universität wien

# DIPLOMARBEIT

Titel der Diplomarbeit

## Unitarity in noncommutative QFTs

angestrebter akademischer Grad

Magister/Magistra der Naturwissenschaften (Mag. rer.nat.)


| | |
|---|---|
| Verfasserin / Verfasser: | Susanne Urach |
| Matrikel-Nummer: | 0404045 |
| Studienrichtung (lt. Studienblatt): | Diplomstudium Physik |
| Betreuerin / Betreuer: | Ao. Univ.-Prof. Dr. Harald Grosse |


Wien 2010

**Abstract**

The unitarity problem of NCQFT is carefully investigated in this work where it is shown that the original negative result for scalar fields with noncommuting spacetime coordinates is a consequence of the time-ordering not commuting with the Moyalproduct. Therefore a new time-ordering is needed, the so-called interaction-point time-ordering (IPTO), which leads to different Feynman rules and renders scalar fields unitary. For gauge fields this new method doesn't work as it is shown that a Ward identity is violated for Compton scattering with spacetime noncommutativity. At the end some possible solutions are mentioned, but aren't conclusively worked out yet.

**Zusammenfassung**

Das Unitaritätsproblem in NCQFT wird in dieser Arbeit ausführlich diskutiert, wobei gezeigt wird, dass das ursprüngliche negative Resultat für skalare Felder mit nichtkommutierenden Raumzeitkoordinaten eine Konsequenz davon ist, dass die Zeitordnung nicht mit dem Moyalprodukt kommutiert und man deshalb eine neue Art der Zeitordnung benötigt, die sogenannte Interaction-point time-ordering (IPTO), welche zu anderen Feynmanregeln führt und bei skalaren Feldern die Unitarität erhält. Diese neue Methode funktioniert für Eichfelder jedoch nicht, weil gezeigt wurde, dass eine Wardidentität für Comptonstreuung mit räumlich und zeitlicher Nichtkommutativität verletzt ist. Zuletzt wurden noch einige Lösungsvorschläge gemacht, jedoch noch nicht vollständig schlüssig ausgearbeitet.

# Contents


**1 Introduction 3**

**2 Unitarity and Causality in QFT 5**
- 2.1 The optical theorem . . . . . . . . . . . . . . . . . . . . . . . . . . 5
- 2.2 Ward identities and Compton scattering . . . . . . . . . . . . . . . 7

**3 Filk's Feynman rules and the Violation of Unitarity of scalar NCQFT in Euclidean ST 9**
- 3.1 Derivation of Filk's Feynman rules . . . . . . . . . . . . . . . . . . 9
- 3.2 Violation of Unitarity for scalar fields in NCQFT . . . . . . . . . . 13

**4 The interaction point time ordering (IPTO) 15**
- 4.1 What is IPTO and why can it be considered the more appropriate time ordering? . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 16
- 4.2 How does IPTO work? . . . . . . . . . . . . . . . . . . . . . . . . . 17
- 4.3 IPTO for scalar field theory . . . . . . . . . . . . . . . . . . . . . . 18
  - 4.3.1 Example for IPTO used on scalar fields: The fishgraph amplitude . . . . . . . . . . . . . . . . . . . . . . . . . . 19
- 4.4 IPTO for gauge fields . . . . . . . . . . . . . . . . . . . . . . . . . 19
  - 4.4.1 Example for IPTO used on gauge fields: cubic interaction in NCQED . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

**5 The modified Feynman rules for scalar field theory by Denk and Schweda 22**
- 5.1 Coordinate space rules . . . . . . . . . . . . . . . . . . . . . . . . . 22
- 5.2 Momentum space rules . . . . . . . . . . . . . . . . . . . . . . . . . 23
- 5.3 Unitarity of the one loop graph in scalar field theory . . . . . . . 24
  - 5.3.1 Calculations for time-like momentum: $\vec{p} = 0$ (CM-frame) 24
  - 5.3.2 Calculations for spacelike momentum: $\mathrm{p}^0 = 0$ . . . . . . . 26

**6 The Hamiltonian approach and the Yang-Feldman formalism 28**
- 6.1 The Hamiltonian approach . . . . . . . . . . . . . . . . . . . . . . 28
  - 6.1.1 Feynman rules in position space for normal ordered $\phi^n$ interaction . . . . . . . . . . . . . . . . . . . . . . . . . . . 29
  - 6.1.2 Feynman rules in momentum space for normal ordered $\phi^n$ interaction . . . . . . . . . . . . . . . . . . . . . . . . . . . 30
- 6.2 The Yang-Feldman formalism . . . . . . . . . . . . . . . . . . . . . 31
  - 6.2.1 Classical perturbation theory . . . . . . . . . . . . . . . . . . 31

6.2.2 The quantum perturbation theory . . . . . . . . . . . . . . 32

**7 Unitarity of gauge fields 34**

7.1 Ward identities for NCQED . . . . . . . . . . . . . . . . . . . . . . 34

7.2 Violation of Ward Identities on Minkowski noncommutative space-time . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36

7.3 Mismatching phases . . . . . . . . . . . . . . . . . . . . . . . . . . 40

**8 Possible ways out 42**

8.1 The generalized first Filk rule . . . . . . . . . . . . . . . . . . . . . 42

8.2 Orthogonality of the electric vector of the deformation matrix . . 43

**9 Conclusions 45**

# Chapter 1

# Introduction

In quantum mechanics noncommutativity is the central concept to describe uncertainty and applies to any conjugate variables such as position and momentum. As momenta fail to commute if there is a magnetic field one can as easily imagine the noncommutativity of position measurements, i.e. that coordinates fail to commute. Noncommutative quantum field theory (NCQFT) received renewed attention as a low-energy-limes of string theory with an electromagnetic background by the possibility of experimental tests, provided the scale of noncommutativity is sufficiently small, i.e. NCQFT is a first step towards a formulation of quantum gravity which avoids the paradoxa of the formation of horizonts when space-time is probed at Planck scales. The starting assumption of NCQFT is that the familiar continuous Minkowski space-time with coordinates $x_\mu$ is the long-distance limit of a space-time geometry with noncommuting coordinates $\hat{x}_\mu$ satisfying commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i\Theta_{\mu\nu}}{\Lambda^2} \tag{1.1}$$

where $\Theta_{\mu\nu}$ may depend on $\hat{x}_\mu$, but like in most studies we assume it to be a constant. Although in most cases where NCQFT is examined the noncommutative coordinates are spacelike

$$\left[\hat{x}^i, \hat{x}^j\right] = \frac{i\Theta^{ij}}{\Lambda^2}; i, j = 1, 2, 3 \tag{1.2}$$

$$\left[\hat{x}^0, \hat{x}^j\right] = \frac{i\Theta^{0j}}{\Lambda^2} = 0, \tag{1.3}$$

we will not do so here, as we are especially interested in space-time noncommutativity. But what's the meaning of the commutation relation above if $\Theta^{0j} \neq 0$, i.e. there exists a space-time noncommutativity? In quantum mechanics the space coordinates of particles are operators whereas the time coordinate isn't, but labels the evolution of the system regarded. Although it isn't quite clear how it's possible that time doesn't commute with space in contradiction with quantum mechanics, there are several reasons why considering such cases gains interest for us [10]:

- Spacetime noncommutativity is a natural extension of space-space noncommutativity.

- Whereas noncommutativity in space occurs when D-branes are placed in a nonzero background magnetic field, spacetime noncommutativity is achieved by placing D-branes in a nonzero background electric field. The commutator leads to an uncertainty relation between time and space of the form $\Delta x^0 \Delta x^i \neq 0$ known as "stringy uncertainty relation" as it is a generic property of string theory even without any electrical field. This also plays an important role regarding the information puzzle in black holes.
- Dealing with space-time noncommutativity may also help us to understand the role of time in string theory.

To realize noncommuting coordinates on an ordinary commuting space-time the associative Moyal $*$-product was introduced

$$(f * g)(x) = \lim_{\xi,\eta \to 0} \left[ e^{i\partial_\xi \wedge \partial_\eta} f(x+\xi) g(x+\eta) \right] \tag{1.4}$$

which results in momentum space in phase factors $e^{ip\wedge q}$ with the antisymmetric product

$$p \wedge q = \frac{1}{2\Lambda^2} p^\mu \Theta_{\mu\nu} q^\nu. \tag{1.5}$$

There exists a correspondence principle according to which all ordinary products of fields in the Lagrangian have to be replaced by Moyal $*$-products to get a NCQFT from a given QFT [9]. So all interaction vertices acquire momentum-dependent phase factors. As the Moyal $*$-product involves derivatives of all orders the resulting theory is non-local, so one has to be careful that the physical interpretation of the theory is not spoiled by this non-locality. For example, it is important for a physical theory that the unitarity of the S-matrix in scattering processes isn't violated. Unfortunately this is only true if only space-like noncommutativity is considered, but not for time-like noncommutativity. Using the Filk Feynman rules for noncommutative Euclidean spacetime in the usual covariant perturbation theory the cutting rules are violated as seen in [2]. But as time doesn't commute with space, one has to change the time ordering as well leading to the interaction point time ordering (IPTO) which we will consider in detail in chapter 3. With the help of IPTO unitarity is reestablished for scalar NCQFT (see chapters 4,5 and 6) with the help of different approaches as the rules developed by Denk and Schweda, the Hamiltonian approach by Bahns etal. and the Yang Feldman formalism, but regretfully not for noncommutative gauge theories (see chapter 7 where we will show that IPTO leads to a violation of the Ward identity in the case of Compton scattering rendering NCQFT with IPTO an unphysical theory). Possible ways out of this dilemma are then mentioned in chapter 8, which are nevertheless not fully satisfactory as one has to assume a generalized first filk rule respectively that the electric vector of the noncommutative matrix $\theta^{\mu\nu}$ stands orthogonal on the scattering momenta which isn't motivated by the theory. Although the problem isn't solved yet, the following provides an exact illustration of where the problem lies for gauge fields with IPTO and conclusions what still needs to be done to save unitarity in NCQFT (see chapter 9).

# Chapter 2

# Unitarity and Causality in QFT

## 2.1 The optical theorem

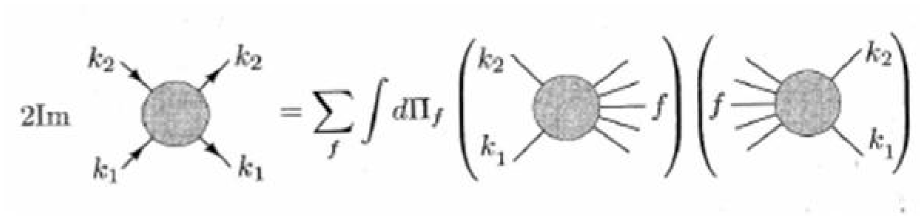


Figure 2.1: The Optical Theorem: The imaginary part of forward scattering arises from the sum of contributions from all possible intermediatestate particles [4].

The scattering amplitude as a function of energy has a branch cut on the positive real axis so that its imaginary part appears as a discontinuity across this branch cut and the imaginary part of the forward scattering amplitude is proportional to the total cross section. The optical theorem is a straightforward consequence of the unitarity of the S-matrix [4]. Inserting $S = 1 + iT$ into $S^\dagger S = 1$ yields $-i(T - T^\dagger) = T^\dagger T$. Consider the result of this equation between two-particle states by inserting a complete set of intermediate states

$$\langle p_1 p_2 | T^\dagger T | k_1 k_2 \rangle = \sum_n \left( \prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \langle p_1 p_2 | T^\dagger | \{q_1\} \rangle \langle \{q_i\} | T | k_1 k_2 \rangle \tag{2.1}$$

Expressing T-matrix elements as invariant matrix elements M times 4-momentumconserving delta functions one gets:

$$-i[M(k_1k_2 \to M^*(p_1p_2 \to k_1k_2)] = \sum_n \left( \prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \tag{2.2}$$

$$M^*(p_1p_2 \to \{q_i\})M(k_1k_2 \to \{q_i\}) \times (2\pi)^4\delta^{(4)}(k_1 + k_2 - \sum_i q_i) \tag{2.3}$$

times an overall delta function $(2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)$. This can be abbreviated by

$$-i[M(a \to b) - M^*(b \to a)] = \sum_f \int d\Pi_f M^*(b \to f)M(a \to f) \tag{2.4}$$

where f stands for allpossible sets of final particles. Although this derivation has been done for two particle states, the above formula is also true for one-particle or multiparticle asymptotic states. In the case of forward scattering one obtains the standard form of the optical theorem by setting $p_i = k_i$ and using the kinematic factors to build the cross section,

$$ImM(k_1, k_2 \to k_1, k_2) = 2E_{cm}p_{cm}\sigma_{tot}(k_1, k_2 \to anything), \tag{2.5}$$

where $E_{cm}$ and $p_{cm}$ are the center of mass quantities. The optical theorem relates the forward scattering amplitude to the total cross section for production of all final states. As the imaginary part of the forward scattering amplitude gives the attenuation of the forward going beam passing through a target, it should be proportional to the probability of scattering. The precise connection arises in the Feyman diagram expansion. An S-matrix element M is purely real unless some denominators vanish, so that the $i\epsilon$ prescription for treating the poles becomes relevant, which is the case when virtual particles in the diagram go onshell. The appearance of an imaginary part of M(s) always requires a branch cut singularity. Let $s_0$ be the threshold energy for production of the lightest multiparticle state. So for real s below $s_0$ the intermediate state cannot go on-shell, so M(s) is real:

$$M(s) = [M(s^*)]^* \tag{2.6}$$

As both sides are analytic functions of s, they can be analytically continued to the entire complex s plane. Near the real axis for $s > s_0$ this implies

$$ReM(s + i\epsilon) = ReM(s - i\epsilon) \tag{2.7}$$

$$ImM(s + i\epsilon) = -ImM(s - i\epsilon). \tag{2.8}$$

So there is a branch cut across the real axis starting at $s_0$. For the discontinuity across the cut one gets

$$DiscM(s) = 2iImM(s + i\epsilon) \tag{2.9}$$

The generalization of this result to multiloop diagrams has been proven by Cutkosky, who showed that the discontinuity of a Feynman diagram across the branch cut is always given by a simple set of cutting rules [4]:

- Cut through the diagram in all possible ways such that the cut propagators can simultaneously be put on shell.
- For each cut replace $1/(p^2 - m^2 + i\epsilon) \to -2\pi i\delta(p^2 - m^2)$ in each cut propagator, then perform the loop integrals.
- Sum the contributions of all possible cuts.

Using these cutting rules, it is possible to prove the optical theorem to all orders in perturbation theory and show that the generalized optical theorem is true not only for S-matrix elements, but for any amplitude M that we can define in terms of Feynman diagrams.

