



universität  
wien

# DIPLOMARBEIT

Titel der Diplomarbeit

## On surface water waves and tsunami propagation

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag. rer. nat.)

Verfasser: Raphael Stuhlmeier  
Matrikel-Nummer: 0547659  
Studienrichtung: Mathematik  
Betreuer: Univ.-Prof. Dr. Adrian Constantin

Wien, am 4. Februar, 2010



2000 *Mathematics Subject Classification.* 35Q51,76D33

*Key words and phrases.* Water waves, Tsunami, Korteweg de-Vries equation

ABSTRACT. This work introduces the inviscid governing equations for water waves from a physically motivated standpoint, in as accessible a manner as possible. From there, certain asymptotic regimes are explored, leading to the Korteweg-de Vries equation. Elaborations are made on applications to tsunami modeling, while taking care to point out shortcomings in the analytical approach as well as unresolved difficulties in reconciling the intriguing nature of water with mathematics.



# Contents

|  |    |
|--|----|
| Chapter 1. Introduction                                  | 1  |
| Chapter 2. The Governing Equations                       | 3  |
| 2.1. The equations of motion                             | 3  |
| 2.2. Boundary conditions                                 | 5  |
| 2.3. Vorticity   | 6  |
| Chapter 3. Asymptotic Models                             | 9  |
| 3.1. Derivation of the Korteweg de-Vries equation        | 9  |
| 3.2. Solitons  | 17 |
| 3.3. A brief look at inverse scattering theory           | 21 |
| 3.4. The Riemann-Hilbert problem for the KdV             | 27 |
| 3.5. Well-posedness and existence for asymptotic regimes | 31 |
| Chapter 4. Tsunami Modelling                             | 35 |
| 4.1. Tsunami   | 35 |
| 4.2. Dynamics near the generation region                 | 37 |
| 4.3. Dynamics in the far field                           | 39 |
| 4.4. Applications to Tsunami                             | 41 |
| 4.5. Measurements  | 46 |
| Appendix: deutsche Zusammenfassung                       | 49 |
| Acknowledgements   | 51 |
| Bibliography   | 53 |
| Index  | 57 |
| Curriculum Vitae   | 59 |



## CHAPTER 1

# Introduction

We have all been fascinated by the strange and wonderful motions of water. We have tried, as children, to dam it - unsuccessfully, as it always found a way out or around, over or under. We swim and dive in it, feeling its pressure and buoyancy, and we slide across its surface on craft great and small, from ocean liners to surfboards.

You will probably have spent time as a child throwing pebbles into it, too. Watching the strange play of waves across its surface. Yet even these simplest of waves on the still surface of a pond, small enough to be caused by surface tension, are somehow ephemeral, elusive. You may follow one with your eyes for a time, watching as it spreads out from where the pebble was thrown. As it makes an ever greater ring, it becomes longer and flatter until, suddenly, it is gone! But just as soon, another wave takes its place, seeming to replicate its form, growing in length and then disappearing too. If you have a steady supply of pebbles, you may replicate this observation, noticing too that there are, between groups of waves regions of still water, which move outwards as well.

Of course, you will have observed that wave motion is not generally a consequence of the water moving. A duck in our pond might move up and down with the passing of a wave, but not appreciably in the direction of the wave. Similarly, a surfer outside of the break is safe and relatively stationary, even on a big day at Waimea Bay, where wave heights can reach 15m. Of course, the situation quickly reverses itself as the wave breaks, and as anyone who has been in the situation can attest, the displaced water surges forward at great speed, swallowing unwary ducks, surfers, and much else.

These examples serve to illustrate a point - wave propagation is, in fact, the propagation of *energy*. This energy can come in wave packets, as in the case of capillary waves. The phenomena we see on a pond - this seeming disappearance of waves near the front of a wave group, and the birth of waves at the back of a wave group - is a consequence of the fact that the waves we see travel faster than the energy! There are really two things propagating here, and of course, this raises the question of what a *wave* is. We have seen that to answer this question is not so simple. Our disturbances are constantly shifting form, making it hard to point at some geometric feature persistent in time and claim it to be “the wave”. While some waves may be periodic in nature, there are numerous examples (which will feature prominently below) where this is not the case. The most famous of these is the solitary wave, but a flood wave, as when a dam breaks, is surely a wave without being periodic. And besides, there are many effects in nature that cause attenuation of amplitudes, elongation of wavelengths, and general breakdown of periodicity. It seems like a lost cause; we might throw our hands in the air and

claim that everything is a wave, but we will not. The salient feature of a wave is that some property propagates through space at a finite speed.

This means, for example, that in the elementary theory of linear partial differential equations, the heat equation is shown to possess infinite propagation speed, while the wave equation propagates disturbances at a finite rate. Thus we do not refer to the flow of heat in terms of a wave, although we do for wave equation - despite the fact that, as we shall see below, the wave equation gives a “wave” of infinitesimally small amplitude.

This is already quite a modern point of view, so we will take a moment to look back. Man has worked with water in practical ways, from irrigation, to drainage and flood control, for millenia. The first scientific investigations, however, stem from Archimedes, who famously (and apocryphally) discovered the principle of buoyancy while in the bath (upon which he is said to have shouted “eureka” while running nude through the streets of Syracuse). Progress then stagnated for nearly 1800 years.

In the late 15<sup>th</sup> and early 16<sup>th</sup> centuries, Leonardo da Vinci spent part of his many talents on observing and formulating laws for the motion of water - such as the principle of continuity that the speed of a stream varies inversely to its cross-sectional area. A century later, Simon Stevin discovered that the pressure at the bottom of a container was equal to the weight of the column of water above it - independent of the shape of the container - the so-called hydrostatic paradox. Finally, after contributions by Galileo, Pascal, and Torricelli, the theory of hydraulics got its first big boost by the establishment of Newtonian mechanics. On this basis, Johann and Daniel Bernoulli (father and son) spent a number of fruitful years working on hydrodynamics, and Leonhard Euler, contemporary of Daniel Bernoulli, was able to derive from physical principles equations for fluid motion that are still used today.

After this point, it is hard to cite all important contributors. Fluid mechanics became a major area of research, stimulating new discoveries in pure and applied mathematics. The new “rational hydrodynamics” based on the study of Euler’s equations generated many important ideas in the theory of partial differential equations, from Green’s functions to eigenfunctions, shock waves, characteristics, even the concept of well-posedness first arose from the study of fluid motion. These rapid advances show no signs of slowing down. Rather, they have been supplemented by numerical methods and become central in many of the technologies we take for granted today. Chief among the successes of numerical experimentation is the discovery of the soliton, of which we will have much to say below.



## CHAPTER 2

# The Governing Equations

In order to study the motion of a fluid physically as well as mathematically (as opposed to simply enjoying the motions of waves in the ocean, ripples in a pond, or eddies in a fast flowing river from an aesthetic standpoint) we need governing equations. In fact, we will neglect something very central, namely viscosity, in our study of fluid flow. This is what caused John von Neumann to quip, quite tellingly, that this study was better termed the study of “dry water”. While there are certainly some fascinating aspects we will miss by neglecting viscosity (e.g. the existence of boundary layers or the fading away of the free surface disturbance due to friction), many useful models can be derived from the Euler equations. We refer to [Joh97, Lig78, MT68] for a discussion of the physical relevance of neglecting viscosity in the study of free surface water waves.

### 2.1. The equations of motion

We begin with water at rest, and will add motion later on, following the elegant development in [FLS63]. First though, we need to clarify what separates a fluid from a solid - what does it mean that something ‘flows’? Physically, a fluid is such that it cannot maintain shear stress. Of course, physically, how much a fluid *resists* this shear (think of honey, or air) depends on viscosity. Also, it is clear that we cannot explain everything with this description - glass, for example is a fluid, however you can clearly slide a glass across a table top.

Now, when a fluid is at rest, there are no shear forces, only forces due to pressure - always acting normal to a given surface. It is then easy to see that the pressure is then the same in any direction - the pressure is *isotropic*. We will go on to make another simplification: that the density of the fluid is constant. To make this plausible, we will concentrate not on any arbitrary fluid (a term which encompasses gases, which are largely compressible) but on water (which is largely incompressible). By this we mean that variations in pressure which we encounter will have negligible effect on the density. (You may contrast this with the law of Boyle-Mariott for ideal gases, where varying the pressure while keeping density constant results in a change in volume.) This simplification means that phenomena which depend on density changes, such as the propagation of sound, will have no place in our equations. (Neglecting compressibility is equivalent to the fact that wave speed  $c$  is small compared to the speed of sound. In fact, for a given water depth  $h$ , the wave speed may be estimated by  $\sqrt{gh}$  ([Lig78] erroneously states that this is the greatest possible wave speed - this is not the case for solitons). Thus  $h \ll c_s^2 g^{-1} = 200\text{km}$ . We note that, on this planet, the deepest part of the ocean is the Challenger Deep of the Mariana Trench, at 11,034 m.)

We also need to take into account forces other than pressure, such as gravity, which will come to play a major role in our treatment of water waves. With

gravitational force per unit mass given by the potential  $\nabla\phi$ , we see that the force per unit volume is simply  $-\rho\nabla\phi$ . Consider now a volume of water in the shape of a small cube, and say we orient the coordinate axes parallel to the edges of the cube. Considering always pairs of faces, so that we may split the pressure force into components along the  $x$ ,  $y$ , and  $z$  axes, we see that the pressure per unit volume is simply  $-\nabla P$ . Thus equilibrium between pressure and gravitational forces will occur for

$$-\nabla P - \rho\nabla\phi = 0.$$

In order for this equation to have a solution, it is clearly necessary that we restrict  $\rho$  in some manner. In our case of constant density, any line of constant  $P + \rho\phi$  is a solution.

**2.1.1. Mass conservation.** We have already explained some of the reasoning behind the *equation of state*

$$\rho = \text{const.}$$

which connects the pressure to the density (in a particularly simple way, here). The next step is to express the conservation of mass. If water flows away from a point, the amount left behind decreases in kind. Thus, denoting the fluid velocity with  $u(x(t), y(t), z(t), t)$ , where  $u = (u, v, w)$ , we see that the mass passing through a unit area of surface is the normal component of  $\rho u$  to the surface. By the divergence theorem, we have

$$0 = - \int_{\partial V} \rho u \cdot n dS = \int_V \nabla \cdot u dx$$

for any volume  $V$ . Therefore

$$(2.1) \quad \nabla \cdot u = 0$$

everywhere, which is the equation of *mass conservation* (or *continuity equation*).

**2.1.2. Euler's equations.** The next equation we shall derive directly from Newton's law  $F = ma$ . That is, the mass of a fluid element multiplied with its acceleration must equal the force acting upon it. As before, we take an element of unit volume, so that its mass is given by  $\rho$ , then

$$\rho a = F.$$

Recall from the discussion of hydrostatics in the beginning of Section 2.1 that we already know what the forces are, namely a force due to pressure and a force due to gravity (having neglected all others). While they are balanced when the water is at rest, now we have

$$\rho a = -\nabla P - \rho\nabla\phi.$$

All that is missing now is the acceleration. But, the acceleration of what? One might be tempted to write simply  $a = \frac{\partial u}{\partial t}$ , but this is not quite right. The partial differential, with respect to  $t$ , say, assumes that all other arguments are constant while  $t$  varies. Writing the acceleration as above will give you only how fast the velocity changes at some fixed point in the fluid. This is not what we need in Newton's law; rather, we need to determine the acceleration for some point as it *moves* through the fluid, which means allowing the other  $t$  dependent arguments to vary. Hence the correct acceleration is given by the total differential

$$a = \frac{du(x(t), y(t), z(t), t)}{dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u.$$

Thus we call

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

the *material derivative* with respect to  $u$ . To demonstrate the fundamental difference between this and the partial derivative mentioned above, consider the example of water flow in an annulus. The acceleration at any fixed point is zero (the partial derivative), while the acceleration of any particle<sup>1</sup> is nonzero everywhere (the material derivative). Putting all this together, we have

$$(2.2) \quad \frac{Du}{Dt} = -\frac{1}{\rho} \nabla P - \nabla \phi,$$

the famed *Euler equations*. Of course, in the case of gravity  $\phi = gz$  and  $-\nabla \phi = (0, 0, -g)$ .

## 2.2. Boundary conditions

So far, we have arrived at the equations for the flow of an inviscid, incompressible fluid, but we want to study waves. We need to differentiate our problems involving wave propagation from others which we could model with the equations derived above - say, the flow of blood through your circulatory system, or oil through a pipeline, air along the wings of an airplane, and many more.<sup>2</sup> The way we undertake this is by prescribing boundary conditions. Consider the case of waves in the open sea: there are essentially two boundaries - the sea-floor, which we can measure (at least macroscopically - if we want to account for sediment that moves with the water, this is another problem), and the surface of the water. The latter is called the *free surface* because its determination is part of the solution. This is the major difficulty in the study of water waves.

On the free surface of the water, we need to account for the fact that we are trying to describe the boundary between two different phases, i.e. an interface. The atmosphere above interacts with the water, though since we are not studying wave generation by means of the wind (cf. [BK75]) we will decouple the motion of the water surface from that of the air in a very simple manner.<sup>3</sup> To this end, the simple decoupling of the atmosphere and the water takes the form

$$(2.3) \quad P = P_{atm} \text{ on the free surface.}$$

The small density of air compared with water makes it reasonable to assume, even for large waves, that the atmospheric pressure  $P_{atm}$  varies little between wave crest and trough. Taking  $z$  the vertical direction, we will mainly consider the case of a flat bed at  $z = 0$ . Considerations of dynamics on a non-flat bed  $z = b(x, y)$  will

---

<sup>1</sup>In the above we have referred variously to water particles, elements, and even points. This may, of course, be a little disquieting, given that water is composed of molecules of finite size. We note (and you may consult [Yih77] for further discussion, and the source of our example) that the number of molecules of a gas in  $1 \mu^3$  is about  $2.69 \times 10^{27}$  (using the Loschmidt constant at 1 atm and 0°C) at standard temperature and pressure. The number for a fluid will be higher still. This means that when we refer to the velocity of a point, it is reasonable to consider this the average velocity of a great number of particles, this in turn as the velocity of their center of mass.

<sup>2</sup>Of course, we would have try to justify  $\rho = const.$  and neglect of viscosity in all these cases. In fact, blood is a non-Newtonian fluid.

<sup>3</sup>The motion of wind across the water can only generate waves if we allow for a shear force. We could conceivably model the passage of storms or pressure-systems over the water by allowing the pressure to vary in time and space.

be mentioned as special cases. We will write the free surface  $z = h(x, y, t)$  where  $h$  stands for “height”.

The next step in our decoupling of the air and water is to make sure that the free surface is composed of water particles at all times - no particles leave the body of water. This implies that velocities along this surface have no normal component. This is called the *kinematic condition*. We can reformulate this to read: the surface,  $S(\mathbf{x}, t) = \text{const.}$  say, moves with the fluid so that it contains always the same fluid particles. This means that the material derivative (2.1.2) of this surface must vanish:

$$\frac{DS}{Dt} = 0$$

Now, taking  $S(x, t) = z - h(x, y, t) = 0$ , we see that

$$\frac{D}{Dt}(z - h(x, y, t)) = 0.$$

which yields

$$(2.4) \quad w = h_t + uh_x + vh_y \text{ on the free surface } z = h(x, y, t).$$

Similarly, we have a kinematic condition for the bottom, the interface between water and the sea-floor, which we find in a manner analogous to the above considerations. In the general case  $z = b(x, y)$ , we again postulate that no water may penetrate the sea-floor, so that

$$\frac{D}{Dt}(z - b(x, y)) = 0.$$

This, in turn, yields

$$(2.5) \quad w = ub_x + vb_y \text{ on the bottom } z = b(x, y).$$

In the simplified case of a flat bed  $z = 0$ , this is clearly  $w = 0$  on  $z = 0$ .

### 2.3. Vorticity

Next, we will introduce a concept central to many questions about the flow of water. This concept is called *vorticity*. Before we get into the details, it is useful to have an idea of what vorticity means. In a sense it is a measure of local spin. A useful analogy (from the classical text [MT68]): if a spherical element of the water were suddenly solidified, would it spin as the water flowed past it. Of course, a uniform velocity field  $u$  has no such local spin. On the other hand, if  $u$ , for example, decreases with depth, we see that there is a local spin; if  $u$  describes a flow of constant angular velocity about an annulus (in 2 dimensions, or a cylinder in 3 dimensions), although there is clearly a global rotation, local rotation is absent. Vorticity is thus simply  $\omega = \nabla \times u$ . We call flows with  $\omega = 0$  *irrotational*, and we will see that these flows have very special and useful properties - especially in that we can thereby reduce the nonlinear Euler equations to the linear Laplace equation  $\nabla^2\phi = 0$  (see below). This is already a significant improvement, although we still have the free boundary to deal with.

Furthermore, if we start with irrotational flow, we need never worry about vorticity cropping up, by virtue of

**THEOREM 2.1** (Lagrange’s Theorem). *A water flow that is irrotational initially will be irrotational at all later times.*

A proof may be found in most texts on classical hydrodynamics, e.g. [Lam95]. Although in real water flows, vorticity is rarely absent, it is usually small enough that it does not play a major role in water wave dynamics unless we wish to account for the presence of underlying non-uniform currents [KS08b, KS08a, CS04, CE04].

