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Quantum Phase and Uncertainty

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Verfasser:	Philipp Thun-Hohenstein
Matrikel-Nummer:	0505979
Studienrichtung:	411 Diplomstudium Physik UniStG
Betreuer:	Ao. Univ.-Prof. Dr. Reinhold Bertlmann

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Contents

Chapter 1

Introduction

The present diploma thesis attempts to provide the reader with a systematic overview over and synthesis of the principal theoretical and experimental efforts conducted thus far to describe the phenomenon of quantum phase in electromagnetic fields. An extensive treatment of this phenomenon is warranted by the fact that it may lead to important applications in information transfer technology and quantum optics, and may harbour far-reaching implications for general quantum theory in its current formulation.

Although quantum phase has never quite been able to generate the intense research efforts associated with, e.g., squeezed light or Bell inequalities, it nonetheless constitutes a highly interesting topic that needs to be addressed by modern quantum physics. Indeed, quantum phase raises many fundamental questions about such basic quantum mechanical heritage as the hermiticity of operators for observables or the definition of Hilbert space for such basic systems as the harmonic oscillator. And although, at first glance, the quantum phase problem may not seem all that hard to approach, this perceived simplicity is quite misleading. To illustrate this point, consider that more than eight decades have been spent searching for a closed quantum phase theory since the problem was initially discovered by Dirac and London [1–3], and although many advances have been made and much has been achieved, still no broadly satisfying solution has been found.

This leads to an interesting first observation about quantum phase: That it constitutes a research area that is extremely fractured and diverse. Before the 1990s, letters discussing quantum phase were essentially singular occurrences, and of the few such occurrences that there were, most of them proposed new approaches that had little to do with any of the previous work and did not care much for compatibility with standard quantum mechanics [4, 5]. It is perhaps characteristic of this early phase that only one serious attempt at a review was made in a timeframe of almost 50 years [6], and that research was often isolated and sometimes redundant[1]. Only in recent years has a certain consolidation of the field

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[1]cf. the example of the Garrison/Wong-operator [4], which was independently rediscovered 14 years later by Galindo [7]

become noticeable, with some excellent reviews [8, 9] and historical overviews [10] being published to provide the different theories with a connecting structure, and with serious efforts being made to relate important approaches among each other (e.g. [11]).

Unfortunately, consolidation alone has not been sufficient to provide some much-needed decluttering to quantum phase theory. And here a second observation about this topic can be made: the extreme scarcity of experiment. Until the 1990's, a few papers by Gerhardt et al [12, 13] provided the only reliable experimental basis against which to test theoretical results predicted by existing approaches, and unfortunately these papers did not provide all too precise measurements, so that none of the different approaches could reasonably be invalidated. But perhaps, this touches upon a more general problem of quantum phase, namely that it has never really been considered a fundamental problem of quantum physics, and has sometimes even been labeled a lost cause[2]. That is striking because, in my opinion, it should be very discomforting that something as basic as the phase of the electromagnetic field does not have a proper description in quantum mechanics. This softly but firmly calls into question whether today's quantum physics, a construct which in many places abuses rather than uses its mathematical foundations, should not be given a general overhaul in order to make sure that abnormalities and difficulties such as quantum phase are not merely artefacts of lacking mathematical rigor, but actual uncertainties yielding insight to (new) natural laws.

With all this in mind, it will come as no surprise to the reader that at present, despite encouraging recent developments which provide more robust experimental data and theoretical digest, it is still impossible to declare any one theory to be the definitive description of quantum phase.

1.1 Outline of the problem and motivation

So what exactly is the problem of quantum phase?

Most briefly put, the quantum phase problem tasks itself with finding a sensible and consistent quantum mechanical description of the classical phase variable φ, which represents the phase of a certain class of oscillators (e.g. electric fields), in a quantum setting (e.g. very low intensity lightbeams). To make some sense of this, it is useful to consider the following example. Given a classical one-dimensional electromagnetic field, one can describe a single mode of this field as

$$E = A \cdot \cos(\varphi) = A \cdot \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2} \right) , \tag{1.1.1}$$

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[2]cf. e.g. `http://www.mth.kcl.ac.uk/~streater/lostcauses.html#VIII` (19.07.2009)

where A is some real-numbered measure of amplitude and φ is the phase. Expanding, we obtain

$$E = \frac{1}{2} \left(r\mathrm{e}^{i\varphi} + r\mathrm{e}^{-i\varphi} \right) = \frac{a + a^*}{2}. \tag{1.1.2}$$

As is well known, we can translate this description of the electromagnetic field into quantum mechanics by replacing a and a^* with the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$, following the bosonic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ (for more detail, see appendix A.1.1). Then the electromagnetic field, which is essentially a generalized position variable, becomes

$$\hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{2}, \tag{1.1.3}$$

(or, more generally, a superposition thereof). The operators $\hat{a}$ and $\hat{a}^\dagger$ are very well-known and versatile operators, and it is very nice to know that they can be used to describe the electromagnetic field. However, they come with one major shortcoming in that they do not contain any easily accessible information about phase. Therefore, to solve the quantum phase problem, we require a quantization of the electromagnetic field that directly reflects the initial equation $E = A \cdot \cos(\varphi)$ instead of circumscribing it with a sum of two exponentials. This would require a quantum mechanical expression for the amplitude A and the phase φ, which would optimally be achieved by introducing hermitian operators for each of these properties.

Such has been tried by effectively reversing the steps taken in equations (1.1.1) and (1.1.2) in order to arrive at a polar decomposition of $\hat{a}$ (cf. [2] and section 3.1.1). This polar decomposition would yield

$$\hat{a} = \sqrt{\hat{n}} \cdot \mathrm{e}^{i\hat{\varphi}} \quad \text{(wrong)}, \tag{1.1.4}$$

where $\hat{n}$ would be the well-known number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ and $\hat{\varphi}$ would be the desired hermitian phase operator. Unfortunately, it was quickly realized that this relation is intrinsically flawed, and eventually, more and more convincing evidence accumulated to compound the suspicion that it is impossible to derive an hermitian phase operator in the conventional Hilbert space of the harmonic oscillator (see section 3.3). Therefore, other approaches were needed and were, in time, provided, but none of them really capture all the aspects of quantum phase while resting within experimental and theoretical boundaries.

That is, in essence, the problem of quantum phase, whose tentative solutions we will consider in the next five chapters.

1.2 Definition of phase

Perhaps at this point, it would be sensible to briefly reflect on what exactly is meant by the term "phase" in order to avoid any confusions about this term which might arise especially in the context of quantum physics, where there exist multiple concepts of phase. We therefore give the following definition:

Definition. *Quantum phase is a measure of the position of a quantum state along the unit circle in phase space, where phase space is the two-dimensional space spanned by the dimensionless position and momentum operators on a given Hilbert space.*

For states such as the coherent states, which are characterized by amplitude and phase and can therefore be described as wave-like, quantum phase essentially describes a wave motion and approaches classical phase especially for high-energy states. But since any state can be decomposed into an integral over coherent states, quantum phase is a meaningful concept even for arbitrary states, including (presumed) random phase states such as the number states (keep in mind that for these latter cases, however, the phase distribution plays a much bigger role than the actual mean phase value).

The above definition sets quantum phase apart from two other kinds of phase commonly encountered in quantum physics: First, the quantum mechanical phase factor which typically appears in solutions to the Schrödinger equation, and second, the geometric phase which some adiabatic processes give rise to when closed-loop pathintegrals are performed. Both only concern the quantum phase problem indirectly, if at all.

However, the distinction between quantum phase and the phase factor is very important. Roughly speaking, the two can be distinguished by the fact that, while the quantum mechanical phase factor has no standalone physical meaning (it vanishes in the modulus squared) and only becomes noticeable as a relative value when it causes interference effects in entangled states, quantum phase is, at least in principle, a real, observable quantity which yields measurable information about certain characteristics of an electromagnetic field state.

Of course, this distinction is sometimes blurry in practice: For example, due to the very high frequencies composing most of the electromagnetic light spectrum, an absolute reference phase for a field state may not always be experimentally feasible, so that in these contexts, quantum phase is reduced to a relative quantity, practically speaking. This has indeed led some authors to conclude that quantum phase may not exist as an absolute quantity at all, rendering the distinction between quantum phase and the phase factor much less clear.

Nonetheless, it remains a mathematical fact that even after the modulus squared is taken, two coherent states described by different parameters α with different phases φ are physically distinct and must therefore be assigned a different phase value; this means that in contrast to phase factors, a quantum phase difference between two coherent states is observable even if they are not entangled. Moreover, multiplying a coherent state with a phase factor will not affect its quantum phase, since its phase information is not contained in a phase factor.[3] This indicates that phase factors and quantum phase are of a different

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[3]For example, in the number state basis, the phase information of a coherent state is expressed in the

quality entirely (for some further reflections on this, cf. chapter 8).

Unfortunately, the mere fact that quantum phase is undoubtedly observable[4] does not render the task of describing it any easier: thus the need to write this thesis!

1.3 Structure of this thesis

Concerning the structure of the present thesis, we will proceed as follows: First, based on such basic notions of quantum theory as the correspondence principle and the (semi)classical limit, the basic requirements which every adequate phase theory should fulfill are defined. Of course, given the complex nature of phase quantum phase, these a priori requirements will not be free of exceptions, but they will serve as a guiding principle to give us a crude estimate of the usefulness of a certain phase representation (chapter 2).

Next, we look at some of the more important historical attempts of deriving a quantum phase theory in chronological order (chapter 3). We will learn that for many decades, the search for a quantum phase theory was dominated by the conviction that phase was an operator-observable described by a hermitian operator, but we will look to identify some of the indications that tell us that this conviction is misguided. The chapter concludes with several (ideas of) proofs showing that the definition of a hermitian phase operator is impossible.

In chapter 4, we recapitulate the major conceptual difficulties that prevent the construction of a hermitian phase operator, and then proceed to examine an approach that tries to circumvent these difficulties by introducing a restricted Hilbert space and constructing the phase operator in this space. It turns out that in such a constricted space, a well-behaved hermitian operator can be defined. We spend several sections investigating and reviewing the formalism based on this operator, thoroughly examining its merits as well as its shortcomings.

In Chapter 5, we finally turn to experiment. We try to relate some of the predictions of the theoretical model presented in chapter 3 to experimental data and make an interesting conclusion about the merits of phase-operator theory.

In Chapter 6, we then briefly summarize the main theoretical results obtained so far.

In Chapter 7, we make an excursus into the realm of uncertainty theory, trying to derive the uncertainty relation between number and phase and spending some time discussing

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phase differences between different orders of number states as described by α, and not in the overall phase factor.

[4]If absolutely or relatively makes not too big a difference

minimum uncertainty states. We also present a wide outlook on potential areas of further research, including several different measures of uncertainty and some new approaches.

Chapter 8 then comprises reflection, conclusion and discussion.

1.4 Remarks on extent of treatment

Two brief remarks are in order to delimit what will be examined in this thesis and what will not:

First, we do concern ourselves with phase difference operators or the description of phase differences in general. This is because it is our postulate that a measure of absolute phase exists, even though its use may be limited in the practical setting of experiment. However, many treatments of phase difference operators exist, and the inclined reader may use many of the papers here as a starting point for further research [6, 14, 15].

Second, we only treat the single mode electric field. Treatments of multi mode electric fields do exist, but these are usually faced with entirely different problems, so that their inclusion would have gone far beyond the scope of this thesis and has therefore been avoided.

With this in mind, we may start the exposition.

Chapter 2

A priori conditions

Based on our understanding of quantum mechanics, of electromagnetic fields and of the common conceptions about what constitutes a desirable physical theory, several criteria that should be fulfilled by a description of quantum phase may be derived. First, the basic quantum mechanical principle whereby in the macroscopic limit, the properties predicted by the quantum description should mirror those derived from classical electrodynamics, must be observed. Second, where phase-relevant predictions can be inferred from other aspects of quantum optics, these should also be reflected in the quantum phase theory. Last, the theory should try to honor established principles of quantum mechanics and refrain from unnecessary reformulations of basic quantum mechanical heritage.

2.1 Correspondence limit

We know from the Hamiltonian $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ of the harmonic oscillator that the energy of an electric field mode is proportional to the average photon-number as expressed by the expectation value of the operator $\hat{n} = \hat{a}^\dagger\hat{a}$. This means that the classical limit is reached for highly intense light where the photon-number tends to infinity. In this limit, we expect the quantum description of phase to yield results that approach the classical predictions. For example, we expect a sine operator $\widehat{\sin(\varphi)}$ to fulfill trigonometric identities

$$\left\langle \widehat{\sin(\varphi)} \right\rangle = \sin(\varphi) \tag{2.1.1a}$$

$$\left\langle \widehat{\cos(\varphi)} \right\rangle = \cos(\varphi) \tag{2.1.1b}$$

$$\left\langle \widehat{\sin^2(\varphi)} \right\rangle = \sin^2(\varphi) \tag{2.1.1c}$$

$$\left\langle \widehat{\cos^2(\varphi)} \right\rangle = \cos^2(\varphi) \tag{2.1.1d}$$

$$\left\langle \widehat{\sin^2(\varphi)} + \widehat{\cos^2(\varphi)} \right\rangle = 1 \tag{2.1.1e}$$

$$\left\langle \left[\widehat{\sin(\varphi)}, \widehat{\cos(\varphi)} \right] \right\rangle = 0 \tag{2.1.1f}$$

and to have an image restricted to the interval $[-1, 1]$. Furthermore, we expect uncertainties to vanish in the large n-limit

$$\lim_{n\to\infty} \frac{\Delta\hat{\varphi}}{\langle\hat{\varphi}\rangle} = 0 \tag{2.1.2a}$$

$$\lim_{n\to\infty} \frac{\Delta\hat{n}}{\langle\hat{n}\rangle} = 0\,. \tag{2.1.2b}$$

Many more examples could be found, but the above should give some idea about what we may reasonably expect from a "good" quantum phase theory.

2.2 Correspondence principle for random phase

Another very important criterion for any successful phase description is its behavior for states of random phase. For example, it is commonly agreed from a phenomenological point of view that the number states, and especially the vacuum state, are states of random phase. The expectation value and second moment of phase in such states should then reflect this randomness, or we may assume that something is wrong. Granted, this approach is somewhat tautological, since we would first need a quantum phase theory to prove that the vacuum state is random in the first place. Nonetheless, it is a reasonable assumption that vacuum phase should be random, and therefore a quantum phase theory should reflect this.

In the literature, the reasoning of the above paragraph has become known as the "Pegg/Barnett acid test", after the authors who first directly proposed it [16, 17]. We present a slightly adapted version here, but the underlying principle is always the same: It consists of first deriving the expectation value and variance for a random phase distribution using classical physics, and then demanding that the calculated results also apply to any quantum mechanical formulation of phase. We start from the obvious prerequisite that a random phase distribution must be flat, since for random phase every phase-value within a window of 2π has an equal likelihood. Taking an interval $[\theta_0, \theta_0 + 2\pi]$, the phase distribution must therefore be

$$P(\theta) = \frac{1}{2\pi} \tag{2.2.1}$$

Using the first mean value theorem, we obtain the average phase and therefore the expectation value

$$\langle\theta\rangle = \frac{\int_{\theta_0}^{\theta_0+2\pi} \theta \cdot P(\theta)\,\mathrm{d}\theta}{\int_{\theta_0}^{\theta_0+2\pi} P(\theta)\,\mathrm{d}\theta} = \frac{1}{2\pi}\left.\frac{\theta^2}{2}\right|_{\theta_0}^{\theta_0+2\pi} = \theta_0 + \pi\,, \tag{2.2.2}$$

from which we can proceed to calculate the variance

$$
\begin{aligned}
\Delta\theta^2 &= \frac{\displaystyle\int_{\theta_0}^{\theta_0+2\pi} (\theta - \langle\theta\rangle)^2 \cdot P(\theta)\,\mathrm{d}\theta}{\displaystyle\int_{\theta_0}^{\theta_0+2\pi} P(\theta)\,\mathrm{d}\theta} = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} (\theta - \theta_0 - \pi)^2\,\mathrm{d}\theta = \\[2ex]
&= \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} (\theta^2 - 2\theta(\theta_0 + \pi) + \theta_0^2 + 2\theta_0\pi + \pi^2)\,\mathrm{d}\theta = \\[2ex]
&= \frac{1}{2\pi}\left[\frac{\theta^3}{3} - \theta^2(\theta_0 + \pi) + \theta(\theta_0^2 + 2\theta_0\pi + \pi^2) \right]\Bigg|_{\theta_0}^{\theta_0+2\pi} = \\[2ex]
&= \frac{1}{2\pi}\left[\frac{6\pi\theta_0^2 + 12\pi^2\theta_0 + 8\pi^3}{3} - 4\pi\theta_0^2 - 8\pi^2\theta_0 - 4\pi^3 + 2\pi\theta_0^2 + 4\pi^2\theta_0 + 2\pi^3 \right] = \\[2ex]
&= \frac{1}{2\pi}\frac{2\pi^3}{3} = \frac{\pi^2}{3}\,.
\end{aligned}
$$

$$(2.2.3)$$

As expected, the variance of a flat probability distribution is independent of the reference phase θ_0; its value is $\pi^2/3$ and we thus expect this value to also apply when a random phase state is evaluated with a phase operator.

2.3 Consistency with modern quantum mechanics

The final criterion is not as easy to express in mathematical expressions as the above two, and it could very well be termed a "soft" or qualitative criterion. But it nonetheless has some importance, demanding that a proposed theory should try to rest within the framework of established quantum mechanics and take special care not to violate any of its basic notions without good reasons. In other words, the theory should fit in.

Of course, some destructive spirit is always necessary, and quantum mechanics would not exist today had it not breached the confines of classical physics, but any significant departure from established theory should be well justified, and authors should avoid proposing new and cumbersome models if it is not clear how such models could serve any purpose besides justifying the author's own assumptions and theories. A paradigmatic example of this is the Hilbert superspace E, which seems to have been constructed by Vaccaro and Bonner [18] merely as an elaborate means to justify the PB-formalism after it had come under criticism for some of its limitations.

Chapter 3

The search for a phase operator

We now turn to the first chapter in the history of quantum phase theory, where we examine the initial efforts made to find a quantum phase operator. We may already reveal here that none of these approaches has solved the quantum phase problem, but very valuable insights about the nature of phase and the typical sources of difficulty associated with it can be gained during their study, which justifies treating them in some length.

3.1 Early attempts

3.1.1 Dirac

3.1.1.1 Dirac phase operator

The initial discussion of quantum phase arose when Dirac introduced a perturbation theory in 1927 with the intent of transferring electrodynamic principles into quantum theory [2]. In the course of this treatment, he tried, among other things, to transpose the (classically straightforward) polar decomposition of the electric field components

$$a = \sqrt{n}\, e^{i\varphi} \tag{3.1.1a}$$

$$a^* = \sqrt{n}\, e^{-i\varphi} \,, \tag{3.1.1b}$$

into a quantum setting (note that we have here reversed Dirac's sign convention to be consistent with later developments). The basic difficulty that arose in this context was that, as we substitute $a \to \hat{a}$ and $a^* \to \hat{a}^\dagger$, the bosonic relation $[\hat{a}, \hat{a}^\dagger] = 1$ must be fulfilled. Dirac tried (in vain, as we shall see below) to overcome this difficulty by using

$$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \tag{3.1.2}$$

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to write[5]

$$\hat{a} = \sqrt{\hat{n}+1}\,\mathrm{e}^{i\hat{\varphi}} = \mathrm{e}^{i\hat{\varphi}}\sqrt{\hat{n}} \tag{3.1.3a}$$

$$\hat{a}^{\dagger} = \sqrt{\hat{n}}\,\mathrm{e}^{-i\hat{\varphi}} = \mathrm{e}^{-i\hat{\varphi}}\sqrt{\hat{n}+1}\,, \tag{3.1.3b}$$

instead of equations (3.1.1), so that the commutation relation was automatically fulfilled:

$$\hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} = \sqrt{\hat{n}+1}\,\mathrm{e}^{i\hat{\varphi}}\,\mathrm{e}^{-i\hat{\varphi}}\sqrt{\hat{n}+1} - \sqrt{\hat{n}}\,\mathrm{e}^{-i\hat{\varphi}}\,\mathrm{e}^{\hat{\varphi}}\sqrt{\hat{n}}$$
$$= \hat{n}+1-\hat{n} = 1\,. \tag{3.1.4}$$

From these relations, he was able to obtain another commutation relation [19]

$$[\mathrm{e}^{i\hat{\varphi}}, \hat{n}] = \mathrm{e}^{i\hat{\varphi}} \tag{3.1.5}$$

which can be verified by direct calculation:

$$[\mathrm{e}^{i\hat{\varphi}}, \hat{n}] = \mathrm{e}^{i\hat{\varphi}}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\,\mathrm{e}^{i\hat{\varphi}} = \mathrm{e}^{i\hat{\varphi}}\mathrm{e}^{-i\hat{\varphi}}\sqrt{\hat{n}+1}\sqrt{\hat{n}+1}\,\mathrm{e}^{i\hat{\varphi}} - \hat{n}\,\mathrm{e}^{i\hat{\varphi}}$$
$$= (\hat{n}+1)\,\mathrm{e}^{i\hat{\varphi}} - \hat{n}\,\mathrm{e}^{i\hat{\varphi}} = \mathrm{e}^{i\hat{\varphi}}\,. \tag{3.1.6}$$

This relation is very interesting and we will encounter it again in later sections, but for now we simply note that we can use it to derive a third commutation relation, this time directly relating $\hat{n}$ to $\hat{\varphi}$. To see this, first decompose the above expression into a Taylor-series

$$[\mathrm{e}^{i\hat{\varphi}}, \hat{n}] = \mathrm{e}^{i\hat{\varphi}} \tag{3.1.7a}$$

$$\Leftrightarrow \sum_{k=0}^{\infty} \frac{i^{k}}{k!}\left[\hat{\varphi}^{k}, \hat{n}\right] = \sum_{k=0}^{\infty} \frac{i^{k}}{k!}\,\hat{\varphi}^{k} \tag{3.1.7b}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \frac{i^{k}}{k!}\left[\hat{\varphi}^{k}, \hat{n}\right] = \sum_{k=1}^{\infty} \frac{i^{k-1}}{(k-1)!}\,\hat{\varphi}^{k-1} \tag{3.1.7c}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \frac{i^{k}}{k!}\left[\hat{\varphi}^{k}, \hat{n}\right] = \sum_{k=1}^{\infty} \frac{i^{k}}{k!}\left(-ik\hat{\varphi}^{k-1}\right)\,. \tag{3.1.7d}$$

Then, postulate that we can equate the individual terms of the sum

$$\left[\hat{\varphi}^{k}, \hat{n}\right] = -ik\hat{\varphi}^{k-1} \tag{3.1.8}$$

---

[5]Note that the following equations of course assume that the operators are sufficiently "good" so that the spectral theorem applies and expressions such as the square root of operators are defined and make sense.

and prove by induction. Equation (3.1.8) is fulfilled for $k = 0$ (since the commutator vanishes, as does the right-hand side), and holds for k given $k - 1$ and lower orders:

$$
\begin{aligned}
\left[\hat{\varphi}^k, \hat{n}\right] &= \hat{\varphi}^k \hat{n} - \hat{n} \hat{\varphi}^k \\
&= \hat{\varphi}^{k-1} \hat{n} \hat{\varphi} + \hat{\varphi}^{k-1} \left[\hat{\varphi}, \hat{n}\right] - \hat{n} \hat{\varphi}^{k-1} \hat{\varphi} \\
&= \left[\hat{\varphi}^{k-1}, \hat{n}\right] \cdot \hat{\varphi} - i \hat{\varphi}^{k-1} \\
&= -i(k-1)\hat{\varphi}^{k-2} \cdot \hat{\varphi} - i \hat{\varphi}^{k-1} \\
&= -ik\hat{\varphi}^{k-1} \,,
\end{aligned}
\tag{3.1.9}
$$

where in the third and fourth equalities, we have applied the induction hypothesis. For $k = 1$, we obtain

$$
[\hat{\varphi}, \hat{n}] = -i \,,
\tag{3.1.10}
$$

which is the desired relation. Switching signs, we note a startling similarity to the case for position and momentum:

$$
[\hat{n}, \hat{\varphi}] = i
\tag{3.1.11}
$$

and

$$
[\hat{x}, \hat{p}] = i\hbar \,.
\tag{3.1.12}
$$

This leads us to expect an uncertainty relation for the supposedly conjugated pair phase and number that should be analogous to the position-momentum uncertainty relation and should look something like

$$
\Delta \hat{\varphi} \cdot \Delta \hat{n} \geq \frac{1}{2} \,,
\tag{3.1.13}
$$

completing the theory.

3.1.1.2 Problems with Dirac's operator

At this point, it would seem that we have found a good description of phase. It is simple, preserves bosonic commutation rules and bears resemblance to the case of position and momentum. Unfortunately, it is also wrong, as Dirac was later himself to find out.

First, observe that evaluating the commutator in the number state basis

$$
\left\langle n' \left| [\hat{n}, \hat{\varphi}] \right| n \right\rangle = \left\langle n' \left| i \right| n \right\rangle
\tag{3.1.14}
$$

leads to

$$
(n' - n) \left\langle n' \left| \hat{\varphi} \right| n \right\rangle = i \, \delta_{nn'} \,,
\tag{3.1.15}
$$

which for $n' = n$ gives

$$
0 = i \,,
\tag{3.1.16}
$$

a clearly nonsensical result.[6] Another problem concerns the fact that in the above uncertainty relation, small variances in the photon-number can cause the variance of the phase to rise above 2π, which is clearly impossible for a variable that is only defined on a 2π interval. These problems arise because Dirac ignored the periodic nature of phase, an easy pitfall in the Dirac bra and ket notation, where domain limitations are not easily visible and quickly ignored. We will take a look at tentative solutions to this periodicity problem in the next section and in 3.2.2.

However, besides the problem of periodicity, there is also a very basic error in the calculation of the commutator in equation (3.1.4), because this equation incorrectly assumed that $\mathrm{e}^{i\hat{\varphi}}$ was unitary and that therefore, the commutator between $\mathrm{e}^{i\hat{\varphi}}$ and its inverse should vanish.[7]

In reality, this is not the case, as a closer look at the matrix elements of $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ and a calculation of the commutator quickly confirms [1, 3]

$$\mathrm{e}^{i\hat{\varphi}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathrm{e}^{-i\hat{\varphi}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (3.1.17)$$

$$\left[\mathrm{e}^{i\hat{\varphi}}, \mathrm{e}^{-i\hat{\varphi}} \right] = 1 - (1 - |0\rangle\langle 0|) = |0\rangle\langle 0| \ . \qquad (3.1.18)$$

Apparently, $\mathrm{e}^{i\hat{\varphi}}$ is only approximately, or more precisely, one-sided unitary. This renders equation (3.1.4) incorrect and thus invalidates the uncertainty relation (3.1.11).

Taking all these problems together, we conclude that Dirac's proposed phase operator model is wrong and that more rigorous considerations are needed to overcome the problems of periodicity and non-unitarity, both of which will be addressed in the next sections.

3.1.2 Tentative solution to the periodicity problem

A first useful solution to the periodicity problem was introduced by Louisell [20] in 1963, shortly before Susskind and Glogower solved (or rather, circumvented) the non-unitarity

---

[6]Granted, the position-momentum commutator seems to cause a similar contradiction. More precisely, the analogous equation for position and momentum is $\langle x'|\hat{p}|x\rangle(x' - x) = i\hbar\delta(x' - x)$. However, since x is continuous, $\hat{p}$ acts as a derivative differentiating $(x' - x)$ to 1, so that we are left with $i\hbar\langle x'|x\rangle = i\hbar\delta(x'-x)$, which is the correct result. For the number-phase commutator, however, such differentiation is not possible since number is discrete

[7]For any unitary operator U, $[U, U^{-1}]$ must be zero because $UU^{-1} = U^{-1}U = \mathbb{1}$

problem with a similar ansatz (see below). Although Louisell's paper lacks a clear derivation and seems to suffer from some omissions in its equations, we can reproduce his solution by taking another look at the relation

$$\left[\hat{\varphi}^k, \hat{n}\right] = -ik\hat{\varphi}^{k-1}\,, \tag{3.1.19}$$

which was derived from the commutator

$$[\mathrm{e}^{i\hat{\varphi}}, \hat{n}] = \mathrm{e}^{i\hat{\varphi}} \tag{3.1.20}$$

in equations (3.1.7a)-(3.1.7d). Reinterpreting $-ik\hat{\varphi}^{k-1}$ to be a derivative $-i\,\partial\hat{\varphi}^k/\partial k$ results in a differential equation for $\hat{\varphi}$, whose validity can be extended to arbitrary functions $f(\hat{\varphi})$ of the phase operator by construction of an infinite power series $\sum c_k\,\hat{\varphi}^k$ over $\hat{\varphi}$. This gives

$$\left[f(\hat{\varphi}^k), \hat{n}\right] = -i\frac{\partial f(\hat{\varphi}^k)}{\partial k} \tag{3.1.21}$$

as a general relation. Inserting $\sin(\hat{\varphi})$ and $\cos(\hat{\varphi})$ for $f(\hat{\varphi})$ leads to commutators

$$[\sin\hat{\varphi}, \hat{n}] = -i\cos\hat{\varphi}\,, \qquad [\cos\hat{\varphi}, \hat{n}] = i\sin\hat{\varphi} \tag{3.1.22}$$

and uncertainty relations

$$\Delta\sin\hat{\varphi}\cdot\Delta\hat{n} \geq \frac{1}{2}\left|\langle\cos\hat{\varphi}\rangle\right|\,, \qquad \Delta\cos\hat{\varphi}\cdot\Delta\hat{n} \geq \frac{1}{2}\left|\langle\sin\hat{\varphi}\rangle\right|\,. \tag{3.1.23}$$

which are not plagued by periodicity inconsistencies. The reason for this seemingly miraculous cure is simple: Relations (3.1.22) and (3.1.23) avoid the contradiction in equation (3.1.15) because the functions $\sin(\hat{\varphi})$ and $\cos(\hat{\varphi})$ are themselves periodic, which at once eliminates any domain definition problems that had to be tackled for the non-periodic $\hat{\varphi}$. Using the sine and cosine relations, we are therefore now in a position to calculate the matrix elements of the phase operator in the number state basis:

$$(n' - n)\left\langle n'\left|\sin\hat{\varphi}\right|n\right\rangle = i\left\langle n'\left|\cos\hat{\varphi}\right|n\right\rangle \tag{3.1.24}$$

$$(n' - n)\left\langle n'\left|\cos\hat{\varphi}\right|n\right\rangle = -i\left\langle n'\left|\sin\hat{\varphi}\right|n\right\rangle\,, \tag{3.1.25}$$

from which follows that

$$\left\langle n'\left|\cos\hat{\varphi}\right|n\right\rangle = \frac{-i}{n'-n}\frac{i}{n'-n}\left\langle n'\left|\cos\hat{\varphi}\right|n\right\rangle\,. \tag{3.1.26}$$

This allows for several remarks about the structure of the sine and cosine operator matrix. First, statement (3.1.26) is only consistent if all matrix elements of sine and cosine are zero except where $|n' - n| = 1$. Where $|n' - n| = 1$, the matrix elements are not determined, allowing for a multiplicative constant as is usual in linear differential equations. Furthermore, if $n' - n = 1$, we see that sine differs from cosine by a factor i, and for $n' - n = -1$

by a factor $-i$. Thus we can conclude that sine and cosine are matrices whose elements are all zero except for the innermost offdiagonals, and that sine differs from cosine by a factor of i in the lower and $-i$ in the upper inner offdiagonal.

Of course, we need to be aware that Louisell's contribution is not a rigorous argument, since the non-unitarity of $e^{i\hat{\varphi}}$ still means that eq. (3.1.4) and therefore also the commutator (3.1.5), from which Louisell's statement ultimately derives, are wrong. Nonetheless, this tentative structure of the sine and cosine operators will be confirmed in the Susskind Glogower formalism to be discussed next.

Before we examine this formalism more closely, however, we give an additional indication that sine and cosine operators might lead to a correct description of phase, which is due to Lerner [6, 21]. Treating the oscillator problem classically, a Jacobi-transformation $\{x, p\} \rightarrow \{J, \varphi\}$ can be introduced as

$$x = (2J/m\omega)^{1/2} \cos \varphi(t) \tag{3.1.27a}$$

$$p = (2m\omega J)^{1/2} \sin \varphi(t) \,, \tag{3.1.27b}$$

where J is an action variable and φ is the angle. The resulting Hamiltonian is

$$H = (p^2/2m) + \tfrac{1}{2}m\omega^2 x^2 = J\omega \tag{3.1.28}$$

Since the Hamiltonian is stationary, J and ω are taken to be constant, while φ, which does not appear explicitly, may depend on time. The standard Poisson-bracket time-derivative $\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}$ then gives

$$\frac{d \sin \varphi(t)}{dt} = -\omega \cos \varphi(t) = \{\sin \varphi(t), H\} \tag{3.1.29a}$$

$$\frac{d \cos \varphi(t)}{dt} = \omega \sin \varphi(t) = \{\cos \varphi(t), H\} \,, \tag{3.1.29b}$$

the time dependence being $\varphi(t) = \varphi - \omega t$. Using Poisson-bracket correspondence

$$\{A, B\} = \tfrac{1}{i\hbar} \left[\hat{A}, \hat{B} \right] \tag{3.1.30}$$

we get

$$[\sin \hat{\varphi}, \hat{n}] = -i \cos \hat{\varphi} \tag{3.1.31a}$$

$$[\cos \hat{\varphi}, \hat{n}] = i \sin \hat{\varphi} \,, \tag{3.1.31b}$$

where we have used that $\hat{H} = \hbar\omega(\hat{n} + \tfrac{1}{2})$. This exactly mirrors the result obtained by Louisell from the commutator (3.1.5).

3.1.3 Susskind Glogower

We now turn to the Susskind Glogower formalism. This formalism was created in 1964, when Susskind and Glogower concisely summarized the problems of the Dirac phase operator and introduced a new formalism of their own to correct these problems [14], thereby formulating a solution which should become the benchmark of quantum phase theories until the late 1980's.[8] Their formalism was also based on sine and cosine operators, and indirectly reflected some of the results obtained by Louisell and Lerner as just shown in section 3.1.2.