## 2.2 Ward identities and Compton scattering

Ward identities express the conservation of the symmetry currents which can be accomplished by putting external charges onshell, but also gauge bosons carry charge and must be put on shell to remove contact terms. Lets consider the example of the lowest order contributing to fermion-antifermion annhililation into a pair of gauge bosons as in [4]. In order $g^2$ there are three diagrams: The

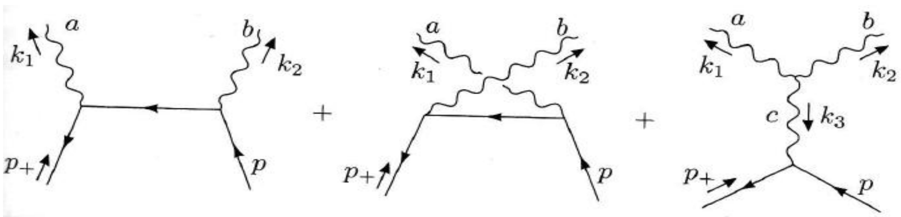


Figure 2.2: Diagrams contributing to fermion-antifermion annhililation to two gauge bosons[4].

first two diagrams sum up to

$$iM^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) = (ig)^2\bar{v}(p_+)\left\{\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\gamma^\nu t^b\right. \tag{2.10}$$

$$\left. + \gamma^\nu t^b \frac{i}{\not{k}_2 - \not{p}_+ - m}\gamma^\mu t^a u(p)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\right. \tag{2.11}$$

The gauge boson polarization vectors satisfy $k_i^\mu \epsilon_\mu(k_i) = 0$ and replacing $\epsilon^*_\nu$ by $k_{2\nu}$ the above takes the following form

$$iM^{\mu\nu}_{1,2}\epsilon^*_{1\mu}k_{2\nu} = (ig)^2\bar{v}(p_+)\left\{\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\not{k}_2 t^b\right. \tag{2.12}$$

$$\left. + \not{k}_2 t^b \frac{i}{\not{k}_2 - \not{p}_+ - m}\gamma^\mu t^a u(p)\epsilon^*_{1\mu}\right. \tag{2.13}$$

Using the Dirac equations $(\not{p} - m)u(p) = 0$ and $\bar{v}(p_+)(-\not{p}_+ - m) = 0$ this gives

$$iM^{\mu\nu}_{1,2}\epsilon^*_{1\mu}k_{2\nu} = (ig)^2\bar{v}(p_+)\left\{-i\gamma^\mu\left[t^a, t^b\right]\right\}u(p)\epsilon^*_{1\mu}. \tag{2.14}$$

In the nonabelian case the residual term is nonzero and depends on the commutator of gauge group generators

$$iM_{1,2}^{\mu\nu}\epsilon_{1\mu}^{*}k_{2\nu} = -g^{2}\bar{v}(p_{+})\gamma^{\mu}u(p)\epsilon_{1\mu}^{*}\cdot f^{abc}t^{c} \tag{2.15}$$

which has the group index structure of a fermion-gauge boson vertex multiplied by a three gauge boson vertex and is therefore identical to the one of the third Feynman graph. For the third diagram we get

$$\begin{aligned} iM_{3}^{\mu\nu}\epsilon_{1\mu}^{*}\epsilon_{2\nu}^{*} &= ig\bar{v}(p_{+})\gamma_{\rho}t^{c}u(p)\frac{-i}{k_{3}^{2}}\epsilon_{\mu}^{*}(k_{1})\epsilon_{\nu}^{*}(k_{2}) \\ \times gf^{abc}&\left[g^{\mu\nu}(k_{2}-k_{1})^{\rho}+g^{\nu\rho}(k_{3}-k_{2})^{\mu}+g^{\rho\mu}(k_{1}-k_{3})^{\nu}\right] \end{aligned} \tag{2.16}$$

Replacing $\epsilon_{\nu}^{*}(k_{2})$ with $k_{2\nu}$ and eliminate $k_{2}$ using energy-momentum conservation $k_{2}=-k_{1}-k_{3}$ the expression in brackets simplify as follows:

$$\begin{aligned} &\epsilon_{\nu}^{*}(k_{2})\left[g^{\mu\nu}(k_{2}-k_{1})^{\rho}+g^{\nu\rho}(k_{3}-k_{2})^{\mu}+g^{\rho\mu}(k_{1}-k_{3})^{\nu}\right] \\ &\to k_{2}^{\mu}(k_{2}-k_{1})^{\rho}+k_{2}^{\rho}(k_{3}-k_{2})^{\mu}+g^{\rho\mu}(k_{1}-k_{3})\cdot k_{2} \\ &= g^{\rho\mu}k_{3}^{2}-k_{3}^{\rho}k_{3}^{\mu}-g^{\rho\mu}k_{1}^{2}+k_{1}^{\rho}k_{1}^{\mu} \end{aligned}$$

Consider the other gauge boson onshell $k_{1}^{2}=0$ and with transverse polarization $k_{1}^{\mu}\epsilon_{\mu}(k_{1})=0$ then the third and fourth terms vanish and the second one is zero when contracted with the fermion current. This leaves us with

$$iM_{3}^{\mu\nu}\epsilon_{1\mu}^{*}k_{2\nu} = g^{2}\bar{v}(p_{+})\gamma^{\mu}u(p)\epsilon_{1\mu}^{*}\cdot f^{abc}t^{c} \tag{2.17}$$

which precisely cancels the remaining term of the other two Feynman graphs.

# Chapter 3

# Filk's Feynman rules and the Violation of Unitarity of scalar NCQFT in Euclidean ST

## 3.1 Derivation of Filk's Feynman rules

The first perturbation scheme for NCQFT on Euclidean spacetime was developed by Filk, who derived NC Feynman rules. These are sometimes called naive Feynman rules as they are a quite simple modification of the ordinary ones for commutative spacetime. The lines are represented by the conventional Feynman propagators, the only difference is that every vertex picks up a factor (twisting, trigonometric function of momenta). The distributional character of the Green functions near conciding points and the locality of the interaction which leads to the problem of multiplying distributions was tried to be cured by regularization of the Green function with the structure of a lattice or by making the interaction nonlocal due to the deformed product for the fields. Originally Filk's reason to study deformed field theories was to free quantum field theory of its singularities, i.e. the UV divergencies, but as shown later on, this wasn't really successful as this approach leads to UV/IR mixing. In Filk's derivation of his NC Feynman rules he introduces the Moyal-star-product, so that he could write classical fields instead of fields on noncommutative coordinates. To begin with he replaces the coordinates on flat space $\{q^\mu\}$ by selfadjoint operators in a Hilbert space which fulfill the following algebra:

$$[q^\mu, q^\nu] = i\Sigma^{\mu\nu} \tag{3.1}$$

$$[\Sigma^{\mu\nu}, q^\lambda] = 0 \tag{3.2}$$

One can write $\Sigma^{\mu\nu}$ as a C-number $\sigma^{\mu\nu}$ as it lies in the center of the generated algebra, for example in the Heisenberg algebra of quantum phase space $\sigma^{\mu\nu} = \hbar\epsilon^{\mu\nu}$. Let's assume that there exists a nondegenerate, but not necessarily positive definite bilinear form which allows to rise and lower indices,

$$T(k) = e^{ik_\mu \hat{x}^\mu}. \tag{3.3}$$

With $q^\mu$ selfadjoint we get the following properties:

$$T^+(k) = T(-k) \tag{3.4}$$

$$T(k)T(k') = T(k+k')e^{-\frac{i}{2}k_\mu k'_\nu \sigma^{\mu\nu}} \tag{3.5}$$

$$trT(k) = \prod_\mu \delta(k_\mu). \tag{3.6}$$

The trace is taken with respect to a basis of the representation space. There exists no finite representation for the algebra of the NC coordinates whereas for the algebra of operators defined above there exists one which can be considered as deformation of lattices. Now we can associate an operator $\Phi$ to a classical function $\phi(x)$:

$$\begin{aligned}\Phi &= \frac{1}{(2\pi)^n}\int dxdkT(k)e^{-ik_\mu x^\mu}\phi(x) \\ &= \frac{1}{(2\pi)^{n/2}}\int dxdkT(k)\tilde{\phi}(x)\end{aligned} \tag{3.7}$$

Using the trace we recover:

$$\phi(x) = \frac{1}{(2\pi)^{n/2}}\int dk e^{ik_\mu x^\mu} tr^+(k). \tag{3.8}$$

So we can define a $*$-product for classical fields:

$$\begin{aligned}(\phi_1 * \phi_2)(x) &= \frac{1}{(2\pi)^{n/2}}\int dk e^{ik_\mu x^\mu} tr[\Phi_1\Phi_2 T^+(k)] \\ &= \int dx_1 dx_2 K(x;x_1,x_2)\phi(x_1)\phi(x_2)\end{aligned} \tag{3.9}$$

with

$$\begin{aligned}K(x;x_1,x_2) &= \frac{1}{(2\pi)^n}\int dk \prod_\mu \delta(x^\mu - x_1^\mu + \frac{1}{2}\theta^{\mu\nu}k_\nu)e^{ik_\nu(x-x_2)^\nu} \\ &= \frac{1}{\pi^n\,|det\theta|}exp(2i[(x-x_1)^\mu \theta^{-1}_{\mu\nu}(x-x_2)^\nu])\end{aligned} \tag{3.10}$$

where $\theta^{\mu\nu} = \hbar\epsilon^{\mu\nu}$ with the deformation parameter $\hbar$. This kernel shows the nonlocality of the Moyal product and as a by product one finds that

$$\int dx\phi_1(x) * \phi_2(x) = \int dx\phi_1\phi_2. \tag{3.11}$$

Under the integral sign the Moyal product of two fields reduces to the ordinary products. In noncommutative situations all ordinary field products are replaced by the corresponding Moyal products, but this replacement doesn't affect the kinetic term which is quadratic. This also implies that the propagators are the same as in commutative quantum field theory, but the vertices are modified by the noncommutativity. To complete the determination of the Feynman rules we will next try to find how the vertices change in momentum space for NCQFT.

Having defined a noncommuative product for ordinary complex fields one may

now consider the deformation of a classical action:

$$S[\phi] = \int dx[(\partial_\mu \bar{\phi})(\partial^\mu \phi) + m^2 \bar{\phi}\phi + g(\bar{\phi}\phi)^p] \tag{3.12}$$

where the interaction part is replaced by the corresponding expression for the fields on non-commutative coordinates, which can be expressed by ordinary fields using the $*$-product:

$$\int dx(\bar{\phi}\phi)^p \to tr(\Phi^+\Phi)^p =$$
$$\int dk_1...dk_{2p}\tilde{V}(k_1,...,k_{2p}) \times \tilde{\bar{\phi}}(-k_1)...\tilde{\phi}(k_{2p}) \tag{3.13}$$

with

$$\tilde{V}(k_1,..,k_n) = \delta(k_1 + ... + k_n) \times exp(\frac{i}{2}\sum_{i<j}^{n} k_i^\mu k_j^\nu \sigma_{\mu\nu}). \tag{3.14}$$

The first term is just the momentum conservation at each vertex, but the additional phase factor renders the interaction nonlocal and the $\tilde{V}$ is only invariant under cyclic permutations.

To calculate the contribution of $\tilde{V}$ to an arbitrary Feynman graph one can use

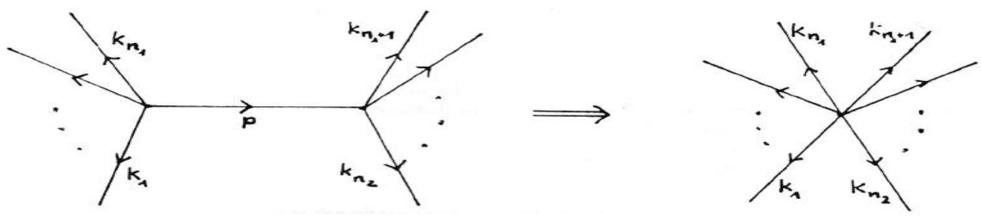


Figure 3.1: Contraction of a line connecting two different points [9]

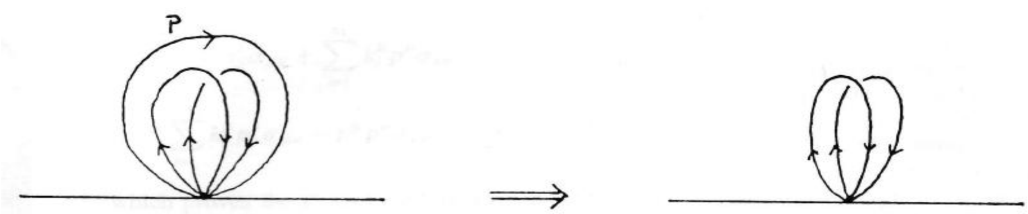


Figure 3.2: Eliminating a closed loop which doesn't intersect other lines [9]

the simplification that one can contract two vertices connected by a line with momentum p

$$\tilde{V}(k_1, ..., k_{n_1}, p)\tilde{V}(-p, k_{n_1+1}, ..., k_{n_2}) = \tilde{V}(k_1, ..., k_{n_2})\delta(k_1 + ... + k_{n_1} + p) \tag{3.15}$$

and that one can eliminate a loop with momentum p which does not cross other lines:

$$\tilde{V}(k_1, ..., k_{n_1}, p, k_{n_1+1}, .., k_{n_2}, -p) = \tilde{V}(k_1, ..., k_{n_1}, k_{n_1+1}, ..., k_{n_2}) \tag{3.16}$$

$$\sum_{i=n_1+1}^{n_2} k_i = 0. \tag{3.17}$$

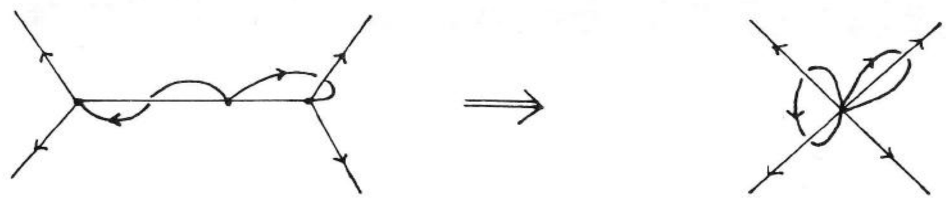

Figure 3.3: Reduced Feynman graph after the succesive contraction of lines connecting different points [9]

The proof of the above statement is straightforward by comparing the delta-functions and the phase factors. For a planar graph this reduction leads to a one vertex graph with only external lines attached and for the phase factor all contributions from internal lines cancel. But for nonplanar graphs loops of the resulting rosette after contacting all vertices will cross other loops or external lines and there is an extra contribution from each crossing. To compute the extra term we introduce the intersection matrix $I_{ij}$ of an oriented graph, which is 1 if j crosses i from right, -1 if j crosses i from the left and 0 if j does not cross i at all. With this convention the contribution of the phase factors for a graph G is given by

$$\Gamma(G) = \tilde{V}(\{\text{external momenta}\}) \times exp\left(\frac{i}{2}\sum_{ij} I_{ij}\sigma_{\mu\nu}k_i^{\mu}k_j^{\nu}\right). \tag{3.18}$$

A graph is planar if the cancellation of twistings from different vertices is such that the resulting twisting does not depend on the internal momenta. For a planar graph the intersection matrix vanishes and one recovers the independence of the internal momenta and therefore their momentum integration is identical to the undeformed theory with the same kind of divergencies occuring. During the whole article by Filk he never mentions that he is working on an Euclidean spacetime, so that he does not have to consider that the time ordering and the Moyal-$*$-product do not commute. The reason for the designation of a Feynman graph as being planar or nonplanar is easily understood when using the double line notation.

