake diagram of $\mathfrak{g}$ by crosses, where two roots connected by an arrow are always either both crossed or both not crossed.
Analog to the complex case, any standard parabolic subalgebra $\mathfrak{p}$ defines a $|k|$-grading by

$$\mathfrak{g}_0 = Z_{\mathfrak{k}}(\mathfrak{a}) \oplus \mathfrak{a} \oplus \bigoplus_{ht_{\Sigma_{\mathfrak{p}}}(\lambda)=0} \mathfrak{g}_\lambda$$

$$\mathfrak{g}_i = \bigoplus_{ht_{\Sigma_{\mathfrak{p}}}(\lambda)=i} \mathfrak{g}_\lambda$$

with $\mathfrak{g}^0 = \mathfrak{p}$ and one can prove that one obtains in this way a correspondence between conjugation classes of parabolic subalgebras of $\mathfrak{g}$ and isomorphism classes of $|k|$-gradings of $\mathfrak{g}$.
Since the complexification of a real semisimple $|k|$-graded Lie algebra is a complex semisimple $|k|$-graded Lie algebra, the subalgebra $\mathfrak{g}_0 \subset \mathfrak{p}$ is as in the complex case reductive and called the *Levi subalgebra* of $\mathfrak{p}$. The dimension of the center of $\mathfrak{g}_0$ thereby exactly equals the number of crosses in the Satake diagram and the Satake diagram of the semisimple part of $\mathfrak{g}_0$ is just obtained by removing all crossed nodes, and all edges and arrow connecting to the crossed nodes.

2.1.4. Natural vector bundles. Suppose $(\mathcal{G} \to M, \omega)$ is a parabolic geometry of some type (G, P) and $\mathbb{V}$ is a representation of P. Then we can form the associated vector bundle V to the principal bundle $\mathcal{G}$ with standard fiber $\mathbb{V}$. Recall that the vector bundle V is defined as

$$V := \mathcal{G} \times_P \mathbb{V} := \mathcal{G} \times \mathbb{V}/\sim,$$

where $\sim$ denotes the equivalence relation $(u, v) \sim (u \cdot p, p^{-1} \cdot v)$ for all $p \in P$. The vector bundles of this form are the natural vector bundles associated to a parabolic geometry $(\mathcal{G} \to M, \omega)$.
Denoting by

$$C^\infty(\mathcal{G}, \mathbb{V})^P := \{f : \mathcal{G} \to \mathbb{V} \quad \text{smooth} : f(up) = p^{-1} f(u) \quad \text{for all} \quad p \in P\}$$

the space of smooth P-equivariant functions, the map which assign to a P-equivariant function f the section $s_f(p(u)) = [u, f(u)]$ of $\mathcal{G} \times_P \mathbb{V}$, where $[u, f(u)]$ denotes the equivalence class of $(u, f(u)) \in \mathcal{G} \times \mathbb{V}$ in V, defines an isomorphism

$$C^\infty(\mathcal{G}, \mathbb{V})^P \cong \Gamma(\mathcal{G} \times_P \mathbb{V}).$$

Let us consider some important examples:
The adjoint representation of G induces a representation of P on $\mathfrak{g}/\mathfrak{p}$. The tangent bundle TM can be viewed as the vector bundle corresponding to this representation, since the Cartan connection induces an isomorphism as follows:

$$\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM$$
$$[u, X + \mathfrak{p}] \mapsto T_u p \omega^{-1}(X).$$

Consequently, for the cotangent bundle we have

$$\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^* \cong T^*M.$$

There is a special class of natural vector bundles, namely those associated vector bundles which correspond to P-representations that are obtained by restricting some G-representation to P. The natural vector bundles obtained in that way are called *tractor bundles* and play an important role in the theory of parabolic geometries, since the always admit natural linear connections.
An important example of a tractor bundle is the associated vector bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ corresponding to the P-representation which is obtained by restricting the adjoint representation of G to P. It is called the *adjoint tractor bundle*. Since the action of P on $\mathfrak{g}$ preserves the filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset ... \supset \mathfrak{g}^k$ associated to the grading on $\mathfrak{g}$, we obtain correspondingly a filtration of the adjoint tractor bundle into subbundles

$$\mathcal{A}M = \mathcal{A}^{-k}M \supset ... \supset \mathcal{A}^0 M \supset ... \supset \mathcal{A}^k M.$$

By definition $\mathcal{A}^0 M = \mathcal{G} \times_P \mathfrak{p}$ and there for we see that the tangent bundle TM can be identified with the quotient $\mathcal{A}M/\mathcal{A}^0 M$ and we have a projection $\mathcal{A}M \to TM$. By contrast, the cotangent bundle can be identified with the subbundle $\mathcal{A}^1 M = \mathcal{G} \times_P \mathfrak{p}_+$, since $\mathfrak{p}_+$ and $(\mathfrak{g}/\mathfrak{p})^*$ are isomorphic as P-representations, see proposition 2.3.

2.1.5. Representations of P. We have seen that the natural vector bundles for parabolic geometries correspond to representations of parabolic subgroups. Therefore let us finish this section by collecting some facts about representations of parabolic subgroups.
Suppose $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a $|k|$-graded semisimple Lie algebra. Starting

with a completely reducible finite dimensional representation of the reductive Levi subalgebra $\mathfrak{g}_0$, we may obtain a completely reducible representation of $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ by trivially extending the action to $\mathfrak{p}$. Let us remark here that a finite dimensional representation of a reductive Lie algebra is completely reducible if and only if its center acts diagonalisably.
Conversely, it is not hard to see that any completely reducible representation of $\mathfrak{p}$ is of this form:

Proposition 2.7. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a $|k|$-graded semisimple Lie algebra with corresponding parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. If $\mathbb{W}$ is a completely reducible finite dimensional representation of $\mathfrak{p}$, then $\mathfrak{p}_+$ acts trivially on $\mathbb{W}$. In particular, irreducible representations of $\mathfrak{p}$ correspond to irreducible representations of $\mathfrak{g}_0$.*

This proposition implies that complex irreducible representations of $\mathfrak{p}$ may be described in terms of highest weights. To explain this, let us recall the representation theory of semisimple Lie algebras.
Suppose first $\mathfrak{g}$ is a complex semisimple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ a be Cartan subalgebra and $\Delta^0 = \{\alpha_1, ..., \alpha_n\}$ be a simple subsystem of the root system Δ corresponding to $\mathfrak{h}$. It is well known that complex irreducible representations of $\mathfrak{g}$ can be described by their highest weights. If $\mathbb{V}$ is a complex irreducible representation, then its highest weight $\lambda \in \mathfrak{h}^*$ is a linear functional on $\mathfrak{h}$, which is dominant and algebraically integral. This means that $2\frac{<\lambda,\alpha>}{<\alpha,\alpha>}$ is a non-negative integer for all $\alpha \in \Delta^0$, where $<,>$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form. In fact, assigning to a complex irreducible representation of $\mathfrak{g}$ its highest weight defines a bijection between isomorphism classes of complex irreducible representations and dominant algebraically integral functionals on $\mathfrak{h}$. The condition for an element in $\mathfrak{h}^*$ to be dominant and integral can be nicely rephrased in terms of fundamental weights. For each simple root α_i the dominant algebraically integral functional ω_i, which is characterised by $2\frac{<\omega_i,\alpha_j>}{<\alpha_j,\alpha_j>} = \delta_{ij}$, is called the *fundamental weight* corresponding to α_i. The fundamental weights form a basis of $\mathfrak{h}^*$ and hence any functional $\lambda \in \mathfrak{h}^*$ can be uniquely written as linear combination of the fundamental weights. Thereby the coefficient of ω_i equals exactly $2\frac{<\lambda,\alpha_i>}{<\alpha_i,\alpha_i>}$. Therefore a functional λ is dominant and algebraically integral if and only if all the coefficients of λ written as linear combination of the fundamental weights are non-negative integers. Hence we can denote a complex irreducible representation with highest weight λ by the Dynkin diagram of $\mathfrak{g}$, where one writes the non-negative integer $2\frac{<\lambda,\alpha_i>}{<\alpha_i,\alpha_i>}$ over the node which corresponds to the simple root α_i.
For complex $|k|$-graded semisimple Lie algebra $\mathfrak{g}$ with corresponding parabolic subalgebra $\mathfrak{p}$, the complex irreducible representations of $\mathfrak{p}$ can be

described very similarly. By the previous proposition, a complex irreducible representation of $\mathfrak{p}$ is the just a complex irreducible representation of the Levi subalgebra $\mathfrak{g}_0$ trivially extended to $\mathfrak{p}$. Since $\mathfrak{g}_0$ is reductive, it decomposes as a direct sum of its center and a complex semisimple Lie algebra $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$. Therefore any complex irreducible representation of $\mathfrak{g}_0$ is given by a complex irreducible representation of the semisimple part $\mathfrak{g}_0^{ss}$ and a linear functional on the center $\mathfrak{z}(\mathfrak{g}_0)$. Now choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a simple subsystem of roots in such way that $\mathfrak{p}$ is standard parabolic and the grading is given by $\Sigma_{\mathfrak{p}}$-height. In proposition 2.6, we have seen that the Cartan algebra naturally splits as $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ is the span of all H_α with $\alpha \in \Delta^0 \backslash \Sigma_{\mathfrak{p}}$. The subalgebra $\mathfrak{h}_0$ is then a Cartan subalgebra of $\mathfrak{g}_0^{ss}$ with corresponding root composition $\mathfrak{g}_0^{ss} = \mathfrak{h}_0 \oplus \bigoplus_{ht_{\Sigma_{\mathfrak{p}}}(\alpha)=0} \mathfrak{g}_\alpha$. Since complex irreducible representations of $\mathfrak{g}_0^{ss}$ are given by dominant, algebraically integral weights, this implies that complex irreducible representations of $\mathfrak{g}_0$ are in one to one correspondence with linear functionals $\lambda : \mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}_0 \to \mathbb{C}$ which are $\mathfrak{p}$-*dominant* and $\mathfrak{p}$-*integral*, i.e. $\frac{2<\lambda,\alpha>}{<\alpha,\alpha>}$ is a non-negative integer for all $\alpha \in \Delta^0 \backslash \Sigma_{\mathfrak{p}}$.

Theorem 2.8. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a complex $|k|$-graded semisimple Lie algebra and choose a Cartan subalgebra $\mathfrak{h}$ and a simple subsystem of roots such that $\mathfrak{p}$ is a standard parabolic subalgebra of $\mathfrak{g}$.*
There is a bijective correspondence between isomorphism classes of complex irreducible representations of $\mathfrak{p}$ and linear functionals $\lambda \in \mathfrak{h}^*$, which are $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral.
We will refer to the $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral linear functional corresponding to a complex irreducible representation $\mathbb{W}$ of $\mathfrak{p}$ as the highest weight of $\mathbb{W}$.

Writing a linear functional $\lambda \in \mathfrak{h}^*$ as a linear combination of fundamental weights, λ is $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral if and only if all the coefficients of the fundamental weights corresponding to roots in $\Delta^0 \backslash \Sigma_{\mathfrak{p}}$ are non-negative integers. In the Dynkin diagram notation, this means that the numbers over the uncrossed nodes are non-negative integers. The coefficients over the crossed nodes can be arbitrary, but if one wants the representation to integrate to a representation of the group P, one has to require that the coefficients over the crossed nodes are integers.
In the case of a real $|k|$-graded semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$, its complexification $\mathfrak{g}^{\mathbb{C}}$ is a complex $|k|$-graded semisimple Lie algebra. We may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and an admissible positive subsystem of the root system corresponding to $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ such that $\mathfrak{p}^{\mathbb{C}}$ is a standard parabolic subalgebra in $\mathfrak{g}^{\mathbb{C}}$. If $\mathbb{W}$ is a complex irreducible representation of $\mathfrak{g}_0$, then it extends to $\mathfrak{g}_0^{\mathbb{C}}$ and can be described by a highest weight. If $\mathbb{W}$ is a real representation, having no invariant complex structure, then the

complexification $\mathbb{W}_{\mathbb{C}}$ is irreducible. By the highest weight of $\mathbb{W}$ we then mean the highest weight of $\mathbb{W}_{\mathbb{C}}$.

2.2. Kostant's version of the Bott-Borel-Weil theorem

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra and suppose $\mathbb{V}$ is a finite dimensional representation of $\mathfrak{g}$. Considering $\mathbb{V}$ as a representation of $\mathfrak{p}_+$ respectively of $\mathfrak{g}_0$ by restriction, we may study the Lie algebra cohomology $H^*(\mathfrak{p}_+, \mathbb{V})$ of $\mathfrak{p}_+$ with values in $\mathbb{V}$. It turns out that the cohomology spaces $H^k(\mathfrak{p}_+, \mathbb{V})$ have naturally the structure of $\mathfrak{g}_0$-modules. In **[27]** Kostant gave a complete description of the cohomology spaces $H^k(\mathfrak{p}_+, \mathbb{V})$ as $\mathfrak{g}_0$ modules in terms of the Hasse diagram of $\mathfrak{p}$. The result allows to compute these cohomology spaces completely algorithmically. The cohomology spaces $H^k(\mathfrak{g}_-, \mathbb{V}^*)$, which are dual to $H^k(\mathfrak{p}_+, \mathbb{V})$, play an important role in the theory of parabolic geometries, which will become manifest in the course of this work. Therefore we will collect in the following section some basic facts about Lie algebra cohomology and will formulate the result of Kostant.

2.2.1. The Hodge theory. Suppose $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a $|k|$-graded semisimple Lie algebra, G is a Lie group with Lie algebra $\mathfrak{g}$ and P a parabolic subgroup corresponding to the given $|k|$-grading. Further, let $\mathbb{V}$ be a finite dimensional representation of G.

Now let us consider the complex for computing the cohomology $H^*(\mathfrak{g}_-, \mathbb{V})$ of $\mathfrak{g}_-$ with values in $\mathbb{V}$. We denote by $C^n(\mathfrak{g}_-, \mathbb{V}) = \Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V}$ the cochain space of n-linear alternating maps from $\mathfrak{g}_-$ to $\mathbb{V}$. Recall that the usual Lie algebra differential $\partial : C^n(\mathfrak{g}_-, \mathbb{V}) \to C^{n+1}(\mathfrak{g}_-, \mathbb{V})$ is given by

$$\partial(\phi)(X_0, ..., X_n) := \sum_{i=0}^{n} (-1)^i X_i \cdot \phi(X_0, ..., \hat{X^i}, ..., X_n)$$
$$+ \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_0, ..., \hat{X^i}, ..., \hat{X^j}, ..., X^n)$$

for $X_0, ..., X_n \in \mathfrak{g}_-$, where the point $\cdot$ denotes the infinitesimal action of $\mathfrak{g}$ on $\mathbb{V}$ and the hat over an argument omission.

Since the Levi subgroup G_0, by its definition, acts on $\mathfrak{g}_-$ via the adjoint action and on $\mathbb{V}$ by restriction, we have an induced action of G_0 on the cochain spaces $C^n(\mathfrak{g}_-, \mathbb{V})$. It is easy to verify that the differentials ∂ are G_0-equivariant and therefore the cohomology spaces

$$H^n(\mathfrak{g}_-, \mathbb{V}) = \ker(\partial)/\mathrm{im}(\partial) = \frac{\ker(\partial : C^n(\mathfrak{g}_-, \mathbb{V}) \to C^{n+1}(\mathfrak{g}_-, \mathbb{V}))}{\mathrm{im}(\partial : C^{n-1}(\mathfrak{g}_-, \mathbb{V}) \to C^n(\mathfrak{g}_-, \mathbb{V}))}$$

are naturally G_0-modules.

Now consider the Lie algebra differential $\partial_{\mathfrak{p}_+} : C^n(\mathfrak{p}_+, \mathbb{V}^*) \to C^{n+1}(\mathfrak{p}_+, \mathbb{V}^*)$

computing the cohomology $H^*(\mathfrak{p}_+, \mathbb{V}^*)$ of $\mathfrak{p}_+$ with values in the dual representation $\mathbb{V}^*$. Since P acts on $\mathfrak{p}_+$ via the adjoint action and on $\mathbb{V}$ by restriction, we obtain induced P-module structures on the spaces $C^n(\mathfrak{p}_+, \mathbb{V}^*)$. It is easy to see that the Lie algebra differential $\partial_{\mathfrak{p}_+}$ in this case is P-equivariant. Dualising $\partial_{\mathfrak{p}_+}$ leads to a P-equivariant map $\partial^* : \Lambda^{n+1}\mathfrak{p}_+ \otimes \mathbb{V} \to \Lambda^n\mathfrak{p}_+ \otimes \mathbb{V}$ which satisfies $\partial^* \circ \partial^* = 0$. Explicitly, ∂^* is given by

$$\partial^*(Z_0 \wedge ... \wedge Z_n \otimes v) = \sum_{i=0}^{n}(-1)^{i+1} Z_0 \wedge .... \wedge \hat{Z}_i \wedge ... \wedge Z_n \otimes Z_n \cdot v$$
$$+ \sum_{i<j}(-1)^{(i+j)}[Z_i, Z_j] \wedge Z_0 \wedge ... \wedge \hat{Z}_i \wedge ... \wedge \hat{Z}_j \wedge ... \wedge Z_n \otimes v$$

and it is called the *Kostant codifferential.*
Since the Killing form induces an isomorphism $\mathfrak{g}_-^* \cong \mathfrak{p}_+$ of G_0-modules, we may identify the G_0-modules $C^n(\mathfrak{g}_-, \mathbb{V}) = \Lambda^n\mathfrak{g}_-^* \otimes \mathbb{V}$ and $\Lambda^n\mathfrak{p}_+ \otimes \mathbb{V}$ and view the codifferential as a map

$$\partial^* : C^n(\mathfrak{g}_-, \mathbb{V}) \to C^{n-1}(\mathfrak{g}_-, \mathbb{V}).$$

Kostant showed that the operators ∂ and ∂^* on $C^n(\mathfrak{g}_-, \mathbb{V})$ are adjoint for some inner product (hermitian in the complex case) on the cochain spaces $C^n(\mathfrak{g}_-, \mathbb{V})$. This implies then that one has a Hodge decomposition:

Proposition 2.9. *Let $\mathbb{V}$ be a finite dimensional representation of G and denote by $\Box$ the G_0-module homomorphism $\partial\partial^* + \partial^*\partial : C^n(\mathfrak{g}_-, \mathbb{V}) \to C^n(\mathfrak{g}_-, \mathbb{V})$. Then we have*

$$C^n(\mathfrak{g}_-, \mathbb{V}) = \mathrm{im}(\partial^*) \oplus \ker(\Box) \oplus \mathrm{im}(\partial).$$

Moreover, $\ker(\partial^*) = \mathrm{im}(\partial^*) \oplus \ker(\Box)$ *and* $\ker(\partial) = \mathrm{im}(\partial) \oplus \ker(\Box)$.

As a consequence, we obtain a G_0-module isomorphism

$$H^n(\mathfrak{g}_-, \mathbb{V}) \cong \ker(\Box)$$

and hence $H^n(\mathfrak{g}_-, \mathbb{V})$ can be naturally viewed as G_0-submodule in $\Lambda^n\mathfrak{g}_-^* \otimes \mathbb{V}$. Moreover, since obviously $\ker(\Box) \cong \ker(\partial^*)/\mathrm{im}(\partial^*)$, we also have

$$H^n(\mathfrak{g}_-, \mathbb{V}) \cong \ker(\partial^*)/\mathrm{im}(\partial^*),$$

which endows $H^n(\mathfrak{g}_-, \mathbb{V})$ with the structure of a P-module. Using the explicit formula of ∂^*, it is not hard to see that P_+ maps $\ker(\partial^*)$ to $\mathrm{im}(\partial^*)$ and therefore the action of P on $\ker(\partial^*)/\mathrm{im}(\partial^*)$ is given by trivially extending the action of G_0 to P.
Since the codifferental is obtained by dualising the differential for computing the cohomology $H^*(\mathfrak{p}_+, \mathbb{V}^*)$, we see that

$$H^n(\mathfrak{g}_-, \mathbb{V}) \cong H^n(\mathfrak{p}_+, \mathbb{V}^*)^*$$

as P-modules.

Remark 2.1. The Killing form induces a G_0-module isomorphism $\mathfrak{g}_-^* \cong \mathfrak{p}_+$ and a P-module isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$. Identifying $\mathfrak{g}_-$ with $\mathfrak{g}/\mathfrak{p}$, we may therefore endow $\mathfrak{g}_-$ with a P-module structure and obtain a duality of the P-modules $C^n(\mathfrak{g}_-, \mathbb{V})$ and $C^n(\mathfrak{p}_+, \mathbb{V}^*)$. However, for the P-module structure on $C^n(\mathfrak{g}_-, \mathbb{V})$ the Lie algebra differential ∂ is *not* equivariant. Therefore, given a parabolic geometry of type (G, P), in contrast to ∂^*, the differential ∂ doesn't induce a vector bundle homomorphism between the associated vector bundles corresponding to $C^n(\mathfrak{g}_-, \mathbb{V})$.

2.2.2. Kostant's result. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a complex graded semisimple Lie algebra and choose a Cartan subalgebra $\mathfrak{h}$ and a simple subsystem of roots $\Delta^0 \subset \Delta$ such that $\mathfrak{p}$ is standard parabolic subalgebra and the grading is given by $\Sigma_{\mathfrak{p}}$-height.
By proposition 2.6 we know that the subalgebra $\mathfrak{g}_0$ is reductive and hence decomposes as a direct sum of its center and a semisimple part $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$. Moreover, we have seen that the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ decomposes naturally as $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0^{ss}$ defined as the span of all H_α with $\alpha \in \Delta^0 \setminus \Sigma_{\mathfrak{p}}$ with the notation as in proposition 2.6. The decomposition of the Cartan algebra induces a decomposition of the real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$, on which all roots are real, which is orthogonal with respect to the inner product induced by the Killing form. Correspondingly, we obtain a orthogonal decomposition of the dual space $\mathfrak{h}_{\mathbb{R}}^*$. This implies that the Weyl group $W_{\mathfrak{p}}$ of the semisimple Lie algebra $\mathfrak{g}_0^{ss}$ can be naturally viewed as a subgroup of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$. Namely, as the subgroup of $W_{\mathfrak{g}}$, which is generated by all root reflections $s_\alpha : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*$ corresponding to roots $\alpha \in \Delta^0 \setminus \Sigma_{\mathfrak{p}}$.
The Hasse diagram of $\mathfrak{p}$ is a distinguished set of representatives of the right coset space $W_{\mathfrak{p}} \backslash W_{\mathfrak{g}}$. To construct these representatives we have to describe the subgroup $W_{\mathfrak{p}}$ more explicitly. Therefore we decompose the space of positive roots $\Delta^+ = \Delta^+(\mathfrak{g}_0) \sqcup \Delta^+(\mathfrak{p}_+)$ according to the subalgebra containing the corresponding root space and set $\phi_w := \{\alpha \in \Delta^+ : w^{-1}(\alpha) \in -\Delta^+\}$ for $w \in W_{\mathfrak{g}}$. Then it turns out that the subgroup $W_{\mathfrak{p}}$ consists exactly of those elements $w \in W_{\mathfrak{g}}$ for which $\phi_w \subset \Delta^+(\mathfrak{g}_0)$. This suggests the following definition for the Hasse diagram of $\mathfrak{p}$:

Definition 2.6. The *Hasse diagram* $W^{\mathfrak{p}}$ of the standard parabolic subalgebra $\mathfrak{p}$ is defined by

$$W^{\mathfrak{p}} := \{w \in W^{\mathfrak{g}} : \phi_w \subset \Delta^+(\mathfrak{p}_+)\}.$$

It can be shown that $W^{\mathfrak{p}}$ is the unique set of representatives of minimal length of the right coset space $W_{\mathfrak{p}} \backslash W^{\mathfrak{g}}$, see e.g. section 3.2.15 in [**15**].

Remark 2.2. It is well known that the Weyl group $W^{\mathfrak{g}}$ acts simply transitive on the set of Weyl chambers. Therefore, choosing a weight λ lying in the interior of the dominant Weyl chamber, the orbit of λ is in bijective correspondence with $W^{\mathfrak{g}}$. For example one can choose λ to be the lowest form ρ, which is the sum of all fundamental weights. Similarly, one can determine the elements of $W^{\mathfrak{p}}$. Denote by $\rho^{\mathfrak{p}}$ the sum of fundamental weights corresponding to $\Sigma_{\mathfrak{p}}$. Then the map $w \mapsto w^{-1}(\rho^{\mathfrak{p}})$ defines a bijection from $W^{\mathfrak{p}}$ to the orbit of $\rho^{\mathfrak{p}}$ under $W^{\mathfrak{g}}$. For details see e.g. section 3.2.16. **[15]**.

Let $\mathbb{V}$ be a complex irreducible representation of $\mathfrak{g}$ and let us consider the cochain space $C^*(\mathfrak{p}_+, \mathbb{V})$ of the cohomology $H^*(\mathfrak{p}_+, \mathbb{V})$. Since the center $\mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}$ acts diagonazible on $\mathfrak{p}_+$ and on $\mathbb{V}$, it does the same on $C^*(\mathfrak{p}_+, \mathbb{V})$. Hence $C^*(\mathfrak{p}_+, \mathbb{V})$ is completely reducible as $\mathfrak{g}_0$-module and we can decompose it into irreducible representations. Since the center $\mathfrak{z}(\mathfrak{g}_0)$ acts by a character on each irreducible component, the decomposition is found by decomposing the representation into irreducible $\mathfrak{g}_0^{ss}$-modules and determining the action of the center on each of these components.
Suppose $\mathbb{W}$ is a completely reducible representation of $\mathfrak{g}_0$, then we can decompose $\mathbb{W}$ also in a coarser way into isotypical components. Recall that a highest weight vector of $\mathbb{W}$ is a weight vector which is killed by all positive root spaces of $\mathfrak{g}_0$. For a wight $\nu \in \mathfrak{h}^* = \mathfrak{z}(\mathfrak{g}_0)^* \oplus \mathfrak{h}_0^*$ of $\mathbb{W}$, the *isotypical component* $\mathbb{W}^\nu \subset \mathbb{W}$ of weight ν is the $\mathfrak{g}_0$-submodule generated by highest weight vectors of weight ν. Choosing a basis, one obtains an isomorphism between the subrepresentation $\mathbb{W}^\nu$ and a direct sum of copies of the irreducible representation with highest weight ν. The number of these copies is called the *multiplicity* of ν in $\mathbb{W}$.
Now we are able to state the theorem of Kostant:

Theorem 2.10. *(Kostant's version of the Bott-Borel-Weil theorem) Let* $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ *be a complex* $|k|$*-graded semisimple Lie algebra and choose a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ *is a standard parabolic subalgebra. Suppose* $\mathbb{V}$ *is a complex irreducible representation of* $\mathfrak{g}$ *with highest weight* λ *and consider the* $\mathfrak{g}_0$*-module* $H^*(\mathfrak{p}_+, \mathbb{V})$*, which is isomorphic to the submodule* $\ker(\Box) \subset C^*(\mathfrak{p}_+, \mathbb{V})$*. Then we have:*

1. *For a functional* $\nu \in \mathfrak{h}^*$ *we have* $H^*(\mathfrak{p}_+, \mathbb{V})^\nu \neq \{0\}$ *if and on if there exists* $w \in W^{\mathfrak{p}}$ *such that* $\nu = \nu_w := w(\lambda + \rho) - \rho$*, where* ρ *is the sum of all fundamental weights of* $\mathfrak{g}$*.*
2. *For* $w \in W^{\mathfrak{p}}$ *the isotypical component* $H^*(\mathfrak{p}_+, \mathbb{V})^{\nu_w}$ *is irreducible and we have a bijection between irreducible components of* $H^*(\mathfrak{p}_+, \mathbb{V})$ *and elements of* $W^{\mathfrak{p}}$*. In addition, even the multiplicity of* $H^*(\mathfrak{p}_+, \mathbb{V})^{\nu_w}$ *as a submodule of* $C^*(\mathfrak{p}_+, \mathbb{V})$ *is one.*

(3) *For $w \in W^{\mathfrak{p}}$ the isotypical component $H^*(\mathfrak{p}_+, \mathbb{V})^{\nu_w}$ is contained in $H^{\ell(w)}(\mathfrak{p}_+, \mathbb{V})$, where $\ell(w)$ is the length of the Weyl group element w. A highest weight vector of $H^*(\mathfrak{p}_+, \mathbb{V})^{\nu_w}$ is given by the cohomology class of $F_{\Phi_w} \otimes v$, where F_{Φ_w} is the wedge product of nonzero elements of each of the root spaces $\mathfrak{g}_{-\alpha}$ with $\alpha \in \Phi_w$ and v is a nonzero weight vector of weight $w(\lambda)$.*

The theorem is formulated for complex semisimple Lie algebras only, but one can deduce immediately information about the real case as well. For a proof see e.g. section 3.3.6. in **[15]**.

Corollary 2.11. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a real $|k|$-graded Lie algebra.*

(1) *If $\mathbb{V}$ is a complex irreducible representation of $\mathfrak{g}$, then the real cohomology spaces $H^*_{\mathbb{R}}(\mathfrak{p}_+, \mathbb{V})$ are naturally complex vector spaces and one has an isomorphism*

$$H^*_{\mathbb{R}}(\mathfrak{p}_+, \mathbb{V}) \cong H^*_{\mathbb{C}}(\mathfrak{p}_+^{\mathbb{C}}, \mathbb{V}).$$

(2) *If $\mathbb{V}$ is a real irreducible representation, then it is easily seen that*

$$H^*_{\mathbb{C}}(\mathfrak{p}_+^{\mathbb{C}}, \mathbb{V} \otimes \mathbb{C}) \cong H^*_{\mathbb{R}}(\mathfrak{p}_+, \mathbb{V}) \otimes \mathbb{C}.$$

2.3. Parabolic geometries and their underlying structures

Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$ and $P \subset G$ a parabolic subgroup corresponding to the $|k|$-grading. Suppose $(p : \mathcal{G} \to M, \omega)$ is a parabolic geometry of type (G, P).
We have seen that the Cartan connection induces an isomorphism of vector bundles

$$\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p} \cong TM.$$

The filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset ... \supset \mathfrak{g}^k$ associated to the grading of $\mathfrak{g}$ induces a filtration of the vector space $\mathfrak{g}/\mathfrak{p}$ given by

$$\mathfrak{g}/\mathfrak{p} = \mathfrak{g}^{-k}/\mathfrak{p} \supset ... \supset \mathfrak{g}^{-1}/\mathfrak{p}.$$

Since the filtration of $\mathfrak{g}/\mathfrak{p}$ is P-invariant, we obtain a filtration of the tangent bundle

$$TM = T^{-k}M \supset ... \supset T^{-1}M$$

by vector subbundles.
Since P acts freely on $\mathcal{G}$, the same holds for the normal subgroup P_+ of P, which we defined in theorem 2.4. Hence the orbit space $\mathcal{G}_0 := \mathcal{G}/P_+$ is a smooth manifold and the projection p factorises to a smooth projection $p_0 : \mathcal{G}_0 \to M$. It is easy to see that $\mathcal{G}_0$ is a principal bundle with structure group $P/P_+ \cong G_0$.

Now consider the associated graded vector bundle $\mathrm{gr}(TM) = \bigoplus_{i=-k}^{-1} \mathrm{gr}_i(TM)$ of the tangent bundle TM. The subbundle $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$ is isomorphic to $\mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{g}^{i+1}$. By definition of the normal subgroup $P_+ \subset P$, this subgroup acts trivially on $\mathfrak{g}^i/\mathfrak{g}^{i+1}$ and hence the P action factorises to an action of $G_0 \cong P/P_+$. Since as G_0-modules we have $\mathfrak{g}^i/\mathfrak{g}^{i+1} \cong \mathfrak{g}_i$ and $P_+ \subset P$ is a normal subgroup, we obtain $\mathrm{gr}_i(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$. Therefore we have

$$\mathrm{gr}(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-.$$

The Lie bracket on $\mathfrak{g}_-$ is G_0-equivariant and hence defines an algebraic bracket on $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$, which gives an algebraic bracket

$$\{\quad,\quad\} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM),$$

which makes each fiber $\mathrm{gr}(T_xM)$ into a nilpotent graded Lie algebra.

Definition 2.7. A parabolic geometry is called *regular*, if M with its canonical filtration of the tangent bundle TM from above is a filtered manifold and the Levi bracket $\mathcal{L}$ of this filtered manifold coincides with $\{\quad,\quad\}$. Otherwise put, this is exactly says, that M with its canonical filtration has to be a regular filtered manifold of type $\mathfrak{g}_-$.

For a parabolic geometry to be regular can be equivalently described in terms of the curvature, see section 3.1.8 in **[15]**:

Proposition 2.12. *A parabolic geometry is regular if and only if the curvature function* $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{g} \otimes \mathfrak{g}$ *satisfies* $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1}$ *for all* $i, j < 0$.

For a regular parabolic geometry the manifold M is a regular filtered manifold of type $\mathfrak{g}_-$. In section 1.1.1 we have seen that in this case we have a natural notion of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of $\mathrm{gr}(TM)$. Recall that the fibers $\mathcal{P}_x(\mathrm{gr}(TM))$ are given by all grading preserving Lie algebra isomorphisms from $\mathfrak{g}_-$ to the symbol algebra $\mathrm{gr}(T_xM)$ and that the structure group of $\mathcal{P}(\mathrm{gr}(TM))$ is $Aut_{gr}(\mathfrak{g}_-)$ acting from the right by composition.
The adjoint action of G induces a homomorphism $G_0 \to Aut_{gr}(\mathfrak{g}_-)$ and we have a reduction $\mathcal{G}_0 \to \mathcal{P}(\mathrm{gr}(TM))$ of the structure group of $\mathcal{P}(\mathrm{gr}(TM))$ to G_0 corresponding to this homomorphism given by $u_0 \mapsto (X \mapsto [u_0, X])$, where $[u_0, X]$ is the equivalence class of (u_0, X) in $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_- \cong \mathrm{gr}(TM)$.
Summerising, we have seen that a regular parabolic geometry induces the following geometric structure on M:

- A filtration of the tangent bundle, which makes M into a regular filtered manifold of type $\mathfrak{g}_-$.
- A reduction of the structure group of $\mathcal{P}(\mathrm{gr}(TM))$ to the structure group G_0 with respect to $Ad : G_0 \to Aut_{gr}(\mathfrak{g}_-)$.

Such a geometric structure is called a *regular infinitesimal flag structure* of type (G,P).

Moreover, for a morphism between parabolic geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ of type (G,P) it is easy to see that the base map is a local isomorphism $f : M \to \tilde{M}$ of filtered manifolds, which is compatible with the G_0-structures on the associated graded bundles $\mathrm{gr}(TM)$ and $\mathrm{gr}(T\tilde{M})$.

So we see that we have a functor from the category of regular parabolic geometries to the category of regular infinitesimal flag structures. There are many parabolic geometries inducing the same infinitesimal flag structure. So to obtain an equivalence of categories one needs a normalisation condition leading to a unique parabolic geometry with a fixed underlying structure.

For $n \geq 0$ consider the vector bundles

$$\Lambda^n T^*M \otimes \mathcal{A}M \cong \mathcal{G} \times_P \Lambda^n(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \mathcal{G} \times_P \Lambda^n \mathfrak{p}_+ \otimes \mathfrak{g}.$$

The natural filtrations of TM and of the adjoint tractor bundle $\mathcal{A}M$ induce a filtration of $\Lambda^n T^*M \otimes \mathcal{A}M$. The filtration is given by the homogeneous degree of multilinear mappings:

$$\phi \in (\Lambda^n T^*M \otimes \mathcal{A}M)^\ell$$

if and only if

$$\phi(T^{i_1}M, ..., T^{i_n}M) \subset \mathcal{A}^{i_1+....+i_n+\ell}M \quad \text{for all} \quad i_1, ..., i_n < 0.$$

For the associated graded vector bundle we have an isomorphism

$$\mathrm{gr}(\Lambda^n T^*M \otimes \mathcal{A}M) \cong \Lambda^n \mathrm{gr}(TM)^* \otimes \mathrm{gr}(\mathcal{A}M)$$

with

$$\mathrm{gr}_\ell(\Lambda^n T^*M \otimes \mathcal{A}M) \cong \bigoplus_{-(i_1+...+i_n)+j=\ell} \mathrm{gr}_{i_1}(TM)^* \wedge .... \wedge \mathrm{gr}_{i_1}(TM)^* \otimes \mathrm{gr}_j(\mathcal{A}M).$$

Since the normal subgroup P_+ acts trivially on $\mathrm{gr}_\ell(\Lambda^n T^*M \otimes \mathcal{A}M)$ and $\Lambda^n(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is a G_0-module isomorphic to $\Lambda^n \mathfrak{g}_-^* \otimes \mathfrak{g}$ we have an isomorphism

$$\mathrm{gr}(\Lambda^n T^*M \otimes \mathcal{A}M) \cong \mathcal{G}_0 \times_{G_0} \Lambda^n \mathfrak{g}_-^* \otimes \mathfrak{g}.$$

Since the differential $\partial : \Lambda^n \mathfrak{g}_-^* \otimes \mathfrak{g} \to \Lambda^{n+1} \mathfrak{g}_-^* \otimes \mathfrak{g}$ for the Lie algebra cohomology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$ is G_0-equivariant, it induces a bundle map

$$\partial : \mathrm{gr}(\Lambda^n T^*M \otimes \mathcal{A}M) \to \mathrm{gr}(\Lambda^{n+1} T^*M \otimes \mathcal{A}M).$$

Note that the definition of ∂ immediately implies that it is compatible with the gradings.