3.1.3.1 Non-hermiticity of Dirac's operator

Susskind and Glogower began their exposition by showing that the Dirac phase operator is not hermitian. In effect, this had inadvertently already been accomplished by London almost 40 years earlier, but it is interesting to show Susskind and Glogower's approach, because it sheds some light on the potential source of the problem of non-unitarity. Consider the definition for the phase operator as given by Dirac in equation (3.1.3b):

$$\hat{a}^\dagger = \sqrt{\hat{n}}\, e^{-i\hat{\varphi}} \tag{3.1.32}$$

We calculate the matrix elements of this equation in the number state basis and obtain

$$\left\langle n' \left| \hat{a}^\dagger \right| n \right\rangle = \left\langle n' \left| \sqrt{\hat{n}}\, e^{-i\hat{\varphi}} \right| n \right\rangle \tag{3.1.33a}$$

$$\Leftrightarrow \sqrt{n+1}\, \delta_{n',n+1} = \sqrt{n'} \left\langle n' \left| e^{-i\hat{\varphi}} \right| n \right\rangle \tag{3.1.33b}$$

$$\Leftrightarrow \begin{cases} \left\langle n' \left| e^{-i\hat{\varphi}} \right| n \right\rangle = 1\,, & n' = n+1 \\ \left\langle n' \left| e^{-i\hat{\varphi}} \right| n \right\rangle = 0\,, & n' \neq n+1;\, n' \neq 0 \end{cases} \tag{3.1.33c}$$

where we have used

$$\sqrt{\hat{n}} = \sum_{n=0}^{\infty} \sqrt{n}\, |n\rangle\langle n|\,. \tag{3.1.34}$$

Because the number operator (or its square-root, respectively) has zero-valued eigenvalues, the matrix elements cannot be determined for $n' = 0$. This means that these matrix elements can be set to arbitrary values without affecting relation (3.1.33c). Therefore, Susskind and Glogower introduced variables $\{r_0, r_1 \cdots\}$ for these undetermined elements

---

[8]Interestingly, Susskind and Glogower were turned onto the problem when Peter Carruthers, their then professor, gave them a homework assignment to investigate whether the quantities introduced by Dirac really exist. For more historical background including some personal anecdotes, compare [10].

and constructed the matrix for $\mathrm{e}^{-i\hat{\varphi}}$ (and analogously, $\mathrm{e}^{i\hat{\varphi}}$):

$$
\mathrm{e}^{-i\hat{\varphi}} = \begin{pmatrix} r_0 & r_1 & r_2 & r_3 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathrm{e}^{i\hat{\varphi}} = \begin{pmatrix} r_0^* & 1 & 0 & 0 & \cdots \\ r_1^* & 0 & 1 & 0 & \cdots \\ r_2^* & 0 & 0 & 1 & \cdots \\ r_3^* & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.1.35}
$$

Multiplication now leads to

$$
\mathrm{e}^{i\hat{\varphi}} \cdot \mathrm{e}^{-i\hat{\varphi}} = \begin{pmatrix} r_0 r_0^* + 1 & r_1 r_0^* & r_2 r_0^* & r_3 r_0^* & \cdots \\ r_0 r_1^* & r_1 r_1^* + 1 & r_2 r_1^* & r_3 r_1^* & \cdots \\ r_0 r_2^* & r_1 r_2^* & r_2 r_2^* + 1 & r_3 r_2^* & \cdots \\ r_0 r_3^* & r_1 r_3^* & r_2 r_2^* & r_3 r_3^* + 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.1.36a}
$$

$$
\mathrm{e}^{-i\hat{\varphi}} \cdot \mathrm{e}^{i\hat{\varphi}} = \begin{pmatrix} \sum r_i r_i^* & r_0 & r_1 & r_2 & \cdots \\ r_0^* & 1 & 0 & 0 & \cdots \\ r_1^* & 0 & 1 & 0 & \cdots \\ r_2^* & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.1.36b}
$$

If $\mathrm{e}^{-i\hat{\varphi}}$ is to be unitary, then equation (3.1.36a) demands that all r_i, r_i^* are zero, so that $UU^{-1} = \mathbb{1}$ is fulfilled. But then, according to equation (3.1.36b), $U^{-1}U = \mathbb{1}$ cannot be fulfilled, since the upper-leftmost matrix element is zero. Therefore, regardless of the choice of the r_i, $\mathrm{e}^{-i\hat{\varphi}}$ cannot be unitary if it is defined as a polar decomposition of $\hat{a}^\dagger$ by

$$
\hat{a}^\dagger = \sqrt{\hat{n}}\,\mathrm{e}^{-i\hat{\varphi}}. \tag{3.1.37}
$$

3.1.3.2 Reasons for the problems with Dirac's operator

Susskind and Glogower's first major conclusion was that the problem of the non-unitarity of $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ arises because the spectrum of the harmonic oscillator is only one-sided infinite. This makes sense when we take another look at eq. (3.1.33c), since we now realize that the reason why the matrix elements could not be determined for $n' = 0$ was that this would imply that $n = -1$, an undefined state in the standard quantum harmonic oscillator formulation (compare appendix A.1.2 for more information). Apparently, the one-sidedness of the spectrum is responsible for an information loss or irreversibility which introduces unwanted vacuum projectors (cf. equation (3.1.18)) and thereby abrogates the

unitarity of $e^{i\hat{\varphi}}$ and $e^{-i\hat{\varphi}}$. Indeed, let us take another look at the product of the matrices (3.1.35) obtained when $r_i, r_i^* = 0$:

$$
e^{i\hat{\varphi}} \cdot e^{-i\hat{\varphi}} =
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\qquad
e^{-i\hat{\varphi}} \cdot e^{i\hat{\varphi}} =
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} .
\tag{3.1.38}
$$

Extending these matrices to $n', n \to -\infty$, we get

$$
e^{i\hat{\varphi}} \cdot e^{-i\hat{\varphi}} =
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\qquad
e^{-i\hat{\varphi}} \cdot e^{i\hat{\varphi}} =
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 1 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} ,
\tag{3.1.39}
$$

a manifestly unitary result. What has happened is that the problematic zero-valued entry in the upper left corner of the matrix to the right in (3.1.38) has been removed to infinity when the Hilbert space was extended to $|-\infty\rangle$. This has many implications, as Susskind and Glogower realized. First, in such an unbounded Hilbert space, phase eigenstates

$$
|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{in\varphi} |n\rangle
\tag{3.1.40}
$$

could be constructed to mirror the impulse eigenstates in the position-basis

$$
|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} |x\rangle \ \mathrm{d}x .
\tag{3.1.41}
$$

Moreover, these phase states, like their momentum analogues, would be orthogonal[9]

$$
\langle \varphi' | \varphi \rangle = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\varphi - \varphi')} = \delta(\varphi' - \varphi)
$$
$$
\langle p' | p \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} \, \mathrm{d}x = \delta(p' - p)
$$
$$\tag{3.1.42}$$

---

[9]Calculation of infinite complex exponential series due to [22], p. 331.

and would therefore allow the definition of a hermitian phase operator as

$$\hat{\varphi} = \int\limits_{-\infty}^{\infty} \mathrm{d}\varphi\, \varphi\, |\varphi\rangle\langle\varphi| \tag{3.1.43}$$

All this is impossible in a one-sided infinite Hilbert space, since in this space, the phase states are not orthogonal and therefore cannot define a hermitian phase operator. Indeed, when we turn to discuss the Pegg/Barnett phase operator in chapter 4, we will see that manipulations of the oscillator Hilbert space play an important role in the derivation of a (quasi)hermitian phase operator $\hat{\varphi}$.

Susskind and Glogower, however, chose to adhere to the conventional one-sided infinite Hilbert space, and therefore had to find some useful means of working with the operators $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ as defined by the matrices (3.1.35). On the basis of these operators, Susskind and Glogower developed a new formalism which produced good results for reasonably large n and immediately supplanted the problematic Dirac formalism introduced in section 3.1.1.

3.1.3.3 Some initial remarks

A few initial remarks have to be made before turning to the details of SG's new formalism. Given that SG's argument showed that the operators $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ are not exponential operator functions of a hermitian phase operator $\hat{\varphi}$, but instead symbolic expressions which are derived from the annihilation and creation operators and whose phase properties are somewhat more subtle, a small adjustment in notation is in order: In the following, whenever an operator is not an operator function of a phase operator, we will extend the operator symbol "^" to the entire expression instead of limiting it to the phase variable. This means that in the following, we will write

$$\widehat{\mathrm{e}^{i\varphi}}, \qquad \widehat{\mathrm{e}^{-i\varphi}} \tag{3.1.44}$$

for the exponential operators. We also note that later authors have often denoted these operators by E_+ and E_-, in an effort to make it even clearer that these are not exponential operator functions. In my opinion, however, the notation in (3.1.44) is clear enough, so we will stand by it.

Finally, we note that $\widehat{\mathrm{e}^{i\varphi}}$ and $\widehat{\mathrm{e}^{-i\varphi}}$ are simply normalized annihilation and creation operators, as can be discerned from the matrix representation:

$$\widehat{\mathrm{e}^{i\varphi}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|\,, \qquad \widehat{\mathrm{e}^{-i\varphi}} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|\,. \tag{3.1.45}$$

With this is mind, we now turn to the formalism.

3.1.3.4 Susskind/Glogower formalism

Susskind's and Glogower's formalism expands on the idea already introduced by Louisell a year earlier [20]. First, the operators $\widehat{e^{i\varphi}}$ and $\widehat{e^{-i\varphi}}$ are used to define sine and cosine operators

$$\widehat{\sin\varphi} = \frac{1}{2i}\left(\widehat{e^{i\varphi}} - \widehat{e^{-i\varphi}}\right) = \frac{1}{2i}\sum_{n=0}^{\infty}|n\rangle\langle n+1| - |n+1\rangle\langle n|\,. \tag{3.1.46a}$$

$$\widehat{\cos\varphi} = \frac{1}{2}\left(\widehat{e^{i\varphi}} + \widehat{e^{-i\varphi}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}|n\rangle\langle n+1| + |n+1\rangle\langle n| \tag{3.1.46b}$$

As already noted by Louisell, this avoids the periodicity problem. Moreover, these operators are hermitian, and they therefore represent an observable. They do not commute with the number operator, but this is, in a way, not surprising given that they are composed of rescaled creation and annihilation operators, which also do not commute with number. The commutators with the number operator turn out to be

$$[\widehat{\cos\varphi}, \hat{n}] = i\,\widehat{\sin\varphi} \tag{3.1.47a}$$

$$[\widehat{\sin\varphi}, \hat{n}] = -i\,\widehat{\cos\varphi}\,, \tag{3.1.47b}$$

and remind us of section 3.1.2. They follow from the commutators

$$[\widehat{e^{i\varphi}}, \hat{n}] = \widehat{e^{i\varphi}} \tag{3.1.48a}$$

$$[\widehat{e^{-i\varphi}}, \hat{n}] = -\widehat{e^{-i\varphi}} \tag{3.1.48b}$$

when the definitions (3.1.46a and 3.1.46b) are inserted. The alert reader may note that we have had some problems with this relation before and may warn us to be wary of using it here; however, in the present case, we did not rely on the unitarity of an exponential phase operator function as in eq. (3.1.5), but instead calculated the commutator of $\widehat{e^{i\varphi}}$ and $\widehat{e^{-i\varphi}}$ with the number operator anew by using the definitions given in eq. (3.1.45):

$$
\begin{aligned}
&[\widehat{e^{i\varphi}}, \hat{n}] = && [\widehat{e^{-i\varphi}}, \hat{n}] = \\
&= \widehat{e^{i\varphi}}\hat{n} - \hat{n}\widehat{e^{i\varphi}} = && = \widehat{e^{-i\varphi}}\hat{n} - \hat{n}\widehat{e^{-i\varphi}} = \\
&= \sum_{n',n}|n'\rangle\langle n'+1| \cdot n\,|n\rangle\langle n| - && = \sum_{n',n}|n'+1\rangle\langle n'| \cdot n\,|n\rangle\langle n| - \\
&\quad - \sum_{n',n}n\,|n'\rangle\langle n'| \cdot |n\rangle\langle n+1| = && \quad - \sum_{n',n}n\,|n'\rangle\langle n'| \cdot |n+1\rangle\langle n| = \\
&= \sum_{n}(n+1)\,|n\rangle\langle n+1| - && = \sum_{n}n\,|n+1\rangle\langle n| - \\
&\quad - \sum_{n}n\,|n\rangle\langle n+1| = && \quad - \sum_{n}(n+1)\,|n+1\rangle\langle n| = \\
&= \sum_{n}|n\rangle\langle n+1| = && = \sum_{n}(-1)\,|n+1\rangle\langle n| = \\
&= \widehat{e^{i\varphi}} && = -\widehat{e^{-i\varphi}}
\end{aligned}
\tag{3.1.49}
$$

This is a nice consistency that speaks for the plausibility of these operators.[10]

Less fortunate is the fact that besides not commuting with the number operator, sine and cosine also do not commute amongst themselves, as the following calculation shows:

$$
\begin{aligned}
[\widehat{\sin\varphi}, \widehat{\cos\varphi}] &= \\
&= \frac{1}{2i}\left(\widehat{e^{i\varphi}} - \widehat{e^{-i\varphi}}\right)\cdot\frac{1}{2}\left(\widehat{e^{i\varphi}} + \widehat{e^{-i\varphi}}\right) - \frac{1}{2}\left(\widehat{e^{i\varphi}} + \widehat{e^{-i\varphi}}\right)\cdot\frac{1}{2i}\left(\widehat{e^{i\varphi}} - \widehat{e^{-i\varphi}}\right) = \\
&= \frac{1}{4i}\left(\widehat{e^{i\varphi}}\widehat{e^{i\varphi}} + \widehat{e^{i\varphi}}\widehat{e^{-i\varphi}} - \widehat{e^{-i\varphi}}\widehat{e^{i\varphi}} - \widehat{e^{-i\varphi}}\widehat{e^{-i\varphi}}\right) - \\
&\quad\quad - \frac{1}{4i}\left(\widehat{e^{i\varphi}}\widehat{e^{i\varphi}} - \widehat{e^{i\varphi}}\widehat{e^{-i\varphi}} + \widehat{e^{-i\varphi}}\widehat{e^{i\varphi}} - \widehat{e^{-i\varphi}}\widehat{e^{-i\varphi}}\right) = \\
&= \frac{1}{2i}\left[\widehat{e^{i\varphi}}, \widehat{e^{-i\varphi}}\right] = \\
&= \frac{1}{2i}\left|0\right\rangle\!\left\langle 0\right| .
\end{aligned} \tag{3.1.50}
$$

This is quite an annoyance, since it disrupts the trigonometric identity $\sin^2\varphi + \cos^2\varphi = 1$, which for the S/G trigonometric operators instead turns out to be

$$
\begin{aligned}
(\widehat{\sin\varphi})^2 + (\widehat{\cos\varphi})^2 &= \\
&= (\widehat{\cos\varphi})^2 + i\,\widehat{\cos\varphi}\,\widehat{\sin\varphi} - i\,\widehat{\cos\varphi}\,\widehat{\sin\varphi} + (\widehat{\sin\varphi})^2 = \\
&= (\widehat{\cos\varphi})^2 + i\,\widehat{\cos\varphi}\,\widehat{\sin\varphi} - i\,\widehat{\sin\varphi}\,\widehat{\cos\varphi} - i[\widehat{\sin\varphi}, \widehat{\cos\varphi}] + (\widehat{\sin\varphi})^2 = \\
&= (\widehat{\cos\varphi} + i\,\widehat{\sin\varphi})(\widehat{\cos\varphi} - i\,\widehat{\sin\varphi}) - \frac{1}{2}\left|0\right\rangle\!\left\langle 0\right| = \\
&= \widehat{e^{i\varphi}}\cdot\widehat{e^{-i\varphi}} - \frac{1}{2}\left|0\right\rangle\!\left\langle 0\right| = \\
&= 1 - \frac{1}{2}\left|0\right\rangle\!\left\langle 0\right| .
\end{aligned} \tag{3.1.51}
$$

It is easy to see that the nonzero commutator $[\widehat{\sin\varphi}, \widehat{\cos\varphi}]$ is at the root of the problem, since without it, the unwanted vacuum projector would vanish. Only in the correspondence limit does the projector $\left|0\right\rangle\!\left\langle 0\right|$ become small enough that the identity $\sin^2\varphi + \cos^2\varphi = 1$ fulfilled.

Nonetheless, the operators $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ have many appreciable properties: They are bounded, their eigenstates are orthogonal and complete, and their eigenvalues are trigonometric functions. To prove this, however, some further investigation is required, to which the next few pages are devoted.

---

[10]Note that the consistency with Louisell's proposition in section 3.1.2 stems from the fact that the Susskind Glogower commutator $[\widehat{e^{i\varphi}}, \hat{n}] = \widehat{e^{i\varphi}}$ is analogous to the Dirac commutator $[e^{i\hat{\varphi}}, \hat{n}] = e^{i\hat{\varphi}}$.

Our first matter of interest are the eigenstates and eigenvalues of $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$, which we calculate using a decomposition in number states. We therefore start with the generalized decomposition

$$|\sin\varphi\rangle = \sum_{n=0}^{\infty} a_n |n\rangle \tag{3.1.52a}$$

$$|\cos\varphi\rangle = \sum_{n=0}^{\infty} b_n |n\rangle\,. \tag{3.1.52b}$$

Applying the operators $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ to these states and remembering their decomposition into exponentials $\widehat{e^{i\varphi}}$ and $\widehat{e^{-i\varphi}}$ (cf. equations (3.1.46a) and (3.1.46b)) yields the eigenvalue equations

$$\widehat{\sin\varphi}\,|\sin\varphi\rangle = \frac{1}{2i}\left[\sum_{n=0}^{\infty} a_{n+1}|n\rangle - \sum_{n=1}^{\infty} a_{n-1}|n\rangle\right] = \lambda_{\sin}\sum_{n=0}^{\infty} a_n|n\rangle \tag{3.1.53a}$$

$$\widehat{\cos\varphi}\,|\cos\varphi\rangle = \frac{1}{2}\left[\sum_{n=0}^{\infty} b_{n+1}|n\rangle + \sum_{n=1}^{\infty} b_{n-1}|n\rangle\right] = \lambda_{\cos}\sum_{n=0}^{\infty} b_n|n\rangle\,, \tag{3.1.53b}$$

where $\lambda_{\sin}$ and $\lambda_{\cos}$ are the sought eigenvalues. These equations can be solved by term-by-term evaluation

$$a_1 = 2i\lambda_{\sin}a_0 \qquad\qquad a_n = 2i\lambda_{\sin}a_{n-1} + a_{n-2} \tag{3.1.54a}$$

$$b_1 = 2\lambda_{\cos}b_0 \qquad\qquad b_n = 2\lambda_{\cos}b_{n-1} - b_{n-2}\,. \tag{3.1.54b}$$

Substituting $2i\lambda_{\sin} = (p - \frac{1}{p})$ and $2\lambda_{\cos} = (q + \frac{1}{q})$ allows us to calculate a_n and b_n (see appendix B.1 for the calculation):

$$a_n = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})}\, p^{2k} a_0 \tag{3.1.55a}$$

$$b_n = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} q^{2k} b_0\,. \tag{3.1.55b}$$

Note that these expressions differ from Susskind and Glogower's result, who incorrectly concluded that $a_n = Ap^n - Bp^{-n}$ and $b_n = Aq^n - Bq^{-n}$ (consult appendix B.1 for why this is incorrect). We therefore cannot follow their approach that p, q should be chosen so that the a_n, b_n are bounded for large n, since the sums do not converge for *any* value of p, q when $n \to \infty$. This means that we must probably drop the requirement that the states $|\sin\varphi\rangle$ and $|\cos\varphi\rangle$ should be normalizable.

However, this not particularly troubling if we compare the sums (3.1.55a) and (3.1.55b) to a free wave particle which is sharp in x and has the form $\Psi(x) = \delta(x)$: This state is also

non-normalizable, but it is nonetheless a decidedly physical state and indeed one of the most basic quantum states we know. Moreover, its decomposition into $|p\rangle$-states, although impossible by strictly orthodox mathematics, turns into

$$\delta(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \mathrm{e}^{ikx} \, \mathrm{d}k \,, \tag{3.1.56}$$

when using distributional mathematics. Plainly speaking, the integral can be evaluated because the rotation of e^{ikx} on the unit circle in the complex plane with rising k is interpreted to cause the exponentials to average out to zero except where $x = 0$.

A comparison of this integral with the sums (3.1.55a) and (3.1.55b) reveals a marked structural similarity, leading us to choose $p, q = \mathrm{e}^{i\varphi}$. The sums then become

$$a_n = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} \, \mathrm{e}^{2ik\varphi} a_0 \tag{3.1.57a}$$

$$b_n = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \mathrm{e}^{2ik\varphi} b_0 \,. \tag{3.1.57b}$$

Setting $a_0 = N \cdot \cos\varphi$ and $b_0 = N \cdot \sin\varphi$ and inserting the appropriate Moivre-identities in the recursion (3.1.54) gives:[11]

$$a_n = \begin{cases} N \cdot \cos[\varphi(n+1)], & n = 0, 2, 4, \cdots \\ N \cdot i\sin[\varphi(n+1)], & n = 1, 3, 5, \cdots \end{cases} \tag{3.1.58a}$$

$$= \frac{N}{2} \cdot \left[\mathrm{e}^{i\varphi(n+1)} - \mathrm{e}^{i\pi(n+1)} \mathrm{e}^{-i\varphi(n+1)} \right] \tag{3.1.58b}$$

$$b_n = N \cdot \sin[\varphi(n+1)] \,, \tag{3.1.58c}$$

where the second equality for a_n may be verified by noting that the factor $\mathrm{e}^{i\pi(n+1)}$ is responsible for alternating between sine and cosine. This allows us to write $|\sin\varphi\rangle$ and $|\cos\varphi\rangle$ as

$$|\sin\varphi\rangle = N \left[\sum_{k=0}^{\infty} \cos[\varphi(2k+1)]|2k\rangle + i \sum_{l=0}^{\infty} \sin[\varphi(2l+2)]|2l+1\rangle \right] \tag{3.1.59a}$$

$$= \frac{N}{2} \cdot \sum_{n=0}^{\infty} \tfrac{1}{2} \left[\mathrm{e}^{i\varphi(n+1)} - \mathrm{e}^{-i(\varphi-\pi)(n+1)} \right] |n\rangle$$

$$|\cos\varphi\rangle = N \cdot \sum_{n=0}^{\infty} \sin[\varphi(n+1)]|n\rangle \,. \tag{3.1.59b}$$

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[11]confer [6], section 6.

Calculating the scalar product, we get

$$\langle \sin\varphi| \sin\varphi'\rangle = \frac{N^2}{4} \sum_{n=0}^{\infty} \left[\mathrm{e}^{-i\varphi(n+1)} - \mathrm{e}^{i(\varphi-\pi)(n+1)} \right] \cdot \left[\mathrm{e}^{i\varphi'(n+1)} - \mathrm{e}^{-i(\varphi'-\pi)(n+1)} \right] =$$

$$= \frac{N^2}{2} \sum_{n=0}^{\infty} \cos[(\varphi'-\varphi)(n+1)] - \cos[(\varphi'+\varphi-\pi)(n+1)] = \qquad (3.1.60)$$

$$= N^2 \frac{\pi}{2} \sum_{n=-\infty}^{\infty} \delta(\varphi'-\varphi-2\pi n) - \delta(\varphi'+\varphi-\pi(2n+1)) \,,$$

and for cosine

$$\langle \cos\varphi| \cos\varphi'\rangle = N^2 \sum_{n=0}^{\infty} \sin[\varphi(n+1)] \sin[\varphi'(n+1)] =$$

$$= \frac{N^2}{2} \sum_{n=0}^{\infty} \cos[(\varphi'-\varphi)(n+1)] - \cos[(\varphi'+\varphi)(n+1)] = \qquad (3.1.61)$$

$$= N^2 \cdot \frac{\pi}{2} \sum_{n=-\infty}^{\infty} \delta(\varphi'-\varphi-2\pi n) - \delta(\varphi'+\varphi-2\pi n) \,,$$

where we have followed [6], who relied on formulas taken from [22]. This result is expected because we already noted above that the structure of the sums for the components a_n and b_n of $|\sin\varphi\rangle$ and $|\cos\varphi\rangle$ (cf. equations (3.1.55a) and (3.1.55b)) implied the presence of delta functions. The periodicity of these delta functions when compared against the integral (3.1.56) arises from the fact that the sums are discrete and therefore many values of φ exist for which $\mathrm{e}^{in\varphi}$ does not rotate in the complex plane as n rises, entailing that the terms do not cancel.

If we now restrict φ, φ' to the range $-\pi/2 \leq \varphi, \varphi' \leq \pi/2$ for sine and $0 \leq \varphi, \varphi' \leq \pi$ for cosine and set $N = (\frac{2}{\pi})^{1/2}$, all deltafunctions except the one within the interval vanish and we are left with the simple orthogonality relation

$$\langle \cos\varphi'| \cos\varphi\rangle = \langle \sin\varphi'| \sin\varphi\rangle = \delta(\varphi'-\varphi) \,, \qquad (3.1.62)$$

which is a standard orthogonality relation for operators with continuous spectra (cf. equations (3.1.42)). We therefore conclude that the eigenstates of the trigonometric operators are orthogonal.

Next, we need to prove the boundedness of the operators, which requires calculation of the eigenvalues. These can be obtained by reverse transformation from $p, q = \mathrm{e}^{i\varphi}$ to $\lambda_{\sin}$ and $\lambda_{\cos}$ according to the substitution made just before eq. (3.1.55a)

$$\lambda_{\sin} = \frac{1}{2i}(p - \frac{1}{p}) = \sin\varphi \qquad \lambda_{\cos} = \frac{1}{2}(q + \frac{1}{q}) = \cos\varphi \,, \qquad (3.1.63)$$

which gives sine and cosine as expected. Therefore,

$$\widehat{\sin\varphi}\,|\sin\varphi\rangle = \sin\varphi\,|\sin\varphi\rangle \qquad \widehat{\cos\varphi}\,|\cos\varphi\rangle = \cos\varphi\,|\cos\varphi\rangle \,, \tag{3.1.64}$$

which leads to the pleasant conclusion that the operators $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ have bounded spectra. This is equivalent to saying that the operators themselves are bounded, since the eigenstates $|\sin\varphi\rangle$ and $|\cos\varphi\rangle$ are orthogonal and complete (this will be shown below), and therefore any function in the Hilbert space may be decomposed into a series of trigonometric eigenstates, for which the boundedness condition $\left\| \hat{A}\,|u\rangle \right\| \leq c\,\|\,|u\rangle\|$ is of course preserved since $\max|sin\varphi| = \max|\cos\varphi| = 1$. Note also that the spectra of $|\sin\varphi\rangle$ and $|\cos\varphi\rangle$ are continuous, since there is no quantum condition limiting φ to discrete values.

The last property to verify is the completeness of $|\sin\varphi\rangle$ and $|\cos\varphi\rangle$. This can be shown by proving that they resolve to the identity. For sine, we recur to the definition in the first line of (3.1.59a) and observe that

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi\, |\sin\varphi\rangle\langle\sin\varphi| =$$

$$= \frac{2}{\pi}\left[\sum_{k',k=0}^{\infty} |2k'\rangle\langle 2k| \int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi\, \cos[\varphi(2k'+1)]\cos[\varphi(2k+1)]\right.$$

$$\left. + \sum_{l',l=0}^{\infty} |2l'+1\rangle\langle 2l+1| \int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi\, \sin[\varphi(2l'+2)]\sin[\varphi(2l+2)]\right] =$$

$$= \frac{2}{\pi}\left[\sum_{k',k=0}^{\infty} |2k'\rangle\langle 2k|\cdot\left\{\begin{array}{ll} 0\,, & k'\neq k \\ \pi/2\,, & k'=k \end{array}\right\} + \sum_{l',l=0}^{\infty} |2l'+1\rangle\langle 2l+1|\cdot\left\{\begin{array}{ll} 0\,, & l'\neq l \\ \pi/2\,, & l'=l \end{array}\right\}\right]$$

$$= \frac{2}{\pi}\sum_{n',n=0}^{\infty} |n'\rangle\langle n|\cdot\left\{\begin{array}{ll} 0\,, & n'\neq n \\ \pi/2\,, & n'=n \end{array}\right\} = \sum_{n',n=0}^{\infty} |n'\rangle\langle n|\,\delta_{n',n} = \mathbb{1} \tag{3.1.65a}$$

$$\int_{0}^{\pi} \mathrm{d}\varphi\, |\cos\varphi\rangle\langle\cos\varphi| = \frac{2}{\pi}\sum_{n',n=0}^{\infty} |n'\rangle\langle n| \int_{0}^{\pi} \mathrm{d}\varphi\, \sin[(n'+1)\varphi]\sin[(n+1)\varphi] =$$

$$= \frac{2}{\pi}\sum_{n',n=0}^{\infty} |n'\rangle\langle n|\cdot\left\{\begin{array}{ll} 0\,, & n'\neq n \\ \pi/2\,, & n'=n \end{array}\right\} = \sum_{n',n=0}^{\infty} |n'\rangle\langle n|\,\delta_{n',n} = \mathbb{1}\,, \tag{3.1.65b}$$

as can easily be verified by consulting any reasonable formulary.[12]

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[12]Note that in the calculation of sine, we used that the modulus square of the first line of equation (3.1.59a) makes imaginary terms disappear, allowing us to ignore terms of mixed sine and cosine.

3.1.3.5 Uncertainty relations

To conclude this section, we give the phase photon-number uncertainties for the Susskind Glogower formalism:

$$\Delta\widehat{\sin\varphi}\cdot\Delta\hat{n} \geq \frac{1}{2}\left|\langle\widehat{\cos\varphi}\rangle\right|, \qquad \Delta\widehat{\cos\varphi}\cdot\Delta\hat{n} \geq \frac{1}{2}\left|\langle\widehat{\sin\varphi}\rangle\right|. \tag{3.1.66}$$

These are based on the commutators

$$\left[\widehat{\sin\varphi},\hat{n}\right] = i\,\widehat{\cos\varphi}, \qquad \left[\widehat{\cos\varphi},\hat{n}\right] = -i\,\widehat{\sin\varphi}, \tag{3.1.67}$$

respectively, which we already know from equations (3.1.47a) and (3.1.47b). An interesting result ensues if we let the phase approach 0, because we may then set $sinx \approx x$ and $cosx \approx 1$, so that

$$\Delta\hat{\varphi}\cdot\Delta\hat{n} \geq \frac{1}{2}. \tag{3.1.68}$$

Note that an equivalent result is obtained as the phase approaches $\pi/2$. This expression bears a striking resemblance to the uncertainty relation resulting from Dirac's approach and is phenomenologically useful. It will be corroborated in an observation due in part to [6] (cf. the second observation in section 3.1.4.2), and will reappear when treating uncertainty in the PB phase formalism (see sections 4.2.6 and 7.1.3).

3.1.4 Discussion of the Susskind/Glogower phase description

Given that the Susskind/Glogower phase description is the first mathematically correct formalism after Dirac's ill-fated phase operator, we briefly review its characteristics in light of the a priori conditions set out in chapter 2.

3.1.4.1 Test of a priori conditions

1. The first a priori condition concerned consistency in the large-n limit. From the examination of the oscillator problem in classical physics as undertaken in section 3.1.2, we note that the sine and cosine operators defined by Susskind and Glogower behave like their classical equivalents when comparing their respective photon-number commutation relations to the corresponding Poisson-brackets. We also note that even though the non-vanishing commutator $[\widehat{\sin\varphi},\widehat{\cos\varphi}]$ contradicts the classical result, it vanishes in the large-n limit since large-n states have negligible overlap with the vacuum state, whence the vacuum projector does not have much influence on the resulting expectation values. This means that those trigonometric relations that deviated from the corresponding classical results because of the vacuum projector will conform in the limit of large n. Thus, it seems that the first apriori requirement is fulfilled.

2. To keep the most interesting for last, we skip right to the third a priori condition set forth in chapter 2 and note that the formulation of Susskind and Glogower rests well within standard quantum mechanics. Although it is unfortunate that an hermitian phase operator was not found, as a manner of compensation, the sine and cosine operators $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ are hermitian and therefore describe a quantum observable. Moreover, the operators are bounded and the eigenstates of sine and cosine are orthogonal and complete when appropriate limitations in the range of the phase-angle are made. The formalism does not require any modification of the oscillator Hilbert space, and it does not presuppose any other non-quantum characteristics or prerequisites. It is therefore a good formalism in light of the third a priori requirement, although some calculational difficulties can at times arise e.g. in the calculation of higher moments with respect to complex states.

3. We now examine whether the SG-formalism fulfills the acid test set forth as the second a priori condition in chapter 2.[13] To this end, we need to calculate the expectation values and variances of sine and cosine in the number state basis, since number states are commonly thought to be states of complete phase uncertainty. Remember that

$$\widehat{\sin\varphi} = \frac{1}{2i}\left[\widehat{e^{i\varphi}} - \widehat{e^{-i\varphi}}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}|n\rangle\langle n+1| - |n+1\rangle\langle n| \tag{3.1.69a}$$

$$\widehat{\cos\varphi} = \frac{1}{2}\left[\widehat{e^{i\varphi}} + \widehat{e^{-i\varphi}}\right] = \frac{1}{2}\sum_{n=0}^{\infty}|n\rangle\langle n+1| + |n+1\rangle\langle n|. \tag{3.1.69b}$$

It is now easy to see that

$$\left\langle n\left|\widehat{\sin\varphi}\right|n\right\rangle = \left\langle n\left|\widehat{\cos\varphi}\right|n\right\rangle = 0 \qquad \forall n \in \mathbb{N} \tag{3.1.70}$$

which is consistent with a randomly distributed phase in an interval of $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ for sine and $0 \leq \varphi \leq \pi$ for cosine. We now turn to the second moments, which we can calculate using the hermiticity of $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$

$$\left\langle n\left|(\widehat{\sin\varphi})^2\right|n\right\rangle = \left\langle\widehat{\sin\varphi}\,n\,\middle|\,\widehat{\sin\varphi}\,n\right\rangle =$$

$$= -\frac{1}{4}\left\{\sum_{n'=0}^{\infty}\langle n|\left(|n'\rangle\langle n'+1| - |n'+1\rangle\langle n'|\right)\right\}$$

$$\times\left\{\sum_{n''=0}^{\infty}\left(|n''\rangle\langle n''+1| - |n''+1\rangle\langle n''|\right)|n\rangle\right\} =$$

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[13]Note that we cannot directly rely on the results obtained for classical phase distributions in section 2.2, since here we have no phase operator, but only sine and cosine operators. The equivalent classic acid test numbers will therefore also have to be calculated

$$= \begin{cases} \frac{1}{4} \langle n+1| \, n+1 \rangle \, , & n = 0 \\ -\frac{1}{4} \left\{ \langle n+1| - \langle n-1| \right\} \left\{ |n-1\rangle - |n+1\rangle \right\} \, , & n \geq 1 \end{cases}$$

$$= \begin{cases} \frac{1}{4} \, , & n = 0 \\ \frac{1}{2} \, , & n \geq 1 \end{cases} \tag{3.1.71a}$$

$$\left\langle n \left| (\widehat{\cos\varphi})^2 \right| n \right\rangle = \left\langle \widehat{\cos\varphi} \, n | \, \widehat{\cos\varphi} \, n \right\rangle =$$

$$= \frac{1}{4} \left\{ \sum_{n'=0}^{\infty} \langle n| \left(|n'\rangle \langle n'+1| + |n'+1\rangle \langle n'| \right) \right\}$$

$$\times \left\{ \sum_{n''=0}^{\infty} \left(|n''\rangle \langle n''+1| + |n''+1\rangle \langle n''| \right) |n\rangle \right\} =$$

$$= \begin{cases} \frac{1}{4} \langle n+1| \, n+1 \rangle \, , & n = 0 \\ \frac{1}{4} \left\{ \langle n+1| + \langle n-1| \right\} \left\{ |n-1\rangle + |n+1\rangle \right\} \, , & n \geq 1 \end{cases}$$

$$= \begin{cases} \frac{1}{4} \, , & n = 0 \\ \frac{1}{2} \, , & n \geq 1 \end{cases} \tag{3.1.71b}$$

The moments are therefore[14]

$$\left\langle n \left| (\widehat{\sin\varphi})^2 \right| n \right\rangle = \left\langle n \left| (\widehat{\cos\varphi})^2 \right| n \right\rangle = \begin{cases} \frac{1}{4} \, , & n = 0 \\ \frac{1}{2} \, , & n \geq 1 \end{cases} \tag{3.1.72}$$

If we compare this to the second moment of a random probability distribution $P(\varphi) = 1/2\pi$

$$\begin{aligned} \left\langle \sin^2 \varphi \right\rangle &= \int_{-\pi}^{\pi} \sin^2 \varphi P(\theta) \, \mathrm{d}\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 \varphi \, \mathrm{d}\varphi = \frac{1}{2} \\ \left\langle \cos^2 \varphi \right\rangle &= \int_{-\pi}^{\pi} \cos^2 \varphi P(\theta) \, \mathrm{d}\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2 \varphi \, \mathrm{d}\varphi = \frac{1}{2} \, , \end{aligned} \tag{3.1.73}$$

we see that the Susskind/Glogower formalism passes the acid test for states $n \geq 1$, but fails it for the vacuum state, where a non-random phase distribution occurs. Moreover, the results get worse for higher moments of $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$, as the calculations in [8] show, so that any states which have considerable overlap with the vacuum state are not described properly by the SG-formalism, at least not according to the a priori acid test.[15] We

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[14]For a calculation of higher orders, consult pages 378 et seq. of [8].

[15]For the possibly limited value of the acid test cf. the last paragraph of subsection 4.3.2.4.

conclude that the culprit lies with the nonzero commutator $[\widehat{\sin\varphi}, \widehat{\cos\varphi}]$ or, equivalently, with the one-sidedness of the spectrum of the harmonic oscillator.

Summarizing the above results, the Susskind/Glogower description is a good formalism in light of the a priori conditions, but fails the acid test for low-n states(i.e. states with non-negligible overlap with the vacuum state). Its main shortcomings are that the use of sine and cosine operators instead of a phase operator leads to a merely indirect description of phase, and that the non-vanishing commutator $[\widehat{\sin\varphi}, \widehat{\cos\varphi}]$ causes simple trigonometric relations not to hold in the low-n limit.

3.1.4.2 Interesting observations

Before finally turning away from the Susskind/Glogower formalism, two interesting observations should still be made:

First observation: The Susskind/Glogower $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ operators have a certain reminiscence of the spin operators $\hat{S}^2$, $\hat{s}_x$, $\hat{s}_y$ and $\hat{s}_z$ in that they do not simultaneously allow precise measurements of the sine and cosine of phase, since they do not commute. Considering that in a phase space representation (cf. figure 3.1), $\widehat{\sin\varphi}$ is a projection to the p-axis and $\widehat{\cos\varphi}$ is a projection to the x-axis, this could imply that a defined phase does not exist and that the phase only assumes a definite value once it is measured against the x-axis or the p-axis.