## 3.2 Violation of Unitarity for scalar fields in NC-QFT

Gomis and Mehen showed in [2] that these NC Feynman rules by Filk applied on the fish graph lead to a violation of unitarity if space-time noncommutativity ( $\Theta^{ij} \neq 0,\, i,j \in \{1,2,3\}$) is involved, but holds for space-space noncommutativity ($\Theta^{ij} \neq 0,\, \Theta^{0i} = 0$). The reason why this result is considered incorrect is that Filk's Feynman rules as mentioned above do not apply on Minkowski spacetime. Nevertheless their proof is stated below as it is still correct for Euclidean spacetime. Mehen and Gomis showed that the cutting rule for the noncommutative $\phi^3$ theory two-point function at lowest order is violated. Using Filk's Feynman

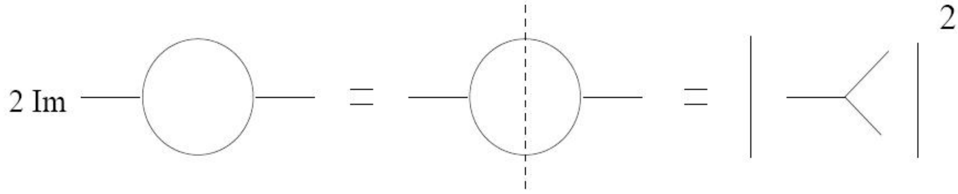


Figure 3.4: The optical theorem applied to the one loop graph [2]

rules the propagators of fields in noncommutative field theories are identical to those of commutative field theory and for each vertex one gets

$$-i\lambda \cos\left(\frac{k \wedge q}{2}\right) \tag{3.19}$$

$$k \wedge q = k_\mu \theta^{\mu\nu} q_\nu \tag{3.20}$$

where k and q are any two of the momenta flowing into the vertex and $\theta^{\mu\nu}$ is a completely antisymmetric matrix. So one gets the following amplitude for the one loop diagram using $\cos^2(x) = (1+\cos(2x))/2$ to separate the planar from the nonplanar term

$$iM = \frac{\lambda^2}{2}\int \frac{d^D l}{(2\pi)^D}\frac{1+\cos(p\wedge l)}{2}\frac{1}{l^2-m^2+i\epsilon}\frac{1}{(l+p)^2-m^2+i\epsilon} \tag{3.21}$$

while the sum over the square amplitudes leads to

$$\sum |M|^2 = \frac{\lambda^2}{2}\frac{1}{(2\pi)^{D-2}}\int \frac{d^{D-1}k}{2k_0}\frac{d^{D-1}q}{2q_0}\delta^D(p-k-q)\frac{1+\cos(p\wedge k)}{2} \tag{3.22}$$

As the planar contribution obviously satisfies unitarity constraints, we will only consider the nonplanar part which looks like

$$M = \frac{\lambda^2}{8}\int \frac{d^D l_E}{(2\pi)^D}\int_0^1 dx \int_0^\infty d\alpha\alpha(exp(-\alpha(l_E^2 + x(1-x)p_E^2 + m^2 - i\epsilon) + il_E \wedge p_E) + c.c.) \tag{3.23}$$

where we combined denominators using Feynman parameters, represented the propagators via Schwinger parameters and performed the analytic continuation

$l^0 = il^0_E$, $p^0 = ip^0_E$ and $\Theta^{0i} \rightarrow -i\Theta^{0i}$. The subscript E denotes Euclidean momenta and the Moyal phase stayed invariant. The integration over the loop momentum $l_E$ gives

$$M = \frac{\lambda^2}{4}\frac{1}{(4\pi)^{D/2}}\int_0^1 dx \int_0^\infty d\alpha\alpha^{1-D/2} exp\left(-\alpha(x(1-x)p_E^2 + m^2 - i\epsilon) - \frac{p_E \circ p_E}{4\alpha}\right) \tag{3.24}$$

For D=4 space-time dimensions and analytically continuing back to Minkowski space the amplitude looks like

$$M = \frac{\lambda^2}{32\pi^2}\int_0^1 dx K_0\left(\sqrt{p \circ p(m^2 - p^2x(1-x) - i\epsilon)}\right) \tag{3.25}$$

where $K_0$ is a modified Bessel function and $p \circ p$ needs to be positive. Let's choose $\Theta^{01} = -\Theta^{10} = \Theta_E$ and $\Theta^{23} = -\Theta^{32} = \Theta_B$ with all other components zero, so that

$$p \circ p = \Theta_E^2(p_0^2 - p_1^2) + \Theta_B^2(p_2^2 + p_3^2). \tag{3.26}$$

For only space-space noncommutativity $p \circ p$ is positive definite, but for space-time noncommutativity $p \circ p$ can be negative, which will lead us to the conclusion that unitarity is satisfied for magnetic theories, but violated for electric field theories.

- For $p \circ p > 0$ the generalized unitarity relation holds:

$$ImM_{D=4} = \frac{\lambda^2}{64\pi}\int_{(1-\gamma)/2}^{(1+\gamma)/2} dx J_0(\sqrt{p \circ p}\sqrt{-m^2 + p^2x(1-x)}) \tag{3.27}$$

$$= \frac{\lambda^2}{32\pi}\frac{\sin(\gamma\sqrt{p^2 p \circ p}/2)}{\sqrt{p^2 p \circ p}} \tag{3.28}$$

$$\sum |M_{D=4}|^2 = \frac{\lambda^2}{4}\frac{\gamma}{32\pi^2}\int d\Omega \cos(p \wedge k) = \frac{\lambda^2}{16\pi}\frac{\sin(\gamma\sqrt{p^2 p \circ p}/2)}{\sqrt{p^2 p \circ p}}. \tag{3.29}$$

- For $p \circ p < 0$ the sum over the final states vanishes because energy-momentum conservation forbids a particle with space-like momenta to decay into two massive onshell paricles, but:

$$ImM_{D=4} = \frac{\lambda^2}{64\pi}\int_0^1 J_0(\sqrt{|p \circ p|\,(m^2 + |p^2|\,x(1-x))}) \tag{3.30}$$

  which is obviously nonzero and the generalized unitarity condition is therefore violated.

# Chapter 4

# The interaction point time ordering (IPTO)

In [12] it is stated that the root of the problem of the violation of unitarity in NCQFT lies in the improper definition of the time-ordered product. If you start to "switch on" interactions, perturbation schemes are no longer equivalent, but may depend on:

- choice of starting point for quantization
- choice of time-ordering (T*-product vs T-product)
- coordinates for quantization
- considered type of functions (correlation function, time-ordered function,...)

To solve the unitarity problem Time-Ordered Perturbation Theory (TOPT) for NCQFT was introduced by Liao and Sibold [6] which was later renamed Interaction Point Time Ordering (IPTO) in [5] to avoid ambiguities. There are different approaches which use IPTO, one of them is the Hamiltonian approach by D. Bahns [3] which we will deal with later on. This perturbation scheme is again unitary, but because of the time-ordering used not covariant. Another one based on IPTO are the modified Feyman rules derived by Denk and Schweda [1] which are the same as the rules by Filk for space-space noncommutativity, but differ if noncommutativity involves time and therefore rander NCQFT unitary. The potential violation of unitarity also doesn't occur if one uses the Yang-Feldman formalism as has been shown by Bahns et al. [3]. This kind of approach would also be covariant and finite.

So why do we believe that NC QFT is indeed unitary and one has to use IPTO instead of the usual time ordering? As is shown by Liao and Sibold [6], although we still define the S-matrix as $S = Texp\left[i\int d^4xL_{int}\right]$ performing the contractions according to Wick's theorem one cannot combine the contraction functions of positive and negative frequency to the causal Feynman propagator. When time doesn't commute with space, the time-ordering procedure doesn't commute with the star multiplication either and therefore Filk's approach isn't well founded. But what do we understand under time-ordering when $\Theta^{0i} \neq 0$, i.e. there exists a nonlocality in time?

## 4.1 What is IPTO and why can it be considered the more appropriate time ordering?

As the naive path integral Feynman rules don't work on noncommutative space-time, it is proposed in [6] that the Gell-Mann-Low formula is used with the time-ordering applied before performing the integrations. The theory is quantized canonically in Minkowski space instead of employing the Euclidean path integral. Starting point is the interaction Hamiltonian on a Fock space:

$$H_I(t) = \int_{x^0=t} d^3x : (\phi * \phi * ... * \phi)(x) : \tag{4.1}$$

Applying the Gell-Mann-Low formula for Green's function one gets:

$$G_n(x_1, ..., x_k) := \frac{i^n}{n!} \int d^4z_1...d^4z_n \langle 0| T\phi(x_1)...\phi(x_k)L_I(z_1)...L_I(z_n) |0\rangle^{con} \tag{4.2}$$

where $L_I$ denotes the interaction Lagrangian and the superscribt means projection onto the connected part. In the interaction picture time ordering is considered for external vertices and interaction points only and not with respect to the actual time-order of the fields in the interaction Lagrangian. Causality is explicitly violated inside the region of interaction, NC scalar field theory is unitary which leads to the assumption that causality and unitarity are mutually exclusive properties of space-time NC geometries. This type of acausal time-ordering has to be explicitly distinguished from a true causal time-ordering. Let's look at the simple case of the two-point function at first order in g for a physical interpretation of the ensuing techniques,

$$G(x, y) = \frac{g}{4!} \int d^4z \langle 0| T(\phi(x)\phi(y)(\phi * \phi * \phi * \phi)(z)) |0\rangle \tag{4.3}$$

where

$$(\phi * \phi * \phi * \phi)(z) = \int \prod_{i=1}^{3} \left( d^4s_i \frac{d^4l_i}{(2\pi)^4} e^{il_is_i} \right)$$

$$\times\phi(z - \frac{1}{2}\tilde{l}_1)\phi(z + s_1 - \frac{1}{2}\tilde{l}_2)\phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3)\phi(z + s_1 + s_2 + s_3) \tag{4.4}$$

$$\tilde{l}^\nu := l_\mu\theta^{\mu\nu}. \tag{4.5}$$

In the true time-ordering we get for this arrangement of fields

$$\begin{aligned} G_{(8)}(x, y) = \int d^4z \int \prod_{i=1}^{3} \left( d^4s_i \frac{d^4l_i}{(2\pi)^4} e^{il_is_i} \right) \tau(s_1^0 + s_2^0 + s_3^0 + \frac{1}{2}\tilde{l}_1^0)\tau(z^0 - \frac{1}{2}\tilde{l}_1^0 - x^0) \\ \times\tau(x^0 - z^0 - s_1^0 + \frac{1}{2}\tilde{l}_2^0)\tau(z^0 + s_1^0 - \frac{1}{2}\tilde{l}_2^0 - y^0)\tau(y^0 - z^0 - s_1^0 - s_2^0 + \frac{1}{2}\tilde{l}_3^0) \\ \times \langle 0| \phi(z + s_1 + s_2 + s_3)\phi(z - \frac{1}{2}\tilde{l}_1)\phi(x)\phi(z + s_1 - \frac{1}{2}\tilde{l}_2)\phi(y)\phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3) |0\rangle \end{aligned} \tag{4.6}$$

where $\tau(t)$denotes the step function. There are 6!=720 different contributions of this type. When the time ordering is defined with respect to the interaction

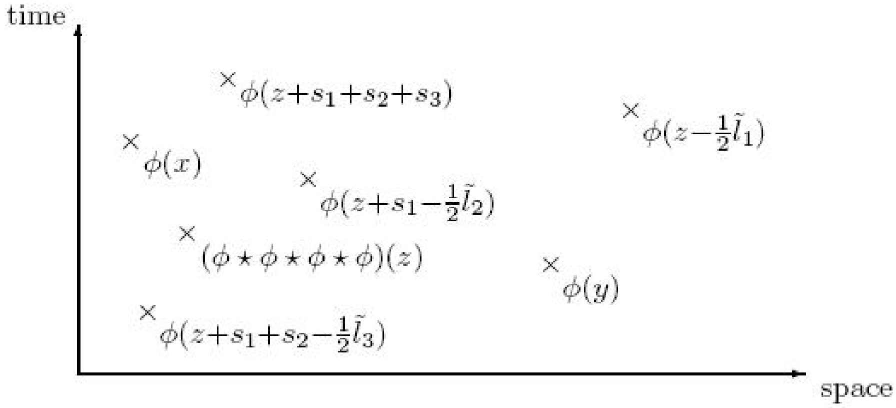


Figure 4.1: Geometrical situation [5]

point we get the following two point function:

$$G'_{(8)}(x,y) = \int d^4z \int \prod_{i=1}^{3} \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{i l_i s_i} \right) \tau(x^0 - z^0)\tau(z^0 - y^0)\times$$
$$\langle 0|\, \phi(z+s_1+s_2+s_3)\phi(z-\frac{1}{2}\tilde{l}_1)\phi(x)\phi(z+s_1-\frac{1}{2}\tilde{l}_2)\phi(y)\phi(z+s_1+s_2-\frac{1}{2}\tilde{l}_3)\,|0\rangle\,. \tag{4.7}$$

There are only 3!=6 different contributions of this type. As in most cases the fields are now at the wrong places with respect to the true time-order, the IPTO interpretation of the Gell-Mann-Low formula violates causality. Both energy solutions propagate in any direction of time. As it is unclear how to derive the Gell-Mann-Low formula in the noncommutative setting, we can't really say which time-ordering is the correct one. An argument in favor of IPTO is that the fields in the Dyson series are ordered with respect to the time stamp of the interaction Hamiltonians, so it doesn't matter how the time-dependence of the interaction Hamiltonian is produced from the time-dependence of the different fields. One can say that noncommutativity "spreads" the interaction over spacetime. The time ordering only acts on the interaction point (IP), but not on the new smeared-out "physical" coordinates of the field operators [14]. The four fields of the interaction point are not time-ordered with respect to each other, but time-oredring is realized between external and interaction points only.