Now suppose ω and ω' are two Cartan connection on $\mathcal{G}$. Since a Cartan connection is P-equivariant and reproduces the generators of fundamental vector fields, their difference $\omega - \omega' \in \Omega^1(\mathcal{G}, \mathfrak{g})$ has to be P-equivariant and horizontal. Therefore it can be viewed as an element of $\Omega^1(M, \mathcal{A}M)$. It turns out that ω and ω' induce the same regular infinitesimal flag structure if and only if $\omega - \omega' \in \Omega^1(M, \mathcal{A}M)^1$, see section 3.1.10. of [**15**]. The curvature of a Cartan connection is P-equivariant and horizontal, hence it can be interpreted as an element of $\Omega^2(M, \mathcal{A}M)$. For the difference of the curvatures of two Cartan connections inducing the same regular infinitesimal flag structure the following holds, for a proof see section 3.1.10. of [**15**] :

Proposition 2.13. *Suppose ω and ω' are Cartan connections satisfying $\omega - \omega' \in \Omega^1(M, \mathcal{A}M)^\ell$ for some $\ell \geq 1$.*
Then the difference of its curvatures $\kappa - \kappa'$ is an element of $\Omega^2(M, \mathcal{A}M)^\ell$ and the induced section $\mathrm{gr}_\ell(\kappa - \kappa')$ of $\mathrm{gr}_\ell(\Lambda^2 T^*M \otimes \mathcal{A}M)$ equals $\partial(\mathrm{gr}_\ell(\omega - \omega'))$, where $\mathrm{gr}_\ell(\omega - \omega') \in \Gamma(\mathrm{gr}_\ell(T^*M \otimes \mathcal{A}M))$ is the section induced by $\omega - \omega'$.

The proposition indicates that a appropriate normalisation condition for a Cartan connection may be found by demanding the curvature to have values in a subbundle of $\Lambda^2 T^*M \otimes \mathcal{A}M$, which after passing to the associated graded is complementary to $\mathrm{im}(\partial) \subset \mathrm{gr}(\Lambda^2 T^*M \otimes \mathcal{A}M)$.
Recall that the Kostant codifferential $\partial^* : \Lambda^{n+1}\mathfrak{p}_+ \otimes \mathfrak{g} \to \Lambda^n \mathfrak{p}_+ \otimes \mathfrak{g}$ is P-equivariant and therefore induces a bundle map

$$\partial^* : \Lambda^{n+1} T^*M \otimes \mathcal{A}M \to \Lambda^n T^*M \otimes \mathcal{A}M.$$

The codifferential ∂^* is easily seen to be filtration preserving. Hence it induces as well a bundle map between the associated graded bundles

$$\mathrm{gr}(\partial^*) : \mathrm{gr}(\Lambda^{n+1} T^*M \otimes \mathcal{A}M) \to \mathrm{gr}(\Lambda^n T^*M \otimes \mathcal{A}M),$$

which is compatible with the gradings.
By proposition 2.9, we have

$$\mathrm{gr}(\Lambda^n T^*M \otimes \mathcal{A}M) = \mathrm{im}(\mathrm{gr}(\partial^*)) \oplus \ker(\Box) \oplus \mathrm{im}(\partial) = \ker(\mathrm{gr}(\partial^*)) \oplus \mathrm{im}(\partial),$$

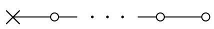
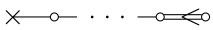
where $\ker(\Box) \cong \mathcal{G}_0 \times_{G_0} H^n(\mathfrak{g}_-, \mathfrak{g})$. This motivates the following definition:

Definition 2.8. A parabolic geometry is called *normal* if its curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ satisfies $\partial^*(\kappa) = 0$.

It can be shown that, given regular infinitesimal flag structure of type (G, P), there always exists a normal regular parabolic geometry of type (G, P) inducing the given infinitesimal flag structure, see section 3.1.14. of [**15**]. To formulate now the equivalence result for normal regular parabolic geometries and regular infinitesimal flag structures, we still have to fix some notation.
Since the differential ∂ is compatible the gradings on the spaces $\Lambda^n \mathfrak{g}_-^* \otimes \mathfrak{g}$, the cohomology spaces are naturally graded as well $H^n(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_\ell H^n(\mathfrak{g}_-, \mathfrak{g})_\ell$. We denote the associated filtration by $H^n(\mathfrak{g}_-, \mathfrak{g})^\ell = \bigoplus_{m \geq \ell} H(\mathfrak{g}_-, \mathfrak{g})_m$. If $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = 0$, it turns out that there is up to isomorphism exactly one

normal regular parabolic geometry of type (G, P) inducing some fixed infinitesimal flag structure. Even more one obtains an equivalence of categories, for a proof see section 3.1.14. of [**15**]:

Theorem 2.14. *Let $\mathfrak{g}$=$\mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra such that none of the simple ideals is contained in $\mathfrak{g}_0$ and such that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = 0$. Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and $P \subset G$ a parabolic subgroup. Then the functor, which assigns to a normal regular parabolic geometry its underlying regular infinitesimal flag structures establishes an equivalence of categories between regular normal parabolic geometries of type (G, P) and regular infinitesimal flag structures of the type (G, P).*

Remark 2.3. The cohomological condition $H^1(\mathfrak{g}_-, \mathfrak{g}) = 0$ can be analysed using Kostant version of the Bott-Borel-Weil theorem [Theorem 2.10]. If $\mathfrak{g}$ is a simple $|k|$-graded Lie algebra, it turns out that the condition is always satisfied except for $\mathfrak{g}$ or $\mathfrak{g}^{\mathbb{C}}$ corresponding to the crossed Dynkin diagrams  or 
It turns out that in these cases the corresponding regular normal parabolic geometries are still determined by some finer underlying structures except for the degenerate case corresponding to the crossed Dynkin diagram $\times$. The underlying structures in these cases are equivalent to classical projective structures and to so called contact projective structures.

Let us finish this chapter by giving an overview of the geometric structures admitting an equivalent description as some parabolic geometries, for more details see [**15**].

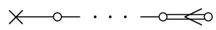
Example 2.2. (Geometries corresponding to $|1|$-graded semisimple Lie algebras) Suppose $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $|1|$-graded semisimple Lie algebra, G a Lie group with Lie algebra $\mathfrak{g}$ and P a parabolic subgroup corresponding to the given grading. For a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), the induced filtration of TM is trivial $TM = T^{-1}M$. Since M is a trivially filtered manifold, the parabolic geometry is automatically regular. Therefore the underlying regular infinitesimal flag structure induced by $(\mathcal{G} \to M, \omega)$ is just a reduction of the structure group of the frame bundle of M to the Levi subgroup G_0 via $Ad : G_0 \to GL(\mathfrak{g}_{-1})$. Under the conditions of theorem 2.14, one obtains an equivalence of regular normal parabolic geometries of type (G, P) and first order G_0-structures.
Some important examples of such geometric structures are conformal structures, almost quaternionic structures and almost Grassmannian structures.
A regular normal parabolic geometry of the exceptional type 

also corresponds to a $|1|$-grading. As mentioned in remark 2.3 such a geometry is equivalent to a finer underlying structure than a infinitesimal flag structure, which can be interpreted as a projective structure.

Example 2.3. (Parabolic geometries determined by its tangential filtrations) A regular infinitesimal flag structure consists of a filtration of the tangent bundle and a reduction of the structure group corresponding to the homomorphism $Ad : G_0 \to Aut_{gr}(\mathfrak{g}_-)$. If this homomorphism is an isomorphism, the reduction of the structure group is an isomorphism between $\mathcal{G}_0$ and the frame bundle of $\mathrm{gr}(TM)$. Therefore theorem 2.14 implies that provided that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = 0$ and $G_0 \cong Aut_{gr}(\mathfrak{g}_-)$ for some parabolic pair (G, P), we have an equivalence of categories between regular normal parabolic geometries of (G, P) and regular filtered manifolds of type $\mathfrak{g}_-$. Since $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$, a regular filtered manifolds of type $\mathfrak{g}_-$ can be as well viewed as a regular bracket generating, whose derived flag of sheaves determines a regular filtered manifold of type $\mathfrak{g}_-$, cf. section 1.1.2.
There is an easy way to construct such structures: It turns out, see [**36**], that if $\mathfrak{g}$ is a $|k|$-graded semisimple Lie algebra with $H^1(\mathfrak{g}_-, \mathfrak{g})^0 = 0$, then setting $G := Aut(\mathfrak{g})$ and $P = \bigcap_{i=-k}^{k} N_G(\mathfrak{g}^i)$ one obtains an isomorphism $G_0 \cong Aut_{\mathrm{gr}}(\mathfrak{g})$. A complete list of $|k|$-graded semisimple Lie algebras satisfying the cohomological condition can be found in [**43**].
The most prominent examples are generic rank 2 distributions on five dimensional manifolds, generic rank 3 distributions on six dimensional manifold as well as quaternionic contact structures, see also section 1.1.2.

Example 2.4. (Parabolic contact structures) A *contact grading* on $\mathfrak{g}$ is a $|2|$-grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_-$ is a Heisenberg Lie algebra. It turns out that such gradings exists only on simple Lie algebras $\mathfrak{g}$ and are unique up to isomorphism. A complete classification in the complex as well as in the real case is known.
Given a real contact grading on a simple Lie algebra $\mathfrak{g}$ and corresponding groups $P \subset G$, then a regular infinitesimal flag structure of type (G, P) is a filtered manifold $TM = T^{-2}M \supset T^{-1}M$ such that the symbol algebra in each point is isomorphic to a Heisenberg Lie algebra together with a reduction of the structure group of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of $\mathrm{gr}(TM)$. The structure group $Aut_{gr}(\mathfrak{g}_-)$ of $\mathcal{P}(\mathrm{gr}(TM))$ can be viewed as a subgroup of $GL(\mathfrak{g}_{-1})$, since any grading preserving automorphism of $\mathfrak{g}_-$ is easily seen to be uniquely determined by its restriction to $\mathfrak{g}_{-1}$. Hence such a regular infinitesimal flag structure is nothing else than a contact manifold $T^{-1}M \subset TM$ together with a reduction of the frame bundle $\mathcal{P}(T^{-1}M)$ of the contact distribution $T^{-1}M$ to G_0 via $Ad : G_0 \to GL(\mathfrak{g}_{-1})$. This reduction can be expressed as an additional structure on the contact subbundle $T^{-1}M$.

Geometric structures of this from are for example partially integrable almost CR-structures, where the additional structure on the contact subbundle is a complex structure, Lagrangean contact structures, where the additional structure is a decomposition of $T^{-1}M$ into isotropic subbundles, and Lie contact structures, where the additional structures is given by two subbundles whose tensor product equals $T^{-1}M$.

A regular normal parabolic geometry of the exceptional type  corresponds as well to a contact grading. Its equivalent underlying structure can be viewed as contact projective structure.

CHAPTER 3

Prolongation of Overdetermined Systems on Regular Infinitesimal Flag Manifolds

Given a regular parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) and an irreducible representation $\mathbb{V}$ of G, one can associate to the tractor bundle V corresponding to $\mathbb{V}$ a sequence of invariant linear differential operators, the so called BGG-sequence associated to V, see **[13]** and **[6]**. The name of this sequence refers to the fact that in the case of the homogeneous model for parabolic geometries of type (G, P) the BGG-sequence associated to V is a complex of invariant differential operators, which corresponds dually to the algebraic Bernstein-Gelfand-Gelfand resolution of the representation $\mathbb{V}$. The first operator occurring in such a sequence gives always rise to an overdetermined system of partial differential equations. Using Weyl structures, the BGG-operators can be naturally viewed as differential operators between natural vector bundles over the underlying regular infinitesimal flag structure on M.

In this chapter we shall study semi-linear differential operators between natural vector bundles over certain regular infinitesimal flag manifolds, which have the same weighted symbol as some first BGG-operator. Given such a semi-linear differential operator D, we will present a conceptual method, using ideas from the construction of BGG-operators, to rewrite the semi-linear system $Ds = 0$ as a system of partial differential equations of the form $\nabla\Sigma + C(\Sigma) = 0$, where ∇ is a linear connection on some vector bundle V over the regular infinitesimal flag manifold M and $C : V \to T^*M \otimes V$ is a bundle map.

If D is linear, then we will see that the associated system $Ds = 0$ is of weighted finite type and the rewriting procedure leads to a system of the form $\nabla\Sigma + C(\Sigma) = 0$, where C is a vector bundle map. Hence $\bar{\nabla} := \nabla + C$ is again a linear connection and we obtain a bijection between solutions of $Ds = 0$ and parallel sections of $\bar{\nabla}$. This implies that the dimension of the solution space is bounded by the rank of V and we will see that the rank of this bundle can be easily computed.

The prolongation procedure, we establish here, will generalise the procedure introduced in **[4]** for overdetermined systems on regular infinitesimal flag manifolds corresponding to $|1|$-graded semisimple Lie algebras to a broader class of regular infinitesimal flag structures by working within the framework of weight jet bundles.

3.1. Regular infinitesimal flag structures

Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ be a $|k|$-graded semisimple Lie algebra and set as in section 2.1.2

$$\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_{-1}$$

and

$$\mathfrak{p}_+ := \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k.$$

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and $P \subseteq G$ a parabolic subgroup with Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. Recall that the Levi subgroup G_0 of P is the subgroup consisting of those elements in P, whose adjoint action preserves the grading on $\mathfrak{g}$:

$$G_0 := \{g \in P : Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for } i = -k, ..., k\}.$$

It is a reductive Lie group with Lie algebra $\mathfrak{g}_0$. Moreover, the definition of G_0 shows that the adjoint action induces a homomorphism

$$Ad : G_0 \to Aut_{\mathrm{gr}}(\mathfrak{g}_-),$$

where $Aut_{\mathrm{gr}}(\mathfrak{g}_-)$ is the group of Lie algebra automorphisms of $\mathfrak{g}_-$, which in addition preserve the grading on $\mathfrak{g}_-$.
Recall from section 2.3. that a regular infinitesimal flag structures of type (G, P) on a manifold M consists of the following data:

- a filtration of the tangent bundle $TM = T^{-k}M \supset ... \supset T^{-1}M$, which makes M into a filtered manifold such that the symbol algebra in each point $\mathrm{gr}(T_xM)$ is isomorphic to the Lie algebra $\mathfrak{g}_-$.
- a reduction $\mathcal{G}_0 \to M$ of the structure group of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of $\mathrm{gr}(TM)$ to the subgroup G_0 with respect to the homomorphism $Ad : G_0 \to Aut_{\mathrm{gr}}(\mathfrak{g}_-)$

Definition 3.1. A smooth manifold M endowed with a regular infinitesimal flag structure of type (G, P) is called a *regular infinitesimal flag manifold of type* (G, P).

In the last chapter, we have seen a variety of examples of geometric structures, which can be viewed as regular infinitesimal flag structures. Moreover, we have noted that for nearly all choices G and P, a regular infinitesimal flag structure of type (G, P) determines a normal regular parabolic geometry of type (G, P), see remark 2.3.

3.1.1. G_0 representations and natural vector bundles for regular infinitesimal flag manifolds. Suppose that M is a regular infinitesimal flag manifold of some type (G, P) and let $\mathbb{E}$ be a representation of G_0. Then we denote by

$$E := \mathcal{G}_0 \times_{G_0} \mathbb{E}$$

the vector bundle associated to the principal bundle $\mathcal{G}_0$ with standard fiber $\mathbb{E}$, see also section 2.1.4. The vector bundles of this form are the natural vector bundles for regular infinitesimal flag structures.
Note that any G_0-equivariant linear map between two G_0-representations $\mathbb{E}$ and $\mathbb{F}$ gives naturally rise to a vector bundle homomorphism between the corresponding associated bundles E and F.
Consider the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of the associated graded bundle

$$\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus ... \oplus \mathrm{gr}_{-1}(TM).$$

Recall that the fiber $\mathcal{P}_x(\mathrm{gr}(TM))$ over a point $x \in M$ consists of all graded Lie algebra isomorphisms $\mathfrak{g}_- \to \mathrm{gr}(T_xM)$. The reduction of the structure group of $\mathcal{P}(\mathrm{gr}(TM))$ to G_0 obviously induces an isomorphism of vector bundles

$$\mathcal{G}_0 \times_{G_0} \mathfrak{g}_- \cong \mathrm{gr}(TM)$$

$$[u_0, X] \mapsto \phi_{u_0}(X),$$

where $u_0 \mapsto \phi_{u_0}$ denotes the reduction $\mathcal{G}_0 \to \mathcal{P}(\mathrm{gr}(TM))$. Restricting this isomorphism to some fiber clearly defines a graded Lie algebra isomorphism. By proposition 2.3 the Killing form induces a duality of G_0-modules between $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ and hence we have isomorphisms of vector bundles

$$\mathcal{G}_0 \times_{G_0} \mathfrak{g}_i \cong \mathrm{gr}_{-i}(TM)^*$$

$$\mathcal{G}_0 \times_{G_0} \mathfrak{p}_+ \cong \mathrm{gr}(TM)^*.$$

Remark 3.1. The filtration of the tangent bundle TM induces a filtration of the cotangent bundle T^*M into vector subbundles $(T^*M)^i$, where $(T^*M)^i$ is the defined as the annihilator of $T^{-i+1}M$. In particular, one obtains an isomorphism of vector bundles $\mathrm{gr}(T^*M) \cong \mathrm{gr}(TM)^*$ mapping $\mathrm{gr}_i(T^*M)$ onto $\mathrm{gr}_{-i}(TM)^*$.

Let us review some basic facts about irreducible representations of the reductive Lie group G_0, as already mentioned in section 2.1.5:
Assume for a moment that $\mathfrak{g}$ is a complex $|k|$-graded semisimple Lie algebra and choose a Cartan subalgebra $\mathfrak{h}$ and a subsystem of simple roots $\Delta^0 = \{\alpha_1, ..., \alpha_n\}$ such that $\mathfrak{p}$ is a standard parabolic subalgebra and the grading is therefore given by $\Sigma_{\mathfrak{p}}$-height, see section 2.1.3.

By proposition 2.7 and theorem 2.8 we know that by assigning to a complex irreducible representations of the reductive Levi subalgebra

$$\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$$

its highest weight, we obtain a bijection between irreducible complex representation of $\mathfrak{g}_0$ and $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral linear functionals on $\mathfrak{h}$. We denote the index set of the simple roots by $I := \{1, ..., n\}$ and define $J \subset I$ as the subset consisting of those elements $i \in I$ with $\alpha_i \in \Sigma_{\mathfrak{p}}$. Recall from section 2.1.5 that writing a linear functional λ on $\mathfrak{h}$ as a sum of fundamental weights the condition to be $\mathfrak{p}$-integral and $\mathfrak{p}$-dominant exactly means that λ is of the following form

$$\lambda = \sum_{i \in J} a_i \omega_i + \sum_{i \in I \setminus J} a_i \omega_i \tag{3.1}$$

where $a_i \in \mathbb{N}_0$ for $i \in I \setminus J$ and $a_i \in \mathbb{R}$ for $i \in J$.
By proposition 2.6 the Cartan subalgebra $\mathfrak{h}$ naturally splits into a direct sum

$$\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}_0,$$

where $\mathfrak{h}_0$ is a Cartan subalgebra for the semisimple part of $\mathfrak{g}_0$

$$\mathfrak{g}_0^{ss} = \mathfrak{h}_0 \oplus \bigoplus_{ht_{\Sigma_{\mathfrak{p}}}(\alpha)=0} \mathfrak{g}_\alpha.$$

By definition of $\mathfrak{h}_0$ the first sum of (3.1) vanishes on $\mathfrak{h}_0$ and hence the irreducible representation of $\mathfrak{g}_0$ with highest weight λ is given by the irreducible representation $\mathbb{E}$ of $\mathfrak{g}_0^{ss}$ with highest weight $\lambda|_{\mathfrak{h}_0}$, where the center $\mathfrak{z}(\mathfrak{g}_0)$ acts on $\mathbb{E}$ by $\lambda|_{\mathfrak{z}(\mathfrak{g}_0)}$.
If $\mathfrak{g}$ is a real $|k|$-graded semisimple Lie algebra, irreducible representations of $\mathfrak{g}_0$ can be as well described by highest weights:
If $\mathbb{E}$ is a complex irreducible representation of the Levi subalgebra $\mathfrak{g}_0$, then it extends uniquely to a complex irreducible representation of the Levi subalgebra $\mathfrak{g}_0^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ and we may describe $\mathbb{E}$ by a highest weight.
If $\mathbb{E}$ is a real irreducible representation of $\mathfrak{g}_0$, having no invariant complex structure, then its complexification $\mathbb{E}^{\mathbb{C}}$ is as well irreducible. By the highest weight of $\mathbb{E}$ we then mean the highest weight of its complexification $\mathbb{E}^{\mathbb{C}}$.

In the sequel we will need the decomposition of the G_0-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ into irreducible components:

Lemma 3.1. *Suppose $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is complex semisimple $|k|$-graded Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and choose a simple subsystem of roots $\Delta^0 = \{\alpha_1, ...\alpha_n\}$ such that the parabolic subalgebra $\mathfrak{p}$ is standard. We define J as the subset of $I := \{1, ..., n\}$ consisting of those elements $i \in I$ with $\alpha_i \in \Sigma_{\mathfrak{p}}$. Then we have:*

(1) *The* $\mathfrak{g}_0$*-module* $\mathfrak{g}_{-1}$ *is completely reducible with decomposition into irreducibles*

$$\mathfrak{g}_{-1} = \bigoplus_{j \in J} \mathfrak{g}_{-1,j},$$

where $\mathfrak{g}_{-1,j}$ *is the unique irreducible representation with highest weight* $-\alpha_j$*.*

(2) *Correspondingly, the dual* $\mathfrak{g}_0$*-module* $\mathfrak{g}_1$ *decomposes into irreducible components*

$$\mathfrak{g}_1 = \bigoplus_{j \in J} \mathfrak{g}_{1,j},$$

where $\mathfrak{g}_{1,j}$ *is the unique irreducible representation of lowest weight* α_j*.*

Proof. By proposition 2.6 the Cartan subalgebra $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ is a Cartan subalgebra of the semisimple part $\mathfrak{g}_0^{ss}$ such that

$$\mathfrak{g}_0^{ss} = \mathfrak{h}_0 \oplus \bigoplus_{ht_{\Sigma_\mathfrak{p}}(\alpha)=0} \mathfrak{g}_\alpha$$

is the corresponding decomposition into root spaces. Since $\mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}$, the center acts diagonalisably on $\mathfrak{g}_{-1}$ and so $\mathfrak{g}_{-1}$ is a completely reducible representation of $\mathfrak{g}_0$. The space $\mathfrak{g}_{-1}$ consists of all root spaces corresponding to roots with $\Sigma_\mathfrak{p}$-height -1. Denoting for $j \in J$ by $\mathfrak{g}_{-1,j}$ the direct sum of all root spaces corresponding to roots of the form

$$-\alpha_j - \sum_{i \in I \setminus J} a_i \alpha_i,$$

where $a_i \in \mathbb{N}_0$, we therefore have

$$\mathfrak{g}_{-1} = \bigoplus_{j \in J} \mathfrak{g}_{-1,j}.$$

Since for $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ one clearly has $ht_{\Sigma_\mathfrak{p}}(\alpha + \beta) = ht_{\Sigma_\mathfrak{p}}(\alpha) + ht_{\Sigma_\mathfrak{p}}(\beta)$, the subspaces $\mathfrak{g}_{-1,j}$ are $\mathfrak{g}_0$-invariant. Moreover, any root space $\mathfrak{g}_\lambda$ lying in $\mathfrak{g}_{-1,j}$ generates $\mathfrak{g}_{-1,j}$ as $\mathfrak{g}_0$-module and hence $\mathfrak{g}_{-1,j}$ is irreducible for all $j \in J$. Obviously, $\mathfrak{g}_{-1,j}$ is the irreducible representation with highest weight $-\alpha_j$.
The assertion for $\mathfrak{g}_1$ follows from the well known fact that an irreducible representation has highest weight λ if and only if the dual representation has lowest weight $-\lambda$. $\square$

We finish this section by introducing the notion of the Cartan product of two irreducible representations of G_0.
Recall that a finite dimensional representation of a reductive Lie algebra is

completely reducible if and only if the center acts diagonalisably. In particular, the tensor product of two completely reducible representations is again completely reducible.

Suppose $\mathbb{E}$ and $\mathbb{F}$ are two complex irreducible representations of $\mathfrak{g}_0$ with highest weight λ respectively μ and consider the completely reducible representation $\mathbb{E} \otimes \mathbb{F}$. It is well known that there exists an irreducible component $\mathbb{E} \circledcirc \mathbb{F}$ of multiplicity one in $\mathbb{E} \otimes \mathbb{F}$, which has highest weight $\lambda + \mu$. In fact, the irreducible component $\mathbb{E} \circledcirc \mathbb{F}$ is generated by the tensor product of highest weight vectors of weight λ respectively μ in $\mathbb{E} \otimes \mathbb{F}$. Up to multiplication by a scalar we also have a unique projection $\mathbb{E} \otimes \mathbb{F} \to \mathbb{E} \circledcirc \mathbb{F}$. For the notion of a highest weight vector and multiplicity see the explanation before theorem 2.10.

We will call

$$\mathbb{E} \circledcirc \mathbb{F}$$

respectively the projection

$$\mathbb{E} \otimes \mathbb{F} \to \mathbb{E} \circledcirc \mathbb{F}$$

the *Cartan product* of $\mathbb{E}$ and $\mathbb{F}$.

Suppose now that $\mathbb{E}$ is a real irreducible representation of $\mathfrak{g}_0$, having no-invariant complex structure, and $\mathbb{F}$ is a complex irreducible representation of $\mathfrak{g}_0$. Then $\mathbb{E}^{\mathbb{C}}$ is irreducible. Moreover, $\mathbb{E} \otimes_{\mathbb{R}} \mathbb{F}$ is a complex representation isomorphic to $\mathbb{E}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{F}$. In this case the *Cartan product* is defined as $\mathbb{E} \circledcirc \mathbb{F} := \mathbb{E}^{\mathbb{C}} \circledcirc \mathbb{F}$.

If $\mathbb{E}$ and $\mathbb{F}$ are two real irreducible representation of $\mathfrak{g}_0$, having no complex invariant structure, then their complexifications $\mathbb{E}^{\mathbb{C}}$ and $\mathbb{F}^{\mathbb{C}}$ are irreducible. The invariant real structures of $\mathbb{E}^{\mathbb{C}}$ and $\mathbb{F}^{\mathbb{C}}$ obviously induce an invariant real structure on $\mathbb{E}^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{F}^{\mathbb{C}} \cong (\mathbb{E} \otimes_{\mathbb{R}} \mathbb{F}) \otimes_{\mathbb{R}} \mathbb{C}$. Since $\mathbb{E}^{\mathbb{C}} \circledcirc \mathbb{F}^{\mathbb{C}}$ is of multiplicity one, this invariant real structure has to map the Cartan product $\mathbb{E}^{\mathbb{C}} \circledcirc \mathbb{F}^{\mathbb{C}}$ to itself and we have an invariant real structure on $\mathbb{E}^{\mathbb{C}} \circledcirc \mathbb{F}^{\mathbb{C}}$. Therefore there exists a unique irreducible component $\mathbb{E} \circledcirc \mathbb{F}$ in $\mathbb{E} \otimes_{\mathbb{R}} \mathbb{F}$, whose complexification is $\mathbb{E}^{\mathbb{C}} \circledcirc \mathbb{F}^{\mathbb{C}}$. We call $\mathbb{E} \circledcirc \mathbb{F}$ the *Cartan product* of $\mathbb{E}$ and $\mathbb{F}$.

3.1.2. Semi-linear differential operators. Suppose that M is a manifold endowed with a regular infinitesimal structure of some type (G, P). We will study in the sequel semi-linear differential operators between sections of natural vector bundles over M.

Definition 3.2. Suppose $\mathbb{E}$ and $\mathbb{F}$ are G_0-representations and write E and F for the corresponding natural vector bundles.

(1) A *semi-linear differential operator* of weighted order r between the vector bundles E and F is a differential operator $D : \Gamma(E) \to \Gamma(F)$,

which can be written as $D = D_1 + D_2$, where D_1 is a linear differential operator of weighted order r and D_2 a differential operator of weighted order $r-1$.

(2) The *weighted symbol* of D is the weighed symbol of D_1

$$\sigma(D_1) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to F.$$

The weighted symbols of semi-linear differential operators between natural vector bundles can be viewed as vector bundle maps between natural vector bundles, since we have:

Proposition 3.2. *Let $\mathcal{U}(\mathfrak{g}_-) = \bigoplus_{i=0}^{\infty} \mathcal{U}_{-i}(\mathfrak{g}_-)$ be the universal enveloping algebra of $\mathfrak{g}_-$ endowed with the weighted grading as in section 1.2.4.*

(1) *For all $r \geq 0$ the space $\mathcal{U}_{-r}(\mathfrak{g}_-)$ admits the structure of a G_0-module and we have an isomorphism of G_0-modules*

$$\mathcal{S}_{-r}(\mathfrak{g}_-) := \bigoplus_{1i_1+...+ki_k=r} S^{i_1}(\mathfrak{g}_{-1}) \otimes ... \otimes S^{i_k}(\mathfrak{g}_{-k}) \cong \mathcal{U}_{-r}(\mathfrak{g}_-).$$

In particular, we have G_0-equivariant inclusions

$$S^{i_1}(\mathfrak{g}_{-1}) \otimes ... \otimes S^{i_k}(\mathfrak{g}_{-k}) \hookrightarrow \mathcal{U}_{-r}(\mathfrak{g}_-)$$

for $i_j \in \mathbb{N}_0$ and $1i_1 + ... + ki_k = r$.

(2) *We have an isomorphism of vector bundles*

$$\mathcal{U}_{-r}(\mathrm{gr}(TM)) \cong \mathcal{G}_0 \times_{G_0} \mathcal{U}_{-r}(\mathfrak{g}_-).$$

and injective vector bundle maps

$$S^{i_1}(\mathrm{gr}_{-1}(TM)) \otimes ... \otimes S^{i_k}(\mathrm{gr}_{-k}(TM)) \hookrightarrow \mathcal{U}_{-r}(\mathrm{gr}(TM))$$

for $i_j \in \mathbb{N}_0$ and $1i_1 + ... + ki_k = r$.

Proof. (1) For $g \in G_0$ the linear map $Ad(g) : \mathfrak{g}_- \to \mathfrak{g}_-$ is a Lie algebra isomorphism. By proposition 1.8 the isomorphism $Ad(g)$ therefore uniquely extends to an isomorphism of unitial associative algebras

$$\mathcal{U}(Ad(g)) : \mathcal{U}(\mathfrak{g}_-) \to \mathcal{U}(\mathfrak{g}_-).$$

Recall, that $\mathcal{U}(Ad(g))$ is explicitly given by

$$\mathcal{U}(Ad(g))(X_1...X_\ell) = Ad(g)(X_1)...Ad(g)(X_\ell) \quad \text{for} \quad X_i \in \mathfrak{g}_-.$$

This shows that, since the isomorphism $Ad(g) : \mathfrak{g}_- \to \mathfrak{g}_-$ preserves the grading, the isomorphism $\mathcal{U}(Ad(\mathfrak{g}_-))$ restricts to an isomorphism

$$\mathcal{U}(Ad(g)) : \mathcal{U}_{-r}(\mathfrak{g}_-) \to \mathcal{U}_{-r}(\mathfrak{g}_-) \quad \text{for all} \quad r \geq 0.$$

By proposition 1.8 $\mathcal{U}$ is a covariant functor and so we have

$$\mathcal{U}(Ad(gg')) = \mathcal{U}(Ad(g)Ad(g')) = \mathcal{U}(Ad(g))\mathcal{U}(Ad(g')).$$

Hence $\mathcal{U}(Ad(\_)) : G_0 \to GL(\mathcal{U}_{-r}(\mathfrak{g}_-))$ is a finite dimensional G_0-representation. The linear isomorphism of proposition 1.10

$$\mathcal{S}(\mathfrak{g}_{-1}) \otimes ... \otimes \mathcal{S}(\mathfrak{g}_{-k}) \cong \mathcal{U}(\mathfrak{g}_-)$$

defined by

$$x_1 \otimes ... \otimes x_k \mapsto \Phi_1(x_1)...\Phi_k(x_k)$$

restricts to a linear isomorphism $\mathcal{S}_{-r}(\mathfrak{g}_-) \cong \mathcal{U}_{-r}(\mathfrak{g}_-)$. By the second part of theorem 1.9, this is even an isomorphism of G_0-modules for the usual G_0-module structure on $\mathcal{S}_{-r}(\mathfrak{g}_-)$.
(2) The isomorphism $\mathrm{gr}(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$ obviously induces an isomorphism $\mathcal{U}_{-r}(\mathrm{gr}(TM)) \cong \mathcal{G}_0 \times_{G_0} \mathcal{U}_{-r}(\mathfrak{g}_-)$, see also proposition 1.14. The inclusions of part (1) are G_0-equivariant and hence induce injective vector bundle maps between the corresponding associated bundles. $\square$

Now let us consider the universal enveloping algebra of the nilpotent graded Lie algebra $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ endowed with its weighted grading

$$\mathcal{U}(\mathfrak{p}_+) = \bigoplus_{i=0}^{\infty} \mathcal{U}_i(\mathfrak{p}_+).$$

Since, by definition of G_0, for each $g \in G_0$ the map $Ad(g) : \mathfrak{p}_+ \to \mathfrak{p}_+$ is a an isomorphism of graded Lie algebras, we obtain as in the proof of proposition 3.2 that $\mathcal{U}_r(\mathfrak{p}_+)$ is a G_0-module for all $r \geq 0$ and that we have an isomorphism of G_0-modules

$$\mathcal{S}_r(\mathfrak{p}_+) := \bigoplus_{1i_1+...+ki_k=r} S^{i_1}(\mathfrak{g}_1) \otimes ... \otimes S^{i_k}(\mathfrak{g}_k) \cong \mathcal{U}_r(\mathfrak{p}_+).$$

Since the Killing form induces a duality between the G_0-modules $\mathfrak{g}_{-i}$ and $\mathfrak{g}_i$, this implies:

Corollary 3.3. *For all $r \geq 0$ the Killing form induces an isomorphism of G_0-modules $\mathcal{U}_{-r}(\mathfrak{g}_-)^* \cong \mathcal{U}_r(\mathfrak{p}_+)$.*

3.2. Prolongation of semi-linear systems of partial differential equations

Let M be a manifold endowed with a regular infinitesimal flag structure corresponding to a $|k|$-graded semisimple Lie algebra $\mathfrak{g}$, where the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Levi subalgebra is one dimensional. We shall study in this section semi-linear differential operators between sections of natural vector bundles over M

$$D : \Gamma(E) \to \Gamma(F),$$

whose associated system of partial differential equations $Ds = 0$ is overdetermined. We will show for a broad class of such differential operators D how to construct a vector bundle V, a linear connection ∇ on V and a bundle map $C : V \to T^*M \otimes V$ such that one has a bijection between the following solution spaces:

$$\{s \in \Gamma(E) : D(s) = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : \nabla\Sigma + C(\Sigma) = 0\}.$$

If D is a linear differential operator, then C will be a vector bundle map and $\nabla + C$ is therefore as well a linear connection. Hence we obtain in this case a bijection between solutions of the studied linear system and parallel sections of the connection $\nabla + C$.

In the last part of this section, we will outline what happens in the case of regular infinitesimal flag manifolds corresponding to $|k|$-graded semisimple Lie algebras $\mathfrak{g}$, where the dimension of $\mathfrak{z}(\mathfrak{g}_0)$ is larger than one.

3.2.1. Semi-linear systems on regular infinitesimal flag structures corresponding to $|k|$-gradings such that $\mathfrak{z}(\mathfrak{g}_0)$ is one dimensional. If not otherwise stated we suppose throughout this section that

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_k$$

is a real or complex $|k|$-graded semisimple Lie algebra, where the center $\mathfrak{z}(\mathfrak{g}_0)$ of its Levi subalgebra

$$\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$$

is of dimension one.

Recall from the description of $|k|$-gradings in terms of roots in section 2.1.3 that this is exactly satisfied, if the $|k|$-grading corresponds in the real case to a crossed Satake diagram respectively in the complex case to a crossed Dynkin diagram where only one root α_j is crossed. We set $\alpha := \alpha_j$.

Regular infinitesimal flag structures corresponding to $|k|$-graded semisimple Lie algebras of this type are for instance all structures corresponding to $|1|$-graded Lie algebras, like conformal and almost quaternionic structures (see also example 2.2), as well as certain types of generic distributions, like generic rank 2 distributions on five dimensional manifolds or generic rank n distributions on manifolds of dimension $\frac{n(n+1)}{2}$, some types of parabolic contact structures, like Lie contact structures and the contact structures associated to the exceptional simple Lie groups (see also example 2.4), and also (split) quaternionic contact structures. For more details on these geometric structures see [**15**].

Given a $|k|$-graded semisimple Lie algebra $\mathfrak{g}$, whose Levi subalgebra has one dimensional center, suppose that G is a simply connected Lie group with Lie algebra $\mathfrak{g}$ and $P \subset G$ the connected parabolic subgroup corresponding to the $|k|$-grading on $\mathfrak{g}$. Note that by theorem 2.4 the connectedness of P implies that the corresponding Levi subgroup $G_0 \subset P$ is as well connected.

Moreover, let M be a manifold endowed with a regular infinitesimal flag structure of type (G, P).

Remark 3.2. We assume G to be simply connected and P to be connected, just to ensure that on one hand representations of the Lie algebra $\mathfrak{g}$ integrate to representations of G and that on the other hand the representation theory of G_0 and $\mathfrak{g}_0$ coincides. This will allow us to formulate the results of this chapter in an uniform way. The conditions can be dropped, whenever one is dealing with some particular representations.

Now suppose that $\mathbb{E}$ is the complex irreducible representation of G_0, whose dual representation $\mathbb{E}^*$ has highest weight

$$\lambda = (r-1)\omega_j + \sum_{i \in I \setminus \{j\}} a_i \omega_i$$

where $r \in \mathbb{N}$ and $a_i \in \mathbb{N}_0$ with the notation as in section 3.1.1.

Remark 3.3. This means that we only consider irreducible representations, whose dual representation corresponds to a $\mathfrak{g}$-dominant and $\mathfrak{g}$-integral weight. As detailed in section 3.1.1 this restriction just concerns the action of the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Levi subalgebra $\mathfrak{g}_0$.