**2.3.1. The irrotational equations of motion.** The equations of motion along with their boundary conditions are commonly expressed as follows:

$$(2.6) \quad \frac{Du}{Dt} = -\nabla P - g$$

$$(2.7) \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$$(2.8) \quad P = P_{atm} \text{ and } w = h_t + uh_x + vh_y \text{ on } z = h(x, y, t)$$

$$(2.9) \quad w = ub_x + vb_y \text{ on } z = b(x, y)$$

There are, however, equivalent formulations, and we will make use of one that is particularly useful in the irrotational case, involving a velocity potential. Assuming  $\omega = \nabla \times u = 0$ , we see that  $u = \nabla \phi$  for a so-called *velocity potential*  $\phi$ , if we consider flows in a simply connected domain, as is the case for water waves. Then, from (2.7),  $\nabla \cdot u = 0$  we see that  $\phi$  satisfies the *Laplace equation*

$$\nabla^2 \phi = 0.$$

Making use of the identity  $\nabla(u \cdot u) = 2(u \cdot \nabla)u + 2u \times (\nabla \times u)$ , we can rewrite (2.6) as

$$\frac{\partial u}{\partial t} + \nabla\left(\frac{1}{2}u \cdot u + P + gz\right) = u \times \omega$$

which, in view of  $u = \nabla \phi$  implies

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + P + gz\right) = 0$$

or

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + P + gz = f(t)$$

for some constant of integration. This is known as *Bernoulli's equation*. The dynamic boundary condition in (2.8) becomes

$$\phi_t + \frac{1}{2}|\nabla \phi|^2 + P + gh = 0 \text{ on } z = h(x, y, t),$$

where we have absorbed  $f(t)$  into  $\phi$ . The kinematic boundary condition of (2.8) becomes

$$\phi_z = h_t + \phi_x h_x + \phi_y h_y \text{ on } z = h(x, y, t),$$

while the bottom condition (2.9) becomes

$$\phi_z = \phi_x b_x + \phi_y b_y \text{ on } z = b(x, y).$$

The well-posedness of the governing equations for irrotational water waves <sup>4</sup> was established in [CS07b, Lan05, Wu97]. This will be addressed in more detail in Section 3.5 below.

<sup>4</sup>Whereby we mean local existence in time with continuous dependence on the initial data for  $u$  and  $h$  in suitable Sobolev spaces, given an initial velocity  $u(x, y, z, 0)$ , an initial wave profile  $h(x, y, 0)$ , and an initial pressure  $P(x, y, z, 0)$  throughout the fluid. The existence of  $P$  is viewed as a closing condition for the system, being determined from a knowledge of  $u$  by means of Bernoulli's equation.



## Asymptotic Models

Although derived more than 200 years ago by Leonhard Euler, the free boundary problem associated with the governing equations for water waves has turned out to be too difficult to allow the development of a direct theory. It is one of the astounding facts about mathematical fluid mechanics that, despite centuries of study, only one explicit solution to the free boundary problem of the Euler equations was ever found. This solution is the so called *Gerstner wave*, discovered in 1802 by its namesake František Josef Gerstner (1756 - 1832). Gerstner described two-dimensional waves where all particles circumscribe circular trajectories, whose radius decreases with depth. The free surface is then in the form of a trochoid, or, in the limit case, a cycloid. A rigorous examination of the Gerstner flow was undertaken in [Con01] (see also [Hen08]). It is, in fact, quite remarkable that such a simple idea, each particle moving in a circular trajectory, would lead to a solution where particles never collide and yet fill out the entire region below the surface. On the other hand, there are many examples throughout history of people who sought to model specific phenomena observable in connection with water waves. Asymptotic methods have always played a central role in this endeavor, as they give us simpler equations that are nevertheless connected to the Euler equations. In what sense we should interpret this connection, we shall see below.

### 3.1. Derivation of the Korteweg de-Vries equation

The Korteweg de-Vries equation, one of the most prominent of the asymptotic models associated with the Euler equations, describes the propagation of plane waves. Thus in what follows, we will use the two-dimensional form of the irrotational Euler equations. Here we use the usual planar coordinates  $x$  and  $y$  to denote the horizontal respectively vertical directions,  $u$  and  $v$  the corresponding velocities. Therefore, mass conservation takes the form

$$(3.1) \quad u_x + v_y = 0.$$

The Euler equations are

$$(3.2) \quad \frac{Du}{Dt} = -\frac{\partial P}{\partial x} \quad \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} - g$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}.$$

The condition of irrotationality is simply

$$(3.3) \quad u_y - v_x = 0.$$

---

Some of the material in this chapter has been adapted from [Stu09]

We write  $y = h_0 + H(x, t)$  for the free surface, where  $h_0$  is an average depth for the water under consideration, and denote the flat bed by  $y = 0$ . Note that it is possible to include an analysis of these dynamics with a non-flat bed as in [CJ08b, Isr] provided the variations in bottom topography are limited appropriately. In order to make the derivations that follow as transparent as possible, we will restrict ourselves to the case of a flat bed. Lastly, the boundary conditions take the form:

$$(3.4) \quad P = P_{atm} \text{ on the free surface } y = h_0 + H(x, t)$$

$$(3.5) \quad v = H_t + uH_x \text{ on } y = h_0 + H(x, t)$$

$$(3.6) \quad v = 0 \text{ on } y = 0$$

**3.1.1. Non-dimensionalization and Scaling.** Roughly speaking, one might say that nonlinear phenomena arise through the interaction of physical parameters on scales of differing magnitude. “In a singled out scale, the same physical quantity may manifest itself linearly. This phenomenon is sometimes referred to as the method of nonlinear separation of variables” [She93]. As such, given that we are working with physical variables (e.g.  $[u]=m/s$ ,  $[\lambda]=m$ , etc.), in order to compare magnitudes meaningfully, the first step is to get rid of their units. Experience has borne out the fact that nonlinear problems are often fruitfully tackled by approximation - by introducing special scales to an otherwise too expansive problem and then considering regimes corresponding to certain values of these scales. Simpler approximate equations in such regimes permit an in-depth analysis of waves enjoying special attributes (such as the existence of solitons [CE07, Joh03] or stability properties [CS07a].)

3.1.1.1. *Non-dimensionalization.* In keeping with this, we introduce  $h_0$  as the typical depth of the water and  $\lambda$  the typical wavelength. These two scales provide the basis for a nondimensional version of the governing equations. The characteristic speed for long gravity waves is taken to be  $\sqrt{gh_0}$ , and together with the wavelength  $\lambda$  this gives us a time scale for horizontal propagation of the wave,  $\lambda/\sqrt{gh_0}$ , as well as horizontal speed. Care must be taken with the vertical speed  $v$  in order to be consistent with (3.1). These considerations lead us to the following non-dimensional variables, which we denote with the usual variable names:

$$(3.7) \quad x \longrightarrow \lambda x \quad y \longrightarrow h_0 y \quad t \longrightarrow \frac{\lambda}{c} t \quad c = \sqrt{gh_0}$$

$$(3.8) \quad u \longrightarrow cu \quad v \longrightarrow cv \frac{h_0}{\lambda}$$

Accordingly, we transform the pressure  $P$  into a perturbation of the hydrostatic pressure as follows

$$(3.9) \quad P = P_{atm} + g(h_0 - y) + gh_0 p$$

where  $p$  is a new non-dimensional pressure variable. Lastly we set

$$(3.10) \quad H(x, t) = a\eta(x, t)$$

where  $\eta$  is the nondimensional surface profile and  $a$  is a typical amplitude.

The components of the Euler equation (3.2) under these transformations become

$$(3.11) \quad \frac{Du}{Dt} = -\frac{\partial p}{\partial x} \quad \delta^2 \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}$$



where  $\delta = h_0/\lambda$  is the *long wavelength* or *shallowness* parameter. Owing to our abuse of notation, the equation of mass conservation (3.1) remains unchanged. The irrotationality condition (3.3) becomes

$$(3.12) \quad u_y - \delta^2 v_x = 0$$

The second characteristic parameter enters via the transformation of the boundary conditions in accordance with (3.7) - (3.10)

$$(3.13) \quad p = \epsilon\eta \text{ on } y = 1 + \epsilon\eta(x, t)$$

$$(3.14) \quad v = \epsilon\left(\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x}\right) \text{ on } y = 1 + \epsilon\eta(x, t)$$

$$(3.15) \quad v = 0 \text{ on } y = 0$$

where the  $\epsilon = a/h_0$  is the so-called *amplitude parameter*.

3.1.1.2. *Scaling.* At this point, we note that  $\epsilon$  and  $\delta$  between them determine the type of water wave problem under consideration. Looking at the surface boundary conditions, we see that  $v$  and  $p$  are both proportional to  $\epsilon$ , the wave amplitude. This is sensible, since as  $\epsilon \rightarrow 0$  the vertical velocity  $v \rightarrow 0$  and of course the pressure perturbation  $p \rightarrow 0$ ; the free surface is perfectly flat. Taking advantage of this, we define a set of scaled variables

$$(3.16) \quad p \rightarrow \epsilon p, \quad v \rightarrow \epsilon v, \quad u \rightarrow \epsilon u$$

where  $u$  is scaled similarly for consistency (note that these formal considerations are supported by rigorous results - see [CSnt]). This leads to the transformation of the system of equations (3.11) with  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \epsilon(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})$ , which we can write explicitly as

$$(3.17) \quad u_t + \epsilon(uu_x + vu_y) = -p_x$$

$$(3.18) \quad \delta^2(v_t + \epsilon(uv_x + vv_y)) = -p_y$$

The equation of mass conservation (3.1) again remains unchanged, as does the nondimensional irrotationality condition (3.12).

The boundary conditions (3.13) - (3.15) become

$$(3.19) \quad p = \eta \text{ on } y = 1 + \epsilon\eta$$

$$(3.20) \quad v = \eta_t + \epsilon\eta_x \text{ on } y = 1 + \epsilon\eta$$

$$(3.21) \quad v = 0 \text{ on } y = 0$$

3.1.1.3. *Approximate Equations.* We now have a series of equations depending on two parameters  $\epsilon$  and  $\delta$  which measure contributions of amplitude, respectively wavelength, to the problem under consideration. The most common approximations made are  $\epsilon \rightarrow 0$  for fixed  $\delta$  and  $\delta \rightarrow 0$  for fixed  $\epsilon$ . These are known as the *linearized problem* and *shallow-water* (or *long wave*) *problem* respectively. As noted above, in the first approach, the hitherto unknown free surface becomes the surface  $y = 1$ , and in a first approximation, we have a linear problem with dispersive effects. In the latter, a glance at (3.18) shows that the pressure becomes independent of  $y$ ; dispersive effects are neglected. For more details see e.g. [Joh97].

While these approximations have been used extensively (and sometimes with some abandon) in the history of water-wave problems, the question remains to what extent any formal asymptotic model can give us relevant results for water waves.

Two questions arise: does an asymptotic model provide a good approximation of a solution to the Euler equations, and is the time scale of the model applicable.

Our goal of understanding the dynamics of the Korteweg-de-Vries equation (henceforth abbreviated KdV) and its possible application to tsunami must therefore proceed cautiously. Roughly, we understand that the KdV describes a balance between nonlinearity and dispersion, which means that we will need to retain both parameters  $\epsilon$  and  $\delta$  to some order in the above equations.

We note that the parameters occur in our equations to order  $\epsilon$  and  $\delta^2$  - it turns out that these are precisely the orders that must be retained. It has been pointed out in [CJ08a] that the KdV may be derived without this assumption by using an additional scaling,

$$\begin{aligned} x &\rightarrow \frac{\delta}{\sqrt{\epsilon}}x, & y &\rightarrow y, & t &\rightarrow \frac{\delta}{\sqrt{\epsilon}}t \\ p &\rightarrow p, & \eta &\rightarrow \eta, & u &\rightarrow u, & v &\rightarrow \frac{\sqrt{\epsilon}}{\delta}v. \end{aligned}$$

As is remarked therein, there is no compelling physical connection between  $\epsilon$  and  $\delta$ , nor more generally between amplitude, wavelength, and water depth. In view of this additional scaling, it is possible to find a KdV balance at some time and place for *any*  $\delta$  provided  $\epsilon \rightarrow 0$ . Unfortunately, no rigorous results exist to corroborate this formal conclusion. Therefore we will retain the classical point of view for which rigorous results are available.

**3.1.2. The Korteweg-de Vries equation (KdV).** Starting from the equations (3.17) - (3.21) we will proceed to derive KdV (introduced in [KDV95]), following the exposition in [Joh97]. As discussed above, we take the classical approach and consider a special choice of parameters, namely  $\delta^2 = O(\epsilon)$  as  $\epsilon \rightarrow 0$ . Then the equations of motion (3.17) - (3.21) along with mass conservation (3.1) and irrotationality (3.12) now appear with  $\delta$  scaled out in favor of  $\epsilon$ :

$$(3.22) \quad u_t + \epsilon(uu_x + vu_y) = -p_x \quad \epsilon(v_t + \epsilon(uv_x + vv_y)) = -p_y$$

$$(3.23) \quad u_x + v_y = 0$$

$$(3.24) \quad u_y - \epsilon v_x = 0$$

$$(3.25) \quad p = \eta \text{ on } y = 1 + \epsilon\eta$$

$$(3.26) \quad v = \eta_t + \epsilon u\eta_x \text{ on } y = 1 + \epsilon\eta$$

$$(3.27) \quad v = 0 \text{ on } y = 0$$

Now we let  $\epsilon \rightarrow 0$  and observe that for a first order approximation (3.22) implies  $u_t + p_x = 0$  as well as that  $p$  is independent of  $y$ . Therefore (3.25) implies that  $p = \eta$  everywhere and (3.24) that  $u$  is independent of  $y$ . It then follows from (3.23) that

$$(3.28) \quad v = -yu_x$$

which satisfies the boundary condition (3.27). Furthermore, since  $v = \eta_t$  on  $y = 1$ , (3.28) implies  $\eta_t = -u_x$  or  $\eta_t + u_x = 0$ . This together with mass conservation (3.23) means that  $\eta$  fulfills the wave equation

$$(3.29) \quad \eta_{tt} - \eta_{xx} = 0$$

We know that the wave equation has right-running as well as left-running solutions (cf. [Eva98]), and will follow the right-running waves, consistent with the introduction of the characteristic variable  $\xi = x - t$ . We also introduce a slow time scale

$$(3.30) \quad \tau = \epsilon t$$

in order to treat the far-field region, where we consider the regime for  $\xi = O(1)$  and  $\tau = O(1)$ . In view of this, we can rewrite the equations of motion (3.22) - (3.27) as follows:

$$(3.31) \quad u_\xi + \epsilon(u_\tau + uu_\xi + vv_y) = -p_\xi \quad \epsilon(-v_\xi + \epsilon(v_\tau + uv_\xi + vv_y)) = -p_y$$

$$(3.32) \quad u_\xi + v_y = 0$$

$$(3.33) \quad u_y - \epsilon v_\xi = 0$$

$$(3.34) \quad p = \eta \text{ on } y = 1 + \epsilon\eta$$

$$(3.35) \quad v = -\eta_\xi + \epsilon(\eta_\tau + u\eta_\xi) \text{ on } y = 1 + \epsilon\eta$$

$$(3.36) \quad v = 0 \text{ on } y = 0$$

We would now like to determine an approximate solution in terms of an asymptotic series Ansatz in  $\epsilon$  (for background cf. [Nay81]) by introducing the series expansions

$$(3.37) \quad \eta(\xi, \tau, \epsilon) \sim \sum_{n \geq 0} \epsilon^n \eta_n(\xi, \tau) \quad u(\xi, \tau, y, \epsilon) \sim \sum_{n \geq 0} \epsilon^n u_n(\xi, \tau, y)$$

$$(3.38) \quad v(\xi, \tau, y, \epsilon) \sim \sum_{n \geq 0} \epsilon^n v_n(\xi, \tau, y) \quad p(\xi, \tau, y, \epsilon) \sim \sum_{n \geq 0} \epsilon^n p_n(\xi, \tau, y)$$

Notice, however, that we have a problem in (3.34) and (3.35):  $\epsilon$  appears both in the coefficients as well as the arguments, making it impossible to equate powers of epsilon as we would like to do. To this end, we perform a transfer of the boundary conditions from  $y = 1 + \epsilon\eta$  to  $y = 1$  by expanding  $p(\xi, \tau, y)$ ,  $u(\xi, \tau, y)$  and  $v(\xi, \tau, y)$  in Taylor series about  $y = 1$  as follows

$$(3.39) \quad p(\xi, \tau, 1 + \epsilon\eta) = p(\xi, \tau, 1) + p_y(\xi, \tau, 1)\epsilon\eta + \frac{1}{2!}p_{yy}(\xi, \tau, 1)\epsilon^2\eta^2 + \dots$$

Substitute this into (3.34) and apply the series expansions for  $\eta$  and  $p$  in (3.37) and (3.38) to get

$$(3.40) \quad p_0 + \epsilon p_1 + \epsilon\eta_0 p_{0y} = \eta_0 + \epsilon\eta_1 + O(\epsilon^2)$$

Analogously substitute the Taylor series for  $u, v$  into (3.35) and expand to get:

$$(3.41) \quad v_0 + \epsilon v_1 + \epsilon\eta_0 v_{0y} = -(\eta_{0\xi} + \epsilon\eta_{1\xi}) + \epsilon(\eta_{0\tau} + u_0\eta_{0\xi}) + O(\epsilon^2)$$

At leading order ( $\epsilon^0$ ) (3.31) - (3.36) then reduces to:

$$(3.42) \quad u_{0\xi} = p_{0\xi} \quad p_{0y} = 0$$

$$(3.43) \quad u_{0\xi} + v_{0y} = 0$$

$$(3.44) \quad u_{0y} = 0$$

$$(3.45) \quad p_0 = \eta_0 \text{ on } y = 1$$

$$(3.46) \quad v_0 = -\eta_{0\xi} \text{ on } y = 1$$

$$(3.47) \quad w_0 = 0 \text{ on } y = 0$$

This system is analogous to that in (3.22) - (3.27) above - and the analogous arguments lead to the fact that  $p_0 = \eta_0$ ,  $u_0 = \eta_0 + C$ ,  $C$  a constant which we may assume to be zero, and  $v_0 = -y\eta_0\xi$ .