## 4.2 How does IPTO work?

In IPTO, each propagator splits into a positive energy (frequency) and a negative energy (frequency) piece: The four-momenta are taken on-shell and look the following way:

$$q^{(\pm)} = (\pm\sqrt{\vec{q}^2 + m^2}, \vec{q}). \tag{4.8}$$

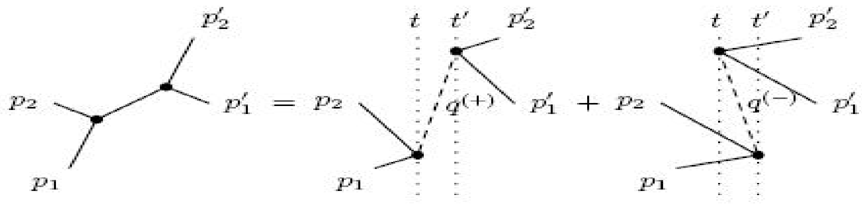


Figure 4.2: IPTO [8]

As the three-momenta are conserved at a vertex, the energy isn't. The Moyal phase $\phi$ of a n-point vertex one gets via Fourier transformation and is given by:

$$(\Phi_1 * \Phi_2 * ... * \Phi_n)(x) \overset{F.T.}{\rightarrow} e^{-i\phi(p_1,p_2,...p_n)}\Phi_1(p_1)\Phi_2(p_2)...\Phi_n(p_n) \tag{4.9}$$

$$\phi(p_1, p_2, ..., p_n) = \sum_{i<j} p_i \wedge p_j \tag{4.10}$$

In general the phase $\phi$ defined above is not cyclically symmetric

$$\phi(p_1, p_2, ..., p_n) \neq \phi(p_2, ..., p_n, p_1). \tag{4.11}$$

Only in the case of four-momentum conservation, the contributions from either the first and the last momentum cancel and cyclical symmetry is recovered:

$$\phi(p_1, p_2, ..., p_n)|_{p_1+p_2+...+p_n=0} = \phi(p_1, p_2, ..., p_{n-1}) = \phi(p_2, ..., p_n). \tag{4.12}$$

This ordering ambiguity caused by the violation of energy conservation has to be taken into account, so the two contributions to the scattering process have different Moyal phases $\phi(q^{(\pm)})$ where the dependence on the external four-momenta is suppressed.

## 4.3 IPTO for scalar field theory

Adding up the scalar propagators $\frac{i}{2q_0^{(\lambda)}} \frac{1}{q_0 - q_0^{(\lambda)} + \lambda i\epsilon}$ with $q_0^{(\pm)} = \pm\sqrt{\vec{q}^2 + m^2}$, which have the same poles as in covariant perturbation theory, leads to

$$\sum_{\lambda=\pm} e^{i\phi(q^{(\lambda)})} \frac{1}{2q_0^{(\lambda)}} \frac{1}{q_0 - q_0^{(\lambda)} + \lambda i\epsilon} = \frac{R(q^{(+)}, q^{(-)})}{q^2 - m^2 + i\epsilon} \tag{4.13}$$

In comparison to the covariant case the residue isn't unity, but a linear combination of the phase factors:

$$R(q^{(+)}, q^{(-)}) = \frac{1}{2} \sum_{\lambda=\pm} e^{i\phi(q^{(\lambda)})} \left(1 + \frac{q_0}{q_0^{(\lambda)}}\right). \tag{4.14}$$

### 4.3.1 Example for IPTO used on scalar fields: The fishgraph amplitude

Consider the fishgraph amplitude of the $\phi^3$ interaction Lagrangian $L_I \propto \lambda\phi * \phi * \phi(x)$: Using the NC Feynman rules developed by Filk one would get:

$$M \propto \lambda^2 \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} (\delta^4(k_1 - p_1 - p_2)(\sum_{sym} e^{k_1,p_1,p_2})) \tag{4.15}$$

$$(\delta^4(k_2 - p_1 - p_2)(\sum_{sym} e^{k_2,p_1,p_2})) \frac{1}{p_1^2 - m^2 + i\epsilon} \frac{1}{p_2^2 - m^2 + i\epsilon} \tag{4.16}$$

$$M \propto \lambda^2 \delta^4(k_1 - k_2) \int \frac{d^4p_1}{(2\pi)^4} \frac{\cos^2(k_1 \wedge p_1)}{(p_1^2 - m^2 + i\epsilon)((p_1 - k_1)^2 - m^2 + i\epsilon)} \tag{4.17}$$

Whereas the NC Feynman rules with IPTO lead to

$$\begin{aligned} M \propto \lambda^2 \sum_{\sigma_{1,2} \in \{-,+\}} & \int \frac{d^3p_1}{E_{\vec{p}_1}} \int \frac{d^3p_2}{E_{\vec{p}_2}} \frac{1}{4}(1 + \sigma_1 \frac{k_1^0}{E_{\vec{k}_1}})(1 + \sigma_2 \frac{k_2^0}{E_{\vec{k}_2}}) \\ & 2\pi\delta(k_1^0 - k_2^0)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{k}_1)\delta^3(\vec{p}_1 + \vec{p}_2 - \vec{k}_2) \\ & \Bigg( \frac{\sum_{sym} e^{-i(-k_{1,\sigma_1},p_{1+},p_{2+})} e^{-i(-k_{2,\sigma_2},p_{1+},p_{2+})}}{k_1^0 - E_{\vec{p}_1} - E_{\vec{p}_2} + i\epsilon} + \\ & \frac{\sum_{sym} e^{-i(-k_{1,\sigma_1},p_{1-},p_{2-})} e^{-i(-k_{2,\sigma_2},p_{1-},p_{2-})}}{-k_2^0 - E_{\vec{p}_1} - E_{\vec{p}_2} + i\epsilon} \end{aligned}$$

The main differences between Filk's Feynman rules and the NC Feynman rules with IPTO are that the four-momenta of intermediate states are on-shell in the second case and that space-time noncommutativity leads to the violation of unitarity for scalar fields in the first case [2], but not in the second as we will prove.

## 4.4 IPTO for gauge fields

But what about propagators with momenta in the numerator as for a spin-1/2 field? As before one again gets positive and negative energy contributions $\frac{i}{2q_0^{(\lambda)}} \frac{\not{q}^{(\lambda)} + m}{q_0 - q_0^{(\lambda)} + \lambda i\epsilon}$ and the poles are the same as in covariant perturbation theory, but the residue is modified and a regular term is added

$$\sum_{\lambda=\pm} e^{i\phi(q^{(\lambda)})} \frac{1}{2q_0^{(\lambda)}} \frac{\not{q}^{(\lambda)} + m}{q_0 - q_0^{(\lambda)} + \lambda i\epsilon} = \frac{R(q^{(+)}, q^{(-)})}{q^2 - m^2 + i\epsilon} - \gamma^0 \frac{R_-(q^{(+)}, q^{(-)})}{q_0^{(+)}} \tag{4.18}$$

with

$$R_\pm(q^{(+)}, q^{(-)}) = \frac{1}{2}(e^{i\phi(q^{(+)})} \pm e^{i\phi(q^{(-)})}). \tag{4.19}$$

The gauge boson propagator in IPTO can be derived analogously and considering Feynman gauge ($\xi = 1$) with no momenta in the numerator one gets $\frac{-ig_{\mu\nu}}{2q_0^{(\lambda)}} \frac{1}{q_0 - q_0^{(\lambda)} + \lambda i\epsilon}$.

### 4.4.1 Example for IPTO used on gauge fields: cubic interaction in NCQED

Consider the cubic interaction in the NCQED $L_3 = e\bar{\psi}*\not{A}*\psi + ei\partial_\mu A_\nu * [A^\mu,^* A^\nu]$. For the $e^+e^-\gamma$-vertex one gets:

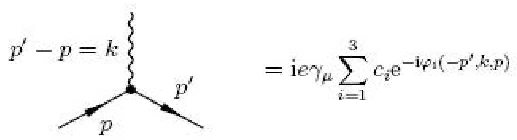


Figure 4.3: $e^+e^-\gamma$-vertex [8]

$$\bar{\psi} * \not{A} * \psi \to (c_1\bar{\psi}_\alpha * A_\mu * \psi_\beta + c_2 A_\mu * \psi_\beta * \bar{\psi}_\alpha + c_3\psi_\beta * \bar{\psi}_\alpha * A_\mu)\gamma^\mu_{\alpha\beta}$$
$$\stackrel{F.T.}{\to} \sum_{i=1}^{3} c_i e^{-i\phi_i(\bar{p},k,p)}\bar{\psi}(\bar{p})\not{A}(k)\psi(p) \quad (4.20)$$

with arbitrary coefficients $c_i$ obeying $c_1 + c_2 + c_3 = 1$ and $\phi_l(k_1, k_2, k_3) = \phi(k_l, k_m, k_n)$ for cyclical permutations $l, m, n$ of $1, 2, 3$.

In the case of the $3\gamma$-vertex the ambiguity in the Moyal phases leads to

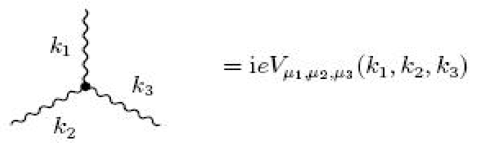


Figure 4.4: $3\gamma$-vertex [8]

$$\partial_\mu A_\nu * [A^\mu,^* A^\nu] \to c'_1 i\partial_\mu A_\nu * A^\mu * A^\nu + c'_2 A^\mu * A^\nu * i\partial_\mu A_\nu + c'_3 A^\nu * i\partial_\mu A_\nu * A^\mu$$
$$-c'_1 i\partial_\mu A_\nu * A^\nu * A^\mu - c'_2 A^\nu * A^\mu * i\partial_\mu A_\nu - c'_3 A^\mu * i\partial_\mu A_\nu * A^\nu \quad (4.21)$$

with $c'_1 + c'_2 + c'_3 = 1$. Fourier transforming the above formula one faces another ambiguity with the derivative couplings. Whereas in covariant perturbation theory derivatives can be shifted by partial integration from one field to the other fields at the same vertex, in TOPT one gets different results because energy isn't conserved. WI's are used to derive an unambiguous prescribtion for the choice of momenta corresponding to the derivatives after Fourier transformation which we will denote by $\bar{k}_i$. So we get for the $3\gamma$-vertex:

$$iV_{\mu_1,\mu_2,\mu_3}(k_1,k_2,k_3) = \sum_{i=1}^{3} c_i' \left(\bar{k}_1^{\mu_2} g^{\mu_1\mu_3} - \bar{k}_1^{\mu_3} g^{\mu_1\mu_2}\right) \times \\ \left(e^{-i\phi_i(k_1,k_2,k_3)} - e^{-i\phi_i(k_1,k_3,k_2)}\right) + cyclic\{1,2,3\}, \tag{4.22}$$

where $k_{1,2,3}$ are the on-shell momenta of IPTO.

# Chapter 5

# The modified Feynman rules for scalar field theory by Denk and Schweda

As noncommutative quantum field theory (NCQFT) is governed by non-local interactions, the usual Feynman rules for time ordered perturbation theory in general don't apply. The first one to study NCQFT perturbatively was T. Filk [9] who obtained the usual Feynman propagator and additional phase factors. However, it is shown by Denk and Schweda in [1] that this treatment is only applicable for cases where the deformation of space-time does not involve time. In the case of time noncommutativity a different kind of time ordering has to be applied called interaction point time ordering (IPTO) which leads to the following modified Feynman rules for scalar field theory.

## 5.1 Coordinate space rules

In the following the diagrammatic rules in coordinate space will be given for calculating

$$G_m^n(x_1, ..., x_n) = \frac{(-i)^m}{m!} \int dt_{n+1}...dt_N \langle 0| T\{\phi(x_1)...\phi(x_n)V(t_{n+1})...V(t_N)\} |0\rangle \tag{5.1}$$

$$V(t) = \int d\lambda v(\lambda, t)\phi(g_1(\lambda, t))\phi(g_2(\lambda, t)), \tag{5.2}$$

where n is the number of external points and m denotes the order of interactions.

- Draw n points and label them with the external coordinates $x_1, ..., x_n$. Their time stamps are $x_1^0, ..., x_n^0$, respectively.
- Draw m circles and label them with the parameters $\lambda_1, ..., \lambda_m$ and the time stamps $t_{n+1}, ..., t_{n+m}$.

- Draw k points into each circle and label them with $g_1, ..., g_k$ where k is the sum of internal and external lines attached to each circle.

- For each possibility of connecting two points pairwise by a line, so that each point is connected to exactly one line, draw a diagram with points and circles as given above.

- For each line connecting two points with coordinates and time stamp x, t and x', t', respectively, write down a contractor

$$-i\Delta(x,t;x',t') = \theta(t-t')\Delta^+(x,x') + \theta^+(t'-t)\Delta^+(x',x), \qquad (5.3)$$

  if the points do not belong to the same circle. If they belong to the same circle, write down either $\Delta^+(x,x')$ or $\Delta^+(x',x)$, depending on whether $\phi(x)$ stands left of $\phi(x')$ within the interaction V(t) or vice versa. External points already carry the 0th componant as time stamp $t_i$, that is to say the time stamp of the circle. The coordinate x of such a point is given by $x = g_j(\lambda, t)$

- For each circle labeled with $\lambda_i$ and $t_{n+i}$ perform the integration

$$(-i)\int dt_{n+i} d\lambda_i v(\lambda_i, t). \qquad (5.4)$$

- Sum up the contributions of all diagrams.

We will mainly consider interactions of the form $V_k(z^0) = \frac{\kappa}{k!}\int d^3z(\phi(z))^{*k}$, which have the following integralrepresentation:

$$V_k(z^0) = \frac{\kappa}{k!}\int d^3z \sum_{i=1}^{k-1}\left(d^4s_i d^4l_i \frac{e^{is_il_i}}{(2\pi)^4}\right)\phi\left(z - \frac{1}{2}l_i + \sum_{j=1}^{i-1}s_j\right)\phi\left(z + \sum_{j=1}^{k-1}s_j\right) \qquad (5.5)$$

where $l_i{}^\nu = l_\mu\theta^{\mu\nu}$. In the interaction point time ordering (IPTO) the time ordering involves only $z^0$. In this case the last but one Feynman rule can be written as

$$\int \sum_{i=1}^{m-1}\left(d^4s_i d^4l_i \frac{e^{is_il_i}}{(2\pi)^4}\right). \qquad (5.6)$$

## 5.2 Momentum space rules

- Draw all possible momentum space Feyman diagrams having n external lines.

- Carefully label each inner and outer line with a four momentum including its flow and make use of the conservation of four momentum at each vertex. External lines are labeled with momenta $p_1, ..., p_n$ with the convention that the $p_is$ are incoming. To every inner line attach also a $\sigma_i$.

- For each inner line write down a propagator of the form

$$\frac{-i}{q^2+m^2-i\epsilon}\frac{\omega_{\mathbf{q}}+\sigma^q q^0}{2\omega_{\mathbf{q}}} \tag{5.7}$$

  where q and $\sigma^q$ represent the labels of the corresponding line.

- For each vertex include a factor $-i\chi(...)$ with the rule to insert $(\pm q_i)^{\sigma_i} = (\pm\mathbf{q}, \pm\sigma_i\omega_{q_i})^T$ into $\chi$ for each line labeled by $q_i$, $\sigma_i$ at the corresponding vertex. Take the + sign for momenta flowing into the vertex otherwise choose the − sign. Due to the symmetry of $\chi$, the order of arguments is not relevant.

- Include the symmetry factor 1/S.

- Assure momentum conservation by $(2\pi)^4\delta^4(p_1+...+p_n)$.

- Integrate over the L independent momenta, which are not fixed by energy-momentum conservation and multiply by $(2\pi)^{-4L}$. Sum over all $\sigma s$.

- Sum up all diagrams in the usual sense.