By lemma 3.1 and the assumption on the grading, we obtain that $\mathfrak{g}_{-1}$ is an irreducible representation of $\mathfrak{g}_0$ with highest weight $-\alpha_j$. Having fixed $\mathbb{E}$, we consider the G_0-representation $\mathbb{F} := \circledcirc^r \mathfrak{g}_{-1}^* \circledcirc \mathbb{E}$.
By proposition 3.2 we have a G_0-equivariant linear projection

$$\mathcal{U}_{-r}(\mathfrak{g}_-)^* \otimes \mathbb{E} \to S^r \mathfrak{g}_{-1}^* \otimes \mathbb{E}.$$

Composing this projection with the projection $S^r \mathfrak{g}_{-1}^* \otimes \mathbb{E} \to \circledcirc^r \mathfrak{g}_{-1}^* \circledcirc \mathbb{E}$, we obtain a G_0-equivariant linear projection

$$\mathcal{U}_{-r}(\mathfrak{g}_-)^* \otimes \mathbb{E} \to \circledcirc^r \mathfrak{g}_{-1}^* \circledcirc \mathbb{E}.$$

Since this map is G_0-equivariant, it induces a surjective vector bundle map between the corresponding associated vector bundles

$$\mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to F = \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E. \tag{3.2}$$

For semi-linear differential operators $D : \Gamma(E) \to \Gamma(F)$ of weighted order r with weighted symbol given by the natural projection (3.2) we will prove the following theorem:

Theorem 3.4. *Suppose that $\mathbb{E}$ is the complex irreducible representation of G_0, which is dual to the irreducible G_0 representation with highest weight*

$$\lambda = (r-1)\omega_j + \sum_{i \in I \setminus \{j\}} a_i \omega_i \text{ with } r \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0$$

and set $\mathbb{F} = \circledcirc^r \mathfrak{g}_{-1}^* \circledcirc \mathbb{E}$.
Then we obtain:
(1) a natural graded vector bundle

$$V = V_0 \oplus ... \oplus V_N$$

over M with $V_0 = E$
(2) for any choice of a principal G_0-connection ∇ on $\mathcal{G}_0 \to M$ and for any choice of a splitting of the filtration of the tangent bundle, i.e. an isomorphism $TM \cong \mathrm{gr}(TM)$ that restricts to a map $T^iM \to \bigoplus_{j \geq i} \mathrm{gr}_j(TM)$ and the component in $\mathrm{gr}_i(TM)$ equals the image of the projection $T^iM \to T^iM/T^{i+1}M$,:

- *a linear connection $\widetilde{\nabla}$ on V*
- *a linear differential operator $L : V_0 \to V$ of weighted order N satisfying $L(s)_0 = s$*

with the following property:
For every semi-linear differential operator $D : \Gamma(E) \to \Gamma(F)$ of weighted order r with symbol given by the natural projection (3.2)

$$\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to F = \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E.$$

the linear differential operator L mapping $s \in \Gamma(E)$ to $L(s) = \Sigma$ induces a bijection between

$$\{s \in \Gamma(E) : D(s) = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : (\widetilde{\nabla} + C)(\Sigma) = 0\}$$

for some bundle map $C : V \to T^*M \otimes V$. The inverse is induced by the projection $V \to V_0 = E$.

To prove this theorem we proceed in three steps.

1. Step - The construction of V. Since we will later consider also the case of overdetermined systems on arbitrary regular infinitesimal flag structures, we drop for a moment the condition that the center of the Levi subalgebra $\mathfrak{z}(\mathfrak{g}_0)$ is one dimensional and consider general $|k|$-graded semisimple Lie algebras.

Proposition 3.5. *Suppose that $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is complex semisimple $|k|$-graded Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and choose a simple subsystem of roots $\Delta^0 = \{\alpha_1, ...\alpha_n\}$ such that the parabolic subalgebra $\mathfrak{p}$ is standard. We define J as the subset of $I := \{1, ..., n\}$ consisting of those*

elements $i \in I$ with $\alpha_i \in \Sigma_{\mathfrak{p}}$.
If $\mathbb{E}$ is the complex irreducible representation of $\mathfrak{g}_0$ whose dual representation has highest weight

$$\lambda = \sum_{j \in J} (r_j - 1)\omega_j + \sum_{i \in I \setminus J} a_i \omega_i \quad r_j \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0,$$

then there exists a unique irreducible complex representation $\mathbb{V}$ of $\mathfrak{g}$ such that

$$H^0(\mathfrak{g}_-, \mathbb{V}) = \mathbb{E} \quad \text{and} \quad H^1(\mathfrak{g}_-, \mathbb{V}) = \bigoplus_{j \in J} \circledcirc^{r_j} \mathfrak{g}^*_{-1,j} \circledcirc \mathbb{E} \tag{3.3}$$

where $\mathfrak{g}_{-1,j}$ is the irreducible representation of $\mathfrak{g}_0$ with highest weight $-\alpha_j$, see also lemma 3.1.

Proof. The functional λ is $\mathfrak{g}$-dominant and $\mathfrak{g}$-integral, hence it corresponds as well to a unique irreducible representation of $\mathfrak{g}$. Let $\mathbb{V}$ be the representation of $\mathfrak{g}$, which is dual to the irreducible representation of $\mathfrak{g}$ with highest weight λ.
By Kostant's version of the Bott-Borel-Weil theorem [theorem 2.10], we obtain that $H^0(\mathfrak{p}_+, \mathbb{V}^*)$ equals the irreducible representation of $\mathfrak{g}_0$ with highest weight λ, which is $\mathbb{E}^*$, and that

$$H^1(\mathfrak{p}_+, \mathbb{V}^*) = \bigoplus_{w \in W^{\mathfrak{p}} : \ell(w) = 1} \mathbb{W}_{w \cdot \lambda},$$

where $\mathbb{W}_{w\cdot\lambda}$ is the irreducible representation of $\mathfrak{g}_0$ with highest weight $w \cdot \lambda := w(\lambda + \rho) - \rho$.
The definition 2.6 of the Hasse diagram $W^{\mathfrak{p}}$ immediately implies that the elements of $W^{\mathfrak{p}}$ of length one are exactly the root reflections s_{α_j} for $j \in J$. Since

$$s_{\alpha_j}(\lambda + \rho) - \rho = \lambda - 2\frac{< \alpha_j, \lambda >}{< \alpha_j, \alpha_j >}\alpha_j - 2\frac{< \alpha_j, \rho >}{< \alpha_j, \alpha_j >}\alpha_j = \lambda - r_j \alpha_j,$$

we have

$$H^1(\mathfrak{p}_+, \mathbb{V}^*) = \bigoplus_{j \in J} \circledcirc^{r_j} \mathfrak{g}_{-1,j} \circledcirc \mathbb{E}^*,$$

where $\mathfrak{g}_{-1,j}$ is the irreducible representation of $\mathfrak{g}_0$ with highest weight $-\alpha_j$. The result know follows from the fact that $H^*(\mathfrak{p}_+, \mathbb{V}^*)$ is dual to $H^*(\mathfrak{g}_-, \mathbb{V})$ as $\mathfrak{g}_0$-module, see section 2.2.1. $\square$

For the real case we deduce from proposition 3.5 and corollary 2.11:

Proposition 3.6. *Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ be a real semisimple Lie algebra. Choose a Cartan involution θ, a θ-stable maximally non-compact Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ and a positive subsystem Δ^+ of the rootsystem Δ corresponding to $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ such that $\mathfrak{p}^{\mathbb{C}}$ is a standard parabolic in $\mathfrak{g}^{\mathbb{C}}$ (see section 2.1.3). Define the set J as in proposition 3.5 with respect to $\Sigma_{\mathfrak{p}^{\mathbb{C}}}$.*

If $\mathbb{E}$ is the complex irreducible representation dual to the complex irreducible representation with highest weight

$$\lambda = \sum_{j \in J} (r_j - 1)\omega_j + \sum_{i \in I \setminus J} a_i \omega_i \quad r_j \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0,$$

then there exists a unique complex irreducible representation $\mathbb{V}$ of $\mathfrak{g}$ such that

$$H^0_{\mathbb{R}}(\mathfrak{g}_-, \mathbb{V}) = \mathbb{E} \quad \text{and} \quad H^1_{\mathbb{R}}(\mathfrak{g}_-, \mathbb{V}) = \bigoplus_{j \in J} \odot^{r_j} (\mathfrak{g}^{\mathbb{C}})^*_{-1,j} \odot \mathbb{E}.$$

PROOF. For a complex representation $\mathbb{V}$ of $\mathfrak{g}$ the real cohomology $H^*_{\mathbb{R}}(\mathfrak{g}_-, \mathbb{V})$ is naturally a complex vector space and $H^*_{\mathbb{R}}(\mathfrak{g}_-, \mathbb{V}) \cong H^*_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}_-, \mathbb{V})$ as complex $\mathfrak{g}_0$ modules by corollary 2.11. Hence the result follows from proposition 3.5. □

Given some $|k|$-graded semisimple Lie algebra $\mathfrak{g}$, let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $P \subset G$ be the connected parabolic subgroup corresponding to the grading on $\mathfrak{g}$. Further, suppose that $\mathbb{E}$ is the complex irreducible representation of the Levi subgroup G_0, which is dual to the irreducible representation with highest weight

$$\lambda = \sum_{j \in J} (r_j - 1)\omega_j + \sum_{i \in I \setminus J} a_i \omega_i \quad r_j \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0$$

and let $\mathbb{V}$ be the irreducible representation of $\mathfrak{g}$ defined in the proposition 3.5 respectively proposition 3.6. Since G is simply connected, $\mathbb{V}$ integrates to a representation of G, which may be viewed as a representation of G_0 by restriction.

Recall that by proposition 2.3 there always exists a unique element $e \in \mathfrak{g}$, whose adjoint action represents the grading on $\mathfrak{g}$:

$$[e, X] = jX \text{ for } X \in \mathfrak{g}_j.$$

In particular, it acts diagonalisably on $\mathfrak{g}$ and therefore on any finite dimensional representation of $\mathfrak{g}$. Now we can decompose $\mathbb{V}$ into eigenspaces for the action of the grading element e on $\mathbb{V}$. Observe that for an eigenvector v with eigenvalue c and $X \in \mathfrak{g}_j$ the vector $X \cdot v$ is eigenvector with eigenvalue $c + j$, since $e \cdot X \cdot v = X \cdot e \cdot v + [e, X] \cdot v$. Therefore, denoting by c the eigenvalue with smallest real part, it follows from the irreducibility of $\mathbb{V}$ that the set of eigenvalues is given by $\{c, c+1, ..., c+N-1\}$ for some $N \geq 1$. For $0 \leq i \leq N$ let $\mathbb{V}_i$ be the eigenspace to the eigenvalue $c+i$ and set $\mathbb{V}_i = 0$ for $i < 0$ or $i > N$. Then we obtain a decomposition of $\mathbb{V}$

$$\mathbb{V} = \mathbb{V}_0 \oplus ... \oplus \mathbb{V}_N \tag{3.4}$$

such that

$$\mathfrak{g}_i \cdot \mathbb{V}_j \subseteq \mathbb{V}_{i+j} \quad \text{for all } i \text{ and } j \tag{3.5}$$

In particular, each subspace $\mathbb{V}_i$ is invariant under the action of $\mathfrak{g}_0$ respectively under the action of G_0.
We denote by upper indices the associated filtration

$$\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset ... \supset \mathbb{V}^N \text{ with } \mathbb{V}^i = \mathbb{V}_i \oplus ... \oplus \mathbb{V}_N. \tag{3.6}$$

Let us describe the components $\mathbb{V}_i$ also in another way:

Lemma 3.7. *Let $\mathbb{V} = \mathbb{V}_0 \oplus ... \oplus \mathbb{V}_N$ be the irreducible representation of $\mathfrak{g}$ whose dual representation has highest weight λ. The grading on $\mathbb{V}$ induces a grading on the dual representation*

$$\mathbb{V}^* = \mathbb{V}^*_{-N} \oplus ... \oplus \mathbb{V}^*_0 \quad \text{with} \quad \mathbb{V}^*_{-i} \cong (\mathbb{V}_i)^*.$$

The component $\mathbb{V}^*_\ell$ consists of all weight spaces of $\mathbb{V}^*$ corresponding to weights of the form

$$\lambda - \sum_{i \in I \setminus J} n_i \alpha_i - \sum_{j \in J} n_j \alpha_j,$$

where $n_i \in \mathbb{N}_0$ for all $i \in I$ and $\sum_{j \in J} n_j = \ell$.

Proof. The result follows immediately from the well known fact that for an irreducible representation $\mathbb{V}^*$ with highest weight λ the weight spaces correspond to weights of the form $\lambda - \sum_{i \in I} n_i \alpha_i$ for $n_i \in \mathbb{N}$ and that $\mathfrak{g}_i$ consists of all roots spaces of Σ_p-height i. (Note that if v is a weight vector of weight μ and $X \in \mathfrak{g}_\alpha$, then Xv is a weight vector of weight $\mu + \alpha$.) □

Let us look more closely at the G_0-equivariant Lie algebra differential ∂ computing the cohomolgy $H^*(\mathfrak{g}_-, \mathbb{V})$:

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathfrak{g}^*_- \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2 \mathfrak{g}^*_- \otimes \mathbb{V} \to ... \to \Lambda^n \mathfrak{g}^*_- \otimes \mathbb{V} \to ...$$

The gradings on $\mathfrak{g}_-$ and $\mathbb{V}$ induce a grading on the cochain spaces $\Lambda^n \mathfrak{g}^*_- \otimes \mathbb{V}$, where i-th grading component is given by

$$(\Lambda^n \mathfrak{g}^*_- \otimes \mathbb{V})_i = \bigoplus_{t=n}^{nk} (\Lambda^n_{-t} \mathfrak{g}_-)^* \otimes \mathbb{V}_{i-t}, \tag{3.7}$$

with

$$\Lambda^n_{-t} \mathfrak{g}_- = \bigoplus_{i_1 + ... + i_n = -t} \mathfrak{g}_{i_1} \wedge ... \wedge \mathfrak{g}_{i_n}.$$

It follows immediately from (3.5) that ∂ is grading preserving, cf. also section 2.3. We will sometimes denote the restriction of ∂ to the i-th grading component by $\partial_i : (\Lambda^n \mathfrak{g}^*_- \otimes \mathbb{V})_i \to (\Lambda^{n+1} \mathfrak{g}^*_- \otimes \mathbb{V})_i$.
Consider the first differential in the complex corresponding to $H^*(\mathfrak{g}_-, \mathbb{V})$:

$$\partial : \mathbb{V} \to \mathfrak{g}^*_- \otimes \mathbb{V}$$

$$\partial(v) = (X \mapsto Xv).$$

By (3.5) the G_0-invariant subspace $\mathbb{V}_0 \subset \mathbb{V}$ is contained in $\ker(\partial)$. Since $\ker(\partial) = H^0(\mathfrak{g}_-, \mathbb{V})$ is an irreducible representation of G_0 we conclude that

$$\mathbb{V}_0 = \ker(\partial) = H^0(\mathfrak{g}_-, \mathbb{V}) \cong \mathbb{E} \tag{3.8}$$

In particular, we see that

$$\partial_i : \mathbb{V}_i \to \bigoplus_{t=1}^{k} \mathfrak{g}^*_{-t} \otimes \mathbb{V}_{i-t}$$

is injective for $i > 0$.
By proposition 2.9 we have a decomposition

$$\mathfrak{g}^*_- \otimes \mathbb{V} = \operatorname{im}(\partial) \oplus \ker(\Box) \oplus \operatorname{im}(\partial^*) = \ker(\partial) \oplus \operatorname{im}(\partial^*),$$

where ∂^* is the Kostant codifferential and $\Box$ the Kostant Laplacian. From the definition ∂^* it follows immediately that ∂^* is as well as ∂ compatible with the gradings on the cochain spaces and hence so is $\Box$. Therefore we obtain

$$(\mathfrak{g}^*_- \otimes \mathbb{V})_i = \operatorname{im}(\partial_i) \oplus \ker(\Box_i) \oplus \operatorname{im}(\partial^*_i) = \ker(\partial_i) \oplus \operatorname{im}(\partial^*_i).$$

Since $H^1(\mathfrak{g}_-, \mathbb{V}) \cong \ker(\Box)$, the first cohomolgy $H^1(\mathfrak{g}_-, \mathbb{V})$ may be viewed as a G_0-submodule of $\mathfrak{g}^*_- \otimes \mathbb{V}$. We know from theorem 2.10 that each irreducible component $\circledcirc^{r_j} \mathfrak{g}^*_{-1,j} \circledcirc \mathbb{E}$ of $H^1(\mathfrak{g}_-, \mathbb{V})$ has multiplicity one in $\mathfrak{g}^*_- \otimes \mathbb{V}$. Using theorem 2.10 we can even determine the grading component, in which $\circledcirc^{r_j} \mathfrak{g}^*_{-1,j} \circledcirc \mathbb{E}$ is lying.
In fact, by part (3) of theorem 2.10 a highest weight vector of the irreducible component $\circledcirc^{r_j} \mathfrak{g}_{-1,j} \circledcirc \mathbb{E}^*$ of $H^1(\mathfrak{p}_+, \mathbb{V}^*)$ viewed as a submodule in $\mathfrak{p}^*_+ \otimes \mathbb{V}^*$ is of the form $X \otimes v$, where $X \in \mathfrak{g}_{-\alpha_j}$ and $v \in \mathbb{V}^*$ is a weight vector of weight $s_{\alpha_j}(\lambda) = \lambda - (r_j - 1)\alpha_j$. Therefore lemma 3.7 implies that the irreducible component $\circledcirc^{r_j} \mathfrak{g}_{-1,j} \circledcirc \mathbb{E}^*$ lies in $\mathfrak{g}_{-1,j} \otimes (\mathbb{V}_{(r_j-1)})^*$. Since we have an isomorphism $H^1(\mathfrak{p}_+, \mathbb{V}^*)^* \cong H^1(\mathfrak{g}_-, \mathbb{V})$, we obtain that

$$\circledcirc^{r_j} \mathfrak{g}^*_{-1,j} \circledcirc \mathbb{E} \subset \mathfrak{g}^*_{-1,j} \otimes \mathbb{V}_{(r_j-1)}.$$

In particular, for $0 < i < \min_{j\in J}\{r_j\}$ we therefore have an exact sequence

$$0 \longrightarrow \mathbb{V}_i \xrightarrow{\partial_i} \bigoplus_{s=1}^{k} \mathfrak{g}^*_{-s} \otimes \mathbb{V}_{i-s} \xrightarrow{\partial_i} \bigoplus_{t=2}^{2k} (\Lambda^2_{-t}\mathfrak{g}_-)^* \otimes \mathbb{V}_{i-t}. \tag{3.9}$$

These observations lead to the following proposition:

Proposition 3.8. *For $0 \le i \le N$ there exists G_0-equivariant inclusions*

$$\phi_i : \mathbb{V}_i \hookrightarrow \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{V}_0.$$

For all $i < q := \min_{j\in J}\{r_j\}$ these inclusions are even isomorphisms

$$\phi_i : \mathbb{V}_i \cong \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{V}_0.$$

PROOF. By means of restriction we can view $\mathbb{V}$ as a representation of $\mathfrak{g}_-$ and hence by the universal property of the universal enveloping algebra [proposition 1.4] also as a $\mathcal{U}(\mathfrak{g}_-)$-module. From (3.5) we conclude that

$$\mathcal{U}_{-i}(\mathfrak{g}_-)\mathbb{V}_i \subseteq \mathbb{V}_0 \qquad \text{for all } 0 \leq i \leq N.$$

Now we define ϕ_i by

$$\phi_i : \mathbb{V}_i \to \mathrm{Hom}(\mathcal{U}_{-i}(\mathfrak{g}_-), \mathbb{V}_0)$$

$$v \mapsto (u \mapsto -u^\top v),$$

where $u \mapsto u^\top$ denotes the principal anti-automorphism of $\mathcal{U}(\mathfrak{g}_-)$, see proposition 1.11.
Observing that

$$\mathcal{U}_{-i}(\mathfrak{g}_-) = \bigoplus_{j=1}^{k} \mathfrak{g}_{-j} \otimes \mathcal{U}_{-(i-j)}(\mathfrak{g}_-)/\mathcal{J}_i \tag{3.10}$$

$$\mathcal{J}_i =< X \otimes Yu - Y \otimes Xu - [X,Y] \otimes u : X \in \mathfrak{g}_{-p}, Y \in \mathfrak{g}_{-q}, u \in \mathcal{U}_{-(i-p-q)}(\mathfrak{g}_-) >$$

we can prove by induction on i that all ϕ_i are injective.
For $i = 0$ the result holds, since $\phi_0 = -id$.
The map $\phi_1 : \mathbb{V}_1 \to \mathrm{Hom}(\mathfrak{g}_{-1}, \mathbb{V}_0)$ equals $\partial_1 : \mathbb{V}_1 \to \mathrm{Hom}(\mathfrak{g}_{-1}, \mathbb{V}_0)$, which is injective by (3.8) and so the result holds also for $i = 1$.
Now suppose that ϕ_j is injective for all $j < i$ and consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{V}_i \arrow[r, "\partial_i"] \arrow[d, "id"'] & \bigoplus_{s=1}^{k} \mathrm{Hom}(\mathfrak{g}_{-s}, \mathbb{V}_{i-s}) \arrow[r, "\partial_i"] \arrow[d, "\imath"'] & \bigoplus_{t=2}^{2k} \mathrm{Hom}(\Lambda^2_{-t}\mathfrak{g}_-, \mathbb{V}_{i-t}) \arrow[d, "\jmath"'] \\
\mathbb{V}_i \arrow[r, "\tilde{\partial}_i"] & \bigoplus_{s=1}^{k} \mathrm{Hom}(\mathfrak{g}_{-s} \otimes \mathcal{U}_{-(i-s)}(\mathfrak{g}_-), \mathbb{V}_0) \arrow[r, "\tilde{\partial}_i"] & \bigoplus_{t=2}^{2k} \mathrm{Hom}(\Lambda^2_{-t}\mathfrak{g}_- \otimes \mathcal{U}_{-(i-t)}(\mathfrak{g}_-), \mathbb{V}_0)
\end{tikzcd}$$

where

$$\imath(f)(X \otimes u) = u^\top f(X) = -\phi_{i-s}(f(X))(u)$$
$$\text{for } X \otimes u \in \mathfrak{g}_{-s} \otimes \mathcal{U}_{-(i-s)}(\mathfrak{g}_-)$$
$$\jmath(g)(X \wedge Y \otimes u) = u^\top g(X \wedge Y) = -\phi_{i-t}(g(X \wedge Y))(u)$$
$$\text{for } X \wedge Y \otimes u \in \Lambda^2_{-t}\mathfrak{g}_- \otimes \mathcal{U}_{-(i-t)}(\mathfrak{g}_-)$$
$$\partial_i(f)(X \wedge Y) = Xf(Y) - Yf(X) - f([X,Y])$$
$$\text{for } X \wedge Y \in \Lambda^2_{-t}\mathfrak{g}_-$$
$$\tilde{\partial}_i(h)(X \wedge Y \otimes u) = h(X \otimes Yu) - h(Y \otimes Xu) - h([X,Y] \otimes u)$$
$$\text{for } X \wedge Y \otimes u \in \Lambda^2_{-t}\mathfrak{g}_- \otimes \mathcal{U}_{-(i-t)}(\mathfrak{g}_-).$$

Since $\partial \circ \partial = 0$, the commutativity of the diagram implies that the composition $\imath \circ \partial_i$ has values in the kernel of $\tilde{\partial}_i$. By (3.10) the kernel $\ker(\tilde{\partial}_i)$ coincides with $\mathrm{Hom}(\mathcal{U}_{-i}(\mathfrak{g}_-), \mathbb{V}_0) \subset \bigoplus_{s=1}^k \mathrm{Hom}(\mathfrak{g}_{-s} \otimes \mathcal{U}_{-(i-s)}(\mathfrak{g}_-), \mathbb{V}_0)$ and so we have

$$\imath \circ \partial_i : \mathbb{V}_i \to \mathrm{Hom}(\mathcal{U}_{-i}(\mathfrak{g}_-), \mathbb{V}_0).$$

Moreover, since $(i \circ \partial_i)(v)(X \otimes u) = u^\top (Xv) = -(Xu)^\top v$, wee see that

$$\imath \circ \partial_i = \phi_i.$$

We know by (3.8) that $\partial_i : \mathbb{V}_i \to \bigoplus_{s=1}^k \mathrm{Hom}(\mathfrak{g}_{-s}, \mathbb{V}_{i-s})$ is injective and by induction hypothesis also $\imath$ is injective. Therefore we have that

$$\phi_i : \mathbb{V}_i \overset{\partial_i}{\cong} \mathrm{im}(\partial_i) \overset{\imath}{\hookrightarrow} \mathrm{Hom}(\mathcal{U}_{-i}(\mathfrak{g}_-), \mathbb{V}_0)$$

is injective and so the first assertion follows.
Since for $0 < i < q$ we have by (3.9) an exact sequence

$$0 \longrightarrow \mathbb{V}_i \xrightarrow{\partial_i} \bigoplus_{s=1}^k \mathrm{Hom}(\mathfrak{g}_{-s}, \mathbb{V}_{i-s}) \xrightarrow{\partial_i} \bigoplus_{t=2}^{2k} \mathrm{Hom}(\Lambda^2_{-t}\mathfrak{g}_-, \mathbb{V}_{i-t}),$$

it follows by induction from the commutative diagram above that

$$\phi_i : \mathbb{V}_i \overset{\partial_i}{\cong} \ker(\partial_i) \overset{\imath}{\cong} \mathrm{Hom}(\mathcal{U}_{-i}(\mathfrak{g}_-), \mathbb{V}_0)$$

is an isomorphism for $0 < i < q$. □

Let us come back to the geometric setting of theorem 3.4. Suppose that $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_k$ is a $|k|$-graded semisimple such that the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Levi subalgebra is one dimensional. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and P be the connected parabolic subgroup corresponding to the grading on $\mathfrak{g}$. Further, assume that M is a manifold endowed with a regular infinitesimal flag structure of type (G, P).
Now let $\mathbb{E}$ be the complex irreducible representation of G_0, whose dual representation has highest weight

$$\lambda = (r-1)\omega_j + \sum_{i \in I \setminus \{j\}} a_i \omega_i \text{ with } r \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0.$$

Then we know by proposition 3.5 respectively 3.6 that there exists an irreducible G-representation

$$\mathbb{V} = \mathbb{V}_0 \oplus ... \oplus \mathbb{V}_N$$

such that

$$H^0(\mathfrak{g}_-, \mathbb{V}) = \mathbb{E} \qquad \text{and} \qquad H^1(\mathfrak{g}_-, \mathbb{V}) = \circledcirc^r \mathfrak{g}_{-1}^* \circledcirc \mathbb{E} = \mathbb{F}.$$

For $i \geq 0$ we have by proposition 3.8 a G_0-equivariant inclusion

$$\phi_i : \mathbb{V}_i \to \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E},$$

which is even an isomorphism

$$\phi_i : \mathbb{V}_i \cong \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E} \quad \text{if} \quad i < r.$$

Since ϕ_i is an isomorphism for $i < r$, the commutative diagram from the proof of proposition 3.8 for $i = r$ looks as follows:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{V}_r \arrow[r, "\partial_r"] \arrow[d, "id"'] & \bigoplus_{s=1}^k \mathrm{Hom}(\mathfrak{g}_{-s}, \mathbb{V}_{r-s}) \arrow[r, "\partial_r"] \arrow[d, "\imath"'] & \bigoplus_{t=2}^{2k} \mathrm{Hom}(\Lambda^2_{-t}\mathfrak{g}_-, \mathbb{V}_{r-t}) \arrow[d, "\jmath"'] \\
\mathbb{V}_r \arrow[r, "\tilde{\partial}_r"] \arrow[d] & \bigoplus_{s=1}^k \mathrm{Hom}(\mathfrak{g}_{-s} \otimes \mathcal{U}_{-(r-s)}(\mathfrak{g}_-), \mathbb{V}_0) \arrow[r, "\tilde{\partial}_r"] \arrow[d] & \bigoplus_{t=2}^{2k} \mathrm{Hom}(\Lambda^2_{-t}\mathfrak{g}_- \otimes \mathcal{U}_{-(r-t)}(\mathfrak{g}_-), \mathbb{V}_0) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

This implies that we have

$$\mathbb{V}_r \overset{\partial_r}{\cong} \mathrm{im}(\partial_r) \subset \ker(\partial_r) = \mathrm{im}(\partial_r) \oplus \ker(\Box_r) \overset{\imath}{\cong} \ker(\tilde{\partial}_r) = \mathcal{U}_{-r}(\mathfrak{g}_-)^* \otimes \mathbb{E}.$$

Note that the map $\imath$ viewed as a map

$$\imath : \bigoplus_{s=1}^{k} \mathfrak{g}^*_{-s} \otimes \mathbb{V}_{r-s} \to \bigoplus_{s=1}^{k} \mathfrak{g}^*_{-s} \otimes \mathcal{U}_{-(r-s)}(\mathfrak{g}_-)^* \otimes \mathbb{V}_0$$

equals

$$\imath = \sum_{s=1}^{k} -id \otimes \phi_{r-s}$$

and therefore the isomorphism induced by $\imath$ between $\ker(\Box_r) = \ker(\Box)$ and $\circledcirc^r \mathfrak{g}^*_{-1} \circledcirc \mathbb{E}$ is given by

$$\ker(\Box) \hookrightarrow \mathfrak{g}^*_{-1} \otimes \mathbb{V}_{r-1} \overset{-id\otimes\phi_{r-1}}{\cong} \mathfrak{g}^*_{-1} \otimes \mathcal{U}_{r-1}(\mathfrak{g}_-)^* \otimes \mathbb{E} \to \mathfrak{g}^*_{-1} \otimes S^{r-1}\mathfrak{g}^*_{-1} \otimes \mathbb{E} \to \circledcirc^r \mathfrak{g}^*_{-1} \circledcirc \mathbb{E}. \tag{3.11}$$

We conclude that we obtain a G_0-equivariant isomorphism

$$\phi_r : \mathbb{V}_r \cong (\mathcal{U}_{-r}(\mathfrak{g}_-)^* \otimes \mathbb{E}) \cap \mathbb{K},$$

where $\mathbb{K} \subset \mathcal{U}_{-r}(\mathfrak{g}_-)^* \otimes \mathbb{E}$ denotes the kernel of the G_0-equivariant projection $\mathcal{U}_{-r}(\mathfrak{g}_-)^* \otimes \mathbb{E} \to \circledcirc^r \mathfrak{g}^*_{-1} \circledcirc \mathbb{E}$. Since $\ker(\Box) = \ker(\Box_r)$, it follows by induction as in the proof of the proposition 3.8 that we have isomorphisms

$$\phi_i : \mathbb{V}_i \cong (\mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E}) \cap (\mathcal{U}_{-(i-r)}(\mathfrak{g}_-)^* \otimes \mathbb{K}) \quad \text{for} \quad i \geq r.$$

Now we define V as the graded vector bundle associated to $\mathbb{V}$:

$$V = V_0 \oplus ... \oplus V_N = \mathcal{G}_0 \times_{G_0} \mathbb{V}_0 \oplus ... \oplus \mathbb{V}_N,$$

where $V_0 = E$. Moreover, we define $K := \mathcal{G}_0 \times_{G_0} \mathbb{K}$ as the natural vector bundle corresponding to $\mathbb{K}$.
Since the isomorphisms ϕ_i are G_0-equivariant, they induce vector bundle isomorphisms between the corresponding vector bundles

$$\begin{aligned} \phi_i &: V_i \cong \mathcal{U}_{-i}(\mathrm{gr}(TM))^* \otimes E && \text{for all } i < r \\ \phi_i &: V_i \cong \mathcal{U}_{-i}(\mathrm{gr}(TM))^* \otimes E \cap \mathcal{U}_{-(i-r)}(\mathrm{gr}(TM))^* \otimes K && \text{for all } i \geq r. \end{aligned}$$

Comparing this with section 1.3. we therefore have:

Corollary 3.9. *A linear differential operator of weighted r*

$$D : \Gamma(E) \to \Gamma(\circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E)$$

with weighted symbol given by the projection 3.2

$$\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E$$

is of weighted finite type with

$$g^i = \mathcal{U}_{-i}(\mathrm{gr}(TM))^* \otimes E \cap \mathcal{U}_{-(i-r)}(\mathrm{gr}(TM))^* \otimes K \cong V_i \textit{ for all } i \geq r,$$

where K is the kernel of the symbol.

Remark 3.4. Let us remark that proposition 3.8 and corollary 3.9 should also be compared with the considerations in [**34**].

2. Step - The construction of the connection $\tilde{\nabla}$ and the differential operator L. Once again consider the Lie algebra differential ∂ and the codifferential ∂^* corresponding to the cohomology $H^*(\mathfrak{g}_-, \mathbb{V})$:

$$\partial : \Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V} \to \Lambda^{n+1} \mathfrak{g}_-^* \otimes \mathbb{V}$$

$$\partial^* : \Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V} \to \Lambda^{n-1} \mathfrak{g}_-^* \otimes \mathbb{V}.$$

Both are G_0-equivariant and compatible with the gradings on the cochain spaces $\Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V}$.
Moreover, we have an algebraic Hodge decomposition of $\Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V}$ given by

$$\Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V} = \mathrm{im}(\partial) \oplus \ker(\Box) \oplus \mathrm{im}(\partial^*)$$

with $\ker(\Box) \cong H^n(\mathfrak{g}_-, \mathbb{V})$.
In particular, restricting ∂ to $\mathrm{im}(\partial^*)$ respectively ∂^* to $\mathrm{im}(\partial)$, we obtain isomorphisms

$$\partial : \mathrm{im}(\partial^*) \cong \mathrm{im}(\partial) \quad \text{ and } \quad \partial^* : \mathrm{im}(\partial) \cong \mathrm{im}(\partial^*).$$

In general, these two maps are not inverse to each other. However, we may define

$$\delta^* : \Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V} \to \Lambda^{n-1} \mathfrak{g}_-^* \otimes \mathbb{V}$$

as the inverse of ∂ on $\mathrm{im}(\partial)$ and zero on $\ker(\partial^*)$. Obviously, we have again $\delta^* \circ \delta^* = 0$ and so δ^* is a differential. Since, by construction, δ^* differs from ∂^* on $\mathrm{im}(\partial)$ just by a G_0-equivariant grading preserving isomorphism of $\mathrm{im}(\partial)$, we conclude that δ^* is as well G_0-equivariant and compatible with the gradings on the cochain spaces. Moreover, it defines the same Hodge decomposition on $\Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V}$. In the sequel, we will use δ^* rather than ∂^* to make certain computations easier.
We will denote the corresponding grading preserving vector bundle maps by the same letters:

$$\partial : \Lambda^n \mathrm{gr}(TM)^* \otimes V \to \Lambda^{n+1} \mathrm{gr}(TM)^* \otimes V$$

$$\delta^* : \Lambda^n \mathrm{gr}(TM)^* \otimes V \to \Lambda^{n-1} \mathrm{gr}(TM)^* \otimes V,$$

where the grading on the vector bundle $\Lambda^n \mathrm{gr}(TM)^* \otimes V$ is induced by the grading on $\Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V}$.
Let us now choose a principal connection on $\mathcal{G}_0 \to M$, then we get induced linear connection on all associated vector bundles and we will denote all of them by ∇.
In particular, we obtain a linear connection $\nabla : \Gamma(V) \to \Gamma(T^*M \otimes V)$ on V. The filtrations on V and TM induces a filtration of $T^*M \otimes V$, where the ℓ-th filtration component $(T^*M \otimes V)^\ell$ consists of all elements in $T^*M \otimes V$ of homogeneity $\geq \ell$, i.e.

$$\phi \in (T^*M \otimes V)^\ell \text{ if and only if } \phi(T^iM) \subset V^{i+\ell} \text{ for } i < 0.$$

Since ∇ is induced from a principal G_0-connection, it has to preserve the grading on V. Hence it raises homogeneity by one:

$$\nabla : \Gamma(V^i) \to \Gamma((T^*M \otimes V)^{i+1}).$$

Remark 3.5. Given two filtered vector bundles $(V, \{V^i\})$ and $(W, \{W^j\})$, we always have a natural filtration on $V^* \otimes W$, where $(V^* \otimes W)^\ell$ consists of all elements of homogeneity $\geq \ell$, i.e. $\phi \in (V^* \otimes W)^\ell$ if and only if $\phi(V^i) \subset W^{i+\ell}$ for all i. In particular, we see that $\phi : V \to W$ is of homogeneity ≥ 0 if and only if it is filtration preserving. Note that an element $\phi \in (V^* \otimes W)^\ell$ induces a bundle map $\mathrm{gr}_\ell(\phi) : \mathrm{gr}(V) \to \mathrm{gr}(W)$ between the associated graded vector bundles $\mathrm{gr}(V) \to \mathrm{gr}(W)$, which is of homogeneity ℓ, meaning that $\mathrm{gr}_\ell(\phi)(\mathrm{gr}_i(V)) \subset \mathrm{gr}_{i+\ell}(W)$ for all i. Mapping an element $\phi \in (V^* \otimes W)^\ell$ to $\mathrm{gr}_\ell(\phi)$ induces an isomorphism of graded vector bundles $\mathrm{gr}(V^* \otimes W) \cong \mathrm{gr}(V)^* \otimes \mathrm{gr}(W)$, with the grading on the latter space given by homogeneous degree.