At order  $\epsilon^1$  we get

$$(3.48) \quad u_{0\tau} + u_0u_{0\xi} + v_0u_{0y} - u_{1\xi} = -p_{1\xi}$$

$$(3.49) \quad -v_{0\xi} = -p_{1y}$$

$$(3.50) \quad u_{1\xi} + v_{1y} = 0$$

$$(3.51) \quad u_{1y} - v_{0\xi} = 0$$

$$(3.52) \quad p_1 + \eta_0p_{0y} = \eta_1 \text{ on } y = 1$$

$$(3.53) \quad v_1 + \eta_0v_{0y} = -\eta_{1\xi} + \eta_{0\tau} + u_0\eta_{0\xi} \text{ on } y = 1$$

$$(3.54) \quad v_1 = 0 \text{ on } y = 0$$

Recall that we know from the first order approximation

$$\begin{aligned} p_0 &= \eta_0 & u_0 &= \eta_0 & v_0 &= -y\eta_0\xi \\ p_{0y} &= 0 & u_{0y} &= 0 & v_{0y} &= -\eta_0\xi \end{aligned}$$

Taking this into account, and in view of the boundary conditions in the second approximation it is easy to see

$$p_1 = \eta_1 \text{ on } y = 1 \text{ and } p_1 = \frac{1-y^2}{2}\eta_{0\xi\xi} + \eta_1$$

Now we would like to eliminate  $\eta_1$  and get an equation solely in  $\eta_0$ . Notice that

$$\begin{aligned} v_{1y} = -u_{1\xi} = -p_{1\xi} - u_{0\tau} - u_0u_{0\xi} &= \frac{y^2-1}{2}\eta_{0\xi\xi\xi} - \eta_{1\xi} - u_{0\tau} - u_0u_{0\xi} = \\ &= \frac{y^2-1}{2}\eta_{0\xi\xi\xi} - \eta_{1\xi} - \eta_{0\tau} - \eta_0\eta_{0\xi} \end{aligned}$$

Integrating with respect to  $y$  yields

$$v_1 = \frac{y^3}{6}\eta_{0\xi\xi\xi} - y\left(\frac{1}{2}\eta_{0\xi\xi\xi} + \eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi}\right)$$

which on the free surface  $y = 1$  is equal to  $-\eta_{1\xi} + \eta_{0\tau} + 2\eta_0\eta_{0\xi}$ , whereupon the factor  $-\eta_{1\xi}$  cancels and we have:

$$(3.55) \quad 2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0$$

the Korteweg-de Vries equation.

**3.1.3. The far-field vs. the near field.** One issue we have glossed over in the above is the rationale behind introducing far-field variables. This is essential, as we will see below, since we must know at what spatial and temporal scales the KdV dynamics become important in order to study their bearing on natural phenomena. We will see that the scaling (3.30) is essential in the KdV by studying the same problem without this scaling. We follow the treatment in [Joh97, Joh05]. Our starting point is again the nondimensionalized, scaled governing equations (3.22) - (3.27). Again we expand  $\eta, u, v$  and  $p$  in asymptotic series, and transfer the boundary conditions, albeit without a change of variables. Note that, at first order,

essentially nothing has changed - we again recover the wave equation. At order  $\epsilon$  we have:

$$(3.56) \quad u_{1t} + u_0 u_{0x} + v_0 u_{0y} = -p_{1x}$$

$$(3.57) \quad v_{0t} = -p_{1y}$$

$$(3.58) \quad u_{1x} + v_{1y} = 0$$

$$(3.59) \quad u_{1y} - v_{0x} = 0$$

$$(3.60) \quad p_1 + \eta_0 p_{0y} = \eta_1 \text{ on } y = 1$$

$$(3.61) \quad v_1 + \eta_0 v_{0y} = \eta_1 + u_0 \eta_{0x} \text{ on } y = 1$$

$$(3.62) \quad v_1 = 0 \text{ on } y = 0$$

We recall some useful facts from the first order expansion:

$$p_0 = \eta_0, \quad u_{0x} = -\eta_{0t}, \quad v_0 = -y u_{0x}, \quad u_{0y} = 0, \quad p_{0y} = 0$$

Thus we see (3.60) implies  $p_1 = \eta_1$  on  $y = 1$ , which, together with (3.57) means

$$-p_{1y} = v_{0t} = -y u_{0xt} \Rightarrow p_1 = -\frac{1-y^2}{2} u_{0xt} + \eta_1$$

and

$$(3.56) \Rightarrow u_{1t} + u_0 u_{0x} = \frac{1-y^2}{2} u_{0xxt} + \eta_{1x}.$$

Then

$$(3.58) \Rightarrow v_{1yt} = -\frac{1-y^2}{2} u_{0xxx} + \eta_{1xx} + (u_0 u_{0x})_x,$$

which, upon integrating with respect to  $y$  gives

$$v_{1t} = \left(-\frac{1}{2} u_{0xxx} + \eta_{1xx} + (u_0 u_{0x})_x\right) y + \frac{y^3}{6} u_{0xxx},$$

where constants of integration are neglected by virtue of (3.62). Now differentiate (3.61) with respect to  $t$

$$v_{1t} - (\eta_0 u_{0x})_t = \eta_{1tt} + u_{0t} \eta_{0x} + u_0 \eta_{0xt}$$

and use the expression for  $v_{1t}$  above on the surface  $y = 1$

$$v_{1t} = -\frac{1}{3} u_{0xxx} + \eta_{1xx} + (u_0 u_{0x})_x$$

to yield finally

$$-\frac{1}{3} u_{0xxx} + \eta_{1xx} + (u_0 u_{0x})_x - (\eta_0 u_{0x})_t = \eta_{1tt} + (u_0 \eta_{0x})_t$$

or equivalently

$$\eta_{1tt} - \eta_{1xx} - \left(\frac{1}{2} \eta_0^2 + u_0^2\right)_{xx} - \frac{1}{3} \eta_{0xxx} = 0.$$

We know from the first order expansion that  $\eta_{0tt} - \eta_{0xx} = 0$ , so that for the asymptotic expansion of  $\eta$

$$\eta \sim \eta_0 + \epsilon \eta_1 + \epsilon^2 \eta_2 + \dots,$$

upon eliminating the term  $u_0$  by virtue of  $u_0 = -\int_{-\infty}^x \eta_{0t} dx$  (making use of the decay assumptions), we have

$$(3.63) \quad \eta_{tt} - \eta_{xx} - \epsilon \left(\frac{1}{2} \eta^2 - \left(\int_{-\infty}^x \eta_t dx\right)^2\right)_{xx} - \frac{\epsilon}{3} \eta_{xxxx} = 0$$

to second order in  $\epsilon$ . This is an integral form of the *Boussinesq equation*. It can be transformed into the conventional form

$$(3.64) \quad \eta_{tt} - \eta_{xx} - \frac{3\epsilon}{2}(\eta^2)_{xx} - \frac{\epsilon}{3}\eta_{xxxx} = 0$$

by

$$\eta \rightarrow \eta - \epsilon\eta^2 \quad x \rightarrow x + \epsilon\left(\int_{-\infty}^x \eta dx\right)$$

(which is equivalent to transforming from Eulerian to Lagrangian coordinates). It is now straightforward to consider the long-time behaviour of (3.64) by making an ansatz

$$\eta(x, t, \epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \eta_n(x, t) \text{ as } \epsilon \rightarrow 0.$$

First, we absorb the constants  $3/2$  and  $1/3$ . Again, at first order, we recover the wave equation, and since we are not interested in full generality but rather in comparing with KdV dynamics, we may follow only the right-going component of the solution,  $\eta_0 = f(x - t)$ . This is consistent with introducing initial conditions

$$\begin{aligned} \eta(x, 0, \epsilon) &= f(x), \\ \eta_t(x, 0, \epsilon) &= -f'(x). \end{aligned}$$

Introducing new variables

$$\xi = x - t \quad \zeta = x + t$$

we have at order  $\epsilon$

$$-4\eta_{1\xi\zeta} = (f^2 + f'')'' \Rightarrow \eta_1 = -\frac{1}{4}\zeta(f^2 + f'')' + C_1(\xi) + C_2(\zeta).$$

By virtue of the initial conditions and the asymptotic expansion, we see that

$$\eta_n = 0, \quad \eta_{nt} = 0 \quad \forall n > 1,$$

which will allow us to determine  $C_1, C_2$ .

$$\eta_1(\xi, \zeta)|_{t=0} = -\frac{1}{4}x(f^2 + f'')' + C_1(x) + C_2(x) = 0$$

and

$$\eta_{1t}|_{t=0} = \left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \zeta}\right)\eta_1|_{t=0} = -\frac{1}{4}(f^2 + f'')' + C_2'(x) + \frac{1}{4}(f^2 + f'')'' - C_1'(x) = 0$$

which may be integrated to give

$$\frac{1}{4}x(f^2 + f'')' - \frac{1}{2}(f^2 + f'') + C_2 - C_1 = C$$

for some constant  $C$ . Together with the condition for  $u_1$  we have

$$\begin{aligned} 2C_2 &= \frac{1}{2}(f^2 + f'') + C \\ 2C_1 &= \frac{1}{2}x(f^2 + f'')' - \frac{1}{2}(f^2 + f'') - C \end{aligned}$$

which gives

$$\eta_1(\xi, \zeta) = \frac{1}{4}t(f^2(\xi) + f''(\xi))' + f^2(\zeta) + f''(\zeta) - f^2(\xi) - f''(\xi),$$

and an asymptotic expansion of  $\eta$

$$\eta(x, t, \epsilon) \sim f(x - t) + \frac{\epsilon}{4}t(f^2(\xi) + f''(\xi))' + f^2(\zeta) + f''(\zeta) - f^2(\xi) - f''(\xi) + O(\epsilon^2).$$

Here we see explicitly the nonuniformity in the expansion when  $t = O(\epsilon^{-1})$  and the source of the long time variable in the KdV. Indeed, we can recover the KdV from the Boussinesq equation (3.64) simply by transforming to far field variables. Again assuming the constants absorbed,

$$\eta_{tt} - \eta_{xx} - \epsilon(\eta^2)_{xx} - \epsilon\eta_{xxxx} = 0$$

under the transformation  $(x, t) \rightarrow (\xi, \tau)$  with  $\xi = x - t$  and  $\tau = \epsilon t$  gives

$$-2\epsilon u_{\xi\tau} + \epsilon^2 u_{\tau\tau} = \epsilon(2u_{\xi}u)_{\xi} + \epsilon u_{\xi\xi\xi\xi}$$

which, to order  $\epsilon$ , after integrating with respect to  $\xi$  gives

$$2u_{\tau} + 2u_{\xi}u + u_{\xi\xi\xi} = 0.$$

### 3.2. Solitons

What is a soliton? It is best to delay answering this question for a time, giving first some historical ideas, and then seeing how the modern treatment provides a comprehensive delineation between soliton and solitary wave. No text treating the Korteweg de-Vries equation is complete without the remarkable discovery made by J. Scott Russell in 1834 of the “great wave of translation” [Rus44].

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.

The existence of solitary waves of translation was at first vehemently denied following J. Scott Russell’s discovery. Speaking to Sir John Herschel, Russell faced the retort that “it is merely half of a common wave that has been cut off”. His rebuttal ([Rus65]):

But it is not so, because the common waves go partly above and partly below the surface level; and not only that but its shape is different. Instead of being half a wave it is clearly a whole wave, with this difference, that the whole wave is not above and below the surface alternately but always above it.

One of Russells most prominent critics was the British Astronomer Royal G.B. Airy, who based his critique on a long wave equation he deduced, with small (but

not vanishing) amplitude:

$$(3.65) \quad \eta_{tt} - gh\eta_{xx} = gh \frac{\partial^2}{\partial x} \left( \frac{3}{2} \frac{\eta^2}{h} \right)$$

This equation has no non-trivial stationary traveling wave solutions, i.e. solutions of the form  $\eta(x, t) = \eta(x - ct)$ . Further, this equation is well known in gas dynamics, where it has been shown that for a simple wave traveling in one direction (such as Russell’s solitary wave), crests travel faster than troughs, which leads to a steepening of the wave front and eventual wave breaking. Russell was able to reproduce these solitary waves of translation in a number of experiments, and eventually gained some insight on their special properties. Writing in 1844, Russell noted “the length, therefore, increases with the depth of fluid directly, being equal to about 6.28 times the depth. The length does not, like the velocity of the wave, increase with the height of the wave in a given depth of fluid. On the contrary, the length appears to diminish as the height of the wave is increased. [...] This extension of length is attended with a diminution of height, and the diminution of length with an increase of height of the wave, so that the change of length and height attend and indicate changes of depth.” In fact, it is easy to demonstrate some of this using the solitary wave solution to the KdV

$$u_1(t, x) = 2A^2 \operatorname{sech}^2 A(x \pm ct).$$

Assuming that the amplitude  $A$  is doubled, the above becomes

$$u_2(t, x) = 4A^2 \operatorname{sech}^2 A(\sqrt{2}(x \pm ct)).$$

We see that the height of  $u_2$  at the position  $(t, x)$  is twice that of  $u_1$  at the position  $(t, \sqrt{2}x)$ , but not at the point  $(t, x)$ . The wave shortens by a factor of  $2^{-1/2}$  as it doubles in height.

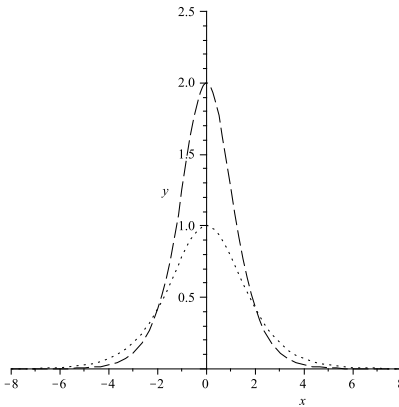


FIGURE 1. Comparisons of  $\operatorname{sech}^2$  solitons  $u_1, u_2$  with  $A = 0.5$

Problems surrounding solitary waves persisted well into the 20th century. In 1953 Ursell, writing on “The long-wave paradox in the theory of gravity waves” [Urs53] highlighted some of these:

The reason for these difficulties is to be sought in the present incomplete state of the theory of non-linear partial differential



equations. In each of the derivations a formal series-expansion in terms of a small parameter is assumed, but the convergence or even the asymptotic nature of the series cannot be proved. However, when the resulting theory contains paradoxes and internal inconsistencies, we must conclude that the original assumption was unjustified.

In other words, the long-wave paradox results from assuming that the conditions  $h/\lambda \ll 1$ ,  $\eta_0/h \ll 1$  must lead to a unique theory, whereas actually the order of magnitude of  $\eta_0\lambda/h^3$  is equally fundamental.

In what does this long-wave paradox consist? Airy, in stating that a solitary wave could not propagate without change of form, relied on an argument that the pressure at any point in a fluid equals the hydrostatic head of water above that point. From this assumption, he derived by formal asymptotic expansion the equation (3.65) to second order. Rayleigh, without using this hydrostatic long-wave assumption, was able to derive the Boussinesq equation

$$(3.66) \quad \eta_{tt} - gh\eta_{xx} = gh \frac{\partial^2}{\partial x} \left( \frac{3}{2} \frac{\eta^2}{h} + \frac{h^2}{3} \frac{\partial^2 \eta}{\partial x^2} \right)$$

which admits solitary wave solutions. Under different assumptions, expanding in terms of  $\epsilon/\delta^2$  Friedrichs obtained Airy's theory as a first approximation. Keller was able to reproduce the solitary wave, but obtained solitary waves of arbitrary amplitude for a given water depth. These problems lacked a satisfactory theoretical framework, one that was only provided by modern developments in the latter half of the 20th century.

At this point, it is helpful to recall what a *traveling wave* refers to. Given a velocity field  $u(x, t)$ , when we look for solutions to some equations describing wave propagation in the form  $u(x, t) = f(x - ct)$  for some constant  $c$ , we can clearly see that this implies that a profile  $f(x)$  is translated to the right at speed  $c$ , if we give  $x$  the dimensions of space and  $t$  dimensions of time. This is exactly the solution to the simplest IVP that can be said to model wave propagation, the transport equation

$$\begin{aligned} u_t + cu_x &= 0 \text{ in } \mathbb{R} \times (0, \infty), \\ u &= f \text{ on } \mathbb{R} \times 0. \end{aligned}$$

The term *solitary wave* often refers to any surface wave profile that dies out at infinity, by which we mean that the profile drops back to the flat water level. We now attempt a first definition of a soliton. The classical text *Solitons* by Drazin [Dra83] provides the following:

A 'soliton' is not precisely defined, but it is used to describe any solution of a nonlinear equation or system which (i) represents a wave of permanent form; (ii) is localized, decaying or becoming constant at infinity; and (iii) may interact strongly with other solitons so that after the interaction it retains its form, almost as if the principle of superposition were valid. [...] a soliton is a localized entity which may keep its identity after an interaction.

A solitary wave may be defined more generally than as a sech-squared solution of the KdV equation. We take it to be any solution

of a nonlinear system which represents a hump-shaped wave of permanent form, whether it is a soliton or not.

Here we see already part of the confusion that is widespread whenever topics of solitons and solitary waves are covered. Often, the differences can get mixed up, and especially as regards the purely applied perspective, it is difficult to distinguish the two. This ties in with our discussion of measurement in 4.5. Drazin follows up with the following caveat:

In the context of the KdV equation, and other similar equations, it is usual to refer to the single-soliton solution as the *solitary wave*, but when more than one of them appear in a solution they are called solitons. Another way of expressing this is to say that the soliton becomes a solitary wave when it is infinitely separated from any other soliton. Also, we must mention the fact that for equations other than the KdV equation the solitary-wave solution may not be a *sech*<sup>2</sup> function; for example, we shall meet a *sech* function and also *arctan*( $e^{\alpha x}$ ). Furthermore, some nonlinear systems have solitary waves but *not* solitons, whereas others (like the KdV equation) have solitary waves which *are* solitons.