## 5.3 Unitarity of the one loop graph in scalar field theory

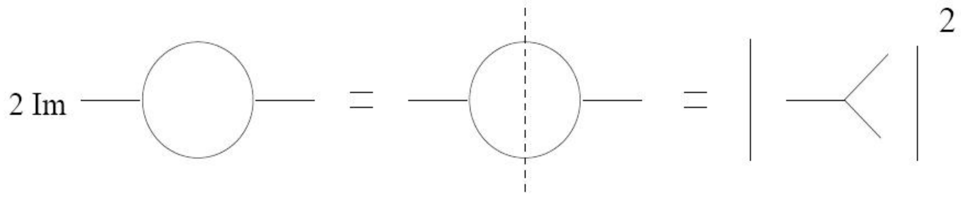


Figure 5.1: The optical theorem applied to the one loop graph [2]

### 5.3.1 Calculations for time-like momentum: $\vec{p} = 0$ (CM-frame)

$$iM = \sum_\sigma \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m^2-i\epsilon}\frac{\omega_{\vec{k}}+\sigma_k k^0}{2\omega_{\vec{k}}}\frac{-i}{(k+p)^2+m^2-i\epsilon}\frac{\omega_{\vec{k}+\vec{p}}+\sigma_{k+p}(k^0+p^0)}{2\omega_{\vec{k}+\vec{p}}}$$
$$(-i)\chi(p,k^{\sigma_k},-(p+k)^{\sigma_{p+k}})(-i)\chi(-p,-k^{\sigma_k},(p+k)^{\sigma_{p+k}}) = \int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2-i\epsilon}\frac{1}{k^2+m^2-i\epsilon}\frac{1}{4\omega_{\vec{k}}^2}$$
$$\big((\omega_{\vec{k}}+k^0)(\omega_{\vec{k}}+(p^0+k^0))\chi^2(p,k^+,-(p+k)^+)+(\omega_{\vec{k}}+k^0)(\omega_{\vec{k}}-(p^0+k^0))\chi^2(p,k^+,-(p+k)^-)$$
$$+(\omega_{\vec{k}}-k^0)(\omega_{\vec{k}}+(p^0+k^0))\chi^2(p,k^-,-(p+k)^+)+(\omega_{\vec{k}}-k^0)(\omega_{\vec{k}}-(p^0+k^0))\chi^2(p,k^-,-(p+k)^-)\big) \tag{5.8}$$

We get the following residues for the integration over the $k^0$-component and closing the integration contour in the lower half plane:

$$-k_0^2 + \vec{k}^2 + m^2 - i\epsilon = 0 \tag{5.9}$$

$$k^0_{1,2} = \pm(\omega_{\vec{k}} - i\epsilon) \tag{5.10}$$

$$-(k_0 + p_0)^2 + \vec{k}^2 + m^2 - i\epsilon = 0 \tag{5.11}$$

$$k^0_{3,4} = -p_0 \pm (\omega_{\vec{k}} - i\epsilon) \tag{5.12}$$

$$Res(k_1^0 = \omega_{\vec{k}} - i\epsilon) = \frac{(2\omega_{\vec{k}} + p^0)\chi^2(p,+,+) - p^0\chi^2(p,+,-)}{p^0(p^0 - 2\omega_{\vec{k}} - i\epsilon)} \tag{5.13}$$

$$Res(k_3^0 = -p^0 + \omega_{\vec{k}} - i\epsilon) = \frac{(2\omega_{\vec{k}} - p^0)\chi^2(p,+,+) + p^0\chi^2(p,-,+)}{p^0(p^0 - 2\omega_{\vec{k}} - i\epsilon)} \tag{5.14}$$

As we are considering a loop in the s-channel and thus $p^0 > 0$ we must only consider the contribution from the pole $k_3^0$. So we are left with:

$$iM = (-2\pi i)\frac{1}{(2\pi)^4}\frac{1}{4}\int d^3k \frac{1}{\omega_{\vec{k}}^2}\frac{(2\omega_{\vec{k}} - p^0)\chi^2(p,+,+) + p^0\chi^2(p,-,+)}{p^0(p^0 - 2\omega_{\vec{k}} - i\epsilon)} \tag{5.15}$$

By going over to polar coordinates and setting $\left|\vec{k}\right| = r$ and $cos(\Theta) = x$ we get the following result:

$$iM = (-2\pi i)\frac{1}{(2\pi)^3}\frac{1}{4p^0}\int_{-1}^{1} dx \int_0^\infty dr \frac{r^2}{r^2+m^2}\frac{(2\sqrt{r^2+m^2} - p^0)\chi^2(p,+,+) + p^0\chi^2(p,-,+)}{p^0 - 2\sqrt{r^2+m^2} - i\epsilon} \tag{5.16}$$

The discontinuity is defined by

$$Disc(f(x)) = lim_{\epsilon\to\infty}(f(x+i\epsilon) - f(x-i\epsilon)) \tag{5.17}$$

Using $\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} = -2i\pi\delta(x)$ we get by setting $y = \sqrt{r^2+m^2}$:

$$\frac{1}{p^0 - 2y - i\epsilon} - \frac{1}{p^0 - 2y + i\epsilon} = -2i\pi\delta(p^0 - 2y) = (-2i\pi)\frac{1}{2}\delta(y - \frac{p^0}{2}) \tag{5.18}$$

Thus the discontinuity is given by

$$Disc(iM) = \frac{1}{2\pi}\frac{1}{4p^0}\int_{-1}^{1} dx \int_m^\infty dy \frac{\sqrt{y^2-m^2}}{y}((2y - p^0)\chi^2(p,+,+) + p^0\chi^2(p,-,+))\delta(p^0 - 2y) \tag{5.19}$$

$$
\begin{aligned}
Disc(iM) = & \quad \frac{1}{2\pi}\frac{1}{4p^0}\frac{1}{2}\frac{\sqrt{p_0^2/4-m^2}}{p_0/2}\int_{-1}^{1} dx p_0 \chi^2(p,-,+) \\
= & \quad \frac{1}{2\pi}\frac{1}{4p^0}\sqrt{p_0^2/4-m^2}\int_{-1}^{1} dx \chi^2(p,-,+)
\end{aligned} \tag{5.20}
$$

To confirm the optical theorem we must now calculate the sum over the absolute amplitudes:

$$
\sum |M|^2 = \frac{1}{(2\pi)^4}\int d^4q d^4l \delta^4(p-q-l)\chi^2(p,-l,-q)(-2i\pi)\delta^4(q^2-m^2)(-2i\pi)\delta^4(l^2-m^2) \tag{5.21}
$$

Using $\frac{1}{2\omega_{\vec{q}}} = \int dq_0 \delta(q^2-m^2)$ the formula above is equivalent to

$$
\sum |M|^2 = \frac{1}{(2\pi)^2}\int \frac{d^3q}{2\omega_{\vec{q}}}\frac{d^3l}{2\omega_{\vec{l}}}\delta^4(p-q-l)\chi^2(p,-l,-q) \tag{5.22}
$$

As $\vec{p}=0$ we get $\omega_{\vec{q}} = \omega_{\vec{l}}$ and $\vec{q} = -\vec{l}$ which results in:

$$
\sum |M|^2 = \frac{1}{(2\pi)^2}\int \frac{d^3l}{4\omega_{\vec{l}}^2}\delta(p^0 - 2\sqrt{l^2+m^2})\chi^2(p,-l,-q) \tag{5.23}
$$

Substituing $r = \left|\vec{l}\right|$ and $x = cos(\Theta)$ and using $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ it follows that the sum of the amplitudes equals:

$$
\begin{aligned}
\sum |M|^2 = \frac{1}{2\pi}\frac{1}{4}\int_0^\infty dr \frac{r^2}{r^2+m^2}\frac{\delta(r-\sqrt{p_0^2/4-m^2})}{4/p_0\sqrt{p_0^2/4-m^2}}\int_{-1}^{1} dx\chi^2(p,-l^+,+l^-) \\
= \frac{1}{2\pi}\frac{1}{4}\sqrt{p_0^2/4-m^2}\int_{-1}^{1} dx\chi^2(p,-l^+,+l^-)
\end{aligned} \tag{5.24}
$$

### 5.3.2 Calculations for spacelike momentum: $\mathbf{p}^0 = 0$

$$
\begin{aligned}
iM = \sum_\sigma \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m^2-i\epsilon}\frac{\omega_{\vec{k}}+\sigma_k k^0}{2\omega_{\vec{k}}}\frac{-i}{(k+p)^2+m^2-i\epsilon}\frac{\omega_{\vec{k}+\vec{p}}+\sigma_{k+p}(k^0+p^0)}{2\omega_{\vec{k}+\vec{p}}} \\
(-i)\chi(p,k^{\sigma_k},-(p+k)^{\sigma_{p+k}})(-i)\chi(-p,-k^{\sigma_k},(p+k)^{\sigma_{p+k}}) \\
= \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2-i\epsilon}\frac{1}{(k+p)^2+m^2-i\epsilon}\frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}+\vec{k}}} \\
\big((\omega_{\vec{k}}+k^0)(\omega_{\vec{k}}+k^0)\chi^2(p,k^+,-(p+k)^+) \\
+(\omega_{\vec{k}}+k^0)(\omega_{\vec{k}}-k^0)\chi^2(p,k^+,-(p+k)^-) \\
+(\omega_{\vec{k}}-k^0)(\omega_{\vec{k}}+k^0)\chi^2(p,k^-,-(p+k)^+) \\
+(\omega_{\vec{k}}-k^0)(\omega_{\vec{k}}-k^0)\chi^2(p,k^-,-(p+k)^-)
\end{aligned} \tag{5.25}
$$

The poles for the $k^0$ integration are the same as before except that we have to put $p^0 = 0$ and by closing the integration contour in the lower half plane we get the following residues:

$$
\begin{aligned}
Res(k_1^0 = \omega_{\vec{k}} - i\epsilon) &= \frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}+\vec{k}}} \frac{1}{(\omega_{\vec{k}} - \omega_{\vec{p}+\vec{k}})(\omega_{\vec{k}} + \omega_{\vec{p}+\vec{k}})} \\
&((\omega_{\vec{k}} + \omega_{\vec{p}+\vec{k}})\chi^2(p,+,+) + (\omega_{\vec{k}} - \omega_{\vec{p}+\vec{k}})\chi^2(p,+,-)) \qquad (5.26) \\
Res(k_3^0 = \omega_{\vec{k}} - i\epsilon) &= \frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}+\vec{k}}} \frac{1}{(\omega_{\vec{k}} - \omega_{\vec{p}+\vec{k}})(\omega_{\vec{k}} + \omega_{\vec{p}+\vec{k}})} \\
&(-(\omega_{\vec{k}} + \omega_{\vec{p}+\vec{k}})\chi^2(p,+,+) - (\omega_{\vec{k}} - \omega_{\vec{p}+\vec{k}})\chi^2(p,+,-)) \qquad (5.27) \\
Res(k_1^0) + Res(k_3^0) &= \frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}+\vec{k}}} \frac{\chi^2(p,+,-) - \chi^2(p,-,+)}{\omega_{\vec{k}} + \omega_{\vec{p}+\vec{k}}} \qquad (5.28)
\end{aligned}
$$

The following amplitude

$$
iM = -i \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_{\vec{k}}\omega_{\vec{p}+\vec{k}}} \frac{\chi^2(p,+,-) - \chi^2(p,-,+)}{\omega_{\vec{k}} + \omega_{\vec{p}+\vec{k}}} \qquad (5.29)
$$

is real, so $Im(M) = 0$ which is equivalent to the sum over the amplitudes.

# Chapter 6

# The Hamiltonian approach and the Yang-Feldman formalism

## 6.1 The Hamiltonian approach

Feynman rules for noncommutative Minkowski spacetime have also been worked out by D. Bahns [3] where the same method of time ordering is used. However, starting point for the derivation of the rules is the S-matrix. But as in the framework worked out by Denk and Schweda unitarity is not violated.

Any S-matrix defined by the limit of the time evolution operator $U(t,s)$ where $t \to \infty$ and $s \to -\infty$,

$$S[g] = 1 + \sum_{r=1}^{\infty} S_r[g] \tag{6.1}$$

$$\begin{aligned} S_r[g] = \frac{(-i)^r}{r!} \sum_{\pi \in P_r} \int dt_1 ... dt_r \theta(t_{\pi_1} - t_{\pi_2}) ... \theta(t_{\pi_{r-1}} - t_{\pi_r}) H_I^g(t_{\pi_1}) ... H_I^g(t_{\pi_r}) \\ = (-i)^r \int dt_1 ... dt_r \theta(t_1 - t_2) ... \theta(t_r - 1 - t_r) H_I^g(t_1) ... H_I^g(t_r) \end{aligned} \tag{6.2}$$

(with $\theta$ denoting the Heaviside step function), is formally unitary (i.e. before renormalization),

$$SS^\dagger = 1 + S_1 + S_1^\dagger + (S_2 + S_1 S_1^\dagger + S_2^\dagger) + ... = 1, \tag{6.3}$$

if the interaction Hamiltonian is symmetric, $H_I(t) = H_I(t)^\dagger$.

*Proof.* The claim is a consequence of the way the time-ordering has been defined.

$$\sum_{N_1+N_2=N} S_{N_1} S_{N_2}^{\dagger} == i^N \sum_{N_1=0}^{N} (-1)^{N_1} \int dt_1 ... dt_N \Theta(t_1 - t_2) ... \Theta(t_{N_1-1} - t_{N_1}) \cdot$$
$$\cdot \Theta(t_{N_1+1} - t_{N_1+2}) ... \Theta(t_{N-1} - t_N) H_I(t_1) ... H_I(t_{N_1}) H_I(t_N) ... H_I(t_{N_1+1})$$
$$= i^N \int dt_1 ... dt_N H_I(t_1) ... H_I(t_N) \sum_{N_1=0}^{N} (-1)^{N_1} \prod_{i=1}^{N_1-1} \Theta(t_i - t_{i+1}) \prod_{i=N_1+1}^{N-1} \Theta(t_{i+1} - t_i) \tag{6.4}$$

where the Heavyside functions with arguments such as $t_N - t_{N+1}$ and empty products such as $\prod_{i=N}^{N-1}$ are set to 1. The above sum is 1 for N=0 and using

$$\sum_{N_1=0}^{N+1} (-1)^{N_1} \prod_{i=1}^{N_1-1} \Theta(t_i - t_{i+1}) \prod_{i=N_1+1}^{N} \Theta(t_{i+1} - t_i)$$
$$= \sum_{N_1=0}^{N} (-1)^{N_1} \prod_{i=1}^{N_1-1} \Theta(t_i - t_{i+1}) \prod_{i=N_1+1}^{N-1} \Theta(t_{i+1} - t_i)$$
$$\cdot \begin{cases} \Theta(t_{N+1} - t_N) & N_1 < N \\ 1 & N_1 = N \end{cases} + (-1)^{N+1} \prod_{i=1}^{N} \Theta(t_i - t_{i+1}) \tag{6.5}$$

together with

$$1 = \Theta(t_{N+1} - t_N) + \Theta(t_N - t_{N+1}) \tag{6.6}$$

one proves by induction that $\sum SS^{\dagger} = 1$. □

Therefore if the time-oredring in the S-Matrix is defined with respect to the parameter t appearing in the Hamiltonians $H_I(t)$, the theory will automatically be unitary.