Choosing a splitting of the filtration of the tangent bundle $TM \cong \mathrm{gr}(TM)$, we can view ∂ and δ^* as grading respectively filtration preserving vector bundle maps on $\Lambda^k T^*M \otimes V$. In particular, the following definition makes sense:

$$\widetilde{\nabla} := \nabla + \partial : \Gamma(V) \to \Gamma(T^*M \otimes V),$$

It is a linear connection on $V = \ker(\partial) \oplus \mathrm{im}(\delta^*) = V_0 \oplus \mathrm{im}(\delta^*)$, which is of homogeneity ≥ 0 and whose lowest homogeneous component is given by the algebraic operator ∂. Now we can construct an operator $L : V_0 \to V$, which splits the projection $\pi : V \to V_0$ and is characterised by having values in the kernel of $\delta^* \circ \widetilde{\nabla}$.
In fact, consider the following linear differential operator

$$\delta^* \circ \tilde{\nabla} : \Gamma(V) \to \Gamma(\mathrm{im}(\delta^*)) \subseteq \Gamma(V).$$

It is of homogeneity ≥ 0 with lowest homogeneous component given by $\delta^* \circ \partial$. If we restrict this operator to $\mathrm{im}(\delta^*)$, the lowest component $\delta^* \circ \partial$ is the identity on $\mathrm{im}(\delta^*)$ and $-(\delta^*\tilde{\nabla} - id)$ is (at most) N-step nilpotent. Therefore $\delta^*\tilde{\nabla}$ is invertible on $\Gamma(\mathrm{im}(\delta^*))$ with inverse given by the von Neumann serie

$$(\delta^*\tilde{\nabla})^{-1} = (id - (-(\delta^*\tilde{\nabla} - id)))^{-1} = \sum_{i=0}^{N} (-1)^i (\delta^*\tilde{\nabla} - id)^i.$$

Now we define the splitting operator L by

$$L(s) = \Sigma - (\delta^*\tilde{\nabla})^{-1}\delta^*\tilde{\nabla}\Sigma,$$

where Σ is a section of V with $\pi(\Sigma) = s$. This is well defined, since Σ is determined up to adding sections of $\mathrm{im}(\delta^*)$ and L is zero on $\mathrm{im}(\delta^*)$. The operator L obviously splits the projection $\pi : V \to V_0 = \ker(\partial)$, i.e. $\pi(L(s)) = s$. In addition, since $\delta^*\tilde{\nabla}(\delta^*\tilde{\nabla})^{-1}$ is the identity on $\Gamma(\mathrm{im}(\delta^*))$, we see that $\delta^*\tilde{\nabla}L = 0$. The splitting operator is uniquely characterised by these two properties, since for a section $\Sigma \in \Gamma(V)$ with $\pi(\Sigma) = s$ and $\delta^*\tilde{\nabla}\Sigma = 0$, we obtain $L(s) = \Sigma - (\delta^*\tilde{\nabla})^{-1}\delta^*\tilde{\nabla}\Sigma = \Sigma$. In particular, this shows that a section Σ of V lies in the image of L if and only if $\delta^*\tilde{\nabla}\Sigma = 0$.
Inserting the formula for $\delta^*\tilde{\nabla}$ and using that $\delta^*\partial$ is the identity on $\mathrm{im}(\delta^*)$, we obtain

$$L(s) = \sum_{i=0}^{N} (-1)^i (\delta^*\nabla)^i(\Sigma) - \sum_{i=0}^{N} (-1)^i (\delta^*\nabla)^i \delta^*\partial(\Sigma).$$

Since the formula of L is independent of the choice of Σ, this implies that

$$L(s) = \Sigma_{i=0}^{N} (-1)^i (\delta^*\nabla)^i(s),$$

where s is viewed as a section of V by trivial extension.
Denoting by L^j the component in $V_0 \oplus ... \oplus V_j$ of L^j, we have:

Proposition 3.10. *There exists a unique linear differential operator* $L : \Gamma(V_0) \to \Gamma(V)$ *such that*

- $\pi(L(s)) = s$, *where* $\pi : V \to V_0$ *is the projection*
- L *has values in the kernel of* $\delta^*\tilde{\nabla}$

In particular, a section $\Sigma \in \Gamma(V)$ *is in* $\operatorname{im}(L)$ *if and only if* $\delta^*\tilde{\nabla}(\Sigma) = 0$. *Moreover, each operator* $L^j : \Gamma(V_0) \to \Gamma(V_0 \oplus ... \oplus V_j)$ *induces a vector bundle map*

$$\mathcal{J}^j(V_0) \to V_0 \oplus ... \oplus V_j$$

which is an isomorphism for $j < r$.

Proof. It only remains to show the last assertion. Note that the principal connection on $\mathcal{G}_0$ induces not only a linear connection ∇ on V, but also a linear connection ∇ on $\operatorname{gr}(TM) \cong TM$, which is compatible with the grading $\nabla : \Gamma(\operatorname{gr}_i(TM)) \to \Gamma(\operatorname{gr}(TM) \otimes \operatorname{gr}_i(TM))$. The G_0-equivariance of $\delta^* : \mathfrak{g}_-^* \otimes \mathbb{V} \to \mathbb{V}$ implies that the corresponding vector bundle map is parallel for the induced linear connection on $\operatorname{gr}(TM)^* \otimes V^* \otimes V$.
Therefore we conclude that $L(s) = \Sigma_{i=0}^N (-1)^i (\delta^*\nabla)^i s$ can be written as

$$L(s) = \sum_{i=0}^{N} (-1)^i (\delta^* \circ (id \otimes \delta^*) \circ ... \circ (\underbrace{id \otimes ... \otimes id}_{i-1} \otimes \delta^*)) \circ \nabla^i s$$

with the convention that the 0-th term is the identity.
Denote by $\mathcal{T}_{-i}(\operatorname{gr}(TM)) = \mathcal{G}_0 \times_{G_0} \mathcal{T}_{-i}(\mathfrak{g}_-)$ the associated vector bundle corresponding to the $-i$-th grading component of the tensor algebra $\mathcal{T}(\mathfrak{g}_-)$ (see (1.6)) and consider the following differential operator

$$D^j : \Gamma(V_0) \to \Gamma(\bigoplus_{i=0}^{j} \mathcal{T}_{-i}(\operatorname{gr}(TM))^* \otimes V_0)$$

$$s \mapsto (\sum_{i=0}^{j} \nabla^i s)_{\leq j}$$

where $(\quad)_{\leq j}$ means that we restrict $\sum_{i=0}^{j} \nabla^i s$ to all grading components of degree $\leq j$ in $\bigoplus_{i=0}^{j} (\operatorname{gr}(TM)^i)^* \otimes V_0$. This operator is obviously of weighted order j. Note that we have

$$\nabla\nabla s(\xi,\eta) - \nabla\nabla s(\eta,\xi) = R(\xi,\eta)(s) + \nabla_{\nabla_\eta \xi} s - \nabla_{\nabla_\xi \eta} s - \nabla_{[\eta,\xi]s} =$$

and so

$$\nabla\nabla s(\xi,\eta) - \nabla\nabla s(\eta,\xi) - \nabla_{\mathcal{L}(\xi,\eta)} s \equiv 0 \text{ mod(terms of lower weighted order in } s).$$

Therefore we conclude that the weighted symbol of D^j is given by the canonical inclusion

$$\sigma(D^j) : \mathcal{U}_{-j}(\operatorname{gr}(TM))^* \otimes V_0 \hookrightarrow \mathcal{T}_{-j}(\operatorname{gr}(TM))^* \otimes V_0 \subset \bigoplus_{i=0}^{j} \mathcal{T}_{-i}(\operatorname{gr}(TM))^* \otimes V_0,$$

which is obtained by dualising the projection $\mathcal{T}_{-j}(\mathrm{gr}(TM)) \to \mathcal{U}_{-j}(\mathrm{gr}(TM))$. Since δ^* is grading preserving, we deduce that

$$L^j(s) = \sum_{i=0}^{j}(-1)^i((\delta^* \circ (id \otimes \delta^*) \circ ... \circ (id \otimes ... \otimes id \otimes \delta^*)) \circ \nabla^i s)_{\leq j} =$$

$$= (\sum_{i=0}^{j}(-1)^i \delta^* \circ (id \otimes \delta^*) \circ ... \circ (id \otimes ... \otimes id \otimes \delta^*)) \circ D^j(s)$$

is of weighted order j and hence induces a vector bundle map

$$L^j : \mathcal{J}^j(V_0) \to V_0 \oplus ... \oplus V_j.$$

Suppose now that $j > 1$ and let us compute the weighted symbol $\sigma(L^j)$ of L^j. It is given by the composition of the weighted symbol of D^j with $\psi_j := \sum_{i=1}^{j}(-1)^i((\delta^* \circ (id \otimes \delta^*) \circ ... \circ (\underbrace{id \otimes ... \otimes id}_{i-1} \otimes \delta^*))$

$$\begin{tikzcd}
\mathcal{U}_{-j}(\mathrm{gr}(TM))^* \otimes V_0 \arrow[r, hook, "\sigma(D^j)"] \arrow[dr, "\sigma(L^j)"'] & \mathcal{T}_{-j}(\mathrm{gr}(TM))^* \otimes V_0 \arrow[d, "\psi_j"] \\
 & V_j
\end{tikzcd}$$

Now consider the injective vector bundle map

$$\phi_j : V_j \to \mathcal{U}_{-j}(\mathrm{gr}(TM))^* \otimes V_0 \subset \mathcal{T}_{-j}(\mathrm{gr}(TM))^* \otimes V_0$$

corresponding to the G_0-equivariant inclusion of proposition 3.8. This vector bundle map can also be written as

$$\phi_j = \sum_{i=1}^{j}(-1)^{i-1} p_0^j \circ (\underbrace{id \otimes ... \otimes id}_{i-1} \otimes \partial) \circ ... \circ (id \otimes \partial) \circ \partial,$$

where $p_0^j : \bigoplus_{i=1}^{j} \mathcal{T}_{-i}(\mathrm{gr}(TM))^* \otimes V_{j-i} \to \mathcal{T}_{-j}(\mathrm{gr}(TM))^* \otimes V_0$ is the projection given by restriction.
Setting $\partial^{(i)} := \underbrace{id \otimes ... \otimes id}_{i-1} \otimes \partial \circ ... \circ (id \otimes \partial) \circ \partial|_{V_j}$ and $\delta^*_{(i)} := \underbrace{id \otimes ... \otimes id}_{i-1} \otimes \delta^*$.
we obtain that that

$$(\sum_{i=1}^{j}(-1)^i(\delta^* \circ ... \circ (\underbrace{id \otimes ... \otimes id}_{i-1} \otimes \delta^*)) \circ (\sum_{i=1}^{j}(-1)^{i-1} p_j^0 \circ (\underbrace{id \otimes ... \otimes id}_{i-1} \otimes \partial) \circ ... \circ \partial)$$

$$= -[...\delta^*_{(j-2)}(p_0^j \circ \partial^{(j-2)} + \delta^*_{(j-1)}(p_0^j \circ \partial^{(j-1)} + \delta^*_{(j)} \circ p_0^j \circ \partial^{(j)}))]. \tag{3.12}$$

Recall that $\delta^* : \mathrm{gr}(TM)^* \otimes V \to V$ is defined as the inverse of ∂ on $\mathrm{im}(\partial) \subset \mathrm{gr}(TM)^* \otimes V$ and zero on the rest. Since $p_0^j \circ \partial^{(j)} = \partial^{(j)}$, we therefore get that

$$\delta^*_{(j)} \circ p_0^j \circ \partial^{(j)} = \delta^*_{(j)} \circ \partial^{(j)} = (id - p_0^j) \circ \partial^{(j-1)}.$$

Hence $p_0^j \circ \partial^{(j-1)} + \delta^*_{(j)} \circ p_0^j \circ \partial^{(j)} = \partial^{(j-1)}$. Since $\delta^*_{(j-1)} \circ \partial^{(j-1)}$ equals again $(id - p_0^j) \circ \partial^{(j-2)}$, we conclude inductively that the composition (3.12) equals $-id$ on V_j.
Since we know by proposition 3.8 that $\phi_j : V_j \to \mathcal{U}_{-j}(\mathrm{gr}(TM))^* \otimes V_0$ is an isomorphism for $j < r$, we therefore conclude that for $j < r$ the weighted symbol of L^j is given by

$$\sigma(L^j) = -\phi_j^{-1} : \mathcal{U}_{-j}(\mathrm{gr}(TM))^* \otimes V_0 \to V_j.$$

Now we can prove the last assertion of the propositon by induction.
The operator L^0 is just the identity on $\Gamma(V_0)$ and identifying $\mathcal{J}^0(V_0) = V_0$ it induces the identity on V_0. Hence we see that the assertion holds for $i = 0$.
Now assume that L^i induces an isomorphism $\mathcal{J}^i(V_0) \to V_0 \oplus ... \oplus V_i$ for all $i < j < r$. Since $\sigma(L^j) : \mathcal{U}_{-j}(\mathrm{gr}(TM))^* \otimes V_0 \to V_j$ is an isomorphism for $j < r$, it follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{U}_{-j}(\mathrm{gr}(TM))^* \otimes V_0 \arrow[r, "\sigma(L^j)=-\phi_j^{-1}"] \arrow[d] & V_j \arrow[d] \\
\mathcal{J}^j(V_0) \arrow[r, "L^j"] \arrow[d] & V_0 \oplus ... \oplus V_j \arrow[d] \\
\mathcal{J}^{j-1}(V_0) \arrow[r, "L^{j-1}"] \arrow[d] & V_0 \oplus ... \oplus V_{j-1} \arrow[d] \\
0 & 0
\end{tikzcd}$$

that L^j induces also an isomorphism $\mathcal{J}^j(V_0) \to V_0 \oplus ... \oplus V_j$. $\square$

3. Step - The construction of the bundle map C. Now we define the following linear differential operator

$$D^\nabla := -(id \otimes \phi_{r-1}) \circ \pi \circ \tilde{\nabla} \circ L : \Gamma(E) \to \Gamma(\circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E),$$

where π denotes the projection

$$\pi : \mathrm{gr}(TM)^* \otimes V \to \mathrm{gr}_{-1}(TM)^* \otimes V_{r-1} \to \ker(\Box).$$

Since the projection π annihilates $\mathrm{im}(\partial)$, we obtain that

$$D^\nabla(s) = -(id \otimes \phi_{r-1})\pi\nabla(Ls)_{r-1},$$

where $(Ls)_{r-1}$ denotes the component in V_{r-1} of $L(s)$. From proposition 3.10 we know that $s \mapsto L(s)_{r-1}$ is a differential operator of weighted order

$r-1$ with weighted symbol given by $-\phi_{r-1}^{-1}$ and so we see that D^{∇} is of weighted order r with weighted symbol given by

$$\sigma(D^{\nabla}) = -id \otimes \phi_{r-1} \circ \pi \circ (-id \otimes \phi_{r-1}^{-1})$$

$$\mathrm{gr}_{-1}(TM)^* \otimes \mathcal{U}_{r-1}(\mathrm{gr}(TM))^* \otimes E \cap \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E.$$

Using (3.11) we conclude that it equals the projection (3.2)

$$\sigma(D^{\nabla}) : \mathcal{U}_{-r}(\mathrm{gr}(T^*M)) \otimes E \to \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E.$$

Similarly as it was done for overdetermined systems on regular flag structures corresponding to $|1|$-graded semisimple Lie algebras in [**4**], we can now start to rewrite the equation $D(s) = 0$.

Proposition 3.11. *Suppose that D is a semi-linear differential operator*

$$D : \Gamma(E) \to \Gamma(\circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E) = \Gamma(F)$$

of weighted order r with weighted symbol given by the projection (3.2)

$$\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E = F.$$

Then there exists a bundle map $A : V_0 \oplus ... \oplus V_N \to F$ such that $s \mapsto Ls$ and the projection $V \to V_0 = E$ induce inverse bijections between the following spaces

$$\{s \in \Gamma(E) : D\sigma = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : \tilde{\nabla}(\Sigma) + A(\Sigma) \in \Gamma(\mathrm{im}(\delta^*))\}.$$

Proof. The operators D and D^{∇} have the same weighted symbol and therefore there exists a bundle map $\psi : \mathcal{J}^{r-1}(E) \to F$ such that

$$D(s) = D^{\nabla}(s) + \psi(j^{r-1}s).$$

By proposition 3.10 the splitting operator L induces an isomorphism

$$L^{r-1} : \mathcal{J}^{r-1}(E) \cong V_0 \oplus ... \oplus V_{r-1}$$

and so there is a unique bundle map

$$A : V_0 \oplus ... \oplus V_{r-1} \to F \quad \text{such that} \quad \psi(j^{r-1}s) = A(Ls),$$

where we view A as map on the whole bundle V by trivial extension.
Since $\tilde{\nabla}Ls$ has values in $\ker(\delta^*)$ by proposition 3.10 and $A(Ls)$ even in $\ker(\Box) \subseteq \ker(\delta^*)$, we obtain that

$$0 = D(s) = \pi(\tilde{\nabla}Ls + A(Ls))$$

if and only if

$$\tilde{\nabla}Ls + A(Ls) \in \Gamma(\mathrm{im}(\delta^*)),$$

where $\pi : T^*M \otimes V \to \ker(\Box)$ is the projection.
Conversely, suppose Σ is a section of V such that $\tilde{\nabla}\Sigma + A(\Sigma) \in \Gamma(\mathrm{im}(\delta^*))$. Then $\delta^*(\tilde{\nabla}\Sigma + A(\Sigma)) = 0$ and since the map A has values in $\ker(\delta^*)$, we get $\delta^*(\tilde{\nabla}\Sigma) = 0$. By proposition 3.10 the equality $\delta^*(\tilde{\nabla}\Sigma) = 0$ implies that $\Sigma = L(\Sigma_0)$ and hence $D(\Sigma_0) = 0$. □

The fact that $A : V \to \ker(\Box) \subset T^*M \otimes V$ is of homogeneity ≥ 1, allows us to compute the section $\tilde{\nabla}\Sigma + A(\Sigma) \in \Gamma(\mathrm{im}(\delta^*))$.

Proposition 3.12. *Let $A : V \to T^*M \otimes V$ be a bundle map of homogeneity ≥ 1. Then there exists a differential operator $B : \Gamma(V) \to \Gamma(T^*M \otimes V)$ such that*

$$\tilde{\nabla}\Sigma + A(\Sigma) \in \Gamma(\mathrm{im}(\delta^*))$$

if and only if

$$\tilde{\nabla}\Sigma + B(\Sigma) = 0.$$

If A is a vector bundle map, then B is a linear differential operator.

Proof. Since we have chosen a splitting of the tangent bundle, we can identify TM with $\mathrm{gr}(TM)$. Therefore we have a grading on differential forms with values in V corresponding to the grading (3.7) on $\Lambda^n \mathfrak{g}_-^* \otimes \mathbb{V}$, which is given by homogeneous degree. As usual we denote by lower indices the grading components

$$(\Lambda^n T^*M \otimes V)_\ell = (\Lambda^n \mathrm{gr}(TM)^* \otimes V)_\ell := \bigoplus_{j=n}^{nk} (\Lambda^n_{-j} \mathrm{gr}(TM))^* \otimes V_{\ell-j}$$

and by upper indices the filtration components $(\Lambda^n T^*M \otimes V)^\ell$ of the associated filtration. The associated filtration is of course exactly the one by homogeneity:

$$\phi \in (\Lambda^n T^*M \otimes V)^\ell \text{ if and only if } \phi(T^{i_1}M, ..., T^{i_n}M) \subset V^{i_1+....+i_n+\ell}.$$

A linear connection on a vector bundle V always extends to a differential operator $\Lambda^k T^*M \otimes V \to \Lambda^{k+1} T^*M \otimes V$ on differential forms with values in V, called the covariant exterior derivative. We denote by $d^{\tilde{\nabla}}$ the covariant exterior derivative corresponding to the linear connection $\tilde{\nabla}$. Recall that for a one form $\phi \in \Gamma(T^*M \otimes V)$ the covariant exterior derivative is given by

$$d^{\tilde{\nabla}}\phi(\xi, \eta) = \tilde{\nabla}_\xi(\phi(\eta)) - \tilde{\nabla}_\eta(\phi(\xi)) - \phi([\xi, \eta]). \tag{3.13}$$

Inserting $\phi = \tilde{\nabla}\Sigma$ into (3.13) we see that $d^{\tilde{\nabla}}\tilde{\nabla}\Sigma(\xi, \eta)$ equals the curvature $\tilde{R}(\xi, \eta)(\Sigma)$ of $\tilde{\nabla}$. We will denote by $\tilde{R}(\Sigma)$ the two form, which is given by $(\xi, \eta) \mapsto \tilde{R}(\xi, \eta)(\Sigma)$.

Now let us consider our equation $\tilde{\nabla}\Sigma + A(\Sigma) = \delta^*\psi$ and show how it can be rewritten. Concerning the bundle map A, we will write $A_i(\Sigma)$ for the i-th grading component and $A^i(\Sigma)$ for the i-th filtration component of $A(\Sigma)$ in $T^*M \otimes V$.
Since δ^* is filtration preserving, we have $\delta^*\psi \in (T^*M\otimes V)^2 \subset (T^*M\otimes V)^1 = T^*M \otimes V$ and we set $B_1(\Sigma) := A_1(\Sigma)$. Then the equation reads as

$$\tilde{\nabla}\Sigma + B_1(\Sigma) + A^2(\Sigma) = \delta^*\psi. \tag{3.14}$$

Since $\tilde{\nabla}$ is of homogeneity ≥ 0 and its lowest homogeneous component is given by ∂, the same is true for $d^{\tilde{\nabla}} : T^*M \otimes V \to \Lambda^2 T^*M \otimes V$. Hence the operator $\delta^* d^{\tilde{\nabla}} : T^*M \otimes V \to T^*M \otimes V$ is also of homogeneity ≥ 0 with lowest homogeneous component $\delta^*\partial$, which by definition of δ^* is the identity on $\mathrm{im}(\delta^*) \subset T^*M \otimes V$. Applying $\delta^* d^{\tilde{\nabla}}$ to the equation (3.14), we can therefore compute the lowest grading component $(\delta^*\psi)_2$. Moving the resulting expression for $(\delta^*\psi)_2$ to the other side of the equation and applying $\delta^* d^{\tilde{\nabla}}$ to the new equation, we can compute $(\delta^*\psi)_3$ and so on until we have computed the whole one form $\delta^*\psi$. More explicitly, if we apply first $d^{\tilde{\nabla}}$ to the equation (3.14), we obtain that

$$\tilde{R}(\Sigma) + d^{\tilde{\nabla}} B_1(\Sigma) + d^{\tilde{\nabla}} A^2(\Sigma) = d^{\tilde{\nabla}} \delta^*\psi.$$

This implies the following equation for the second grading component $\partial((\delta^*\psi)_2)$ of $(d^{\tilde{\nabla}}\delta^*\psi)$

$$(\tilde{R}(\Sigma) + d^{\tilde{\nabla}} B_1(\Sigma))_2 + \partial(A_2(\Sigma)) = \partial((\delta^*\psi)_2).$$

Applying now δ^* we see that

$$\delta^*((\tilde{R}(\Sigma) + d^{\tilde{\nabla}} B_1(\Sigma))_2 + \partial A_2(\Sigma)) = \delta^*\partial((\delta^*\psi)_2) = (\delta^*\psi)_2,$$

since $\delta^*\partial$ is the identity on $\mathrm{im}(\delta^*)$.
If we set $B_2(\Sigma) := A_2(\Sigma) - \delta^*((\tilde{R}(\Sigma) + d^{\tilde{\nabla}} B_1(\Sigma))_2 + \partial A_2(\Sigma))$, the equation (3.14) can be now written as

$$\tilde{\nabla}(\Sigma) + B_1(\Sigma) + B_2(\Sigma) + A^3(\Sigma) = (\delta^*\psi)^3.$$

Applying again $\delta^* d^{\tilde{\nabla}}$, we can compute $(\delta^*\psi)_3$ and define B_3 by substracting the resulting expression for $(\delta^*\psi)_3$ from $A_3(\Sigma)$. In this way, we can inductively define

$$B_i(\Sigma) := A_i(\Sigma) - \delta^*([\tilde{R}(\Sigma) + d^{\tilde{\nabla}}(B_1(\Sigma) + ... + B_{i-1}(\Sigma))]_i + \partial(A_i(\Sigma)).$$

Defining the differential operator B as $B(\Sigma) := \sum_{i=1}^{N+k} B_i(\Sigma)$, it has by construction the property required in the proposition and we are done. $\square$

To compute the weighted order of the differential operator B, we need a bit of information about the curvature $\tilde{R}$ of $\tilde{\nabla}$.

Lemma 3.13. *Let $R \in \Lambda^2 T^*M \otimes V$ be the curvature of the connection ∇ on V and T the torsion of the connection ∇ on $TM \cong \mathrm{gr}(TM)$. Then the curvature of $\tilde{\nabla}$ is given by*

$$\tilde{R}(\xi,\eta)(\Sigma) = R(\xi,\eta)(\Sigma) + \partial(\Sigma)(T(\xi,\eta) + \{\xi,\eta\}).$$

Moreover, the map $\Sigma \mapsto \tilde{R}(\Sigma)$ is of homogeneity ≥ 1.

Proof. The curvature of $\tilde{\nabla} = \nabla + \partial$ is given by

$$\tilde{R}(\xi,\eta)(\Sigma) = \tilde{\nabla}_\xi \tilde{\nabla}_\eta \Sigma - \tilde{\nabla}_\eta \tilde{\nabla}_\xi \Sigma - \tilde{\nabla}_{[\xi,\eta]}\Sigma.$$

For the first term we have

$$\tilde{\nabla}_\xi \tilde{\nabla}_\eta \Sigma = \nabla_\xi \nabla_\eta \Sigma + \nabla_\xi(\partial(\Sigma)(\eta)) + \partial(\nabla_\eta \Sigma)(\xi) + \partial(\partial\Sigma(\eta))(\xi). \tag{3.15}$$

The second summand of (3.15) can be written as

$$\nabla_\xi(\partial(\Sigma)(\eta)) = (\nabla_\xi(\partial\Sigma))(\eta) + \partial\Sigma(\nabla_\xi \eta).$$

Since $\partial : \mathbb{V} \to \mathfrak{g}_-^* \otimes \mathbb{V}$ is G_0-equivariant, the induced vector bundle map is parallel and so we have $(\nabla_\xi(\partial\Sigma))(\eta) = \partial(\nabla_\xi \Sigma)(\eta)$. Putting this together, we obtain that

$$\tilde{\nabla}_\xi \tilde{\nabla}_\eta \Sigma = \nabla_\xi \nabla_\eta \Sigma + \partial(\nabla_\xi \Sigma)(\eta) + \partial\Sigma(\nabla_\xi \eta) + \partial(\nabla_\eta \Sigma)(\xi) + \partial(\partial\Sigma(\eta))(\xi).$$

Therefore we have

$$\begin{aligned}\tilde{R}(\xi,\eta)(\Sigma) &= R(\xi,\eta)(\Sigma) + \partial\Sigma(T(\xi,\eta)) + \partial(\partial\Sigma(\eta))(\xi) - \partial(\partial\Sigma(\xi))(\eta)\\ &= R(\xi,\eta)(\Sigma) + \partial(\Sigma)(T(\xi,\eta) + \{\xi,\eta\}),\end{aligned} \tag{3.16}$$

since

$$0 = \partial(\partial\Sigma)(\xi,\eta) = \partial(\partial\Sigma(\eta))(\xi) - \partial(\partial\Sigma(\xi))(\eta) - \partial\Sigma(\{\xi,\eta\}).$$

Since $\tilde{R}(\Sigma)$ equals $d^{\tilde{\nabla}}\tilde{\nabla}(\Sigma)$, the map $\Sigma \mapsto \tilde{R}(\Sigma)$ is at least of homogeneity ≥ 0. To see that is actually of homogeneity ≥ 1 we consider the formula (3.16).
The curvature $\Sigma \mapsto R(\Sigma) = d^\nabla \nabla(\Sigma)$ of ∇ is of homogeneity ≥ 2, since ∇ and d^∇ both are of homogeneity ≥ 1. Now consider the second term of (3.16) given by $\partial(\Sigma)(T(\xi,\eta) + \{\xi,\eta\})$. We have

$$T(\xi,\eta) + \{\xi,\eta\} = \nabla_\xi \eta - \nabla_\eta \xi - [\xi,\eta] + \{\xi,\eta\}$$

and under the identification of $\mathrm{gr}(TM)$ with TM we can view $\{\xi,\eta\}$ as the grading component of lowest degree $-(ord(\xi) + ord(\eta))$ of $[\xi,\eta]$. Therefore the two form $T + \{\quad,\quad\}$ is of homogeneity ≥ 1. This implies that

$$\Sigma \mapsto \partial(\Sigma)(T + \{\quad,\quad\})$$

is of homogeneity ≥ 1, since ∂ is filtration preserving. $\square$

Using lemma 3.13 we can determine the weighted order of B:

Proposition 3.14. *The differential operator B is of weighted order $N+k-1$, where k is the depth of the filtration of TM. Therefore it defines a bundle map*

$$B : \mathcal{J}^{N+k-1}(V) \to T^*M \otimes V.$$

Moreover, the component B_i factors through

$$\mathcal{J}^{i-1}(V_0) \oplus \mathcal{J}^{i-2}(V_1) \oplus ... \oplus \mathcal{J}^1(V_{i-2}) \oplus V_{i-1}.$$

Proof. Let us write a section $\Sigma \in \Gamma(V)$ as $\Sigma = (\Sigma_0, ..., \Sigma_N)$. We shall prove the proposition by induction on i.
Since $A : V \to T^*M \otimes V$ is of homogeneity ≥ 1, $B_1(\Sigma) = A_1(\Sigma)$ just depends on Σ_0 and so the assertion holds for $i = 1$.
Now consider $B_2(\Sigma) = A_2(\Sigma) - \delta^*((\tilde{R}(\Sigma) + d^{\tilde{\nabla}} B_1(\Sigma))_2 + \partial A_2(\Sigma))$.
The component $A_2(\Sigma)$ depends on Σ_0 and Σ_1, since A is of homogeneity ≥ 1. By the lemma 3.13 we know that $\Sigma \mapsto \tilde{R}(\Sigma)$ is also of homogeneity ≥ 1 and therefore $\tilde{R}(\Sigma)_2$ only depends on Σ_0 and Σ_1. So it remains to look at the term

$$(d^{\tilde{\nabla}} B_1(\Sigma))_2 \in \Gamma(\mathrm{gr}_{-1}(TM)^* \wedge \mathrm{gr}_{-1}(TM)^* \otimes V_0).$$

For $\xi, \eta \in \Gamma(T^{-1}M)$ we have

$$\begin{aligned} &d^{\tilde{\nabla}} B_1(\Sigma)(\xi,\eta) = \tilde{\nabla}_\xi(B_1(\Sigma)(\eta)) - \tilde{\nabla}_\eta(B_1(\Sigma)(\xi)) - B_1(\Sigma)([\xi,\eta]) \\ &= \nabla_\xi(B_1(\Sigma)(\eta)) - \nabla_\eta(B_1(\Sigma)(\xi)) + \partial(B_1(\Sigma)(\eta))(\xi) - \partial(B_1(\Sigma)(\xi))(\eta) - B_1(\Sigma)([\xi,\eta]) \\ &= \nabla_\xi(B_1(\Sigma)(\eta)) - \nabla_\eta(B_1(\Sigma)(\xi)) + \partial(B_1(\Sigma))(\xi,\eta) - B_1(\Sigma)([\xi,\eta] - \{\xi,\eta\}). \end{aligned}$$

Since ∂ is grading preserving, ∂ has to annihilate $B_1(\Sigma)$ and we obtain that

$$d^{\tilde{\nabla}} B_1(\Sigma)(\xi,\eta) = \nabla_\xi(B_1(\Sigma)(\eta)) - \nabla_\eta(B_1(\Sigma)(\xi)) - B_1(\Sigma)([\xi,\eta] - \{\xi,\eta\}).$$

Since the component $B_1(\Sigma) \in \mathrm{gr}_{-1}(TM)^* \otimes V_0$ just depends on Σ_0, we therefore conclude that $(d^{\tilde{\nabla}} B_1(\Sigma))_2$ depends on the weighted one jet of Σ_0. In total, we see that B_2 induces a bundle map $\mathcal{J}^1(V_0) \oplus V_1 \to (T^*M \otimes V)_2$.
Now assume the statement is true for B_i with $i < N + k$. The $i+1$-th component is given by

$$B_{i+1}(\Sigma) := A_{i+1}(\Sigma) - \delta^*([\tilde{R}(\Sigma) + d^{\tilde{\nabla}}(B_1(\Sigma) + ... + B_i(\Sigma))]_{i+1} + \partial(A_{i+1}(\Sigma)).$$

Again, since A and $\tilde{R}$ are of homogeneity ≥ 1, $A_{i+1}(\Sigma)$ and $(\tilde{R}(\Sigma))_{i+1}$, depends only on $\Sigma_0, ..., \Sigma_i$. So it remains to study the term

$$(d^{\tilde{\nabla}}(B_1(\Sigma) + ... + B_i(\Sigma)))_{i+1}.$$

For $j < i+1$ consider

$$B_j(\Sigma) \in \Gamma(\bigoplus_{\ell=1}^{j} \mathrm{gr}_{-\ell}(TM)^* \otimes V_{j-\ell}).$$

We know that the operator $d^{\tilde{\nabla}}$ is of homgeneity ≥ 0 and hence we have $d^{\tilde{\nabla}}(B_j(\Sigma)) \in \Gamma((\Lambda^2 \mathrm{gr}(TM)^* \otimes V)^j)$. Since ∂ is grading preserving, we obtain for vector fields $\xi, \eta \in \Gamma(TM)$ with $\mathrm{ord}(\xi) + \mathrm{ord}(\eta) = i+1$ that

$$\begin{aligned}
&(d^{\tilde{\nabla}} B_j(\Sigma))_{i+1}(\xi, \eta) = \\
&= \nabla_\xi(B_j(\Sigma)(\eta)) - \nabla_\eta(B_j(\Sigma)(\xi)) + \partial(B_j(\Sigma))(\xi, \eta) - B_j(\Sigma)([\xi, \eta] - \{\xi, \eta\}) \\
&= \nabla_\xi(B_j(\Sigma)(\eta)) - \nabla_\eta(B_j(\Sigma)(\xi)) - B_j(\Sigma)([\xi, \eta] - \{\xi, \eta\}).
\end{aligned}$$

This implies that $d^{\tilde{\nabla}}(B_j(\Sigma))_{i+1}$ depends on $B_j(\Sigma)$ and derivatives of $B_j(\Sigma)$ in direction of vector fields of order $i+1-j$. The claim now follows from the assumption that $B_j(\Sigma)$ factors through $\mathcal{J}^{j-1}(V_0) \oplus ... \oplus V_{j-1}$ for all $j < i+1$. □

Now one can do the last step in rewriting the equation $D(s) = 0$ by solving $\tilde{\nabla}\Sigma + B(\Sigma) = 0$ component by component.

Proposition 3.15. *Suppose that $B : \mathcal{J}^{N+k-1}(V) \to T^*M \otimes V$ is a bundle map such that its i-th component*

$$B_i : \mathcal{J}^{N+k-1}(V) \to (T^*M \otimes V)_i$$

factors through

$$\mathcal{J}^{i-1}(V_0) \oplus \mathcal{J}^{i-2}(V_1) \oplus ... \oplus \mathcal{J}^1(V_{i-2}) \oplus V_{i-1}.$$

Then there exists a bundle map $C : V \to T^*M \otimes V$ such that

$$\tilde{\nabla}\Sigma + B(\Sigma) = 0$$

is equivalent to

$$\tilde{\nabla}\Sigma + C(\Sigma) = 0.$$

If B is a vector bundle homomorphism, then also C can be chosen to be a vector bundle homomorphism.

Proof. The linear connection $\tilde{\nabla} = \nabla + \partial$ is of homogeneity ≥ 0 with lowest homogeneous component given by the vector bundle map ∂. Since we have a linear connection ∇ on $TM \cong \mathrm{gr}(TM)$, we can from iterated covariant derivatives $\tilde{\nabla}^i$. We know that the linear connection on TM is of homogeneity ≥ 1, since $\nabla : \Gamma(\mathrm{gr}_i(TM)) \to \Gamma(\mathrm{gr}(TM)^* \otimes \mathrm{gr}_i(TM))$, and hence, since $\tilde{\nabla}$ is of homogeneity ≥ 0 with lowest homogeneous component ∂, we conclude that the iterated covariant derivative $\tilde{\nabla}^i$ is also of homogeneity ≥ 0 and that its lowest homogeneous component is algebraic. By assumption on B we therefore deduce that the component B_i just depends on $\Sigma_{\leq i-1}$, $(\tilde{\nabla}\Sigma)_{\leq i-1}$,...,$(\tilde{\nabla}^{i-1}\Sigma)_{\leq i-1}$ and we may write

$$B_i(\Sigma) = B_i(\Sigma_{\leq i-1}, (\tilde{\nabla}\Sigma)_{\leq i-1}, ..., (\tilde{\nabla}^{i-1}\Sigma)_{\leq i-1}),$$

where $(\_)_{\leq i-1}$ means that we restrict to grading components of degree $\leq i-1$. Let us now consider the equation $\tilde{\nabla}(\Sigma)+B(\Sigma) = 0$ grading component by grading component.

For the first component we get

$$(\tilde{\nabla}\Sigma)_1 + B_1(\Sigma) = 0$$

and we set $C_1(\Sigma) := B_1(\Sigma_0)$.

For the second component we have

$$(\tilde{\nabla}\Sigma)_2 + B_2(\Sigma_0, \Sigma_1, (\tilde{\nabla}\Sigma)_1) = 0$$

and we define $C_2(\Sigma_0, \Sigma_1) := B_2(\Sigma_0, \Sigma_1, -C_1(\Sigma_0))$.

By construction we have

$$((\tilde{\nabla}\Sigma) + B(\Sigma))_{\leq 2} = 0 \quad \text{if and only if} \quad ((\tilde{\nabla}\Sigma) + C(\Sigma))_{\leq 2} = 0, \qquad (3.17)$$

where $C = C_1 + C_2$.

Since $\tilde{\nabla}$ is of homogeneity ≥ 0, we obtain for a section Σ satisfying (3.17) that also $(\tilde{\nabla}(\tilde{\nabla}\Sigma + C(\Sigma)))_{\leq 2} = 0$. In addition, we have

$$(\tilde{\nabla}C(\Sigma))_{\leq 2} = (\tilde{\nabla}C(\Sigma))_2 = [\nabla C_1(\Sigma) + id \otimes \partial(C_2(\Sigma))]_2$$

and so we see that $(\tilde{\nabla}C(\Sigma))_2$ depends on the weighted one jet of $C_1(\Sigma)$ and algebraic on $C_2(\Sigma)$ and therefore it just depends on $\Sigma_{\leq 1}$ and $(\tilde{\nabla}\Sigma)_1$. By (3.17) this implies that we obtain an algebraic expression for $(\tilde{\nabla}^2\Sigma)_2 = -(\tilde{\nabla}C(\Sigma))_2$ in terms of Σ_0 and Σ_1. So we can express all terms occurring in $B_3(\Sigma_{\leq 2}, (\tilde{\nabla}\Sigma)_{\leq 2}, ..., (\tilde{\nabla}^2\Sigma)_{\leq 2})$ in terms of Σ_0, Σ_1 and Σ_2 and inserting the resulting expressions into B_3 we obtain a bundle map $C_3(\Sigma_0, \Sigma_1, \Sigma_2)$ that satisfies

$$(\tilde{\nabla}\Sigma + B(\Sigma))_{\leq 3} = 0 \quad \text{if and only if} \quad (\tilde{\nabla}\Sigma + C(\Sigma))_{\leq 3} = 0,$$

where $C = C_1 + C_2 + C_3$.