That could be a first indication that phase measurements depend on the measuring apparatus employed, as is proposed by Nohs, Fougéres and Mandel (cf. section 5.1.2). This would work in a way similar e.g. to the Stern-Gerlach experiment where

Figure 3.1: An exemplary coherent state and its x- and p-axis projections in phase space

all spin-vectors are digitally assigned either the value "up" or the value "down" on the z-axis upon passing through the inhomogeneous magnetic field, which leads to the loss of their prior x- and y-axis values (since $\hat{s}_z$ does not commute with $\hat{s}_x$ and $\hat{s}_y$). In the same

vein, a measurement of the phase by an x-axis projection would seem to preclude an ancillary measurement of the p-axis projection.[16] This implied "fuzziness" of the phase variable will be a recurring topic in the remainder of this thesis.

Second observation: The second interesting observation is that, upon closer examination, it can be shown that the Susskind/Glogower formalism does not prohibit the definition of an hermitian phase operator as such. Instead, as noted by Carruthers and Nieto in [6], the formalism actually allows for *two* (!) equally valid definitions of an hermitian phase operator. These result from a power series development of the inverse of sine and cosine (arcsin and arccos, respectively), applied to the $\widehat{\sin \varphi}$ and $\widehat{\cos \varphi}$ operators. This is possible because these operators have bounded spectra, and gives

$$\hat{\varphi}_{\sin} \equiv \arcsin(\widehat{\sin \varphi}) = \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k \cdot (2k+1)} \cdot (\widehat{\sin \varphi})^{2k+1} \tag{3.1.74a}$$

$$\hat{\varphi}_{\cos} \equiv \arccos(\widehat{\cos \varphi}) = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2k \cdot (2k+1)} \cdot (\widehat{\cos \varphi})^{2k+1} \tag{3.1.74b}$$

$\hat{\varphi}_{\sin}$ and $\hat{\varphi}_{\cos}$ are obviously hermitian since all the constituting moments of $\widehat{\sin \varphi}$ and $\widehat{\cos \varphi}$ are hermitian, and therefore define unitary exponentials

$$U_{\sin} = \mathrm{e}^{i\hat{\varphi}_{\sin}} \tag{3.1.75a}$$

$$U_{\cos} = \mathrm{e}^{i\hat{\varphi}_{\cos}} \tag{3.1.75b}$$

for which

$$U_{\sin}U_{\sin}^{-1} = U_{\sin}^{-1}U_{\sin} = 1 \tag{3.1.76a}$$

$$U_{\cos}U_{\cos}^{-1} = U_{\cos}^{-1}U_{\cos} = 1 \tag{3.1.76b}$$

is fulfilled. However, the two operators do not commute, since their constituent $\widehat{\sin \varphi}$ and $\widehat{\cos \varphi}$ operators also do not commute. This unfortunately means that these phase operators, even though hermitian, are useless, since they give an ambiguous representation of phase, where the phase measurement is dependent on an arbitrary choice between $\hat{\varphi}_{\sin}$ or $\hat{\varphi}_{\cos}$ and therefore does not reflect any physical property. Therefore, we note the operators $\hat{\varphi}_{\sin}$ and $\hat{\varphi}_{\cos}$ as peculiarities which, although interesting, cannot be used in any useful way to improve upon the Susskind/Glogower formalism.

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[16] And conversely, if an hermitian phase operator would exist, the commutator $[\widehat{\sin \varphi}, \widehat{\cos \varphi}]$ would be zero, and the phase would, depending on the state it is in, have a more or less precisely defined value within phase space, of which the p- as well as the x-axis projection could be measured.

3.2 Further developments

We now turn away from the Susskind-Glogower formalism and briefly address the developments which bridge the gap between SG-theory and the finite-dimensional Hilbert-space ansatz that would later lead to the PB phase formalism. The most important of these developments is due to Lévy-Leblond, and will be treated in the next section. A noteworthy minor result concerning a refinement of the solution to the periodicity problem will then be presented in section 3.2.2.

3.2.1 Lévy-Leblond

In 1976, Lévy-Leblond [23] advocated a new approach to quantum phase under the enigmatic title "Who is afraid of non-hermitian operators?". Unsatisfied with the non-zero commutator between sine and cosine, which meant being unable to make measurements of the x- as well as the p-projection of the phase variable simultaneously, Lévy-Leblond took a closer look at the properties of the $\widehat{e^{i\varphi}}$ operator as defined in eq. (3.1.45). He concluded that this operator should be able to provide a satisfactory description of phase even without the use of the hermitian $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$, if one were to dismiss the common dogma that every observable should have a corresponding hermitian operator. That such a dismissal would not be accepted easily, he was aware of.

3.2.1.1 General properties of non-hermitian operators

It is indeed one of the most sacred principles in quantum mechanics that every observable is represented by an hermitian operator which produces the observable as an eigenvalue for appropriate eigenfunctions. The reasoning leading to this principle is simple: only hermitian operators produce real-valued eigenvalues, and only real-valued eigenvalues can possibly represent a real, measurable observable. This reasoning is of course self-explanatory to most of us, but Lévy-Leblond prompts us to ask ourselves if we really know enough about the world to definitely exclude the possibility that imaginary numbers exist. After all, he provokes,[17] for all we know, the world consists only of the rational numbers, since the inexactitudes involved in all measurement processes only allow us to measure such numbers; indeed, no apparatus has ever measured an irrational number. Yet we do not doubt the existence of irrational numbers, and no one proposes to use only rational operators. By the same reasoning, we should at least consider non-hermitian operators.

This argument is certainly simplistic, but it gets the point across. And there are better and much more scientific arguments to confirm the plausibility of non-hermitian operators,

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[17]Cf. [23], footnote 4

too. One such argument underlines that some of the most important states in quantum optics, the coherent states, are eigenstates of a non-hermitian operator. Another argument is the widespread use of operator functions. For example, if we take an hermitian operator $\hat{a}$

$$\hat{a} = \sum_k a_k \, |a_k\rangle\langle a_k| \, , \tag{3.2.1}$$

operator functions of this operator can be formed simply by replacing the eigenvalues a_k in the sum with some function $f(a_k)$, yielding a new operator $f(\hat{a})$

$$f(\hat{a}) = \sum_k f(a_k) \, |a_k\rangle\langle a_k| \, . \tag{3.2.2}$$

Such operator functions are employed, for example, if the sum (3.2.1) for the original operator does not converge well enough[18] or has other undesirable properties; whatever the ultimate reason, it is important to note that such operator functions are usually non-hermitian. Nonetheless, they are employed in many physical problems without hesitation. For example, it is a well-known result in quantum physics that every unitary operator $\hat{U}$ may be rewritten as the exponential of some hermitian operator $\hat{a}$, using $\hat{U} = \mathrm{e}^{i\hat{a}}$, but often, use of the unitary instead of the hermitian operator is more common!

We conclude from this that it is not the use of a non-hermitian operator per se that is most commonly rejected, but instead the use of a non-hermitian operator which cannot be expressed as the operator function of some hermitian operator. There is some justification to this intuition, but Lévy-Leblond proposes that where an hermitian operator cannot be found, it is decidedly better to work with a non-hermitian operator than not being able to define an operator at all. With Lévy-Leblond, we shall pursue this proposition a little further.

There is one big difference between non-hermitian operator functions and non-hermitian operators, though: Whenever we are dealing with an operator function $f(\hat{a})$, this means that the eigenstates $|a_k\rangle$ of $\hat{a}$ exist, even if $\hat{a}$ is for some reason inaccessible. But where $\hat{a}$ is not only inaccessible, but either does not exist or is entirely unknown, we cannot proceed that easily, since we have no information about the presumed eigenstates $|a_k\rangle$. This means that any eigenstates we find for a non-hermitian operator $\widehat{f(a)}$ are not the eigenstates $|a_k\rangle$ of some hermitian operator, but different eigenstates altogether. This causes us to depart slightly from the notation used by Lévy-Leblond: To indicate that $\widehat{f(a)}$ cannot simply be an operator function of $\hat{a}$ in cases where $\hat{a}$ is not well-defined, we denote the eigenstates in

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[18]Such is the case for hermitian phase operators defined on finite Hilbert spaces, which converge only weakly in the one-sided infinite Hilbert space. Cf. section 4

such a case as $|f(a_k)\rangle$, giving the definition:

$$\widehat{f(a)} = \sum_k f(a_k) \, |f(a_k)\rangle\langle f(a_k)| \, , \tag{3.2.3}$$

But this is more than a mere notational difference! The states $|a_k\rangle$, being eigenstates of a hermitian operator, are automatically orthogonal and complete. But the same cannot be said of the $|f(a_k)\rangle$, indeed, they are neither necessarily orthogonal, nor necessarily complete. Therefore, they are not necessarily useful as a quantum representation of the (presumed) observable a, unless, and this is one of Lévy-Leblond's central assertions, they allow a resolution to the identity

$$\int \mathrm{d}a \, |f(a)\rangle\langle f(a)| = \mathbb{1} \, , \tag{3.2.4}$$

since this then makes it possible to calculate expectation values and variances (Note that the integral is still over a, since this is the quantum variable that we want to describe).[19]

3.2.1.2 Expectation values and variance of non-hermitian operators

For any operator that resolves to the identity, an arbitrary state $|\psi\rangle$ may be decomposed as

$$|\psi\rangle = \int \mathrm{d}a \, |f(a)\rangle\langle f(a)|\,\psi\rangle \tag{3.2.5}$$

so that expectation values may be calculated as

$$\left\langle \psi \left| \widehat{f(a)} \right| \psi \right\rangle = \int \mathrm{d}a \, f(a) \, |\langle f(a)|\,\psi\rangle|^2 . \tag{3.2.6}$$

This works even if the operator behaves nonnormal in the sense that

$$\widehat{f(a)}\,\widehat{f(a)}^\dagger \neq \widehat{f(a)}^\dagger\,\widehat{f(a)} \, , \tag{3.2.7}$$

all that is required is a resolution to the identity.[20]

We can also define uncertainty relations. Lévy-Leblond gives these in a generalized form for the operator $\widehat{f(a)}$ in the state $|\psi\rangle$ as

$$\Delta\widehat{f(a)} = \left\langle \psi \left| \widehat{f(a)}^\dagger \widehat{f(a)} \right| \psi \right\rangle - \left| \left\langle \psi \left| \widehat{f(a)} \right| \psi \right\rangle \right|^2 = \left\| (\widehat{f(a)} - \left\langle \widehat{f(a)} \right\rangle \mathbb{1}) \, |\psi\rangle \right\|^2 , \tag{3.2.8}$$

---

[19] Also, compare this to Susskind and Glogower's sine and cosine operators, which were also resolved to the identity by integrating over φ in equations (3.1.65), since this is the variable to be described.

[20] However, note that if the states are overcomplete, repeated measurements of $f(a)$ may not repeatedly give the same result, violating the projection hypothesis.

however, care must be taken not to confuse $\widehat{f(a)}^\dagger \widehat{f(a)}$ with $\left|\widehat{f(a)}\right|^2$, since the latter only applies for hermitian operators.

3.2.1.3 Non-hermitian phase operator

With the above preliminaries in mind, one may proceed to define the non-hermitian phase operator. Here, Lévy-Leblond uses an interesting approach that is based on an analogy to the description of position and momentum.[21] Reflecting on the fact that x is canonically conjugate to p and that φ is expected to be canonically conjugate to the photon-number n, he observes the relation

$$\langle p|\,x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\mathrm{e}^{ipx/\hbar} \tag{3.2.9}$$

and searches for an analogous expression such that

$$\langle n|\,\varphi\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{in\varphi}\,. \tag{3.2.10}$$

This is accomplished by introducing the states

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\mathrm{e}^{in\varphi}\,|n\rangle \tag{3.2.11}$$

which we already know in a similar form from equation (3.1.40), although there, we were assuming a doubly infinite Hilbert space. Here, no such assumption is made, and we stay in conventional Hilbert space, so it turns out that these states are not orthogonal

$$\langle\varphi'|\,\varphi\rangle = \sum_{n=0}^{\infty}\mathrm{e}^{-in(\varphi-\varphi')} \neq \delta(\varphi-\varphi')\,, \tag{3.2.12}$$

since the equality would require the spectrum to include negative values. The non-orthogonality means that these states cannot possibly define a hermitian phase operator, since hermitian operators have orthogonal eigenstates, and we therefore change the notation of these states from $|\varphi\rangle$ to $\left|\mathrm{e}^{i\varphi}\right\rangle$. However, the states are complete in that

$$\int_0^{2\pi}\mathrm{d}\varphi\,\left|\mathrm{e}^{i\varphi}\right\rangle\!\left\langle\mathrm{e}^{i\varphi}\right| = \frac{1}{2\pi}\sum_{n',n=0}^{\infty}|n'\rangle\langle n|\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{i(n'-n)\varphi} = \tag{3.2.13a}$$

$$= \sum_{n',n=0}^{\infty}|n'\rangle\langle n|\left\{\begin{array}{ll}1\,, & n'=n \\ 0\,, & n'\neq n\end{array}\right\} = \tag{3.2.13b}$$

---

[21]Similar considerations are also employed in [6], however, Carruthers and Nieto do not see any physical value in the resulting states and term them mere "auxiliary states".

$$= \sum_{n',n=0}^{\infty} |n'\rangle\langle n| \, \delta_{n'n} = \mathbb{1} \, , \tag{3.2.13c}$$

so they do allow the construction of a non-hermitian exponential phase operator $\widehat{e^{i\varphi}}$ (note again that we have already renamed the states $\left|e^{i\varphi}\right\rangle$ instead of $|\varphi\rangle$):

$$\widehat{e^{i\varphi}} = \int_0^{2\pi} \mathrm{d}\varphi \, e^{i\varphi} \left|e^{i\varphi}\right\rangle\left\langle e^{i\varphi}\right| = \frac{1}{2\pi} \sum_{n',n=0}^{\infty} |n'\rangle\langle n| \int_0^{2\pi} \mathrm{d}\varphi \, e^{i\varphi} \, e^{i(n'-n)\varphi} = \tag{3.2.14a}$$

$$= \frac{1}{2\pi} \sum_{n',n=0}^{\infty} |n'\rangle\langle n| \int_0^{2\pi} \mathrm{d}\varphi \, e^{i(n'+1-n)\varphi} = \sum_{n',n=0}^{\infty} |n'\rangle\langle n| \left\{ \begin{array}{ll} 1, & n'+1 = n \\ 0, & n'+1 \neq n \end{array} \right\} = \tag{3.2.14b}$$

$$= \sum_{n',n=0}^{\infty} |n'\rangle\langle n| \, \delta_{n'+1,n} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \, . \tag{3.2.14c}$$

We already know this operator from the Susskind-Glogower formalism, but back then we saw it merely as a means to define a hermitian operator. Now, we are more sensitive to the possible merits of this operator: Since the eigenstates $\left|e^{i\varphi}\right\rangle$ resolve to the identity, we may apply all the results obtained in subsections 3.2.1.1 and 3.2.1.2, including a (complicated) uncertainty relation and probabilistic phase measurements of the form

$$\left\langle \psi \left|\widehat{e^{i\varphi}}\right| \psi \right\rangle = \int_0^{2\pi} \mathrm{d}\varphi \, e^{i\varphi} \left|\left\langle e^{i\varphi}| \psi \right\rangle\right|^2 . \tag{3.2.15}$$

This is a very important result! Indeed, we may already reveal here that expression (3.2.15) will recur later on in a generalization of SG-theory (cf. section 4.3.2).

Other advantages to Lévy-Leblond's approach are the fact that we only need one operator instead of two to describe phase, and that we avoid the awkward non-commuting sine and cosine operators of Suskind/Glogower's formalism. Moreover, the operator and its eigenstates have transformation properties characteristic of phase. For example, an angular shift of λ in the operator simply induces a corresponding shift in the eigenvalues and vice versa, while leaving the eigenstates untouched:

$$\widehat{e^{i(\varphi+\lambda)}} = e^{i\lambda} \widehat{e^{i\varphi}} \, , \tag{3.2.16a}$$

$$\widehat{e^{i(\varphi+\lambda)}} \left|e^{i\varphi}\right\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n',n=0}^{\infty} e^{i\lambda} |n'\rangle\langle n'+1| \cdot e^{in\varphi} |n\rangle = \tag{3.2.16b}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{i\lambda} e^{i(n+1)\varphi} |n\rangle = e^{i(\varphi+\lambda)} \left|e^{i\varphi}\right\rangle \tag{3.2.16c}$$

This is in contrast to the $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ operators, where this is not the case:

$$\widehat{\sin(\varphi+\lambda)} = \frac{1}{2i}\left[e^{i\lambda}\widehat{e^{i\varphi}} - e^{-i\lambda}\widehat{e^{-i\varphi}}\right] = \tag{3.2.17a}$$

$$= \frac{1}{4i}\left[\left(e^{i\lambda}+e^{-i\lambda}\right)\left(\widehat{e^{i\varphi}}-\widehat{e^{-i\varphi}}\right) + \left(e^{i\lambda}-e^{-i\lambda}\right)\left(\widehat{e^{i\varphi}}+\widehat{e^{-i\varphi}}\right)\right] = \tag{3.2.17b}$$

$$= \cos\lambda\,\widehat{\sin\varphi} + \sin\lambda\,\widehat{\cos\varphi}\,, \tag{3.2.17c}$$

$$\widehat{\cos(\varphi+\lambda)} = \frac{1}{2}\left[e^{i\lambda}\widehat{e^{i\varphi}} + e^{-i\lambda}\widehat{e^{-i\varphi}}\right] = \tag{3.2.17d}$$

$$= \frac{1}{4}\left[\left(e^{i\lambda}+e^{-i\lambda}\right)\left(\widehat{e^{i\varphi}}+\widehat{e^{-i\varphi}}\right) - \left(-e^{i\lambda}+e^{-i\lambda}\right)\left(\widehat{e^{i\varphi}}-\widehat{e^{-i\varphi}}\right)\right] = \tag{3.2.17e}$$

$$= \cos\lambda\,\widehat{\cos\varphi} - \sin\lambda\,\widehat{\sin\varphi}\,. \tag{3.2.17f}$$

One last advantage of using $\widehat{e^{i\varphi}}$ is that the time development of the states $\left|e^{i\varphi}\right\rangle$ follows naturally as:

$$e^{-i\frac{\hat{H}}{\hbar}t}\left|e^{i\varphi}\right\rangle = \frac{1}{\sqrt{2\pi}}e^{-i\hat{n}\omega t}\sum_{n=0}e^{in\varphi}\left|n\right\rangle = \frac{1}{\sqrt{2\pi}}\sum_{n=0}e^{in\varphi}\,e^{-in\omega t}\left|n\right\rangle = \left|e^{i(\varphi-\omega t)}\right\rangle\,. \tag{3.2.18}$$

3.2.1.4 Discussion

Lévy-Leblond formulates an interesting approach which seems to be ahead of its time. As will be seen in a later sections, he has effectively transformed the description of quantum phase from an operator formalism to a probability-operator measure formalism constructed from the eigenstates of the exponential phase operator $\widehat{e^{i\varphi}}$ (see section 4.3.2). This comes with the caveat, however, that the probability-operator measure employed by Lévy-Leblond follows naturally from the SG-formalism, meaning that, ironically, Lévy-Leblond's solution was contained in SG's equations all along.

The similarities between the SG and the LL-formalism become clearer when we take a more critical look at the advantages Lévy-Leblond tries to convince us his operator has. Sure, employment of $\widehat{e^{i\varphi}}$ does avoid the awkward non-zero commutator $[\widehat{\sin\varphi}, \widehat{\cos\varphi}]$, but on the other hand, the loss of hermiticity and the nonorthogonality of the eigenstates means that the state projection hypothesis no longer applies. Thus, subsequent measurements of phase will always be fuzzy, which is, in essence, the same problem that already plagued the SG-formalism. To see this more clearly, recall that $\widehat{e^{i\varphi}} = \widehat{\cos\varphi} + i\widehat{\sin\varphi}$. Therefore,

$$\int_0^{2\pi} d\varphi\, e^{i\varphi}\left|\left\langle e^{i\varphi}|\psi\right\rangle\right|^2 = \left\langle\psi\left|\widehat{e^{i\varphi}}\right|\psi\right\rangle = \left\langle\psi\left|\widehat{\cos\varphi}\right|\psi\right\rangle + i\left\langle\psi\left|\widehat{\sin\varphi}\right|\psi\right\rangle\,. \tag{3.2.19}$$

Nonetheless, Lévy-Leblond's interpretation is an important contribution because it foreshadows the malleability of the SG formalism, which will turn out to be equivalent, besides

to Lévy-Leblond's formalism, also to the PB phase operator formalism to be discussed in the next chapter.

3.2.2 Periodicity: revisited

We now address the apparent solution to the periodicity problem, found some time after Lévy-Leblond, which avoids ambiguity problems by a simple extension of the reasoning that led to the introduction of sine and cosine operators in section 3.1.2. It was proposed to directly render φ periodic by modifying it into a sawtooth function (cf. overview in [9]).

Revisiting the standard Poisson bracket time-derivative introduced in section 3.1.2, we have

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\} + \frac{\partial F}{\partial t} \,. \tag{3.2.20}$$

For a continuous phase $\varphi = -\omega t$, this becomes

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \{\varphi, H\} = -\omega \,, \tag{3.2.21}$$

leading to the quantum-mechanically incorrect Dirac commutator

$$[\hat{\varphi}, \hat{n}] = -i \tag{3.2.22}$$

However, if we restrict the phase to the 2π-interval $\varphi_0 \leq \varphi \leq \varphi_0 + 2\pi$, which is accomplished by introducing a sawtooth function

$$\varphi = -i \ln \left[\mathrm{e}^{i(\varphi - \varphi_0)} \right] \,, \tag{3.2.23}$$

we have

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \{\varphi, H\} = -\omega \left[-i(i - i2\pi\delta(\varphi - \varphi_0 + 2\pi k)) \right] = -\omega \left[1 - 2\pi\delta(\varphi - \varphi_0 + 2\pi k) \right] \,. \tag{3.2.24}$$

In operator language, i.e. after the substitution $\{A, B\} = \frac{1}{i\hbar}[\hat{A}, \hat{B}]$, this leads to a new commutator which circumvents the problems experienced at the end of section 3.1.1 while obviating the need for a description by trigonometric operator functions. The new commutator is given by

$$[\hat{\varphi}, \hat{n}] = -i \left[1 - 2\pi\delta(\hat{\varphi} - \varphi_0 + 2\pi k) \right] \tag{3.2.25}$$

and we see that now, the periodicity problem (3.1.15) in Dirac's initial commutator is attenuated by the presence of an extra term

$$\left\langle n' \left| [\hat{n}, \hat{\varphi}] \right| n \right\rangle = \left\langle n' \left| -i \left[1 - 2\pi\delta(\hat{\varphi} - \varphi_0 + 2\pi k) \right] \right| n \right\rangle \tag{3.2.26}$$

leading to

$$(n' - n) \left\langle n' \left| \hat{\varphi} \right| n \right\rangle = -i\, \delta_{nn'} - \left\langle n' \left| i2\pi\delta(\hat{\varphi} - \varphi_0 + 2\pi k) \right| n \right\rangle , \qquad (3.2.27)$$

which for $n' = n$ no longer automatically implies that $0 = i$.

3.3 Proofs of the impossibility of a hermitian phase operator

To complete this chapter, we delve a little deeper into the intrinsic properties of the harmonic oscillator description and give three (ideas of) proofs to illustrate the difficulties encountered when pursuing hermitian phase operators on the standard harmonic oscillator Hilbert space.

3.3.1 Index theorem

The first proof [24] is rather straightforward, although we will revisit its implications later on when discussing the weak operator topology (see section 4.3). It consists of showing from an index theoretical viewpoint that the harmonic oscillator Hilbert space cannot accommodate an hermitian phase operator if this phase operator is derived from the number operator, especially via polar decomposition as used by Dirac. This is verified by calculating the dimension of the kernel of the number-operator and of its hermitian conjugate, and applying these results to possible phase operators. We thus first want to investigate the quantity

$$\dim \ker \hat{a}^\dagger \hat{a} - \dim \ker \hat{a}\hat{a}^\dagger . \qquad (3.3.1)$$

Proposition 3.3.1. *The index difference between the number operator and its hermitian conjugate in standard oscillator Hilbert space is equal to* 1.

Proof. The index of an operator is calculated by evaluating its trace [24]. Then

$$\dim \ker \hat{a}^\dagger \hat{a} - \dim \ker \hat{a}\hat{a}^\dagger = \mathrm{Tr}(\mathrm{e}^{-\hat{a}^\dagger \hat{a}/M^2}) - \mathrm{Tr}(\mathrm{e}^{-\hat{a}\hat{a}^\dagger/M^2}) \qquad (3.3.2)$$

leads to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{e}^{-n/M^2} - \sum_{n=0}^{\infty} \mathrm{e}^{-(n+1)/M^2} \\
&= 1 + \sum_{n=1}^{\infty} \mathrm{e}^{-n/M^2} - \sum_{n=1}^{\infty} \mathrm{e}^{-n/M^2} = 1 ,
\end{aligned} \qquad (3.3.3)$$

since the trace is independent of the chosen basis and can therefore be evaluated in the number representation as the sum of the exponentiated eigenvalues (n for $\hat{a}^\dagger \hat{a}$ and $n + 1$

for $\hat{a}\hat{a}^\dagger$). The above holds independently of the value of the positive constant M^2. This means that

$$\dim \ker \hat{a}^\dagger \hat{a} - \dim \ker \hat{a}\hat{a}^\dagger = 1 \, . \tag{3.3.4}$$

$\square$

Proposition 3.3.2. *Given the above index difference, the definition of an hermitian phase operator by polar decomposition of the annihilation operator is impossible in the standard oscillator Hilbert space.*

Proof. We evaluate the polar decomposition

$$\hat{a} = \mathrm{e}^{i\hat{\varphi}} \hat{n} \, , \tag{3.3.5}$$

where $\hat{\varphi}$ is supposed to be a hermitian operator. Using the well-known property that the exponent of a hermitian operator is unitary, and that the number operator is hermitian, we can rewrite the above equation as

$$\hat{a} = \hat{U}\hat{H} \, , \tag{3.3.6}$$

where $\hat{U}$ denotes a unitary operator and $\hat{H}$ denotes an hermitian operator. Then we can write

$$\hat{a}^\dagger \hat{a} = \hat{H}\hat{U}^\dagger \hat{U}\hat{H} = \hat{H}^2 \qquad \hat{a}\hat{a}^\dagger = \hat{U}\hat{H}^2\hat{U}^\dagger \, . \tag{3.3.7}$$

The utility of the index theorem now consists in the fact that the index is invariant under unitary transformations (cf. [25]). Thus

$$\dim \ker \hat{A} = \dim \ker \hat{V}\hat{A}\hat{U}^\dagger \, , \tag{3.3.8}$$

which, applied to the present problem, yields

$$\dim \ker \hat{H}^2 - \dim \ker \hat{U}\hat{H}^2\hat{U}^\dagger = 0 \tag{3.3.9}$$

But on the other hand, we know from our calculation in (3.3.3) that

$$\dim \ker \hat{H}^2 - \dim \ker \hat{U}^\dagger \hat{H}^2\hat{U} = \dim \ker \hat{a}^\dagger \hat{a} - \dim \ker \hat{a}\hat{a}^\dagger = 1 \, , \tag{3.3.10}$$

This contradictory result proves that the derivation of an hermitian phase operator by polar decomposition of the annihilation operator is impossible. $\square$

The index relation already foreshadows the developments that will be examined in the next chapter — namely the postulation of a hermitian phase operator on a finite Hilbert space. To see how a limitation to a finite Hilbert space could help, consider that if the oscillator

spectrum were bounded from above, i.e. if it terminated with a state $|s\rangle$, we would have

$$
\begin{aligned}
\sum_{n=0}^{s} \mathrm{e}^{-n/M^2} - \sum_{n=0}^{s} \mathrm{e}^{-(n+1)/M^2} &= \\
&= 1 + \sum_{n=1}^{s} \mathrm{e}^{-n/M^2} - \sum_{n=1}^{s} \mathrm{e}^{-n/M^2} - 1 = 0 \,,
\end{aligned}
\tag{3.3.11}
$$

where it must be kept in mind that the eigenvalue $s+1$ in the sum above is zero, whence the last term of the left-hand side arises after the index shift. Another way of putting this is that the annihilation operator annihilates the vacuum ($\hat{a}\,|0\rangle = 0$) while the creation operator annihilates the state $|s\rangle$ ($\hat{a}^\dagger\,|s\rangle = 0$) because a higher state does not exist. We can thus even explicitly specify the kernels:

$$
\ker \hat{a}^\dagger \hat{a} = \{|0\rangle\} \,, \qquad \ker \hat{a}\hat{a}^\dagger = \{|s\rangle\} \,.
\tag{3.3.12}
$$

The new index relation for the finite-dimensional Hilbert space now easily allows for a hermitian phase operator, since

$$
\dim \ker \hat{H}^2 - \dim \ker \hat{U}^\dagger \hat{H}^2 \hat{U} = \dim \ker \hat{a}^\dagger \hat{a} - \dim \ker \hat{a}\hat{a}^\dagger = 0 \,,
\tag{3.3.13}
$$

resolving the contradiction. Finally, we note that an hermitian phase operator defined on the finite-dimensional Hilbert-space cannot be extended to the one-sided infinite-dimensional Hilbert space by a simple limiting process, since for $s \to \infty$, the kernel $\ker \hat{a}\hat{a}^\dagger = \{|s\rangle\}$ becomes ill-defined and the initial index relation

$$
\dim \ker \hat{a}^\dagger \hat{a} - \dim \ker \hat{a}\hat{a}^\dagger = 1 \,.
\tag{3.3.14}
$$

again applies.

3.3.2 Garrison/Wong operator

We now examine a second proof for the absence of a hermitian phase operator, which is based on abstract notions of conjugate variables in the form of Heisenberg pairs, although it is useful to keep in mind from the start that this proof, strictly speaking, only shows that the photon-number operator cannot have a hermitian operator as its canonic conjugate.

The proof is really a misappropriation of a paper by Garrison and Wong [4] where these two authors have proposed a new derivation of a hermitian phase operator which is conjugate to photon-number (cf. also [7, 26, 27]). We first examine their approach and then show that it inevitably leads to results that are incompatible with the standard hermitian oscillator formulation, thereby proving that within that formulation, an hermitian phase operator conjugate to number cannot exist.

3.3.2.1 Canonical commutation relations

Consider any two canonically conjugate operators $\hat{Q}, \hat{P}$. The canonical commutation relation (henceforth: CCR) between these operators can be expressed in two forms: the Weyl form and the Heisenberg form [4]. The Weyl form relates two one-parameter unitary groups $U(\alpha)$ and $V(\beta)$, which are also known as a Weyl pair, so that

$$U(\alpha)V(\beta) = e^{i\alpha\beta}V(\beta)U(\alpha), \tag{3.3.15}$$

where $U(\alpha)$ and $V(\beta)$ can be represented as complex exponentials of dense, unbounded self-adjoint operators

$$U(\alpha) = e^{i\alpha P}, \qquad V(\beta) = e^{i\beta Q}. \tag{3.3.16}$$

The Weyl form is a strong CCR because it imposes very strict requirements on the spectra of the operators which satisfy it. These strong conditions originate from the fact that the standard position and momentum operators may be used as generators for $U(\alpha)$ and $V(\beta)$, and that by the von Neumann-theorem, any other operators satisfying (3.3.16) are unitarily equivalent to the $U(\alpha)$ and $V(\beta)$ first generated from the position and momentum operators. This means that the possible generators of all $U(\alpha)$ and $V(\beta)$ are unitarily equivalent, and accordingly must possess, like the position and momentum operators, unbounded and continuous spectra. This is obviously impossible for a hypothetical CCR between phase and photon-number, since the photon-number spectrum is discrete and the phase spectrum is periodic.[22]

In contrast, the weaker Heisenberg form is related to the well-known commutator

$$[\hat{Q}, \hat{P}] = i, \tag{3.3.17}$$

and is defined as

$$[\hat{Q}, \hat{P}]\,|\psi\rangle = i\,|\psi\rangle\,, \qquad |\psi\rangle \in \mathscr{C}\,, \tag{3.3.18}$$

where $\hat{P}, \hat{Q}$ are two self-adjoint operators and $\mathscr{C}$ is a dense subspace of the Hilbert space $\mathscr{H}$ which contains all $|\psi\rangle$ for which the commutation relation above is valid. As can be seen by a formal power series expansion of (3.3.16) analogous to the calculations in (3.1.7a) et seq, the Heisenberg form is formally contained in the Weyl form. But the same is not true of the reverse: a Heisenberg form does not imply the existence of an equivalent Weyl form, and therefore the existence of a Weyl form is stronger.

Conversely, however, the Heisenberg form is much more flexible and is the only form applicable when the spectra of the operators in question are discrete, bounded and/or periodic [4, 7, 26], as long as special care is taken in such cases in the definition of $\mathscr{C}$. For

---

[22]This was also realized by Lévy-Leblond [23] and proven rigorously by Rocca and Sirugue [28].

example, if $\hat{P}$ is discrete and $\hat{Q}$ is periodic in the sense that

$$\mathscr{D}(\hat{P}) = \left\{ |\psi\rangle \,\middle|\, |\psi\rangle = \sum a_n \,|p_n\rangle \,, |p_n\rangle \in \mathscr{M} \subset \mathscr{H} \right\} \tag{3.3.19a}$$

$$\mathscr{D}(\hat{Q}) = \left\{ |\psi\rangle \,\middle|\, |\psi(x)\rangle = |\psi(x + L)\rangle \right\} \,, \tag{3.3.19b}$$

where $\mathscr{M}$ is a discrete subspace of $\mathscr{H}$, then for all $|\psi\rangle \in \mathscr{C}$, $|\psi\rangle$ as well as the state after interaction with one operator must lie in the domain of the other operator

$$|\psi\rangle \,, \hat{Q} \,|\psi\rangle \in \mathscr{D}(\hat{P}) \tag{3.3.20a}$$

$$|\psi\rangle \,, \hat{P} \,|\psi\rangle \in \mathscr{D}(\hat{Q}) \,. \tag{3.3.20b}$$

For the resulting subspace $\mathscr{C}$, the Heisenberg form then leads to the well known uncertainty relation

$$\Delta\hat{Q} \cdot \Delta\hat{P} \geq \frac{1}{2} \tag{3.3.21}$$

for all $|\psi\rangle \in \mathscr{C}$.

This is a mathematical relation which always holds (provided $\hat{Q}$ and $\hat{P}$ are dimensionless). The physical suitability of such a relation, however, may differ wildly depending on whether all physically relevant states are part of the subspace $\mathscr{C}$ or not. That is because for states outside of $\mathscr{C}$, the Heisenberg form becomes invalid, and therefore, no canonic conjugation relation exists.

3.3.2.2 Proof of the impossibility of a physically useful hermitian phase operator

Our proof now goes as follows. First, define an hermitian phase operator and determine the subspace $\mathscr{C}$ in which a CCR with the number operator applies. Then, show that the number states $|n\rangle$ are not contained in $\mathscr{C}$ and that an approximation of the number states $|n\rangle$ by states that are contained in $\mathscr{C}$ produces absurd results. From this, conclude that a hermitian phase operator which satisfies a CCR is incompatible with the standard harmonic oscillator formulation. The first part is formulated in much more detail in [4, 7, 8, 26], the second part in [8, 27].

First step

Proposition 3.3.3. *We define an hermitian operator $\hat{\varphi}$ (Garrison/Wong operator) as*

$$\langle n' \,|\hat{\varphi}|\, n\rangle = \frac{(-1)^{n'-n}}{i(n'-n)}(1 - \delta_{n',n}) \,. \tag{3.3.22}$$

This operator fulfills the Heisenberg form

$$[\hat{n}, \hat{\varphi}] \left| \psi \right\rangle = i \left| \psi \right\rangle \tag{3.3.23}$$

for the dense subset

$$\mathscr{C} = \left\{ \left| \psi \right\rangle \ \middle| \ \left| \psi \right\rangle = \sum_{n=0}^{\infty} a_n \left| n \right\rangle \ \text{and} \ \sum_{n=0}^{\infty} (-1)^n a_n = 0 \right\}, \tag{3.3.24}$$

which is at the same time the largest such subspace that can be constructed for any hermitian phase operator.

No proof of this statement will be given here, since its demonstrative value is limited; let it just be said that the structure of $\mathscr{C}$ is the result of a recurrence to Hardy space $\mathcal{H}^2$ representations of operators in the form of holomorphic functions on the unit disk, which provide an infinite series condition for the coefficients of the number state expansion of states that adhere to $\mathscr{C}$. Concerning the last sentence of the statement, this superficially follows from the fact that $\hat{\varphi}$ is essentially the analogue of the angle operator which is used in the derivation of the (well-behaved) Heisenberg pair of angle and angular momentum $\hat{\Theta}$ and $\hat{L}_z$, with minimal restrictions introduced in order to create compatibility with the bounded spectrum of $\hat{n}$, thereby making sure that $\hat{\varphi}$ is sufficiently widely defined. A more abstract and thorough reasoning notes that the Garrison-Wong operator represents the most general Toeplitz operator for phase (for this and other reasons, cf. [9], p. 146 with further references). All this is shown more thoroughly in the already cited references [4, 7, 26], which the reader may refer to at his leisure.