(The other solitons he mentions are those of the sine-Gordon equation  $\psi_{tt} - \psi_{xx} + \sin \psi = 0$ .) An illustrative example is that of the Burgers equation

$$u_t + uu_x = \nu u_{xx},$$

which has a traveling wave solution of the form

$$u(x, t) = c(1 - \tanh(c(x - ct)/2\nu))$$

for all  $c$ , but these solutions do not interact as solitons. Newell, one of the architects of inverse scattering theory (and the N of AKNS) in his book *Solitons in Mathematics and Physics* [New85] writes:

The fact that two solitary waves of an equation preserve their form through nonlinear interaction is often taken to be both the acid test for and the definition of the soliton. I want to warn the reader that this condition is only necessary. There are equations [...] which admit two-phase solitary wave solutions, and therefore the asymptotic form of each individual solitary wave is preserved through collision, which do not possess all the ingredients for the admission to the soliton class. The proper definition of a soliton involves its identification with certain of the scattering data of an eigenvalue problem.

One such equation to which Newell refers is given by

$$2(\tau\tau_{tx} - \tau_x\tau_t) + (\partial_x^8\tau)\tau - 13(\partial_x^7\tau)\tau_x + 44\tau_{xx}(\partial_x^6\tau) - 119\tau_{xxx}(\partial_x^5\tau) + 77\tau_{xxxx}^2 = 0.$$

In fact, the above equation has only 2-soliton solutions. (This equation did not appear out of thin air; using the Hirota differential operator  $D$  the above can be written as  $(D_x D_t + D_x^8)\tau \cdot \tau = 0$ .)

### 3.3. A brief look at inverse scattering theory

*Fermi expressed often a belief that future fundamental theories in physics may involve non-linear operators and equations, and that it would be useful to attempt practice in the mathematics needed for the understanding of non-linear systems. **Stansilaw Ulam***

Having discussed solitons in nature and soliton solutions to certain equations, the question arises what these have to do with one another. Can we establish when and whether solitons will arise from some disturbance in a fluid? It seems that for displacements of a net positive volume, there are experimental results that support the eventual emergence of solitons[**HS74**]. The theoretical background for these results [**Zab68**] goes back to inverse scattering theory and the work of Zabusky, Kruskal, Gardner, Greene, Miura, and others[**GGKM67**]. We will give a short description of the development of this remarkable theory of eigenvalue problems associated with certain, so-called completely integrable equations, looking specifically at the KdV.

In order to understand the background that led to these developments, we will consider the problem of heat conduction in solids. The classical heat equation

$$u_t - \Delta u = 0$$

is derived from the law of conservation of heat

$$u_t = -\nabla \cdot F$$

(cf. [**FLS63**, **Eva98**]) under the assumption that the flux density  $F$  is linear, i.e.

$$F = -\kappa \nabla u,$$

for some thermal conductivity  $\kappa$ . You may recall (or cf. [**Eva98**]) that the heat equation forces infinite propagation speed for disturbances. Clearly this is not physically meaningful. An analogue of this equation can be derived by assuming the material modeled by a set of masses coupled by springs - a lattice - as opposed to a continuous medium. Still the problem of infinite heat conductivity remains. To go about resolving this, the suggestion was made to assume the coupling in the lattice to be weakly nonlinear. It is easy to appreciate that this makes the problem considerably more difficult, and it was not until the advent of the modern computer that it became possible to treat this problem numerically. The expectation was clearly to see the heat in a localized initial state flow into the other masses until a thermal equilibrium was reached. (Note that this is not exactly the vantage point taken by the experimenters Fermi, Pasta, and Ulam [**FPU74**] (using the MANIAC at Los Alamos - see [**Ula76**] for an engaging account). In fact their idea was a much more general one, namely to get a preliminary idea for the rate of mixing or “thermalization” in nonlinear problems. At the time, the prevalent belief was clearly in the universality of such mixing in non-linear systems.) Instead of seeing this energy distributed among the 64 modes in their lattice, however, the experimenters found that it flowed back and forth, eventually reproducing the initial conditions.

This aroused considerable interest - in fact, Newell [**New85**] compares it to the Michelson-Morley experiment in having challenged “the basic thinking of physicists of the day”. Before we delve into the details, we emphasize again the main point, from [**AS81**]:

Certain nonlinear problems have a surprisingly simple underlying structure, and can be solved by essentially linear methods.

We begin by noting that the KdV

$$(3.67) \quad u_t - 6uu_x + u_{xxx} = 0$$

is invariant under Galilean transformations  $u(x - 6ct, t) - c$ . Next we apply the *Miura transformation*, which takes  $u$  into the Riccati equation

$$(3.68) \quad u = v^2 + v_x.$$

Substituting this into the KdV and rearranging terms, we see that we recover an operator plus a modified KdV equation (henceforth mKdV)

$$u_t - 6uu_x + u_{xxx} = \left(2v + \frac{\partial}{\partial x}\right)(v_t + 6v^2v_x + v_{xxx}).$$

Therefore we see that for  $v$  a solution of the mKdV, (3.68) yields a solution of the KdV (3.67). It may seem that we are simply going in circles, but this is not so. It is well known that Riccati equations may be linearized under the transformation  $v = \psi_x/\psi$  for some differentiable  $\psi \neq 0$ . This linearization gives us

$$(3.69) \quad \psi_{xx} - u\psi = 0,$$

and upon introducing a Galilean transformation for  $u$  which we now denote by  $u \rightarrow u - \lambda$ , we see that

$$(3.70) \quad \psi_{xx} + (\lambda - u)\psi = 0,$$

where  $\psi$  and  $\lambda$  depend parametrically on time. This is, in fact, the stationary Schrödinger equation. At first glance, this may look entirely unhelpful - but luckily Gardner, Greene, Kruskal and Miura were familiar with quantum physics and the well-established scattering theory of the Schrödinger equation. The idea is as follows: if we have an initial condition  $u(x, 0)$  for  $t = 0$  which we call a *potential* evolving according to (3.67), we can determine the time-evolution of  $\psi$  and  $\lambda$ . Motivated by the physical scattering of, e.g. an electron in a crystal lattice with an impurity, the following idea is helpful: we imagine an incident wave  $e^{-ikx}$  (the electron) coming from the right ( $+\infty$ ) towards the potential  $u(x)$  (the impurity). The result of this interaction is a transmitted wave  $Te^{-ikx}$  traveling towards  $-\infty$  and a reflected wave  $Re^{ikx}$  traveling towards  $+\infty$ . The amplitudes of the reflected and transmitted waves,  $R$  and  $T$  constitute part of the *scattering data* for this potential. We will follow the development in [Dra83].

Since we want to solve a Cauchy initial value problem for  $t > 0$  and  $x \in \mathbb{R}$ , we let

$$u(x, 0) = g(x).$$

In the context of water waves, this could mean that we prescribe an initial displacement, for which we further require square-integrability of  $g$  and its derivatives

$$\int_{-\infty}^{\infty} \left| \frac{d^n g}{dx^n} \right|^2 < \infty \text{ for } n \in \{1, \dots, 4\},$$

to ensure the existence of unique smooth solutions to the IVP. We will also require stronger decay in terms of the *Faddeev condition*

$$\int_{-\infty}^{\infty} (1 + x^2)|g(x)|dx < \infty.$$

This ensures that  $u$  decays rapidly enough for  $|x| \rightarrow \infty$  that we may write

$$\psi'' \sim -\lambda\psi,$$

the free Schrödinger equation. Consequently, the further asymptotic behavior of  $\psi$  depends on the sign of  $\lambda$ . We consider now the so-called *bound states* corresponding to  $\lambda < 0$ . Define  $\kappa = \sqrt{-\lambda} > 0$ . We consider first  $x \rightarrow -\infty$ , where, in order for  $\psi$  to remain bounded we require  $\psi(x) \sim \alpha e^{\kappa x}$ . What happens to this bounded solution as  $x \rightarrow +\infty$ ? In general, unfortunately, we will have

$$\psi \sim \beta e^{\kappa x} + \gamma e^{-\kappa x} \text{ as } x \rightarrow +\infty$$

which incorporates both exponential and decaying terms. Of course, this is unbounded unless  $\beta = 0$ , and those distinguished  $\lambda$  for which  $\beta = 0$  are called the discrete spectrum. (Note that we could fix the coefficients for  $x \rightarrow \infty$  in place of those for  $x \rightarrow -\infty$ .) For such  $\lambda$  we see  $\psi \rightarrow 0$  as  $|x| \rightarrow \infty$ . Such  $\lambda$  need not exist, such as in the case  $u(x) \geq 0$  for all  $x \in \mathbb{R}$ .

It is useful to note that (3.70) is a Sturm-Liouville equation. Thus we can bring the well-developed spectral theory for Sturm-Liouville operators (such as  $-\frac{\partial^2}{\partial x^2} + u$ ) to bear (see, e.g. [CL84]).

We will be particularly interested in the case where  $u \leq 0$  for all  $x \in \mathbb{R}$  and  $u$  decays sufficiently fast at infinity, which yields only a finite number of discrete eigenvalues  $\lambda_1, \dots, \lambda_N$  which may be ordered so that  $\kappa_1 < \dots < \kappa_N$  - this is guaranteed by the Faddeev condition above. Otherwise we could also encounter infinitely many bound states, as is the case for the *Coulomb potential*  $u(x) = -\alpha/x$  or for  $u(x) = \alpha/x^2$ .

We denote the corresponding eigenfunctions  $\psi_1, \dots, \psi_N$  where

$$\psi_n(x) \sim c_n e^{-\kappa_n x} \text{ as } x \rightarrow \infty.$$

We will find it convenient to have  $\psi_n \in L^2$ , which fixes the constants  $c_n$ . The alternative case for  $\lambda > 0$ , the *unbound states* poses no problems with exponential terms, and we have

$$(3.71) \quad \hat{\psi} \sim T e^{-ikx} \text{ for } x \rightarrow -\infty$$

and

$$(3.72) \quad \hat{\psi} \sim e^{-ikx} + R e^{ikx} \text{ for } x \rightarrow +\infty$$

in analogy with the above, where we have  $k = \sqrt{\lambda}$ .  $T$  and  $R$  are called the *transmission* respectively *reflection coefficients*.

It will be useful in some of the calculations that follow to differentiate (3.70) with respect to  $x$  to yield

$$(3.73) \quad \psi_{xxx} - u_x \psi + (\lambda - u) \psi_t = 0,$$

and with respect to  $t$  to yield

$$(3.74) \quad \psi_{xxt} + (\lambda_t - u_t) \psi + (\lambda - u) \psi_t = 0.$$

Now for general  $\psi, \kappa$  let

$$(3.75) \quad R = \psi_t + u_x \psi - 2(u + 2\lambda) \psi_x$$

so that

$$(3.76) \quad \frac{\partial}{\partial x} (\psi R_x - \psi_x R) = \psi^2 (u_t - \lambda_t + u_{xxx} - 6u u_x).$$

So we see by virtue of (3.70) and (3.67) that

$$\psi^2 \lambda_t + \frac{\partial}{\partial x}(\psi R_x - \psi_x R) = 0.$$

At this point, we recall the above results about specific eigenvalues  $\lambda$  in order to progress. For the bound states  $\lambda < 0$  we assumed that  $\psi_n \in L^2$  decays as  $|x| \rightarrow \infty$ , integrating the above w.r.t.  $x$  yields the surprising conclusion that  $\lambda_{nt} = 0$ , i.e. the eigenvalues are independent of time. Hence we also deduce that

$$\frac{\partial}{\partial x}(\psi_n R_{nx} - \psi_{nx} R_n) = 0,$$

whereby we see that

$$\psi_n^2 \frac{\partial}{\partial x} \left( \frac{R_n}{\psi_n} \right) = \psi_n R_{nx} - \psi_{nx} R_n = \lim_{x \rightarrow \infty} \psi_n R_{nx} - \psi_{nx} R_n = 0.$$

As before, we see that  $R_n/\psi_n$  depends only on time, and thus equate it with its limit for  $x \rightarrow \infty$ . Recall (3.75) so that

$$(3.77) \quad \lim_{x \rightarrow \infty} \frac{R_n}{\psi_n} = \lim_{x \rightarrow \infty} \frac{\psi_{nt} + u_x \psi_n - 2(u + 2\lambda_n) \psi_{nx}}{\psi_n} = -4\kappa_n^3$$

since  $u$  and its derivatives decay at infinity, and  $\psi_n \sim c_n e^{-\kappa_n x}$ . Substituting this into (3.75) we get the *evolution equation* for the bound states

$$(3.78) \quad \psi_{nt} + u_x \psi_n - 2(u + 2\lambda) \psi_{nx} + 4\kappa_n^3 = 0,$$

whereby we can deduce (letting  $x \rightarrow \infty$ , since  $\kappa_n$  is independent of  $t$ ) that

$$c'_n(t) = 4\kappa_n^3 c_n \Rightarrow c_n(t) = c_n(0) e^{4\kappa_n^3 t}.$$

We apply an analogous procedure to the unbound states  $\lambda > 0$ . We define

$$(3.79) \quad \hat{R} = \hat{\psi}_t + u_x \hat{\psi} - 2(u + 2\lambda) \hat{\psi}_x$$

and, since any real  $k$  is admissible, we consider the problem for fixed  $k$ . Therefore  $\hat{\psi}^2 \lambda_t + \frac{\partial}{\partial x}(\hat{\psi} \hat{R}_x - \hat{\psi}_x \hat{R}) = 0$  (and  $\lambda_t = 0$  by virtue of  $k$  fixed) yields the fact that  $\hat{\psi} \hat{R}_x - \hat{\psi}_x \hat{R}$  depends only on time and  $k$ . Therefore,

$$\lim_{x \rightarrow -\infty} \hat{R} = \lim_{x \rightarrow -\infty} \hat{\psi}_t + u_x \hat{\psi} - 2(u + 2\lambda) \hat{\psi}_x = \left( \frac{dT}{dt} + 4\lambda T i k \right) e^{-ikx},$$

since  $u$  and  $u_x$  decay, and using (3.71), so that

$$\begin{aligned} & \lim_{x \rightarrow -\infty} \hat{\psi} \hat{R}_x - \hat{\psi}_x \hat{R} = \\ & = T e^{-ikx} \left( \frac{dT}{dt} + 4\lambda T i k \right) (-ik) e^{-ikx} + \\ & + (ik) e^{-ikx} \left( \frac{dT}{dt} + 4\lambda T i k \right) e^{-ikx} = 0. \end{aligned}$$

and thus  $\hat{\psi} \hat{R}_x - \hat{\psi}_x \hat{R} = 0$  for all  $t$ . Again we see that  $0 = \hat{\psi} \hat{R}_x - \hat{\psi}_x \hat{R} = \hat{\psi}^2 \frac{\partial}{\partial x} \left( \frac{\hat{R}}{\hat{\psi}} \right) \Rightarrow \frac{\partial}{\partial x} \left( \frac{\hat{R}}{\hat{\psi}} \right) = 0$ . Now, using the asymptotic behavior of  $\hat{\psi}, \hat{R}$  as  $x \rightarrow -\infty$  and the fact that

$$\hat{R}/\hat{\psi} = f(t)$$

we see that

$$(3.80) \quad \frac{dT}{dt} + 4T i k^3 = T f(t).$$

Now, as  $x \rightarrow +\infty$ , we have

$$\lim_{x \rightarrow -\infty} \hat{R} = \lim_{x \rightarrow -\infty} \hat{\psi}_t + u_x \hat{\psi} - 2(u + 2\lambda) \hat{\psi}_x = \frac{dR}{dt} e^{ikx} - 4k^3 i (R e^{ikx} - e^{-ikx})$$

whereby

$$\hat{R}/\hat{\psi} = f(t) \text{ so that } \frac{dR}{dt} e^{ikx} - 4k^3 i (R e^{ikx} - e^{-ikx}) = f(t)(e^{-ikx} + R e^{ikx}).$$

Since  $e^{ikx}$  and  $e^{-ikx}$  are linearly independent, we match the coefficients to give

$$(3.81) \quad \frac{dR}{dt} - 4k^3 i R = R f(t),$$

$$(3.82) \quad 4k^3 i = f(t).$$

Using (3.82) in (3.80) we see

$$\frac{dT}{dt} = 0$$

and then from (3.81) we note

$$\frac{dR}{dt} = 8k^3 i R$$

allowing us to solve for  $T$  and  $R$ :

$$(3.83) \quad T(k, t) = T(k, 0)$$

$$(3.84) \quad R(k, t) = R(k, 0) e^{8k^3 i t}$$

We now have all the scattering data needed for a complete picture of the asymptotic behavior of  $\psi$ . The crucial next step is finding  $u(x, t)$  from this data, the *inverse scattering problem*. This can be done via the Gel'fand-Levitan-Marchenko integral equation

$$K(x, y, t) + B(x + y, t) + \int_x^\infty K(x, z, t) B(y + z, t) dz = 0 \text{ for } y > x$$

where  $B$  is defined by

$$B(x + y, t) = \sum_{n=1}^N c_n(t) e^{-\kappa_n(x+y)} + \frac{1}{2\pi} \int_{-\infty}^\infty R(k, t) e^{ik(x+y)} dk.$$