Suppose now inductively that we have found bundle maps $C_1, ..., C_i$ for $i < N + k$ such that

$$(\tilde{\nabla}\Sigma + B(\Sigma))_{\leq i} = 0 \quad \text{if and only if} \quad (\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i} = 0, \qquad (3.18)$$

where $C = C_1 + ... + C_i$ and C_j depends only on $\Sigma_{\leq j-1}$. Assume further that for any section Σ satisfying (3.18) we have derived algebraic expressions in terms of $\Sigma_0, ..., \Sigma_{\leq i-1}$ for all $(\tilde{\nabla}^\ell\Sigma)_{\leq i}$ with $\ell = 1, ..., i$.

Inserting these expressions into $B_{i+1}(\Sigma)$, we obtain a bundle map $C_{i+1}(\Sigma_0, ..., \Sigma_i)$ such that

$$(\tilde{\nabla}\Sigma + B(\Sigma))_{\leq i+1} = 0 \quad \text{if and only if} \quad (\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i+1} = 0, \tag{3.19}$$

$C = C_1 + ... + C_{i+1}$.
It remains to show that for any section Σ satisfying (3.19) we can deduce algebraic expressions in terms of $\Sigma_0, ..., \Sigma_i$ for all $(\tilde{\nabla}^\ell \Sigma)_{\leq i+1}$ occurring in B_{i+2}, where $\ell = 1, ..., i+1$. Since $\tilde{\nabla}^j$ is of homogeneity ≥ 0, $(\tilde{\nabla}\Sigma + C(\Sigma))_{\leq i+1} = 0$ implies that $(\tilde{\nabla}^j(\tilde{\nabla}\Sigma + C(\Sigma)))_{\leq i+1} = 0$. The differential operator

$$((\tilde{\nabla}^1 C(\Sigma))_{i+1}, ..., (\tilde{\nabla}^i C(\Sigma))_{i+1})$$

depends on the weighted i-jet of $C_1(\Sigma)$, on the weighted $i-1$-jet of $C_2(\Sigma)$,...,on the weighted one jet of $C_i(\Sigma)$ and algebraic on $C_{i+1}(\Sigma)$. Therefore it just depends on $\Sigma_{\leq i}$, $(\tilde{\nabla}\Sigma)_{\leq i}$..., $(\tilde{\nabla}^i\Sigma)_{\leq i}$, for which we have by induction hypothesis algebraic formulae in terms of $\Sigma_0, ..., \Sigma_i$. Hence we get formulae in terms of $\Sigma_0, .., \Sigma_i$ for $(\tilde{\nabla}^{j+1}\Sigma)_{\leq i+1}$ with $j = 0, ..., i$ and we are done.
If B is a linear differential operator, C will be a vector bundle map by construction. $\square$

Summing up, we have seen by propositions 3.11, 3.12, 3.14 and 3.15 that given a semi-linear differential operator $D : \Gamma(E) \to \Gamma(\circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E)$ of weighted order r with weighted symbol given by the projection $\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E$, we can construct a linear connection $\tilde{\nabla}$ on some vector bundle V, a linear differential operator $L : \Gamma(E) \to \Gamma(V)$ and a bundle map $C : V \to T^*M \otimes V$ such that the operator L induces a bijection

$$\{s \in \Gamma(E) : Ds = 0\} \overset{L}{\leftrightarrow} \{\Sigma \in \Gamma(V) : \widetilde{\nabla}(\Sigma) + C(\Sigma) = 0\}.$$

Therefore we have proved theorem 3.4. As an immediate consequence of theorem 3.4 we obtain:

Corollary 3.16. *Let $\mathbb{E}$ be the complex irreducible representation of G_0, whose dual representation $\mathbb{E}^*$ has highest weight*

$$\lambda = (r-1)\omega_j + \sum_{i \in I \setminus \{j\}} a_i \omega_i \text{ with } r \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0.$$

For a linear differential operator $D : \Gamma(E) \to \Gamma(\circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E)$ of weighted order r with weighted symbol given by the projection

$$\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to \circledcirc^r \mathrm{gr}_{-1}(TM)^* \circledcirc E$$

the solution space of the linear system $Ds = 0$ is finite dimensional and bounded by the dimension of the irreducible G representation of highest weight λ.

PROOF. Let V be the natural vector bundle corresponding to the irreducible G representation $\mathbb{V}$, whose dual representation has highest weight λ. By the proof of theorem 3.4 there is a linear connection $\tilde{\nabla}$ on V and a vector bundle map $C : V \to T^*M \otimes V$ such that solutions of $Ds = 0$ are in bijective correspondence with solutions of $\tilde{\nabla}\Sigma + C(\Sigma) = 0$. Since C is a vector bundle map, $\bar{\nabla} := \tilde{\nabla} + C$ is linear connection on V and hence solutions of $Ds = 0$ correspond to parallel sections of $\bar{\nabla}$. Since a parallel section of a linear connection is already determined by its value in a single point, we see that the dimension of the solution space is bounded by the rank of V. The result now follows from the fact that the standard fiber of the vector bundle V is given by the G representation $\mathbb{V}$. □

Using standard tools from the representation theory of semisimple Lie groups the dimension of the representation $\mathbb{V}$ can always be easily computed from its highest weight. Hence we see that for a linear differential operator D as in the corollary, we can read off directly from λ a bound for the dimension of the solution space of the system $Ds = 0$.

Remark 3.6. (Sharpness of the bounds)
By remark 2.3 we know that in nearly all cases a regular infinitesimal flag structure on a manifold M corresponding to $|k|$-graded semisimple Lie algebra $\mathfrak{g}$, where the center of $\mathfrak{g}_0$ is one dimensional, determines a regular normal parabolic geometry $(\mathcal{G} \to M, \omega)$ of the same type. As already mentioned in the introduction a large class of invariant differential operators for $(\mathcal{G} \to M, \omega)$ occur as differential operators in some BGG-sequence, see **[13]** and **[6]**. Choosing a Weyl structure, see section 4.1.4, these operators can be interpreted as differential operators between natural vector bundles associated to the underlying infinitesimal flag structure on M. The first operator in a BGG-sequence is always a linear differential operator of the form described in theorem 3.4 and hence the prolongation procedure presented here applies to them. On the one hand this shows that theorem 3.4 covers a lot of geometrically interesting equations, like the equation for the infinitesimal automorphism of $(\mathcal{G} \to M, \omega)$ or in the case of conformal geometries the equations for conformal Killing tensors, the equation for twistor spinors and the equation for Einstein scales. On the other hand it shows that the bound in corollary 3.16 is sharp. In fact, considering the homogenous model of the parabolic geometry in question $(G \to G/P, \omega_{MC})$ and the first BGG-operator D^V associated to some G representation $\mathbb{V}$. It turns out that in the case of the homogeneous model D^V equals D^{∇}. Recall from section 2.1.4 that a Cartan connection induces a linear connection, called tractor connection, on any natural vector bundle associated to a G representation. The prolongation procedure described in this section identifies solution of

the overdetermined system $D^V(s) = 0$ on G/P with parallel section of the tractor connection on V. Since in the case of the homogeneous model, the tractor connection on the associated vector bundle V has vanishing curvature, the dimension of the solution space equals the rank of V and so we see that the bound in corollary 3.16 is sharp . For more information on tractor connections see [**9**].

3.2.2. Semi-linear systems on regular infinitesimal flag structures - the general case. Suppose that

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$$

is a complex $|k|$-graded semisimple such that the center $\mathfrak{z}(\mathfrak{g}_0)$ of the Levi subalgebra has dimension $d > 1$. Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and $P \subset G$ be the connected parabolic subgroup corresponding to the grading on $\mathfrak{g}$. As usual we denote by $G_0 \subset P$ the corresponding Levi subgroup. Further, assume that M is a manifold endowed with a regular infinitesimal flag structure of type (G, P).
Now let $\mathbb{E}$ be the complex irreducible representation of G_0 whose dual representation has highest weight

$$\lambda = \sum_{j\in J}(r_j - 1)\omega_i + \sum_{i\in I\setminus J} a_i\omega_i \qquad r_j \in \mathbb{N} \text{ and } a_i \in \mathbb{N}_0 \tag{3.20}$$

Recall that the number of elements in J exactly equals d.
From lemma 3.1 we know that $\mathfrak{g}_{-1}$ is a completely reducible G_0-module whose decomposition into irreducible submodules is given by

$$\mathfrak{g}_{-1} = \bigoplus_{j\in J} \mathfrak{g}_{-1,j}$$

where $\mathfrak{g}_{-1,j}$ is the irreducible representation with highest weight $-\alpha_j$. By proposition 3.5 there exists a unique irreducible representation $\mathbb{V}$ of G satisfying that

$$H^0(\mathfrak{g}_-, \mathbb{V}) = \mathbb{E} \quad \text{and} \quad H^1(\mathfrak{g}_-, \mathbb{V}) = \bigoplus_{j\in J} \circledcirc^{r_j} \mathfrak{g}^*_{-1,j} \circledcirc \mathbb{E}. \tag{3.21}$$

Moreover, we observed that $\mathbb{V}$ admits a decomposition

$$\mathbb{V} = \mathbb{V}_0 \oplus ... \oplus \mathbb{V}_N \qquad \text{with} \quad \mathbb{V}_0 = \mathbb{E}$$

such that

$$\mathfrak{g}_i\mathbb{V}_j \subset \mathbb{V}_{i+j}.$$

In particular, each $\mathbb{V}_i$ is invariant under the action of G_0. As usual we denote the corresponding natural vector bundle by $V = V_0 \oplus ... \oplus V_N = \mathcal{G}_0 \times_{G_0} \mathbb{V}$,

where $V_0 = E$.
We have seen in proposition 3.8 that there are G_0-equivariant inclusions

$$\phi_i : \mathbb{V}_i \to \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E},$$

which are isomorphism for $i < \min\{r_j : j \in J\}$.
Now let us relabel the d elements in $\{r_j : j \in J\}$ and denote them by $r_1 \leq ... \leq r_d$. Correspondingly, we write the decomposition of $\mathfrak{g}_{-1}$ into irreducible components as

$$\mathfrak{g}_{-1} = \mathfrak{g}_{-1,1} \oplus ... \oplus \mathfrak{g}_{-1,d},$$

where $\mathfrak{g}_{-1,i}$ is the irreducible G_0-module with highest weight minus the simple root, where the corresponding fundamental weight has coefficient $r_i - 1$ in the decomposition (3.20) of λ.
Using proposition 3.2 we conclude that we have G_0-equivariant projection

$$\mathcal{U}_{-r_i}(\mathfrak{g}_-)^* \otimes \mathbb{E} \to S^{r_i}(\mathfrak{g}_-)^* \otimes \mathbb{E} \to S^{r_i}(\mathfrak{g}_{-1,i})^* \otimes \mathbb{E} \to \circledcirc^{r_i} \mathfrak{g}^*_{-1,i} \circledcirc \mathbb{E}. \quad (3.22)$$

Remark 3.7. Denoting by $\mathbb{K}_{r_i}$ the kernel of the projection (3.22) one can prove similarly as in the previous section that

$$\phi_i : \mathbb{V}_i \cong \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E} \quad \text{for } 0 \leq i < r_1$$

$$\phi_i : \mathbb{V}_i \cong \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E} \cap (\mathcal{U}_{-(i-r_1)}(\mathfrak{g}_-)^* \otimes \mathbb{K}_{r_1}) \quad \text{for } r_1 \leq i < r_2$$

$$\vdots$$

$$\phi_i : \mathbb{V}_i \cong \mathcal{U}_{-i}(\mathfrak{g}_-)^* \otimes \mathbb{E} \cap (\mathcal{U}_{-(i-r_1)}(\mathfrak{g}_-)^* \otimes \mathbb{K}_{r_1}) \cap ... \cap (\mathcal{U}_{-(i-r_d)}(\mathfrak{g}_-)^* \otimes \mathbb{K}_{r_d}) \text{ for } r_d \leq i \leq N.$$

Choosing a principal G_0-connection on $\mathcal{G}_0 \to M$, we obtain linear connections on all natural vector bundles associated to $\mathcal{G}_0$. In particular, we obtain a linear connection $\nabla : \Gamma(V) \to \Gamma(T^*M \otimes V)$ on V, which is of homogeneity ≥ 1, cf. the first step of the proof of theorem 3.4.
If we choose a splitting of the filtration of the tangent bundle $TM \cong \mathrm{gr}(TM)$, we can interpret ∂ and δ^* as grading preserving bundle maps on differential forms with values in V and in particular we can define another linear connection on V by setting $\tilde{\nabla} := \nabla + \partial$. By construction $\tilde{\nabla}$ is of homogeneity ≥ 0 with lowest homgeneous component given by ∂.
As in the proof of theorem 3.4 one then obtains the following proposition:

Proposition 3.17. *The differential operator*

$$L(s) = \sum_{i=0}^{N} (-1)^i (\delta^* \nabla)^i (s)$$

is the unique linear differential operator $L : \Gamma(V_0) \to \Gamma(V)$ such that

- *$\pi(L(s)) = s$, where $\pi : V \to V_0$ is the projection*

- *L has values in the kernel of $\delta^*\tilde{\nabla}$*

In particular, a section $\Sigma \in \Gamma(V)$ is in $\mathrm{im}(L)$ *if and only if $\delta^*\tilde{\nabla}(\Sigma) = 0$. Moreover, each operator $L^j : \Gamma(V_0) \to \Gamma(V_0 \oplus ... \oplus V_j)$ induces a vector bundle map*

$$\mathcal{J}^j(V_0) \to V_0 \oplus ... \oplus V_j,$$

which is an isomorphism for $j < r_1$.

Denote by $\mathrm{gr}_{-1,i}(TM)$ the natural vector bundle corresponding to $\mathfrak{g}_{-1,i}$ and consider the linear differential operator given by

$$D^\nabla = (D_1^\nabla, ..., D_d^\nabla) : \Gamma(E) \to \Gamma(\circledcirc^{r_1}\mathrm{gr}_{-1,1}(TM)^* \circledcirc E \oplus ... \oplus \circledcirc^{r_d}\mathrm{gr}_{-1,d}(TM)^* \circledcirc E),$$

where

$$D_i^\nabla := -id \otimes \phi_i \circ \pi_i \circ \tilde{\nabla} \circ L : \Gamma(E) \to \Gamma(\circledcirc^{r_i}\mathrm{gr}_{-1,i}(TM)^* \circledcirc E)$$

and $\pi_i : \mathrm{gr}(TM)^* \otimes V \to \mathrm{gr}_{-1,i}(TM)^* \otimes V_{r_i-1} \to \ker(\Box_i)$ is the projection with the notation of the previous section. It is not hard to see that D_i^∇ is of weigted order r_i and its weighted symbol is induced by the G_0-equivariant projection (3.22)

$$\sigma(D_i^\nabla) : \mathcal{U}_{-r_i}(\mathrm{gr}(TM))^* \otimes E \to \circledcirc^{r_i}\mathrm{gr}_{-1,i}(TM)^* \circledcirc E.$$

Since, by proposition 3.17, the splitting operator L induces a vector bundle isomorphism $\mathcal{J}^{r_1-1}(E) \to V_0 \oplus ... \oplus V_{r_1-1}$ we can prove analogously to proposition 3.11 the following:

Proposition 3.18. *Suppose that*

$$D = (D_1, ..., D_d) : \Gamma(E) \to \Gamma(\circledcirc^{r_1}\mathrm{gr}_{-1,1}(TM)^* \circledcirc E \oplus ... \oplus \circledcirc^{r_d}\mathrm{gr}_{-1,d}(TM)^* \circledcirc E)$$

is a differential operator, which differs from D^∇ by a bundle map

$$\psi : \mathcal{J}^{r_1-1}(E) \to \circledcirc^{r_1}\mathrm{gr}_{-1,1}(TM)^* \circledcirc E \oplus ... \oplus \circledcirc^{r_d}\mathrm{gr}_{-1,d}(TM)^* \circledcirc E,$$

then there exists a bundle map

$$A : V_0 \oplus ... \oplus V_N \to \circledcirc^{r_1}\mathrm{gr}_{-1,1}(TM)^* \circledcirc E \oplus ... \oplus \circledcirc^{r_d}\mathrm{gr}_{-1,d}(TM)^* \circledcirc E$$

such that $s \mapsto Ls$ induces a bijection

$$\{s \in \Gamma(E) : D\sigma = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : \widetilde{\nabla}(\Sigma) + A(\Sigma) \in \mathrm{im}(\delta^*)\}.$$

Since A is of homogeneity ≥ 1, we can proceed as in the previous section and construct a bundle map $C : V \to T^*M \otimes V$ such that the splitting operator $s \mapsto Ls$ induces a bijection

$$\{s \in \Gamma(E) : D(s) = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : (\widetilde{\nabla} + C)(\Sigma) = 0\}.$$

Remark 3.8. From the observations of this section we see that a result analogously to theorem 3.4 for overdetermined systems on general regular infinitesimal flag manifolds, like a prolongation procedure for all differential operators of the form

$$D = (D_1, ..., D_d) : \Gamma(E) \to \Gamma(\circledcirc^{r_1}\mathrm{gr}_{-1,1}(TM)^*\circledcirc E \oplus ... \oplus \circledcirc^{r_d}\mathrm{gr}_{-1,d}(TM)^*\circledcirc E),$$

where D_i is a semi-linear differential operator of weighted order r_i whose weighted symbol is induced by the G_0-equivariant projection (3.22), can not be obtained in the same way as for semi-linear system on regular infinitesimal flag manifolds corresponding to $|k|$-graded semisimple Lie algebra $\mathfrak{g}$, where $\mathfrak{z}(\mathfrak{g}_0)$ is one dimensional. The problem is that L^j just induces an isomorphism $\mathcal{J}^j(V_0) \to V_0 \oplus ... \oplus V_j$, if $j < r_1$.

Remark 3.9. Let us mentioned that the approach of this section can be adapted to prolong the first operator D^V occurring in the BGG-sequence associated to some G representation $\mathbb{V}$, which in the case of the homogeneous model coincide with the differential operator D^∇. By definition of D^V, see [**13**] and [**6**], there is a differential operator $L : \Gamma(V_0) \to \Gamma(V)$, which induces a bijection

$$\{s \in \Gamma(E) : D^V\sigma = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : \nabla^V(\Sigma) \in \mathrm{im}(\partial^*)\},$$

where ∇^V is the tractor connection on V and then one can proceed as in the previous section to construct a linear connection on V such that its parallel sections correspond bijectively to solutions of $D^V s = 0$. Hence one can deal in this way with a broad class of interesting geometric equations on regular infinitesimal flag structures.

3.3. Prolongation on contact manifolds

In this section we want to show how theorem 3.4 can be applied to contact manifolds.

3.3.1. Contact structures. Suppose that M is a smooth manifold of dimension $2n+1$. By section 1.1.3 a contact structure on M is a vector subbundle $H \subset TM$ of rank $2n$ such that in each point $x \in M$ the Levi bracket $\mathcal{L}_x : H_x \times H_x \to T_xM/H_x$ is non-degenerate.

Definition 3.3. A *(local) contact form* for a contact manifold (M, H) is a (local) section α of T^*M such that $\ker(\alpha_x) = H_x$ for all $x \in M$ lying in the domain of α.

The line bundle $Q = TM/H$ is a locally trivial. Any local trivialisation of Q can of course be viewed as a local contact form and conversely any local contact form factorises to a local trivialisation of Q.This shows that local

contact forms always exist and are in bijective correspondence to local triviallisation of Q. In particular, a local contact form α is uniquely determined up to multiplication by a nowhere vanishing smooth function on the domain of α and all contact manifolds of the same dimension are locally isomorphic, cf. proposition 1.1. Note that for a (local) contact form we always have $d\alpha|_{\Lambda^2 H} = -\alpha \circ \mathcal{L}$.
Since a line bundle is trivial if and only if it is orientable, we see that there exists a global contact for a contact manifold (M, H) if and only if the line bundle Q is orientable. Given a global contact form α, the differential form $\alpha \wedge (d\alpha)^n$ is a volume form on M. Therefore a contact manifold (M, H) is orientable if and only if the quotient bundle $Q = TM/H$ is orientable. As usual, we call a orientable contact manifold (M, H) together with the choice of an orientation on $\mathrm{gr}(TM) = H \oplus Q = H \oplus \mathbb{R}$ an oriented contact manifold. Note that for a orientable contact manifold one may choose as an orientation on $\mathrm{gr}(TM)$ the orientation induced by a global contact form. For an orientable contact manifold the following proposition is well known:

Proposition 3.19. *Let (M, H) be a orientable contact manifold and suppose that $\alpha \in \Gamma(T^*M)$ is a contact form for $H \subset TM$.*
Then there exists a unique vector field r on M such that $\alpha(r) = 1$ and $i_r d\alpha = d\alpha(r, \_) = 0$. It is called the Reeb vector field associated to α.
In particular, α induces a splitting of the filtration of the tangent bundle given by

$$TM \cong \mathrm{gr}(TM) = H \oplus \mathbb{R}$$

$$\xi \mapsto (\xi - \alpha(\xi)r, \alpha(\xi)).$$

Proof. The contact form α is a nowhere vanishing section of T^*M and hence we can locally find a vector field ξ such that $\alpha(\xi)$ is nowhere vanishing. By multiplying ξ with an appropriate smooth function, we can assume that $\alpha(\xi) = 1$. Now consider the restriction of the one form $i_\xi d\alpha$ to H. Since $d\alpha|_{\Lambda^2 H}$ is non-degenerate, there is a section $\eta \in \Gamma(H)$ such that $i_\xi d\alpha = i_\eta d\alpha$. Putting $r := \xi - \eta$, we see that r is a vector field with the required properties. If r' is a vector field with the same properties, then $\alpha(r - r') = 0$ and hence $r - r' \in \Gamma(H)$. Since $i_{r-r'} d\alpha = 0$, the non-degeneracy of $d\alpha|_{\Lambda^2 H}$ therefore implies that $r' = r$. $\square$

For $n \geq 1$ consider now $\mathbb{R}^{2n+2}$ endowed with the skew-symmetric non-degenerate bilinear form

$$< (x_0, ..., x_{2n+1}), (y_0, ..., y_{2n+1}) >= x_0 y_{2n+1} - y_0 x_{2n+1} + \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i).$$

Moreover, let

$$\mathfrak{g} = \mathfrak{sp}(2n{+}2, \mathbb{R}) = \{A \in \mathrm{End}(\mathbb{R}^{2n+2}) :< Ax, y >= - < x, Ay > \text{ for all} x, y \in \mathbb{R}^{2n+2}\}$$

be the symplectic Lie algebra with respect to $<,>$.
It turns out that $\mathfrak{g}$ is given by block matrices of block sizes 1, n, n and 1 of the following form:

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & W & z \\ X & A & B & W^t \\ Y & C & -A^t & -Z^t \\ x & Y^t & -X^t & -a \end{pmatrix} : B^t = B, C^t = C \right\}.$$

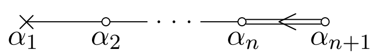
This realisation of $\mathfrak{g}$ defines a $|2|$-grading on $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ given by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}.$$

The Lie bracket restricted to $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is given by

$$[(X_1, Y_1), (X_2, Y_2)] = -2(X_1^t Y_2 - Y_1^t X_2).$$

Hence it is just -2 two times the standard symplectic from on $\mathbb{R}^{2n}$ and so we see that $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a Heisenberg algebra and the $|2|$-grading on $\mathfrak{g}$ is a contact grading.
Let us choose as Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the diagonal matrices in $\mathfrak{g}$. Denoting by ϵ_i the linear functional on $\mathfrak{h}$, which extracts the i-th entry of a diagonal matrix, then the roots corresponding to the Cartan subalgebra $\mathfrak{h}$ are given by $\Delta = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n+1\} \cup \{\pm 2\epsilon_i : 1 \leq i \leq n+1\}$. As simple subsystem of Δ we choose $\Delta^0 = \{\alpha_1, ..., \alpha_{n+1}\}$, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n$ and $\alpha_{n+1} = 2\epsilon_{n+1}$. The $|2|$-grading on $\mathfrak{g}$ is then given by Σ-height, where $\Sigma = \{\alpha_1\}$ and we may refer to this grading by the crossed Satake diagram



Note that $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathfrak{sp}(2n, \mathbb{R})$, where $\mathfrak{sp}(2n, \mathbb{R})$ is the symplectic Lie algebra with respect of the standard symplectic form on $\mathbb{R}^{2n}$. By proposition 2.3 the Killing form on $\mathfrak{g}$ induces $\mathfrak{g}_0$-module isomorphisms $\mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$ and $\mathfrak{g}_{-2} \cong \mathfrak{g}_2^*$. From lemma 3.1 follows that $\mathfrak{g}_{-1}$ is an irreducible representation of $\mathfrak{g}_0$ with highest weight $-\alpha_1$ and correspondingly $\mathfrak{g}_1$ is irreducible with lowest weight α_1.
Let $G = Sp(2n + 2, \mathbb{R})$ be the symplectic Lie group consisting of linear symplectic automorphisms of $(\mathbb{R}^{2n+2}, <,>)$ and let $P \subset G$ be the parabolic

subgroup with Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ given by the connected component of the identity of all block upper triangular matrices in G with block sizes 1, n, n and 1. The corresponding Levi subgroup $G_0 \subset P$ is the given by all the block diagonal matrices in P

$$G_0 = \left\{ \begin{pmatrix} e & & \\ & D & \\ & & e^{-1} \end{pmatrix} : D \in Sp(2n, \mathbb{R}), e \in \mathbb{R}_{>0} \right\},$$

where $Sp(2n, \mathbb{R})$ is the symplectic Lie group wit respect to the standard symplectic form on $\mathbb{R}^{2n}$.
A regular infinitesimal flag structure of type (G, P) on a manifold M consists on one hand of a filtration of the tangent bundle, which makes M into a filtered manifold whose symbol algebra in each point is isomorphic to the Heisenberg algebra $\mathfrak{g}_-$. So it consists of a filtration $TM = T^{-2}M \supset T^{-1}M$ such that the Levi bracket $\mathcal{L} : T^{-1}M \times T^{-1}M \to TM/T^{-1}M$ is non-degenerate in each point, meaning that $H := T^{-1}M$ is a contact distribution on M. On the other hand, we have a reduction of the structure group $\mathcal{G}_0 \to \mathcal{P}(\mathrm{gr}(TM))$ of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of $\mathrm{gr}(TM)$ with respect to the homomorphism $Ad : G_0 \hookrightarrow Aut_{gr}(\mathfrak{g}_-)$. As already mentioned in example 2.4, any element in $Aut_{\mathrm{gr}}(\mathfrak{g}_-)$ is determined by its restriction to $\mathfrak{g}_{-1}$, since $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$. Therefore we have an inclusion $Aut_{\mathrm{gr}}(\mathfrak{g}_-) \hookrightarrow GL(\mathfrak{g}_{-1})$ and $\mathcal{P}(\mathrm{gr}(TM))$ can be viewed as a subbundle of the frame bundle $\mathcal{P}(H)$ of H. Hence the reduction of the structure group can be seen as a reduction of the structure group $\mathcal{G}_0 \to \mathcal{P}(\mathrm{gr}(TM)) \subset \mathcal{P}(H)$ of the frame bundle $\mathcal{P}(H)$ with respect to the homomorphism $G_0 \overset{Ad}{\to} Aut_{\mathrm{gr}}(\mathfrak{g}_-) \subset GL(\mathfrak{g}_{-1})$ and hence the reduction can be interpreted as an additional structure on the distribution H. To interpret the reduction we just need to analyse the inclusion $Ad : G_0 \hookrightarrow Aut_{gr}(\mathfrak{g}_-) \subset GL(\mathfrak{g}_{-1})$.
Writing an element of G_0 as (D, e) and an element in $\mathfrak{g}_-$ as $(x, Z = X + Y)$ the action of G_0 on $\mathfrak{g}_-$ is given by

$$(e^{-2}x, e^{-1}DZ).$$

This immediately implies that the adjoint action of G_0 on $\mathfrak{g}_-$ identifies G_0 with group of grading preserving Lie algebra automorphisms of $\mathfrak{g}_-$, which in addition preserve an orientation on $\mathfrak{g}_-$. Note that G_0 can identified with the conformal symplectic group $CSp(2n, \mathbb{R})$. Therefore we obtain:

Proposition 3.20. *Suppose that M is a manifold of dimension $2n + 1$.*
A regular inifinitesimal flag structure on M of type $(Sp(2n + 2, \mathbb{R}), P)$ is an oriented contact manifold (M, H).

Remark 3.10. By proposition 3.20 we see that a regular infinitesimal flag structure of type (G, P) doesn't determine a regular normal parabolic geometry of type (G, P). Regular normal parabolic geometries of type (G, P) provide exactly one of the two exceptional classes of parabolic geometries, which are not determined by their underlying flag structures, but by finer underlying structures, which in this case can be interpreted as contact projective structures, see also remark 2.3.

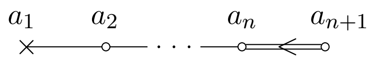
To formulate theorem 3.4 for oriented contact structures let us fix some notation. It is well known that the fundamental weight $\omega_i : \mathfrak{h} \to \mathbb{R}$ corresponding to the simple root α_i is given by

$$\omega_i = \epsilon_1 + ... + \epsilon_i.$$

The highest weight λ of an irreducible representation $\mathbb{W}$ of $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{sp}(2n, \mathbb{R})$ is a $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral linear functional on $\mathfrak{h}$. Hence λ is of the form

$$\lambda = a_1\omega_1 + \sum_{i=2}^{n+1} a_i\omega_i,$$

where $a_i \in \mathbb{N}_0$ for $2 \leq i \leq n+1$ and $a_1 \in \mathbb{R}$ arbitrary, see also section 3.1.1. We refer to the irreducible representation $\mathbb{W}$ of $\mathfrak{g}_0$ with highest weight λ by the diagram:



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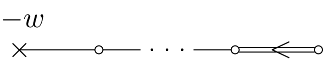
The completely reducible natural vector bundles for oriented contact structures are the vector bundles associated to $\mathcal{G}_0$, which correspond to completely reducible representations of $G_0 = CSp(2n, \mathbb{R})$. Since G_0 has a one dimensional center, there is a one parameter family of one dimensional representations and therefore a one parameter family of natural line bundles. For $w \in \mathbb{R}$ we denote by $\mathbb{R}[w]$ the one dimensional representation

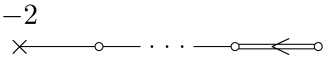
$\mathbb{R}[w]$



.

Of course we have $\mathbb{R}[w]^* \cong \mathbb{R}[-w]$.
The grading component $\mathfrak{g}_{-2}$ consists of the root space corresponding to the root $-2\epsilon_1 = -2(\alpha_1 + ... + \alpha_{n+1})$ and hence we obtain

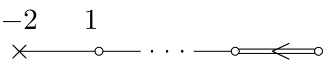
$\mathfrak{g}_{-2} = \mathbb{R}[2]$



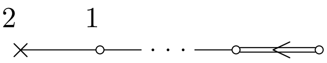
.

The corresponding vector bundle is $\mathrm{gr}_{-2}(TM) = Q \cong \mathcal{G}_0 \times_{CSp(2n,\mathbb{R})} \mathfrak{g}_{-2}$, which is trivial, since (M, H) is oriented.
The irreducible $CSp(2n, \mathbb{R})$ representation $\mathfrak{g}_{-1}$ has highest weight $-\alpha_1 = -2\omega_1 + \omega_2$ and so we get

$\mathfrak{g}_{-1}$



and since $\mathfrak{g}_{-1}^* \cong \mathfrak{g}_1$ as $CSp(2n, \mathbb{R})$-modules we have

$\mathfrak{g}_1$  .

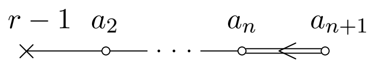
The corresponding natural vector bundles are

$$\mathrm{gr}_{-1}(TM) = H \cong \mathcal{G}_0 \times_{CSp(2n,\mathbb{R})} \mathfrak{g}_{-1}$$

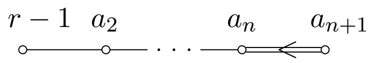
$$\mathrm{gr}_{-1}(TM)^* = H^* \cong \mathcal{G}_0 \times_{CSp(2n,\mathbb{R})} \mathfrak{g}_1.$$

The semisimple part of G_0 is given by $Sp(2n, \mathbb{R})$. Viewing $\mathfrak{g}_{-1}$ as representation of $Sp(2n, \mathbb{R})$ we see from its highest weight that it equals the standard representation $\mathbb{R}^{2n}$. Hence all completely reducible vector bundles for oriented contact structures can be obtained from tensor bundles of H and natural line bundles.
Applying theorem 3.4 to the case of regular infinitesimal flag structures of type $(Sp(2n+2), P)$ we obtain:

Theorem 3.21. *Let (M, H) be an oriented contact manifold. Suppose that $\mathbb{E}$ is the irreducible representation of $CSp(2n, \mathbb{R})$ which is dual to the representation*

 .

For any choice of a contact form α and for any choice of a principal $CSp(2n, \mathbb{R})$-connection ∇ on $\mathcal{G}_0 \to M$, there exists a linear connection $\widetilde{\nabla}$ on V, where V is the natural vector bundle associated to the irreducible representation of $Sp(2n+2, \mathbb{R})$, which is dual to the representation

 ,

with the following property:
For every semi-linear differential operator $D : \Gamma(E) \to \Gamma(S^r H^* \circledcirc E)$ of weighted order r with symbol given by the natural projection (3.2)

$$\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to S^r H^* \circledcirc E$$

there is a bijection between

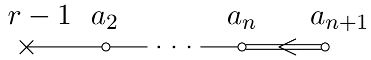
$$\{s \in \Gamma(E) : D(s) = 0\} \leftrightarrow \{\Sigma \in \Gamma(V) : (\widetilde{\nabla} + C)(\Sigma) = 0\}$$

for some bundle map $C : V \to T^*M \otimes V$.

Remark 3.11. The choice of a contact form for an oriented contact manifold (M, H) reduces the structure group of $\mathcal{P}(H)$ further to $Sp(2n, \mathbb{R})$. Denoting the corresponding principal $Sp(2n, \mathbb{R})$ bundle by $\mathcal{G}_0'$, the reduction $\mathcal{G}_0' \to \mathcal{G}_0$ induces an isomorphism $\mathcal{G}_0' \times_{Sp(2n,\mathbb{R})} \mathbb{E} \cong \mathcal{G}_0 \times_{CSp(2n,\mathbb{R})} \mathbb{E}$ for all representations $\mathbb{E}$ of $CSp(2n, \mathbb{R})$. Hence, having choose a contact form, it would be more natural in theorem 3.21 to work with a principal $Sp(2n,\mathbb{R})$-connection on $\mathcal{G}'_0$. Note that the proof of the theorem remains valid in this case.

As a corollary we have:

Corollary 3.22. *Let (M,H) be a oriented contact manifold. Suppose that $\mathbb{E}$ is the irreducible representation of $CSp(2n,\mathbb{R})$ which is dual to the representation*

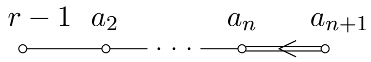


.

Then for every linear differential operator $D : \Gamma(E) \to \Gamma(S^rH^* \circledcirc E)$ of weighted order r with symbol given by the natural projection (3.2)

$$\sigma(D) : \mathcal{U}_{-r}(\mathrm{gr}(TM))^* \otimes E \to S^rH^* \circledcirc E.$$

the associated linear system $Ds = 0$ is of weighted finite type and the dimension of the solution space is bounded by the dimension of the $Sp(2n+2,\mathbb{R})$ representation

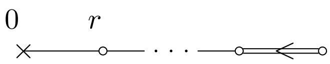


.

Let us consider one basic example of such a system.

Example 3.1. Suppose that (M,H) is a oriented contact manifold of dimension $2n+1$ and let ∇ be a principal $CSp(2n,\mathbb{R})$-connection on $\mathcal{G}_0$. Denote by $S^rH^*[-2r]$ the natural vector bundle associated to the irreducible $CSp(2n,\mathbb{R})$ representation

$$S^r\mathfrak{g}_1 \otimes \mathbb{R}[-2r]$$



.

The principal connection ∇ induces a partial connection on $S^rH^*[-2r]$

$$\nabla : \Gamma(S^rH^*[-2r]) \to \Gamma(H^* \otimes S^rH^*[-2r])$$

and so we can consider the linear differential operator $D : \Gamma(S^rH^*[-2r]) \to \Gamma(S^{r+1}H^*[-2r])$ of weighted order one given by the composition

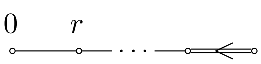
$$D : \Gamma(S^rH^*[-2r]) \xrightarrow{\nabla} \Gamma(H^* \otimes S^rH^*[-2r]) \xrightarrow{sym} S^{r+1}H^*[-2r],$$

where $sym : H^* \otimes S^rH^* \to S^{r+1}H^*$ denotes the symmetrisation. Obviously the weighted symbol of D is given by the symmetrisation

$$\sigma(D) : H^* \otimes S^rH^*[-2r] \xrightarrow{sym} S^{r+1}H^*[-2r]$$

and hence we can apply corollary 3.22:

$$\dim(\{s \in \Gamma(S^rH^*[-2r]) : Ds = 0\}) \leq \dim($$



$$) = \frac{(2n+2r+1)(2n+r-1)!(2n+r)!}{(2n-1)!(2n+1)!r!(r+1)!}.$$

Remark 3.12. The results of this section fit together with the results obtained by Eastwood and Gover in [**21**]. There also some other examples of differential operators can be found to which theorem 3.21 respectively corollary 3.22 applies.