Here, the most important thing to remember is just that for any number state expansion

$$\left| \psi \right\rangle = \sum_{n=0}^{\infty} a_n \left| n \right\rangle, \tag{3.3.25}$$

the coefficients f_n must obey

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0. \tag{3.3.26}$$

Second step If any useful hermitian operator is to exist on standard Hilbert space as a canonic conjugate of $\hat{n}$, the subspace $\mathscr{C}$ should be large enough to describe or at least approximate all physical properties of harmonic oscillator states. However, it turns out that this is not the case for even the simplest states on the oscillator Hilbert space, the number states $\left| n \right\rangle$.

Proposition 3.3.4. *The number states $\left| n \right\rangle$ are not part of the subspace $\mathscr{C}$.*

Proof. This can be seen when using the number state expansion of the number states:

$$|n'\rangle = \sum_{n=0}^{\infty} a_n |n\rangle \,. \tag{3.3.27}$$

Here, it is not possible to simply set $a_n = \delta_{n',n}$, since this would not satisfy the requirement in (3.3.24) that all states of the subspace $\mathscr{C}$ fulfill

$$\sum_{n=0}^{\infty} (-1)^n a_n = 0 \,. \tag{3.3.28}$$

$\square$

The fact that the number states are not represented in $\mathscr{C}$ is, by itself, not severely troubling, since in the corresponding description for angular momentum (which is well-behaved), the eigenstates of angular momentum are also not part of the subspace and accordingly also violate the CCR. However, we *do* expect approximations to the number states in $\mathscr{C}$ to at least converge to the number states in $\mathscr{H}$. More importantly, we expect such convergence with respect to all important characteristics, of which the low variance in photon-number (zero for exact number states) is a key element. However, again, this turns out not to be the case, as was proven by [27]:

Proposition 3.3.5. *There exists a parametrized approximation of the number states that is composed of states $|\psi\rangle \in \mathscr{C}$ and approximates the number states arbitrarily well. However, the properties of such states are in direct contradiction with the properties of the number states they intend to describe.*

Proof. Consider the approximation $(0 \leq \varepsilon \leq 1)$ [27]

$$|n, \varepsilon\rangle = \sqrt{\frac{1+\varepsilon}{2}} \left[|n\rangle - \frac{1-\varepsilon}{\varepsilon} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k |n+k\rangle \right] \,, \tag{3.3.29}$$

which satisfies

$$\langle n | n, \varepsilon\rangle = \sqrt{\frac{1+\varepsilon}{2}} \to 1 \,, \quad \varepsilon \to 1$$

$$\langle n, \varepsilon | n, \varepsilon\rangle = \frac{1+\varepsilon}{2} \left[1 + \left(\frac{1-\varepsilon}{\varepsilon}\right)^2 \sum_{k=1}^{\infty} \varepsilon^{2k} \right] = \tag{3.3.30}$$

$$= \frac{1+\varepsilon}{2} \left[1 + \left(\frac{1-\varepsilon}{\varepsilon}\right)^2 \frac{\varepsilon^2}{1-\varepsilon^2} \right] = \frac{1+\varepsilon}{2} \left[\frac{2-2\varepsilon}{1-\varepsilon^2} \right] = 1$$

and is part of the subspace $\mathscr{C}$ because it satisfies $\sum_{m=0}^{\infty} (-1)^m a_m = 0$:

$$\sum_{m=0}^{\infty} (-1)^m a_m = \sqrt{\frac{1+\varepsilon}{2}} \left[(-1)^n - \frac{1-\varepsilon}{\varepsilon} \sum_{m=n+1}^{\infty} (-1)^m (-1)^{m-n} \varepsilon^{m-n} \right] =$$

$$= \sqrt{\frac{1+\varepsilon}{2}} \left[(-1)^n - (-1)^n \frac{1-\varepsilon}{\varepsilon} \sum_{k=1} \varepsilon^k \right] = \tag{3.3.31}$$

$$= \sqrt{\frac{1+\varepsilon}{2}} \left[(-1)^n - (-1)^n \frac{1-\varepsilon}{\varepsilon} \frac{\varepsilon}{1-\varepsilon} \right] = 0$$

We now calculate expectation value and variance of the number operator in these states, which will have to approximate $\langle \hat{n} \rangle = n$ and $\Delta \hat{n} = 0$. We start with the expectation value:

$$\langle n, \varepsilon \left| \hat{n} \right| n, \varepsilon \rangle = \langle n, \varepsilon | \cdot \left\{ \sqrt{\frac{1+\varepsilon}{2}} \left[\hat{n} \left| n \right\rangle - \frac{1-\varepsilon}{\varepsilon} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \, \hat{n} \left| n+k \right\rangle \right] \right\} =$$

$$= \langle n, \varepsilon | \cdot \left\{ \sqrt{\frac{1+\varepsilon}{2}} \left[n \left| n \right\rangle - \frac{1-\varepsilon}{\varepsilon} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \, (n+k) \left| n+k \right\rangle \right] \right\} =$$

$$= n \langle n, \varepsilon | n, \varepsilon \rangle + \frac{1+\varepsilon}{2} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2 \sum_{k=1}^{\infty} k \, \varepsilon^{2k} = \tag{3.3.32}$$

$$= n + \frac{1+\varepsilon}{2} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2 \frac{\varepsilon^2}{(1-\varepsilon^2)^2} =$$

$$= n + \frac{1}{2(1+\varepsilon)} \rightarrow n + \frac{1}{4} \,, \quad \varepsilon \rightarrow 1 \,,$$

where in the third equality, we have separated all terms with coefficient n, and where in the fourth equality we have used the formula for the infinite arithmetic-geometric series

$$\sum_{k=1}^{\infty} k q^k = \frac{q}{(1-q)^2} \,. \tag{3.3.33}$$

While the approximation is not perfect, this could be overcome. We now turn to the calculation of the variance

$$\left\langle n, \varepsilon \left| (\hat{n} - \langle \hat{n} \rangle)^2 \right| n, \varepsilon \right\rangle =$$

$$= \langle n, \varepsilon | \cdot \left\{ \sqrt{\frac{1+\varepsilon}{2}} \left[\hat{n}\hat{n} \left| n \right\rangle - \frac{1-\varepsilon}{\varepsilon} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \, \hat{n}\hat{n} \left| n+k \right\rangle \right] \right\} - \langle n, \varepsilon \left| \hat{n} \right| n, \varepsilon \rangle^2 =$$

$$= \langle n, \varepsilon | \cdot \left\{ \sqrt{\frac{1+\varepsilon}{2}} \left[n^2 \left| n \right\rangle - \frac{1-\varepsilon}{\varepsilon} \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \, (n+k)^2 \left| n+k \right\rangle \right] \right\} - \langle n, \varepsilon \left| \hat{n} \right| n, \varepsilon \rangle^2 =$$

$$= n^2 + 2n \left\{ \frac{1}{2(1+\varepsilon)} \right\} + \left\{ \frac{1+\varepsilon}{2} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2 \sum_{k=1}^{\infty} k^2 \, \varepsilon^{2k} \right\} - \left\{ n + \frac{1}{2(1+\varepsilon)} \right\}^2 =$$

$$= - \left\{ \frac{1}{2(1+\varepsilon)} \right\}^2 + \left\{ \frac{1+\varepsilon}{2} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2 \sum_{k=1}^{\infty} k^2 \, \varepsilon^{2k} \right\} = \tag{3.3.34}$$

$$= - \left\{ \frac{1}{2(1+\varepsilon)} \right\}^2 + \left\{ \frac{1+\varepsilon}{2} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2 \frac{\varepsilon^4 + \varepsilon^2}{(1-\varepsilon^2)^3} \right\}$$

$$= -\left\{\frac{1}{2(1+\varepsilon)}\right\}^2 + \left\{\frac{\varepsilon^2+1}{2(1+\varepsilon)^2(1-\varepsilon)}\right\} \to \infty\,, \quad \varepsilon \to 1\,,$$

where in the third equality, we have separated everything we already know from the calculation of the expectation value, leaving the evaluation of a quadratic arithmetic-geometric series according to the formula

$$\sum_{k=1}^{\infty} k^2 q^k = \frac{q^2+q}{(1-q)^3}\,. \tag{3.3.35}$$

As we see, the second moment diverges as $\varepsilon \to 1$ because it has an unremovable pole at $\varepsilon = 1$. This leads to the absurd result that the closer the $\mathscr{C}$-space approximation of the number states approximates the real number states, the larger the variance in $\hat{n}$ becomes, which is in direct contradiction with the expected physical behavior, where $\Delta\hat{n} = 0$. Therefore, a Garrison/Wong type operator (i.e. a hermitian operator defined so that a CCR with the number operator is fulfilled) leads to unphysical results. $\qquad\square$

3.3.3 Orthogonality considerations

The last proof is at the same time the simplest and perhaps the most general (similar [9], p. 146). Its ansatz is to show the impossibility of any hermitian operator that describes certain basic characteristic elements of phase. Note that this proof applies irrespective of whether the phase operator is supposed to fulfill a CCR or whether it should follow from a polar decomposition of the operator $\hat{a}$ or not; it simply proves the impossibility of representing phase properties with an hermitian operator.

The phase properties we demand in addition to the standard hermitian operator requirements of orthogonality etc are
(i) A unitary phase shift operator shifts the phase
(ii) 2π-Periodicity
(iii) A continuous range of eigenvalues in the interval $\varphi_0 \leq \varphi \leq \varphi_0 + 2\pi$

The first of these is the one which will cause trouble. We may derive the unitary phase shift operator from the time development operator. The latter is given by

$$e^{i\frac{\hat{H}}{\hbar}t} = e^{i\hat{n}\omega t}\,. \tag{3.3.36}$$

Setting $\omega t = \lambda$, we obtain the unitary phase shift operator $e^{i\hat{n}\lambda}$ and require

$$e^{i\hat{n}\lambda}\ket{\varphi} = \ket{\varphi + \lambda}\,. \tag{3.3.37}$$

We now look at a continuous superposition of phase eigenstates around the center of the

interval $\varphi_0 - \pi \leq \varphi \leq \varphi_0 + \pi$:

$$
\begin{aligned}
|\psi\rangle &= \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \, |\varphi_0 + \mu\rangle = \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \sum_{n=0}^{\infty} |n\rangle\langle n|\, \varphi_0 + \mu\rangle \\
&= \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \sum_{n=0}^{\infty} |n\rangle\big\langle n \big| \mathrm{e}^{i\hat{n}\mu} \big| \varphi_0 \big\rangle = \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \sum_{n=0}^{\infty} \mathrm{e}^{in\mu} \langle n|\, \varphi_0\rangle |n\rangle = \qquad (3.3.38) \\
&= 2 \sum_{n=0}^{\infty} \frac{\sin(n\pi/3)}{n} c_n \, |n\rangle \; .
\end{aligned}
$$

If the states $|\varphi\rangle$ really are supposed to be eigenstates of a hermitian operator, this means that we must have

$$
\big\langle \psi \big| \mathrm{e}^{i\hat{n}\lambda} \big| \varphi_0 \big\rangle = \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \, \langle \varphi_0 + \mu|\, \varphi_0 + \lambda\rangle = 0 \, , \quad \lambda \in \left(-\pi, -\frac{\pi}{3} \right) \cup \left(\frac{\pi}{3}, \pi \right) \, . \qquad (3.3.39)
$$

because of the orthogonality condition, i.e. while $|\psi\rangle$ only has phase contributions inside the interval $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, $\mathrm{e}^{i\hat{n}\lambda} |\varphi_0\rangle$ only has contributions outside this interval when λ is an element of the specified interval, so that the total overlap is zero.

But expanding the above orthogonality condition in terms of number states with the help of calculation (3.3.38) gives

$$
\begin{aligned}
\big\langle \psi \big| \mathrm{e}^{i\hat{n}\lambda} \big| \varphi_0 \big\rangle &= \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \, \langle \varphi_0 + \mu|\, \varphi_0 + \lambda\rangle = \\
&= \int\limits_{-\pi/3}^{\pi/3} \mathrm{d}\mu \sum_{n=0}^{\infty} \langle \varphi_0 + \mu|\, n\rangle\langle n|\, \varphi_0 + \lambda\rangle = \\
&= \sum_{n=0}^{\infty} \frac{\sin(n\pi/3)}{n} |\langle n|\, \varphi_0\rangle|^2 \mathrm{e}^{in\lambda} \qquad (3.3.40) \\
&= \sum_{n=0}^{\infty} \frac{\sin(n\pi/3)}{n} |c_n|^2 \mathrm{e}^{in\lambda} = 0 \, , \quad \lambda \in \left(-\pi, -\frac{\pi}{3} \right) \cup \left(\frac{\pi}{3}, \pi \right) \, .
\end{aligned}
$$

But basic analysis tells us that this orthogonality condition cannot be fulfilled for every λ from a set of nonzero Lebesgue measure (cf. [9], page 147), the construction of an hermitian operator with an orthogonal set of eigenstates is incompatible with the basic requirements expected from any phase observable.

3.4 Summary

We have come quite a long way in this chapter. We have seen that the construction of a phase operator is no simple task, and that Dirac's overly simplistic approach in this matter was doomed to fail. We have seen that there are two main problems facing any potential theory of quantum phase, those two being, first, the periodicity of the phase variable, and second, the non-unitarity of the exponential operator arising from a polar decomposition of the annihilation operator. Both these problems can be solved or circumvented in different ways.

Susskind and Glogower, for their part, came up with the ingenious idea of abandoning the search for an hermitian phase operator altogether, reasoning that trigonometric functions offered a better alternative since, first, they intrinsically incorporate a solution to the periodicity problem, being themselves periodic, and second, they are hermitian even though the exponential operator by way of which they are defined is not.

We have also seen that the operators $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ allow the first mathematically correct description of phase, since they are bounded and their eigenstates orthogonal and complete, so that they are in every sense mathematically well-behaved. However, we have also seen that $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ do not commute, that they therefore do not allow a precise measurement of the full phase information, and that they give wrong values for higher moments if there is significant overlap with the vacuum state.

This has caused us to turn away from the SG-formalism and to investigate further developments, of which the most important is the Lévy-Leblond formalism, which in a slightly different form will recur in the next chapter. We closed our exposition on the historical development of phase operator theory with several proofs that compound the overall conclusion of this chapter: that it is impossible to define a well-behaved hermitian operator on the standard oscillator Hilbert space.

Chapter 4

The Pegg/Barnett phase operator

We now turn to an interpretation of quantum phase that has stirred up the quantum optics community at the end of the 1980's and caused a veritable revival of the topic, which had already appeared to die down:[23] the Pegg/Barnett phase operator [15, 29–32]. This formalism was first introduced in its present form in 1988 and has led to a marked polarization of the quantum phase debate, leading to the formation of a pro- and an anti-PB camp. On the one hand, Pegg and Barnett are convinced that their operator has solved the quantum phase problem — a conviction that they and their collaborators have not been shy to emphasize in a plethora of papers and reviews (cf. e.g. [18, 33–36]) - while many others are not so impressed by the PB formalism or have even fiercely criticized it (cf. [27, 37, 38]). As always, the truth will be situated somewhere along the middle, so we will lend an ear to both camps, reviewing first the advantages and capabilities of the PB-formalism and then going into some of the criticism that has been voiced (starting in section 4.3).

To give a brief preview of this debate: The main advantages that Pegg and Barnett claim of their formalism is that a solution is found for the non-unitarity problem that plagued the Susskind-Glogower formalism and caused the appearance of unwanted vacuum projectors in the commutators of the exponential, sine and cosine operators. As a result, a full phase measurement is possible and operator functions of the phase can be defined which mirror the properties of their classical equivalents by construction and not just in the large-n limit. However, the Pegg/Barnett formalism comes with the big catch that it is only indirectly connected to the standard oscillator Hilbert space, which questions the validity of the results obtained even if they may at first glance appear to be plausible.

Whatever the case may be, one thing is for sure: the Pegg/Barnett formalism is an important contribution to the quantum phase problem, and regardless of its ultimate merit,

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[23]Compare the surely less than 100 papers covering the subject prior to 1988 with the about 400 contributions made only between 1988 and 1996.

its impact on the discussion of phase operators and quantum phase in general has been so intense that no review of quantum phase theory would be complete without it.

4.1 Concept and preliminary efforts

The basic principle of the Pegg/Barnett formalism is very simple to describe: Instead of trying to construct an hermitian operator in the standard oscillator Hilbert space $\mathscr{H}_\infty$, where this has sufficiently often been shown to be impossible (cf. section 3.3, [14, 24] etc), Pegg and Barnett construct their phase operator on a limited subspace $\mathscr{H}_{s+1} \in \mathscr{H}_\infty$ which terminates with the state $|s\rangle$ and, as we have already seen in section 3.3.1, has an index of 0. This means that the subspace $\mathscr{H}_{s+1}$ allows the construction of an hermitian phase operator in a polar decomposition of $\hat{a}_s$. All relevant physical properties such as expectation values and higher moments are then calculated in the $\mathscr{H}_{s+1}$ subspace, and finally, the limit $s \to \infty$ is taken to port the results to the standard Hilbert space $\mathscr{H}_\infty$.

It is important to note here that none of the concepts introduced by Pegg and Barnett are essentially new. It was already Susskind and Glogower who remarked that tinkering with the Hilbert space might allow one to avoid the non-unitarity problem,[24] and the idea of introducing supplementary terms in a finite Hilbert space in order to make the relevant operators cyclical was introduced as early as 1961 by Louisell (cf. [10]).

Even the phase operators themselves are not unique: When the limit $s \to \infty$ is taken prematurely in the hope that the series of finite Hilbert space operators $\{\hat{\varphi}_s\}$ might converge to a hermitian phase operator $\hat{\varphi}$ in $\mathscr{H}_\infty$, one recovers the operators of Popov and Yarunin [38, 39], which are also equivalent to the Garrison/Wong operator [4, 7, 26] as shown in [40].

But the Susskind/Glogower just like the Popov/Yarunin and Garrison/Wong description suffer from the problem that they do not pass the acid test, that is, they attribute anisotropic phase properties to the number states. Consequently, the main novelty introduced by Pegg and Barnett is not the operator, but the way in which physical properties are calculated. As will be seen, and PB never tire of repeating this, the defining element of the PB formalism is that expectation values are calculated *first* using the finite-dimensional operators, and *only then* is the limit $s \to \infty$ taken. This, at least ostensibly, does away with many of the difficulties encountered by earlier approaches, leading, among other things, to correct second moments for the phase of number states that pass the acid test.

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[24]Note also that Pegg and Barnetts early efforts [29] still focused on an infinite-dimensional Hilbert space, the same space already investigated by Susskind/Glogower (cf. section 3.1.3.2).

4.2 Formalism

As already announced, we now turn first to the Pegg/Barnett formalism as originally introduced by these two authors and amended in collaborations with Vaccaro, including the advantages that it encompasses, before turning in the later sections to the critique this formalism has received in the years following its publication.

4.2.1 Construction of phase eigenstates

The first step towards defining the PB-phase operator is to define the Hilbert space that it operates in. As was already indicated, we choose an $s + 1$-dimensional subspace $\mathscr{H}_{s+1} \subset \mathscr{H}_\infty$ which is spanned by the states $\{|0\rangle, |1\rangle, \cdots, |s\rangle\}$ (Pegg and Barnett also call this space Ψ). Next, we define appropriate eigenstates on this space. Examine

$$|\varphi\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \mathrm{e}^{in\varphi} |n\rangle \qquad (4.2.1)$$

These states are states of defined phase [41] and represent an uncountable infinity of different phase states depending on the value of $\varphi \in (\varphi_0, \varphi_0 + 2\pi)$. They are overcomplete and, correspondingly, not orthogonal. However, an orthogonal basis of $s + 1$ phase states which spans $\mathscr{H}_{s+1}$ can be obtained by specifying a reference angle φ_0 and partitioning the interval $(\varphi_0, \varphi_0 + 2\pi)$ into $s + 1$ angles φ_m where

$$\varphi_m = \varphi_0 + \frac{2\pi m}{s+1} \,. \qquad (4.2.2)$$

We can then use these φ_m to restrict the definition of the phase states to

$$|\varphi_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \mathrm{e}^{in\varphi_m} |n\rangle \qquad (4.2.3)$$

These states are orthogonal and complete:

$$\begin{aligned}
\langle \varphi_m | \varphi_{m'} \rangle &= \frac{1}{s+1} \sum_{n=0}^{s} \mathrm{e}^{in(\varphi_{m'} - \varphi_m)} = \frac{1}{s+1} \sum_{n=0}^{s} \mathrm{e}^{in2\pi(m'-m)/(s+1)} = \\
&= \frac{1}{s+1} \left[\frac{1 - \mathrm{e}^{i2\pi(m'-m)}}{1 - \mathrm{e}^{i2\pi(m'-m)/(s+1)}} \right] = \delta_{mm'} \,,
\end{aligned} \qquad (4.2.4)$$

where we have used a standard geometric series expansion and the fact that for all $m \neq m' + k(s+1)$, the numerator vanishes but the denominator does not, proving orthogonality

modulo $s + 1$. The completeness follows as

$$
\sum_{m=0}^{s} |\varphi_m\rangle\langle\varphi_m| = \frac{1}{1+s} \sum_{m,n',n=0}^{s} e^{i(n'-n)\varphi_m} |n'\rangle\langle n| =
$$

$$
= \frac{1}{1+s} \sum_{n',n=0}^{s} |n'\rangle\langle n| \, e^{i(n'-n)\varphi_0} \sum_{m=0}^{s} e^{i(n'-n)2\pi m/(s+1)} =
$$

$$
= \frac{1}{1+s} \sum_{n',n=0}^{s} |n'\rangle\langle n| \, e^{i(n'-n)\varphi_0} \left[\frac{1 - e^{i(n'-n)2\pi}}{1 - e^{i(n'-n)2\pi/(s+1)}} \right] =
$$

$$
= \sum_{n',n=0}^{s} |n'\rangle\langle n| \, e^{i(n'-n)\varphi_0} \delta_{n',n} = \sum_{n=0}^{s} |n\rangle\langle n| = \mathbb{1}
$$

(4.2.5)

The orthogonality and completeness of the states $|\varphi\rangle_m$ are prerequisites for the resulting operator to be hermitian. However, we also want the operator to describe phase, and this is reflected in several interesting properties which justify calling the phase states phase states.

First, a unitary shift operator can be introduced which shifts the angle of a phase state by an amount λ

$$
e^{i\hat{n}\lambda} |\varphi\rangle = \sum_{m=0}^{s} e^{in(\varphi+\lambda)} |n\rangle = |\varphi + \lambda\rangle \ .
$$

(4.2.6)

Note that according to the general postulate whereby the unitary exponential of one conjugate operator modifies the eigenvalues of the other, we expect that the phase operator constructed from the phase states will be conjugate to $\hat{n}$. Note also that when we choose $\lambda = -\omega t$, the unitary shift operator becomes the time-evolution operator (propagator)

$$
e^{-i\frac{\hat{H}}{\hbar}t} |\varphi\rangle = e^{-i\hat{n}\omega t} |\varphi\rangle = |\varphi - \omega t\rangle \ .
$$

(4.2.7)

Second, ladder operators can be found which allow jumping from $|\varphi_m\rangle$ to $|\varphi_{m\pm 1}\rangle$

$$
\hat{\varphi}_\pm |\varphi_m\rangle = e^{i\hat{n}(\pm 2\pi/(s+1)} |\varphi_m\rangle = \sum_{m=0}^{s} e^{in(\varphi_0 + 2\pi(m\pm 1)/(s+1))} |n\rangle = |\varphi_{m\pm 1}\rangle \ ,
$$

(4.2.8)

so that the $|\varphi_m\rangle$ may also be defined as

$$
|\varphi_m\rangle = (\hat{\varphi}_\pm)^m |\varphi_0\rangle = e^{i\hat{n}(2\pi m/(s+1))} |\varphi_0\rangle
$$

(4.2.9)

These properties convince us that the phase states are well capable of reflecting phase properties.

4.2.2 Construction of the hermitian phase operator

Using the now defined complete orthonormal basis $\{|\varphi_0\rangle, |\varphi_1\rangle, \cdots, |\varphi_m\rangle\}$, we define the *Pegg/Barnett hermitian phase operator* as

$$\hat{\varphi}_{\varphi_0} = \sum_{m=0}^{s} \varphi_m |\varphi_m\rangle\langle\varphi_m| \,, \tag{4.2.10}$$

where the subscript φ_0 stands for the angle at which the partition of the unit circle into (s+1) intervals begins. The phase operator may be expressed in the number state basis as

$$\hat{\varphi}_{\varphi_0} = \sum_{m=0}^{s} \varphi_m |\varphi_m\rangle\langle\varphi_m| = \varphi_0 + \frac{2\pi}{s+1} \sum_{m=0}^{s} m |\varphi_m\rangle\langle\varphi_m| = \tag{4.2.11a}$$

$$= \varphi_0 + \frac{2\pi}{(s+1)^2} \sum_{n',n=0}^{s} |n'\rangle\langle n| \, e^{i(n'-n)\varphi_0} \sum_{m=0}^{s} m \, e^{i(n'-n)2\pi m/(s+1)} = \tag{4.2.11b}$$

$$= \varphi_0 + \frac{s\pi}{s+1} + \frac{2\pi}{(s+1)^2} \sum_{n'\neq n}^{s} |n'\rangle\langle n| \, e^{i(n'-n)\varphi_0} \tag{4.2.11c}$$

$$\times \left[-\frac{1}{1 - e^{i(n'-n)2\pi/(s+1)}} - \frac{s}{1 - e^{i(n'-n)2\pi/(s+1)}} \right] \tag{4.2.11d}$$

$$= \varphi_0 + \frac{s\pi}{s+1} + \frac{2\pi}{s+1} \sum_{n'\neq n}^{s} |n'\rangle\langle n| \, \frac{e^{i(n'-n)\varphi_0}}{e^{i(n'-n)2\pi/(s+1)} - 1} \,, \tag{4.2.11e}$$

where in the second-to-last equality we have used the standard summation formula for arithmetic series

$$\sum_{m=0}^{s} m = \frac{s(s+1)}{2} \tag{4.2.12a}$$

to obtain the term $\frac{s\pi}{s+1}$ for the sum of the diagonal elements and the summation formula for the arithmetic-geometric series

$$\sum_{m=0}^{s} m q^m = \frac{q}{(1-q)^2}(1 - q^s) - \frac{sq^{s+1}}{1-q} \tag{4.2.12b}$$

in combination with the fact that $q^{s+1} = \left(e^{i(n'-n)2\pi m/(s+1)} \right)^{s+1} = 1$ to obtain the remaining terms.

It is interesting to note that in $\mathscr{H}_{s+1}$, $\hat{n}$ and $\hat{\varphi}_{\varphi_0}$ are amazingly similar because of the discreteness, completeness and orthogonality of the eigenstates $|\varphi_m\rangle$. For example, we

may derive the following phase state expansion of the number states

$$|n\rangle = \sum_{m=0}^{s} \langle \varphi_m | n \rangle \cdot |\varphi_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} e^{-in\varphi_m} |\varphi_m\rangle \ , \qquad (4.2.13)$$

which allows us to find an expression for the number operator $\hat{n}$ in the phase basis

$$\hat{n} = \sum_{n=0}^{s} n \, |n\rangle\langle n| = \frac{1}{1+s} \sum_{m',m=0}^{s} |\varphi_{m'}\rangle\langle\varphi_m| \sum_{n=0}^{s} n e^{-in(\varphi'_m - \varphi_m)} = \qquad (4.2.14a)$$

$$= \frac{s}{2} + \sum_{m' \neq m}^{s} |\varphi_{m'}\rangle\langle\varphi_m| \frac{1}{e^{-i(m'-m)2\pi/(s+1)} - 1} \qquad (4.2.14b)$$

using the same steps as above. Note the perfect symmetry between the number operator in phase states and the phase operator in number states if we choose $\varphi_0 = 0$ and multiply $\hat{n}$ with a factor $2\pi/(s+1)$:

$$\hat{\varphi}_{\varphi_0} = \frac{s\pi}{s+1} + \frac{2\pi}{s+1} \sum_{n' \neq n}^{s} |n'\rangle\langle n| \frac{1}{e^{i(n'-n)2\pi/(s+1)} - 1} \qquad (4.2.15a)$$

$$\frac{2\pi}{s+1}\hat{n} = \frac{s\pi}{s+1} + \frac{2\pi}{s+1} \sum_{m' \neq m}^{s} |\varphi_{m'}\rangle\langle\varphi_m| \frac{1}{e^{-i(m'-m)2\pi/(s+1)} - 1} \qquad (4.2.15b)$$

As startling as this similarity is, it can, of course, quickly be explained by the particular construction of the PB-formalism. In $\mathscr{H}_{s+1}$, both $|n\rangle$ and $|\varphi\rangle$ may take on $s + 1$ values. For $|n\rangle$, these span the interval $(0, s)$, and for $|\varphi\rangle$ the interval $(\varphi_0, \varphi_0 + 2\pi s/(s+1))$. Thus, setting $\varphi_0 = 0$ and introducing a factor of $2\pi s/(s + 1)$ to the spectral interval of the number states, the operators $\hat{n}$ and $\hat{\varphi}$ become mathematically identical because they are now defined on the same interval and possess an equal amount of eigenstates.

Now that we have defined a phase operator, we may go on to calculate the commutator of $\hat{n}$ and $\hat{\varphi}_{\varphi_0}$ in the number state basis (an equivalent expression for phase states can analogously be derived):

$$[\hat{n}, \hat{\varphi}_{\varphi_0}] = \frac{2\pi}{s+1} \sum_{n' \neq n}^{s} |n'\rangle\langle n| \frac{(n'-n)e^{i(n'-n)\varphi_0}}{e^{i(n'-n)2\pi/(s+1)} - 1} \qquad (4.2.16)$$

We see that the commutator has a trace of zero

$$\langle n \, | [\hat{n}, \hat{\varphi}_{\varphi_0}] | \, n \rangle = 0 \qquad (4.2.17)$$

while the diagonal elements are all nonzero and also differ from each other. This is surprising since from the Poisson bracket correspondence, we instead expected

$$[\hat{n}, \hat{\varphi}_{\varphi_0}] = i \, , \qquad (4.2.18)$$

i.e. where all off-diagonal elements are zero, but the trace does not vanish. We will make more sense of this in sections 4.2.4 and 4.2.6.

4.2.3 Completing the formalism

Having defined the phase operator and examined some of its properties, there are several important matters that now need to be addressed. Foremost, based on our bad experiences with limited subspaces in section 3.3.2, we are interested in the consistency of the representations of basic operators and properties of the quantum harmonic oscillator in the $\mathscr{H}_{s+1}$-subspace.

We first examine the action of the unitary exponential phase operator on the phase eigenvalues and eigenstates

$$\mathrm{e}^{i\hat{\varphi}}\ket{\varphi} = \sum_{k=0}^{\infty} \frac{i^k (\hat{\varphi}_{\varphi_0})^k}{k!} \ket{\varphi} = \sum_{k=0}^{\infty} \frac{i^k \varphi^k}{k!} \ket{\varphi} = \mathrm{e}^{i\varphi}\ket{\varphi} \ , \tag{4.2.19}$$

where the power series is allowed because $\hat{\varphi}_{\varphi_0}$ is a bounded operator. We then infer the effect of $\mathrm{e}^{i\hat{\varphi}}$ on the number states by invoking the phase state expansion (4.2.13):

$$\mathrm{e}^{i\hat{\varphi}}\ket{n} = \frac{1}{\sqrt{s+1}}\mathrm{e}^{i\hat{\varphi}} \sum_{m=0}^{s} \mathrm{e}^{-in\varphi_m}\ket{\varphi_m} = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} \mathrm{e}^{-i(n-1)\varphi_m}\ket{\varphi_m}$$
$$= \ket{n-1} \ , \quad \forall n \geq 1 \ . \tag{4.2.20}$$

This is a very satisfying result since it reproduces the action of the exponentials already defined by Susskind and Glogower. However, we are also interested in finding out how $\mathrm{e}^{i\hat{\varphi}}$ affects the vacuum state:

$$\mathrm{e}^{i\hat{\varphi}}\ket{0} = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} \mathrm{e}^{i\varphi_m}\ket{\varphi_m} = \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} \mathrm{e}^{i[\varphi_0 + 2\pi m/(s+1)]}\ket{\varphi_m} =$$
$$= \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} \mathrm{e}^{i[\varphi_0 + 2\pi m/(s+1) - 2\pi m(s+1)/(s+1)]}\ket{\varphi_m} =$$
$$= \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} \mathrm{e}^{i[\varphi_0 + s\varphi_0 - s\varphi_0 - 2\pi ms/(s+1)]}\ket{\varphi_m} = \tag{4.2.21}$$
$$= \frac{1}{\sqrt{s+1}}\mathrm{e}^{i(s+1)\varphi_0} \sum_{m=0}^{s} \mathrm{e}^{-is\varphi_m}\ket{\varphi_m} =$$
$$= \mathrm{e}^{i(s+1)\varphi_0}\ket{s}$$

This is interesting. Apparently, the finite dimension of the Hilbert subspace $\mathscr{H}_{s+1}$ and the symmetry between $\hat{n}$ and $\hat{\varphi}$ has caused the photon-number to become cyclic. Therefore,

$\mathrm{e}^{-i\hat{\varphi}}$ and $\mathrm{e}^{i\hat{\varphi}}$ may be written as

$$\mathrm{e}^{i\hat{\varphi}} = \sum_{n=0}^{s-1} |n\rangle\langle n+1| + \mathrm{e}^{i(s+1)\varphi_0} |s\rangle\langle 0| \qquad (4.2.22\mathrm{a})$$

$$\mathrm{e}^{-i\hat{\varphi}} = \sum_{n=0}^{s-1} |n+1\rangle\langle n| + \mathrm{e}^{-i(s+1)\varphi_0} |0\rangle\langle s| \qquad (4.2.22\mathrm{b})$$

This is a very important result, since it is responsible for the unitarity of $\mathrm{e}^{i\hat{\varphi}}$, as evidenced by the vanishing commutator with its hermitian adjoint $\mathrm{e}^{-i\hat{\varphi}}$

$$[\mathrm{e}^{i\hat{\varphi}}, \mathrm{e}^{-i\hat{\varphi}}] = \left[\sum_{n=0}^{s-1} |n\rangle\langle n|\right] + |s\rangle\langle s| - \left[\sum_{n=0}^{s-1} |n+1\rangle\langle n+1|\right] - |0\rangle\langle 0| = 0. \qquad (4.2.23)$$

Unfortunately, this unitarity is specific to the $\mathscr{H}_{s+1}$ Hilbert space and is lost for $s \to \infty$, since in this limit, $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ cannot be described by eq. (4.2.22) because the state $|\infty\rangle$ is not defined. This explains why Susskind and Glogower were not able to derive a unitary exponential phase operator from the polar composition of the annihilation operator in section 3.1.3.1: Remember that SG had deduced that the polar decomposition of $\hat{a}$ ($\hat{a}^\dagger$) is not unique since it contains an arbitrary matrix element $|\psi\rangle\bar{0}$ ($|0\rangle\bar{\psi}$) such that

$$\mathrm{e}^{i\hat{\varphi}} = \left[\sum_{n=0}^{\infty} |n\rangle\langle n+1|\right] + |\psi\rangle\langle 0| \qquad (4.2.24\mathrm{a})$$

$$\mathrm{e}^{-i\hat{\varphi}} = \left[\sum_{n=0}^{\infty} |n+1\rangle\langle n|\right] + |0\rangle\langle \psi| . \qquad (4.2.24\mathrm{b})$$

SG were, as we remember, unable to find any $|\psi\rangle$ to render $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ unitary, and we now know why: while in the finite dimensional Hilbert space $\mathscr{H}_{s+1}$, we could simply set $|\psi\rangle = \mathrm{e}^{i(s+1)\varphi_0} |s\rangle$ to obtain the unitary operators of eq. (4.2.22), this is impossible in the infinite dimensional Hilbert space, since we cannot set $|\psi\rangle = |\infty\rangle$. Therefore, in a one-sided infinite Hilbert space, there is no way to choose $|\psi\rangle$ in a way to render the operators $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ unitary.

Another way of visualizing this is to consider the matrix representation of $\mathrm{e}^{i\hat{\varphi}}$ and $\mathrm{e}^{-i\hat{\varphi}}$ in

$\mathscr{H}_{s+1}$.

$$\mathrm{e}^{i\hat{\varphi}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \mathrm{e}^{i(s+1)\varphi} & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \quad \mathrm{e}^{-i\hat{\varphi}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \mathrm{e}^{-i(s+1)\varphi} \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

$$(4.2.25)$$

In the limit $s \to \infty$, the cyclical terms on the lower left and upper right, respectively, which are responsible for providing unitarity, vanish into infinity and become undefined. As an aside, note that the eigenvalue equations for the above matrices are polynomials of degree $s + 1$ whose roots turn out to be the φ_m, confirming the consistency of the formalism.