Then, for  $K(x, y, t)$  the unique solution to this linear ordinary integral equation, we find the time evolution of  $u$  by

$$(3.85) \quad u(x, t) = -2 \frac{\partial}{\partial x} K(x, x, t).$$

This is still a non-trivial problem, but it is important to note that we have gone from a nonlinear partial differential equation to needing to solve only a linear second order ODE for  $u(x, 0) = g(x)$  and a linear ordinary integral equation. The crucial step in this development was that we were able to show that the bound-state eigenvalues were independent of time. There is a more general method used to establish this fact, due to Lax [Lax68]. Our presentation follows that in [Dra83]. In this development, we formulate our nonlinear IVP in the general form

$$(3.86) \quad u_t = N(u)$$

for  $N : X \rightarrow X$  some nonlinear operator on a function space, e.g.  $N(u) = 6uu_x - u_{xxx}$ . Assume we can show that (3.86) is equivalent to

$$(3.87) \quad L_t = BL - LB$$

for some linear operators  $L$  and  $B$  depending on  $u$  on a Hilbert space  $\mathfrak{H}$ , and assume that  $L$  is symmetric. We wish to treat the eigenvalue problem

$$L\psi = \lambda\psi$$

for all  $t \geq 0$ ,  $\psi \in \mathfrak{H}$ , and differentiate it with respect to  $t$  to yield

$$L\psi_t + L_t\psi = \lambda_t\psi + \lambda\psi_t.$$

Using (3.87)

$$\begin{aligned} L\psi_t + (BL - LB)\psi &= \lambda_t\psi + \lambda\psi_t \\ \Rightarrow L\psi_t + B\lambda\psi - LB\psi &= \lambda_t\psi + \lambda\psi_t \\ \Rightarrow \lambda_t\psi &= (L - \lambda)(\psi_t - B\psi) \end{aligned}$$

so that, by symmetry

$$\lambda_t \langle \psi | \psi \rangle = \langle \psi | (L - \lambda)(\psi_t - B\psi) \rangle = \langle (L - \lambda)\psi | \psi_t - B\psi \rangle = \langle 0 | \psi_t - B\psi \rangle = 0.$$

So we see  $\lambda_t = 0$ , and

$$L(\psi_t - B\psi) = \lambda(\psi_t - B\psi).$$

Since  $B$  and  $L$  are not unique in (3.87), we may redefine  $B$  so that  $\psi_t = B\psi$  for all  $t > 0$  without changing (3.87). To summarize: If (3.86) can be expressed with a so called *Lax pair*  $L, B$  according to (3.87), and if  $L\psi = \lambda\psi$ , then the eigenvalues are independent of time, and  $\psi$  evolves according to

$$\psi_t = B\psi.$$

In the context of the KdV, the eigenvalue problem was simply that of the Schrödinger equation, so we see

$$L = \left(-\frac{\partial^2}{\partial x^2} + u\right)$$

and we had an evolution equation derived from  $R$ . Indeed, setting

$$B = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x + \alpha$$

we recover (for any constant  $\alpha$ )

$$BL - LB = -u_{xxx} + 6uu_x$$

and  $L_t = BL - LB$  gives the KdV. Now we may apply the scattering and inverse scattering theory as above. Note that there is a systematic way to find suitable  $B$  when  $L$  is the Schrödinger operator. This has to do with making the commutator  $[L, B]$  a multiplication operator and the fact the  $B$  must necessarily be anti-symmetric. Therefore  $B$  may be composed of odd-order differential operators, giving rise to a *Lax hierarchy* for the KdV. The major obstacle in this method is, of course, finding the Lax pair! Generally, this is far from easy, although for linear PDE, a general method for constructing Lax pairs can be found in [Fok97]. However, the advantage of this more general approach is that we can see that a whole class of equations can be treated with scattering and inverse scattering theory, e.g. the transport equation can be shown to have constant eigenvalues by virtue of the choice  $B = c\partial/\partial x$ ,  $L = -\partial^2/\partial x^2 + u$ . This is the first equation in the Lax hierarchy for the KdV.



### 3.4. The Riemann-Hilbert problem for the KdV

The Riemann-Hilbert problem originated in the 19<sup>th</sup> century with the work of Bernhard Riemann on a certain extension of the Poisson formula, which for the half-plane is a consequence of Cauchy's integral formula. This generalization consists of finding an analytic function within a closed contour  $C$  fulfilling

$$\alpha(t)u(t) + \beta(t)v(t) = \gamma(t) \text{ on the contour}$$

for  $w = u + iv$  analytic and given real functions  $\alpha, \beta, \gamma$ . Hilbert later identified this problem with that of finding two analytic functions  $\Phi^+, \Phi^-$  defined within respectively without a closed contour  $C$  such that

$$(3.88) \quad \Phi^+(t) - g(t)\Phi^-(t) = f(t) \text{ on the contour}$$

for given  $g, f$ . Hilbert related this problem to certain singular integral equations, and Plemelj was able, in 1908, to make major inroads in treating what has come to be known as the Riemann-Hilbert problem. A closed form solution was first given by Muskhelishvili [Mus53].

Just as the Poisson formula follows from Cauchy's integral formula

$$\phi(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\zeta)}{\zeta - z} d\zeta$$

for  $\phi$  analytic on and within a simple closed contour  $C$ , we shall first investigate generalizations in the form of *Cauchy-type integrals*

$$(3.89) \quad \Phi(z) = \frac{1}{2\pi i} \int_L \frac{\phi(\tau)}{\tau - z} d\tau$$

for  $\phi$  Hölder continuous on a simple smooth arc  $L$  of finite length; that is:

there is a  $\kappa > 0$ ,  $0 < \lambda \leq 1$  such that  $|\phi(\tau_1) - \phi(\tau_2)| \leq \kappa|\tau_1 - \tau_2|^\lambda$  for all  $\tau_1, \tau_2 \in L$

For infinite arcs, such as  $L$  the real axis, we additionally require that, for  $t \rightarrow \pm\infty$ ,  $\phi(\tau) \rightarrow \phi(\infty)$  with

$$|\phi(\tau) - \phi(\infty)| \leq \frac{M}{|\tau|^\mu}, \quad M > 0, \quad \mu > 0.$$

Under this assumption, (3.89) defines a function analytic off  $L$ . As  $z$  approaches some  $t \in L$ , we define the Cauchy principal value of (3.89) as the symmetric limit

$$\lim_{\epsilon \rightarrow 0} \int_{L-L_\epsilon} \frac{\phi(\tau)}{\tau - t} d\tau =: p.v. \int_L \frac{\phi(\tau)}{\tau - t} d\tau$$

where  $L_\epsilon$  is the part of the contour within a circle of radius  $\epsilon$  centered at  $t$ .

We will use the convention in what follows that, for  $L$  oriented, the  $\oplus$  side lies to the left in direction of orientation, while the  $\ominus$  side lies to the right. We now give the Sokhotski-Plemelj formulae for the limits of  $\Phi$  (above) as  $z$  approaches  $t$  from the  $\oplus$  resp.  $\ominus$  side, denoted by  $\Phi^+$  resp.  $\Phi^-$ . A proof may be found in [Mus53] or [AF03].

**THEOREM 3.1** (Sokhotski-Plemelj).

$$\Phi^\pm(t) = \pm \frac{1}{2} \phi(t) + \frac{1}{2\pi i} p.v. \int_L \frac{\phi(\tau)}{\tau - z} d\tau$$

or

$$\begin{aligned}\Phi^+(t) - \Phi^-(t) &= \phi(t) \\ \Phi^+(t) + \Phi^-(t) &= \frac{1}{\pi i} p.v. \int_L \frac{\phi(\tau)}{\tau - z} d\tau\end{aligned}$$

We may now restate the Riemann-Hilbert problem in terms of finding a sectionally analytic function  $\Phi$ , that is, analytic on  $D - L$  for some domain  $D \subset \mathbb{C}$ ,  $L \subset D$ , continuous on  $L$  from the  $\oplus$  and  $\ominus$  side, and with limits  $\Phi^+$ ,  $\Phi^-$  such that

$$\Phi^+(t) = g(t)\Phi^-(t) + f(t)$$

where  $g, f$  satisfy Hölder conditions on  $L$ ,  $g(t) \neq 0 \quad \forall t \in L$ . This *inhomogeneous scalar Riemann-Hilbert problem* always has a solution. If we specify the degree of  $\Phi$  at infinity, i.e.  $\Phi(z) \sim z^n, \quad |z| \rightarrow \infty$ , then we may also establish uniqueness of this solution.

We can also formulate so-called *vector Riemann Hilbert problems* of the same form

$$\Phi^+(t) = g(t)\Phi^-(t) \text{ for } t \in C$$

where  $C$  is a contour,  $G$  a nonsingular matrix whose entries satisfy a Hölder condition. The task then is to find a sectionally analytic vector function  $\Phi(t)$  whose components tend from the  $\oplus$  side to those of  $\Phi^+(t)$  and from the  $\ominus$  side to those of  $\Phi^-(t)$ . Furthermore we ask that the components of  $\Phi$  have some finite degree at infinity.

In this case, existence of solutions with a given degree is considerably more difficult than in the inhomogeneous scalar problem (where solutions always exist) or the homogeneous scalar problem (where nontrivial solutions exist if the index of  $g$  is positive). In general, we need to take into account the partial indices of the matrix  $G$ , of which there are  $n$  if  $G$  is  $n \times n$ . At best, we can give an iterative procedure for the fundamental solution matrix [**Vek67**].

We turn our attention now directly to the Riemann-Hilbert problem for the KdV. Our point of departure is again the time independent Schrödinger equation

$$(3.90) \quad -\psi_{xx} + (u - k^2)\psi = 0$$

(where we write  $k^2 = \lambda$ , which physically corresponds to using the momentum  $k$  in place of the energy  $\lambda$ ). For each real  $k \neq 0$  the solutions of (3.90) form a two dimensional vector space. We consider two bases in this space

$$\begin{aligned}\psi_1(x, k) &\sim e^{-ikx} \\ \psi_2(x, k) &\sim e^{ikx}\end{aligned} \quad x \rightarrow +\infty$$

and

$$\begin{aligned}\phi_1(x, k) &\sim e^{-ikx} \\ \phi_2(x, k) &\sim e^{ikx}\end{aligned} \quad x \rightarrow -\infty$$

Since  $u(x)$  is real valued, we have

$$\phi_1 = \bar{\phi}_2, \quad \psi_1 = \bar{\psi}_2$$

and

$$(3.91) \quad \phi_1(x, k) = \phi_2(x, -k), \quad \psi_1(x, k) = \psi_2(x, -k).$$

Since  $\psi_1, \psi_2$  are linearly independent (consider the Wronskian  $W(\psi_1, \psi_2)$ ), we may write

$$\phi_i(x, k) = \sum_{n=1}^2 T_{in}(k) \psi_n(x, k), \quad i = 1, 2$$

where we see that the matrix  $T(k)$  must have the form

$$T = \begin{pmatrix} a(k) & b(k) \\ \bar{a}(k) & \bar{b}(k) \end{pmatrix}$$

We will now single out  $\phi_1 =: \phi$ . Considering  $W(\phi, \bar{\phi}) = W(\psi, \bar{\psi}) = 2ik$ , we see that  $|a(k)|^2 - |b(k)|^2 = 1$ .

Now consider the asymptotic behaviour of  $\phi(x, k)/a(k)$  for  $x \rightarrow \infty$ :

$$\frac{\phi(x, k)}{a(k)} \sim e^{-ikx} + \frac{b(k)}{a(k)} e^{ikx}$$

and for  $x \rightarrow -\infty$

$$\frac{\phi(x, k)}{a(k)} \sim \frac{e^{-ikx}}{a(k)}.$$

We can identify  $\phi/a$  with  $\hat{\psi}$  in 3.3 above and see that the coefficient of reflection  $R = \frac{b(k)}{a(k)}$  and the transmission coefficient  $T = 1/a(k)$ . We also see that  $|R|^2 + |T|^2 = 1$ .

We have already determined the time evolution of  $T$  and  $R$  above. Now, however, we depart from the classical approach, where we suppressed analytical properties of the transmission coefficient. We will be interested in studying these properties and formulating the inverse scattering problem in terms of a Riemann-Hilbert problem.

To this end, let us consider  $Im(k) > 0$  above. Here we can identify  $\phi$  with  $\psi$  in the preceding section, where we see

$$\begin{aligned} a(k) e^{-iRe(k)x} &= \beta \\ b(k) e^{-iRe(k)x} &= \gamma \end{aligned}$$

where we recall that  $\beta$  vanishes at the eigenvalues  $\kappa_1, \dots, \kappa_n$ , i.e.  $a(k)$  has zeroes at these points [DJ89]. The expression  $W(\phi_1, \psi_1) = 2ika$  when differentiated with respect to  $k$  can be used to show that the poles of  $T(k) = 1/a(k)$  are simple.  $T(k)$  is otherwise analytic in the upper half  $k$ -plane. Since  $|a(k)|^2 = (1 - |R(k)|^2)^{-1}$  and we know the zeroes of  $a(k)$ , we can recover  $a$  from  $R$ . We need to supplement this data with the norming constant for the bound states derived above. Recall that for eigenvalues  $-\kappa_n^2$  we had eigenfunctions

$$\phi_n \sim c_n e^{-\kappa_n x} \quad x \rightarrow \infty.$$

We fixed the constants  $c_n$  via the  $L^2$ -norm of  $\phi_n$ , and derived their time evolution.

Now we will construct the Riemann-Hilbert problem. Note that

$$(3.92) \quad \phi_1(x, k) e^{ikx} \sim 1, \quad |k| \rightarrow \infty, \quad Im(k) > 0$$

$$(3.93) \quad \psi_1(x, k) e^{ikx} \sim 1, \quad |k| \rightarrow \infty, \quad Im(k) < 0$$

where (3.92) is analytic in the upper half  $k$ -plane and (3.93) is analytic in the lower half  $k$ -plane, while the reverse holds for  $\phi_2(x, k) = \phi_1(x, -k)$ ,  $\psi_2(x, k) = \psi_1(x, -k)$

when these are multiplied by  $e^{-ikx}$ . Notice also that we have

$$\begin{aligned}\frac{\phi_1 e^{ikx}}{a(k)} &= e^{ikx} \psi_1(x, k) + \frac{b(k)}{a(k)} e^{ikx} \psi_2(x, k) \\ \frac{\phi_2 e^{-ikx}}{\bar{a}(k)} &= \frac{\bar{b}}{\bar{a}} e^{-ikx} \psi_1(x, k) + e^{-ikx} \psi_2(x, k)\end{aligned}$$

using the linear combinations with the matrix  $T(k)$ . Therefore

$$\begin{aligned}T(k)\phi_1(x, k) &= (1 - R(k)\bar{R}(k))\psi_1 e^{ikx} + R(k)(e^{ikx}\psi_2 + \bar{R}(k)e^{ikx}\psi_1) = \\ &= (1 - R(k)\bar{R}(k))\psi_1 e^{ikx} + (R(k)e^{-2ikx})\phi_2 e^{-ikx} \bar{T}(k)\end{aligned}$$

and

$$\psi_2(x, k)e^{-ikx} = -\bar{R}(k)e^{-ikx}\psi_1(x, k) + \bar{T}(k)\phi_2 e^{-ikx}.$$

Now, using (3.91) we rewrite this as

$$(3.94) \quad \begin{pmatrix} T(k)\phi_1(x, k)e^{ikx} \\ \psi_1(x, -k)e^{-ikx} \end{pmatrix} = \begin{pmatrix} 1 - R(k)\bar{R}(k) & R(k)e^{2ikx} \\ -\bar{R}(k)e^{-2ikx} & 1 \end{pmatrix} \begin{pmatrix} \psi_1(x, k)e^{ikx} \\ \bar{T}(k)\phi_1(x, -k)e^{-ikx} \end{pmatrix}$$

which defines a matrix Riemann-Hilbert problem on the line  $k \in \mathbf{R}$  of the form

$$\Phi^+ = G\Phi^-$$

where  $\Phi^+$  is analytic in the upper half plane except for the poles of  $T(k)$ , and  $\Phi^-$  likewise in the lower half plane, and owing to our normalization  $\Phi^+$  and  $\Phi^-$  approach  $(1, 1)^T$  as  $|k| \rightarrow \infty$ . Also note  $\det G = 1 \neq 0$ .

While this is a generalized Riemann-Hilbert problem of index zero (owing to the fact that  $G + G^*$  is positive definite, cf. [AF03]), uniqueness of the solution can be established and it can be transformed into a conventional vector Riemann-Hilbert problem by supplementing the above with certain pole conditions around  $\pm\kappa_i$  (cf. [GT09]).

In order to recover the potential, we will need to derive some integral equations for the expression  $\phi_1(x, k)e^{ikx} =: M$ , which solves the ODE

$$M_{xx} - 2ikM_x = -u(x)M$$

(obtained by transforming the Schrödinger equation  $\psi \rightarrow Me^{ikx}$ ). This yields the following representation as an integral equation (see [AF03])

$$M(x, k) = 1 + \int_{-\infty}^{+\infty} G(x - \xi, k)u(\xi)M(\xi, k)d\xi$$

(where we have suppressed the parametric dependence on time). The kernel  $G(x, k)$  is the Green's function of  $M$ , and satisfies

$$G_{xx} - 2ikG_x = -\delta(x).$$

We find that

$$G(x, k) = \frac{1}{2ik}(1 - e^{2ikx})H(x)$$

for the Heaviside function  $H$ . Thus

$$M(x, k) = 1 + \frac{1}{2ik} \int_{-\infty}^x (1 - e^{2ik(x-\xi)})u(\xi)M(\xi, k)d\xi.$$

Now consider the limit as  $|k| \rightarrow \infty$ , where we make the time dependence explicit. The term

$$\int_{-\infty}^x e^{2ik(x-\xi)} u(\xi, t) M(\xi, k, t) d\xi$$

vanishes by virtue of the Riemann-Lebesgue lemma, so that

$$M(x, k, t) \sim 1 - \frac{1}{2ik} \int_{-\infty}^x u(\xi, t) d\xi$$

where we have used the asymptotic behavior of  $M = \phi_1(x, k)e^{ikx} \rightarrow 1$ . We write  $M \sim 1 + \frac{\Theta(x, t)}{k}$  as the large  $k$  asymptotics, and, finding an analogous expression for the lower half plane (using  $\psi_1(x, k)e^{ikx}$ ) we have

$$\Theta(x, t) = \begin{cases} \frac{-1}{2ik} \int_{-\infty}^x u(\xi, t) d\xi & \text{Im}(k) > 0 \\ \frac{1}{2ik} \int_x^{\infty} u(\xi, t) d\xi & \text{Im}(k) < 0 \end{cases}$$

So that we can recover the potential via

$$u(x, t) = -2ik \frac{\partial}{\partial x} \Theta(x, t).$$

### 3.5. Well-posedness and existence for asymptotic regimes

We now need to address a question posed above: what do these asymptotic models tell us about actual water waves. Although we have derived a number of asymptotic equations from the water wave equations, the central question is whether the properties of these equations carry over to properties of the full water wave equations. (It might be interjected that it is naive to assume that the Euler equations will tell us everything about fluid motion, as Lagrange did in writing ‘‘By this discovery, all fluid mechanics was reduced to a single point of analysis, and if the equations involved were integrable, one could determine completely, in all cases, the motion of a fluid moved by any forces...’’ (translated by G. Birkhoff in [Bir60]. Birkhoff goes on to describe many of the paradoxes resulting from solutions to the Euler equations (with known boundary) which ‘‘disagree grossly with observation, flagrantly contradicting the opinion of Lagrange’’. We remain cautiously optimistic.) In order to compare solutions of an asymptotic model and those of the governing equations, the first question is whether the water wave equations are well posed. We say, following [Eva98] that a certain problem for a PDE is *well-posed* if

- the problem has a solution,
- this solution is unique, and
- the solution depends continuously on the initial or boundary data given.