CHAPTER 4

Construction of invariant operators for Lagrangean contact structures via curved Casimir operators

Suppose that $(\mathcal{G} \to M, \omega)$ is a parabolic geometry of type (G, P) and let $\mathbb{V}$ be a finite dimensional representation of P. On the natural vector bundle V corresponding to $\mathbb{V}$ there exists a basic invariant differential operator, called the curved Casimir operator on V. In **[10]** it was shown how curved Casimir operators can be used to conceptually construct invariant operators acting between sections of natural vector bundles corresponding to completely reducible representations of P. The only problem one has to deal with is that it is not apparent from the construction that the obtained operator is non-zero. In a forthcoming article Čap and Gover therefore developed in the case of parabolic geometries corresponding to $|1|$-graded semisimple Lie algebras a method for computing the principal symbol of an operator constructed in this way. By constructing invariant operators for Lagrangean contact structures, which are related to the square of a Sub-Laplacian, we will demonstrate that in the case of parabolic geometries correponding to $|k|$-graded semisimple Lie algebras for $k > 1$ one should compute the weighted symbol of the operators, constructed by means of curved Casimir operators, rather than the usual principal symbol.

4.1. Curved Casimir operators for parabolic geometries

Curved Casimir operators for parabolic geometries have been first introduced in **[16]**. We collect in this section their basic properties.

4.1.1. The adjoint tractor bundle. Suppose $(\mathcal{G} \to M, \omega)$ is a regular parabolic geometry of type (G, P) and consider the adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$. We observed in section 2.1.4 that the P-invariant filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset ... \supset \mathfrak{g}^k$ induces a filtration by subbundles of the adjoint tractor bundle:

$$\mathcal{A}M = \mathcal{A}^{-k}M \supset ... \supset \mathcal{A}^0M \supset ... \supset \mathcal{A}^kM$$

with $\mathcal{A}^iM = \mathcal{G} \times_P \mathfrak{g}^i$. In particular, $\mathcal{A}^0M = \mathcal{G} \times_P \mathfrak{p}$ and hence $\mathcal{A}M/\mathcal{A}^0M \cong TM$. So we have a natural projection from the adjoint tractor bundle to the tangent bundle

$$\Pi : \mathcal{A}M \to TM.$$

The Lie bracket $[\quad,\quad]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is P-equivariant and therefore induces a vector bundle map

$$\{\quad,\quad\}:\mathcal{A}M\times\mathcal{A}M\to\mathcal{A}M,$$

which makes each fiber $\mathcal{A}_xM$ into a filtered Lie algebra isomorphic to $\mathfrak{g}$.
The Killing form is a non-degenerate G-invariant bilinear form on $\mathfrak{g}$ and therefore it induces a non-degnerate vector bundle map

$$B:\mathcal{A}M\times\mathcal{A}M\to\mathbb{K}.$$

By the second part of proposition 2.3, the isomorphism $\mathcal{A}M\cong\mathcal{A}^*M$ defined by B induces a duality between the vector bundles $\mathcal{A}M/\mathcal{A}^{-i+1}M$ and $\mathcal{A}^iM$. In particular, we have an isomorphism between $(\mathcal{A}M/\mathcal{A}^0M)^*\cong T^*M$ and $\mathcal{A}^1M=\mathcal{G}\times_P\mathfrak{p}_+$.

Remark 4.1. Consider the associated graded bundle of the adjoint tractor bundle $\mathrm{gr}(\mathcal{A}M)=\mathrm{gr}_{-k}(\mathcal{A}M)\oplus...\oplus\mathrm{gr}_0(\mathcal{A}M)\oplus...\oplus\mathrm{gr}_k(\mathcal{A}M)$. Since P_+ acts trivially on $\mathfrak{g}^i/\mathfrak{g}^{i+1}$, we obtain $\mathrm{gr}_i(\mathcal{A}M)=\mathcal{G}_0\times_{G_0}\mathfrak{g}_i$. Therefore we can decompose $\mathrm{gr}(\mathcal{A}M)$ as follows:

$$\mathrm{gr}(\mathcal{A}M)=\mathrm{gr}(TM)\oplus\mathrm{gr}_0(\mathcal{A}M)\oplus\mathrm{gr}(T^*M).$$

The algebraic bracket from above induces a bracket on $\mathrm{gr}(\mathcal{A}M)$, which extends the algebraic bracket $\{\quad,\quad\}$ on $\mathrm{gr}(TM)$ defined in section 2.3.

Suppose $\mathbb{V}$ is a representation of P and denote as usual by $V=\mathcal{G}\times_P\mathbb{V}$ the corresponding natural vector bundle. Then the infinitesimal representation $\mathfrak{p}\times\mathbb{V}\to\mathbb{V}$ induces a bundle map

$$\bullet:\mathcal{A}^0M\times V\to V,$$

which makes each fiber V_x into a module over the Lie algebra $\mathcal{A}^0_xM$.
If $\mathbb{V}$ is a representation of G, then the infinitesimal action $\mathfrak{g}\otimes\mathbb{V}\to\mathbb{V}$ even induces a bundle map

$$\bullet:\mathcal{A}M\times V\to V,$$

which makes each fiber V_x into a module over the Lie algebra $\mathcal{A}_xM$.

4.1.2. The fundamental derivative. Let $\mathbb{V}$ be representation of P with corresponding natural vector bundle $V=\mathcal{G}\times_P\mathbb{V}$. We have seen in section 2.1.4 that its space of sections $\Gamma(V)$ can be identified with the space $C^\infty(\mathcal{G},\mathbb{V})^P$ of P-equivariant smooth functions.
The Cartan connection ω induces a linear isomorphism $\omega:T_u\mathcal{G}\cong\mathfrak{g}$ and therefore the space of sections of the adjoint tractor bundle $\Gamma(\mathcal{A}M)\cong C^\infty(\mathcal{G},\mathfrak{g})^P$ may be identified with the space of P-invariant vector fields $\mathfrak{X}(\mathcal{G})^P$ on $\mathcal{G}$. Differentiating a P-equivariant function $f:\mathcal{G}\to\mathbb{V}$ with respect

to a P-invariant vector field $\xi \in \mathfrak{X}(\mathcal{G})^P$, the resulting function $\xi \cdot f : \mathcal{G} \to \mathbb{V}$ is again P-equivariant:

$$(\xi \cdot f)(up) = T_u(f \circ r^p)(T_u r^p)^{-1}\xi(up) = T_u(p^{-1}f(u))\xi(u) = p^{-1}(\xi \cdot f(u)).$$

So we obtain a well defined bilinear differential operator

$$D = D^V : \Gamma(\mathcal{A}M) \times \Gamma(V) \to \Gamma(V)$$

$$(s, v) \mapsto D_s(v),$$

where $D_s(v) \in \Gamma(V)$ is the section corresponding to the P-equivariant function $\xi_s \cdot f_v : \mathcal{G} \to \mathbb{V}$. The operator D is called the *fundamental derivative* or *the fundamental D-operator.*

We collect the basic properties of the fundamental derivative in the following proposition, for a proof see [**9**]:

Proposition 4.1. *Suppose $\mathbb{V}$ is a representation of P and denote by $V = \mathcal{G} \times_P \mathbb{V}$ the corresponding natural vector bundle. Then we have:*

(1) *The fundamental derivative $D : \Gamma(\mathcal{A}M) \times \Gamma(V) \to \Gamma(V)$ is linear over smooth functions $C^\infty(M, \mathbb{K})$ in the $\mathcal{A}M$-entry and satisfies a Leibniz rule in the W-entry:*

$$D_s(fv) = fD_s(v) + (\Pi(s) \cdot f)v,$$

where $f \in C^\infty(M, \mathbb{K})$ and $\Pi : \mathcal{A}M \to TM$ is the natural projection.

(2) *Suppose $\mathbb{V}'$ is another representations of P and $\mathbb{V} \to \mathbb{V}'$ is a P-equivariant map. Denoting by $\Phi : V \to V'$ the corresponding bundle map, we have:*

$$D_s^{V'}(\Phi(v)) = \Phi(D_s^V(v)).$$

Remark 4.2. The naturality of the fundamental derivative, stated in part (2) of the proposition, justifies to denote the fundamental derivative for any bundle just by D.

4.1.3. Curved Casimir operators. By part (1) of the proposition the fundamental derivative can be viewed as a differential operator

$$D : \Gamma(V) \to \Gamma(\mathcal{A}^*M \otimes V).$$

Since $\mathcal{A}^*M \otimes V$ is as well a natural vector bundle, we can iterate the fundamental derivative. In particular, we have the differential operator

$$D^2 : \Gamma(V) \to \Gamma(\otimes^2\mathcal{A}^*M \otimes V).$$

The bundle map B induced by the Killing form can be used to identify $\mathcal{A}M$ and $\mathcal{A}^*M$. In particular, we may view B as bundle map $\otimes^2\mathcal{A}^*M \to \mathbb{K}$.

Definition 4.1. Let $\mathbb{V}$ be a representation of P and set $V = \mathcal{G} \times_P \mathbb{V}$. The linear differential operator $\mathcal{C}_V : \Gamma(V) \to \Gamma(V)$ defined by

$$\mathcal{C}_V = B \otimes id \circ D^2$$

is called the *curved Casimir operator* on V.

Remark 4.3. The name curved Casimir operator for $\mathcal{C}$ is due to the fact that on natural vector bundles associated to the homogeneous model $(G \to G/P, \omega_{MC})$ the differential operator $\mathcal{C}$ is given by the action of the Casimir element. Choose a basis $\{X_i\}$ of $\mathfrak{g}$ and denote by $\{X^i\}$ the dual basis with respect to the Killing form, then this means that $\mathcal{C}(f) = \sum R_{X_i} R_{X^i} f$, where $R_X \in \mathfrak{X}(G)^P$ is the right invariant vector field generated by $X \in \mathfrak{g}$ and f is a section of $V = G \times_P \mathbb{V}$.

The naturality of the fundamental derivative immediately implies the naturality of the Casimir operator:

Proposition 4.2. *Suppose $\mathbb{V}$ and $\mathbb{V}'$ are two P representations and let $\Phi : V \to V'$ be a vector bundle map induced by some P-equivariant linear map $\mathbb{V} \to \mathbb{V}'$. Then we have:*

$$\mathcal{C}_{V'}(\Phi(v)) = \Phi(\mathcal{C}_V(v)).$$

In particular, the Casimir operator preserves sections of natural subbundles of V and the restriction of the Casimir operator to sections of a natural subbundle coincides with the Casimir of that subbundle. Similarly, the induced operator acting on sections of a natural quotient bundle coincides with the Casimir operator of that quotient bundle.

For the adjoint tractor bundle $\mathcal{A}M$ one can always find special local frames, see [**16**]:

Definition 4.2. An *adapted local frame* for $\mathcal{A}M$ is a local frame

$$\{X_i, A_r, Z^i : i = 1, ..., \dim(\mathfrak{p}_+), r = 1, ..., \dim(\mathfrak{g}_0)\}$$

such that:

- $Z^i \in \Gamma(\mathcal{A}^1M)$ for all i and $A_r \in \Gamma(\mathcal{A}^0M)$ for all r
- $B(X_i, X_j) = 0$, $B(X_i, A_r) = 0$ and $B(X_i, Z^j) = \delta_i^j$ for all j, i and r

where $B : \mathcal{A}M \times \mathcal{A}M \to \mathbb{K}$ is the bundle map induced by the Killing form.

The second part of proposition 2.3 implies that B satisfies $B(Z^i, Z^j) = 0$ and $B(Z^i, A_r) = 0$ for all i, j and r. Using this one can show, see [**16**]:

Corollary 4.3. *If $\{X_i, A_r, Z^i\}$ is an adapted local frame for $\mathcal{A}M$, then the dual frame with respect to B is given by $\{Z^i, A^r, X_i\}$ for certain sections $A^r \in \Gamma(\mathcal{A}^0M)$. In addition, A_r and A^r project to dual local frames of $\mathcal{A}^0M/\mathcal{A}^1M$.*

In terms of an adapted local frame the curved Casimir operator looks as follows, for a proof see [**16**]:

Proposition 4.4. *Suppose $V = \mathcal{G} \times_P \mathbb{V}$ is a natural vector bundle and $\mathcal{C} : \Gamma(V) \to \Gamma(V)$ the curved Casimir operator on V. Choose an adapted local frame $\{X_i, A_r, Z^i\}$ over some open subset $U \subset M$ with dual frame $\{Z^i, A^r, X_i\}$. Then the curved Casimir operator is given by*

$$C(v)|_U = -2\sum_i Z^i \bullet D_{X_i} v - \sum_i \{Z^i, X_i\} \bullet v + \sum_r A^r \bullet A_r \bullet v \tag{4.1}$$

where v is a section of V.

We have seen in section 2.1.5 that $P_+ \subset P$ acts trivially on an irreducible representation of P and so an irreducible representation actually comes from an irreducible representation of G_0 trivially extended to P. Therefore the first sum of (4.1) vanishes, which shows that the curved Casimir operator on a vector bundle corresponding to an irreducible representation is algebraic. Recall from section 2.1.5 that irreducible representation representations of P can be described by their highest respectively by their lowest weight. It was proved in [**16**] that the curved Casimir operator on an irreducible natural vector bundle acts as follows:

Proposition 4.5. *Let $\mathbb{V}$ be an irreducible representation of P with lowest weight $-\lambda$. Then the Casimir operator $\mathcal{C} : \Gamma(V) \to \Gamma(V)$ acts by multiplication with the scalar*

$$\beta_V := <\lambda, \lambda + 2\rho>,$$

where $<,>$ is the inner product induced by the Killing form and ρ the sum of the fundamental weights. The number β_V is called the Casimir eigenvalue of $\mathbb{V}$.

By proposition 4.4 we see that, if the Casimir operator is not algebraic, it is differential operator of weighted order at most k. By corollary 3.3 the G_0-modules $\mathcal{U}_{-k}(\mathfrak{g}_-)^*$ and $\mathcal{U}_k(\mathfrak{p}_+)$ can be identified via the Killing form. The spaces $\mathcal{U}_{-k}(\mathfrak{g}_-)^*$ and $\mathcal{U}_k(\mathfrak{p}_+)$ can be also seen as isomorphic P-module by trivially extending the action of G_0 to P and we have

$$\mathcal{U}_{-k}(\mathrm{gr}(TM))^* = \mathcal{G} \times_P \mathcal{U}_{-k}(\mathfrak{g}_-)^* \cong \mathcal{G} \times_P \mathcal{U}_k(\mathfrak{p}_+) = \mathcal{U}_k(\mathrm{gr}(T^*M)).$$

The projection $\mathcal{U}_k(\mathfrak{p}_+) \to \mathfrak{g}_k$, see section 3.1.2, is then P-equivariant and we obtain a vector bundle map between the corresponding vector bundles.
Suppose $\mathcal{C}$ is not algebraic, then it follows from proposition 4.4 that its weighted symbol of order k

$$\sigma_k(\mathcal{C}) : \mathcal{U}_k(\mathrm{gr}(T^*M)) \otimes V \to V$$

is induced by the P-equivariant map by -2Ω, where Ω is the composition of the projection $\mathcal{U}_k(\mathfrak{p}_+)\otimes\mathbb{V} \to \mathfrak{g}_k\otimes\mathbb{V}$ with the action of $\mathfrak{g}_k$ on the representation

$$\Omega : \mathcal{U}_k(\mathfrak{p}_+) \otimes \mathbb{V} \to \mathfrak{g}_k \otimes \mathbb{V} \to \mathbb{V}.$$

If the action of $\mathfrak{g}_k$ on $\mathbb{V}$ is trivial, the weighted symbol of order k vanishes and the Casimir operator is of weighted order at least $k-1$. Inductively one can see that the weighted order of $\mathcal{C}$ is the largest i such that the action of $\mathfrak{g}_i$ on $\mathbb{V}$ is non-trivial and that its weighted symbol comes from the action of $\mathfrak{g}_i$ on $\mathbb{V}$.

4.1.4. Weyl structures for parabolic geometries. A regular parabolic geometry $(p : \mathcal{G} \to M, \omega)$ induces a regular infinitesimal flag structure on M of the same type, consisting of a filtration of the tangent bundle of M and the G_0-principal bundle $p_0 : \mathcal{G}_0 \to M$. Assuming that the regular parabolic geometry is normal and $H^1(\mathfrak{g}_-, \mathfrak{g}) = 0$, we know by theorem 2.14 that the underlying regular infinitesimal flag structure already determines the regular normal parabolic geometry. Weyl structures for parabolic geometries provide a tool to describe the Cartan connection of a regular normal parabolic geometry in terms of objects defined on $\mathcal{G}_0$ and lead to preferred principal connections on $\mathcal{G}_0$.
For a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), consider the G_0-principal bundle $p_0 : \mathcal{G}_0 \to M$, where $\mathcal{G}_0$ was defined as the quotient $\mathcal{G}/P_+$. Hence we also have a natural projection $\pi : \mathcal{G} \to \mathcal{G}_0$, which is easily seen to be a trivial principal bundle with structure group P_+.

Definition 4.3. A (local) *Weyl structure* for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ is a (local) smooth G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$.

In **[14]** it was shown that global Weyl always exists and form an affine space modeled on $\Gamma(\mathrm{gr}(T^*M))$. Note that $\mathrm{gr}(T^*M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ and hence sections of $\mathrm{gr}(T^*M)$ may be viewed as smooth G_0-equivariant functions $\Upsilon = (\Upsilon_1, ..., \Upsilon_k) : \mathcal{G}_0 \to \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$.

Theorem 4.6. *Suppose $(p : \mathcal{G} \to M, \omega)$ is a parabolic geometry. Then there exists a global Weyl structure and the space of Weyl structures is a affine space modeled on $\Gamma(\mathrm{gr}(T^*M))$. Explicitly, if σ and $\hat{\sigma}$ are two Weyl structures, then there exists a section $\Upsilon \in \Gamma(\mathrm{gr}(T^*M))$ such that*

$$\hat{\sigma}(u) = \sigma(u)exp(\Upsilon_1(u))...exp(\Upsilon_k(u)) \quad \textit{for all } u \in \mathcal{G}_0.$$

Suppose $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is a Weyl structure for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then one can consider the pullback $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$ of the Cartan connection ω with respect to the Weyl structure σ. The equivariancy of σ immediately implies the equivariancy of $\sigma^*\omega$:

$$(r^g)^*(\sigma^*\omega) = Ad(g^{-1})\sigma^*\omega \text{ for all } g \in G_0.$$

Hence we may decompose $\sigma^*\omega$ with respect to the G_0-invariant decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus ... \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_k$ and write this decomposition as

$$\sigma^*\omega = \sigma^*\omega_{-k} + ... + \sigma^*\omega_0 + ... + \sigma^*\omega_k.$$

For $X \in \mathfrak{g}_0$ consider the corresponding fundamental vector field $\zeta_X^{\mathcal{G}_0}(u) = \frac{d}{dt}|_{t=0} uexp(tX)$ on $\mathcal{G}_0$. Since σ is G_0-equivariant, we have $\sigma(uexp(tX)) = \sigma(u)exp(tX)$ and hence $\sigma^*\omega(\zeta_X^{\mathcal{G}_0}(u)) = \omega(\zeta_X(\sigma(u))) = X$.
Therefore $\sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ is a principal connection on $\mathcal{G}_0 \to M$. It is called the *Weyl connection* associated to the Weyl structure σ. The principal connection $\sigma^*\omega_0$ induces linear connections on all natural vector bundles associated to $\mathcal{G}_0$. These are also called Weyl connections.
The fact that $\sigma^*\omega$ reproduces the generators of fundamental vector fields also shows that for $i \neq 0$ the G_0-equivariant one form $\sigma^*\omega_i$ is horizontal. So it can be interpreted as an element in $\Omega^1(M, \mathcal{G}_0 \times_{G_0} \mathfrak{g}_i)$.
The positive components of $\sigma^*\omega$ define a one form with values in $\mathrm{gr}(T^*M)$

$$\sigma^*\omega_+ := \sigma^*\omega_1 + ... + \sigma^*\omega_k \in \Omega^1(M, \mathrm{gr}(T^*M)),$$

which is called the *Rho tensor* associated to the Weyl structure σ.
The negative components of $\sigma^*\omega$ define a one form with values in $\mathrm{gr}(TM)$

$$\sigma^*\omega_- := \sigma^*\omega_{-1} + ... + \sigma^*\omega_{-k} \in \Omega^1(M, \mathrm{gr}(TM)),$$

which is called the *soldering form* associated to the Weyl structure σ.
It is not hard to see that the soldering form is an isomorphism $TM \to \mathrm{gr}(TM)$, which defines a splitting of the filtration of TM, meaning that for all $i < 0$ the isomorphism restricts to a map $T^iM \to \bigoplus_{j \geq i} \mathrm{gr}_j(TM)$ and the component in $\mathrm{gr}_i(TM)$ equals the image of the projection $T^iM \to T^iM/T^{i+1}M$.
Observe that a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ may be viewed as a reduction of the structure group from P to G_0 and we have:

Proposition 4.7. *Let $(p : \mathcal{G} \to M, \omega)$ be a parabolic geometry and suppose $\mathbb{V}$ is a representation of P. The choice of a Weyl structure σ induces an isomorphism*

$$\mathcal{G} \times_P \mathbb{V} \cong \mathcal{G}_0 \times_{G_0} \mathbb{V}.$$

Moreover, the Weyl connection induces a linear connection on $V = \mathcal{G} \times_P \mathbb{V}$.

PROOF. Consider the map $\mathcal{G}_0 \times \mathbb{V} \to \mathcal{G} \times_P \mathbb{V}$ given by $(u, v) \mapsto [\sigma(u), v]$. The equivariancy of σ implies that it induces a well defined smooth map

$\mathcal{G}_0 \times_{G_0} \mathbb{V} \to \mathcal{G} \times_P \mathbb{V}$, which is a fiber bundle map over the identity on M and restricts to a linear isomorphism between the fibers. $\square$

Suppose $\mathbb{V}$ is a representation of P. Then $\mathbb{V}$ admits a P-invariant filtration

$$\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset ... \supset \mathbb{V}^N,$$

which is inductively defined by

$$\mathbb{V}^N = \{v \in \mathbb{V} : Zv = 0 \text{ for all } Z \in \mathfrak{p}_+\}$$

$$\mathbb{V}^i = \{v \in \mathbb{V} : Zv \in \mathbb{V}^{i+1} \text{ for all } Z \in \mathfrak{p}_+\}.$$

By construction of the filtration we have $P_+\mathbb{V}^i \subset \mathbb{V}^{i+1}$ and hence P_+ acts trivially on the quotients $\mathbb{V}^i/\mathbb{V}^{i+1}$. Therefore $\mathbb{V}^i/\mathbb{V}^{i+1}$ is a representation of G_0 trivially extended to P. Accordingly, we have a filtration of the corresponding vector bundle $V = \mathcal{G} \times_P \mathbb{V}$ into subbundles

$$V = V^0 \supset ... \supset V^N$$

and we can consider the associated graded vector bundle

$$\mathrm{gr}(V) = \mathrm{gr}_0(V) \oplus ... \oplus \mathrm{gr}_N(V)$$

with $\mathrm{gr}_i(V) = V^i/V^{i+1}$.
Since $P_+ \subset P$ is a normal subgroup and P_+ acts trivially on $\mathbb{V}^i/\mathbb{V}^{i+1}$, we obtain an isomorphism $\mathrm{gr}(V) \cong \mathcal{G}_0 \times_{G_0} \mathrm{gr}(\mathbb{V}) = \mathcal{G}_0 \times_{G_0} \mathbb{V}$. The proposition 4.7 therefore implies:

Corollary 4.8. *A Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ induces an isomorphism of vector bundles $V \cong \mathrm{gr}(V)$, which defines a splitting of the filtration.*

In **[14]** one can find the transformation rules for the Weyl connection, the Rho tensor and the soldering form respectively the splittings of filtered vector bundles under the change of a Weyl structure.

Remark 4.4. Suppose $D : \Gamma(V) \to \Gamma(W)$ is a differential operator between sections of natural vector bundles associated to $\mathcal{G}$. Choosing a Weyl structure, one can interpret D as a differential operator between sections of natural vector bundles associated to $\mathcal{G}_0$ and write down an expression for D in terms of the data associated with the choosen Weyl structure. Invariance of D can then be phrased as the fact that this expression doesn't change, if one changes the Weyl structure.

4.1.5. A formula for the curved Casimir operator. Suppose $(p : \mathcal{G} \to M, \omega)$ is a parabolic geometry corresponding to a $|k|$-graded semisimple Lie algebra and choose a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. Let $\mathbb{V}$ be a representation of P endowed with its P-invariant filtration $\mathbb{V} = \mathbb{V}^0 \supset ... \supset \mathbb{V}^N$ as defined in section 4.1.4. By corollary 4.8, the Weyl structure induces an isomorphism $V \cong \mathrm{gr}(V) = \mathrm{gr}_0(V) \oplus ... \oplus \mathrm{gr}_N(V)$. We identify a section $v \in \Gamma(V)$ with the corresponding section $(v)_\sigma = (v_0, ..., v_N)$ of $\mathrm{gr}(V)$. The Weyl connection on $\mathcal{G}_0 \to M$ induces a linear connection on $V \cong \mathrm{gr}(V)$, which we denote by $\nabla = \nabla^\sigma$. Moreover, we write $\mathsf{P} \in \Omega^1(M, \mathrm{gr}(T^*M))$ for the Rho-tensor associated to the Weyl structure. Then we have the following formula for the fundamental derivative, for a proof see section 5.1. in [**15**].

Proposition 4.9. *Let $\sigma : \mathcal{G}_0 \to \mathcal{G}$ be a Weyl structure for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then the fundamental derivative on $V = \mathcal{G} \times_P \mathbb{V}$ is given by*

$$D_s(v)_i = \nabla_{\Pi(s)} v_i - s_0 \bullet v_i + \sum_{j=1}^{k} (\mathsf{P}_j(\Pi(s)) - s_j) \bullet v_{i-j},$$

where $s \in \Gamma(\mathcal{A}M)$ with $(s)_\sigma = (s_{-k}, ..., s_k)$, $v \in \Gamma(V)$ with $(v)_\sigma = (v_0, .., v_N)$ and $\mathsf{P}_j(\Pi(s))$ denotes the component in $\mathrm{gr}_j(T^*M)$ of $\mathsf{P}(\Pi(s))$.

By proposition 2.7 we know that a representation of P is completely reducible if and only if it comes from a completely reducible representation of G_0 via the quotient map $P \to P/P_+ = G_0$. Recall that a representation of G_0 is completely reducible if and only if the center of G_0 acts diagonalisably. Assuming that we are dealing with a representation $\mathbb{V}$ of P such that the center of G_0 acts diagonalisably, the center of G_0 also acts diagonalisably on the quotients $\mathbb{V}^i/\mathbb{V}^{i+1}$ and so the quotients $\mathbb{V}^i/\mathbb{V}^{i+1}$ are completely reducible G_0-representations. We obtain the following formula for the Casimir operator on natural vector bundles associated to such representations, see also [**10**]:

Proposition 4.10. *Suppose $(p : \mathcal{G} \to M, \omega)$ is a parabolic geometry and choose a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. Let $\mathbb{V}$ be a representation of P such that center of G_0 acts diagonalisably and set $V = \mathcal{G} \times_P \mathbb{V}$. Choosing an adapted local frame $\{Z^i, A_r, X_i\}$ for the adjoint tractor bundle, which is compatible with the splitting of the adjoint tractor bundle induced by σ, the curved Casimir operator is locally given by:*

$$\mathcal{C}(v) = \beta(v) - 2 \sum_{i=1}^{\dim(\mathfrak{p}_+)} Z^i \bullet (\nabla_{X_i} v + \mathsf{P}(X_i) \bullet v),$$

where $\beta : V \to V$ is the bundle map defined by $\beta_W id$ on each irreducible component $W \subset V \cong \mathrm{gr}(V)$.

PROOF. Via the Weyl structure σ, we have an isomorphism $\mathcal{A}M \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$. Hence we have $\mathcal{A}M \cong \mathrm{gr}(TM) \oplus \mathrm{gr}_0(\mathcal{A}M) \oplus \mathrm{gr}(T^*M)$ and by assumption $\{Z^i\}$ is a local frame of $\mathrm{gr}(T^*M)$, $\{A_r\}$ a local frame of $\mathrm{gr}_0(\mathcal{A}M)$ and $\{X_i\}$ a local frame of $\mathrm{gr}(TM)$. Since the center of G_0 acts diagonalisably on $\mathbb{V}$, the vector bundle $V \cong \mathrm{gr}(V)$ splits into a direct sum of irreducible bundles.
Considering the formula for the Casimir operator in proposition 4.1, we see that $\mathcal{C}$ is the sum of $-2\sum_i Z^i \bullet D_{X_i} v$ and a tensorial term depending just on the action of $\mathrm{gr}_0(\mathcal{A}M) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_0$ on V. Hence the tensorial term has to preserve each irreducible component $W \subset V$. The naturality of the Casimir operator (proposition 4.2) and proposition 4.5 imply that the Casimir operator $\mathcal{C}$ acts on $\Gamma(W)$ by multiplication with β_W and so the tensorial term has to coincide with β. The result follows now directly from proposition 4.10. □

4.1.6. Construction of invariant operators for parabolic geometries via curved Casimir operators. We have seen that on any natural vector bundle V we have a basic invariant differential operator given by the curved Casimir operator $\mathcal{C} = \mathcal{C}_V : \Gamma(V) \to \Gamma(V)$. Curved Casimir operators can be used to construct other invariant operators for parabolic geometries. In [**16**] it was shown how to construct splitting operators via curved Casimir operators and in [**10**] it was demonstrated how to construct invariant differential operator between irreducible natural vector bundles. Let us explain this:
Suppose $\mathbb{V}$ is a representation of P such that the center of the Levi subgroup G_0 acts diagonalisably and denote by $\mathbb{V} = \mathbb{V}^0 \supset \mathbb{V}^1 \supset ... \supset \mathbb{V}^N$ its natural P-invariant filtration.
Accordingly, we write the filtration by vector subbundles of the corresponding natural vector bundle as

$$V = V^0 \supset V^1 \supset ... \supset V^N,$$

where by assumption on $\mathbb{V}$ the quotient bundles V^i/V^{i+1} correspond to completely reducible representations. From the naturality of the Casimir operator $\mathcal{C}$ (proposition 4.2) and from proposition 4.5 it follows that the Casimir operator acts by a scalar on section of each irreducible component of V^i/V^{i+1}. We denote the different scalars occurring in this manner by $\beta_i^1, ..., \beta_i^{n_i}$.
Now consider the following differential operator acting on sections of V

$$L_i = \prod_{\ell=1}^{n_i} (\mathcal{C} - \beta_i^\ell).$$

Observe that all factors in this product obviously commute.
The crucial observation for constructing natural differential operators by means of Casimir operators is the following:

Lemma 4.11. *The differential operator $L_i : \Gamma(V) \to \Gamma(V)$ satisfies that:*

- $L_i(\Gamma(V^j)) \subset \Gamma(V^j)$ *for all* $j = 0, ..., N$
- $L_i(\Gamma(V^i)) \subset \Gamma(V^{i+1})$

Proof. The naturality of the Casimir operator, formulated in proposition 4.2, implies that $\mathcal{C}$ preserves sections of V^j and hence so does L_i. Moreover, this shows that L_i induces a differential operator on sections of V^j/V^{j+1} and the naturality of $\mathcal{C}$ implies that it is given by $\prod_{\ell=1}^{n_i}(\mathcal{C}_{V^j/V^{j+1}} - \beta_i^\ell)$. In particular, L_i induces the zero operator on $\Gamma(V^i/V^{i+1})$ and so we must have that $L_i(\Gamma(V^i)) \subset \Gamma(V^{i+1})$. □

Fix now two indices $i < j$ and let W be an irreducible subbundle of V^i/V^{i+1}. Denoting by $\pi : V^i \to V^i/V^{i+1}$ the natural projection, the preimage $\pi^{-1}(W) \subset V^i \subset V$ is a natural subbundle.
We write $\pi_j : V^i \to V^i/V^{j+1}$ for the natural projection. By lemma 4.11 the composition

$$\pi_j \circ L_j \circ ... \circ L_{i+1} : \Gamma(V^i) \to \Gamma(V^i/V^{j+1})$$

is a well defined invariant differential operator.
Again using lemma 4.11, we conclude that the operator $L_j \circ ... \circ L_{i+1}$ maps section of V^{i+1} to section of V^{j+1} and therefore the above composition induces a differential operator $\Gamma(V^i/V^{i+1}) \to \Gamma(V^i/V^{j+1})$. The naturality of the Casimir operator implies that $L_j \circ ... \circ L_{i+1}$ preserves sections of natural subbundles of V^i. In particular, it preserves the space $\Gamma(\pi^{-1}(W))$. Since $W \cong \pi^{-1}(W)/(V^{i+1} \cap \pi^{-1}(W))$, the composition $\pi_j \circ L_j \circ ... \circ L_{i+1}$ induces an invariant differential operator

$$L : \Gamma(W) \to \Gamma(\pi^{-1}(W)/(V^{j+1} \cap \pi^{-1}(W)).$$

Assume that β is the Casimir eigenvalue corresponding to the irreducible bundle W.
In **[16]** it was shown that if $\beta \neq \beta_m^\ell$ for $i < m \leq j$ and all ℓ, then L composed with the projection

$$\pi^{-1}(W)/(V^{j+1} \cap \pi^{-1}(W)) \to \pi^{-1}(W)/(V^{i+1} \cap \pi^{-1}(W)) \cong W$$

is a non-zero multiple of the identity and so L defines a splitting operator for the projection $\pi^{-1}(W)/(V^{j+1} \cap \pi^{-1}(W)) \to W$.
Conversely, suppose now that β coincides with one of the Casimir eigenvalues β_j^ℓ. Without loss of generality we assume that $\beta = \beta_j^1$ and let $\tilde{W}$ be the direct

sum of all irreducible subbundles of V^j/V^{j+1} corresponding to β_j^1. Consider $L_j = (\mathcal{C} - \beta)\tilde{L}_j$, where $\tilde{L}_j = \prod_{\ell=2}^{n_i}(\mathcal{C} - \beta_i^\ell)$. Since all the factors commute, we have

$$L_j \circ .. \circ L_{i+1} = \tilde{L}_j \circ .. \circ L_{i+1} \circ (\mathcal{C} - \beta).$$

Since $(\mathcal{C} - \beta)$ maps sections of $\pi^{-1}(W)$ to sections of V^{i+1}, we conclude using lemma 4.11 that $L_{j-1} \circ .. \circ L_{i+1}$ maps sections of $\pi^{-1}(W)$ to sections of V^j and $\tilde{L}_j$ preserves $\Gamma(V^j)$ by naturality. Therefore the differential operator L has values in $\Gamma(V^j/V^{j+1})$. Again, by naturality we have

$$(C - \beta)\pi_j \circ L_j \circ ... \circ L_{i+1} = \pi_j \circ L_j \circ ... \circ L_{i+1} \circ (\mathcal{C} - \beta)$$

and since this composition induces the zero operator on W, we obtain that L has values in $\Gamma(\tilde{W})$. Hence we have constructed an invariant differential operator $L : \Gamma(W) \to \Gamma(\tilde{W})$, cf. section 2.3. of [**10**].
This argumentation shows that curved Casimir operators provide an efficient tool to conceptually construct invariant differential operators for parabolic geometries between natural completely reducible vector bundles. However, it is not visible from this construction, if the obtained operator L is zero or not. In a forthcoming article Čap and Gover therefore established in the case of parabolic geometries corresponding to $|1|$-gradings a systematical method for computing the principal symbol of L.
Dealing with parabolic geometries corresponding to $|k|$-gradings for $k > 1$, it turns out that one should study the weighted symbol rather than the usual one. Similarly as in the $|1|$-graded case it should be possible to find a systematical way to compute the weighted symbol of L. We will demonstrate this in the next sections by constructing examples of invariant differential operators for Langrangean contact structures.

4.2. Construction of invariant operators for Lagrangean contact structures

In this section we will show how curved Casimir operators can be used to construct invariant differential operators for Langrangean contact structures related to the square of a Sub-Laplacian.

4.2.1. Lagrangean contact structures. For $n \geq 1$ consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ of trace free endomorphisms of $\mathbb{R}^{n+2}$. We can decompose $\mathfrak{g}$ into blocks of size 1, n and 1 as follows:

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & \gamma \\ X & A & W \\ \beta & Y & b \end{pmatrix} : a, b, \beta, \gamma \in \mathbb{R}; X, W \in \mathbb{R}^n; Z, Y \in \mathbb{R}^{n*}; a + b + tr(A) = 0 \right\}.$$

This decomposition defines a $|2|$-grading on $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ given by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^E & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1^F \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^F & \mathfrak{g}_0 \end{pmatrix},$$

with $\mathfrak{g}_{\pm 1} = \mathfrak{g}_{\pm 1}^E \oplus \mathfrak{g}_{\pm 1}^F$.

Since the restriction of the Lie bracket to $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is just the negative of the standard symplectic form on $\mathbb{R}^{2n}$, the grading is a contact grading, meaning that $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a Heisenberg algebra. In this picture, the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ corresponds to the decomposition of the symplectic vector space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^{n*}$ into a direct sum of Lagrangean subspaces. Moreover, the restriction of the Lie bracket to $\mathfrak{g}_{-1}^E \times \mathfrak{g}_{-1}^F$ is non-degenerate and hence defines an isomorphism $\mathfrak{g}_{-1}^F \cong L(\mathfrak{g}_{-1}^E, \mathfrak{g}_{-2})$.