One last matter of interest is the derivation of the $(s + 1)$-dimensional annihilation and creation operators:

$$\hat{a} = \mathrm{e}^{i\hat{\varphi}}\sqrt{\hat{n}} = \sum_{n=0}^{s-1} \sqrt{n+1}\,|n\rangle\langle n+1| \tag{4.2.26a}$$

$$\hat{a}^{\dagger} = (\mathrm{e}^{i\hat{\varphi}}\sqrt{\hat{n}})^{\dagger} = \sqrt{\hat{n}}\,\mathrm{e}^{-i\hat{\varphi}} = \sum_{n=0}^{s-1} \sqrt{n+1}\,|n+1\rangle\langle n| \tag{4.2.26b}$$

Note that all dependencies on s within the series terms have been eliminated here, and that only the sum limits depend on s. This is of course necessary in order for $\hat{a}$ and $\hat{a}^{\dagger}$ to be well-defined in $\mathscr{H}_{\infty}$. With this, we have expressed all relevant operators in the $\mathscr{H}_{s+r}$ Hilbert space. Next, we examine how these operators behave when the limit $s \to \infty$ is taken.

4.2.4 Taking the limit

Pegg and Barnett never seem to tire of emphasizing that in their formalism, all physical properties such as expectation values, probability distributions and higher moments must be evaluated in the $s + 1$-dimensional Hilbert space $\mathscr{H}_{(s+1)}$, and only then may the limit $s \to \infty$ be taken. There is a very simple reason for this seemingly overboarding pertinacity, and this reason quickly becomes apparent if we ignore PB's instructions and try to take the limit of the phase operator matrix elements directly:

$$\lim_{s \to \infty} \langle n\,|\hat{\varphi}_{\varphi_0}|\,n\rangle = \varphi_0 + \pi \tag{4.2.27a}$$

$$\lim_{s\to\infty} \left\langle n' \left| \hat{\varphi}_{\varphi_0} \right| n \right\rangle = \lim_{s\to\infty} \frac{2\pi}{s+1} \frac{e^{i(n'-n)\varphi_0}}{e^{i(n'-n)2\pi/(s+1)} - 1} \qquad (4.2.27b)$$

$$= \frac{e^{i(n'-n)\varphi_0}}{i(n'-n)} \,, \qquad n' \neq n \,. \qquad (4.2.27c)$$

This is, in fact, the Garrison/Wong operator that we already met in section 3.3.2 in proposition 3.3.3 (to see this, simply set the reference phase φ_0 to $-\pi$ and include the distinction between $n = n'$ and $n \neq n'$ as a Kronecker-delta) [9, 10]. But we know from that section (cf. also [9, 27]) that the Garrison/Wong operator does not offer a satisfactory description of phase for several reasons, among which figure its anisotropy and the fact that the number space cannot be approximated in the subspace $\mathscr{C}$.[25] For now, we will thus observe PB's insistence that the limit is to be taken only after the relevant properties are calculated, although we will come back to the plausibility of this order of limit-taking in section 4.3.

But the phase operator is not the only element of the subspace $\mathscr{H}_{s+1}$ which causes problems when it is directly transferred to the standard Hilbert space $\mathscr{H}_\infty$. The phase states also cause problems, given that their expectation value diverges for $s \to \infty$

$$\left\langle \varphi \left| \hat{n} \right| \varphi \right\rangle = \frac{s}{2} \,, \qquad (4.2.28)$$

as can be seen by inserting the definition (4.2.14b). Lastly, the commutator between the annihilation and creation operators is also undefined for $s \to \infty$

$$[\hat{a}, \hat{a}^\dagger] = \sum_{n=0}^{s-1} (n+1) \left| n \right\rangle\!\left\langle n \right| - \sum_{n=0}^{s-1} (n+1) \left| n+1 \right\rangle\!\left\langle n+1 \right| = \qquad (4.2.29a)$$

$$= \sum_{n=0}^{s-1} \left| n \right\rangle\!\left\langle n \right| - s \left| s \right\rangle\!\left\langle s \right| = \qquad (4.2.29b)$$

$$= \mathbb{1} - (s+1) \left| s \right\rangle\!\left\langle s \right| \qquad (4.2.29c)$$

4.2.5 Physical states

At first, the problems involved in the limit-taking process seem prohibitive, since it is very cumbersome to work in $\mathscr{H}_{s+1}$ given that every state and operator one wants to work with has to be ported to $\mathscr{H}_{s+1}$, the expectation values calculated, and the result then re-ported back into $\mathscr{H}_\infty$ by taking the limit. PB realized this, and invented the notion of the so-called "physical states" to alleviate the problem. "Physical states" are states that are created by interactions of finite time with a finite energy source, which means that their number-expectation value $\langle \hat{n} \rangle$ is finite so that we can always find a number

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[25]More information in that section

state $|m\rangle$ upwards of which the overlap of the physical state $|p\rangle$ with higher number states $\langle p|\,m+j\rangle\,, \quad j=1,2,3\cdots$ becomes negligible. For such a physical states, we can choose s high enough that the differences between results in $\mathscr{H}_{s+1}$ and $\mathscr{H}_\infty$ become negligible, and we shall denote this by giving the evaluated quantity a subscript p.

With the help of physical states, many of the above difficulties disappear. For example, the commutator $[\hat{a},\hat{a}^\dagger]$ simply becomes

$$[\hat{a},\hat{a}^\dagger]_p = \mathbb{1} \tag{4.2.30}$$

since we may simply ignore the $|s\rangle\langle s|$ projector for lack of an exited state $|s\rangle$ it could interact with.

However, it must be clear that the physical states do not work magic. They do not enable us to transfer our entire calculation into $\mathscr{H}_\infty$ and avoid the limiting process altogether (at least conceptually), but instead work the opposite way: they greatly facilitate porting the states we want to work with into $\mathscr{H}_{s+1}$, since these states show almost no difference in behaviour in $\mathscr{H}_{s+1}$ and $\mathscr{H}_\infty$. Staying in the subspace $\mathscr{H}_{s+1}$ to treat physical states then comes with the double advantage that unitarity is preserved while the physical situation is adequately described. And if we formally remain in $\mathscr{H}_{s+1}$, the limit may even be taken prematurely in special cases. For example, choosing s so that the highest relevant $n',n \ll s$, we again recover the matrix elements of the Garrison/Wong operator, but here the matrix is formally of finite dimension

$$\langle n\,|\hat{\varphi}_{\varphi_0}|\,n\rangle = \varphi_0 + \pi\frac{s}{s+1} \tag{4.2.31a}$$

$$\left\langle n'\,|\hat{\varphi}_{\varphi_0}|\,n\right\rangle = \frac{\mathrm{e}^{i(n'-n)\varphi_0}}{i(n'-n)}\,, \qquad n' \neq n\,. \tag{4.2.31b}$$

This leads to a simplified traceless commutator

$$\left\langle n'\,|[\hat{n},\hat{\varphi}_{\varphi_0}]|\,n\right\rangle = \frac{(n'-n)\mathrm{e}^{i(n'-n)\varphi_0}}{i(n'-n)} \tag{4.2.32a}$$

$$= -i(1-\delta_{n',n})\mathrm{e}^{i(n'-n)\varphi_0}\,. \tag{4.2.32b}$$

or, more generally,

$$[\hat{n},\hat{\varphi}_{\varphi_0}]_p = -i\sum_{n',n=0}^{s}\big|n'\big\rangle\langle n|\,(1-\delta_{n',n})\mathrm{e}^{i(n'-n)\varphi_0} = \tag{4.2.33a}$$

$$= i - i\left[\sum_{n'=0}^{s}\mathrm{e}^{in'\varphi_0}\,\big|n'\big\rangle\right]\left[\sum_{n=0}^{s}\mathrm{e}^{-in\varphi_0}\,\langle n|\right] = \tag{4.2.33b}$$

$$= i\left[1 - (s+1)\,|\varphi_0\rangle\langle\varphi_0|\right]\,. \tag{4.2.33c}$$

4.2.6 Uncertainty relations

We now evaluate the expectation value for the simplified commutator for physical states $[\hat{n}, \hat{\varphi}_{\varphi_0}]_p$, whose matrix elements we take from (4.2.33c).

$$\langle p \left| [\hat{n}, \hat{\varphi}_{\varphi_0}] \right| p \rangle = i \left[1 - (s+1) \left| \langle p | \varphi_0 \rangle \right|^2 \right] . \tag{4.2.34a}$$

Remembering that the $|\varphi_m\rangle$ are orthogonal and complete, $|\langle p | \varphi_0 \rangle|^2$ is simply the probability that the state $|p\rangle$ has the phase φ_0. For very large s, this may be written as $P(\varphi_0)2\pi/(s+1)$ where $P(\varphi_0)$ is a probability distribution and $2\pi/(s+1)$ is the density of states. This means that surprisingly, we recover from the complicated-looking commutator (4.2.33c), which is already a simplification of (4.2.16), the Poisson-bracket correspondence result deduced in section 3.2.2 for a saw-toothed periodic phase function. This leads to the uncertainty relation

$$\Delta \hat{n} \cdot \Delta \hat{\varphi}_{\varphi_0} \geq \frac{1}{2} \left| 1 - 2\pi P(\varphi_0) \right| \tag{4.2.35}$$

for physical states (cf. section 7.1.3 for a consolidated derivation) [15].

4.2.7 Operator functions

By now, it should have become sufficiently clear that the main asset of the PB formalism is the existence of an hermitian phase operator on the restricted $\mathscr{H}_{s+1}$. One of the biggest advantages of hermitian operators is that they allow operator functions to be easily defined as follows:

$$f(\hat{\varphi}_{\varphi_0}) = \sum_{m=0}^{s} f(\varphi_m) \, |\varphi_m\rangle\langle\varphi_m| \tag{4.2.36}$$

In this way, cosine and sine functions may be constructed besides the exponential ladder operators we already know

$$\sin \hat{\varphi}_{\varphi_0} = \sum_{m=0}^{s} \sin \varphi_m \, |\varphi_m\rangle\langle\varphi_m| \tag{4.2.37a}$$

$$\cos \hat{\varphi}_{\varphi_0} = \sum_{m=0}^{s} \cos \varphi_m \, |\varphi_m\rangle\langle\varphi_m| , \tag{4.2.37b}$$

for which by construction

$$\sin \hat{\varphi}_{\varphi_0} \, |\varphi\rangle = \sin \varphi \, |\varphi\rangle \tag{4.2.38a}$$

$$\cos \hat{\varphi}_{\varphi_0} \, |\varphi\rangle = \cos \varphi \, |\varphi\rangle \tag{4.2.38b}$$

applies, i.e. the eigenvalues are just the trigonometric functions.

This is a very pleasant simplification compared to the Susskind/Glogower and Lévy-Leblond formalisms (sections 3.1.3.3 and 3.2.1.3), where for lack of a hermitian phase

operator, each operator had to receive an individual set of eigenstates because the operator equivalents of sine, cosine and the exponential did not commute:

$$\widehat{\sin\varphi}\,|\sin\varphi\rangle = \sin\varphi\,|\sin\varphi\rangle \tag{4.2.39a}$$

$$\widehat{\cos\varphi}\,|\cos\varphi\rangle = \cos\varphi\,|\cos\varphi\rangle \tag{4.2.39b}$$

$$\widehat{\mathrm{e}^{\pm i\varphi}}\left|\mathrm{e}^{i\varphi}\right\rangle = \mathrm{e}^{i\varphi}\left|\mathrm{e}^{i\varphi}\right\rangle \tag{4.2.39c}$$

$$|\sin\varphi\rangle \neq |\cos\varphi\rangle \neq \left|\mathrm{e}^{i\varphi}\right\rangle \quad \text{(in general)}. \tag{4.2.39d}$$

Furthermore, it lies in the nature of operator functions that identities fulfilled by the constituent functions are also fulfilled by the operators, e.g.

$$\sin^2\hat{\varphi}_{\varphi_0} + \cos^2\hat{\varphi}_{\varphi_0} = \sum_{m=0}^{s}\left(\sin^2\varphi_m + \cos^2\varphi_m\right)|\varphi_m\rangle\langle\varphi_m| = \mathbb{1} \tag{4.2.40a}$$

$$[\sin\hat{\varphi}_{\varphi_0}, \cos\hat{\varphi}_{\varphi_0}] = \sum_{m=0}^{s}[\sin\varphi_m, \cos\varphi_m]\,|\varphi_m\rangle\langle\varphi_m| = 0 \tag{4.2.40b}$$

4.2.8 Working with the formalism

Pegg and Barnett also supply a powerful calculation mechanism for computation of the expectation values and variance of states that they call partial phase states. These states have the form

$$|b\rangle = \sum_{n=0}^{s} b_n\,\mathrm{e}^{in\beta}\,|n\rangle \tag{4.2.41}$$

and are very similar to the phase states. Indeed, the phase states themselves result for $b_n = (s+1)^{-1/2}$ and $\beta = \varphi$, but number states may also be described as partial phase states with $b_n = \delta_{n',n}$ and $\beta = 0$ yielding $|n'\rangle$. Another important class of partial phase states are the coherent states with $b_n = |\alpha| \cdot \mathrm{e}^{-|\alpha|^2/2}/\sqrt{n!}$ and $\beta = \arg\alpha$.

Observe that the partial phase states are normalized

$$\langle b|\,b\rangle = \sum_{n=0}^{s} b_n^2 = 1 \tag{4.2.42}$$

if the sum of the b_n squared is equal to one (true for number and phase states and arbitrarily close to true for coherent states). We can now calculate the expectation value for $|b\rangle$ by using the representation of $\hat{\varphi}_{\varphi_0}$ in phase states

$$\langle b\,|\hat{\varphi}_{\varphi_0}|\,b\rangle = \sum_{m=0}^{s}\varphi_m\,|\langle\varphi_m|b\rangle|^2 = \sum_{m=0}^{s}\varphi_m\left|\frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}b_n\,\mathrm{e}^{-in(\varphi_m-\beta)}\right|^2 = \tag{4.2.43a}$$

$$= \sum_{m=0}^{s}\varphi_m\left[\frac{1}{s+1}\sum_{n',n=0}^{s}b_n b_{n'}\,\mathrm{e}^{-i(n-n')(\varphi_m-\beta)}\right] = \tag{4.2.43b}$$

$$= \sum_{m=0}^{s} \varphi_m \left[\frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'}^{s} b_n b_{n'} \cos \left[(n-n')(\beta - \varphi_m) \right] \right] . \qquad (4.2.43c)$$

Setting

$$\varphi_0 = \beta - \frac{\pi s}{s+1} \qquad (4.2.44)$$

and offsetting the index

$$\mu = m - \frac{s}{2} \qquad (4.2.45)$$

yields

$$\langle b \,|\hat{\varphi}_{\varphi_0}|\, b \rangle = \sum_{\mu=-\frac{s}{2}}^{\frac{s}{2}} \varphi_\mu \left[\frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'}^{s} b_n b_{n'} \cos \left[(n-n')\mu 2\pi/(s+1) \right] \right] = \quad (4.2.46a)$$

$$= \sum_{\mu=-\frac{s}{2}}^{\frac{s}{2}} \beta \left[\frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'}^{s} b_n b_{n'} \cos \left[(n-n')\mu 2\pi/(s+1) \right] \right] = \quad (4.2.46b)$$

$$= \sum_{\mu=-\frac{s}{2}}^{\frac{s}{2}} \beta \left[\frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'}^{s} b_n b_{n'} \, \mathfrak{Re} \left\{ \mathrm{e}^{i(n-n')\mu 2\pi/(s+1)} \right\} \right] = \quad (4.2.46c)$$

$$= \sum_{\mu=-\frac{s}{2}}^{\frac{s}{2}} \beta \frac{1}{s+1} = \beta \,, \qquad (4.2.46d)$$

where we have used in the second equality that the sum over a symmetrical component times an asymmetrical component is zero, and in the fourth equality that the sum over a regular $s+1$-sided polygon in the complex plane is identically equal to zero.

The result

$$\langle b \,|\hat{\varphi}_{\varphi_0}|\, b \rangle = \beta \qquad (4.2.47)$$

arises independently of the choice of φ_0[26] and is very useful because it does not depend on s and therefore may be applied directly to any partial phase state without bothering with any limiting process.[27]

Next, we calculate the variance for partial phase states

$$\Delta \hat{\varphi}_{\varphi_0}^2 = \left\langle b \left| (\hat{\varphi}_{\varphi_0} - \langle \hat{\varphi}_{\varphi_0} \rangle)^2 \right| b \right\rangle = \sum_{\mu=-\frac{s}{2}}^{\frac{s}{2}} \left(-\frac{\mu 2\pi}{s+1} \right)^2 |\langle \varphi_\mu | b \rangle|^2 = \qquad (4.2.48a)$$

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[26]Eq. (4.2.44) only facilitates the calculation since it symmetrizes the cosine

[27]Note that this corroborates our quasidefinition in section 1.2 where we have stated that phase is not merely a phase factor, but rather a series of interference effects between number states. Indeed, the β that is recovered as the phase expectation value is none other than that which determines the phase difference between each successive pair of number states.

$$= \frac{4\pi^2}{(s+1)^2} \sum_{\mu=-\frac{s}{2}}^{\frac{s}{2}} \mu^2 \left[\frac{1}{s+1} + \frac{2}{s+1} \sum_{n>n'}^{s} b_n b_{n'} \cos\left[(n-n')\frac{\mu 2\pi}{(s+1)}\right] \right] . \quad (4.2.48\text{b})$$

Unfortunately, the trick with $\mathfrak{Re}\{e^{i\phi}\} = 0$ for regular polygons, which was used in the fourth equality of (4.2.46), does not work here, since the sum contains a term μ^2 and therefore the polygon is irregular. However, since we are dealing with a c-number which does not contain any states or operators anymore, we may, according to the prime directive of the PB formalism, take the limit as $s \to \infty$. This yields an integral where we replace $\mu 2\pi/(s+1)$ by φ, $2\pi/(s+1)$ by $\mathrm{d}\varphi$ and integrate from $-\pi$ to π:

$$\Delta\hat{\varphi}_{\varphi_0}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2\sum_{n>n'}^{s} b_n b_{n'} \cos\left[(n-n')\varphi\right] \right] \varphi^2 \, \mathrm{d}\varphi = \quad (4.2.49\text{a})$$

$$= \frac{\pi^2}{3} + \left[\sum_{n>n'}^{s} b_n b_{n'} \frac{4\varphi}{2\pi(n-n')^2} \cos\left\{(n-n')\varphi\right\} \right] \Bigg|_{-\pi}^{\pi} = \quad (4.2.49\text{b})$$

$$= \frac{\pi^2}{3} + 4\sum_{n>n'}^{s} b_n b_{n'} \frac{1}{(n-n')^2}(-1)^{(n-n')} \quad (4.2.49\text{c})$$

We have thus found an expression that allows the calculation of the variance of arbitrary partial phase states if the coefficients b_n are known. This very handy result will be employed in the next section to apply the acid test. However, note here already that there are some justified reservations as to the validity of this result, which will be addressed in section 4.4.2.

4.2.9 Acid test

We now carry out the Pegg/Barnett acid test, already anticipating the not-so-surprising result that the PB-phase operator passes the PB-acid test. As test states, we again use the number states.

Expressing these as partial phase states

$$|n\rangle = \sum_{k=0}^{s} b_k \, e^{ik\beta} \, |k\rangle \quad (4.2.50)$$

gives us $b_k = \delta_{nk}$, $\beta = 0$. Using

$$\beta = \varphi + \frac{s\pi}{s+1} \quad (4.2.51)$$

and taking the limit $s \to \infty$ we obtain from eq. (4.2.47) that

$$\langle \hat{\varphi}_{\varphi_0} \rangle = \varphi_0 + \pi , \quad (4.2.52)$$

which is consistent with the classical description (cf. eq. (2.2.2)). Using equation (4.2.49c),

the variance follows trivially as

$$\Delta\hat{\varphi}_{\varphi_0}^2 = \frac{\pi^2}{3} \,, \tag{4.2.53}$$

also consistent with the classical result (cf. eq. 2.2.3). We also briefly calculate the mean and variance of the sine and cosine of the phase operator in order to be able to make a direct comparison to the Susskind/Glogower results. We have

$$\langle n \, | \cos\hat{\varphi}_{\varphi_0} | \, n \rangle = \frac{1}{s+1} \sum_{m=0}^{s} \cos\varphi_m \, |\langle\varphi_m | \, n\rangle|^2 = \frac{1}{s+1} \sum_{m=0}^{s} \cos\varphi_m \, \left| e^{in\varphi_m} \right|^2 = \tag{4.2.54a}$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} \cos\varphi_m = \frac{1}{s+1} \sum_{m=0}^{s} \mathfrak{Re}\left\{ e^{i\varphi_m} \right\} = 0 \tag{4.2.54b}$$

$$\langle n \, | \sin\hat{\varphi}_{\varphi_0} | \, n \rangle = \frac{1}{s+1} \sum_{m=0}^{s} \sin\varphi_m \, |\langle\varphi_m | \, n\rangle|^2 = \frac{1}{s+1} \sum_{m=0}^{s} \sin\varphi_m \, \left| e^{in\varphi_m} \right|^2 = \tag{4.2.54c}$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} \sin\varphi_m = \frac{1}{s+1} \sum_{m=0}^{s} \mathfrak{Im}\left\{ e^{i\varphi_m} \right\} = 0 \,, \tag{4.2.54d}$$

where we have again used that the sum over a regular polygon in the complex plane is identically zero. Calculating the variances ($\hat{=}$ the second moments, since the expectation values are zero) proceeds in a similar fashion:

$$\left\langle \cos^2\hat{\varphi}_{\varphi_0} \right\rangle = \frac{1}{s+1} \sum_{m=0}^{s} \cos^2\varphi_m \, |\langle\varphi_m | \, n\rangle|^2 = \frac{1}{s+1} \sum_{m=0}^{s} \cos^2\varphi_m \, \left| e^{in\varphi_m} \right|^2 = \tag{4.2.55a}$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} \frac{1}{2} + \frac{1}{2}\cos\varphi_m = \frac{1}{2} + \frac{1}{s+1} \sum_{m=0}^{s} \mathfrak{Re}\left\{ e^{2i\varphi_m} \right\} = \frac{1}{2} \tag{4.2.55b}$$

$$\left\langle \sin^2\hat{\varphi}_{\varphi_0} \right\rangle = \frac{1}{s+1} \sum_{m=0}^{s} \sin^2\varphi_m \, |\langle\varphi_m | \, n\rangle|^2 = \frac{1}{s+1} \sum_{m=0}^{s} \sin^2\varphi_m \, \left| e^{in\varphi_m} \right|^2 = \tag{4.2.55c}$$

$$= \frac{1}{s+1} \sum_{m=0}^{s} \frac{1}{2} - \frac{1}{2}\cos\varphi_m = \frac{1}{2} - \frac{1}{s+1} \sum_{m=0}^{s} \mathfrak{Im}\left\{ e^{2i\varphi_m} \right\} = \frac{1}{2} \,, \tag{4.2.55d}$$

These results are consistent with their classical equivalents as calculated in (3.1.73), and this applies even for the vacuum state, which is an appreciable improvement over the Susskind/Glogower results, where the vacuum state had an incorrect variance of 1/4 in sine and cosine, implying an anisotropic phase distribution.

4.2.10 Interim discussion

So far, the Pegg/Barnett formalism seems to provide a very robust framework. In a limited but large Hilbert space which is apparently sufficient for physically accessible states, a Hermitian phase operator is defined which yields an uncertainty relation corresponding to that which we expect from the Poisson-bracket correspondence, avoids the periodicity

problem, and solves the non-unitarity problem by construction since the phase operator is Hermitian. Moreover, sine, cosine and exponential operators can be defined as simple operator functions instead of needing to be tediously derived, which allows them to fulfill all known functional identities by construction, besides also commuting with each other. In addition, the notion of partial phase states yields a simple but powerful calculational technique with which expectation values and variances of a wide class of physical states, including number states and coherent states, can be obtained very easily.

However, there are several caveats to observe. First, the PB phase operator cannot be rendered mathematically equivalent to an infinite-dimensional phase operator on the standard oscillator Hilbert space, since the operations of calculating the physical properties and taking the limit do not commute [10, 17]. For example, calculating the variance of the number state only after the limits of the operators are taken (cf. the matrix elements calculated in eq. (4.2.27)) yields (we set $\varphi_0 = -\pi$ for simplicity of calculation):

$$\langle n \, | \hat{\varphi}_{\text{lim}} | \, n \rangle = \varphi_0 + \pi = 0 \tag{4.2.56a}$$

$$\Delta \hat{\varphi}_{\text{lim}}^2 = \left\langle n \left| (\hat{\varphi}_{\text{lim}} - \langle \hat{\varphi}_{\text{lim}} \rangle)^2 \right| n \right\rangle = \left\langle n \left| \hat{\varphi}_{\text{lim}}^2 \right| n \right\rangle = \tag{4.2.56b}$$

$$= \sum_{n'=0}^{\infty} \left\langle n \, | \hat{\varphi}_{\text{lim}} | \, n' \right\rangle \left\langle n' \, | \hat{\varphi}_{\text{lim}} | \, n \right\rangle = \sum_{\substack{n'=0 \\ n' \neq n}}^{\infty} \frac{(-1)^{(n'-n)}}{i(n'-n)} \frac{(-1)^{(n-n')}}{i(n-n')} = \tag{4.2.56c}$$

$$= \sum_{\substack{n'=0 \\ n' \neq n}}^{\infty} \frac{1}{(n-n')^2} = \sum_{n'=n+1}^{\infty} \frac{1}{(n-n')^2} + \sum_{n'=0}^{n-1} \frac{1}{(n-n')^2} = \tag{4.2.56d}$$

$$= \frac{\pi^2}{6} + \sum_{k=1}^{n} \frac{1}{k^2} \, . \tag{4.2.56e}$$

This is not equal to the classical variance except if we take the limit $n \to \infty$. But that means that calculating the limit of the operator first and then finding the variance gives wrong results for all but very high average photon-numbers $\langle \hat{n} \rangle$, a result that is much worse than that of the SG-formalism! Conversely, the reverse procedure, as prescribed by PB, gives the correct results even for vacuum states. The underlying reason for this strange behavior lies with the weak operator topology, which governs the PB-operator and will be examined more closely in section 4.3.1.

Second, the PB phase operator does not fix the core problem associated with the SG-formalism - the non-commuting sine and cosine operators. To appreciate this, note that the PB formalism does not allow for precise phase measurement because the phase states themselves are unphysical (their expectation value diverges in $\mathscr{H}_\infty$). Thus, a phase measurement cannot project the measured state into a phase state, but only into a physical partial phase state, such as a coherent state or a highly phase squeezed state. But with these states, a certain amount of uncertainty in the phase always remains, meaning that

repeated phase measurements do not necessarily produce the same results. Consequently, in the PB formalism exact phase measurement is just as impossible as it is in the SG formalism; the only difference is the source of the problem, which has shifted from the non-commuting sine and cosine operators to the non-physical phase eigenstates. This allows us to suspect that the differences between the PB- and the SG-formalism may not be so great as they are commonly portrayed, a conclusion which will turn out to be correct (cf. section 4.3.2).

4.3 Observations about the PB phase operator

In this section, which can be described as an inlay between the treatment of advantages in the previous section and of criticism in the next section, we will try to develop a better grasp of the nature of the PB-phase operator by investigating its convergence properties from a functional analytic viewpoint, and will then try to relate it to the SG-formalism using the concept of generalized measurement and probability operator measures.

4.3.1 Taking the limit: revisited

It will be shown that the PB-hermitian phase operator is a weakly convergent operator [18, 42], which explains many of the awkward properties of the limiting process as observed in the last several sections.

4.3.1.1 Weak operator topology

But first, we need to introduce the concept of strongly and weakly convergent operators:

Strongly convergent operators are defined by their effect on strongly convergent sequences of states. A Cauchy sequence of states $\{|f\rangle_1, |f\rangle_2, \cdots\}$ is strongly convergent and is said to have the limit $|f\rangle$ if and only if

$$\||f\rangle - |f\rangle_n\| \to 0, \qquad n \to \infty, \tag{4.3.1}$$

i.e. if it converges in the norm. Then, we have the following definition

Definition. *A sequence of operators $\left\{\hat{A}_1, \hat{A}_2, \cdots\right\}$ converges strongly to an operator $\hat{A}$ if and only if the sequence of states $\left\{\hat{A}_n |g\rangle = |h\rangle_n\right\}$ converges strongly towards the state $\hat{A}|g\rangle = |h\rangle$.*

Another way to put this is that for strongly convergent operators

$$\left\| (\hat{A} - \hat{A}_n) \ket{g} \right\| \to 0 \,, \qquad n \to \infty \tag{4.3.2}$$

for all states $\ket{g} \in \mathscr{H}$.

The main benefit of strongly convergent operators is that multiplication is jointly continuous, i.e. for any two strongly convergent sequences of operators $\left\{ \hat{A}_1, \hat{A}_2, \cdots \right\}$ and $\left\{ \hat{B}_1, \hat{B}_2, \cdots \right\}$ which converge to $\hat{A}$ and $\hat{B}$, the product sequence $\left\{ \hat{B}_1 \hat{A}_1, \hat{B}_2 \hat{A}_2, \cdots \right\}$ is also strongly convergent and approaches $\hat{B}\hat{A}$ so that

$$\left\| (\hat{B}\hat{A} - \hat{B}_n \hat{A}_n) \ket{g} \right\| \to 0 \,, \qquad n \to \infty \,. \tag{4.3.3}$$

This follows because

$$\left\| (\hat{B}\hat{A} - \hat{B}_n \hat{A}_n) \ket{g} \right\| = \left\| \hat{B}\hat{A} \ket{g} - \hat{B}_n \hat{A}_n \ket{g} \right\| = \tag{4.3.4a}$$

$$= \left\| \hat{B} \ket{h} - \hat{B}_n \ket{h}_n \right\| = \tag{4.3.4b}$$

$$= \left\| \hat{B} \ket{h} - \hat{B}_n \ket{h} + \hat{B}_n \ket{h} - \hat{B}_n \ket{h}_n \right\| = \tag{4.3.4c}$$

$$= \left\| (\hat{B} - \hat{B}_n)(\ket{h}) + \hat{B}_n(\ket{h} - \ket{h}_n) \right\| \to 0 \,, \qquad n \to \infty \,. \tag{4.3.4d}$$

Weakly convergent operators on the other hand, do not converge in the norm. Commonly, weak convergence is defined as follows:

Definition. *The sequence of operators $\left\{ \hat{C}_1, \hat{C}_2, \cdots \right\}$ converges weakly to an operator $\hat{C}_w$ if*

$$\left\langle h \left| (\hat{C}_w - \hat{C}_n) \right| g \right\rangle \to 0 \,, \qquad n \to \infty \tag{4.3.5}$$

for all $\ket{h}, \ket{g} \in \mathscr{H}$.

Every strongly convergent operator also converges weakly, but the same is not true of the reverse (cf. page 44 et seq of [43]).

The most important *disadvantage* of weakly convergent operators is that the product of two weakly convergent operators is *not* jointly continuous. Consider that

$$\left\langle h \left| (\hat{D}_w \hat{C}_w - \hat{D}_n \hat{C}_n) \right| g \right\rangle = \left\langle \hat{D}_w h \left| \hat{C}_w \right| g \right\rangle - \left\langle \hat{D}_n h \left| \hat{C}_n \right| g \right\rangle \tag{4.3.6a}$$

$$= \left\langle k \left| \hat{C}_w \right| g \right\rangle - \left\langle k_n \left| \hat{C}_n \right| g \right\rangle = \tag{4.3.6b}$$

$$= \left\langle k \left| \hat{C}_w \right| g \right\rangle - \left\langle k \left| \hat{C}_n \right| g \right\rangle + \left\langle k \left| \hat{C}_n \right| g \right\rangle - \left\langle k_n \left| \hat{C}_n \right| g \right\rangle = \tag{4.3.6c}$$

$$= \left\langle k \left| \hat{C}_w - \hat{C}_n \right| g \right\rangle + \left\langle k - k_n \left| \hat{C}_n \right| g \right\rangle \nrightarrow 0 \,, \qquad n \to \infty \tag{4.3.6d}$$

does not converge to 0 because weakly convergent operators do not converge in the norm

$$\left\| (\hat{D}_w - D_n) \left| g \right\rangle \right\| = \left\| \left| k \right\rangle_n - \left| k \right\rangle \right\| \nrightarrow 0, \qquad n \to \infty. \tag{4.3.7}$$

As an aside, note that this last inequality means that eigenfunctions will in general not be preserved when taking the weak limit of an operator.

4.3.1.2 Application to PB-formalism

Now, we want to figure out how the weak operator topology relates to the limit taking process. We therefore investigate which topology is required in order for the operations of taking the limit and calculating the expectation value to commute. This can be expressed mathematically as

$$\lim_{s \to \infty} \left\langle f_s \left| \hat{A}_s \right| f_s \right\rangle = \left\langle f \left| \lim_{s \to \infty} \hat{A}_s \right| f \right\rangle, \tag{4.3.8}$$

where

$$\left| f_s \right\rangle = \sum_{n=0}^{s} f_n \left| n \right\rangle \tag{4.3.9}$$

and

$$\left| f \right\rangle = \sum_{n=0}^{\infty} f_n \left| n \right\rangle. \tag{4.3.10}$$

It turns out that relation (4.3.8) is true, unsurprisingly, for strongly convergent operators and their higher moments, but it is also true for the first moments of weakly convergent operators, since

$$\lim_{s \to \infty} \left\langle f_s \left| \hat{A}_s \right| f_s \right\rangle = \lim_{s \to \infty} \left\langle f \left| \hat{A}_s \right| f \right\rangle = \left\langle f \left| \lim_{s \to \infty} \hat{A}_s \right| f \right\rangle, \tag{4.3.11}$$

where we have used that $A_s \left| f_s \right\rangle = A_s \left| f \right\rangle$ because A_s only operates in the subspace $\mathscr{H}_{s+1}$. The equivalence then follows from the definition of weakly convergent operators as given in eq. (4.3.5).

However, because the product of two weakly convergent operators is not jointly continuous, we conclude that higher moments of weakly convergent operators *do not necessarily* fulfill the above relation:

$$\lim_{s \to \infty} \left\langle f_s \left| \hat{A}_s^k \right| f_s \right\rangle \neq \left\langle f \left| \lim_{s \to \infty} \hat{A}_s^k \right| f \right\rangle. \tag{4.3.12}$$

4.3.1.3 Application to PB operators

Applied to the individual operators of the PB formalism, this means that all strongly convergent operators, including all standard hermitian operators and the annihilation and creation operators, are well-behaved in the standard Hilbert space $\mathscr{H}_\infty$ as well as in the subspace $\mathscr{H}_{s+1}$. On the other hand, weakly convergent operators exhibit limit invariance

only for their first moments, but not for higher moments. It is not surprising that the PB-phase operator is only weakly convergent [42], given its strange behavior under limit-taking operations.

We may now establish a direct relation between the Susskind/Glogower formalism and the Pegg/Barnett formalism:

$$\lim_{s \to \infty} F(\hat{\varphi}_{\varphi_0}) = \widehat{F(\varphi)}, \tag{4.3.13}$$

where $F(\hat{\varphi}_{\varphi_0})$ is a true operator function of the PB phase operator for the limited subspace $\mathscr{H}_{s+1}$ and $\widehat{F(\varphi)}$ is the Susskind/Glogower equivalent of this operator function, such as $\widehat{\sin \varphi}$, $\widehat{\cos \varphi}$ or $\widehat{e^{i\varphi}}$. Indeed, it can be shown [42] that

$$\lim_{s \to \infty} \sin \hat{\varphi}_{\varphi_0} = \widehat{\sin \varphi} \tag{4.3.14a}$$

$$\lim_{s \to \infty} \cos \hat{\varphi}_{\varphi_0} = \widehat{\cos \varphi} \tag{4.3.14b}$$

$$\lim_{s \to \infty} e^{i\hat{\varphi}_{\varphi_0}} = \widehat{e^{i\varphi}} \tag{4.3.14c}$$

An exception to this rule is the function $F(\hat{\varphi}_{\varphi_0}) = \hat{\varphi}_{\varphi_0}$, i.e. the phase itself, since no Susskind/Glogower equivalent exists for the phase operator itself. However, the phase operator converges weakly to the Garrison/Wong operator.