Since we are concerned only with water waves, this last point is especially important to ensure that we are working with a physically relevant solution. For the governing equations, this problem was solved during the latter half of the twentieth century, starting with the case of 1-dimensional propagation for irrotational flow over finite depth in [Yos82], albeit for small displacements and short time. Major breakthroughs were made by [Wu97] for the case of infinite depth by removing the restrictions to small displacements, and [Lan05] who treated the finite depth case with varying bed in 3-dimensions. Newer results exist for cases of rotational flow, but for our purposes - understanding the dynamics in the shallow-water, small amplitude regime as exemplified by the KdV and Boussinesq equations - it is enough

to know of the well-posedness as presented in [Lan05]. More recently, Coutand and Shkoller [CS07b] were able to remove the condition of irrotationality under some additional hypotheses on the behavior of the surface at the water/air interface.

We will be asking in what sense solutions of an asymptotic model like the KdV have something in common with solutions to the water wave equations. Do solutions of the governing equations exist on the time scales of the asymptotic models, and are the models good approximations of these solutions? For example, do N-soliton-like solutions of the governing equations exist? We should not expect too much, as the following quote from [Wri06] demonstrates.

The most notable deviation between true solutions and the KdV approximation is the size of the phase shift after a collision. In addition, soliton-like solutions to the type of systems we study frequently develop a very small amplitude dispersive wave train behind each soliton, which moves in the same direction[...]. The KdV approximation does not predict the existence of these dispersive wave trains. As these sorts of discrepancies are observed even in the case where there is only one wave train moving unidirectionally, we believe that they are, loosely, independent of interactions between the left- and right-moving wave trains. They reflect intrinsic differences between the approximation and the original system.

Nevertheless, we can answer the questions of existence time and quality of approximation largely positively. We present a brief sketch of the results as they relate to the KdV. Some of the first work in this direction was done by Kano and Nishida [KN86] in 1985. Unfortunately, this was predicated on the assumption of analytic initial conditions and presented results for too short a time scale to see KdV dynamics. Schneider and Wayne [SW00] were able to improve upon this result, achieving an approximation on the slow time scale of order  $\epsilon^{-1}$ , albeit with an error estimate of order  $\epsilon^{1/4}$ . Unfortunately, for large times, this means that the error is of the order  $\epsilon^{-3/4}$ , which is not desirable for small  $\epsilon$ . The most recent results are due to Lannes [Lan05] and Alvarez-Samaniego and Lannes [ASL08], where the latter work gives the same estimates as the former using a more general method which is also used to treat the KP, Green-Naghdi, and other equations.

The main thread of [Lan05] is to explore systems of Boussinesq-type introduced by Bona, Chen, and Saut in [BCS02]. A subclass related to these systems is derived which is symmetric, bringing to bear many desirable properties of such systems. The relations between these systems are explored, and they are shown to be consistent. The main result of interest for the KdV approximation is

**THEOREM 3.2** (Theorem 7). *Let  $s \in \mathbb{R}$ . For  $\sigma$  large enough, if  $(v_0, \eta_0) \in (H^\sigma(\mathbb{R}))^2$ , then there exists  $T_0 > 0$  such that for all  $t \in [0, \frac{T_0}{\epsilon}]$ ,*

$$|(v^\epsilon, \eta^\epsilon) - (v_{KdV}^\epsilon, \eta_{KdV}^\epsilon)|_{L^\infty([0,t], H^s(\mathbb{R})^2)} \leq \text{Const.} \epsilon^2 t^{3/2}$$

Herein we have used  $v$  as the gradient of the velocity potential at the free surface, and it is understood that  $(v^\epsilon, \eta^\epsilon)$  is a solution to the Euler equations, and  $(v_{KdV}^\epsilon, \eta_{KdV}^\epsilon)$  a solution to the KdV. If, in addition, the initial conditions satisfy a decay assumption such that there exists  $\alpha > \frac{1}{2}$  and

$$\sup_{x \in \mathbb{R}} |(1+x^2)^\alpha (\partial_x^\beta v_0(x), \partial_x^\beta \eta_0(x))| < \infty, \quad \beta = 0, \dots, s$$

then the approximation can be improved to

$$|(v^\epsilon, \eta^\epsilon) - (v_{KdV}^\epsilon, \eta_{KdV}^\epsilon)|_{L^\infty([0,t], H^s(\mathbb{R}^2))} \leq \text{Const.} \epsilon^2 t.$$

The long-time nature of this existence result is especially important for the KdV, since time therein is slowed in comparison to physical or Boussinesq time. We also note that the error never grows beyond  $O(\epsilon)$ .





## Tsunami Modelling

### 4.1. Tsunami

What is a tsunami? I am sure you already have a picture in your mind, perhaps of a towering wave ready to crash on some unsuspecting coastline, perhaps something resembling “The Great Wave off Kanagawa” of Katsushika Hokusai. Tsunami is a Japanese word meaning “harbor wave”, however, it is not in general one wave, but rather a series of waves generated by impulsive vertical displacement of the ocean surface. The first tsunami in recorded history occurred in 2000 B.C. off the coast of Syria, but instrumental earthquake recording and the accurate identification of tsunami associated therewith has only been used since 1900.

Most tsunami are caused by rapid vertical movement along breaks in the Earth’s crust. A large mass of earth rising or falling imparts energy to the column of water situated above it, resulting in wave generation. The areas most prone to such movements are called *subduction zones*, where an oceanic plate collides with and dips under a continental plate. These zones are prevalent along much of the Pacific rim (excepting the west coast of the United States and Canada) as well as along many island arcs, such as New Guinea, Japan, Kamchatka, and the Solomon Islands. There are only small subduction zones in the Atlantic Ocean, along the Carribean and Scotia arcs, while the Indian Ocean houses the major subduction of the Indo-Australian plate beneath the Eurasian plate - the characteristics of which mean that most tsunami generated here propagate towards Java and Sumatra, rather than into the Indian Ocean.

Other prominent sources of tsunami include volcanoes - generally through collapse of the volcanic edifice, subsidence, landslides or earthquakes associated with the eruption. Tsunami may also be caused by submarine landslides, rockslides, or, more rarely, meteors, or deep-focus earthquakes with no surface rupture.

While no direct observation or measurement of the generation of tsunami has been undertaken, studies of the data suggest that several factors correlate with the size of a tsunami. Among these are the size of the (shallow-focus) earthquake, the area and shape of the rupture zone, rate and type of displacement of the sea-floor, magnitude of displacement, and the water depth in the generation region. Many of these factors are also discussed in [Ham73] (see below). It may also be observed that tsunamis generated on the continental shelf tend to be of longer period, while those generated in the deep waters beyond the continental shelf tend to have shorter periods. It is important to note that, because of their long period compared with

---

Some of the material in this chapter has been adapted from [Stu09]

wind waves, tsunami waves can bend around obstacles and enter bays and gulfs readily.



**FIGURE 1. Destruction wrought by the Chilean tsunami of 1960 at Hilo, HI**

Also note that radiation of tsunami is directional. Source regions are generally elliptical, with a major axis as long as 600 km corresponding to the active part of the fault. The majority of tsunami energy is transmitted at right angles to the major axis, which allows us to consider some tsunami models two-dimensional for simplicity (at least as long as the propagation is over the open ocean of relatively uniform depth). Tsunami may propagate in deep water at speeds in the neighborhood of 1,000 km per hour, but their height in the open ocean is generally 1 m or less, with wavelengths in the hundreds of kilometers. Borrowing an example from [Seg07], sitting in a boat in the Pacific, the great Chilean tsunami of 1960 would have taken between 45 min to an hour to pass one by while raising the boat by less than one centimeter per minute - hardly noticeable on the open sea. Nevertheless, the tsunami reached run-ups of 7 m in Kamchatka and 10.7 m in Hilo, Hawai'i<sup>1</sup> where it caused widespread destruction after traveling 10,000 km in just under 15 hours. This is due to the fact that, as the tsunami enters coastal waters, its velocity is reduced while the height of each wave increases. This “pile up” of waves can then produce a devastating local tsunami. (Perhaps a word on the terminology run-up is in order: the *run up* is defined as the maximum height of the water observed above a reference sea level.) Locally, “run ups” of 30 to 50 m have occurred, generally in close proximity to the generation region of the tsunami. In the far field, the severity of a tsunami decreases only slowly with distance. As for the behaviour of tsunami near the coast, a tsunami wave may break on the beach, or appear as a flood, a bore, or an undular bore. However, it is possible that the trough of the wave arrives first, in which case the water level recedes rapidly [MR99]. This is an unfortunate contribution to fatalities, as curious inhabitants explore the offshore area, unaware

Figure 1 reprinted from [LL89]. Photo Credit: Sunset Newspaper

<sup>1</sup><http://wcatwc.arh.noaa.gov/web.tsus/19600522/runups.htm>

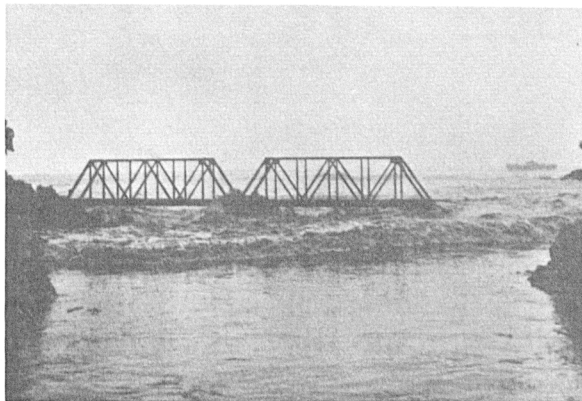


FIGURE 2. **Bore formed by the tsunami of April 1, 1946 at Hilo, HI**

that this phenomenon is caused by a tsunami, or that a tsunami consists of several waves.

One of the primary problems of mathematical tsunami modeling is to reconcile the dynamics near the generation-region with those of the far field (and, in turn, to match these with the near-shore dynamics of shoaling and inundation). An engaging historical account of tsunami modelling can be found in [Ham73]. Initially, because of the difficulty of the nonlinear models, solutions were sought for the linear theory for specific bed deformations. Attempts have been made to evaluate the resulting integrals asymptotically [Kel63][Kaj63], but as a consequence of the purely linear theory used, the far field behavior was determined to be solely an oscillatory wave train which continues to disperse into its harmonic components. As a result, the amplitude of the leading wave decays, and no waves of permanent form develop, something clearly at odds with real world tsunami dynamics where nonlinear effects eventually become apparent.

#### 4.2. Dynamics near the generation region

The KdV gives a balance of nonlinearity and dispersion for the far-field region of a tsunami, meaning that an initial disturbance must travel long distances until these dynamics can become important. What happens for shorter distances? This is by no means a simple question, since near the tsunami source we must take into account the complex impulsive deformation of the bottom. We look at the lucid presentation in [Ham73] for possible answers. Initially, for the irrotational Euler equations given in terms of a stream function, it is possible to incorporate a prescribed bed movement  $\zeta(x, t)$  into the linearized boundary conditions. This yields the system:

$$(4.1) \quad \phi_y(x, -h, t) = \zeta_t(x, t)$$

$$(4.2) \quad \phi_{tt}(x, 0, t) + g\phi_y(x, 0, t) = 0$$

---

Figure 2 reprinted from [LL89]. Photo Credit: Univ. of California at Berkeley

Upon using the Laplace transform in  $t$  and the Fourier transform in  $x$ , we are able to get a rather complicated expression for the free surface,

$$\eta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \lim_{\Gamma \rightarrow \infty} \frac{1}{2\pi i} \int_{\mu - i\Gamma}^{\mu + i\Gamma} \frac{s^2 e^{-ikx} e^{st} \bar{\zeta}(k, s)}{(s^2 + \omega^2) \cosh kh} ds \right) dk,$$

where  $\omega^2 = gk \tanh kh$ . Hammack then poses the question under what conditions the linearized governing equations are applicable to the region near the tsunami source. He proposes a scaling and nondimensionalization similar to that done for flat beds in Chapter 3, with the difference that the scales of the bottom deformation are included and play a major role. Examining these new scaled equations, we differentiate two cases depending on the rupture speed. In case of impulsive deformation, i.e. cases where the rupture is fast enough that the displaced water has not had the time to leave the generation region, linear theory is seen to be appropriate when  $\zeta_0/h \ll 1$ . This means that the rupture size  $\zeta_0$  is small compared to water depth  $h$ . On the other hand, for creeping ruptures, it is seen that linear theory is always applicable near the generation region. Taking into account the generation of tsunami along large subduction zones where we may expect typical water depths to be in the range of kilometers, the bed displacement is largely in the range of meters to tens of meters. In fact, it is unlikely to be much more than 30 m even for the largest tsunami [Bar05]. Thus it appears we may always take linear theory to be relevant near the generation region for physical tsunami generated by displacement of the sea-floor (of course, tsunami generated by landslides, submarine landslides, rockfalls, or meteors could have different dynamics). In the subsequent experimental examination, Hammack determined that for  $\zeta_0/h > 0.2$  significant nonlinear effects were apparent in the impulsive deformation case, as well as in the case of an intermediate-speed displacement, starting from the edge of the displaced bed onwards. As predicted, in the case of creeping displacements, nonlinear effects were absent even when  $\zeta_0/h \approx 1$ . It is interesting to note that the numerical simulations based on MOST and SIFT algorithms currently used in tsunami warning systems incorporate only instantaneous rupture. Therefore the question of whether different dynamics result with variations in rupture speed is of great practical importance. Hammack's experimental results [Ham73] once again provide stimulating information. For impulsive rupture, near the generation region the wave profile depends primarily on the shape of the final displacement. However, as the rupture time increases, the wave signatures become more and more dependent on the time-displacement history of the movement - which means that slower ruptures might result in tsunami dynamics unaccounted for by current numerical models.

There is another perspective we might bring to bear on near-field dynamics - that of the Boussinesq model. Recall that we derived the Boussinesq equation (3.63) in an attempt to elucidate the nonuniformity that caused us to consider the KdV. In light of this, we see that the Boussinesq equation

$$\eta_{tt} - \eta_{xx} - \epsilon(\eta^2)_{xx} - \epsilon\eta_{xxxx} = 0$$

is accurate to order  $\epsilon$  in the near field  $x = O(1)$ ,  $t = O(1)$  for long waves ( $\epsilon = O(\delta^2)$ ,  $\epsilon \rightarrow 0$ ), but breaks down for long times  $t = O(1/\epsilon)$ . For  $\epsilon$  very small in a given physical problem, as is the case for tsunami, it seems the near-field might be described essentially by the classical wave equation - which is the view taken in [Seg07]. On the other hand, [MFS08] argue that, for rectangular displacements,

dispersion plays a role from the very beginning, and the appropriate leading order equation is simply a linearized KdV (or *Airy equation*)

$$\eta_t + \frac{1}{6}\eta_{xxx} = 0.$$

In this case, it might be argued that a linearization of the Boussinesq equation could be a better choice, as we have seen that KdV arises for long times, whereas the Boussinesq model more aptly describes the near source region.

### 4.3. Dynamics in the far field

**4.3.1. The Ursell number - an approach to far field dynamics.** The theory surrounding the Ursell number  $U$  was first established in [Urs53] and further developed in [Ham73]. In the main, it revolves around a few simple observations. The terms  $\epsilon$  and  $\delta$  that we have used in the process of nondimensionalization and scaling have been called the amplitude parameter and long wavelength (or shallowness) parameter respectively. They can also, and are often, interpreted differently - namely as the relative magnitudes of the nonlinear and linear terms respectively. Hammack writes:

the relative importance of nonlinear and linear effects in a long wave propagating in a two-dimensional fluid domain is indicated by the ratio

$$(4.3) \quad \frac{\text{nonlinear effects}}{\text{linear effects}} \propto \frac{a/h_0}{(h_0/\lambda)^2} = \frac{a\lambda^2}{h_0^3} = U$$

As mentioned, this stems from considering the nondimensional scaled Euler equations

$$(4.4) \quad \phi_{zz} + \delta^2\phi_{xx} = 0,$$

$$(4.5) \quad \phi_z = \delta^2(\eta_t + \epsilon\phi_x\eta_x) \text{ on } z = 1 + \epsilon\eta,$$

$$(4.6) \quad \phi_t + \eta + \frac{1}{2}\epsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = 0 \text{ on } z = 1 + \epsilon\eta,$$

$$(4.7) \quad \phi_z = 0 \text{ on } z = 0,$$

where we see that the magnitude of the nonlinear terms is given by the parameter  $\epsilon$ , while the linear contribution is measured in terms of  $\delta^2$  (cf. [Ham73]). We might thus assume that, so long as  $\epsilon$  is small enough, linear theory yields an appropriate model. Recall that  $\epsilon \rightarrow 0$  leads to the linearised problem (which is dispersive),  $\delta \rightarrow 0$  to the long-wave problem (nonlinear and nondispersive).