### 6.1.1 Feynman rules in position space for normal ordered $\phi^n$ interaction

- Draw all ordinary connencted Feynman graphs of the process under consideration, characterized by the number of vertices and external momenta. Consider all possibilities to distribute the external momenta to the vertices.
- Pick one of the above graphs and assign vectors to its vertices.
- Choose one particular time order and write down the Heaviside functions

$$\int d^4x_1 ... d^4x_r \Theta(...) ... \Theta(...). \tag{6.7}$$

- For every internal line write down a mass-shell integral

$$\frac{1}{(2\pi)^3} \int \frac{dk}{2\omega_k} e^{-ik(x_i - x_j)} \tag{6.8}$$

where $x_{i,0} > x_{j,0}$ and the internal momentum k labels a directed line leading from $x_j$ to $x_i$.

- For an external momentum q leaving the vertex $x_i$ multiply with $(2\pi)^{-3/2}e^{iqx_i}$. For an external momentum q entering the vertex $x_i$ multiply with $(2\pi)^{-3/2}e^{-iqx_i}$.
- At each vertex the twisting is determined by the following rules: an external momentum leaving the vertex enters with a - sign; an external momentum flowing into the vertex enters with a + sign; an internal momentum enters with a + sign, if the vertex is the endpoint of the momentum's line, and it enters with a - sign if the vertex is the starting point of the momentum's line.
- For each vertex multiply with a factor $\frac{g}{n!}$. Multiply the expression with the symmetry factor and $\frac{(-i)^r}{r!}$, if there are r vertices.

### 6.1.2 Feynman rules in momentum space for normal ordered $\phi^n$ interaction

- Draw all ordinary connected Feynman graphs of the process under consideration, characterized by the number of vertices and external momenta. Consider all possibilities to distribute the external momenta to the vertices.
- Pick one of the above graphs and assign times $t_1, ..., t_r$ to its vertices. Choose a particular time-ordering.
- For every internal line write down a mass-shell integral

$$\frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k} \tag{6.9}$$

where k labels the directed line connecting the earlier vertex with the later one.

- For each vertex j apart from the earliest one write down the following energy factors,

$$\frac{i}{2\pi}\frac{1}{-\sum_i k_{i,0} + i\epsilon} \tag{6.10}$$

where the sum runs over the 0-components of the internal momenta flowing into the vertex, the internal momenta flowing into any of the later vertices, provided they start at earlier vertices than the vertex under consideration, and external momenta, which flow into or out of the vertex under consideration or any of the later vertices of the graph.

- At each vertex impose 3-momentum conservation $(2\pi)^3\delta^{(3)}(...)$ and overall energy conservation of the external momenta $2\pi\delta(...)$. For every external momentum multiply with a factor $(2\pi)^{-3/2}$.
- At each vertex the twisting is determined by the following rules: an external momentum leaving the vertex enters with a - sign;

an external momentum flowing into the vertex enters with a + sign;
an internal momentum enters with a + sign, if the vertex is te endpoint of the momentum's line, and it enters with a - sign if the vertex is the starting point of the momentum's line.

- For each vertex multiply with a factor $\frac{g}{n!}$. Multiply the expression with the Symmetry factor and $\frac{(-i)^r}{r!}$, if there are r vertices.

It remains to show that the amplitudes in both frameworks agree on-shell, which would be the proposed way to get out of the disagreement with the Feynman rules of Denk and Schweda.

## 6.2 The Yang-Feldman formalism

Another possibility to define quantum field theory on noncommutative Minkowski space is based on the Yang-Feldman equation, which turns out to be inequivalent to the Hamiltonian approach. Here the field equation is used as a starting point and the interaction field is constructed iteratively. This formalism is Lorentz-covariant and works exclusively in the Heisenberg picture.The initial conditions are not given at a fixed instant in time, but asymptotically at infinite times [3].

### 6.2.1 Classical perturbation theory

We are now considering the field equation of a classical field on NC Minkowski space with a self-interaction given by $\phi^{n-1}(q)$,

$$(\Box_q - m^2)\phi(q) = -g\phi^{n-1}(q) \tag{6.11}$$

The field equation can be solved recursively by the ansatz

$$\phi(q) = \sum_{\kappa=0}^{\infty} g^{\kappa}\phi_{\kappa(q)} \tag{6.12}$$

where

$$\phi_\kappa(q) = \sum_{\kappa_1+\ldots\kappa_{n-1}=\kappa-1} (G \times \phi_{\kappa_1}\ldots\phi_{\kappa_{n-1}})(q) \tag{6.13}$$

$$= \sum_{\kappa_1+\ldots\kappa_{n-1}=\kappa-1} \int dx G(x)\phi_{\kappa_1}(q-x)\ldots\phi_{\kappa_{n-1}}(q-x) \tag{6.14}$$

with some ordinary Green function G of the Klein-Gordan equation, which is fixed by initial conditions given at infinite times.

*Proof.* The field equation at order $\kappa$ reads

$$g^{\kappa}(\Box_a - m^2)\phi_\kappa(q+a)\big|_{a=0} = -g^{\kappa} \sum_{\sum \kappa_i=\kappa-1} \phi_{\kappa_1}\ldots\phi_{\kappa_{n-1}}(q) \tag{6.15}$$

and is solved by

$$(\Box_a - m^2) \sum_{\sum \kappa_i = \kappa - 1} \int dx G(x)\ \phi_{\kappa_1}(q+a-x)...\phi_{\kappa_{n-1}}(q+a-x)\big|_{a=0}$$

$$= (\Box_a - m^2) \sum_{\sum \kappa_i \kappa - 1} \int dx G(x+a)\ \phi_{\kappa_1}(q-x)...\phi_{\kappa_{n-1}}(q-x)\big|_{a=0}$$

$$= - \sum_{\sum \kappa_i = \kappa - 1} \phi_{\kappa_1}...\phi_{\kappa_{n-1}}(q) \quad (6.16)$$

Initial conditions at infinity apply in the following sense:

$$\lim_{t \to -\infty} \phi(q + te_0) = \lim_{t \to -\infty} \sum_{\kappa} g^{\kappa} \sum_{\sum \kappa_i \kappa - 1} \int dx G(x + te_0) \phi_{\kappa_1}(q-x)...\phi_{\kappa_{n-1}}(q-x) \quad (6.17)$$

□

On the level of the interacting field unitarity means that the field is Hermitean. It can be easily seen by construction that when the incoming field is Hermitean so is the interacting field $\phi_\kappa$ at any order $\kappa$.

*Proof.* Let $\phi_0$, the field at zeroth order, be the incoming field, then the interacting field at the order $\kappa$ is

$$\phi_\kappa(q) = \sum_{\sum \kappa_i = \kappa - 1} \int dx \Delta_{ret}(x) \phi_{\kappa_1}(q-x)...\phi_{\kappa_{n-1}}(q-x) \quad (6.18)$$

where $\Delta_{ret} = \theta(x_0)\Delta(x)$ is called the retarded propagator. So if the zeroth order field is Hermitean, $\phi_0 = \phi_0$ and the propagator real as in our case, the interacting field is Hermitean too:

$$\phi_\kappa(q) = \sum_{\sum \kappa_i = \kappa - 1} \int dx \Delta_{ret}(x) \phi^{deggar}_{\kappa_{n-1}}(q-x)...\phi^{deggar}_{\kappa_1}(q-x) = \phi_\kappa(q) \quad (6.19)$$

□

### 6.2.2 The quantum perturbation theory

In the case of quantum fields on the noncommutative Minkowski space starting point is the free quantum field

$$\phi(q+x) = (2\pi)^{-3/2} \int \frac{dk}{.2\omega_k \left(a(k) \otimes e^{-ik(q+x)} + a(k) \otimes e^{ik(q+x)}\right)\big|_{k \in H_m^+}} \quad (6.20)$$

where $H_m^+$ denotes the positive mass shell and $a$ respectively $a^t$ are the ordinary annihilation and creation operators on the symmetric Fock space. A q-field is then associated to a linear map from testfunctions to closable operators as an adiabatic infrared cutoff function g is needed to make the convolution with Green functions well-defined. This yields

$$\phi^g_\kappa(q) = \sum_{\kappa_1 + ... + \kappa_{n-1} = \kappa - 1} \int dx g(x) \Delta_{ret}(x) \phi_{\kappa_1}(q-x)...\phi_{\kappa_{n-1}}(q-x). \quad (6.21)$$

The proof that the interacting field is Hermitean and therefore the theory unitary follows by induction from the classical case because normal ordering does not spoil Hermiticity as it is defined as a substraction of Hermitean terms.

# Chapter 7

# Unitarity of gauge fields

Above we have proven the unitarity of scalar fields as the unitarity of the one loop graph can be generalized to the unitarity in any order of perturbation theory via the cutting rules. The unitarity of time-like noncommutative gauge theories (NCGT) has first been studied in [11] by Liao and Dehne for the case of noncommutative Quantum Electrodynamis (NCQED) without external bosons where unitarity has been established. Nevertheless TOPT isn't the answer because Thorsten Ohl, Reinhold Rückl and Jörg Zeiner [8] found a violation of Ward identities in time-ordered perturbation theory in the case of simple processes with external gauge bosons when all orders of the noncommutative parameters $\Theta^{i0}$ are taken into account. So at last time-ordered perturbation theory cannot solve the unitarity problem of timelike noncommutative quantum field theories. In the following we want to look at this violation of Ward identities, which lead to an invalidation of the cutting rules for loops involving gauge bosons and therefore rander NCQFT nonunitary in this case. We will only look at NCQED because the simplest example for the violation of WIs in TOPT is provided by Compton scattering $e^-\gamma \to e^-\gamma$ but the argumentation is valid for arbitrary U(N) NCGT.

## 7.1 Ward identities for NCQED

In NCQED the Lagrangian looks like

$$L = -\frac{1}{4}F_{\mu\nu} * F_{\mu\nu} + \bar{\psi} * (i\not{D} - m) * \psi \tag{7.1}$$

with

$$D_\mu = \partial_\mu - ieA_\mu \tag{7.2}$$

$$F_{\mu\nu} = \frac{i}{e}[D_\mu,^* D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu,^* A_\nu] \tag{7.3}$$

and is invariant under the gauge transformations

$$\delta_\eta \psi = ie\eta * \psi \tag{7.4}$$

$$\delta_\eta \bar{\psi} = -ie\bar{\psi} * \eta \tag{7.5}$$

$$\delta_\eta A_\mu = [D_\mu,^* \eta] \tag{7.6}$$

where the commutator is defined by

$$[f,^* g] = f * g - g * f. \tag{7.7}$$

This Lagrangian together with an appropriate gauge fixing term,

$$L_{g.f.} = \delta_{BRST}\left(\bar{c} * \left(\frac{\xi}{2}B + \partial_\mu A^\mu\right)\right) \tag{7.8}$$

with c the Faddeev-Popov ghosts, $\bar{c}$ the antighosts and B the Nakanishi-Lautrup field, yields an invariant under the following BRST transformations:

$$\delta_{BRST}\psi = ie\eta * \psi \tag{7.9}$$

$$\delta_{BRST}\bar{\psi} = -ie\bar{\psi} * \eta \tag{7.10}$$

$$\delta_{BRST}A_\mu = [D_\mu,^* \eta] \tag{7.11}$$

$$\delta_{BRST}c = i\,[c,^* c] \tag{7.12}$$

$$\delta_{BRST}\bar{c} = B \tag{7.13}$$

$$\delta_{BRST}B = 0 \tag{7.14}$$

This invariance leads to relations among the Green functions known as Slavnov-Taylor identities (STIs). one of them can be derived from

$$\langle 0|\, T\delta_{BRST}\left(\bar{c}(x)\Phi_1(x_1)\Phi_2(x_2)...\Phi_n(x_n)\right)|0\rangle = 0. \tag{7.15}$$

Using the equations of motion for $B = -\partial_\mu A^\mu/\xi$ we get:

$$\begin{aligned} &\frac{\partial}{\partial x^\mu}\langle 0|\, TA^\mu(x)\Phi_1(x_1)...\delta_{BRST}\Phi_i(x_i)...\Phi_n(x_n)|0\rangle = \\ \xi\sum_i(\pm)&\langle 0|\, T\bar{c}(x)\Phi_1(x_1)\Phi_2(x_2)...\delta_{BRST}\Phi_i(x_i)...\Phi_n(x_n)|0\rangle. \end{aligned} \tag{7.16}$$

The sign of each summand is fixed by the anticommuting nature of the BRST transformation. As the BRST transformations of the physical degrees of freedom with $\partial_\mu A^\mu = 0$ are bilinear in these fields and the ghost c, their contributions to the STIs are cancelled when matrix elements of physical fields are amputated on-shell. The on-shell STIs therefore reduce to the Ward Identities

$$\left\langle 0\left|T\frac{\partial}{\partial x^\mu}A^\mu(x)\phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)\right|0\right\rangle_{amputated,onshell} = 0. \tag{7.17}$$

This condition is equivalent to the following in momentum space for external photons

$$k_{1\alpha}M^{\alpha\beta...} = k_{2\beta}M^{\alpha\beta...} = ... = 0 \tag{7.18}$$

which is derived in ordinary QED directly from the gauge invariance which we will shortly demonstrate [14].In ordinary QED the gauge field transforms as

$$A_\mu \rightarrow A_\mu + \partial_\mu\alpha \tag{7.19}$$

In Lorentz gauge the gauge field can be described by a plane wave

$$A_\mu \approx \epsilon^*_\mu(k)e^{\pm ikx} \tag{7.20}$$

With $\alpha \approx \tilde{\alpha}(k)e^{\pm ikx}$ the gauge invariance translates in the following transformation of the polarization vector

$$\epsilon^*_\mu(k) \rightarrow \epsilon^*_\mu(k) \pm k_\mu \tilde{\alpha}(k) \tag{7.21}$$

Thus, the invariance of the amplitude M which looks like

$$M = \epsilon^*_\alpha(k_1)\epsilon^*_\beta(k_2)...M^{\alpha\beta\cdots}(k_1, k_2, ...) \tag{7.22}$$

for processes with external photons leads to the constraint

$$k_{1\alpha}M^{\alpha\beta\cdots} = k_{2\beta}M^{\alpha\beta\cdots} = ... = 0. \tag{7.23}$$

## 7.2 Violation of Ward Identities on Minkowski noncommutative spacetime

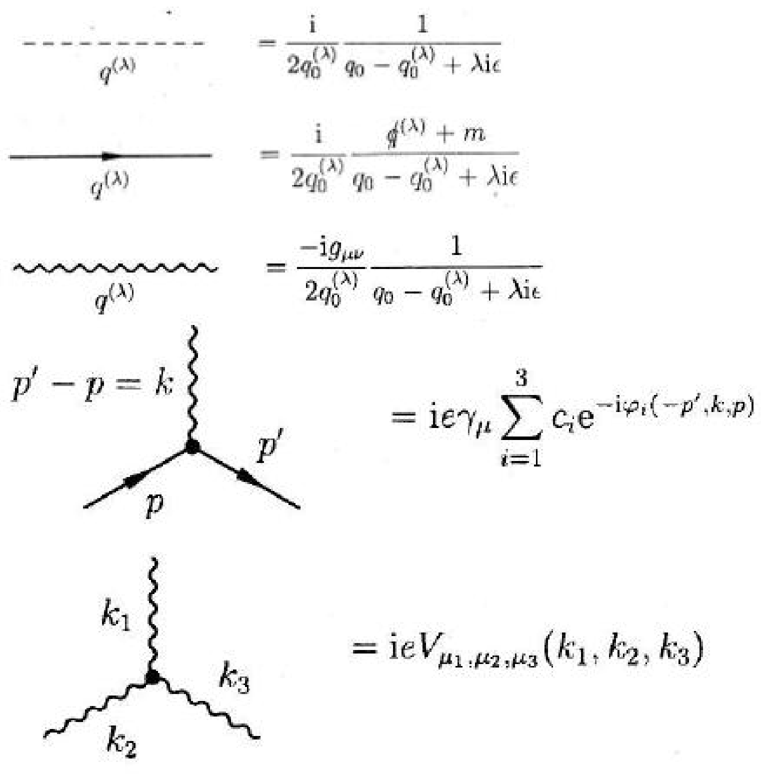


Figure 7.1: Feynman rules by Ohl, Rückl and Zeiner[8]

Can IPTO also render time-like NC gauge theories unitary? The answer given in an article by Ohl, Rückl and Zeiner [8] is NO: In the following we will prove that the use of IPTO for a proof of unitarity in NCQED will lead automatically to a violation of the above WIs for Compton scattering and therefore to a theory which doesn't make any physical sense. This is because in this case the charge generating the BRST transformation called BRST charge then won't be conserved and we won't be able to construct a positive norm Hilbert space for the physical asymptotic states from the cohomology of the BRST operator using the condition $Q_{BRST}\,|phys\rangle = 0$. Also this violation of the tree level WIs will invalidate the cutting rules for loops involving gauge bosons and therefore the optical theorem isn't fulfilled and unitarity is violated. In the following we will use the already derived Feynman rules for IPTO which can be seen as a generalization of the Feynman rules by Denk and Schweda to gauge fields.