For all the above, the first moments are identical in $\mathscr{H}_\infty$ and $\mathscr{H}_{s+1}$, in agreement with our predictions of the last section. Conversely, higher moments are not identical in these two Hilbert space, and therefore they depend on the order in that the limit is taken, as we could expect from (4.3.12).[28] ´This means that for higher moments, the weak operator topology forces us to make a choice as to which side of (4.3.12) should be chosen to represent the true expectation values.

In this context, Pegg and Barnett and with them Vaccaro [42] have impliedly expressed a preference for choosing the left-hand-side of (4.3.12) by demanding that the limit be taken after expectation values are calculated in $\mathscr{H}_{s+1}$. However, this cannot really mask the fact that the decision between the left-hand-side and the right-hand-side of (4.3.12) is principally arbitrary. Moreover, as we will see in the next sections, there is no experimental or even theoretical ground for decidedly preferring one side over the other. Instead, it turns out that the PB acid test is not a good benchmark for the quality of a phase theory, as becomes apparent when we consider the result of the next section, which will show that the SG formalism (a formalism which fails the acid test) and the PB formalism (a formalism which passes it) are quantum mechanically *equivalent*.

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[28] Recall that we have already seen all this in section 4.2.10 for the phase itself, and in sections 3.1.4.1 and 4.2.9 when comparing expectation values for $\widehat{\sin \varphi}$ and $\sin(\hat{\varphi}_{\varphi_0})$

4.3.2 Equivalence of the SG formalism

In this section, we evaluate the representation of the PB phase operator as a probability-operator measure (POM) and show that this measure is equivalent to a POM introduced from the eigenstates $\left|e^{i\varphi}\right\rangle$ of the SG exponential operator $\widehat{e^{i\varphi}}$.

4.3.2.1 Probability-operator measures

Probability-operator measures (or positive-valued operator measures) are an element of generalized measurement theory and relax the requirements imposed on the quantum measurement of observables. Introductions to this concept can be found in [44, 45] (chapters 4 and 4, respectively).

Recall that in standard quantum theory, the measurement process is limited to observables which are described by hermitian operators and have complete orthonormal base sets. Generalized measurement theory allows us to dismiss this requirement and conduct measurements of properties that are not described by a quantum observable. Very simplistically put, this is possible because the real world does not "care" about the hermitian operator formalism, it only cares about probability distributions, which are needed to calculate expectation values and higher moments of a given variable. For example, for number states we have

$$P(n)_\psi = |\langle n| \psi\rangle|^2 \,, \tag{4.3.15}$$

for which we merely require

$$0 \le P(n)_\psi \le 1 \tag{4.3.16a}$$

$$\sum_{n=0}^{\infty} P(n)_\psi = 1 \,. \tag{4.3.16b}$$

It is easy to see that these requirements are fulfilled because

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \,, \tag{4.3.17}$$

so that

$$\sum_{n=0}^{\infty} P(n)_\psi = \sum_{n=0}^{\infty} \langle\psi| n\rangle\langle n| \psi\rangle = \langle\psi| \psi\rangle = 1 \,. \tag{4.3.18}$$

We now define a probability-operator $\hat{\Pi}_n = |n\rangle\langle n|$ for which holds

$$\sum_{n=0}^{\infty} \hat{\Pi}_n = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \,, \tag{4.3.19}$$

so that the probability distribution becomes

$$P(n)_\psi = \langle\psi\,|\Pi_n|\,\psi\rangle \tag{4.3.20}$$

This probability-operator is a *measure* which can be used to evaluate expectation values etc, thus the name probability-operator measure (POM). In this special case, it is also a projector operator measure, since all the $\Pi_n = |n\rangle\langle n|$ are projectors.

But this is not necessary — a POM preserves its measure properties even if its constituent operators are not projectors. This is one of the main results of generalized measurement theory, which postulates that probability operator measures can be defined for much wider classes of states than the complete orthonormal base states. The only property that is required for this is that the states in question resolve to the identity.

As an example, we consider the coherent states, which are the eigenkets of the annihilation operator $\hat{a}$:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{4.3.21}$$

As we know, $\hat{a}$ is not hermitian, and the states $|\alpha\rangle$ are not orthogonal, i.e.

$$\begin{aligned}
\langle\beta|\,\alpha\rangle &= \mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n=0}^{\infty}\frac{(\beta^*\alpha)^n}{n!} = \\
&= \mathrm{e}^{-(|\alpha|^2+|\beta|^2)/2}\mathrm{e}^{\beta^*\alpha} = \mathrm{e}^{-|\alpha-\beta|^2/2} \neq 0\,,
\end{aligned} \tag{4.3.22}$$

but the coherent states $\left|\alpha = r\mathrm{e}^{i\varphi}\right\rangle$ do resolve to the identity

$$\begin{aligned}
\frac{1}{\pi}\int\limits_{\alpha\in\mathbb{C}}\mathrm{d}\alpha\,|\alpha\rangle\langle\alpha| &= \frac{1}{\pi}\int\limits_{\alpha\in\mathbb{C}}\mathrm{d}\alpha\,\mathrm{e}^{-|\alpha|^2}\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha^*)^{n'}\alpha^n}{\sqrt{n'!n!}}\,\left|n'\right\rangle\langle n| = \\
&= \frac{1}{\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{\infty}d\phi\,r\mathrm{d}r\,\mathrm{e}^{-r^2}\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}\mathrm{e}^{i(n'-n)}\frac{r^{n'+n}}{\sqrt{n'!n!}}\,\left|n'\right\rangle\langle n| = \\
&= 2\int\limits_{0}^{\infty}r\mathrm{d}r\,\mathrm{e}^{-r^2}\sum_{n=0}^{\infty}\frac{r^{2n}}{n!}\,|n\rangle\langle n| = \mathbb{1}
\end{aligned} \tag{4.3.23}$$

where in the last equality, we have used that the radial integration turns into the gamma function for the substitution $\rho = r^2$, so that $2\int_0^\infty r\mathrm{d}r\,\mathrm{e}^{-r^2}r^{2n} = n!$, leaving (after cancelling with $n!$ in the denominator) $\sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1}$. According to generalized measurement theory, we may now write the probability-operator measure for the coherent states as

$$\mathrm{d}\hat{\Pi}(\alpha) = \frac{\mathrm{d}\alpha}{\pi}\,|\alpha\rangle\langle\alpha| \tag{4.3.24}$$

where we have used the differential form to reflect that we are dealing with integrals instead of sums. Note here that POMs are also referred to as positive operator-valued measures

(POVM), and the two terms are interchangeable. Also note that, apparently, we have

$$\int\limits_{\alpha \in \mathbb{C}} \mathrm{d}\hat{\Pi}(\alpha) = \mathbb{1} \tag{4.3.25}$$

The POM can now be used to calculate the probability distribution

$$P(\alpha)_\psi = \frac{\left\langle \psi \left| \mathrm{d}\hat{\Pi}(\alpha) \right| \psi \right\rangle}{\mathrm{d}\alpha} = \frac{\langle \psi | \alpha \rangle \langle \alpha | \psi \rangle}{\pi} = \frac{1}{\pi} \left| \langle \alpha | \psi \rangle \right|^2 \tag{4.3.26a}$$

where we have (formally) taken the derivative of the POM to be able to make the calculation

$$\frac{\mathrm{d}\hat{\Pi}(\alpha)}{\mathrm{d}\alpha} = \frac{\mathrm{d}\alpha}{\mathrm{d}\alpha} \frac{|\alpha\rangle\langle\alpha|}{\pi} = \frac{|\alpha\rangle\langle\alpha|}{\pi} \tag{4.3.26b}$$

where the seemingly superfluous factor $\frac{1}{\pi}$ reflects the overcompleteness of the coherent states, which requires a rescaling of the usual expression $\left| \langle \alpha | \psi \rangle \right|^2$ in order to guarantee that the integral over the probability distribution adds up to one. This probability distribution contains the complete physical information about the state, obviating the need for a hermitian operator to execute measurement.

We now leave our example of the coherent states and make some general remarks about the capabilities of the POM formalism. Probability-operator measures are amazingly versatile, since all they require is a set of parametrized states which resolve to the identity. It is not even necessary that these states be normalizable, or that they are physically accessible. Moreover, once such states are found, they can be used to define an arbitrary multitude of operators

$$F(\hat{A}) = \int F(a) \, \mathrm{d}\Pi(a). \tag{4.3.27}$$

These operators may not be hermitian, but they nonetheless define observable quantities, i.e. "generalized" observables, since the expectation values of $F(\hat{A})$ may be calculated as

$$\left\langle F(\hat{A}) \right\rangle = \left\langle \psi \left| \int F(a) \, \mathrm{d}\Pi(a) \right| \psi \right\rangle = \int F(a) \left\langle \psi \left| \mathrm{d}\Pi(a) \right| \psi \right\rangle, \tag{4.3.28}$$

and all physical properties are therein reflected [46]. Simply put, this works because the POM, i.e. the eigenstates of the measured operator, generate the measurement statistics, while the eigenvalues represent the corresponding measurement quantities.

We close this section with a short conceptual remark about operators. The POM formalism makes it clear that in general, observables do not necessarily have to correlate with any physical information. This is because we may recur to any set of states which resolve to the identity to generate our measurement statistics, which we then supply with arbitrary eigenvalues to generate the operators. The challenge lies in finding eigenvalues and

measurement statistics that make physical sense, and this is exactly what we will try to do in the next section for phase.

4.3.2.2 The SG-POM

We now calculate the SG probability-operator measure. For this, we first have to find a complete set of states that we can associate with phase. We have already met such states and discussed them in some detail when describing the Lévy-Leblond formalism in section 3.2.1. In fact, when reviewing that section, we note that Lévy had already proposed using these states for probabilistic phase measurement and even pointed out the similarity to the coherent states. The states in question are, of course, the eigenkets of the lowering operator $\widehat{e^{i\varphi}}$. As Lévy-Leblond already pointed out, this operator possesses the eigenstates (up to a constant)

$$\left| e^{i\varphi} \right\rangle = \sum_{n=0}^{\infty} e^{in\varphi} \left| n \right\rangle . \tag{4.3.29}$$

So what makes $\left| e^{i\varphi} \right\rangle$ a good "phase state"? First, the states $\left| e^{i\varphi} \right\rangle$ resolve to the identity in the interval 2π, i.e. in a full rotation of the unit circle (cf. 3.2.1 for the calculation), consistent with phase periodicity:

$$\frac{1}{2\pi} \int\limits_{\varphi_0}^{\varphi_0+2\pi} d\varphi \left| e^{i\varphi} \right\rangle\!\left\langle e^{i\varphi} \right| = \mathbb{1} \tag{4.3.30}$$

Furthermore, the states $\left| e^{i\varphi} \right\rangle$ are invariant under a (one-sided) unitary number shift transformation and their phase is shifted under the action of a unitary phase shift operator generated by the number operator. This means that these states have a full phase component and (almost) no number component[29], so that they may describe phase with minimal interference from the photon-number statistics [47]:

$$\widehat{e^{i\varphi}} \left| e^{i\varphi} \right\rangle\!\left\langle e^{i\varphi} \right| \widehat{e^{-i\varphi}} = e^{i\varphi} \left| e^{i\varphi} \right\rangle\!\left\langle e^{i\varphi} \right| e^{-i\varphi} = \left| e^{i\varphi} \right\rangle\!\left\langle e^{i\varphi} \right| \tag{4.3.31a}$$

$$e^{i\hat{n}\lambda} \left| e^{i\varphi} \right\rangle\!\left\langle e^{i\varphi} \right| e^{-i\hat{n}\lambda} = \left[\sum_{n'=0} e^{i\hat{n}\lambda} e^{in\varphi} \left| n \right\rangle \right] \left[\sum_{n'=0} e^{-i\hat{n}\lambda} e^{-in\varphi} \left\langle n \right| \right] = \tag{4.3.31b}$$

$$= \left| e^{i(\varphi+\lambda)} \right\rangle\!\left\langle e^{i(\varphi+\lambda)} \right| , \tag{4.3.31c}$$

where the second result corresponds to a shift by λ in the probability distribution.

Given these pleasant properties, we hope that the states $\left| e^{i\varphi} \right\rangle$, even though reflecting something akin to an exponential of the phase, may be used to directly generate a POM

---

[29] "Almost" because $\widehat{e^{i\varphi}}$ is only one-sided unitary, making it an incomplete number shift operator.

for phase. We therefore define the SG-POM as

$$d\Pi(\varphi) = d\varphi \, \frac{\left| e^{i\varphi} \right\rangle \left\langle e^{i\varphi} \right|}{2\pi} \tag{4.3.32}$$

and use it to generate the phase operator $\hat{\varphi}$

$$\hat{\varphi} = \int\limits_{\varphi_0}^{\varphi_0 + 2\pi} \varphi \, d\Pi(\varphi) = \frac{1}{2\pi} \int\limits_{\varphi_0}^{\varphi_0 + 2\pi} \varphi \, d\varphi \, \left| e^{i\varphi} \right\rangle \left\langle e^{i\varphi} \right| \tag{4.3.33}$$

This is extremely useful! First, note that this is the first usable expression for a phase operator on $\mathscr{H}_\infty$ that we have found in this entire paper! Second, we may now calculate the expectation value of any function of φ, including higher moments, trigonometric functions etc simply by evaluating the corresponding integral with the expectation value of the POM

$$\langle F(\hat{\varphi}) \rangle = \int\limits_{\varphi_0}^{\varphi_0 + 2\pi} F(\varphi) \, \langle \psi \, | \, d\Pi(\varphi) | \, \psi \rangle \tag{4.3.34}$$

Inserting $F(\hat{\varphi}) = \hat{\varphi}^2$, we obtain

$$\left\langle \hat{\varphi}^2 \right\rangle = \int\limits_{-\pi}^{\pi} \varphi^2 \, \langle \psi \, | \, d\Pi(\varphi) | \, \psi \rangle \, , \tag{4.3.35}$$

where we have chosen the window $-\pi \leq \varphi \leq \pi$ for simplicity of calculation. Inserting a number state $|\psi\rangle = |n\rangle$ gives

$$\begin{aligned}
\left\langle \hat{\varphi}^2 \right\rangle &= \int\limits_{-\pi}^{\pi} \varphi^2 \, \langle n \, | \, d\Pi(\varphi) | \, n \rangle = \\[2mm]
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi^2 \, \left| \left\langle n | \, e^{i\varphi} \right\rangle \right|^2 \, d\varphi = \\[2mm]
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi^2 \, \left| e^{in\varphi} \right|^2 \, d\varphi = \\[2mm]
&= \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \varphi^2 \, d\varphi = \frac{\pi^2}{3} \, ,
\end{aligned} \tag{4.3.36}$$

Apparently, the SG-POM formalism passes the acid test! In fact, it has even been shown that it represents an ideal phase measurement [48]. This is interesting — didn't we show in section 3.1.4.1 that the SG formalism failed the acid test, because the SG-sine and cosine operators formed from the exponentials $\widehat{e^{i\varphi}}$ and $\widehat{e^{-i\varphi}}$ produced wrong second moments for the vacuum state? The answer is, in fact, that yes, we did!

The resolution to this apparent contradiction, first made explicit in [49], lies in a miscon-

ception about what exactly constitutes an ideal phase measurement. It has been shown that the SG-*POM*, and therefore the operator $\widehat{e^{i\varphi}}$, satisfy this quality, meaning that they have an a priori connection to phase measurement. But the same cannot be said of the $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$ operators, since they are but the real and imaginary components of $\widehat{e^{i\varphi}}$, which, although hermitian, do not have this a priori connection to ideal phase measurement. Instead, they have a connection to ideal *cosine/sine-of-phase* measurement (cf. on a related note [11]), but such measurement is not necessarily related to the ideal *phase* measurement. Therefore, there is no reason to expect that the phase statistics generated by the operator $\widehat{e^{i\varphi}}$ agree with the statistics of the operators $\widehat{\sin\varphi}$ and $\widehat{\cos\varphi}$.

To make this connection a little clearer, we calculate the second moment of the sine of the phase using the SG-POM (setting again $\varphi_0 = -\pi$ for ease of calculation) and compare it to the SG-sine operator:

$$
\begin{aligned}
\left\langle \sin^2 \hat{\varphi} \right\rangle &= \int_{-\pi}^{\pi} \sin^2 \varphi \left\langle \psi \,|\, \mathrm{d}\Pi(\varphi)|\, \psi \right\rangle = \\[2mm]
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\varphi \, \sin^2 \varphi \left| \left\langle e^{i\varphi} |\, \psi \right\rangle \right|^2 = \\[2mm]
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\varphi \, \sin^2 \varphi \sum_{n'=0}^{\infty} \sum_{n=0}^{\infty} e^{i(n'-n)\varphi} \left\langle \psi |\, n' \right\rangle \left\langle n |\, \psi \right\rangle = \\[2mm]
&= \frac{1}{2\pi} \frac{2\pi}{2} \left| \left\langle n |\, \psi \right\rangle \right|^2 = \\[2mm]
&= \left\langle \widehat{\sin\varphi}^2 \right\rangle + \frac{1}{4} \left| \left\langle 0 |\, \psi \right\rangle \right|^2 ,
\end{aligned}
\tag{4.3.37}
$$

where we have used in the second-to-last equality that the integral is equal to zero except if $n = n'$, and that for $n = n'$, we have a regular $\sin^2 x$ integration yielding π over the unit circle. The last equality is not a calculation, but simply the relation between the second moment calculated from $\sin\hat{\varphi}$ and that calculated from $\widehat{\sin\varphi}$. Here it becomes obvious that $\sin\hat{\varphi}$, which only exists in the sense of generalized measurements, and $\widehat{\sin\varphi}$ are two different operators possessing different higher moments, even though they both somehow describe the sine of the phase. But for phase measurement, the POM-formulation is the right one to choose.

4.3.2.3 The PB-POM

It now remains to be shown that the PB formalism is simply a different formulation of the SG-POM [18, 42, 46, 47]. In section 4.2.7, we have seen that in the truncated Hilbert

space $\mathscr{H}_{s+1}$, the following holds:

$$F(\hat{\varphi}_{\varphi_0}) = \sum_{m=0}^{s} F(\varphi_m) \, |\varphi_m\rangle\langle\varphi_m| \qquad (4.3.38)$$

This leads to a POM

$$\hat{\Pi}(\Delta) = \sum_{\Delta} |\varphi_m\rangle\langle\varphi_m| \qquad (4.3.39)$$

where Δ is an interval within $\varphi_0 \leq \varphi \leq \varphi_0 + 2\pi$. For physical states, the (weak) integral limit of this sum becomes

$$\hat{\Pi}(\Delta) = \lim_{s\to\infty} \sum_{\Delta} |\varphi_m\rangle\langle\varphi_m| = \qquad (4.3.40\text{a})$$

$$= \lim_{s\to\infty} \sum_{\Delta} \frac{1}{s+1} \sum_{n',n=0}^{s} \mathrm{e}^{i(n'-n)\varphi_m} \, |n'\rangle\langle n| = \qquad (4.3.40\text{b})$$

$$= \int_{\Delta} \frac{\mathrm{d}\varphi}{2\pi} \sum_{n',n=0}^{\infty} \mathrm{e}^{i(n'-n)\varphi_m} \, |n'\rangle\langle n| = \qquad (4.3.40\text{c})$$

$$= \int_{\Delta} \frac{\mathrm{d}\varphi}{2\pi} \, \left|\mathrm{e}^{i\varphi}\right\rangle\left\langle\mathrm{e}^{i\varphi}\right| \qquad (4.3.40\text{d})$$

as calculated by [42, 46] in making the substitutions $\varphi_m = \varphi_0 + 2\pi m/(s+1) \rightarrow \varphi$, $2\pi/(s+1) \rightarrow \mathrm{d}\varphi$. Setting finally $\Delta \rightarrow \varphi$, we have

$$\mathrm{d}\Pi(\varphi) = \frac{\mathrm{d}\varphi}{2\pi} \, \left|\mathrm{e}^{i\varphi}\right\rangle\left\langle\mathrm{e}^{i\varphi}\right| , \qquad (4.3.41)$$

which is equal to the POM generated from the SG-formalism

4.3.2.4 Conclusion

As we see, the limit $s \to \infty$ yields the SG-POM. The fact that the integral POM is only the weak limit of the discrete POM in $\mathscr{H}_{s+1}$ does not seem to be problematic, since we do not expect to take any powers of the POM itself, only of the functions it is associated with. Then, by the definition of weak convergence, the expectation values of the POM in $\mathscr{H}_{s+1}$ and $\mathscr{H}_{\infty}$ are equal and we may use the POM generated by the phase eigenstates $|\varphi\rangle$ analogously to the Susskind/Glogower POM.

As an aside, note that Vaccaro and Pegg have actually calculated that not only does the PB-POM converge to the SG-POM, but all operator functions of the PB-phase operator also converge independently to the corresponding SG operator functions when interpreted in the POM sense [42].

Several important conclusions arise from the equivalence of the SG-POM and the PB-POM. First, and this conclusion is quite drastic, we must ask whether the PB formalism

really has any merit. After all, the PB formalisms main advantage was supposed to be the hermiticity of its phase operator, which was supposed to give a natural description of phase. However, any such advantages only exist in the subspace $\mathscr{H}_{s+1}$, and the phase is definitely not a normal quantum observable in $\mathscr{H}_\infty$. This is illustrated most simply be the fact that the projection measurement hypothesis cannot be fulfilled, given that the phase states are not physically accessible states. A consequence of this inaccessibility is that precise phase measurement is impossible, and that therefore, only fuzzy measurements can be made. But fuzzy measurements are already fully described by the SG-POM-formalism, which seems to obviate the need for any theory beyond the Susskind/Glogower exponentials.

One aspect which can be put in favor of the PB theory is that the notion of a POM arises naturally in that context, since it is the limit of projection-valued operator measures describing a hermitian phase distribution, whereas the SG-POM has to rely on results from generalized measurement theory [46]. However, the difficulty of justifying POMs only seems to be replaced by the difficulty of justifying the limit.

A second conclusion that arises concerns the PB acid test. The validity of this test as a means of evaluating the quality of a phase theory is severely put into question [46, 49], since the SG formalism, which fails the acid test, nonetheless produces a probability-operator measure which gives a satisfactory description of phase.

4.4 Critique of the PB formalism

Having now gained a slightly more thorough understanding of the PB phase operator, we may proceed to describe two major criticisms that have been voiced against the PB phase operator. Pegg and Barnett and their collaborators have for the most part dismissed these criticisms and continue to insist that their formalism solves the quantum phase problem [36]. But especially the two criticisms presented in the following sections raise very valid points, and Pegg and Barnett have not been able to do away with all of them. The problems we speak of are related to the cyclical definition of the phase operator in the $s + 1$-dimensional Hilbert space $\mathscr{H}_{s+1}$ (subsection 4.4.1) and to the calculation of the variance of the phase operator for physical states (subsection 4.4.2).

4.4.1 Problems with cyclical ladder operators

The first criticism was raised by Vorontsov and Rembovsky [50] and exposes problems in the limiting process stemming from the cyclicality of the phase operator, i.e. from the fact that

$$\widehat{e^{i\varphi}} |0\rangle = e^{i(s+1)\varphi_0} |s\rangle \tag{4.4.1}$$

In short, using generalized measurement theory, Vorontsov shows that any "good" phase measurement, i.e. any measurement which measures a finite phase window, causes states in the vicinity of the measurement to become excited. For low photon-number states, this may lead to states of the order $|s\rangle, |s-1\rangle$ etc to be excited because of relation 4.4.1, rendering the limiting process impossible. To see this [50], consider again the probability distribution arising for some initial state $|f\rangle$, which may also be written as the trace of a density matrix

$$P(\varphi_m)_f = \frac{1}{2\pi} |\langle \varphi_m | f \rangle|^2 = \sum_{m=0}^{s} \langle f | \varphi_m \rangle \langle \varphi_m | f \rangle = \mathrm{Tr}(\hat{\rho}_f \hat{\Pi}_{\varphi_m}) \qquad (4.4.2)$$

General measurement theory now provides an analog of the projection operator for measurement which is called the reduction operator $\hat{R}$, which for the phase operator has the form

$$\hat{R}(\tilde{\varphi}) = \sum_{m=0}^{s} \sqrt{w(\tilde{\varphi}|\varphi_m)} \, |\varphi_m\rangle\langle\varphi_m| \qquad (4.4.3)$$

where $\tilde{\varphi}$ is the measured value and $w(\tilde{\varphi}|\varphi_m)$ is a device function which emulates the projection operator (for hermitian observables, $w(\tilde{\varphi}|\varphi_m)$ simply becomes the Kronecker delta or the delta function). The density matrix of the system after measurement has taken place can then be described as

$$\hat{\rho}_g = \frac{\hat{R}^\dagger(\tilde{\varphi}) \hat{\rho}_f \hat{R}(\tilde{\varphi})}{\mathrm{Tr}(\hat{\rho}_f \hat{\Pi}_{\varphi_m})} \qquad (4.4.4)$$

Evaluating the photon-number probability distribution of this new state yields

$$P(n)_g = |\langle n | g \rangle|^2 = \mathrm{Tr}(\hat{\rho}_g \hat{\Pi}_n) = \frac{\sum_{n=0}^{\infty} \langle n | \hat{R}^\dagger(\tilde{\varphi}) \rho_f \hat{R}(\tilde{\varphi}) | n \rangle\langle n | n \rangle}{\mathrm{Tr}(\hat{\rho}_f \hat{\Pi}_{\varphi_m})} = \qquad (4.4.5\mathrm{a})$$

$$= \frac{\sum_{m',m=0}^{s} \left\langle n \left| \sqrt{w(\tilde{\varphi}|\varphi_m)} \right| \varphi_m \right\rangle \langle \varphi_m | \hat{\rho}_f | \varphi_{m'} \rangle \left\langle \varphi_{m'} \left| \sqrt{w(\tilde{\varphi}|\varphi_m)} \right| n \right\rangle}{\mathrm{Tr}(\hat{\rho}_f \hat{\Pi}_{\varphi_m})} \qquad (4.4.5\mathrm{b})$$

So far, all is well, but problems arise when we try to measure the phase of a single number state $|n_0\rangle$ and then calculate its photon-number probability distribution ($\rho_f = |n_0\rangle\langle n_0|$):

$$P(n)_g = \frac{\sum_{m',m=0}^{s} \left\langle n \left| \sqrt{w(\tilde{\varphi}|\varphi_m)} \right| \varphi_m \right\rangle \langle \varphi_m | n_0 \rangle\langle n_0 | \varphi_{m'} \rangle \left\langle \varphi_{m'} \left| \sqrt{w(\tilde{\varphi}|\varphi_m)} \right| n \right\rangle}{\mathrm{Tr}(|n_0\rangle\langle n_0 | \varphi_m \rangle\langle \varphi_m |)} = \qquad (4.4.6\mathrm{a})$$

$$= \frac{\left| \sum_{m=0}^{s} \left\langle n \left| \sqrt{w(\tilde{\varphi}|\varphi_m)} \right| \varphi_m \right\rangle \langle \varphi_m | n_0 \rangle \right|^2}{|\langle n_0 | \varphi_m \rangle|^2} = \qquad (4.4.6\mathrm{b})$$

$$= \frac{\left| \sum_{m=0}^{s} \sqrt{w(\tilde{\varphi}|\varphi_m)} \, \mathrm{e}^{i(n-n_0)2\pi m/(s+1)} \right|^2}{|\langle n_0 | \, \varphi_m \rangle|^2} \tag{4.4.6c}$$

The result is a discrete Fourier transform of the periodic function $\sqrt{w(\tilde{\varphi}|\varphi_m)}$ around n_0. If the measuring function is reminiscent of a projection, i.e. if it is rectangularly shaped and narrow, then the Fourier-transform gives a function of the general form $\sin(x)/x$. This means that several states around $|n_0\rangle$ will also excited by the measuring process (for a phase projection, each number state would be equally excited, leading to full uncertainty in number and an exact value for the phase).

However, this is a problem if the number states are defined cyclically like in the PB-formalism. Consider that $n_0 = 1$, then several states in the vicinity of $|1\rangle$ will also be excited, such as $|0\rangle, |s\rangle, |s-1\rangle$. The problem quickly becomes apparent when trying to take the limit $s \to \infty$, since the measuring apparatus can never produce the infinite energy needed to excite the uppermost number states as it should according to the PB formalism.

The causes a change of perspective. Might it not be, as PB have tried to convince us in subsection 4.3.2.3, that the PB formalism justifies the SG-POM formalism, but instead that the latter justifies the former? After all, the cyclical representation of the number states in $\mathscr{H}_{s+1}$ space wreaks havoc on the energy distribution after a measurement, making it seem implausible to be able to measure any states in the $\mathscr{H}_{s+1}$ subspace with any physical sense, even if they are physical states, i.e. states for which the highest excited number state $n \ll s$. There would then a priori be no reason to expect the limit to converge to an ideal phase measurement, and the only reason why we do expect it to converge *in fact* is because we know that it is equivalent to the SG-POM formalism.

4.4.2 Problems with the limits

Another problem with the PB formalism that has often been pointed out [27, 38, 46, 51] also concerns its limiting procedure, and it implies again that the PB formalism as initially introduced in [15, 30] may not reflect an ideal phase measurement, ironically acquiring this property only once it has been approximated to the SG-POM. In this case, the problem lies in all equations where a sum over $s + 1$ is replaced by a continuous integral, such as in the calculations for the partial phase states in section 4.2.8. Note that this also affects the calculation of the variance of the number states, meaning that the PB-phase operator may even fail its own acid test.

Basically, the argument [27, 38, 51] goes that the limit of a sum such as

$$\left\langle \widehat{F(\varphi)} \right\rangle_n = \lim_{s \to \infty} \sum_{m=0}^{s} F(\varphi) \, |\langle \varphi_m | n \rangle|^2 = \lim_{s \to \infty} \sum_{m=0}^{s} F(\varphi) \frac{1}{s+1} \sum_{n',n=0}^{s} \mathrm{e}^{i(n'-n)\varphi_m} \tag{4.4.7}$$

cannot simply be taken to be ($\varphi_m \to \varphi$, $2\pi/(s+1) \to \mathrm{d}\varphi$):

$$\left\langle \widehat{F(\varphi)} \right\rangle_n = \int\limits_{-\pi}^{\pi} F(\varphi) \frac{\mathrm{d}\varphi}{2\pi} \sum_{n',n=0}^{\infty} \mathrm{e}^{i(n'-n)\varphi} \tag{4.4.8}$$

as it could be in a real-valued integral. Instead, complex integration may only be approximated in this way if a sufficient amount of residues are found that allow an integral representation of the sum as a sum of residues [38]. This is the case for the functions $F(\varphi) = \varphi$ and $F(\varphi) = 1$, but in general not for any other functions. In these functions, complicated additional terms arise, so that the integral must really be written as

$$\left\langle \widehat{F(\varphi)} \right\rangle_n = \int\limits_{-\pi}^{\pi} F(\varphi) \frac{\mathrm{d}\varphi}{2\pi} \sum_{n',n=0}^{\infty} \mathrm{e}^{i(n'-n)\varphi} + \mathscr{D}_n^{(F)}, \quad \begin{cases} \mathscr{D}_n^{(F)} = 0 & F(\varphi) = \varphi, F(\varphi) = 1 \\ \mathscr{D}_n^{(F)} \neq F & \text{otherwise} \end{cases}$$
$$\tag{4.4.9}$$

The $\mathscr{D}_n^{(F)}$-terms are very hard to calculate [8], but it seems likely that they are not all equal to zero. The implication of this is that any higher moments of $\hat{\varphi}_{\varphi_0}$ as well as any general functions of $\hat{\varphi}_{\varphi_0}$ do not yield the correct expectation value when the limit is taken first. This could only be guaranteed if the continuum limit of going from the exact sum to the integral were made rigorous. One way in which this could be done is to take the continuum limit only for $F(\varphi) = 1$ to obtain the PB-POM, and then to apply the axioms of generalized measurement theory to calculate the expectation values of higher moments and arbitrary functions with the SG/PB-POM formalism.

But then, the use of the PB formalism is rather marginal, given that now, it does not present an alternative justification for the use of the SG-POMs anymore, as Vaccaro and Pegg purport it to do [42]: The fact that only the zeroth and first order of the SG-POM description are equivalently obtained from the PB approach means that the application of the PB-POM to generalized functions of phase also has to rely on the axioms of generalized measurement theory.

4.5 Conclusion

The results obtained in these sections are very important because they provide some deeper insights into the nature of phase measurement. First, we have examined the PB formalism as initially introduced by Pegg and Barnett, which has some very pleasant properties and seems very well-behaved at first glance. Unfortunately, more profound considerations about the assumptions supporting the formalism show that its use may be limited. The main problems of the PB hermitian phase operator include weak convergence on $\mathscr{H}_\infty$, difficulties with the limiting process, unphysical results stemming from the cyclicality of the number operator, and several other quirks.

The difficulties of the PB phase operator are perhaps symptomatic: by now, it should have become clear that phase is not an observable in the same category as number, position and momentum, and that this needs to be accepted. After all, it has been shown several times, and even acknowledged by PB, that precise (i.e. projection-valued) phase measurement is impossible because it leads to divergent phase states, and we must therefore conclude that phase is a fuzzy observable.

But this clues us in to several interesting observations if we compare phase to another fuzzy variable we already know: the parameter α of the coherent states. We note several interesting analogies. First, the operator corresponding to the phase exponential $\widehat{e^{i\varphi}}$ is the annihilation operator $\hat{a}$. Both these operators are non-hermitian, but can be decomposed into hermitian operators $\widehat{e^{i\varphi}} = \widehat{\cos\varphi} + i\widehat{\sin\varphi}$; $\hat{a} = \hat{x} + i\hat{p}$. We may note that these similarities qualitatively follow from the fact that $\widehat{e^{i\varphi}}$ is a normalized version of $\hat{a}$.

Second, we can observe that fuzzy measurement of α is possible, and that the non-commuting nature of $[\hat{x}, \hat{p}] \neq 0$ does not prevent this measurement. Analogously, we conclude that φ may be measured even though $[\widehat{\sin\varphi}, \widehat{\cos\varphi}] \neq 0$. The formalism to conduct this measurement is the probability-operator measure (SG-POM), which allows measurement of the expectation values of arbitrary functions of φ and therefore yields a full description of the phase properties.

As an additional bonus, the SG-POM represents an (almost) canonical phase distribution[30], i.e. even though it is impossible to find a hermitian operator for φ, φ and n are canonical in the sense that one variable can be isolated from the other. This becomes apparent when the SG-POM is operated on by unitary phase and number shift operators, where it turns out that the phase shift leads to a phase offset in the phase distribution, while the number shift operator leaves the phase probability distribution untouched, as long as special care is taken to respect the one-sided unitarity of the number shift operator.

We therefore consider the SG/PB-POM formalism to be the most useful of the phase operator related formalisms introduced thus far, and treat it as a benchmark when examining the adequacy of such formalisms in general. In the next chapter, we will evaluate whether such a benchmark position is experimentally justified.

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[30]Almost because the number shift operator $\widehat{e^{i\varphi}}$ is only one sided unitary

Chapter 5

Experimental data

In this chapter we take a much-needed look at some experimental data and contrast this data to the quantum phase theories we have encountered so far. Recall that we already noted in the introduction how the treatment of quantum phase was remarkably devoid of experimental verification, allowing almost a dozen partially contradicting theories to remain virtually unchallenged for the better part of a century.

The Noh, Fougères and Mandel experiments [52–55] have brought a welcome reversal of trends to this rather unproductive state of affairs. Although isolated experiments had been conducted before, e.g. by Gerhardt et al [12, 13], these were either so non-specific or so limited in resolution that a falsification of any of the then prevalent theories was practically impossible, while new theories needed little modification to comply with the available results. Noh, Fougères and Mandel have changed this by conducting a rather rigorous experimental procedure whose precise results have, finally, brought discredit on several of the operator-based phase theories, including, surprisingly, the PB-formalism. At the same time, some justification has been provided for the less abstract approaches such as those involving quasiprobability distributions.[31].

Unfortunately, no definitive conclusions have been drawn from these results. Pegg, Barnett and Vaccaro continue to insist that their operator is the true solution to the phase problem, recently having published a review of the quantum phase problem which comprises their own papers virtually exclusively [36]. And ironically, Noh et al's results have not only failed to settle the feud between the PB-camp and the quasiprobability camp, but Noh et al have appended their own interpretation of quantum phase to an already confused field that is in desperate need of decluttering.

In the following, we will focus primarily on Noh et al's experimental results and less on their phase description, although we will briefly touch on the latter, too.

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[31]Quasiprobability distributions are also called measured phase distributions, but their treatment is unfortunately beyond the scope of this book. Some literature about quasiprobability distributions in general includes [56, 57], while phase-specific treatments are given in [8, 11, 47, 58–63]

5.1 The NFM experiment

The experiment conducted by Noh, Fougères and Mandel is a high precision homodyne phase measurement scheme measuring the phase difference between two inputs derived from the same initial source. Although it is not an absolute phase measurement *stricto sensu*, it approaches absolute phase measurement if one of the inputs is set up to be a highly coherent, strong local oscillator.