However, Ursell [Urs53] states “that the linear theory of surface waves is valid only if  $a\lambda^2/h_0^3 \ll 1$ , the well-known condition  $a/\lambda \ll 1$  not being sufficient”.

Looking at this problem, it seems that a measurement of the initial profile of a wave, including all the data needed to determine  $\epsilon$  and  $\delta$  (namely amplitude, wavelength, and water depth) would determine the Ursell number and thereby the applicability of either linear or nonlinear theory. However, here we run into complications. An initial disturbance must propagate somehow, and this could lead to changes in all these parameters with time. Concentrating on the long-wave region, Ursell assumes initial conditions that were used historically to generate solitary waves: a partition in a tank of water, on one side of which the water level is higher, was to be suddenly removed. This may be likened (cf. [Ham73]) to an impulsive deformation  $\eta(|x| < b, 0) = \text{const.}$ ,  $\eta(|x| > b, 0) = 0$  (it is also analogous

to the dam break problem). Following Jeffreys and Jeffreys [JJ46], who present the long-time behavior for this initial condition via an Airy integral, Ursell derives a relation wherein  $U$  grows with time like  $t^{\frac{1}{3}}$ . On the basis of this, it is argued that nonlinear effects will always come to be important given long enough times; Ursell writes that it may be seen “in a rough way how a non-linear solitary wave can emerge after a time  $\epsilon^{-3}$ ”. It seems a shortcoming of this approach, however, that once a wave of permanent form has been established, the Ursell number no longer changes (insofar as the water depth remains constant). Therefore this evolution of  $U$  as  $t^{1/3}$  must come to an end somewhere. Indeed, the experimental results of Hammack [Ham73] corroborate this.  $U$  was computed for measured wave profiles generated by an impulsive exponential deformation, and an initial growth in nonlinear effects was established until the amplitude and frequency dispersion (contributions of  $\epsilon$  and  $\delta$  respectively) were about equal. A balance of these effects was maintained during further propagation.

This breakdown in the growth of the Ursell number suggests that there may be shortcomings to this approach. Indeed, the hydrodynamical relevance of solving the dam-break problem by linear theory and then extrapolating the results is uncertain. Ursell must nevertheless be credited for highlighting the importance of the regime  $\epsilon = O(\delta^2)$ .

#### 4.3.2. The KdV balance - another approach to far field propagation.

It is certainly sensible to ask why the KdV equation should be chosen to model natural phenomena. In most instances in nature, when we observe a solitary wave, that is in fact all we are observing. As we have pointed out, there is no reason to think that this solitary wave must be a soliton. Further, there are a number of model equations that allow solitary wave solutions. The BBM model, for example, describes shallow water waves to the same order as the KdV, and allows solitary wave solutions that are not solitons. The important point, though, is that solitary waves can be generated in the laboratory, and their interaction seems to fit very well that of soliton solutions of the KdV. The BBM model, on the other hand, has a number of properties that make it more accessible to computation, in particular the simple numerical solution developed in [Per66]. Of course, there are a number of other completely integrable equations that may be used to model certain aspects of water wave propagation, such as the Camassa-Holm, Degasperis-Procesi, and others [CL09]. We will focus exclusively on the KdV.

A central question is where the notion of a KdV balance comes from. The idea is, of course, that, since the KdV contains both weakly nonlinear and dispersive terms, and since the soliton solutions of the KdV are clearly of permanent form, nonlinearity and dispersion must be holding each other in check. Intuitively, it is said that the nonlinearity of the KdV causes waves to break (or, cf. [Ham73], nonlinearity is referred to as causing amplitude dispersion) while dispersion causes components of different frequencies to travel at different speeds. Clearly, neither of these forces has the upper hand in the motion of solitons.

Two mechanisms are at work when solitons arise, one being the balance of nonlinearity and dispersion, the other the actual separation of solitons. One of the remarkable things inverse scattering theory for the KdV tells us is that for essentially any localized positive disturbance, we can always decompose it into a number of solitons and an oscillatory tail. If the initial disturbance is exactly of the form  $n(n+1)sech^2$ , we get  $n$  solitons and no oscillatory tail (i.e. the one soliton solution

of the KdV is reproduced by a  $2\text{sech}^2$  initial profile, the two soliton solution by a  $6\text{sech}^2$ , and so on), otherwise we always have some oscillatory components. In this sense, it bears some resemblance to the Fourier transform.

The balance of nonlinearity and dispersion goes back to the slow time scale  $\tau$  and the new distance  $\xi$  that were introduced in the derivation of the KdV. Thus, in looking for relevant effects in these new variables, i.e. when  $\xi, \tau$  are of order 1, we need first to return to the original variables. Recall that we transformed  $\tau = \epsilon t$ , and the non-dimensionalisation (3.7) performed in 3.1.1.1. Then

$$(4.8) \quad x - t = O(1) \quad \tau = O(1)$$

means

$$(4.9) \quad \frac{x - t\sqrt{gh_0}}{\lambda} = O(1), \quad \frac{\epsilon t\sqrt{gh_0}}{\lambda} = O(1).$$

which gives a length scale for the KdV balance of

$$(4.10) \quad x = O\left(\frac{\lambda}{\epsilon}\right).$$

We note that this length scale was long thought to be  $x = O(h_0/\epsilon)$ , (cf. the classical results [Ham73], [HS78]) as is also espoused in the recent survey [Seg07], but (4.10) provides the correct scale - see also the discussion in [Con09].<sup>2</sup>

Thus far we have identified a regime,  $\epsilon = O(\delta^2)$ , but what does this mean in practical terms? Given that we need to check whether this regime holds based on real-world data, we take the approach that  $O(1)$  allows for deviation by a factor of ten in either direction, as is usually assumed in the hydrodynamical literature (see e.g. [Lig78]). Thus

$$10^{-1} \leq \frac{\epsilon}{\delta^2} \leq 10$$

is a good realization of the KdV regime. Recall that the definitions  $\epsilon = a/h_0$ ,  $\delta = h_0/\lambda$  mean that the above is

$$(4.11) \quad \frac{10^{-1}h_0^3}{a} \leq \lambda^2 \leq \frac{10h_0^3}{a}.$$

#### 4.4. Applications to Tsunami

We have established length scales for a KdV balance, but it remains to be seen whether such scales are applicable to real world tsunami. In doing so, we will focus on two of the largest tsunami of recorded history. We will look at the first, generated by a series of earthquakes in southern Chile on May 22, 1960 - as it propagated from Chile to Hawai'i. These earthquakes, among them the largest ever recorded, resulted from a rupture about 1000 km long and 150 km wide along the fault between the Nazca and South American plates, at a focal depth of 33 km. The principal shock occurring on May 22 at 19:11 GCT registered at 9.5 on the moment magnitude scale, and led to changes in land elevation ranging from 6 m of uplift to 2 m of subsidence - which has been modeled to correspond to an

<sup>2</sup>The previously mentioned work of Constantin and Johnson [CJ08a], which uses a formal additional scaling in place of the relation  $\epsilon = O(\delta^2)$  derives the length and time scales for a KdV balance of

$$x = O(\lambda\delta\epsilon^{-3/2}) \text{ and } t = O\left(\frac{\lambda}{\sqrt{gh_0\delta\epsilon^{-3/2}}}\right)$$

for any  $\delta$  as  $\epsilon \rightarrow 0$ . This is consistent with our results, as the scale (4.9) can be recovered simply by returning to  $\epsilon = O(\delta^2)$ , i.e. rendering the scaling meaningless.

average dislocation of 20 m along the fault, with peaks of more than 30 m [Bar05]. This subsidence extended as far as 29 km inland, resulting in some 10 km<sup>2</sup> of forest around the Río Maullín being submerged by the tides and consequently defoliated [CA<sup>+</sup>05].

Not only was the principal earthquake at 39.5°S, 74.5°W especially powerful, it generated tsunami with an average run-up of 12.2 m and a maximal run-up on the adjacent Chilean coast of 25 m. Over the course of the next day, a number of tsunami wreaked havoc upon the Pacific, taking the lives of more than 2000 people and causing millions of dollars in damages. The initial wave traveled between 670 and 740 km/h, with a wavelength of between 500 - 800 km and a height in the open ocean of only 40 cm [Bry08], [HGR96]. The propagation distances involved in the 1960 Chilean tsunami are among the largest possible on earth, making it one of the best candidates among teleseismic tsunami for the appearance of a KdV balance.

The second tsunami we focus on is the devastating Boxing Day (December 26) tsunami of 2004, generated by a magnitude 9.0 earthquake off the coast of northern Sumatra which killed more than 290,000 people and displaced millions. Wave heights of 30 m were reported along the west coast of Sumatra, 5-10 m along the east coast of India and 3-5 m around Phuket, Thailand. For the first time, heights in the open ocean were measured by satellite. Two hours after the earthquake, the open water height was 60 cm, while by 3 hours and 15 minutes, it had dropped to about 40 cm.<sup>3</sup> Traveling westward towards India, the tsunami



FIGURE 3. **The Boxing Day tsunami reaching the coast of Thailand.** Note the initial negative wave, followed by two essentially two-dimensional waves, both of which have broken further out, the larger behind the smaller.

had a speed of about 620 km per hour in waters about 3 km deep, while the eastward wave towards Thailand travelled at a slower rate of 350 km per hour in about 1 km deep waters [Seg07] (note that these are essentially the shallow water speeds  $\sqrt{gh}$ ). The wavelengths westwards are said to have been about 100 km,

<sup>3</sup><http://www.noaa.gov/news/stories2005/s2365.htm>  
Figure 3 reproduced courtesy of A. Constantin



and those eastwards slightly smaller [Seg07]. It is also important to note that the rupture was roughly 100 km in the east west direction and 900 km in the north south direction. This makes it feasible to expect some directivity in the tsunami energy, and gives a rationale for using the two-dimensional KdV model initially for east-west propagation.

**4.4.1. Indonesia, 2004.** We may treat the eastward and westward propagation as separate cases. Since the tsunami reached Thailand to the east after about 1 hour, we assume  $a = 1$  m and  $h_0 = 1$  km, with  $\lambda \leq 100$  km. We see that (4.11) yields a realistic range of wavelengths between 10 and 100 km. So far we are in the right regime to see KdV dynamics, but we need to find a bound for the distance in which we expect a balance of nonlinearity and dispersion to appear. In its eastward propagation, the tsunami travelled about 600 km before reaching Thailand, thus (4.10) means that our length scale for KdV balance must fulfill

$$(4.12) \quad \frac{\lambda}{\epsilon} < 600 \text{ km.}$$

Together with (4.11) we can eliminate  $a$  herein to get

$$\lambda^3 < 6000h_0^2 \text{ km}$$

which means

$$(4.13) \quad \lambda < 18 \text{ km,}$$

clearly an unrealistic figure. In Figure 3, we see that an initial negative wave was followed by a series of waves, the first considerably smaller than the second, which can be seen some distance behind it. Had soliton theory played a role, we would expect the largest wave to be in front, as larger solitons travel faster than smaller.

We make the same calculations for the westward propagation, with a distance of 1500 km across the Bay of Bengal. Considering the travel time, we know from satellite data that the amplitude was roughly 60 cm. Therefore we have

$$(4.14) \quad \lambda < 51 \text{ km,}$$

which is also at odds with the calculated wavelengths.

**4.4.2. Chile, 1960.** For the Chilean tsunami of 1960, taking  $a = 0.4$  m and  $h_0 = 4.3$  km (cf. [Bry08]), we see that (4.11) yields a range of wavelengths between 140 and 1400 km. Given that we consider the 1960 tsunami only between Chile and Hawai'i, a distance of about  $10^4$  km, (4.10) means that we need

$$(4.15) \quad \frac{\lambda}{\epsilon} < 10^4 \text{ km.}$$

Again using (4.11) to eliminate  $a$  herein to get

$$\lambda^3 < 10^5 h_0^2 \text{ km} \approx 1.8 \times 10^6 \text{ km}$$

or

$$(4.16) \quad \lambda < 121 \text{ km.}$$

However, measurements place the wavelength of the tsunami of May 22, 1960 between 500 - 800 km [Bry08] making it unlikely that KdV dynamics played a role. This is further supported by the fact that the first two tsunami waves reaching Hilo, Hawai'i were smaller than the third, most destructive wave - something which should not occur if KdV dynamics were significant for the leading waves of the

tsunami. We have deliberately used the wavelength  $\lambda$ , because of the relative ease of measurement and error tolerance compared with measuring amplitude. An argument based on the amplitude can be found in [CH09] and [Con09].

#### 4.4.3. Is there hope for the KdV paradigm in tsunami modelling?

It seems from the above considerations that KdV theory is unlikely to play a role in tsunami dynamics in the open ocean. A look at the formula for identifying the KdV regime (4.11) shows that for tsunami in most oceans, we have  $\epsilon = O(\delta^2)$ , but the length scales required to see KdV dynamics are too long. This is represented below (cf. [CJ08a] where a similar table is presented for the KdV balance in terms of amplitude and depth):

TABLE 1. Upper bounds on wavelengths necessary for KdV length scales by propagation distance and water depth, under the assumption  $\epsilon = O(\delta^2)$

| Propagation distance (km) | 1      | 4      | 100    | 1,000  | 10,000 |
|---------------------------|--------|--------|--------|--------|--------|
| Depth 5 m                 | 63m    | 100m   | 292m   | 629m   | 1.3 km |
| Depth 10 m                | 100m   | 159m   | 464m   | 1 km   | 2.2 km |
| Depth 100 m               | 464m   | 737m   | 2.2 km | 4.6 km | 10 km  |
| Depth 1000 m              | 2.2 km | 3.4 km | 10 km  | 22 km  | 46 km  |
| Depth 2000 m              | 3.4 km | 5.4 km | 16 km  | 34 km  | 74 km  |
| Depth 4000 m              | 5.4 km | 8.6 km | 25 km  | 54 km  | 117 km |
| Depth 4300 m              | 5.7 km | 9 km   | 26 km  | 57 km  | 122 km |

We see that, for Scott Russell’s soliton, described to be 30 ft (9 m) long and between a foot and a foot and a half in height (30 - 45 cm) in a channel perhaps 5 m deep, it would certainly be possible for a KdV balance to occur over the distance he followed it (“one or two miles”, or between 1.5 and 3 km). And although it seems that tsunami in the open ocean generally have wavelengths too long for this balance to occur, the possibility remains that in their near-shore dynamics, perhaps following shoaling, solitons could be generated. We will explore this possibility for one event in 1986, the tsunami of May 7th, as it impacted Kaiaka Bay on the north shore of Oahu, HI. An earthquake of magnitude 7.7 occurred off the Andreanov Islands, Alaska and caused a tsunami that was felt in many places along the Pacific. In Hawai’i, though a costly evacuation was ordered, abnormally small run-ups were recorded. These are attributed to the short fault length and the shallow angle of subduction [LL89]. Despite this, in Kaiaka Bay a significant solitary wave was generated. The best DEM (Digital Elevation Model) data for the bay suggests that it is no more than two meters deep [fTR]. This is by the fact that Kaiaka Bay is the drainage basin for the Paukauila and Kiikii streams which carry sediment down from approximately 610 m above mean sea level [oLNR02]. While the bay is approximately 500m wide, the areas seen in the photos are in a section where it narrows to between 100 (left photo) and 125 m (right photo). Cane Haul Rd. bridge seen to the right in Figure 3 is about 8m wide, while the Haleiwa Road bridge crossing Paukauila Stream in the left photograph is about 11 m wide. In the upper left corner of the right photograph, one can see Kiikii Stream entering the bay. Given this, it seems feasible to estimate the effective wavelength as in the area of 3 m.(cf. [MFS08]). Since there is only a single crest, wavelength in the classical



FIGURE 4. Solitary wave on May 7, 1986 at Kaiaka Bay, Oahu, HI

sense does not apply. [MFS08] refer to “effective duration in time and space”, but it seems to us more succinct to say simply effective wavelength. Hammack suggests in [Ham73] that an appropriate definition of  $\lambda$  in a region of complex waves is  $\lambda = O(a/a_x)$ , where he takes  $a_x$  to be the slope of the wave. He further suggests that, to establish a numerical value for the characteristic length, the operational definition  $\lambda = |a_0|/|(a_x)_{max}|$  where  $a_0$  is the total change in wave amplitude within a region and  $(a_x)_{max}$  is the maximum slope of the wave in the region. Of course, for a tsunami in the open ocean, this would be very difficult to implement, seeing as, if we have a wavelength of some 500 km, and an elevation of 0.5 m, the slope is very near to zero ( $10^{-6}$ .) It is difficult to ascertain the amplitude of the wave. It appears in the photo as a solitary bore, lacking a significant oscillatory tail (as opposed to the case of undular bores). The wake seen behind the wave suggests that some turbulent mixing is going on. The strength threshold  $\beta$  at which such turbulent bores appear is said to be approximately 0.3 [Lig78]. Thus a crude calculation of bore strength

$$\beta = \frac{A_1 - A_0}{A_0},$$

where  $A_0$  is the initial cross sectional area and  $A_1$  is the cross sectional area behind the bore, using an average water depth of 1m, a width of 100m, yields that the wave should be some 30 cm or more in height. This seems reasonable given the other dimensions of the wave. We may take  $a = 0.5m$ . We see with this data,  $\lambda = 3m$ ,  $a = 0.5m$ , and  $h_0 = 2m$  that we are indeed in an appropriate regime to see KdV dynamics, as

$$\frac{\epsilon}{\delta^2} = \frac{a\lambda^2}{h_0^3} \approx 0.56$$

---

Figure 4 courtesy of the Department of Ocean and Resources Engineering, School of Ocean and Earth Science and Technology, University of Hawaii at Manoa

which is within the tolerance range  $10^{-1} < \epsilon/\delta^2 < 10$ . The distance between deeper waters and the location of the wave in the photographs is about 800m, so the length scale for KdV dynamics yields

$$\frac{\lambda}{\epsilon} < 800m$$

or

$$\lambda < 31m,$$

which appears to be the case. Although we find that KdV dynamics could be relevant in a wave such as that discussed, some cautionary remarks are in order. Clearly the wave pictured does not look much like a KdV solitary wave - but it is important to note that, outside of controlled experiments, or in the confines of a relatively uniform canal (in which we may expect a relatively uniform bottom topography, few if any existing waves or currents, etc.) we cannot expect a solitary wave in nature to meet the standards of mathematical rigor. It nevertheless seems appropriate to consider the wave at Kaiaka Bay as a proof of concept - that tsunami in their near shore dynamics could very well generate solitary waves that might be modeled with the KdV. It does not seem that this is simply an artifact of the particular numbers involved, as changing them by a factor of 2 does not impact the conclusion. (It is also interesting to note that the dam break problem from which classical models for bores may be derived is the same that was used experimentally to generate solitary waves [Urs53]. Ursell writes “since the days of Scott Russell it has been customary to generate the solitary wave by putting a water-tight partition in the canal, raising the level on one side of the partition, and then removing the partition suddenly”.)