*Proof.* There are three Feynman graphs contributing to the Compton scattering. Out of the three graphs we get the following derivative of the Greenfunction in

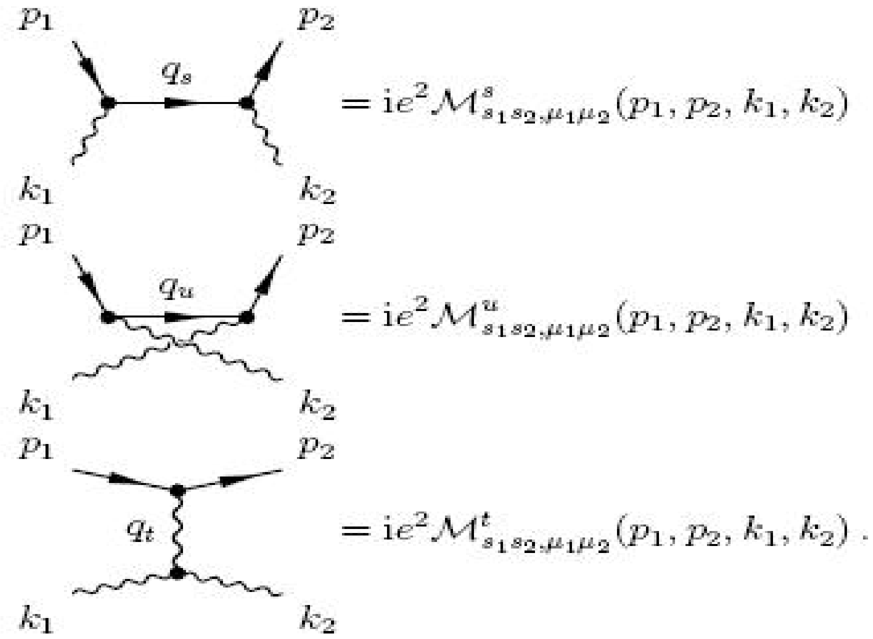


Figure 7.2: Graphs for Compton scattering[8]

momentum coordinates, which is the sum of the derivative of the Greenfunction for each graph and must be zero so that the above WI is satisfied:

$$k_1^{\mu_1}\,\langle 0|\,TA_{\mu_1}(k_1)\left(\epsilon^{\mu_2}_{(\kappa)}(k_2)A_{\mu_2}(k_2)\right)\psi_1(p_1)\overline{\psi}_2(p_2)\,|0\rangle_{amputated,on-shell} = W^s_{(\kappa)} + W^u_{(\kappa)} + W^t_{(\kappa)} \quad (7.24)$$

Now compute the derivatives of the Greenfunctions of the different channels using:

$$q_s = p_1 + k_1 = p_2 + k_2 \tag{7.25}$$

$$q_u = p_1 - k_2 = p_2 - k_1 \tag{7.26}$$

$$q_t = p_2 - p_1 = k_1 - k_2 \tag{7.27}$$

s-channel:
With

$$\frac{1}{\not{q}_s^{(+)} - m + i\epsilon} \not{k}_1 u_{s_1}(p_1) = \frac{1}{\not{q}_s^{(+)} - m + i\epsilon} (\not{p}_1 + \not{k}_1 - m) u_{s_1}(p_1) = u_{s_1}(p_1) \tag{7.28}$$

we get

$$W^s_{(\kappa)} = k_1^{\mu_1} \epsilon^{\mu_2}_{(\kappa)}(k_2) M^s_{s_1 s_2, \mu_1 \mu_2}(p_1, p_2, k_1, k_2) = W^{s,1}_{(\kappa)} + W^{s,0}_{(\kappa)} \tag{7.29}$$

$$W^{s,1}_{(\kappa)} = \sum_{i,j=1}^{3} c_i c_j R^{s,ij}(q_s^{(+)}, q_s^{(-)}) \bar{u}_{s_2}(p_2) \not{\epsilon}_{(\kappa)}(k_2) u_{s_1}(p_1) \tag{7.30}$$

$$W^{s,0}_{(\kappa)} = -\sum_{i,j=1}^{3} c_i c_j R^{s,ij}_{-}(q_s^{(+)}, q_s^{(-)}) \frac{1}{q^{(+)}_{s,0}} \bar{u}_{s_2}(p_2) \not{\epsilon}_{(\kappa)}(k_2) \gamma_0 \not{k}_1 u_{s_1}(p_1) \tag{7.31}$$

where

$$R^{s,ij}(q_s^{(+)}, q_s^{(-)}) = \sum_{\lambda=\pm} e^{-i\phi_i(-p_2, -k_2, q_s^{(\lambda)})} e^{-i\phi_j(-q_s^{(\lambda)}, k_1, p_1)} \frac{1}{2} \left(1 + \frac{q_{s,0}}{q^{(\lambda)}_{s,0}}\right) \tag{7.32}$$

$$R^{s,ij}_{-}(q_s^{(+)}, q_s^{(-)}) = \sum_{\lambda=\pm} e^{-i\phi_i(-p_2, -k_2, q_s^{(\lambda)})} e^{-i\phi_j(-q_s^{(\lambda)}, k_1, p_1)} \frac{\lambda}{2} \tag{7.33}$$

Similarly for the u-channel with $-k_2 \leftrightarrow k_1$:

$$W^u_{(\kappa)} = k_1^{\mu_1} \epsilon^{\mu_2}_{(\kappa)}(k_2) M^u_{s_1 s_2, \mu_1 \mu_2}(p_1, p_2, k_1, k_2) = W^{u,1}_{(\kappa)} + W^{u,0}_{(\kappa)} \tag{7.34}$$

$$W^{u,1}_{(\kappa)} = \sum_{i,j=1}^{3} c_i c_j R^{u,ij}(q_u^{(+)}, q_u^{(-)}) \bar{u}_{s_2}(p_2) \not{\epsilon}_{(\kappa)}(k_2) u_{s_1}(p_1) \tag{7.35}$$

$$W^{u,0}_{(\kappa)} = -\sum_{i,j=1}^{3} c_i c_j R^{u,ij}_{-}(q_u^{(+)}, q_u^{(-)}) \frac{1}{q^{(+)}_{u,0}} \bar{u}_{s_2}(p_2) \not{k}_1(k_2) \gamma_0 \not{\epsilon}_{(\kappa)} u_{s_1}(p_1) \tag{7.36}$$

where

$$R^{u,ij}(q_u^{(+)}, q_u^{(-)}) = \sum_{\lambda=\pm} e^{-i\phi_i(-p_2, -k_2, q_s^{(\lambda)})} e^{-i\phi_j(-q_s^{(\lambda)}, k_1, p_1)} \frac{1}{2} \left(1 + \frac{q_{u,0}}{q^{(\lambda)}_{u,0}}\right) \tag{7.37}$$

$$R^{u,ij}_{-}(q_u^{(+)}, q_u^{(-)}) = \sum_{\lambda=\pm} e^{-i\phi_i(-p_2, -k_2, q_u^{(\lambda)})} e^{-i\phi_j(-q_u^{(\lambda)}, k_1, p_1)} \frac{\lambda}{2} \tag{7.38}$$

t-channel:

$$W^t_{(\kappa)} = k_1^{\mu_1}\epsilon^{\mu_2}_{(\kappa)}(k_2)M^t_{s_1s_2,\mu_1\mu_2}(p_1,p_2,k_1,k_2)$$
$$= \sum_{i=1}^{3} c_i \sum_{\lambda=\pm} e^{-i\phi_i(-p_2,q_t^{(\lambda)},p_1)}\frac{1}{2}\left(1+\frac{q_{t,0}}{q^{(\lambda)}_{t,0}}\right)\times$$
$$\bar{u}_{s_2}(p_2)\gamma^{\mu_3}u_{s_1}(p_1)\frac{1}{q_t^2}V_{\mu_1\mu_2\mu_3}(k_1,-k_2,-q_t^{(\lambda)})k_1^{\mu_1}\epsilon^{\mu_2}_{(\kappa)}(k_2) \tag{7.39}$$

If we are considering only space-like noncommutativity, we can remove all dependence of the Moyal phases on the internal momenta, yielding

$$W^s_{(\kappa)} = e^{-i\phi(-p_2,-k_2,k_1,p_1)}\bar{u}_{s_2}(p_2)\not{\epsilon}_{(\kappa)}(k_2)u_{s_1}(p_1), \tag{7.40}$$

$$W^u_{(\kappa)} = -e^{-i\phi(-p_2,k_1,-k_2,p_1)}\bar{u}_{s_2}(p_2)\not{\epsilon}_{(\kappa)}(k_2)u_{s_1}(p_1) \tag{7.41}$$

and

$$W^t_{(\kappa)} = e^{-i\phi(p_1,-p_2)}\bar{u}_{s_2}(p_2)\gamma^{\mu_3}u_{s_1}(p_1)\frac{1}{q_t^2}V_{\mu_1\mu_2\mu_3}(k_1,-k_2,-q_t)k_1^{\mu_1}\epsilon^{\mu_2}_{(\kappa)}(k_2). \tag{7.42}$$

Combining the s- and u-channel contributions as the overall energy conservation makes the phases cyclically symmetric

$$W^s_{(\kappa)} + W^u_{(\kappa)} = -2i\sin(k_1\wedge k_2)e^{ip_1\wedge p_2}\bar{u}_{s_2}(p_2)\not{\epsilon}_{(\kappa)}(k_2)u_{s_1}(p_1) \tag{7.43}$$

and using

$$k_1^{\mu_1}\epsilon^{\mu_2}_{(\kappa)}(k_2)V_{\mu_1\mu_2\mu_3}(k_1,-k_2,-q_t) = i\left(q_t^2\not{\epsilon}_{(\kappa)}(k_2) - (q_t\epsilon^{\mu_3}_{(\kappa)}(k_2))q_{t,\mu_3}\right)2\sin(k_1\wedge k_2) \tag{7.44}$$

one gets the required result $W^s_{(\kappa)} + W^t_{(\kappa)} + W^u_{(\kappa)} = 0$. But in general $W^s_{(\kappa)} + W^u_{(\kappa)}$ and $W^t_{(\kappa)}$ with its $1/q_t^2$ pole cancel each other only if

$$k_1^{\mu_1}\epsilon^{\mu_2}_{(\kappa)}(k_2)V_{\mu_1\mu_2\mu_3}(k_1,-k_2,-q_t^{(\lambda)}) = \alpha_1 q_t^2\epsilon_{(\kappa),\mu_3} + \alpha_2 q_{t,\mu_3}. \tag{7.45}$$

The $\alpha_2$-term is allowed since current conservation implies $\bar{u}(p_2)\not{q}_t u(p_1) = 0$. So we can make the following ansatz:

$$\tilde{V}_{\mu_1\mu_2\mu_3}(b_1,b_2,b_3 \mid k_1,k_2,k_3) = (b_1\bar{k}_{1,\mu_3} - b_2\bar{k}_{2,\mu_3})g_{\mu_1\mu_2}+$$
$$(b_2\bar{k}_{2,\mu_1} - b_3\bar{k}_{3,\mu_1})g_{\mu_2\mu_3} + (b_3\bar{k}_{3,\mu_2} - b_1\bar{k}_{1,\mu_2})g_{\mu_3\mu_1} \tag{7.46}$$

where the coefficients $b_i$ can contain momentum dependent phase factors. For the Compton scattering processes we can put $\bar{k}_1 = k_1$ and $\bar{k}_2 = k_2$ as they are external onshell momenta ($k_1^2 = k_2^2 = 0$). With $\epsilon^\mu_{(\kappa)}(k_2)k_{2,\mu}$ and $\delta k = k_1 + k_2 + k_3$ we obtain

$$k_1^{\mu_1}\epsilon^{\mu_2}_{(\kappa)}(k_2)\tilde{V}_{\mu_1\mu_2\mu_3}(b_1,b_2,b_3 \mid k_1,k_2,k_3) = \frac{b_3+b_2}{2}\bar{k}_3^2\epsilon_{(\kappa),\mu_3}(k_2)-$$
$$b(\bar{k}_3\epsilon_{(\kappa)}(k_2))\bar{k}_{3,\mu_3} + (\bar{k}_3\epsilon_{(\kappa)}(k_2))(b_3k_1 + b_2k_2 + b\bar{k}_3)_{\mu_3}$$
$$-b_2(\delta\bar{k}\epsilon_{(\kappa)})k_{2,\mu_3} + (\delta\bar{k}(b_3k_2 - b_2\bar{k}_3))\epsilon_{(\kappa),\mu_3} + \frac{b_2-b_3}{2}(\delta\bar{k})^2\epsilon_{(\kappa),\mu_3} \tag{7.47}$$

where the term proportional to b has been added and subtracted. The pole is cancelled by the first term which corresponds to $\alpha_1$ and the second term proportional to $\alpha_2$ doesn't contribute to the WI. As the remaining terms have to vanish we derive the following conditions

$$\delta\bar{k} = k_1 + k_2 + \bar{k}_3 = 0 \tag{7.48}$$

$$b_3 k_1 + b_2 k_2 + b\bar{k}_3 = 0 \tag{7.49}$$

which can be satisfied simultanously for $k_i \neq 0$, if and only if

$$b_3 = b_2 = b \tag{7.50}$$

This leads to energy-momentum conservation which doesn't hold if the propagator momentum with the index $\lambda$ is involved in IPTO as then the sum over the energies must be positive and negative and therefore the WI can't be satisfied in the case of timelike NC. $\square$