5.1.1 Setup and measurement

Noh et al propose two measuring schemes, one of which allows simultaneous measurement of the sine and cosine of the phase difference between the two inputs, and one of which does not. We focus here on the one which does allow such simultaneous measurement, and which is shown in figure 5.1. As we see, the two inputs (each consisting of a coherent state/superposition of coherent states and a vacuum state) are duplicated through beam splitters and led to two sets of detectors D_3 through D_6, where interference measurements are conducted via additional beam splitters. Additionally, one of the paths leading to the second set of detectors contains a $\lambda/4$ phase shift, which causes detectors D_5 and D_6 to measure sine instead of cosine.
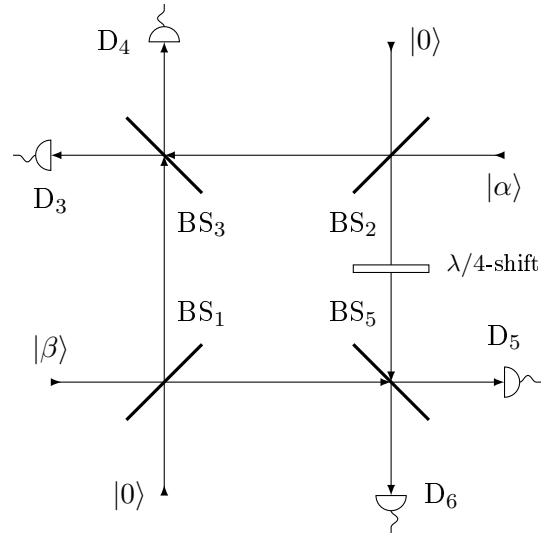
Figure 5.1: The measurement scheme used by Noh et al.

The detectors measure counting rates n_j which can be turned into sine and cosine measurements of the phase difference as follows [8, 55]:

$$\cos_M(\varphi_2 - \varphi_1) = \frac{n_4 - n_3}{[(n_4 - n_3)^2 + (n_6 - n_5)^2]^{1/2}} \tag{5.1.1a}$$

$$\sin_M(\varphi_2 - \varphi_1) = \frac{n_6 - n_5}{[(n_4 - n_3)^2 + (n_6 - n_5)^2]^{1/2}} \tag{5.1.1b}$$

5.1.2 The NFM operational phase

To explain these counting rates quantum mechanically, Noh, Fougères and Mandel's introduced the so-called concept of "operational phase". This is supposed to express that the operators which optimally describe a measurement do not exist independently of that measurement, but must instead be sought and found individually for each experimental setup or "operation".

Indeed, for the eight-port homodyne measurement, the operators that describe the measurement are uncannily similar to equations (5.1.1a) and (5.1.1b) describing the photon-counts at detectors D_3 through D_6:

$$\hat{C}_M = \frac{\hat{n}_4 - \hat{n}_3}{[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2]^{1/2}} \tag{5.1.2a}$$

$$\hat{S}_M = \frac{\hat{n}_6 - \hat{n}_5}{[(\hat{n}_4 - \hat{n}_3)^2 + (\hat{n}_6 - \hat{n}_5)^2]^{1/2}} \cdot \tag{5.1.2b}$$

From these operators, NFM are able to calculate all relevant quantum mechanical figures in impressive agreement with their experimental data.

Obviously, such a direct reliance on the experimental setup is uncomforting and has very serious implications for our understanding of quantum phase. We will, however, not delve more deeply into these problems for two reasons (for more information on those who will, cf. [59]): First, our focus is on quantum phase *theory* and therefore we do not gain much by surrendering to the idea that an abstract concept of phase does not exist independently of any specific measurement. Second and more importantly, there are two ways to circumvent the daunting conclusion reached by Noh, Fougères and Mandel, which will briefly be presented in sections 5.2 and 5.3.

5.1.3 NFM and the SG/PB-POM formalism

We now want to know how the SG/PB-POM formalism holds up against the experimental results. Luckily, such a comparison has already been executed Noh et al., who fed the setup in fig. 5.1 with two coherent states, the first of which had a mean photon number ranging from $\langle n_1 \rangle = 0,01 \cdots 30$ and the second of which (the reference beam) was fixed at $\langle n_2 \rangle = 50$. They then plotted the operator cosine expectation value against the average photon number of the weaker beam and compared this data to the theoretical values calculated using the SG/PB-POM and their own interpretation of quantum phase (cf. figure 5.2, taken from [53]).

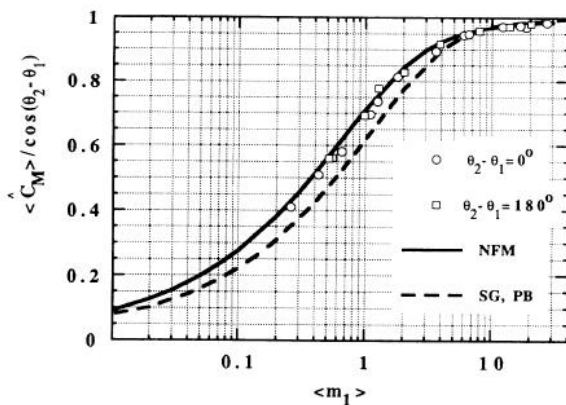**Figure 5.2:** Comparison between the results measured and predicted in the NFM-experiment with the corresponding results obtained via the SG/PB-POM formalism. Taken from [53]

Unsurprisingly, NFM's operational phase describes the experiment beautifully. Unfortunately the same can not be said of the SG/PB-POM formalism, which consistently undershoots the data. Given that the measurements conducted by Noh et al are very precise

and the error bars therefore very small, this shows that the SG/PB-POM is incompatible at least with the eight-port homodyne measurement shown above in figure 5.1!

5.2 Connections to quasiprobability distributions

In 1993, Freyberger, Vogel and Schleich as well as Leonhardt and Paul introduced several papers in direct response to the NFM experiments [63–66], in which they posited that the NFM-phase distribution could be written as the radial integration of the Q-function of the measured state. This would mean that the experimental data of NFM could be explained within the theoretical framework of the area-of-overlap approach in phase-space, thus obviating the need for "operational phase" as proposed by NFM, and bridging the divide between theory and experiment that NFM's results threatened to introduce into the quantum phase discussion. Unfortunately, the treatment of this approach is beyond the scope of this thesis, but it is presented in more than sufficient detail in the papers just cited in this paragraph.

5.3 Optical homodyne tomography (OHT)

We make one more minor digression before finally proceeding to the conclusion of this chapter. Smithey et al. have developed another experimental procedure for measuring phase distributions which, similar to medical imaging, they call optical homodyne tomography, or OHT for short [8, 11]. We do not go into the details of this contribution, but note that OHT is a procedure that permits a direct measurement of the Wigner function of a given state, from which all relevant quantum physical data can be calculated.

What is interesting about this is that each of the theories we have encountered so far (SG-formalism, PB-formalism) can be used to explain the experiment. This is because the formalism employed leads to different experimental phase distribution resulting from the raw data, and each of these experimental phase theories is in excellent agreement with its respective theory. The interested reader is referred here to the original paper [11].

5.4 Conclusion

Unfortunately, we have to conclude this section by noting that the experiments conducted thus far have not been able to decide decisively in favor of one or another phase theory. It is, however, encouraging that NFM's conclusion on the impossibility of finding one unifying abstract phase operator is faulty, since the NFM data can be directly explained by a variant of the area-of-overlap approach and does not unequivocally contradict the SG/PB-POM formalism.

Chapter 6

Summary of theoretical framework

We have come quite a long way since the beginning of this thesis. Starting from Dirac's first attempt of a phase description in analogy to the position and momentum operators, which was derived from a polar decomposition of the destruction operator, we have seen the SG formalism, determined the reasons for the awkward behavior of the proposed phase observable, proved that an hermitian phase operator cannot exist, have nonetheless found one, albeit in a truncated Hilbert space, have met the PB formalism and reviewed some of the criticisms voiced against it, generalized it to the SG/PB-POM formalism, and finally took a look at available experimental data.

Before we proceed to the last two chapters, we now briefly take time to resume the current state of quantum phase theory by summarizing the essentials of the phase operator approach, where the SG/PB-POM formalism seemed to afford the most reasonable description.

Remember that we started from a rescaled annihilation operator which we denoted $\widehat{e^{i\varphi}}$

$$\widehat{e^{i\varphi}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \ .$$

(6.0.1)

We used the eigenstates generated by this operator

$$\left|e^{i\varphi}\right\rangle = \sum_{n=0}^{\infty} e^{in\varphi} |n\rangle$$

(6.0.2)

to define a probability operator measure

$$d\Pi(\varphi) = \frac{d\varphi}{2\pi} \left|e^{i\varphi}\right\rangle\left\langle e^{i\varphi}\right|$$

(6.0.3)

which resolved to the identity

$$\int_{\varphi}^{\varphi_0+2\pi} d\Pi(\varphi) = \int_{\varphi}^{\varphi_0+2\pi} \frac{d\varphi}{2\pi} \left|e^{i\varphi}\right\rangle\left\langle e^{i\varphi}\right| = \mathbb{1} \ .$$

(6.0.4)

Using this probability operator measure, we were able to define a non-hermitian phase operator

$$
\hat{\varphi} = \int\limits_{\varphi_0}^{\varphi_0+2\pi} \varphi \, \mathrm{d}\Pi(\varphi) = \frac{1}{2\pi} \int\limits_{\varphi_0}^{\varphi_0+2\pi} \varphi \, \mathrm{d}\varphi \, \left|\mathrm{e}^{i\varphi}\right\rangle\!\left\langle\mathrm{e}^{i\varphi}\right| \tag{6.0.5}
$$

which offered a complete description of phase because it allowed the calculation of the phase probability distribution for arbitrary states $|\psi\rangle$

$$
P(\varphi) = \frac{1}{2\pi} \left|\left\langle\mathrm{e}^{i\varphi}\,\middle|\,\psi\right\rangle\right|^2 \tag{6.0.6}
$$

and the calculation of expectation values of arbitrary functions of phase as

$$
\langle F(\hat{\varphi})\rangle = \int\limits_{\varphi_0}^{\varphi_0+2\pi} F(\varphi) \, \langle \psi \,|\, \mathrm{d}\Pi(\varphi)| \, \psi \rangle \;. \tag{6.0.7}
$$

The SG/PB-POM consolidated approach just summarized comes with the several pleasant characteristics that higher moments of the phase are consistent with the phenomenologically expected higher moments (so-called "acid test"), and that because of the possibility to calculate arbitrary functions of phase with formula (6.0.7), we dispose over maximum flexibility that the initial SG-formalism was unable to provide us with. The only (expected) caveat lies in the fact that exact phase measurement is impossible, as the eigenkets of the exponential phase operator (lowering operator) are overcomplete and therefore not orthogonal, yielding a residual fuzziness that violates the projection hypothesis and leads to the possibility that repeated measurements of phase yield different results. This we can live with, since the practical value of the projection hypothesis is debatable at best.

Chapter 7

Some theory on uncertainty relations

Having resumed the state of affairs in quantum phase in the previous chapter, this chapter will be the last theoretical excursion before we finally continue to conclusions and discussion. The purpose of this chapter is twofold: on the one hand, it combines the results concerning quantum uncertainties that arose in the text thus far into a consolidated treatment with respect to the SG/PB-formalism. On the other hand, it explores some additional concepts that have not thus far been reviewed, including some observations on the nature of uncertainty relations in general.

The chapter will be divided into two sections, the first dealing with a supposed uncertainty relation between number and phase, and the second focusing on phase uncertainty and examining the maximum resolution of phase measurements, which, given that phase eigenstates are held to be physically inaccessible, will be seen to be finite.

7.1 The phase-number uncertainty relation

In this section, we examine the justification of the empirical result

$$\Delta \hat{n} \cdot \Delta \hat{\varphi} \geq \tfrac{1}{2} \tag{7.1.1}$$

by uncertainty-theoretic arguments followed by an application of the formalisms we have described in chapters 3-4.

7.1.1 Heisenberg uncertainty

In general physics, the Heisenberg formulation of uncertainty is the commonly accepted canon. We recall that according to this formulation, for any operators $\hat{A}$ and $\hat{B}$, we have

$$\Delta \hat{A} \cdot \Delta \hat{B} \geq \tfrac{1}{2} \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right| , \tag{7.1.2}$$

which follows rather straightforwardly from the definition of the variance and the Cauchy-Schwarz inequality. In the case of the position and momentum operators, this simplifies to

$$\Delta \hat{x} \cdot \Delta \hat{p} \geq \frac{\hbar}{2} \tag{7.1.3}$$

since $[\hat{x}, \hat{p}] = i\hbar$. This is a ubiquitously known expression which motivated the search for an equivalent number-phase relation

$$\Delta \hat{n} \cdot \Delta \hat{\varphi} \geq \frac{1}{2} \qquad \text{(wrong).} \tag{7.1.4}$$

At this point, there are two questions to ask: First, how do we adapt the Heisenberg formulation to periodic variables measured in angular coordinates, such as phase? And second, is a root-mean-square measure really the optimal way to measure phase uncertainty? Both will be dealt with in the following subsections.

7.1.2 Derivation using generalized coordinates

First, we will expand the Heisenberg formulation of uncertainty to generalized coordinates [67] in order to be able to integrate periodicity into the uncertainty model. To this end, we define generalized coordinates for the position and momentum variables q_l and $\hat{p}_l$, where $\hat{p}_l$ and q_l are related such that [68]

$$\hat{p}_l = -i\hbar \left(\frac{\partial}{\partial q_l} + \frac{1}{2w} \frac{\partial w}{\partial q_l} \right) \tag{7.1.5}$$

with $w = \sqrt{g}$ and g the elements of the metric tensor G_{ik}. We introduce a wave function ψ and use the well-known fact that the integral over a modulus squared is necessarily greater than or equal to zero to obtain

$$I(\xi) = \int |\xi(q_l - \langle q_l \rangle)\psi + i(\hat{p}_l - \langle \hat{p}_l \rangle)\psi|^2 \, d\tau \geq 0 \,. \tag{7.1.6}$$

ξ is an arbitrary real parameter and the integration is conducted over the entire space of the variables q_l. Expanding expression (7.1.6) yields

$$I(\xi) = \int \left[\xi^2 (q_l - \langle q_l \rangle)^* \psi^* (q_l - \langle q_l \rangle)\psi - i(\hat{p}_l - \langle \hat{p}_1 \rangle)^* \psi^* \xi (q_l - \langle q_l \rangle)\psi + \right. \tag{7.1.7}$$
$$\left. + i\xi(q_l - \langle q_l \rangle)^* \psi^* (\hat{p}_l - \langle \hat{p}_1 \rangle)\psi + (\hat{p}_l - \langle \hat{p}_1 \rangle)^* \psi^* (\hat{p}_l - \langle \hat{p}_1 \rangle)\psi \right] d\tau \geq 0 \,,$$

and with

$$\int \hat{A}^* \psi^* \hat{A}\psi \, d\tau = \left\langle A^2 \right\rangle \tag{7.1.8}$$

as well as the definitions $(\Delta q_l)^2 = \left\langle (q_l - \langle q_l \rangle)^2 \right\rangle$ and $(\Delta \hat{p}_l)^2 = \left\langle (\hat{p}_l - \langle \hat{p}_l \rangle)^2 \right\rangle$ we finally obtain

$$
I(\xi) = \xi^2 (\Delta q_l)^2 + \xi \hbar \int \left[\left(\frac{\partial}{\partial q_l} + \frac{1}{2w}\frac{\partial w}{\partial q_l} - \left\langle \frac{\partial}{\partial q_l} \right\rangle - \left\langle \frac{1}{2w}\frac{\partial w}{\partial q_l} \right\rangle \right) \psi^*(q_l - \langle q_l \rangle)\psi + \right.
$$
$$
\left. + (q_l - \langle q_l \rangle)^* \psi^* \cdot \left(\frac{\partial}{\partial q_l} + \frac{1}{2w}\frac{\partial w}{\partial q_l} - \left\langle \frac{\partial}{\partial q_l} \right\rangle - \left\langle \frac{1}{2w}\frac{\partial w}{\partial q_l} \right\rangle \right) \psi \right] d\tau + (\Delta \hat{p}_l)^2 \geq 0 \,,
$$
$$ \tag{7.1.9} $$

or for short

$$
I(\xi) = \xi^2 (\Delta q_l)^2 - \xi \hbar J + (\Delta \hat{p}_l)^2 \geq 0 \,. \tag{7.1.10}
$$

The middle integral can be integrated by parts [67] and yields

$$
J = 1 - \int \prod_{i \neq l}^{n} dq_i \left. \left[(q_l - \langle q_l \rangle)w\psi\psi^* \right] \right|_a^b \,, \tag{7.1.11}
$$

where (a, b) is the interval on which the generalized variable q_l is defined. We now rearrange the inequality (7.1.10) as follows

$$
\xi(\Delta q_l)^2 + \frac{1}{\xi}(\Delta \hat{p}_l)^2 \geq \hbar J \tag{7.1.12a}
$$

$$
\Leftrightarrow \quad \xi^2 (\Delta q_l)^4 + \frac{1}{\xi^2}(\Delta \hat{p}_l)^4 + 2(\Delta \hat{q}_l)^2 (\Delta \hat{p}_l)^2 \geq \hbar^2 J^2 \tag{7.1.12b}
$$

$$
\Leftrightarrow \quad \left[\xi(\Delta q_l)^2 - \frac{1}{\xi}(\Delta \hat{p}_l)^2 \right]^2 + 4(\Delta \hat{q}_l)^2 (\Delta \hat{p}_l)^2 \geq \hbar^2 J^2 \tag{7.1.12c}
$$

$$
\Leftrightarrow \quad 4(\Delta \hat{q}_l)^2 (\Delta \hat{p}_l)^2 \geq \hbar^2 J^2 \,, \tag{7.1.12d}
$$

where the last line follows from the fact that we have derived equation (7.1.10) for arbitrary ξ, and that therefore we can always find a ξ which causes $\left[\xi(\Delta q_l)^2 - \xi^{-1}(\Delta \hat{p}_l)^2 \right]^2$ to vanish. Taking the square root and rearranging gives the well-known Heisenberg formulation plus an extra term describing the metric:

$$
\Delta \hat{q}_l \cdot \Delta \hat{p}_l \geq \tfrac{1}{2}\hbar \left| 1 - \int \prod_{i \neq l}^{n} dq_i \left. \left[(q_l - \langle q_l \rangle)w\psi\psi^* \right] \right|_a^b \right| \tag{7.1.13}
$$

For linear cartesian coordinates, this simplifies to the well-known

$$
\Delta \hat{x} \cdot \Delta \hat{p} \geq \tfrac{1}{2}\hbar \,. \tag{7.1.14}
$$

But for the phase-number uncertainty, we need to explicitly calculate the right-hand-side of eq. (7.1.13). We note that since $\varphi = q_l$ is the only degree of freedom, $\prod_{i \neq l}^{n} w\, dq_i = 0$

and we may drop the integral, calculating only $\left[(\varphi - \langle\varphi\rangle)\psi\psi^*\right]\big|_0^{2\pi}$.

$$\left[(\varphi - \langle\varphi\rangle)\psi\psi^*\right]\big|_0^{2\pi} = (2\pi - \langle\varphi\rangle)\,|\psi(2\pi)|^2 - (0 - \langle\varphi\rangle)\,|\psi(0)|^2 = 2\pi\,|\psi(2\pi)|^2 \ . \qquad (7.1.15)$$

This leads to the uncertainty relation

$$\Delta\hat{\varphi} \cdot \Delta\hat{n} \geq \left|1 - 2\pi\,|\psi(2\pi)|^2\right| \ , \qquad (7.1.16)$$

which is in agreement with phenomenological observations.

It is important, however, to note that this uncertainty relation is intrinsically approximate, since - in marked contrast to the case of angle and angular momentum operators treated in [67] - a reasonably well-behaved phase operator does not share a canonic commutation relation with the number operator (cf. 3.3.2) and therefore, eq. (7.1.5) does not apply but approximately.

7.1.3 Derivation using PB-formalism

Several papers corroborate that the expression for the number-phase uncertainty derived above is reasonable by comparing it to the uncertainty relation arising from the PB-formalism [15, 69]. This is an important display of consistency in quantum phase operator theory, since there is no a priori reason to expect these results to be equal, as they are relying on different assumptions (a similar case of such unexpected connections was the proof of equality of the SG-POM and the PB-POM).

We remember that in section 4.2.6, we derived in passing that

$$\Delta\hat{n} \cdot \Delta\hat{\varphi}_{\varphi_0} \geq \frac{1}{2}\left|1 - 2\pi P(\varphi_0)\right| \ . \qquad (7.1.17)$$

This is equal to eq. (7.1.16) because we can modify eq. (7.1.15) to incorporate a reference phase φ_0 as follows

$$\left[(\varphi - \langle\varphi\rangle)\psi\psi^*\right]\big|_{\varphi_0}^{2\pi+\varphi_0} = (2\pi + \varphi_0 - \langle\varphi\rangle)\,|\psi(2\pi + \varphi_0)|^2 - (\varphi_0 - \langle\varphi\rangle)\,|\psi(\varphi_0)|^2 =$$
$$= 2\pi\,|\psi(\varphi_0)|^2 = 2\pi P(\varphi_0) \ . \qquad (7.1.18)$$

We used here that by periodicity, $|\psi(2\pi + \varphi_0)|^2 = |\psi(\varphi_0)|^2$, and that $|\psi(\varphi_0)|^2 = P(\varphi_0)$.

Scattered throughout chapter 4, we covered all that is needed to derive this uncertainty relation, but we repeat the most important steps here in a consolidated version. To this end, recall the definition of the phase operator in terms of number states in equations (4.2.11), which we used in eq. (4.2.16) to calculate the commutator as

$$[\hat{n}, \hat{\varphi}_{\varphi_0}] = \frac{2\pi}{s+1}\sum_{n'\neq n}^{s} |n'\rangle\langle n|\,\frac{(n'-n)\mathrm{e}^{i(n'-n)\varphi_0}}{\mathrm{e}^{i(n'-n)2\pi/(s+1)} - 1} \ . \qquad (7.1.19)$$

Note that the sum lacks a term for $n' = n$, which means that the trace $\langle n\,|[\hat{\varphi}_{\varphi_0}, \hat{n}]|\,n \rangle \equiv 0$.

Expression (7.1.19) is not particularly helpful, and does not, in fact, directly lead to the uncertainty relation derived from the metric tensor and the supposed CCR (canonic commutation relation) above. However, a very pragmatic simplification does: With Pegg and Barnett, we argue that any physically accessible state arises from an interaction that is finite in time and intensity, usually with an energy source that is finite in the sense that the highest excitable number state is bounded. Requiring finite interaction time and intensity is a weaker condition than also requiring a finite energy source - the latter excludes, for example, the coherent states - but any state satisfying the stronger condition can be approximated arbitrarily well to a state which satisfies the weaker condition.

Calculating the commutator for physical states is then equal to setting s very large but not infinite, which yields the following approximation

$$
\begin{aligned}
[\hat{n}, \hat{\varphi}_{\varphi_0}]_p &= \lim_{s \gg \langle n \rangle} \frac{2\pi}{s+1} \sum_{n' \neq n}^{s} |n'\rangle\langle n| \, \frac{(n'-n)\mathrm{e}^{i(n'-n)\varphi_0}}{\mathrm{e}^{i(n'-n)2\pi/(s+1)} - 1} = \\
&\approx \frac{2\pi}{s+1} \sum_{n' \neq n}^{s} |n'\rangle\langle n| \, \frac{(n'-n)\mathrm{e}^{i(n'-n)\varphi_0}}{i(n'-n)2\pi/(s+1)} = \\
&= -i \sum_{n' \neq n}^{s} |n'\rangle\langle n| \, \mathrm{e}^{i(n'-n)\varphi_0} = \\
&= -i \sum_{n',n=0}^{s} |n'\rangle\langle n| \, (1 - \delta_{nn'})\mathrm{e}^{i(n'-n)\varphi_0} \,.
\end{aligned}
\tag{7.1.20}
$$

We now factor the sum

$$
\begin{aligned}
[\hat{n}, \hat{\varphi}_{\varphi_0}]_p &= -i \sum_{n',n=0}^{s} |n'\rangle\langle n| \, (1 - \delta_{nn'})\mathrm{e}^{i(n'-n)\varphi_0} = \\
&= i \sum_{n=0}^{s} |n\rangle\langle n| - i \left[\sum_{n'=0}^{s} \mathrm{e}^{in'\varphi_0} |n'\rangle \right] \cdot \left[\sum_{n=0}^{s} \mathrm{e}^{-in\varphi_0} \langle n| \right] = \\
&= i \big[1 - (s+1)\, |\varphi_0\rangle\langle\varphi_0| \big] \,,
\end{aligned}
\tag{7.1.21}
$$

where for the last equality, we have used that

$$
\sum_{n}^{s} |n\rangle\langle n| \equiv 1
\tag{7.1.22}
$$

and

$$
|\varphi\rangle = (s+1)^{-1/2} \sum_{n=0}^{s} \mathrm{e}^{in\varphi} |n\rangle \,.
\tag{7.1.23}
$$

Evaluating for the physical states, which we will now simply call $|\psi\rangle$, yields

$$\langle\psi\,|[\hat{n},\hat{\varphi}_{\varphi_0}]|\,\psi\rangle = i\Big[1 - (s+1)\,|\langle\psi|\,\varphi_0\rangle|^2\Big]\,, \qquad (7.1.24)$$

which for large s approaches the probability distribution

$$\langle\psi\,|[\hat{n},\hat{\varphi}_{\varphi_0}]|\,\psi\rangle = i\Big[1 - 2\pi P(\varphi_0)\Big]\,, \qquad (7.1.25)$$

yielding the sought-after relation

$$\Delta\hat{n}\cdot\Delta\hat{\varphi}_{\varphi_0} \geq \frac{1}{2}\,|1 - 2\pi P(\varphi_0)|\,. \qquad (7.1.26)$$

According to the weak operator topology and the conclusions reached about the PB-POM (cf. sections 4.3.1 and 4.3.2.3), this result should be permissible because the calculation of the commutator only requires the first moment of the operator $\hat{\varphi}_{\varphi_0}$ and not any higher moments.

7.1.4 Different measures of uncertainty

We conclude our section on the number-phase-uncertainty relation with a short primer on other possible measures of uncertainty besides the simple rms-variance that has been used above. The importance of such alternative measures is rising, since the second moment of an operator is admittedly a very good measure for single hump distributions in linear coordinates, but performs rather poorly on multiple hump distributions or on periodic variables such as phase, where we must accept an unwieldy dependence of the uncertainty on an arbitrarily defined reference phase, rendering the uncertainty itself a tad arbitrary.

For the case of quantum phase, Bialynicki-Birula, Freyberger and Schleich have compiled a brief overview of useful measures besides rms, which we will examine briefly in turn [70].

7.1.4.1 Overview

The somewhat remarkable dominance of the second moment of operator-observables as a measure of uncertainty is most certainly due to the importance of variance in statistics and stochastic calculus. However, there are several measures which are just as adequate and often serve as a useful alternative in the context of unconventional wave-functions (i.e. multiple humps, curvilinear coordinates, periodic boundaries etc).

An uncertainty measure may most generally be expressed as a function of some functional of φ:

$$\Delta_u\varphi = F_u(X), \qquad X = \int\limits_{\varphi_0}^{\varphi_0+2\pi} \mathrm{d}\varphi\, P(\varphi) f_u[\varphi, P]\,. \qquad (7.1.27)$$

For the variance, it is easy to see that

$$F_v(X) = X \tag{7.1.28a}$$

$$f_v[\varphi, P] = (\varphi - \langle\varphi\rangle)^2 \tag{7.1.28b}$$

yielding

$$\Delta_v\varphi = \int\limits_{\varphi_0}^{\varphi_0+2\pi} \mathrm{d}\varphi\, P(\varphi)(\varphi - \langle\varphi\rangle)^2 \,. \tag{7.1.28c}$$

But it is also easy to surmise that for different values of φ_0, the integral will oscillate between a maximum and a minimum value, yielding only a vague estimate of the uncertainty instead of a precise value.

7.1.4.2 Inverse of maximal value

The inverse of maximal value measure is at once an improvement and a setback when compared to the variance. It is defined as

$$F_m(X) = X^{-1} \tag{7.1.29a}$$

$$f_m[\varphi, P] = \delta(\varphi - \varphi_{max}) \tag{7.1.29b}$$

yielding

$$\Delta_m\varphi = \left[\int\limits_{\varphi_0}^{\varphi_0+2\pi} \mathrm{d}\varphi\, P(\varphi)\delta(\varphi - \varphi_{max}) \right]^{-1} = [P(\varphi_{max})]^{-1} \,, \tag{7.1.29c}$$

where φ_{max} is the value which maximizes $P(\varphi)$. Obviously, this measure is invariant under rotations since the probability distribution on which this measure directly depends is itself rotationally invariant under the operation $\max |P(\varphi)|$.[32] On the other hand, it suffers from the big problem that it originates from the evaluation of a single value instead of a distribution, causing this measure to be unable to differentiate between say a triangular and a rectangular function if their maxima are equal. Also, it is very difficult to calculate this measure from experimental data, since the measurement of the maximum value is much less precise than the measurement of an entire distribution.

---

[32]Note that the authors of [70] apparently do not share this opinion and hold the inverse of maximal phase measure not to be rotationally invariant, but I do not see why.

The main reason that this rather awkward measure is in any use at all probably stems from some well-known papers by Shapiro and others (e.g. [71]) which rely on it. For most intents and purposes, however, it is of rather limited use.

7.1.4.3 Inverse of averaged distribution

The inverse of averaged distribution is yet another slight improvement over the inverse of maximal value measure and is defined as

$$F_a(X) = X^{-1} \tag{7.1.30a}$$

$$f_a[\varphi, P] = P(\varphi) \tag{7.1.30b}$$

yielding

$$\Delta_a \varphi = \left[\int\limits_{\varphi_0}^{\varphi_0 + 2\pi} \mathrm{d}\varphi \, P(\varphi) P(\varphi) \right]^{-1}. \tag{7.1.30c}$$

Unlike the inverse of maximal value measure, it takes the global behavior of the phase distribution into account, but unlike the variance, it does not have any nonlinear direct dependence on φ, making it rotationally invariant. It is therefore better suited to the description of uncertainty than either of these, although its practical use seems to be limited.

7.1.4.4 Dispersion

We introduce, finally, the dispersion, which is the average value of the phase factor, as follows

$$F_d(X) = 1 - X^2 \tag{7.1.31a}$$

$$f_d[\varphi, P] = \mathrm{e}^{i\varphi} \tag{7.1.31b}$$

which yields

$$\Delta_d \varphi = 1 - \left[\int\limits_{\varphi_0}^{\varphi_0 + 2\pi} \mathrm{d}\varphi \, P(\varphi) \mathrm{e}^{i\varphi} \right]^2. \tag{7.1.31c}$$

The use of the 2π-periodic function $\mathrm{e}^{i\varphi}$ in this equation best reflects the periodicity of the probability distribution, therefore making it naturally suited to the description of phase, and for sharp peaks at $\varphi = 0$, this measure approaches the variance, as can be

qualitatively ascertained by noting the small-value Taylor approximation of the exponential which approximately yields $1 - [e^{i\varphi}]^2 \approx \varphi^2$.

Note that Hradil, Rehacek et al [72] have also found dispersion to be a good uncertainty measure for the observables angle and angular momentum and have used the product of the second moment of angular momentum and the dispersion of angle to calculate minimum uncertainty light polarization states. Although we do not treat angle and angular momentum here, we note that a similar relation might be useful in the case of phase and number when we turn to minimum uncertainty states, see section 7.2

7.1.4.5 Entropic measure of uncertainty

We briefly note one final measure of uncertainty which has attracted some attention, and which is based on entropy considerations. In this case, we have

$$F_e(X) = X \tag{7.1.32a}$$

$$f_e[\varphi, P] = -\ln P(\varphi) \tag{7.1.32b}$$

yielding

$$\Delta_e \varphi = - \int\limits_{\varphi_0}^{\varphi_0 + 2\pi} \mathrm{d}\varphi \, P(\varphi) \ln P(\varphi) \,. \tag{7.1.32c}$$

We will not go any further into this measure, but note that it sometimes offers a viable alternative when the usual uncertainty measures fail.

7.2 Minimum uncertainty states

Having spent enough time on the notion of uncertainty and the calculation of uncertainty relations, we now delve into the subject of minimum uncertainty states. Minimum uncertainty states are states that minimize a given uncertainty relation; minimum uncertainty states that reach the equality between the left and right hand sides of an uncertainty relation are called intelligent states, and an example of the latter would be the coherent states for the position-momentum uncertainty relation.

7.2.1 Calculational tools

From a conceptual point of view, the determination of minimum uncertainty states is mathematically straightforward, even though the concrete calculation may turn out to be

quite complex and may sometimes require numeric methods. However, the usability of the states thus derived varies widely, ranging from well understood (coherent states) to nonsensical (most of the phase-number minimum uncertainty states).

In this section, we will give the basic calculus needed to determine minimum uncertainty states [73] and apply it exemplarily to the uncertainty relation between position and momentum. In the next section, we will then list (since the actual calculation is tedious and not specifically revealing) some of the results that the procedure gives for phase-number minimum uncertainty states.

We start with the uncertainty product for a normalizable state $|\psi\rangle$ and use the Schwartz inequality [73]

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 = \left\langle \hat{A}'^2 \right\rangle \left\langle \hat{B}'^2 \right\rangle \geq \left| \left\langle \hat{A}'\hat{B}' \right\rangle \right|^2 , \tag{7.2.1}$$

where $\hat{A}' = \hat{A} - \left\langle \hat{A} \right\rangle$. We can now make two rearrangements. On the one hand, a simple calculation shows that we can partition $\left| \left\langle \hat{A}'\hat{B}' \right\rangle \right|^2$ into a commutator and an anticommutator as follows

$$P(\psi) = \tfrac{1}{4} \left| \left\langle [\hat{A}', \hat{B}'] \right\rangle \right|^2 \tag{7.2.2a}$$

$$Q(\psi) = \tfrac{1}{4} \left| \left\langle \{\hat{A}', \hat{B}'\} \right\rangle \right|^2 \tag{7.2.2b}$$

$$P(\psi) + Q(\psi) = \left| \left\langle \hat{A}'\hat{B}' \right\rangle \right|^2 , \tag{7.2.2c}$$

yielding

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 = P(\psi) + Q(\psi) + R(\psi) \tag{7.2.3}$$

with $R(\psi)$ an additional remainder term resulting from the application of the Schwartz inequality. On the other hand, we know that quite generally,

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \tfrac{1}{4} \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right|^2 = \tfrac{1}{4} \left| \left\langle [\hat{A}', \hat{B}'] \right\rangle \right|^2 = P(\psi) , \tag{7.2.4}$$

This means that in order to minimize the uncertainty relation, we need to find a $|\psi\rangle$ so that $Q(\psi) + R(\psi) = 0$. Moreover, since $Q(\psi)$ is positive semidefinite, both $Q(\psi)$ and $R(\psi)$ need to vanish individually. This yields a special type of eigenvalue equation whereby

$$\hat{A}' |\psi\rangle + i\gamma\hat{B}' |\psi\rangle = 0 \tag{7.2.5}$$

or

$$(\hat{A} + i\gamma\hat{B}) |\psi\rangle = (\left\langle \hat{A} \right\rangle + i\gamma \left\langle \hat{B} \right\rangle) |\psi\rangle . \tag{7.2.6}$$

The solution to this differential equation gives all states that minimize the uncertainty relation. These states are determined by the operators $\hat{A}$ and $\hat{B}$ and by the three free parameters γ, $\left\langle \hat{A} \right\rangle$ and $\left\langle \hat{B} \right\rangle$. Moreover, the above equation gives us a simple test to see whether a state is a minimum uncertainty state because all minimum uncertainty states

are eigenkets of the $(\hat{A} + i\gamma\hat{B})$ operator.

For example, if we want to figure out the minimum uncertainty states of the position and momentum operators, we need merely find the eigenstates of $\hat{X} + i\gamma\hat{P}$. For $\gamma = 1$, $\hat{X} + i\gamma\hat{P}$ simply becomes the annihilation operator $\hat{a}$, identifying the vacuum state and the coherent states as minimum uncertainty states since

$$\hat{a}\left|0\right\rangle = 0\left|0\right\rangle = 0 \tag{7.2.7a}$$

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle . \tag{7.2.7b}$$

We also see that $\left\langle\hat{X}\right\rangle$ and $\left\langle\hat{P}\right\rangle$ correctly turn out to be the coordinates of the coherent state in the complex plane, and upon closer examination, it turns out that $\gamma \neq 1$ leads to a squeezed annihilation operator which has the squeezed states as eigenstates, which are thus also minimum uncertainty states.