Choose as Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the diagonal matrices and denote by $\epsilon_i : \mathfrak{h} \to \mathbb{R}$ the linear functional, which extracts the i-th entry of a matrix in $\mathfrak{h}$. It is well known that the root system corresponding to $\mathfrak{h}$ is given by $\Delta = \{\epsilon_i - \epsilon_j : i \neq j\}$. Let $\Delta^0 = \{\alpha_1, ..., \alpha_{n+1}\}$ be the simple subsystem of roots, where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Then the $|2|$-grading is given by Σ-height with $\Sigma = \{\alpha_1, \alpha_{n+1}\}$ and we may refer to this grading by the crossed Satake diagram

$$\underset{\alpha_1}{\times} \!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \!-\!\!-\! \cdots \!-\!\!-\! \underset{\alpha_n}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_{n+1}}{\times} \quad .$$

Note that $\mathfrak{g}_0 \cong \mathbb{R}^2 \oplus \mathfrak{sl}(n, \mathbb{R})$. We know from proposition 2.3 that the Killing form induces isomorphisms of $\mathfrak{g}_0$-modules $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}^*$ and $\mathfrak{g}_2 \cong \mathfrak{g}_{-2}^*$. The decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ coincides with decomposition of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ into irreducible components, where $\mathfrak{g}_{-1}^E$ has highest weight $-\alpha_1$ and $\mathfrak{g}_{-1}^F$ has highest weight $-\alpha_{n+1}$, see also lemma 3.1. Correspondingly, $\mathfrak{g}_1 = \mathfrak{g}_1^E \oplus \mathfrak{g}_1^F$ is the decomposition into irreducible components of the dual module $\mathfrak{g}_1$, where $\mathfrak{g}_1^E \cong (\mathfrak{g}_{-1}^E)^*$ has lowest weight α_1 and $\mathfrak{g}_1^F \cong (\mathfrak{g}_{-1}^F)^*$ has lowest weight α_{n+1}.

Let G be the special linear group $SL(n+2, \mathbb{R})$ consisting of volume preserving automorphisms of $\mathbb{R}^{n+2}$. As a parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ we choose the connected component of the identity of the subgroup of block upper triangular matrices with block size 1, n and 1. The corresponding Levi subgroup G_0 is then the subgroup of block diagonal matrices of P

$$G_0 = \left\{ \begin{pmatrix} c & & \\ & C & \\ & & e \end{pmatrix} : C \in GL_+(n, \mathbb{R}), c, e \in \mathbb{R}_{>0} \text{ with } c\det(C)e = 1 \right\},$$

where $GL_+(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) : \det(A) > 0\}$ is the group of orientation preserving automorphisms of $\mathbb{R}^n$.
By remark 2.3, we know that $H^1(\mathfrak{g}_-,\mathfrak{g})^1 = 0$ and so we deduce from theorem 2.14 that regular normal parabolic geometries of type (G,P) are determined by their underlying regular infinitesimal flag structure of type (G,P).
A regular infinitesimal flag structure of type (G,P) on a manifold M consists on one hand of a filtration of the tangent bundle, which makes M into a filtered manifold with symbol algebra in each point isomorphic to the Heisenberg Lie algebra $\mathfrak{g}_-$. Hence it consists of a filtration of the tangent bundle of M

$$TM = T^{-2}M \supset T^{-1}M$$

such that the Levi bracket $\mathcal{L} : T^{-1}M \times T^{-1}M \to TM/T^{-1}M$ is non-degenerate in each point, which means that $H := T^{-1}M$ is a contact distribution.
On the other hand, one has a reduction $\mathcal{G}_0 \to \mathcal{P}(\mathrm{gr}(TM))$ of the structure group of the frame bundle $\mathcal{P}(\mathrm{gr}(TM))$ of the associated graded $\mathrm{gr}(TM) = \mathrm{gr}_{-2}(TM) \oplus \mathrm{gr}_{-1}(TM)$ corresponding to the homomorphism $Ad : G_0 \hookrightarrow Aut_{gr}(\mathfrak{g}_-)$. To interpret this reduction as an additional structure on the contact manifold M, one has to analyse the inclusion $G_0 \hookrightarrow Aut_{gr}(\mathfrak{g}_-)$.
Denoting elements of G_0 by (c,C,e), the action of (c,C,e) on $(\beta,X,Y) \in \mathfrak{g}_{-2} \oplus \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$ is given by

$$(c,C,e)\cdot(\beta,X,Y) = (ec^{-1}\beta, c^{-1}CX, eYC^{-1}) =$$
$$= (c^{-2}\det(C)^{-1}\beta, c^{-1}CX, c^{-1}\det(C)^{-1}YC^{-1}).$$

Hence the Levi subgroup G_0 acts on $\mathfrak{g}_-$ by grading preserving Lie algebra automorphisms, which in addition preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$ and preserve an orientation on $\mathfrak{g}_-$.
Conversely, assume that $\Phi \in Aut_{gr}(\mathfrak{g}_-)$ preserves the decomposition of $\mathfrak{g}_{-1}$ and preserves an orientation on $\mathfrak{g}_-$ and denote by $C \in GL_+(\mathfrak{g}^E_{-1})$ the restriction of Φ to $\mathfrak{g}^E_{-1}$. The compatibility of Φ with the Lie bracket immediately implies that Φ equals $(\beta,X,Y) \mapsto (e\beta, CX, eYC^{-1})$ for some positive number $e \in \mathbb{R}_{>0}$. The element $(e\beta, CX, eYC^{-1})$ equals

$$(\sqrt[n+2]{\det(C)^{-1}e^{-1}},\ \sqrt[n+2]{\det(C)^{-1}e^{-1}}C,\ \sqrt[n+2]{\det(C)^{-1}e^{-1}}e)\cdot(X,Y,\beta).$$

Therefore the Levi subgroup G_0 can be identified via the adjoint action with the subgroup of those automorphisms in $Aut_{gr}(\mathfrak{g}_-)$, which preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$ and preserve an orientation on $\mathfrak{g}_-$. So the reduction $\mathcal{G}_0 \to \mathcal{P}(\mathrm{gr}(TM))$ can be interpreted as the decomposition of the contact distribution H into the direct sum

$$H = E \oplus F,$$

where $E = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}^E$ and $F = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}^F$ and the choice of an orientation on $\mathrm{gr}(TM) = TM/H \oplus H$.
The line bundle $Q := TM/H$ is orientable if and only if it is trivial. Recall from section 3.3 that the line bundle Q associated to a contact structure is trivial if and only if there exists a global contact form. This implies that a manifold endowed with a contact distribution is orientable if and only if Q is orientable. Given an orientable contact manifold one can always choose as an orientation on $\mathrm{gr}(TM)$ the orientation induced by the choice of a global contact form.

Definition 4.4. Suppose M is manifold of dimension $2n + 1$.

(1) A *Lagrangean contact structure* on M is a contact structure $H \subset TM$ together with a decomposition into a direct of sum $H = E \oplus F$, where E and F are subbundles of rank n and where the Levi bracket restricted to $E \times E$ as well as to $F \times F$ vanishes.
(2) An orientable Lagrangean contact manifold together with the choice of an orientation on the bundle $\mathrm{gr}(TM) = TM/H \oplus H$ is called an *oriented Lagrangean contact manifold.*

By the description of regular infinitesimal flag structures of type (G, P) and theorem 2.14 we therefore have:

Theorem 4.12. *There is an equivalence of categories between regular normal parabolic geometries of type (G, P) and oriented Lagrangean contact manifolds $(M, H = E \oplus F)$.*

Remark 4.5. The name Lagrangean contact structures for these geometric structures goes back to the article **[39]** by Takeuchi. In this article the author shows that there is a correspondence between a certain class of parabolic geometries and Lagrangean contact structures.

A Lagrangean contact structure naturally occurs on the projectivised cotangent bundle $\mathcal{P}(T^*\mathbb{R}P^{n+1})$ of the projective space of dimension $\mathbb{R}P^{n+1}$. Let us shortly explain this, for details see **[39]**.

Example 4.1. Let us denote by

$$F_{1,n+1}(\mathbb{R}^{n+2}) = \{V_0 \subset V_1 \subset \mathbb{R}^{n+2} : V_0, V_1 \subset \mathbb{R}^{n+2}; \dim(V_0) = 1, \dim(V_1) = n+1\}$$

the flag manifold of lines in hyperplanes in $\mathbb{R}^{n+2}$. Moreover, let $\tilde{G} = PGL(n+2, \mathbb{R})$ be the Lie group with Lie algebra $\mathfrak{g}$, which is defined as the quotient of $GL(n+2, \mathbb{R})$ by its center and let $\tilde{P} \subset G$ be the parabolic subgroup with Lie algebra $\mathfrak{p}$ given by the equivalence classes of block upper triangular matrices in $\tilde{G}$ with block sizes 1, n and 1. The Lie group $\tilde{G}$ obviously acts transitively on $F_{1,n+1}(\mathbb{R}^{n+2})$ and $\tilde{P}$ is the stabiliser in $\tilde{G}$ of

the standard flag $\mathbb{R}^1 \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$. Therefore $\tilde{G}/\tilde{P}$ is diffeomorphic to $F_{1,n+1}(\mathbb{R}^{n+2})$.
There are two natural smooth projections

$$\pi_0 : F_{1,n+1}(\mathbb{R}^{n+2}) \to \mathbb{R}P^{n+1}$$

$$\pi_1 : F_{1,n+1}(\mathbb{R}^{n+2}) \to \mathbb{R}P^{(n+1)*},$$

where π_0 is given by mapping a flag to its line and π_1 is given by mapping a flag to its hyperplane and identifying the hyperplane with a line in $(\mathbb{R}^{n+2})^*$.
The projection π_0 is a fiber bundle, where the fiber over some fixed line $[x]$ is the space of all hyperplanes in $\mathbb{R}^{n+2}$ containing $[x]$, which can be identified with the space of all hyperplanes in $\mathbb{R}^{n+2}/[x]$. Identifying a hyperplane with a line in the dual space, we obtain that the fiber is isomorphic to $\mathbb{R}P^n$. It is not hard to see that the vertical bundle $\ker(T\pi_0)$ of π_0 exactly corresponds to the $\tilde{G}_0$ representation $\mathfrak{g}^F_{-1}$.
The projection π_1 is a fiber bundle, where the fiber over a line $[\lambda]$ is the space of all hyperplanes in $(\mathbb{R}^{n+2})^*$ containing $[\lambda]$, which is therefore isomorphic to $\mathbb{R}P^n$. The vertical subbundle $\ker(T\pi_1)$ of π_1 corresponds to $\tilde{G}_0$ representation $\mathfrak{g}^E_{-1}$.
The subbundle $\ker(T\pi_1) \oplus \ker(T\pi_0) \subset TF_{1,n+1}(\mathbb{R}^{n+2})$ defines a Lagrangean contact structure on $F_{1,n+1}(\mathbb{R}^{n+2})$. Note that the projection π_0 identifies the flag manifold $F_{1,n+1}(\mathbb{R}^{n+2})$ with the projectivised cotangent bundle $\mathcal{P}(T^*\mathbb{R}P^{n+1})$ and under this identification the Lagrangean contact structure is just the canonical contact structure on $\mathcal{P}(T^*\mathbb{R}P^{n+1})$.

Remark 4.6. The inclusion $Ad : \tilde{G}_0 \hookrightarrow Aut_{gr}(\mathfrak{g}_-)$ maps the Levi subgroup to the subgroup of $Aut_{gr}(\mathfrak{g}_-)$, consisting of all grading preserving Lie algebra automorphisms, which in addition preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$. Hence by theorem 2.14 one obtains an equivalence of categories between regular normal parabolic geometries of type $(\tilde{G}, \tilde{P})$ and Lagrangean contact structures $(M, H = E \oplus F)$. The flag manifold $F_{1,n+1}(\mathbb{R}^{n+2}) \cong \tilde{G}/\tilde{P}$ is the homogeneous model for these geometries. Moreover, let us remark that the projectivised cotangent bundle of any manifold, which has a projective structure, admits canonically a Lagrangean contact stuctures, see section 4.4.2 in **[15]**.

4.2.2. Completely reducible natural vector bundles for oriented Lagrangean contact structures. Suppose $\mathfrak{g}$ is the Lie algebra $\mathfrak{sl}(n+2, \mathbb{R})$ endowed with the contact grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ corresponding to Lagrangean contact structures. Choose as Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the diagonal matrices and let $\Delta^0 = \{\alpha_1, ..., \alpha_{n+1}\}$ be the subset of simple roots with $\alpha_i = \epsilon_i - \epsilon_{i+1}$ as in section 4.2.1. Then it is well known that the fundamental weight $\omega_i : \mathfrak{h} \to \mathbb{R}$ corresponding to simple root α_i is given by

$$\omega_i = \epsilon_1 + ... + \epsilon_i - \frac{i}{n+2}(\epsilon_1 + ... + \epsilon_{n+2}).$$

The highest weight of an irreducible representation $\mathbb{W}$ of the Levi subalgebra $\mathfrak{g}_0 = \mathbb{R}^2 \oplus \mathfrak{sl}(n, \mathbb{R})$ is a $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral functional $\lambda : \mathfrak{h} \to \mathbb{R}$. Written as a linear combination of fundamental weights λ is of the form $\lambda = a_1\omega_1 + a_{n+1}\omega_{n+1} + \sum_{i=2}^{n} a_i\omega_i$, where $a_i \in \mathbb{N}_0$ for $i = 2, ..., n$, cf. also section 3.1.1. We refer to the irreducible representation $\mathbb{W}$ of $\mathfrak{g}_0$ with highest weight λ by the diagram

$$\overset{a_1}{\times}\!\!\text{---}\!\!\overset{a_2}{\circ}\!\!\text{---} \cdots \text{---}\!\!\overset{a_n}{\circ}\!\!\text{---}\!\!\overset{a_{n+1}}{\times} .$$

The natural completely reducible vector bundles for oriented Lagrangean contact structures are the associated vector bundles corresponding to completely reducible representations of the Levi subgroup

$$G_0 = \left\{ \begin{pmatrix} c & & \\ & C & \\ & & e \end{pmatrix} : C \in GL_+(n, \mathbb{R}), c, e \in \mathbb{R}_{>0} \text{ with } c\det(C)e = 1 \right\}.$$

Since G_0 has a two dimensional center, we have a two parameter family of one dimensional representations and therefore a two parameter family of natural line bundles. For $w, w' \in \mathbb{Z}$ we denote by $\mathbb{R}[w, w']$ the one dimensional representation corresponding to the highest weight

$$\mathbb{R}[w, w'] \qquad \overset{-w}{\times}\!\!\text{---}\!\!\circ\!\!\text{---} \cdots \text{---}\!\!\circ\!\!\text{---}\!\!\overset{-w'}{\times} .$$

Obviously, we have $\mathbb{R}[w, w']^* \cong \mathbb{R}[-w, -w']$.

We denote the natural line bundle corresponding to $\mathbb{R}[w, w']$ by

$$\mathcal{E}[w, w'] := \mathcal{G}_0 \times_{G_0} \mathbb{R}[w, w'].$$

In particular, we deduce from the description of the grading in terms of roots that $\mathfrak{g}_{-2}$ coincides with the root space corresponding to the root $-(\alpha_1 + ... + \alpha_{n+1}) = -\omega_1 - \omega_{n+1}$ and so we have:

$$\mathfrak{g}_{-2} = \mathbb{R}[1, 1] \qquad \overset{-1}{\times}\!\!\text{---}\!\!\circ\!\!\text{---} \cdots \text{---}\!\!\circ\!\!\text{---}\!\!\overset{-1}{\times} .$$

Therefore we have $\mathrm{gr}_{-2}(TM) = Q = \mathcal{E}[1, 1]$ and $Q^* = \mathcal{E}[-1, -1]$. Since we are dealing with oriented Lagrangean contact structures, these bundles are trivial.

We already know that the G_0-representation $\mathfrak{g}_{-1}$ splits into irreducible components $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$, where $\mathfrak{g}_{-1}^E$ has highest weight $-\alpha_1$ and $\mathfrak{g}_{-1}^F$ has highest weight $-\alpha_{n+1}$. Written as a sum of fundamental weights we have $-\alpha_1 = -2\omega_1 + \omega_2$ and $-\alpha_{n+1} = \omega_n - 2\omega_{n+1}$ and so we obtain:

$$\mathbb{E} := \mathfrak{g}_{-1}^E \qquad \overset{-2}{\times}\!\!-\!\!-\!\!\overset{1}{\circ}\!\!-\!\!-\cdots-\!\!-\!\!\circ\!\!-\!\!-\!\!\times \quad .$$

and

$$\mathbb{F} := \mathfrak{g}_{-1}^F \qquad \times\!\!-\!\!-\!\!\circ\!\!-\!\!-\cdots-\!\!-\!\!\overset{1}{\circ}\!\!-\!\!-\!\!\overset{-2}{\times} \quad .$$

These representations give rise to the natural vector bundles E and F. Recall that the Levi bracket $\mathcal{L} : E \times F \to Q$ induces an isomorphism $E \cong F^* \otimes Q$. We denote the tensor product of $\mathbb{E}$ or $\mathbb{F}$ with the one dimensional representation $\mathbb{R}[w, w']$ by

$$\mathbb{E}[w, w'] := \mathbb{E} \otimes \mathbb{R}[w, w'] \qquad \mathbb{F}[w, w'] := \mathbb{F} \otimes \mathbb{R}[w, w'].$$

Remark 4.7. The semisimple part of G_0 is $SL(n, \mathbb{R})$ and $\mathfrak{g}_{-1}^E$ viewed as a representation of $SL(n, \mathbb{R})$ is the standard representation $\mathbb{R}^n$. Therefore all completely reducible natural bundles for oriented Lagrangean contact structures can in fact be constructed from tensor bundles of E and natural line bundles.

In the sequel we will use the following abstract index notation:
We set

$$\mathcal{E}^\alpha := E = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}^E \qquad \mathcal{E}_\alpha := E^* = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1^E$$

and

$$\mathcal{E}^{\overline{\alpha}} := F = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}^F \qquad \mathcal{E}_{\overline{\alpha}} := F = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_1^F.$$

Correspondingly, sections of these vector bundles will be written by attaching the suitable index: we write ϕ^α for a section of $\mathcal{E}^\alpha$ and $\phi_{\overline{\alpha}}$ for a section of $\mathcal{E}_{\overline{\alpha}}$ and so on.
Tensor products of the above vector bundles will abbreviated by attaching for each factor occurring in the tensor product to $\mathcal{E}$ a suitable index: For instance

$$\mathcal{E}^{\alpha\beta} := E \otimes E$$
$$\mathcal{E}^\alpha{}_{\overline{\beta}} := E \otimes F^*.$$

Moreover, enclosing to indices by round brackets means symmetrisation

$$\mathcal{E}^{(\alpha\beta)} := S^2 E \qquad \mathcal{E}^{(\overline{\alpha}\overline{\beta})} := S^2 F.$$

Further, we abbreviate the tensor product of a tensor product above with a line bundle $\mathcal{E}[w, w']$ by omitting the tensor product sign and $\mathcal{E}$:
For instance we write

$$\mathcal{E}^\alpha[w, w'] := \mathcal{E}^\alpha \otimes \mathcal{E}[w, w'] \qquad \mathcal{E}^{\overline{\alpha}}[w, w'] := \mathcal{E}^{\overline{\alpha}} \otimes \mathcal{E}[w, w'].$$

and

$$\mathcal{E}^{\alpha\beta}[w, w'] := \mathcal{E}^{\alpha\beta} \otimes \mathcal{E}[w, w'].$$

Taking traces is indicated by using the same symbol for two indices, meaning that for a section $\phi_\alpha{}^\beta \in \Gamma(\mathcal{E}_\alpha{}^\beta)$ we will write $\phi_\alpha{}^\alpha \in C^\infty(M)$ for its trace. Observe that the Levi bracket can be seen as an invertible section $\mathcal{L}_{\alpha\overline{\beta}} \in \Gamma(\mathcal{E}_{\alpha\overline{\beta}}[1,1])$ and we denote its inverse by $\mathcal{L}^{\alpha\overline{\beta}} \in \Gamma(\mathcal{E}^{\alpha\overline{\beta}}[-1,-1])$. The Levi bracket $\mathcal{L}_{\alpha\overline{\beta}}$ induces an isomorphism $\mathcal{E}^\alpha \cong \mathcal{E}_{\overline{\alpha}}[1,1]$ and likewise $\mathcal{L}^{\alpha\overline{\beta}}$ induces an isomorphism $\mathcal{E}_\alpha \cong \mathcal{E}^{\overline{\alpha}}[-1,-1]$. Hence we can lower and raise indices at the expense of a weight.

4.2.3. An invariant operator for oriented Lagrangean contact structures. Suppose that we have given some parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G,P) with $G = SL(n+2,\mathbb{R})$ and $P \subset G$ as in the previous sections.

Let $\mathbb{T} := \mathbb{R}^{n+2}$ be the standard representation of G and denote by $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+2}$ the corresponding associated vector bundle, which is called the *standard tractor bundle.*

From the matrix description of $\mathfrak{g} = \mathfrak{g}_{-2} \oplus ... \oplus \mathfrak{g}_2$ in section 4.2.1 it follows immediately that the natural P-invariant filtration of $\mathbb{T}$ (see section 4.1.4) is of the form

$$\mathbb{T} = \mathbb{T}^0 \supset \mathbb{T}^1 \supset \mathbb{T}^2,$$

where the iterated quotients are the irreducible G_0-representations

$$\mathbb{T}/\mathbb{T}^0 = \mathbb{R}[0,1] \quad \mathbb{T}^1/\mathbb{T}^2 = \mathbb{E}[-1,0] \quad \text{ and } \quad \mathbb{T}^2 = \mathbb{R}[-1,0].$$

Therefore the standard tractor bundle admits a filtration

$$\mathcal{T} = \mathcal{T}^0 \supset \mathcal{T}^1 \supset \mathcal{T}^2$$

with associated graded vector bundle $\mathrm{gr}(\mathcal{T}) = \mathcal{G}_0 \times_{G_0} \mathbb{R}^{n+2}$ given by

$$\mathrm{gr}(\mathcal{T}) = \mathrm{gr}_0(\mathcal{T}) \oplus \mathrm{gr}_1(\mathcal{T}) \oplus \mathrm{gr}_2(\mathcal{T}) = \mathcal{E}[0,1] \oplus \mathcal{E}^\alpha[-1,0] \oplus \mathcal{E}[-1,0].$$

We deduce from the P-invariant filtration of $\mathbb{R}^{n+2}$ that the G-representation $\tilde{\mathbb{V}} := S^2\mathbb{R}^{n+2}$ admits a P-invariant filtration of the form

$$\tilde{\mathbb{V}} = \tilde{\mathbb{V}}^0 \supset \tilde{\mathbb{V}}^1 \supset \tilde{\mathbb{V}}^2 \supset \tilde{\mathbb{V}}^3 \supset \tilde{\mathbb{V}}^4,$$

where

$$\tilde{\mathbb{V}}/\tilde{\mathbb{V}}^1 = \mathbb{R}[0,2] \quad \tilde{\mathbb{V}}^1/\tilde{\mathbb{V}}^2 = \mathbb{E}[-1,1] \quad \tilde{\mathbb{V}}^2/\tilde{\mathbb{V}}^3 = S^2\mathbb{E}[-2,0] \oplus \mathbb{R}[-1,1]$$
$$\tilde{\mathbb{V}}^3/\tilde{\mathbb{V}}^4 = \mathbb{E}[-2,0] \quad \tilde{\mathbb{V}}^4 = \mathbb{R}[-2,0].$$

Hence we have a filtration of the corresponding vector bundle $\tilde{V} = \mathcal{G} \times_P \mathbb{W}$ given by

$$\tilde{V} = \tilde{V^0} \supset \tilde{V^1} \supset \tilde{V^2} \supset \tilde{V^3} \supset \tilde{V^4}$$

with associated graded bundle

$$\mathrm{gr}(\tilde{V}) = \mathrm{gr}_0(\tilde{V}) \oplus \mathrm{gr}_1(\tilde{V}) \oplus \mathrm{gr}_2(\tilde{V}) \oplus \mathrm{gr}_3(\tilde{V}) \oplus \mathrm{gr}_4(\tilde{V}) =$$

$$= \mathcal{E}[0,2] \oplus \mathcal{E}^\alpha[-1,1] \oplus (\mathcal{E}^{(\alpha\beta)}[-2,0] \oplus \mathcal{E}[-1,1]) \oplus \mathcal{E}^\alpha[-2,0] \oplus \mathcal{E}[-2,0].$$

Now consider the representation $\mathbb{V} = S^2\mathbb{R}^{n+2}[w, w'-2]$ of G given by the tensor product of $\tilde{\mathbb{V}}$ with the one dimensional representation $\mathbb{R}[w, w'-2]$. By the considerations above the corresponding vector bundle V admits a filtration

$$V = V^0 \supset V^1 \supset V^2 \supset V^3 \supset V^4$$

with associated graded vector bundle given by

$$\mathrm{gr}(V) = \mathrm{gr}_0(V) \oplus \mathrm{gr}_1(V) \oplus \mathrm{gr}_2(V) \oplus \mathrm{gr}_3(V) \oplus \mathrm{gr}_4(V) =$$

$$\mathcal{E}[w,w'] \oplus \mathcal{E}^\alpha[w-1,w'-1] \oplus (\mathcal{E}^{(\alpha\beta)}[w-2,w'-2] \oplus \mathcal{E}[w-1,w'-1]) \oplus \mathcal{E}^\alpha[w-2,w'-2] \oplus \mathcal{E}[w-2,w'-2].$$

We want now to construct an invariant operator for Lagrangean contact structures

$$L : \Gamma(\mathcal{E}[w,w']) \to \Gamma(\mathcal{E}[w-2,w'-2])$$

using the curved Casimir operator $\mathcal{C} : \Gamma(V) \to \Gamma(V)$ as explained in section 4.1.6. Therefore we compute the Casimir operator in terms of data associated to a Weyl structure, see proposition 4.10.
Choose a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. By corollary 4.8 the Weyl structure σ induces isomorphism between a natural vector bundle and its associated graded, hence in particular we have an isomorphism $\mathrm{gr}(V) \cong V$ and we will write sections of V as

$$\begin{pmatrix} \sigma \\ \mu^\alpha \\ R^{\alpha\beta} \quad | \quad \rho \\ \nu^\alpha \\ \theta \end{pmatrix},$$

with the convention that σ is a section of $\mathcal{E}[w,w']$ and so on. The Weyl connection $\sigma^*\omega_0$ induces a linear connection on V, which we denote by ∇. Moreover, we decompose the Rho-tensor $\mathsf{P} \in \Gamma(\mathrm{gr}(T^*M) \otimes \mathrm{gr}(T^*M))$ associated to σ with respect to decomposition

$$\mathrm{gr}(T^*M) = \mathrm{gr}_1(T^*M) \oplus \mathrm{gr}_2(T^*M) = (\mathcal{E}_\alpha \oplus \mathcal{E}_{\overline{\alpha}}) \oplus \mathcal{E}[-1,-1] :$$

the four components of homogeneity two are denoted by

$$A_{\alpha\beta} \in \Gamma(\mathcal{E}_{\alpha\beta}) \quad A_{\overline{\alpha}\overline{\beta}} \in \Gamma(\mathcal{E}_{\overline{\alpha}\overline{\beta}}) \quad \mathsf{P}_{\alpha\overline{\beta}} \in \Gamma(\mathcal{E}_{\alpha\overline{\beta}}) \quad \text{and} \quad \mathsf{P}_{\overline{\alpha}\beta} \in \Gamma(\mathcal{E}_{\overline{\alpha}\beta})$$

the four components of homogeneity three are denoted by

$$T_\alpha \in \Gamma(\mathcal{E}_\alpha[-1,-1]) \quad T_{\overline{\alpha}} \in \Gamma(\mathcal{E}_{\overline{\alpha}}[-1,-1])$$

$$S_\alpha \in \Gamma(\mathcal{E}[-1,-1] \otimes \mathcal{E}_\alpha) \quad S_{\overline{\alpha}} \in \Gamma(\mathcal{E}[-1,-1] \otimes \mathcal{E}_{\overline{\alpha}})$$

and the single component of homogeneity four is denote by

$$S \in \Gamma(\mathcal{E}[-2,-2]).$$

With regard to proposition 4.10 we have to determine the vector bundle map $\bullet : \mathrm{gr}(T^*M) \times V \to V$ induced from the action of $\mathfrak{p}_+$ on $S^2\mathbb{R}^{n+2}[w, w'-2]$. Observe that the action of $\mathfrak{p}_+$ on the standard representation $\mathbb{T} = \mathbb{R}^{n+2}$ is given by

$$\begin{pmatrix} 0 & Z & \gamma \\ 0 & 0 & W \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_2 \\ t_1 \\ t_0 \end{pmatrix} = \begin{pmatrix} Zt_1 + \gamma t_0 \\ Wt_0 \\ 0 \end{pmatrix}.$$

From this one deduces easily that

$$\phi_\alpha \bullet \begin{pmatrix} \sigma \\ \mu^\beta \\ R^{\beta\gamma} \quad | \quad \rho \\ \nu^\beta \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \quad | \quad \phi_\alpha \mu^\alpha \\ 2\phi_\alpha R^{\alpha\beta} \\ \phi_\alpha \nu^\alpha \end{pmatrix}$$

$$\phi_{\bar\alpha} \bullet \begin{pmatrix} \sigma \\ \mu^\beta \\ R^{\beta\gamma} \quad | \quad \rho \\ \nu^\beta \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 2\phi^\beta \sigma \\ \phi^{(\beta}\mu^{\gamma)} \quad | \quad 0 \\ \phi^\beta \rho \\ 0 \end{pmatrix}$$

$$\bar{s} \bullet \begin{pmatrix} \sigma \\ \mu^\beta \\ R^{\beta\gamma} \quad | \quad \rho \\ \nu^\beta \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \quad | \quad 2\bar{s}\sigma \\ \mu^\beta \bar{s} \\ \rho\bar{s} \end{pmatrix}$$

for sections $\phi_\alpha \in \Gamma(\mathcal{E}_\alpha)$, $\phi_{\overline{\alpha}} \in \Gamma(\mathcal{E}_{\overline{\alpha}})$ and $\bar{s} \in \Gamma(\mathcal{E}[-1,-1])$.
The chosen Weyl structure σ induces isomorphisms $TM \cong \mathrm{gr}(TM)$ and $\mathcal{A}M \cong \mathrm{gr}(\mathcal{A}M) = \mathrm{gr}(TM) \oplus \mathrm{gr}_0(\mathcal{A}M) \oplus \mathrm{gr}(T^*M)$. Via σ we can therefore view $TM \cong \mathrm{gr}(TM)$ as a subbundle of the adjoint tractor bundle $\mathcal{A}M$. Hence we can consider the restriction of the fundamental derivative D to $TM \cong \mathrm{gr}(TM) = (\mathcal{E}^\alpha \oplus \mathcal{E}^{\overline{\alpha}}) \oplus \mathcal{E}[1,1]$ and using proposition 4.9 we obtain :

$$D_\alpha \begin{pmatrix} \sigma \\ \mu^\beta \\ R^{\beta\gamma} \quad | \quad \rho \\ \nu^\beta \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_\alpha \sigma \\ \nabla_\alpha \mu^\beta + 2\mathsf{P}_\alpha{}^\beta \sigma \\ \nabla_\alpha R^{\beta\gamma} + \mathsf{P}_\alpha{}^{(\beta}\mu^{\gamma)} \quad | \quad \nabla_\alpha \rho + A_{\alpha\beta}\mu^\beta + 2T_\alpha \sigma \\ \nabla_\alpha \nu^\beta + 2A_{\alpha\gamma}R^{\gamma\beta} + \mathsf{P}_\alpha{}^\beta \rho + T_\alpha \mu^\beta \\ \nabla_\alpha \theta + A_{\alpha\beta}\nu^\beta + T_\alpha \rho \end{pmatrix}$$

$$D_{\overline{\alpha}}\begin{pmatrix} \sigma \\ \mu^\beta \\ R^{\beta\gamma} \quad | \quad \rho \\ \nu^\beta \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_{\overline{\alpha}}\sigma \\ \nabla_{\overline{\alpha}}\mu^\beta + 2A_{\overline{\alpha}}{}^\beta\sigma \\ \nabla_{\overline{\alpha}}R^{\beta\gamma} + A_{\overline{\alpha}}{}^{(\beta}\mu^{\gamma)} \quad | \quad \nabla_{\overline{\alpha}}\rho + \mathsf{P}_{\overline{\alpha}\beta}\mu^\beta + 2T_{\overline{\alpha}}\sigma \\ \nabla_{\overline{\alpha}}\nu^\beta + 2\mathsf{P}_{\overline{\alpha}\gamma}R^{\gamma\beta} + A_{\overline{\alpha}}{}^\beta\rho + T_{\overline{\alpha}}\mu^\beta \\ \nabla_{\overline{\alpha}}\theta + \mathsf{P}_{\overline{\alpha}\beta}\nu^\beta + T_{\overline{\alpha}}\rho \end{pmatrix}$$

$$D_s\begin{pmatrix} \sigma \\ \mu^\beta \\ R^{\beta\gamma} \quad | \quad \rho \\ \nu^\beta \\ \theta \end{pmatrix} = \begin{pmatrix} \nabla_s\sigma \\ \nabla_s\mu^\beta + 2S^\beta\sigma \\ \nabla_sR^{\beta\gamma} + S^{(\beta}\mu^{\gamma)} \quad | \quad \nabla_s\rho + S_\beta\mu^\beta + 2S\sigma \\ \nabla_s\nu^\beta + 2S_\gamma R^{\gamma\beta} + S^\beta\rho + S\mu^\beta \\ \nabla_s\theta + S_\beta\nu^\beta + S\rho \end{pmatrix}$$

where $D_\alpha(\_)$ (resp. $D_{\bar{\alpha}}(\_)$) denotes the fundamental derivative in direction of sections of $\mathcal{E}^\alpha$ (resp. $\mathcal{E}^{\bar{\alpha}}$) and $D_s(\_)$ denotes the fundamental derivative in direction of sections s of $\mathcal{E}[1,1]$.
According to the formula for the curved Casimir operator in proposition 4.10, the curved Casimir operator $\mathcal{C} : \Gamma(V) \to \Gamma(V)$ is given by the sum of the bundle map $\beta : V \to V$ with -2 times the action of the index α (resp. $\overline{\alpha}$, resp. s) on $D_\alpha(v)$ (resp. $D_{\overline{\alpha}}(v)$, resp. $D_s(v)$).

Proposition 4.13. *Let $(\mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P) and suppose that V is the natural vector bundle associated to the G representation $S^2(\mathbb{R}^{n+2})[w, w'-2]$. Choosing some Weyl structure $\mathcal{G}_0 \to \mathcal{G}$ the curved Casimir operator $\mathcal{C} : \Gamma(V) \to \Gamma(V)$ is given by*

$$\mathcal{C}\begin{pmatrix} \sigma \\ \mu^\alpha \\ R^{\alpha\beta} \quad | \quad \rho \\ \nu^\alpha \\ \theta \end{pmatrix} =$$

$$= \begin{pmatrix} \beta_0\sigma \\ \beta_1\mu^\alpha - 4\nabla^\alpha\sigma \\ \beta_2^1R^{\alpha\beta} - 2\nabla^{(\alpha}\mu^{\beta)} - 4A^{(\alpha\beta)}\sigma \quad | \quad \beta_2^2\rho - 2\nabla_\alpha\mu^\alpha - 4(P_\alpha{}^\alpha\sigma + \nabla_s\sigma) \\ \beta_3\nu^\alpha - 4\nabla_\beta R^{\beta\alpha} - 2\nabla^\alpha\rho - 4P_\beta{}^{(\beta}\mu^{\alpha)} - 2P^\alpha{}_\beta\mu^\beta - 2\nabla_s\mu^\alpha - 4(T^\alpha\sigma + S^\alpha\sigma) \\ \beta_4\theta - 2\nabla_\alpha\nu^\alpha - 4A_{\alpha\beta}R^{\alpha\beta} - 2(\nabla_s\rho + P_\alpha{}^\alpha\rho) - 2(T_\alpha\mu^\alpha + S_\alpha\mu^\alpha) - 4S\sigma \end{pmatrix},$$

where $\beta_0, \beta_1, \beta_2^1, \beta_2^2, \beta_3, \beta_4$ are the Casimir eigenvalues of the irreducible bundles occurring in $\mathrm{gr}(V)$.

It remains to determine the Casimir eigenvalues $\beta_0, \beta_1, \beta_2^1, \beta_2^2, \beta_3, \beta_4$ of the irreducible bundles occurring in $\mathrm{gr}(V)$. By proposition 4.5 the Casimir eigenvalue $\beta_{\mathbb{W}}$ of an irreducible representation $\mathbb{W}$ of G_0 is given by

$$< \lambda, \lambda + 2\rho >,$$

where λ is the highest weight of the dual representation $\mathbb{W}^*$ and $\rho = \omega_1 + ... + \omega_{n+1} = \sum_{i=1}^{n+2} \frac{1}{2}(n+1-2(i-1))\epsilon_i$ is the lowest form. The easiest way to compute the Casimir eigenvalues is to express all weights in terms of the functionals ϵ_i, which are orthogonal with respect to the trace form of $\mathfrak{sl}(n+2,\mathbb{R})$, then to calculate the inner product of λ and $\lambda+2\rho$ with respect to the trace form and to use the well known fact that the Killing form $<,>$ is just $2(n+2)$ times the trace form. Doing so, one directly calculates:

Proposition 4.14. *The Casimir eigenvalues corresponding to the irreducible components of* $\mathrm{gr}(V)$ *are given by*

$$\beta_0 = 2((n+1)w'^2 + w'(2w + (n+1)(n+2)) + (n+1)w(w+n+2))$$

$$\beta_1 = 2((n+1)w'^2 + w'(2w + (n-1)(n+2)) + (n+1)w(w+n+2))$$

$$\beta_2^1 = 2((n+1)w'^2+w'(2w+(n-3)(n+2))+(n+1)w^2+(n+1)(n+2)w+4(n+2))$$

$$\beta_2^2 = 2((n+1)w'^2+w'(2w+(n-1)(n+1))+(n+1)w^2+(n-1)(n-2)w-2n(n+2))$$

$$\beta_3 = 2((n+1)w'^2+w'(2w+(n-3)(n+2))+(n+1)w^2+(n-1)(n+2)w-2(n-1)(n+2))$$

$$\beta_4 = 2((n+1)w'^2+w'(2w+(n-3)(n+2))+(n+1)w^2+(n-3)(n+2)w-4(n-1)(n+2)).$$

For the differences $c_i^j := \beta_0 - \beta_i^j$ *one has*

$$c_1 = 4(n+2)w'$$

$$c_2^1 = 8(n+2)(w'-1)$$

$$c_2^2 = 4(n+2)(w+w'+n)$$

$$c_3 = 4(n+2)(2w'+w+n-1)$$

$$c_4 = 8(n+2)(w'+w+n-1).$$

Now consider the differential operator

$$(\mathcal{C}-\beta_4)(\mathcal{C}-\beta_3)(\mathcal{C}-\beta_2^1)(\mathcal{C}-\beta_2^2)(\mathcal{C}-\beta_1) : \Gamma(V) \to \Gamma(V). \tag{4.2}$$

As explained in section 4.1.6 the operator (4.2) induces an invariant differential operator from the top slot $\mathcal{E}[w,w']$ to the bottom slot $\mathcal{E}[w-2,w'-2]$, if $c_4 = 0$. For $w'+w+n-1 = 0$ we therefore obtain a differential operator

$$L : \Gamma(\mathcal{E}[w,w']) \to \Gamma(\mathcal{E}[w-2,w'-2]).$$

Since the filtration of V is of the form $V \supset V^1 \supset ... \supset V^4$, the differential operator L is of weighted order at most 4. However, it is not apparent from the construction, if it is zero or not. Therefore we will in the sequel compute its weighted symbol.