## 7.3 Mismatching phases

Another argument as to why $W^s_\kappa + W^u_\kappa + W^t_\kappa = 0$ can't hold is the general structure of the phase factors. The violation of energy conservation constrained by the overall momentum conservation leads to the following parametrisations:

$$\delta q_s^{(\lambda)} = q_s^{(\lambda)} - p_1 - k_1 = q_s^{(\lambda)} - p_1 - k_1 \tag{7.51}$$

$$\delta q_u^{(\lambda)} = q_u^{(\lambda)} - p_1 + k_2 = q_u^{(\lambda)} - p_2 + k_1 \tag{7.52}$$

$$\delta q_t^{(\lambda)} = q_t^{(\lambda)} - p_1 + p_2 = q_s^{(\lambda)} + k_1 - k_2 \tag{7.53}$$

The phase factors are for the s-channel

$$e^{-i(\phi_i(-p_2,-k_2,q_s^{(\lambda)})+\phi_j(-q_s^{(\lambda)},k_1,p_1))} =$$

$$e^{ip_1\wedge p_2} e^{-ik_1\wedge k_2} \begin{bmatrix} 1 & e^{2i\delta q_s^{(\lambda)}\wedge p_2} & e^{2i\delta q_s^{(\lambda)}\wedge q_s} \\ e^{-2i\delta q_s^{(\lambda)}\wedge q_s} & e^{-2i\delta q_s^{(\lambda)}\wedge k_2} & 1 \\ e^{-2i\delta q_s^{(\lambda)}\wedge p_1} & e^{2i\delta q_s^{(\lambda)}\wedge q_s} & e^{2i\delta q_s^{(\lambda)}\wedge k_1} \end{bmatrix} \tag{7.54}$$

and for the u-channel ($k_1 \leftrightarrow -k_2$)

$$e^{-i(\phi_i(-p_2,k_1,q_u^{(\lambda)})+\phi_j(-q_u^{(\lambda)},-k_2,p_1))} =$$

$$e^{ip_1\wedge p_2} e^{ik_1\wedge k_2} \begin{bmatrix} 1 & e^{2i\delta q_s^{(\lambda)}\wedge p_2} & e^{2i\delta q_u^{(\lambda)}\wedge q_u} \\ e^{-2i\delta q_u^{(\lambda)}\wedge q_u} & e^{2i\delta q_u^{(\lambda)}\wedge k_1} & 1 \\ e^{-2i\delta q_u^{(\lambda)}\wedge p_1} & e^{2i\delta q_u^{(\lambda)}\wedge q_t} & e^{-2i\delta q_u^{(\lambda)}\wedge k_2} \end{bmatrix} \tag{7.55}$$

For the t-channel the phase factors are with

$$iV_{\mu_1,\mu_2,\mu_3}(k_1,k_2,k_3) = \sum_{i=1}^{3} c'_i \left(\bar{k}_1^{\mu_2} g^{\mu_1\mu_3} - \bar{k}_1^{\mu_3} g^{\mu_1\mu_2}\right) \times$$

$$\left(e^{-i\phi_i(k_1,k_2,k_3)} - e^{-i\phi_i(k_1,k_3,k_2)}\right) + cyclic\{1,2,3\} \tag{7.56}$$

a combination of

$$e^{-i(\phi_i(-p_2,q_t^{(\lambda)},p_1)+\phi_j(k_1,-k_2,-q_t^{(\lambda)}))} =$$

$$e^{ip_1\wedge p_2}e^{-ik_1\wedge k_2}\left[\begin{array}{ccc} e^{-2i\delta q_t^{(\lambda)}\wedge p_1} & e^{2i\delta q_t^{(\lambda)}\wedge q_s} & e^{2i\delta q_t^{(\lambda)}\wedge p_2} \\ 1 & e^{2i\delta q_t^{(\lambda)}\wedge k_1} & e^{2i\delta q_t^{(\lambda)}\wedge q_t} \\ e^{-2i\delta q_t^{(\lambda)}\wedge q_t} & e^{2i\delta q_t^{(\lambda)}\wedge k_2} & 1 \end{array}\right] \tag{7.57}$$

and the same with $k_1 \leftrightarrow k_2$. In order to preserve the WI the factors $e^{\pm ik_1\wedge k_2}$ must combine to a factor $\sin(k_1 \wedge k_2)$, which isn't possible as the factors depend on the IPTO momenta $\delta q_s^{(\lambda)}$, $\delta q_s^{(\lambda)}$ and $\delta q_s^{(\lambda)}$. Or to put it another way the frequency components can never compensate each other as in the case of the fermion propagator it depends on the mass whereas the energy of the photon propagator is independent of mass. So IPTO cannot be used to cure the unitarity problem for timelike NCGT with external photons.

# Chapter 8

# Possible ways out

## 8.1 The generalized first Filk rule

In this chapter I want to offer a possible solution of the problem that IPTO violates the above Ward identity by restricting the phase combinations with the help of a generalized first Filk rule. It still has to be proven that this rule follows out of the Wick theorem in Minkowski spacetime as was shown by Filk to be the case for Euclidean spacetime to make this restriction physically plausible instead of just a mere wish of us. The generalised first Filk rule would be as already mentioned in the chapter about Filk's Feynman rules that to calculate the phase factor one has to contract to vertices connected by a propagator, such that it isn't possible that phases appear that cross the propagator and therefore contain it's impulse. It is intuitively clear that then the Ward identity must be fulfilled as the critical propagator impulses caused by IPTO won't appear.

From each vertex point one gets three possible phases which can be combined with each other leading in the case with no restriction to a $3\times3$ matrix. For $W_1$ one gets from one vertex point the following possible phase factors

$$e^{i(p_1\wedge k_1+p_1\wedge k_3^{\pm}+k_1\wedge k_3^{\pm})} \tag{8.1}$$

$$e^{i(k_1\wedge k_3^{\pm}+k_1\wedge p_1+k_3^{\pm}\wedge p_1)} \tag{8.2}$$

$$e^{i(k_3^{\pm}\wedge p_1+k_3^{\pm}\wedge k_1+p_1\wedge k_1)} \tag{8.3}$$

and from the other

$$e^{i(k_2\wedge p_2+k_2\wedge(-k_3^{\pm})+p_2\wedge(-k_3^{\pm}))} \tag{8.4}$$

$$e^{i(p_2\wedge(-k_3^{\pm})+p_2\wedge k_2+(-k_3^{\pm})\wedge k_2)} \tag{8.5}$$

$$e^{i((-k_3^{\pm})\wedge k_2+(-k_3^{\pm})\wedge p_2+k_2\wedge p_2)}. \tag{8.6}$$

Applying our generalized first Filk rule there are only two combinations leading to

$$e^{i(p_2\wedge p_1+k_1\wedge k_2)}. \tag{8.7}$$

The same is true for $W_2$ as the above is invariant under the change $k_1 \leftrightarrow -k_2$. For $W_3$ one has to use the following phase factors

$$e^{i(p_2 \wedge p_2 + p_2 \wedge k_3^{\pm} + p_1{}_3^{\pm})} \tag{8.8}$$

$$e^{i(p_1{}_3^{\pm} + p_1 \wedge p_2 + k_3^{\pm} \wedge p_2)} \tag{8.9}$$

$$e^{i(k_3^{\pm} \wedge p_2 + k_3^{\pm} \wedge p_1 + p_2 \wedge p_1)} \tag{8.10}$$

and

$$e^{i(k_1 \wedge k_2 + k_1 \wedge (-k_3^{\pm}) + k_2 \wedge (-k_3^{\pm}))} \tag{8.11}$$

$$e^{i(k_2 \wedge (-k_3^{\pm}) + k_2 \wedge k_1 + (-k_3^{\pm}) \wedge k_1)} \tag{8.12}$$

$$e^{i((-k_3^{\pm}) \wedge k_1 + (-k_3^{\pm}) \wedge k_2 + k_1 \wedge k_2)}. \tag{8.13}$$

Now that the phases aren't mismatched anymore, let's look if also the rest works out. First consider the following term from the third Feynman graph with the boson-boson vertex:

$$\sum_{\pm} \frac{(k_1 - k_2)^{\rho} g^{\mu\nu} + (k_2 - k_3^{\pm}) g^{\nu\rho} + (k_3^{\pm} - k_1)^{\nu} g^{\rho\mu}}{2\omega_3 (k_3^0 \mp \omega_3 + i\epsilon)} \cdot \epsilon_{k_1}^{\mu} \cdot k_2^{\nu}. \tag{8.14}$$

Using that $k_2$ is onshell, i.e. $k_2^2 = 0$, $k_1$ is transverse, i.e. $(k_1 \cdot \epsilon_{k_1}) = 0$, $\not{k}_1 = -\not{k}_2$ and that $k_3$ contracted with the fermion current is zero, this leads to

$$\frac{-2(k_1 k_2) + (\omega_3 - k_3^0)\omega_2}{2\omega_3 (k_3^0 - \omega_3)} - \frac{-2(k_1 \cdot k_2) + (-\omega_3 - k_3^0)\omega_2}{2\omega_3(\omega_3 + k_3^0)} \tag{8.15}$$

and after a short calculation to

$$\frac{2(k_1 k_2)}{2(k_1 k_2) + m^2 - (k_3^0)^2} = 1 \tag{8.16}$$

. This is exacly what we need to make $W_1 + W_2 + W_3 = 0$ when considering the calculations in the chapter before. The only flaw in the argument is that we imposed the first Filk rule generalized on Minkowski spacetime without deriving it physically, so that this all may be only wishful thinking.

## 8.2 Orthogonality of the electric vector of the deformation matrix

Another possible solution would be to consider all 18 phases and add the complex conjugated then everthing would be independent of the sign of the energy of the intermediate particle, but this would be the electron in one case and the photon in the other which would again not match. Like above the phases must be the same which is possible if the electric vector of the deformation matrix is orthogonal to the particle impulses in the scattering process, so that all unwanted phase factors cancel. In this case one doesn't need the generalized first

Filk rule as the seven unwanted combinations from each Feybman graph vanish considering that

$$k_3^{\pm} \wedge ... = (k_3^{\pm} - k_3 + k_3) \wedge ... = (k_3^{\pm} - k_3) = (\pm\omega_3 - k_3^0)\Theta^{0j}(...) = 0 \quad (8.17)$$

where we have used that the term with $k_3$ is zero because of energy-momentum conservation and the zero component of the deformation matrix stands orthogonal on the plane where the scattering process takes place.

# Chapter 9

# Conclusions

The above shows that although there propably doesn't exist any problem with unitarity on an Euclidean spacetime which is wrongly shown in [13] as Euclidean and Minkowski rules have been mixed in this paper, still much has to be done to render Minkowski spacetime unitary. We have shown, that the unitarity problem for scalar field theories in deformed Minkowski space-time can be solved by using the apprpriate Denk-Schweda rules. As for the nc gauge field model we offered a possible solution using the generalized Filk rule. Whether this rule can be derived from a carfull study of the asymptotic LSZ type condition needs further studies. This can be considered one of the main problems of NCQFT besides UV/IR mixing and the renormalisation problem. Other possible solutions than the one above to save unitarity for gauge fields in NCQFT on Minkowski spacetime would be:

- The Feynman rules for NCQFT with IPTO weren't derived for three different fields yet, which might be the reason why unitarity is violated. So new Feynman rules have to be developed and tested if the Ward identity for Compton scattering is fulfilled. Then there could be a try to prove the unitarity in general for gauge fields with IPTO.
- The WIs could be changed or the BRST invariance is broken, then an anormaly appears and renormalizability is lost.
- Maybe IPTO is the wrong way to solve the unitarity problem as it doesn't go conform with the BRST transformation. There have been attemps to solve the renormalzation problem differently with the help of coordinate coherent states. Unitarity of scalar fields has already been shown in this framework [15].
- LSZ reduction formulae would abrogate the relation between Green function and scattering amplitude.
- NCQFT on Minkowski spacetime isn't unitary and therefore no physical theory.

# Bibliography


[1] S.Denk and M.Schweda *Time ordered perturbation theory for non-local interactions; applications to NCQFT*, quant-ph/0101032

[2] Jaume Gomis and Thomas Mehen *Space-Time Noncommutative Field Theories And Unitarity*, quant-ph/0110078

[3] Karoline Dorothea Bahns , Dissertation (2003), *Perturbative Methods on the Noncommutative Minkowski Space*

[4] Michael E. Peskin and Daniel V. Schroeder *An Introduction to Quantum Field Theory*

[5] H. Bozkaya, P.Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda, R. Wulkenhaar *Space/time noncommutative field theories and causality*

[6] Yi Liao, Klaus Sibold *Time-ordered perturbation theory on noncommutative spacetime: basic rules*

[7] D.Bahns, S.Doplicher, K.Fredenhagen and G.Piacitelli *On the unitarity problem in space-time noncommutative theories*, Phys.Lett. B533 (2002) 178

[8] Ohl, Rueckl, Zeiner *Unitarity of Time-Like Noncommutative Gauge Theories: The Violation of Ward Identities in Time-Ordered Perturbation Theory*

[9] Thomas Filk *Divergencies in NCQFT*

[10] N. Seiberg, L. Susskind, N. Toumbas *Space/Time Non-Commutativity and Causality*

[11] Yi Liao, Christoph Dehne *Some Phenomenological Consequences of the Time-Ordered Perturbation Theory of QED on Noncommutative Spacetime*

[12] Chaiho Rim and Jae Hyung Yee *Unitarity in space-time noncommutative field theories*

[13] T. Mariz,C.A. de S. Pires and R.F. Ribeiro *Ward identity in noncommutative QED*

[14] P. Fischer and V. Putz *Ward identity in noncommutative QED*

[15] Anais Smailagic and Euro Spallucci *Lorentz invariance, Unitarity and UV-finiteness of QFT on noncommutative spacetime*

# Curriculum vitae

**Personal data:**

Name: Susanne Urach
Date of birth: April $27^{th}$, 1986
Place of birth: Krems an der Donau, Austria
Nationality: Austria

**Education**

- 1992-1996 Primary school in Hadersdorf am Kamp, Lower Austria
- 1996-1997 BG Piaristengasse, Krems an der Donau, Lower Austria
- 1997-2004 BG Rechte Kremszeile, Krems an der Donau, Lower Austria
- June 2004 Matura (school leaving exam)
- 2004-2010 Undergraduate studies in physics and mathematics at the University of Vienna

**Employment**

October 2009-January 2010: Tutor at the Faculty of Physics of the University of Vienna