7.2.2 Calculation of optimal states

Unfortunately, in the case of the number-phase uncertainty relation, the minimum uncertainty states (MUS) are much less well defined, because the variance causes certain problems in the derivation of such states owing to the periodicity of phase, while another uncertainty measures has not yet unequivocally been agreed upon, so that many parallel formulations exist (e.g. using dispersion [74] or the inverse of squared probability measure [75]).

We will therefore restrict ourselves to a short summary of the variance-based minimum uncertainty states that have been found by Vaccaro and Pegg in applying their formalism [33], and briefly compare them to the Bandilla Paul dispersion type uncertainty states [74].

Concerning, first, the variance minimizing number-phase minimum uncertainty states, it has been determined that *sensu stricto*, these are limited to the number states. This follows from the general solution to the number-phase minimum uncertainty state differential equation

$$(\hat{n} + i\gamma\hat{\varphi}_{\varphi_0})\left|\psi\right\rangle = (\langle\hat{n}\rangle + i\gamma\langle\hat{\varphi}_{\varphi_0}\rangle)\left|\psi\right\rangle , \tag{7.2.8}$$

which gives a wave function for the MUS as follows [33]

$$\psi(\varphi) = \delta^{-1/2} A(\gamma)\mathrm{e}^{-\frac{1}{2}\gamma(\varphi + 2\pi k - \langle\varphi\rangle)^2 - i\langle n\rangle(\varphi + 2\pi k)} . \tag{7.2.9}$$

Here, γ is the phase weight parameter from the differential equation and $A(\gamma)$ a normalization coefficient. The periodicity of $\psi(\varphi)$ is reflected in appropriate choices of k for appropriate intervals, but this will in general lead to discontinuities in function and slope that are impermissible for a wave function, unless $\gamma = 0$, giving the number states.

Nonetheless, it is possible to give states that are approximately minimum uncertainty states. More precisely, if γ is very large, then all values in which $\psi(\varphi)$ differs significantly from zero are within the main 2π-interval, and the boundaries therefore become approximately continuous. For such states, it turns out that

$$(\Delta \hat{n})^2 \approx \tfrac{1}{2}\gamma \tag{7.2.10a}$$

$$(\Delta \hat{\varphi}_{\varphi_0})^2 \approx (2\gamma)^{-1} \,, \tag{7.2.10b}$$

giving a squared uncertainty product of approximately $\frac{1}{4}$ and thereby satisfying the uncertainty product

$$\Delta \hat{n} \cdot \Delta \hat{\varphi}_{\varphi_0} \geq \frac{1}{2} \left| 1 - 2\pi P(\varphi_0) \right| \,, \tag{7.2.11}$$

where by construction, $P(\varphi_0)$ is very small.

One very important result to remember is that intense coherent states and certain classes of squeezed states satisfy the requirements of continuity well enough that they may be considered minimum uncertainty states of phase, as has been shown in [16, 33].

We now briefly turn to the Bandilla/Paul MUS [74] to show the importance that the choice of measure has for the calculation of minimum uncertainty states.

Using the dispersion measure

$$(\Delta \hat{\varphi})^2 = 1 - \left| \left\langle \widehat{e^{i\varphi}} \right\rangle \right|^2 \tag{7.2.12}$$

Bandilla and Paul are led to the maximization problem

$$\sum_{n=0}^{\infty} c_n c_{n+1} = \max \tag{7.2.13}$$

with constraints

$$\sum_{n=0}^{\infty} c_n^2 = 1 \tag{7.2.14a}$$

$$\sum_{n=0}^{\infty} n\, c_n^2 = N \quad \text{(fixed)} \tag{7.2.14b}$$

The solution of this variational problem gives a complicated expression in Bessel functions for the number state coefficients, which is markedly different from the photon statistics the coherent states exhibit. A marked difference can also be found between the phase statistics of the two states, showing that the (high intensity) coherent states are approximate MUS for variance, but not for dispersion (cf. also the graphs in [74]).

7.3 Fuzzy measurement in the phase variable

The fact that phase states are not physical states cues us in to suspect that there may be another aspect of uncertainty besides the complementarity between number and phase to look out for. More concretely, we have seen many times that phase is not a classic hermitian observable and therefore it is impossible to prepare any states that give a sharp phase distribution.

We therefore ask if there is a way to predict how sharp such a measurement can become? Note that for the position-momentum uncertainty relation, this problem does not exist, since in principle, we may approximate the position and momentum eigenstates as closely as we like, yielding no restriction on the possible sharpness of one isolated variable if the uncertainty in the other is arbitrary.

Ou [76] has made a very simple argument for the fundamental limit of quantum phase precision measurement which corroborates fuzzy measurement, but does not rely on any considerations concerning phase operators. Instead, Ou recurs to the quantum nature of light to find a lower bound for the detectability of phase differences. More precisely, taking any Mach/Zehnder-interferometer, we have for the interference

$$I_{Out} = I_{In}(1 - \cos \varphi)/2 \,. \tag{7.3.1}$$

Sensitivity here is highest around $\varphi = \pi/2$, were $\cos \varphi \approx -(\varphi - \pi/2)$ so that there we have

$$\Delta I_{Out} = I_{In}\Delta\varphi/2 \,. \tag{7.3.2}$$

In principle, this would set no limit for the fineness of $\Delta\varphi$, but the intensity is quantized and thus if we replace the above equation by the corresponding expression for photon numbers

$$\Delta N_{Out} = N_{In}\Delta\varphi/2 \,. \tag{7.3.3}$$

and set $\Delta N_{Out} \geq 1$, i.e. a measurable phase shift requires at least one jump in quantum number, we have

$$\Delta\varphi \geq 2/N_{In} = 1/N \tag{7.3.4}$$

where $N = N_{In}/2$ corresponds to the beam in the interferometer that experiences the phase shift. For classical states of light which exhibit at best Poissonian statistics, this leads to

$$\Delta\varphi \geq 1/\langle N \rangle \,. \tag{7.3.5}$$

This is of course not a rigorous argument, since states could be found that exhibit sub-Poissonian statistics, or that circumvent the above reasoning in other ways. However, it turns out, as Ou shows by employing various other gedankenexperimente [76], that the limit $\Delta\varphi \geq 1/\langle N \rangle$ seems to be coercive, and we are of course left to wonder if that is not

related to the fuzziness of the phase observable as already stated multiple times in this thesis.

7.4 Generalized uncertainty

We close this chapter on uncertainty with a reference to an interesting spin on uncertainty that could be used to understand phase more thoroughly. This approach was examined by Ozawa [77, 78] and recurs to the realm of generalized measurement to obtain a more differentiated view about uncertainty than the simple Heisenberg relation (recall that we have already met generalized measurement theory, albeit in a very simple sense, when deriving the SG-POM, see section 4.3.2.2). Unfortunately, the treatment of generalized uncertainty is beyond the scope of this thesis, but the inclined reader is encouraged to peruse Ozawa's papers.

Chapter 8

Reflections and conclusion

We are now at the end of our journey through the realm of quantum phase, and here we would like to think about why we encountered so many problems. After all, from a neutral, common sense perspective and a pictorial motivation from phase space, it would make obvious sense that number and phase should be conjugated to each other, that the description of phase and number should be analogous to the description of angle and angular momentum, and that a standard Heisenberg uncertainty relation holds.

What the last seven chapters have done is to show us most of the *mechanics* of the problem. But they have not shown us why the mechanics cause problems. For example, we know that phase is periodic, that the decomposition of the destruction operator is not unitary, that a strong convergence to the one-sided infinite Hilbert space is impossible, etc. etc., and that these combined facts make the definition of a classical phase observable impossible. But we do not know the ulterior *reason* for this behavior! This is why in the present chapter, the question is: *Why does this have to be so?*.

One possibility why the definition of a hermitian phase operator is confronted with so many difficulties at least in the standard formulation of quantum mechanics has been proposed by Mendas [79], who warned that the existence of an hermitian phase observable would allow the definition of a time operator $\hat{T}$, making time itself an observable and thereby depriving quantum mechanics of its smooth unitary evolution parameter t. The problems plagueing the phase operator may then be seen as an analogue of the problems preventing the definition of a time operator.

Another observation that could be made is that if phase were conjugate to number, we could violate the uncertainty relation for position and momentum. This is because the position-momentum uncertainty relation demands that if x is sharp, p must be infinite. But for a phase state at $\varphi = 90°$, while position would be sharply equal to zero, momentum would not be entirely indefinite, since the sign of p would be known to be positive. This effectively halves the uncertainty in p, which is not allowed by the canonic commutation relation of position and momentum, causing problems.

It may also be stated that phase measurement must inherently be fuzzy since it involves a simultaneous measurement of position and momentum. This may be valid and would warrant further investigation also in light of the uncertainty theory introduced by Ozawa, but the number operator constitutes a *prima facie* counterargument to this consideration, since number is sharply defined in the number states even though it also seems to require the simultaneous measurement of two quadrature components.

Despite these difficulties, possible remedies may exist if we choose to look at the proverbial "big picture". For example, we may have overidealized the absolute quality of phase, and need to turn instead to a model which has more intrinsic physical meaning, e.g. by looking at phase difference measurement. Or, possibly, we need a change in scope and have to include at least a second mode in order to attenuate the vacuum destruction behavior of the annihilation operator (i.e. "think outside the box", literally).

Whatever the real explanation for the encountered problems may ultimately be, phase is certainly an area that has the potential to bring forth many interesting applications, and it is with this outlook that we finish.

Appendix A

Quantum mechanical background

A.1 The harmonic oscillator model

A.1.1 Quantization of the EM-field

Given Maxwell's equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = q \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \text{(A.1.1)}$$

we examine a vector potential of the form $\boldsymbol{A}(\boldsymbol{r}, t)$. Then we have

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \qquad \text{(A.1.2)}$$

and $\boldsymbol{A}(\boldsymbol{r}, t)$ fulfills the wave equation $\Box \boldsymbol{A} = 0$. It follows that $\boldsymbol{A}(\boldsymbol{r}, t)$ must be a combination of Fourier-terms such as $\sin(\omega_k t)$.

If we now observe a limited volume of space, $\boldsymbol{A}(\boldsymbol{r}, t)$ has a Fourier-decomposition with a limited spectrum, so that

$$\boldsymbol{A}(\boldsymbol{r}, t) = \boldsymbol{A}^{(+)}(\boldsymbol{r}, t) + \boldsymbol{A}^{(-)}(\boldsymbol{r}, t) \qquad \text{(A.1.3)}$$

$$\boldsymbol{A}^{(+)}(\boldsymbol{r}, t) = \sum_k^\infty c_k \boldsymbol{u_k}(\boldsymbol{r}) \mathrm{e}^{-i\omega_k t} \qquad \text{(A.1.4)}$$

$$\boldsymbol{A}^{(+)}(\boldsymbol{r}, t) = \left(\boldsymbol{A}^{(-)}(\boldsymbol{r}, t) \right)^* \qquad \text{(A.1.5)}$$

Note that $\boldsymbol{u_k}(\boldsymbol{r})$ is a quantized representation of the position of the electromagnetic wave. Replacing c_k and c_k^* by the operators $\hat{a}_k$ und $\hat{a}_k^\dagger$, the quantized electromagnetic field results:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_k^\infty \left(\frac{\hbar}{2\omega_k \epsilon_0} \right) \left[\hat{a}_k \boldsymbol{u_k}(\boldsymbol{r}) \mathrm{e}^{-i\omega_k t} + \hat{a}_k^\dagger \boldsymbol{u_k^*}(\boldsymbol{r}) \mathrm{e}^{i\omega_k t} \right] \qquad \text{(A.1.6)}$$

The operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ fulfill the bosonic relation

$$[\hat{a}_k, \hat{a}_k'] = [\hat{a}_k^\dagger, \hat{a}_k'^\dagger] = 0 \tag{A.1.7}$$

$$[\hat{a}_k, \hat{a}_k'^\dagger] = \delta_{kk'} . \tag{A.1.8}$$

The appropriate differentiations of $\boldsymbol{A}(\boldsymbol{r}, t)$ then lead, according to equation (A.1.2), to $\boldsymbol{E}(\boldsymbol{r}, t)$ und $\boldsymbol{B}(\boldsymbol{r}, t)$. Combining this with the commutator relations in equations (A.1.7) and (A.1.8), the Hamiltonian of the electromagnetic field

$$H = \frac{1}{2} \int (\epsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2) \, dx \tag{A.1.9}$$

may be expressed as

$$\hat{H} = \sum_k \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right) . \tag{A.1.10}$$

Thus, the electromagnetic field may be described by a superposition of quantum harmonic oscillators.

A.1.2 Fock states

Restricting ourselves to a single mode k and dropping the index, we may go on to define the Hamiltonian of the electric field mode as:

$$\hat{H} = \hbar \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \tag{A.1.11}$$

Using the bosonic commutator

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{A.1.12}$$

we may make the following observations about the eigenstates of the Hamiltonian:

$$H \ket{n} = \hbar \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \ket{n} = E_n \ket{n} \tag{A.1.13}$$

Applying $\hat{a}$ from the left side and using the commutator:

$$\hat{a} \, H \ket{n} = \hat{a} \, E_n \ket{n} \tag{A.1.14}$$

$$H \, \hat{a} \ket{n} + \hbar \omega \, \hat{a} \ket{n} = E_n \, \hat{a} \ket{n} \tag{A.1.15}$$

$$H \, \hat{a} \ket{n} = (E_n - \hbar \omega) \, \hat{a} \ket{n} \tag{A.1.16}$$

$$H \ket{n-1} = E_{n-1} \ket{n-1} \tag{A.1.17}$$

Using $\langle n-1 | \, n-1 \rangle = 1$ we obtain:

$$\Rightarrow \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{A.1.18}$$

$$\Rightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{A.1.19}$$

Apparently, the operators $\hat{a}$ und $\hat{a}^\dagger$ shift the energy eigenvalues n down or up, respectively, by $\hbar\omega$. This has earned them the names "annihilation operator" and "creation operator".

We call the eigenstates $|n\rangle$ number states or Fock states to the eigenvalue n. They are eigenstates of the number operator $\hat{a}^\dagger\hat{a}$ (or short: $\hat{n}$) which is itself directly derived from the Hamiltonian, and constitute a complete orthonormal system.

$$\sum_n |n\rangle\langle n| = 1 \qquad \langle n | \, m \rangle = \delta_{nm} \tag{A.1.20}$$

The number states measure the amplitude of an harmonic oscillator. The amplitude is sharply defined, but conversely, the number states have maximum uncertainty in phase (cf. the phase space representation in figure A.1).
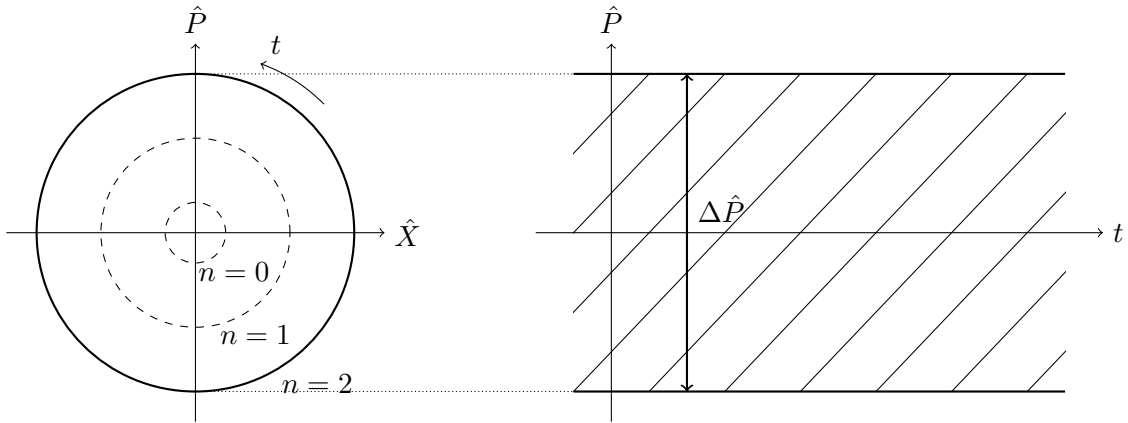

Figure A.1: Phase space view of a number state

A.1.3 Coherent states

Because the number states exhibit complete uncertainty in phase, they are usually not adequate to describe individual light beams. Instead, this purpose is filled by the coherent states, which can most precisely be described (and pictured) as displaced vacuum states (cf. figure A.2). It is thus no coincidence that they are defined by the action of the so-called displacement operator $\hat{D}(\alpha)$

$$\hat{D}(\alpha) \equiv \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) \tag{A.1.21}$$

on the vacuum

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle \,. \tag{A.1.22}$$

Figure A.2: Phase space view of a coherent state

An equivalent definition is that the coherent states are eigenstates of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{A.1.23}$$

Finally, a third definition is in terms of number states

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{A.1.24}$$

All of these formulations are equivalent, as we can appreciate by showing that the state defined in A.1.24 is

(i) an eigenstate of the annihilation operator, and

(ii) fulfills $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$.

ad (i): Apply $\hat{a}$ to $|\alpha\rangle$:

$$
\begin{aligned}
\hat{a}|\alpha\rangle &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}\,\hat{a}|n\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}\,\sqrt{n}|n-1\rangle \\
&= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \left(\sum_n \frac{\alpha^{n+1}}{\sqrt{n!}}\,|n\rangle + \hat{a}|0\rangle \right) = \alpha \left(\mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \right) \\
&= \alpha|\alpha\rangle
\end{aligned}
\tag{A.1.25}
$$

ad (ii): Express $|n\rangle$ in terms of $|0\rangle$ as

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{A.1.26}$$

and rearrange

$$\Rightarrow |\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n (\hat{a}^\dagger)^n}{n!}\,|0\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{\alpha \hat{a}^\dagger}\,|0\rangle \tag{A.1.27}$$

Now insert a term $\exp\{-\alpha^*\hat{a}\}$ just before the vacuum, which is possible since

$$\mathrm{e}^{-\alpha^*\hat{a}}\,|0\rangle = \sum_n \frac{(-\alpha^*\hat{a})^n}{n!}\,|0\rangle = |0\rangle \tag{A.1.28}$$

Thus

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\hat{a}^\dagger}\mathrm{e}^{-\alpha^*\hat{a}}\,|0\rangle\,, \tag{A.1.29}$$

which using a variant of the Baker-Hausdorff-formula may be rearranged to obtain

$$\mathrm{e}^{-[A,B]/2}\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B} \tag{A.1.30}$$

Therefore

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha\hat{a}^\dagger}\mathrm{e}^{-\alpha^*\hat{a}}\,|0\rangle = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)|0\rangle = \hat{D}(\alpha)|0\rangle \tag{A.1.31}$$

as desired.

A.1.4 Squeezed states

A variation on the coherent states are the squeezed states (for a comprehensive overview, cf., e.g., [80]), which are squeezed coherent states for which the uncertainty in one quadrature component is reduced (squeezed), while a corresponding increase is observed in the other quadrature. For example, if the coherent states satisfied

$$\Delta A \cdot \Delta B = 1 \tag{A.1.32a}$$

$$\Delta A = \Delta B = 1\,, \tag{A.1.32b}$$

where $\hat{A}$ and $\hat{B}$ are rescaled and possibly rotated position and momentum operators, a squeezed state will satisfy

$$\Delta A \cdot \Delta B = 1 \tag{A.1.33a}$$

$$\Delta A < 1 \text{ or } \Delta B < 1 \tag{A.1.33b}$$

(in a weaker formulation, the first line may be omitted). The squeeze may occur in any direction, leading to a general distinction between number squeezed states (which exhibit less number uncertainty) and phase squeezed states (which exhibit less phase uncertainty). To get a better feel for squeezed states, cf. figures A.3 and A.4.

A closed expression for the squeezed states in terms of number states is unwieldy, but in analogy to the coherent states, squeezed states may be obtained from the vacuum states by the action of a specialized operator called the squeeze operator, which is applied to a

displaced vacuum (i.e. a coherent state):

$$|\alpha, \varepsilon\rangle = \hat{S}(\varepsilon)\, \hat{D}(\alpha)\, |0\rangle \ , \tag{A.1.34}$$

where $\varepsilon = r \cdot e^{i\theta}$ is the squeeze parameter denoting the angle and magnitude of the squeeze. The squeeze operator is defined as

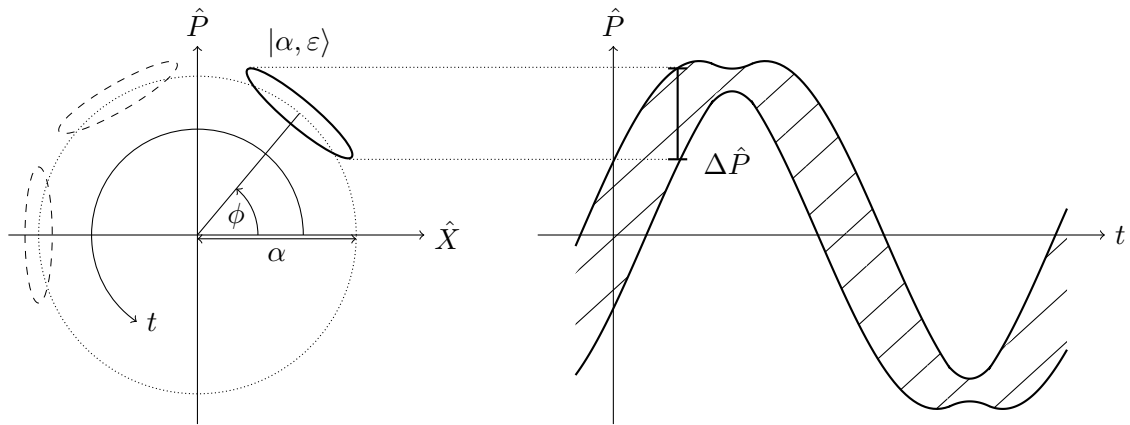$$\hat{S}(\varepsilon) = e^{\frac{1}{2}(\varepsilon^* \hat{a}^2 - \varepsilon(\hat{a}^\dagger)^2)} \tag{A.1.35}$$
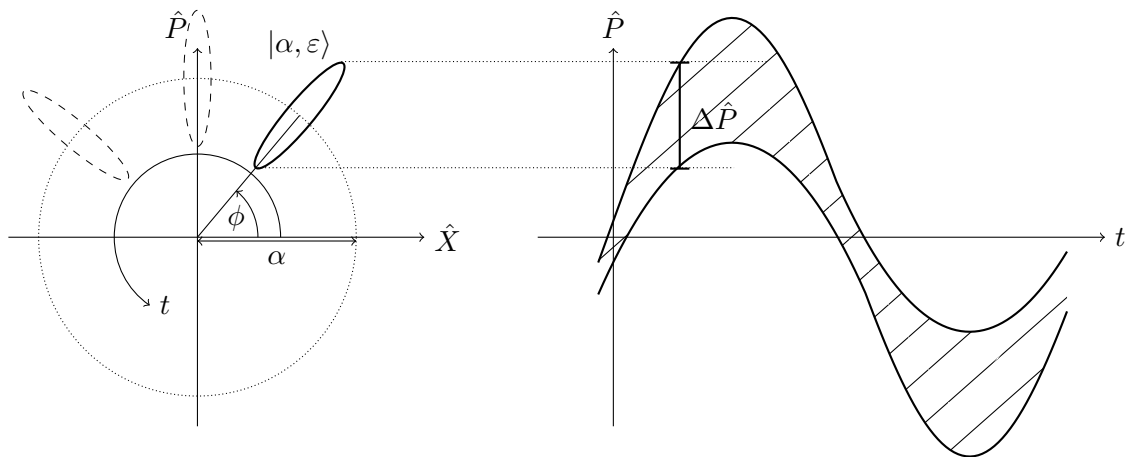

Figure A.3: Phase space view of an exemplary number squeezed state

Figure A.4: Phase space view of an exemplary phase squeezed state

Appendix B

Miscellaneous

B.1 Susskind Glogower recursion relation

This appendix refers to the calculation of the eigenvalues and eigenstates of the $\widehat{\sin \varphi}$ and $\widehat{\cos \varphi}$ operators in section 3.1.3. The recursion relation between the coefficients of the number state decomposition was

$$a_1 = 2i\lambda_{\sin} a_0 \qquad\qquad a_n = 2i\lambda_{\sin} a_{n-1} + a_{n-2} \qquad (\text{B.1.1a})$$

$$b_1 = 2\lambda_{\cos} b_0 \qquad\qquad b_n = 2\lambda_{\cos} b_{n-1} - b_{n-2} \qquad (\text{B.1.1b})$$

Susskind and Glogower [14] and with them Carruthers and Nieto [6] then proposed substituting $2i\lambda_{\sin} = (p - \frac{1}{p})$ and $2\lambda_{\cos} = (q + \frac{1}{q})$ (their notation was slightly different), which was supposed to give

$$a_n = Ap^n + Bp^{-n} \qquad\qquad (\text{B.1.2a})$$

$$b_n = Aq^n + Bq^{-n} . \qquad\qquad (\text{B.1.2b})$$

Testing this against the recursion relation apparently checks out

$$
\begin{aligned}
a_n &= 2i\lambda a_{n-1} + a_{n-2} = \\
&= (p - \frac{1}{p})(Ap^{n-1} + Bp^{-n+1}) + Ap^{n-2} + Bp^{-n+2} \\
&= Ap^n - Ap^{n-2} + Bp^{-n} - Bp^{-n+2} + Ap^{n-2} + Bp^{-n+2} = \\
&= Ap^n + Bp^{-n} = a_n
\end{aligned}
\qquad (\text{B.1.3})
$$

and

$$b_n = 2\lambda b_{n-1} - b_{n-2} =$$
$$= (q + \frac{1}{q})(Aq^{n-1} + Bq^{-n+1}) - (Aq^{n-2} + Bq^{-n+2})$$
$$= Aq^n + Aq^{n-2} + Bq^{-n} + Bq^{-n+2} - Aq^{n-2} - Bq^{-n+2} =$$
$$= Aq^n + Bq^{-n} = b_n \,.$$

(B.1.4)

However, this is only half of a full induction proof. When we test whether the initial recursions for a_n and b_n are right, we find

$$a_1 = 2i\lambda_{\sin}a_0 = (p - \frac{1}{p})(A + B) =$$
$$= Ap - Ap^{-1} + Bp - Bp^{-1} \neq Ap + Bp^{-1} = a_1$$

(B.1.5)

and

$$b_1 = 2\lambda_{\cos}b_0 = (q + \frac{1}{q})(A + B) =$$
$$= Aq + Aq^{-1} + Bq + Bq^{-1} \neq Aq + Bq^{-1} = b_1 \,.$$

(B.1.6)

The problem therefore lies with not making a full induction. But the substitutions $2i\lambda_{\sin} = (p - \frac{1}{p})$ and $2\lambda_{\cos} = (q + \frac{1}{q})$ can still be used to arrive at a sensible result, namely

$$a_n = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k} a_0$$

(B.1.7a)

$$b_n = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} q^{2k} b_0 \,,$$

(B.1.7b)

where if n is odd, the sum runs over $\{\cdots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\}$. We first test whether the relation is fulfilled for the initial coefficients and get

$$a_1 = 2i\lambda_{\sin}a_0 = (p - \frac{1}{p})a_0 = a_1$$

(B.1.8)

and

$$b_1 = 2\lambda_{\cos}b_0 = (q + \frac{1}{q})b_0 = b_1$$

(B.1.9)

Now we make the inductions:

$$a_{n+1} = 2i\lambda_{\sin}a_n + a_{n-1} =$$

$$(p - \frac{1}{p})\left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k}a_0\right\} + \left\{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (-1)^{(k-\frac{n-1}{2})} p^{2k}a_0\right\} =$$

$$= \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k+1}a_0\right\} - \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k-1}a_0\right\} +$$

$$+ \left\{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} (-1)^{(k-\frac{n-1}{2})} p^{2k}a_0\right\} =$$

$$= \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k+1}a_0\right\} - \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k-1}a_0\right\} + \qquad \text{(B.1.10)}$$

$$+ \left\{\sum_{k=-\frac{n-2}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k-1}a_0\right\} =$$

$$= \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} (-1)^{(k-\frac{n}{2})} p^{2k+1}a_0\right\} - (-1)^{(-n)} p^{-n-1} =$$

$$= \left\{\sum_{k=-\frac{n+1}{2}}^{\frac{n+1}{2}} (-1)^{(k-\frac{n+1}{2})} p^{2k}a_0\right\} = a_{n+1}$$

and

$$b_{n+1} = 2\lambda_{\cos}b_n - b_{n-1} = (q + \frac{1}{q})\left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} q^{2k}b_0\right\} - \left\{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} q^{2k}b_0\right\} =$$

$$= \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} q^{2k+1}b_0\right\} + \left\{\sum_{k=-\frac{n}{2}}^{\frac{n}{2}} q^{2k-1}b_0\right\} - \left\{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} q^{2k}b_0\right\} =$$

$$\text{(B.1.11)}$$

$$= \left\{\sum_{k=-\frac{n+1}{2}}^{\frac{n+1}{2}} q^{2k}b_0\right\} + \left\{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} q^{2k}b_0\right\} - \left\{\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} q^{2k}b_0\right\} =$$

$$= \left\{\sum_{k=-\frac{n+1}{2}}^{\frac{n+1}{2}} q^{2k}b_0\right\} = b_{n+1}$$

and we see that the recursion relations are fulfilled.

List of Figures

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Abstract

In the present thesis, a problem almost as old as quantum physics itself is critically reviewed from a modern point of view. That problem is the so-called "quantum-phase problem", which was discovered as early as 1926 by Dirac and remains puzzling in some aspects to this day; its subject is a theoretical description of quantum phase, i.e. the phase of a quantized electromagnetic or other field described by harmonic oscillators.

While in this thesis, this problem is not solved (it has indeed reached a magnitude that by far eludes the scope of any reasonably-sized diploma thesis), a coherent and consolidated derivation of the two main operator-based phase theories developed to date, the Susskind/Glogower- and the Pegg/Barnett-formalism, is provided, by means of which key insights into the nature of phase and the unique difficulties plagueing its quantum description are gained. These insights include a systematic identification of the main issues at hand, an abstract reasoning about the existence of phase-operators and crosslinks to functional analysis and generalized measurement theory.

These theoretical parts are then followed by a brief primer on the experiments that have been conducted so far, and by an overview over the topic of uncertainty and uncertainty relations in the specific context of quantum phase. We close with general observations and remarks. Throughout the thesis, special emphasis is put on providing the intermediate steps of most of the derivations instead of just reproducing the results given in the literature, a useful process which at once operates as a sanity-check and illustrates how to work with the quantities involved in any treatment of quantum phase.

Zusammenfassung

In der vorliegenden Diplomarbeit wird aus heutiger Sicht ein Problem behandelt, dass beinahe so alt ist wie die Quantenphysik selbst: das sogenannte Problem der Quantenphase. Dieses Problem wurde bereits 1926 von Dirac entdeckt und stellt bis heute einen Aspekt der Quantenphysik dar, der nicht vollständig verstanden ist; das Problem betrifft eine theoretische Beschreibung von Quantenphase, d.h. des quantenmechanischen Äquivalents der Phase eines Feldes, welches sich durch harmonische Oszillatoren beschreiben lässt.

Das Problem wird in der vorliegenden Arbeit zwar keiner Lösung zugeführt (dazu ist das Thema viel zu groß und der vorhandene Platz viel zu gering), doch wird ein durchstrukturierter und konsolidierter Überblick über die zwei hauptsächlichen auf einer Beschreibung durch Operatoren beruhenden Quantenphasentheorien vorgenommen: den Susskind/Glogower- und den Pegg/Barnett-Formalismus. Mithilfe dieser beiden Ansätze werden wichtige Einblicke in die Natur der Quantenphase und die spezifischen Probleme, die sich ihrer Beschreibung in den Weg stellen, gewonnen. Diese Einblicke umfassen beispielsweise eine systematische Beschreibung der hauptsächlichen Problemkreise, eine abstrakte Diskussion darüber, ob die Konstruktion eines Quantenphasenoperators überhaupt möglich ist, und Querverbindungen etwa zur Funktionalanalysis oder der verallgemeinerten Messtheorie.

Auf diese eher theoretisch gehaltenen Arbeitsabschnitte folgt eine kurze Vorstellung der bislang durchgeführten Experimente und ein Überblick über Unschärfe und Unschärferelationen aus dem spezifischen Blickwinkel der Quantenphase. Die Arbeit schließt mit allgemeinen Beobachtungen und Bemerkungen ab. Durchgängig wird dabei besondere Aufmerksamkeit darauf gelegt, Zwischenschritte in wichtigen Ableitungen anzuführen anstatt Ergebnisse der Literatur einfach zu übernehmen, um einerseits eine Art Konsistenzprüfung zu unternehmen und andererseits ein Gefühl dafür zu erhalten, wie die typischen Größen einer Quantenphasentheorie miteinander interagieren.

Resume

Personal Information:

Full Name:	**Philipp Thun-Hohenstein**
Address:	Pötzleinsdorferstraße 52 A-1180 Wien, Austria
Mobile number:	+43 676 965 9047
E-mail:	p.thun-hohenstein@gmx.net
Nationality:	Austrian
Date of Birth:	May 30th, 1987
Place of Birth:	Geneva, Switzerland
Marital status:	Single
Academic degree:	Mag. iur.
Compulsory military service:	completed, 08/2006-01/2007

Education:

Since 04/2009	**Doctorate studies in Law (Dr. iur.)** University of Vienna, Expected graduation date 12/2010 Dissertation topic: „International commercial arbitration and insolvency"
Since 10/2007	**Undergraduate studies in International Business (B.Sc)** Vienna University of Economics and Business, Expected graduation date 10/2010 Specializations: Finance, Entrepreneurship & Innovation
Since 03/2007	**Undergraduate studies in Art History (Mag. phil.)** University of Vienna, Expected graduation date 03/2011 Thesis topic to be determined
Since 10/2005	**Undergraduate studies in Physics (Mag. rer. nat.)** University of Vienna, Expected graduation date 02/2010 Thesis topic: „Quantum Phase and Uncertainty"
10/2005-09/2008	**Undergraduate studies in Law (Mag. iur.)** University of Vienna, completed Specializations: International Commercial Arbitration, Intellectual Property, Commercial Law
06/2005	**High School Graduation (Abitur)** German School New York (GSNY), New York Numerus Clausus: 1,0 (equivalent to GPA 4.0)

International Experience:

Spring term of 2008	Student exchange (Erasmus) at the Université Paris II (Panthéon-Assas), Paris
2004-2005	Science Honors Program, Columbia University, New York
1999-2005	Secondary education abroad: German School New York (GSNY), New York

Professional Experience:	
12/2008-01/2009 03/2009-09/2009	**District Court of the Inner City of Vienna (Bezirksgericht Innere Stadt Wien) and Commercial Court of Vienna (Handelsgericht Wien), Vienna** Position: Judicial Clerk. Occupation: Research and Drafting for Court Orders and Decisions.
07/2008	**Schönherr Rechtsanwälte GmbH (law firm), Vienna** Position: Summer Associate. Occupation: Summer Associate Program; Practice Groups: Competition, M&A, Banking & Finance and Real Estate
02/2006-11/2007	**Hausmaninger Kletter RAe GmbH (law firm), Vienna** Position: Contractor. Occupation: Intense contribution to the publication of *Fasching/Konecny*, Zivilprozessgesetze[2] Band IV/2, Manz 2007 (Commentary of Austrian Arbitration Law)
07/2007-08/2007	**Patentanwaltskanzlei Sonn & Partner (patent attorney's office), Vienna** Position: Intern. Occupation: Research, Drafting and Translation Services in Patent and Trademark Law
07/2003	**Argos Capital Management, Inc., New York** Position: Intern. Occupation: IT-Administration for Portfolio Management Software
02/2003	**Propeller Z (architect's office), Vienna** **Hausmaninger Herbst RAe GmbH (law firm), Vienna**
Awards and Certificates:	
10/2009	2nd place (of 546) in official ranking of best Vienna University Law School Graduates
04/2009	Willem C. Vis International Commercial Arbitration Moot Court: Pieter Sanders Award, Honorable Mention Frédéric Eisemann Award (Orals), 9th place (of 228; tied)
10/2008	2nd place (of 647) in official ranking of best Vienna University Second/Third Year Law Students
06/2008	Certificates obtained during Student Exchange in Paris: Certificat élémentaire de droit français et européen Certificat de français de spécialité
2005-2009	University of Vienna Scholarships
2008-2009	Vienna University of Economics Scholarships
Language Skills:	
Fluent	German (native), English
Proficient	French
Basic	Italian, Russian
Miscellaneous:	
Computer skills	Microsoft Office, C++, Mathematica, LaTeX, Law Databases
Leisure Activities	Art and Art History, Foreign Languages, Piano, Tennis