The Weyl connection induces linear connections on all natural vector bundles, in particular also on $T^*M \cong \mathrm{gr}(T^*M)$, therefore we can from iterated covariant derivatives.

Lemma 4.15. *For a section $\sigma \in \mathcal{E}[w, w']$ we have*

$$\nabla_\alpha \nabla_\beta \sigma - \nabla_\beta \nabla_\alpha \sigma \equiv 0 \bmod(\text{terms of weighted order} \leq 1)$$
$$\nabla_\alpha \nabla_s \sigma - \nabla_s \nabla_\alpha \sigma \equiv 0 \bmod(\text{terms of weighted order} \leq 2)$$
$$\nabla_\alpha \nabla_{\overline{\beta}} \sigma - \nabla_{\overline{\beta}} \nabla_\alpha \sigma \equiv \mathcal{L}_{\alpha\overline{\beta}} \nabla_s \sigma \bmod(\text{terms of weighted order} \leq 1).$$

Moreover, we have $\mathcal{L}(\phi^\alpha, \phi^{\overline{\beta}}) = -\phi^\alpha \phi_\alpha s$ and so $\mathcal{L}_{\alpha\overline{\beta}} = -\delta_\alpha{}^\beta s$ respectively $\mathcal{L}_{\overline{\alpha}\beta} = \delta^\alpha{}_\beta s$.

Proof. One always has $\nabla^2\sigma(\xi, \eta) = \nabla_\xi \nabla_\eta \sigma - \nabla_{\nabla_\xi \eta}\sigma$ and therefore we obtain

$$\nabla^2\sigma(\xi, \eta) - \nabla^2\sigma(\eta, \xi) = \nabla_\xi \nabla_\eta \sigma - \nabla_{\nabla_\xi \eta}\sigma - \nabla_\eta \nabla_\xi \sigma + \nabla_{\nabla_\eta \xi}\sigma =$$
$$= R(\xi, \eta)\sigma + \nabla_{[\xi,\eta]}\sigma + \nabla_{\nabla_\eta \xi}\sigma - \nabla_{\nabla_\xi \eta}\sigma.$$

This equation immediately implies the commutator relations above.
The last statement follows directly from the fact that the restriction of the Lie bracket to $\mathfrak{g}^E_{-1} \times \mathfrak{g}^F_{-1} \to \mathfrak{g}_{-2}$ is given by the negative of the standard symplectic form on $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^{n*}$, see section 4.2.1. □

We now compute the weighted symbol of the differential operator L. Note that for $w' + w + n - 1 = 0$ we obtain

$$\begin{pmatrix} & c_0 & \\ & c_1 & \\ c_2^1 & \mid & c_2^2 \\ & c_3 & \\ & c_4 & \end{pmatrix} = \begin{pmatrix} & 0 & \\ & 4(n+2)w' & \\ 8(n+2)(w'-1) & \mid & 4(n+2) \\ & 4(n+2)w' & \\ & 0 & \end{pmatrix}.$$

In particular, we have that $\beta_0 = \beta_4$ and $\beta_1 = \beta_3$. We know that all factors in (4.2) commute. To compute the weighted symbol of L the most convenient way is to apply the factors in (4.2) in the opposite order. Since we are just interested in the highest order part of L, we will freely rearrange terms in accordance with lemma 4.15.
Since $\beta_0 = \beta_4$, applying $(\mathcal{C} - \beta_4)$ to a section $\sigma \in \Gamma(\mathcal{E}[w, w'])$ we obtain:

$$(\mathcal{C}-\beta_4)\begin{pmatrix} & \sigma & \\ & 0 & \\ 0 & \mid & 0 \\ & 0 & \\ & 0 & \end{pmatrix} = \begin{pmatrix} & 0 & \\ & -4\nabla^\alpha\sigma & \\ 0 & \mid & -4\nabla_s\sigma \\ & 0 & \\ & 0 & \end{pmatrix} + \begin{matrix}\text{terms of lower weighted order} \\ \text{in each component.}\end{matrix}$$

Since $\beta_1 = \beta_3$, applying $(\mathcal{C} - \beta_3)$ to the result leads to

$$\begin{pmatrix} 0 \\ 0 \\ 8\nabla^{(\alpha}\nabla^{\beta)}\sigma \quad | \quad -4(\beta_2^2 - \beta_3)\nabla_s\sigma + 8\nabla_\alpha\nabla^\alpha\sigma \\ 16\nabla^\alpha\nabla_s\sigma \\ 8\nabla_s\nabla_s\sigma \end{pmatrix} + \begin{matrix}\text{terms of lower weighted order} \\ \text{in each component.}\end{matrix}$$

Applying $(\mathcal{C} - \beta_2^2)$ we obtain

$$\begin{pmatrix} 0 \\ 0 \\ 8(\beta_2^1 - \beta_2^2)\nabla^{(\alpha}\nabla^{\beta)}\sigma \quad | \quad 0 \\ 8(\beta_3 - \beta_2^2)\nabla^\alpha\nabla_s\sigma - 32\nabla_\beta\nabla^{(\beta}\nabla^{\alpha)}\sigma - 16\nabla^\alpha\nabla_\beta\nabla^\beta\sigma \\ 8(\beta_4 - \beta_3)\nabla_s\nabla_s\sigma - 48\nabla_\alpha\nabla^\alpha\nabla_s\sigma \end{pmatrix} + \begin{matrix}\text{terms of lower weighted order} \\ \text{in each component.}\end{matrix}$$

Applying $(\mathcal{C} - \beta_2^1)$ leads to

$$\begin{pmatrix} 0 \\ 0 \\ 0 \quad | \quad 0 \\ 8(\beta_3 - \beta_2^1)(\beta_3 - \beta_2^2)\nabla^\alpha\nabla_s\sigma - 32(\beta_3 - \beta_2^2)\nabla_\beta\nabla^{(\beta}\nabla^{\alpha)}\sigma - 16(\beta_3 - \beta_2^1)\nabla^\alpha\nabla_\beta\nabla^\beta\sigma \\ 8(\beta_4 - \beta_2^1)(\beta_4 - \beta_3)\nabla_s\nabla_s\sigma - 16(3\beta_4 - 3\beta_2^1 + \beta_3 - \beta_2^2)\nabla_\alpha\nabla^\alpha\nabla_s\sigma + 64\nabla_\alpha\nabla_\beta\nabla^{(\beta}\nabla^{\alpha)}\sigma + \\ +32\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta\sigma \end{pmatrix}$$

$$+\text{terms of lower weighted order in each component.}$$

Applying $(\mathcal{C} - \beta_3)$, we obtain in the bottom slot

$$64c_2^2\nabla_\alpha\nabla_\beta\nabla^{(\beta}\nabla^{\alpha)}\sigma + 32c_2^1\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta\sigma - 16[(c_2^1 - c_3)(c_2^2 - c_3) + c_3(3c_2^1 + c_2^2 - c_3)]\nabla_\alpha\nabla^\alpha\nabla_s\sigma +$$
$$+8c_3c_3c_2^1\nabla_s\nabla_s\sigma + \text{ terms of weighted lower order} =$$
$$= 64c_2^2\nabla_\alpha\nabla_\beta\nabla^{(\beta}\nabla^{\alpha)}\sigma + 32c_2^1\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta\sigma - 16c_2^1(c_2^2 + 2c_3)\nabla_\alpha\nabla^\alpha\nabla_s\sigma +$$
$$+8c_3c_3c_2^1\nabla_s\nabla_s\sigma + \text{ terms of weighted lower order.}$$

By lemma 4.15 we have

$$32c_2^1\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta\sigma =$$
$$= 32c_2^1\nabla_\alpha\nabla_\beta\nabla^{(\alpha}\nabla^{\beta)}\sigma + 32c_2^1\nabla_\alpha\mathcal{L}^\alpha{}_\beta\nabla_s\nabla^\beta\sigma + \text{ terms of lower weighted order}$$
$$= 32c_2^1\nabla_\alpha\nabla_\beta\nabla^{(\alpha}\nabla^{\beta)}\sigma + 32c_2^1\nabla_\alpha\nabla_s\nabla^\alpha\sigma + \text{ terms of lower weighted order}$$
$$= 32c_2^1\nabla_\alpha\nabla_\beta\nabla^{(\alpha}\nabla^{\beta)}\sigma + 32c_2^1\nabla_\alpha\nabla^\alpha\nabla_s\sigma + \text{ terms of lower weighted order.}$$

and therefore the term of highest weighted order of L is given by

$$32(2c_2^2+c_2^1)\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta\sigma-16(c_2^1c_2^2+2c_2^1c_3-2c_2^1)\nabla_\alpha\nabla^\alpha\nabla_s\sigma+8c_3c_3c_2^1\nabla_s\nabla_s\sigma =$$

$$\begin{aligned}
&= 128(n+2)w'\nabla_\alpha\nabla_\beta\nabla^{(\beta}\nabla^{\alpha)}\sigma\\
&- 512(n+2)^2(2w'^2 - \frac{w'+2(n+2)w'}{2(n+2)} + \frac{1-2(n+2)}{2(n+2)})\nabla_\alpha\nabla^\alpha\nabla_s\sigma\\
&+ 1024(n+2)^3w'^2(w'-1)\nabla_s\nabla_s\sigma
\end{aligned} \tag{4.3}$$

We conclude that L is always non-zero, since the highest order term never vanishes completely. Note that the usual principal symbol vanishes for $w'=0$ and so it was in fact crucial to compute the weighted symbol of L. Summing up, we have the following theorem:

Theorem 4.16. *For $w+w'+n-1=0$ there is an invariant differential operator of weighted order four*

$$L : \mathcal{E}[w,w'] \to \mathcal{E}[w-2,w'-2]$$

whose weighted symbol $\mathcal{U}_{-4}(\mathrm{gr}(TM))^* \otimes \mathcal{E}[w,w'] \to \mathcal{E}[w-2,w'-2]$ is induced by the appropriate multiples (see (4.3)) of the contractions

$$S^2\mathfrak{g}_1^E \otimes S^2\mathfrak{g}_1^F \otimes \mathbb{R}[w,w'] \to \mathbb{R}[w-2,w'-2]$$
$$\mathfrak{g}_1^E \otimes \mathfrak{g}_1^F \otimes \mathfrak{g}_2 \otimes \mathbb{R}[w,w'] \to \mathbb{R}[w-2,w'-2]$$

and the appropriate multiple (see (4.3)) of the identity

$$\mathfrak{g}_2 \otimes \mathfrak{g}_2 \otimes \mathbb{R}[w,w'] = \mathbb{R}[w-2,w'-2] \to \mathbb{R}[w-2,w'-2].$$

Now we may also consider the dual representation $\mathbb{T}^* = (\mathbb{R}^{n+2})^*$ of the standard representation of $SL(n+2,\mathbb{R})$. From the filtration of the standard representation we deduce that the dual representation admits filtration of the form

$$\mathbb{T}^* = (\mathbb{T}^*)^0 \supset (\mathbb{T}^*)^1 \supset (\mathbb{T}^*)^2$$

where

$$(\mathbb{T}^*)^0/(\mathbb{T}^*)^1 = \mathbb{R}[1,0] \quad (\mathbb{T}^*)^1/(\mathbb{T}^*)^2 = \mathbb{F}[0,-1] \quad (\mathbb{T}^*)^2 = \mathbb{R}[0,-1].$$

Therefore the corresponding vector bundle $\mathcal{T}^*$ has a filtration of the form

$$\mathcal{T}^* = \mathcal{T}^{*0} \supset \mathcal{T}^{*1} \supset \mathcal{T}^{*2}$$

with associated graded vector bundle $\mathrm{gr}(\mathcal{T}) = \mathcal{G}_0 \times_{G_0} \mathbb{T}^*$ given by

$$\mathrm{gr}(\mathcal{T}^*) = \mathrm{gr}_0(\mathcal{T}^*) \oplus \mathrm{gr}_1(\mathcal{T}^*) \oplus \mathrm{gr}_2(\mathcal{T}^*) = \mathcal{E}[1,0] \oplus \mathcal{E}^{\overline{\alpha}}[0,-1] \oplus \mathcal{E}[0,-1].$$

We conclude that the natural vector bundle V corresponding to the representation $\mathbb{W} = S^2\mathbb{R}^{n+2*}[w-2,w']$ admits a filtration

$$W = W^0 \supset W^1 \supset W^2 \supset W^3 \supset W^4$$

with associated graded vector bundle given by

$$\mathrm{gr}(W) = \mathrm{gr}_0(W) \oplus \mathrm{gr}_1(W) \oplus \mathrm{gr}_2(W) \oplus \mathrm{gr}_3(W) \oplus \mathrm{gr}_4(W) =$$

$$\mathcal{E}[w,w'] \oplus \mathcal{E}^{\overline{\alpha}}[w-1,w'-1] \oplus (\mathcal{E}^{(\overline{\alpha}\overline{\beta})}[w-2,w'-2] \oplus \mathcal{E}[w-1,w'-1]) \oplus \mathcal{E}^{\overline{\alpha}}[w-2,w'-2] \oplus \mathcal{E}[w-2,w'-2].$$

Since $\mathcal{E}[w,w']$ and $\mathcal{E}[w-2,w'-2]$ occur as well as irreducible components of $\mathrm{gr}(W)$, we can use also the Casimir operator on W to construct an invariant operator

$$\overline{L} : \mathcal{E}[w,w'] \to \mathcal{E}[w-2,w'-2].$$

Choosing a Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, we identify W and $\mathrm{gr}(W)$ and write sections of $\mathrm{gr}(W)$ as

$$\begin{pmatrix} \sigma \\ \mu^{\overline{\alpha}} \\ R^{\overline{\alpha}\overline{\beta}} \quad | \quad \rho \\ \nu^{\overline{\alpha}} \\ \theta \end{pmatrix},$$

with the convention that σ is a section of $\mathcal{E}[w,w']$.
From the action of $\mathfrak{p}_+$ on $\mathbb{R}^{n+2^*}$ one easily deduces that the bundle map $\bullet : \mathrm{gr}(T^*M) \times W \to W$ is given by:

$$\phi_\alpha \bullet \begin{pmatrix} \sigma \\ \mu^{\overline{\beta}} \\ R^{\overline{\beta}\overline{\gamma}} \quad | \quad \rho \\ \nu^{\overline{\beta}} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ -2\phi^{\overline{\beta}}\sigma \\ -\phi^{(\overline{\beta}}\mu^{\overline{\gamma})} \quad | \quad 0 \\ -2\phi^{\overline{\beta}}\rho \\ 0 \end{pmatrix}$$

$$\phi_{\bar{\alpha}} \bullet \begin{pmatrix} \sigma \\ \mu^{\bar{\beta}} \\ R^{\bar{\beta}\bar{\gamma}} \quad | \quad \rho \\ \nu^{\bar{\beta}} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \quad | \quad -\phi_{\bar{\alpha}}\mu^{\bar{\alpha}} \\ -2\phi_{\bar{\alpha}}R^{\bar{\alpha}\bar{\beta}} \\ -\phi_{\bar{\alpha}}\nu^{\bar{\alpha}} \end{pmatrix}$$

$$\overline{s} \bullet \begin{pmatrix} \sigma \\ \mu^{\bar{\beta}} \\ R^{\bar{\beta}\bar{\gamma}} \quad | \quad \rho \\ \nu^{\bar{\beta}} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \quad | \quad -2\overline{s}\sigma \\ -\overline{s}\mu^{\bar{\beta}} \\ -\overline{s}\rho \end{pmatrix}$$

for sections $\phi_\alpha \in \Gamma(\mathcal{E}_\alpha)$, $\phi_{\overline{\alpha}} \in \Gamma(\mathcal{E}_{\overline{\alpha}})$ and $\overline{s} \in \Gamma(\mathcal{E}[-1,-1])$.
The Casimir operator $\mathcal{C} : \Gamma(W) \to \Gamma(W)$ on W is now computed as before and we obtain:

Proposition 4.17. *Let $(\mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P) and suppose that W is the natural vector bundle associated to the G representation $S^2(\mathbb{R}^{n+2})^*[w-2, w']$. Choosing some Weyl structure $\mathcal{G}_0 \to \mathcal{G}$ the curved Casimir operator $\mathcal{C} : \Gamma(W) \to \Gamma(W)$ is given by*

$$\mathcal{C}\begin{pmatrix} \sigma \\ \mu^{\bar{\alpha}} \\ R^{\bar{\alpha}\bar{\beta}} \quad | \quad \rho \\ \nu^{\bar{\alpha}} \\ \theta \end{pmatrix} =$$

$$= \begin{pmatrix} \beta_0 \sigma \\ \beta_1 \mu^{\bar{\alpha}} + 4\nabla^{\bar{\alpha}}\sigma \\ \beta_2^1 R^{\bar{\alpha}\bar{\beta}} + 2\nabla^{(\bar{\alpha}}\mu^{\bar{\beta})} - 4A^{(\bar{\alpha}\bar{\beta})}\sigma \quad | \quad \beta_2^2\rho + 2\nabla_{\bar{\alpha}}\mu^{\bar{\alpha}} - 4(\mathsf{P}_{\bar{\alpha}}{}^{\bar{\alpha}}\sigma - \nabla_s\sigma) \\ \beta_3\nu^{\bar{\alpha}} + 4\nabla_{\bar{\beta}}R^{\bar{\beta}\bar{\alpha}} + 2\nabla^{\bar{\alpha}}\rho - 4\mathsf{P}_{\bar{\beta}}{}^{(\bar{\beta}}\mu^{\bar{\alpha})} - 2\mathsf{P}^{\bar{\alpha}}{}_{\bar{\beta}}\mu^{\bar{\beta}} + 2\nabla_s\mu^{\bar{\alpha}} - 4(T^{\bar{\alpha}}\sigma + S^{\bar{\alpha}}\sigma) \\ \beta_4\theta + 2\nabla_{\bar{\alpha}}\nu^{\bar{\alpha}} - 4A_{\bar{\alpha}\bar{\beta}}R^{\bar{\alpha}\bar{\beta}} + 2(\nabla_s\rho - \mathsf{P}_{\bar{\alpha}}{}^{\bar{\alpha}}\rho) - 2(T_{\bar{\alpha}}\mu^{\bar{\alpha}} + S_{\bar{\alpha}}\mu^{\bar{\alpha}}) - 4S\sigma \end{pmatrix},$$

where $\gamma_0, \gamma_1, \gamma_2^1, \gamma_2^2, \gamma_3, \gamma_4$ are the Casimir eigenvalues of the irreducible bundles occurring in $\mathrm{gr}(W)$.

The Casimir eigenvalues of the irreducible bundles occurring in $\mathrm{gr}(W)$ are computed straightforward.

Proposition 4.18. *The Casimir eigenvalues corresponding to the irreducible components of* $\mathrm{gr}(W)$ *are given by*

$$\gamma_0 = 2((n+1)w'^2 + w'(2w + (n+1)(n+2)) + (n+1)w(w+n+2))$$

$$\gamma_1 = 2((n+1)w'^2 + w'(2w + (n+1)(n+2)) + w((n+1)w + (n-1)(n+2))$$

$$\gamma_2^1 = 2((n+1)w'^2 + w'(2w+(n+1)(n+2)) + (n+1)w^2 + (n-3)(n+2)w + 4(n+2))$$

$$\gamma_2^2 = 2((n+1)w'^2 + w'(2w+(n-1)(n+1)) + (n+1)w^2 + (n-1)(n-2)w - 2n(n+2))$$

$$\gamma_3 = 2((n+1)w'^2 + w'(2w+n^2+n-2) + (n+1)w^2 + (n^2-n-6)w - 2(n-1)(n+2))$$

$$\gamma_4 = 2((n+1)w'^2 + w'(2w+(n-3)(n+2)) + (n+1)w^2 + (n-3)(n+2)w - 4(n-1)(n+2)).$$

For the differences $c_i^j := \gamma_0 - \gamma_i^j$ one has

$$c_1 = 4(n+2)w$$

$$c_2^1 = 8(n+2)(w-1)$$

$$c_2^2 = 4(n+2)(w+w'+n)$$

$$c_3 = 4(n+2)(w'+2w+n-1)$$

$$c_4 = 8(n+2)(w'+w+n-1).$$

Again, for $w + w' + n - 1 = 0$ the invariant differential operator

$$(\mathcal{C} - \gamma_4)(\mathcal{C} - \gamma_3)(\mathcal{C} - \gamma_2^1)(\mathcal{C} - \gamma_2^2)(\mathcal{C} - \gamma_1) : \Gamma(W) \to \Gamma(W)$$

may induces an invariant differential operator of weighted order four

$$\bar{L} : \mathcal{E}[w, w'] \to \mathcal{E}[w - 2, w' - 2].$$

To see that it is in fact nonzero we compute in the same way as before its weighted symbol. The highest order term of $\bar{L}$ turns out to be given by

$$64c_2^2\nabla_{\bar{\alpha}}\nabla_{\bar{\beta}}\nabla^{(\bar{\beta}}\nabla^{\bar{\alpha})}\sigma + 32c_2^1\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}\nabla_{\bar{\beta}}\nabla^{\bar{\beta}}\sigma + 16c_2^1(c_2^2 + 2c_3)\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}\nabla_s\sigma +$$
$$+ 8c_3c_3c_2^1\nabla_s\nabla_s\sigma.$$

Using lemma 4.15, this equals

$$32(2c_2^2 + c_2^1)\nabla_{\bar{\alpha}}\nabla_{\bar{\beta}}\nabla^{(\bar{\beta}}\nabla^{\bar{\alpha})}\sigma + 16(c_2^1c_2^2 + 2c_2^1c_3 - 2c_2^1)\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}\nabla_s\sigma + 8c_3c_3c_2^1\nabla_s\nabla_s\sigma. \tag{4.4}$$

For $w + w' + n - 1 = 0$ the differences of the Casimir eigenvalues are

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2^1 \quad | \quad c_2^2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4(n+2)w \\ 8(n+2)(w-1) \quad | \quad 4(n+2) \\ 4(n+2)w \\ 0 \end{pmatrix}.$$

and inserting them into (4.4) we obtain that the highest order term of $\bar{L}$ in the weighted sense is given by

$$\begin{aligned} &128(n+2)w\nabla_{\bar{\alpha}}\nabla_{\bar{\beta}}\nabla^{(\bar{\beta}}\nabla^{\bar{\alpha})}\sigma \\ &+ 512(n+2)^2(2w^2 - \frac{w + 2(n+2)w}{2(n+2)} + \frac{1 - 2(n+2)}{2(n+2)})\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}\nabla_s\sigma \\ &+ 1024(n+2)^3w^2(w-1)\nabla_s\nabla_s\sigma. \end{aligned} \tag{4.5}$$

Therefore we have the following theorem:

Theorem 4.19. *For $w + w' + n - 1 = 0$ there is an invariant differential operator of weighted order four*

$$\bar{L} : \mathcal{E}[w, w'] \to \mathcal{E}[w - 2, w' - 2]$$

whose weighted symbol $\mathcal{U}_{-4}(\mathrm{gr}(TM))^* \otimes \mathcal{E}[w, w'] \to \mathcal{E}[w-2, w'-2]$ is induced by the appropriate multiples (see (4.5)) of the contractions

$$S^2\mathfrak{g}_1^F \otimes S^2\mathfrak{g}_1^E \otimes \mathbb{R}[w, w'] \to \mathbb{R}[w - 2, w' - 2]$$

$$\mathfrak{g}_1^F \otimes \mathfrak{g}_1^E \otimes \mathfrak{g}_2 \otimes \mathbb{R}[w, w'] \to \mathbb{R}[w - 2, w' - 2]$$

and the appropriate multiple (see (4.5)) of the identity

$$\mathfrak{g}_2 \otimes \mathfrak{g}_2 \otimes \mathbb{R}[w, w'] = \mathbb{R}[w - 2, w' - 2] \to \mathbb{R}[w - 2, w' - 2].$$

Using lemma 4.15 we can write the highest order term of L (4.3) respectively of $\bar{L}$ (4.5) also as

$$
\begin{aligned}
&= 128(n+2)w'\nabla_\alpha\nabla^\alpha\nabla_\beta\nabla^\beta\sigma \\
&- 512(n+2)^2(2w'^2 - w' + \frac{1-2(n+2)}{2(n+2)})\nabla_\alpha\nabla^\alpha\nabla_s\sigma \\
&+ 1024(n+2)^3 w'^2(w'-1)\nabla_{s'}\nabla_s\sigma
\end{aligned} \tag{4.6}
$$

respectively as

$$
\begin{aligned}
&= 128(n+2)w\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}\nabla_{\bar{\beta}}\nabla^{\bar{\beta}}\sigma \\
&+ 512(n+2)^2(2w^2 - w + \frac{1-2(n+2)}{2(n+2)})\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}\nabla_s\sigma \\
&+ 1024(n+2)^3 w^2(w-1)\nabla_{s'}\nabla_s\sigma.
\end{aligned} \tag{4.7}
$$

From (4.6) and (4.7) we see that at least for certain weights w resp. w' the invariant operators L and $\bar{L}$ give rise to invariant differential operators, whose usual principal symbol coincides with the usual principal symbol of the Sub-Laplacian $\Delta = -(\nabla_\alpha\nabla^\alpha + \nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}})$.
The results of this chapter are closely related to the article **[24]** of Gover and Graham, where the authors construct CR-invariant powers of Sub-Laplacians on manifolds endowed with partially integrable almost CR-manifolds. Further studies will be needed to understand the precise relation of the results of this chapter to the ones in **[24]**.

In summary, we have seen in this chapter that curved Casimir operators provide an efficient tool to construct invariant operators for parabolic geometries. The only problem is that it is not apparent from the construction that the obtained operator is nonzero. The examples above show that to prove that a constructed differential operator is in fact nonzero, one has to compute the weighted symbol rather than the usual principal symbol of the operator. In the case of $|1|$-graded parabolic geometries Čap and Gover worked out in a forthcoming article a uniform method to compute the principal symbol of an invariant operator constructed via curved Casimir operators. It should be possible to adapt this method to compute the weighted symbol of an invariant operator constructed via Casimir operators in the case of general $|k|$-graded parabolic geometries.

Bibliography

[1] Beals, R., Greiner, P.C.: *Calculus on Heisenberg manifolds.* Annals of Math. Studies, vol. 119. Princeton Univ. Press, Princeton NJ. 1988.

[2] Bernstein, I.N. , Gelfand, I.M., Gelfand, I.S.: *Differential operators on the base affine space and a study of g-modules.* In: Lie Groups and their Representations. Editor I.M. Gelfand, Adam Hilger. 1975. p. 21-64.

[3] Biquard, O.: *Métriques d'Einstein asymptotiquement symétriques.* Astérisque 265, 2000.

[4] Branson, T., Čap, A., Eastwood, M., Gover, A.R.: *Prolongations of Geometric Overdetermined Systems.* Intern. Jour. of Math. vol. 17. no. 6. 2006. p. 641-664.

[5] Bryant, R.: *Conformal geometry and 3-plane fields on 6-manifolds.* In: RIMS Symposium Proceedings: Developments of Cartan Geometry and Related Mathematical Problems, vol. 1502 . July 2006. p. 1-15.

[6] Calderbank, D.M.J., Diemer, T.: *Differential invariants and curved Bernstein-Gelfand-Gelfand sequences.* J. Reine Angew. Math. 537. 2001. p. 67-103.

[7] Čap, A.: *Two constructions with parabolic geometries.* In:*Proceedings of the 25th Winter School on Geometry and Physics, Srni 2005.* Rend. Circ. Mat. Palermo Suppl. ser. II, 79. 2006. p.11-37

[8] Čap, A.: *Overdetermined Systems, Conformal Differential Geometry and the BGG Complex.* In: M.G. Eastwood, W. Miller (eds.): *Symmetries and Overdetermined Systems of Partial Differential Equations.* The IMA Volumes in Mathematics and its Applications 144, Springer 2008.

[9] Čap, A., Gover, A.R.: *Tractor calculi for parabolic geometries.* Trans. Amer. Math. Soc. 354. no. 4. 2002. p. 1511-1548.

[10] Čap, A., Gover, A.R., Souček, V.: *Conformally invariant operators via curved Casimirs: examples.* To appear in Pure Appl. Math. Q.

[11] Čap, A., Neusser, K.: *On automorphism groups of certain types of generic distributions.* Differential Geom. Appl. 27. 2009. p. 769-779.

[12] Čap, A., Schichl, H.: *Parabolic Geometries and Canonical Cartan Connections.* Hokkaido Math. J. 29 No.3 .2000. p. 453-505.

[13] Čap, A., Slovák, J., Souček, V.: *Bernstein-Gelfand-Gelfand sequences.* Annals of Math. 154. 2001. p. 97-113.

[14] Čap, A., Slovák, J.: *Weyl structures for parabolic geometries.* Math. Scand. 93. 2003. p. 53-90.

[15] Čap, A., Slovák, J.: *Parabolic Geometries I: Background and General Theory.* Mathematical Surveys and Monographs Vol. 154, American Mathematical Society.

[16] Čap, A., Souček, V.: *Curved Casimir operators and the BGG machinery.* SIGMA 3. 2007.

[17] Cartan, É.: *Les systèmes de Pfaff à cinq variables et les équations aux derivées partielles du second order.* Ann. École Normale 27. 1910. p. 263-355.

[18] Dixmier, J.: *Enveloping algebras.* Graduate Studies in Mathematics, Vol. 11. American Mathematical Society.

[19] van Erp, E.: *The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part 1.* To appear in Annals of Math. Available as preprint arXiv: 0804.2490.

[20] Eastwood, M.: *Prolongation of Linear Overdetermined Systems on Affine and Riemannian Manifolds.* Rend. Circ. Mat. Palermo Suppl. 75. 2005. p. 89-108.

[21] Eastwood, M., Gover, A.R.: *Prolongation on Contact Manifolds.* Available as preprint arXiv: 0910.5519.

[22] Goldschmidt, H.: *Existence theorems for analytic linear partial differential equations.* Annals of Math., vol.86. 1967. p. 246-270.

[23] Goldschmidt, H.: *Prolongations of linear partial differential equations: A conjecture of Élie Cartan.* Ann scient. École Norm. Sup. 4 série. t. 1. 1968. 417-444.

[24] Gover, A.R., Graham, C.R.: *CR invariant powers of the sub- Laplacian.* J. Reine Angew. Math. 583. 2005. p. 1-27.

[25] Hammerl, M., Somberg, P., Soucek, V., Silhan, J.: *On a new normalization on tractor covariant derivatives.* Available as preprint arxiv 1003.6090.

[26] Knapp, A. W.: *Lie groups beyond an introduction.* Birkhäuser 2004.

[27] Kostant, B.: *Lie algebra cohomology and the generalized Borel-Weil theorem.* Annals of Math. 74. 1961. p. 329-387.

[28] Lepowsky, J.: *A generalization of the Bernstein-Gelfand-Gelfand resolution.* J. Algebra 49 no. 2 .1977.p. 496-511.

[29] Montgomery, R.: *A Tour of Subriemannian Geometries, their Geodesics and Applications.* Mathematical Surveys and Monographs Vol. 91, American Mathematical Society.

[30] Morimoto, T.: *Théorème de Cartan-Kähler dans une classe de fonctions formelles Gevrey.* C. R. Acad. Sci. Paris. 311, série A. 1990. p. 443-436.

[31] Morimoto, T.: *Théorème d'existence de solutions analytiques pour des systèmes d'équations aux dérivées partielles non-linéaires avec singularités.* C.R. Acad. Sci. Paris. 321, série 1. 1995. p.1491-1496.'

[32] Morimoto, T.: *Lie algebras, geometric structures and differential equations on filtered manifolds.* In "Lie Groups Geometric Structures and Differential Equations - One Hundred Years after Sophus Lie", Adv. Stud. Pure Math., Math. Soc. of Japan, Tokyo. 2002. 205-252.

[33] Morimoto, T.: *private communication.*

[34] Morimoto, T.: *Differential Equations Associated to a Representation of a Lie algebra from the Viewpoint of Nilpotent Analysis.* RIMS Kokyuroku 1502, Kyoto University. 2006/07.p. 238-250.

[35] Neusser, K.: *Universal Prolongation of Linear Partial Differential Equations on Filtered Manifolds.* Arch. Math. (Brno) 45, issue 4. 2009. p. 289-300.

[36] Sagerschnig, K.: *Parabolic geometries determined by filtrations of the tangent bundle.* Rend. Circ. Mat. Palermo Suppl. ser. II, 79. 2006. p. 175-181.

[37] Sharpe, R.W.: *Differential Geometry-Cartan's Generalization of Klein's Erlangen Program.* Graduate Texts in Mathematics 166, Springer-Verlag 1997.

[38] Spencer, D.C.: *Overdetermined systems of linear partial differential equations.* Bulletin of Amer. Math. Soc. 75. 1969. p. 179-239.

[39] Takeuchi, M.: *Lagrangean contact structures on projective cotangent bundles.* Osaka J. Math. 31. 1994. p. 837-860.

[40] Tanaka, N.: On differential systems, graded Lie algebras and pseudo-groups. J. Math. Kyoto Univ. 10. 1970. p. 1-82.

[41] Taylor, M.E.: *Noncommutative microlocal analysis I.* Mem. Amer. Math. Soc. 52, no. 313. 1984.

[42] Varadarajan, V. S.: *Lie groups, Lie algebras and Their Representations.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey. 1974

[43] Yamaguchi, K: *Differential Systems Associated with Simple Graded Lie Algebras.* Advanced Studies in Pure Math. 22 .1993. p. 413-494.

Abstract (deutsch)

Eine filtrierte Mannigfaltigkeit ist eine glatte Mannigfaltigkeit M, deren Tangentialbündel TM eine Filtrierung in Teilvektorbündel $TM = T^{-k}M \supset ... \supset T^{-1}M$ besitzt, die mit der Lie Klammer von Vektorfeldern verträglich ist. Wenn man Differentialoperatoren auf filtrierten Mannigfaltigkeiten studiert, stellt sich heraus, dass man den Begriff der Ordnung eines Differentialoperators an die Filtrierung des Tangentialbündels anpassen sollte. Die Veränderung des Ordnungsbegriffs führt zu einem Konzept von gewichteten Jetbündeln von Schnitten von Vektorbündeln über filtrierten Mannigfaltigkeiten, das einen geeigneten Rahmen bildet um Differentialgleichungen auf filtrierten Mannigfaltigkeiten zu untersuchen.
Eine interessante Klasse von filtrierten Mannigfaltigkeiten sind reguläre infintesimale Flaggenstrukturen, die als Parabolischen Geometrien zugrunde liegende geometrische Strukturen auftreten. In der vorliegenden Arbeit werden wir Differentialoperatoren auf regulären infinitesimalen Flaggenmannigfaltigkeiten im Rahmen von gewichteten Jetbündeln studieren. Im ersten Teil der Arbeit werden wir uns mit dem Problem der Prolongation von Differentialgleichungen auf filtrierten Mannigfaltigkeiten beschäftigen. Hierbei werden wir zunächst zeigen, dass ein lineares System von Differentialgleichungen von endlichem gewichtetem Typ auf einer filtrierten Mannigfaltigkeit immer kanonisch äquivalent zu einem bestimmten linearem System von gewichteter Ordnung eins ist. Das impliziert, dass der Lösungsraum eines solchen Systems immer von endlicher Dimension ist. Dann werden wir zeigen, wie man für eine gewisse Klasse von semi-linearen Systemen von Differentialgleichungen auf bestimmten regulären infinitesimalen Flaggenmannigfaltigkeiten eine lineare Konnexion ∇ auf einem Vektorbündel V über der regulären infinitesimalen Flaggenmannigfaltigkeit M sowie eine Bündelabbildung $C : V \to T^*M \otimes V$ konstruieren kann, so dass Lösungen des untersuchten semi-linearem Systems in Bijektion zu Lösungen des Systems $\nabla\Sigma + C(\Sigma) = 0$ stehen. Insbesondere werden wir dadurch scharfe Schranken für die Dimension des Lösungsraums für eine große Klasse von linearen Systemen von gewichtetem endlichem Typ auf bestimmten regulären infinitesimalen Flaggenmannigfaltigkeiten erhalten. Im zweiten Teil der Arbeit, werden wir uns mit der Konstruktion invarianter Differentialoperatoren für parabolische

Geometrien mit Hilfe von gekrümmten Casimir-Operatoren auseinandersetzen. Wir werden in diesem Zusammenhang Casimir-Operatoren verwenden um invariante Differentialoperatoren für Lagrange-Kontakt-Strukturen zu konstruieren.

Curriculum vitae

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Publications

- *On automorphisms groups of some types of generic distributions* (joint with A. Čap). Differential Geom. Appl. Vol. 27. 2009. p. 769-779.
- *Universal prolongation of linear partial differential equations on filtered manifolds.* Arch. Math. (Brno) 45, issue 4. 2009. p. 289-300.