#### 4.5. Measurements

The problem of accurate measurement of the many parameters involved in developing a theory for shallow-water waves is now apparent. While we generally take a typical wavelength  $\lambda$  in our nondimensionalization procedure, it is not quite clear how typical any such length can be. Strictly speaking, wavelengths are defined for wave trains, which tsunami are not, for the most part. Further, a look at the period data for most tsunami will typically reveal a very large variance.<sup>4</sup> In

TABLE 2. Periods for the 3.01.2009 Tsunami

| Location                         | Period(min) |
|----------------------------------|-------------|
| Tosa-Shimizu                     | 14          |
| Ishigakiko                       | 18          |
| Naha                             | 24          |
| Omaezaki                         | 12          |
| Yap Island, Caroline Islands     | 6           |
| Malakal Island, Caroline Islands | 14          |

Table 2, even within Japan (Omaezaki and Naha) there is a 2-fold difference in periods. Taking into account the data from Malakal in the Federated States of Micronesia, we see a four-fold difference. In Table 3 it is interesting to note that Luganville, Vanuatu is the location closest to the tsunami source (294 km) for

<sup>4</sup>Data from [http://www.ngdc.noaa.gov/hazard/tsu\\_db.shtml](http://www.ngdc.noaa.gov/hazard/tsu_db.shtml)

TABLE 3. Periods for the Samoan tsunami of 7.10.2009

| Location              | Period(min) |
|-----------------------|-------------|
| Port San Luis, CA     | 9           |
| Hilo, Hawaii, HI      | 10          |
| Honolulu, Oahu, HI    | 7           |
| Kawaihae, Hawaii, HI  | 6           |
| Nawiliwili, Kauai, HI | 4           |
| Luganville            | 20          |

which period data is available. Hawaii is roughly 5500 km distant; nevertheless, the variability of tsunami period for different locations in Hawaii is 2.5-fold. Clearly many phenomena could play a central role in determining this variability, among them reflection, resonance, refraction or diffraction. It seems that the Hawaiian Islands, due to their proximity, make an accurate period difficult to discern. It is well known that Hilo, Hawaii, due to the shape of its bay is particularly prone to large run-ups<sup>5</sup>, perhaps due to its natural resonant frequencies. At any rate, what we must take away from this data is the knowledge that caution must be exercised when trying to apply our mathematical methods to real world tsunami. This is especially difficult for historical tsunami for which few accurate measurements exist.

Attempts have been made to reconcile variability of tsunami periods with the mathematical theory. Initial attempts were made to explain this on the basis of viscosity, the internal friction of the fluid. Later theories focused on the effects of dispersion [Mun46, Kaj63]. Therein, for long distances from the tsunami source and constant depth, the period increase was noted to be proportional to  $t^{1/3}$  (for one dimensional wave propagation). This does not seem to be borne out by our data, suggesting that other effects than those considered may play a role. (There may certainly be good criteria for discarding certain data points, and the two representative events we have chosen were selected at random from events of the past year. Munk [Mun46] finds favourable comparisons between this theory and data for the tsunami of April 1, 1946. Unfortunately, the in-depth look at global tsunami data necessary to corroborate this, while interesting, is outside the scope of this manuscript.) Most significantly, in the above studies the equations of motion are linearized in order that disturbances in the bottom topography may be modeled. On the basis of this, the far field behavior of these disturbances is investigated. Kajiura in [Kaj63] lists his main concerns as the assumptions of 1) linear approximation, 2) constant depth and no lateral boundary, 3) the leading wave at long distances from the source, and 4) time dependence of the source of delta-function type.

Further problems exist, of course, in the measurement of tsunami amplitudes, which are typically very small in the open ocean. This problem is almost universally circumvented simply by assuming the amplitude to be 1m. Especially in the case of older tsunami data when amplitudes were not recorded by satellite but solely by tide gauges, the design of these gauges introduces errors into the data. The problem lies in the fact that tide gauges are designed to measure a phenomenon with a period of twelve hours, and not a tsunami with a period measured in minutes. In order to eliminate short period waves of no interest, these tide gauges have wells

---

<sup>5</sup>Dr. G. Fryer, personal communication

which dampen these disturbances. In general, the larger the wave and the shorter the period, the greater is the discrepancy between real and recorded amplitudes in such gauges [Noy76]. There are a number of linear models that have been used to explain attenuation of tsunami waves in the far field (cf. the discussion in [Kaj63], who proposes an attenuation proportional to powers of the distance from source  $r$ ;  $r^{-1/3}$  for 1 dimensional propagation, and either  $r^{-2/3}$  or  $r^{-1}$  based on source characteristics for 2 dimensional propagation), but these models do not seem to account well for observations made in laboratory experiments, and it may be assumed that, due to the difficulties sketched above, it will be difficult to verify their agreement with actual tsunami.

## Appendix: deutsche Zusammenfassung

In dieser Arbeit werden die reibungslosen Bewegungsgleichungen für Wasserwellen mit physikalischer Motivation eingeführt. Es folgt ein Studium der Eigenschaften dieser Gleichungen, die durch Anwendung asymptotischer Näherungen zur Korteweg-de Vries Gleichung führen. Schließlich wird die Korteweg-de Vries Gleichung hinsichtlich ihrer Anwendung im Bereich der Tsunami Modellierung untersucht.





## Acknowledgements

First and foremost, the author is grateful to Prof. Adrian Constantin for his help with the preparation of this manuscript. The author would also like to thank Prof. K.F. Cheung and Dr. Volker Roeber of the University of Hawai'i, Manoa and Dr. Gerard Fryer of the NOAA Pacific Tsunami Warning Center for the interesting and helpful discussions, as well as Prof. Gerald Teschl and Prof. Iryna Egorova for their suggestions on literature. Finally, the author would like to thank Univ.-Doz. Karl M. Stuhlmeier for his support.



## Bibliography

- [AF03] M. J. Ablowitz and A. S. Fokas. *Complex variables: introduction and applications*. Cambridge University Press, Cambridge, UK, 2nd ed edition, 2003.
- [AS81] M. J. Ablowitz and Harvey Segur. *Solitons and the inverse scattering transform*, volume 4. SIAM, Philadelphia, 1981.
- [ASL08] B. Alvarez-Samaniego and D. Lannes. Large time existence for 3D water waves and asymptotics. *Invent. Math.*, 171(3):485–541, 2008.
- [Bar05] S. Barrientos. Earthquakes - Giant returns in time. *Nature*, 437(7057):329–329, 2005.
- [BCS02] J.L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory. *J. Nonlinear Sci.*, 12(4):283–318, 2002.
- [Bir60] G. Birkhoff. *Hydrodynamics: a study in logic, fact, and similitude*. Princeton University Press, Princeton, N.J., rev. ed edition, 1960.
- [BK75] T. P. Barnett and K. E. Kenyon. Recent advances in the study of wind waves. *Rep. Prog. Phys.*, 38:667–729, 1975.
- [Bry08] E. Bryant. *Tsunami: the underrated hazard*. Springer-Praxis books. Springer, Berlin, 2008.
- [CA<sup>+</sup>05] M Cisternas, BF Atwater, et al. Predecessors of the giant 1960 Chile earthquake. *Nature*, 437(7057):404–407, 2005.
- [CE04] A. Constantin and J. Escher. Symmetry of steady periodic surface water waves with vorticity. *J. Fluid Mech.*, 498:171–181, 2004.
- [CE07] A. Constantin and J. Escher. Particle trajectories in solitary water waves. *Bull. Amer. Math. Soc.*, 44(3):423–431, 2007.
- [CH09] A. Constantin and D. Henry. Solitons and Tsunamis. *Z. Naturforsch.*, 64a:65–68, 2009.
- [CJ08a] A. Constantin and R.S. Johnson. On the Non-Dimensionalisation, Scaling and Resulting Interpretation of the Classical Governing Equations for Water Waves. *J. Nonlinear Math. Phys.*, 15(Suppl. 2):58–73, Aug 2008.
- [CJ08b] A. Constantin and R.S. Johnson. Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis. *Fluid Dyn. Res.*, 40(3):175–211, 2008.
- [CL84] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. R.E. Krieger, Malabar, Fla., 1984.
- [CL09] A. Constantin and D. Lannes. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. *Arch. Ration. Mech. Anal.*, 192(1):165–186, 2009.
- [Con01] Adrian Constantin. On the deep water wave motion. *J. Phys. A, Math. Gen.*, 34(7):1405–1417, 2001.
- [Con09] A. Constantin. On the relevance of soliton theory to tsunami modelling. *Wave Motion*, 2009. DOI: 10.1016/j.wavemoti.2009.05.002.
- [CS04] A. Constantin and W. Strauss. Exact steady periodic water waves with vorticity. *Commun. Pure Appl. Math.*, 57(4):481–527, 2004.
- [CS07a] A. Constantin and W. Strauss. Stability properties of steady water waves with vorticity. *Commun. Pure Appl. Math.*, 60(6):911–950, 2007.
- [CS07b] D. Coutand and S. Shkoller. Well-posedness of the free-surface incompressible Euler equations with or without surface tension. *J. Am. Math. Soc.*, 20(3):829–930, 2007.
- [CSnt] A. Constantin and W. Strauss. Pressure beneath a Stokes wave. *Commun. Pure Appl. Math.*, in print.

- [DJ89] P. G Drazin and R. S Johnson. *Solitons: an introduction*. Cambridge University Press, Cambridge, 1989.
- [Dra83] P. G Drazin. *Solitons*. Cambridge University Press, Cambridge, 1983.
- [Eva98] L. C. Evans. *Partial differential equations*. American Mathematical Society, Providence, R.I., 1998.
- [FLS63] R. P. Feynman, R. B. Leighton, and M. L. Sands. *The Feynman lectures on physics*. Addison-Wesley Pub. Co, Reading, Mass., 1963.
- [Fok97] A.S. Fokas. A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. Lond., Ser. A*, 453(1962):1411–1443, 1997.
- [FPU74] E. Fermi, J. Pasta, and S. Ulam. Studies of nonlinear problems. I. Nonlin. Wave Motion, Proc. Summer Sem. Potsdam (New York) 1972, 143-156 (1974)., 1974.
- [fTR] NOAA Center for Tsunami Research. Oahu, hi 1 arc-second tsunami inundation dem.
- [GGKM67] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura. Method for solving the korteweg-devries equation. *Phys. Rev. Lett.*, 19(19):1095–1097, Nov 1967.
- [GT09] L. Grunert and G. Teschl. Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent. *Math. Phys. Anal. Geom.*, 12(3):287–324, 2009.
- [Ham73] J. L. Hammack. A note on tsunamis: their generation and propagation in an ocean of uniform depth. *J. Fluid Mech.*, 60:769–799, 1973.
- [Hen08] D. Henry. On Gerstner’s Water Wave. *Journal of Nonlinear Mathematical Physics*, 15(Suppl. 2):87–95, Aug 2008.
- [HGR96] Ph. Heinrich, S. Guibourg, and R. Roche. Numerical modeling of the 1960 Chilean tsunami. Impact on French Polynesia. *Phys. Chem. Earth*, 21(Issues 1-2):19–25, 1996.
- [HS74] J. L. Hammack and H. Segur. The Korteweg-de Vries equation and water waves. Part 2: Comparison with experiments. *J. Fluid Mech.*, 65:289–314, 1974.
- [HS78] J. L. Hammack and H. Segur. The Korteweg-de Vries equation and water waves. Part 3: Oscillatory waves. *J. Fluid Mech.*, 84:337–358, 1978.
- [Isr] S. Israwi. Variable depth KdV equations and generalizations to more nonlinear regimes. arXiv:0901.3201.
- [JJ46] H. Jeffreys and B. S. Jeffreys. *Methods of mathematical physics*. The University press, Cambridge, 1946.
- [Joh97] R. S Johnson. *A modern introduction to the mathematical theory of water waves*. Cambridge University Press, Cambridge, 1997.
- [Joh03] R. S Johnson. The classical problem of water waves: a reservoir of integrable and nearly-integrable equations. *J. Nonlinear Math. Phys.*, 10(suppl. 1):72–92, 2003.
- [Joh05] R. S Johnson. *Singular perturbation theory: mathematical and analytical techniques with applications to engineering*. Springer, New York, 2005.
- [Kaj63] K. Kajiwara. The leading wave of a tsunami. *Bull. Earthquake Res. Inst. Tokyo University*, 41:535–571, 1963.
- [KDV95] D. J. Korteweg and G. De Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.*, 39:422–443, 1895.
- [Kel63] J. B. Keller. *Tsunamis: water waves produced by earthquakes*, volume 24, pages 154–166. Int. Un. Geodesy & Geophys. Monograph, 1963.
- [KN86] T. Kano and T. Nishida. A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves. *Osaka J. Math.*, 23:389–413, 1986.
- [KS08a] J. Ko and W. Strauss. Effect of vorticity on steady water waves. *J. Fluid Mech.*, 608:197–215, 2008.
- [KS08b] J. Ko and W. Strauss. Large-amplitude steady rotational water waves. *Eur. J. Mech. B*, 27(2):96–109, 2008.
- [Lam95] H. Lamb. *Hydrodynamics*. University Press, Cambridge, 1895.
- [Lan05] D. Lannes. Well-posedness of the water-waves equations. *J. Am. Math. Soc.*, 18(3):605–654, 2005.
- [Lax68] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.*, 21:467–490, 1968.
- [Lig78] J. Lighthill. *Waves in fluids*. Cambridge University Press, Cambridge, 1978.
- [LL89] J. F. Lander and P. A. Lockridge. United states tsunamis (including united states possessions) 1690-1988. Technical Report Publication 41-2, National Geophysical Data Center, August 1989.

- [MFS08] P. A. Madsen, D. R. Fuhrman, and H. A. Schäffer. On the solitary wave paradigm for tsunamis. *J. Geophys. Res.*, 113, 12 2008.
- [MR99] R. K. Mazova and J. Ramirez. Tsunami waves with an initial negative wave on the Chilean coast. *NATURAL HAZARDS*, 20(1):83–92, 1999.
- [MT68] L. M Milne-Thomson. *Theoretical hydrodynamics*. Macmillan, London, 5th ed., revised and enlarged edition, 1968.
- [Mun46] W. H. Munk. *Increase in the period of waves traveling over large distances: with applications to tsunamis, swell, and seismic surface waves*. PhD thesis, UC San Diego: Scripps Institute of Oceanography, 1946.
- [Mus53] N. I Muskhelishvili. *Singular integral equations: boundary problems of function theory and their application to mathematical physics*. P. Noordhoff, Groningen, 1953.
- [Nay81] A. H. Nayfeh. *Introduction to perturbation techniques*. Wiley, New York, 1981.
- [New85] A. C. Newell. *Solitons in mathematics and physics*, volume 48. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1985.
- [Noy76] B.J. Noye. Recordings of tsunamis by tide wells. In R.A. Heath and M.M. Crisswell, editors, *Tsunami Research Symposium, 1974*, volume Bulletin 15, pages 87–94. Royal Society of New Zealand, 1976.
- [oLNR02] Department of Land and State of Hawaii Natural Resources. North shore paukaui streambank erosion and riparian area community project. Technical report, Annual Report to the Twenty-Second Legislature, Regular Session of 2003, December 2002.
- [Per66] D. H. Peregrine. Calculations of the development of an undular bore. *Journal of Fluid Mechanics Digital Archive*, 25(02):321–330, 1966.
- [Rus44] J. Scott Russell. Report on waves. *Rept. Fourteenth Meeting of the British Association for the Advancement of Science*, pages 311–390+57 plates, 1844.
- [Rus65] J. Scott Russell. *The Modern System of Naval Architecture*, volume 1, page 208. Day and Son, London, 1865.
- [Seg07] H. Segur. Waves in shallow water, with emphasis on the tsunami of 2004. In A. Kundu, editor, *Tsunami and Nonlinear Waves*, pages 3–29. Springer, 2007.
- [She93] S. S. Shen. *A course on nonlinear waves*, volume v. 3. Kluwer Academic, Dordrecht, 1993.
- [Stu09] R. Stuhlmeier. KdV theory and the Chilean tsunami of 1960. *Discrete Contin. Dyn. Syst., Ser. B*, 12(3):623–632, 2009.
- [SW00] G. Schneider and C. E. Wayne. The long-wave limit for the water wave problem. I: The case of zero surface tension. *Commun. Pure Appl. Math.*, 53(12):1475–1535, 2000.
- [Ula76] S. M. Ulam. *Adventures of a mathematician*. Scribner, New York, 1976.
- [Urs53] F. Ursell. The long-wave paradox in the theory of gravity waves. *Math. Proc. Cambridge Philos. Soc.*, 49(4):685–694, 1953.
- [Vek67] N. P Vekua. *Systems of singular integral equations*. P. Noordhoff, Groningen, 1967.
- [Wri06] J. D. Wright. Corrections to the KdV approximation for water waves. *SIAM J. Math. Anal.*, 37(4):1161–1206, 2006.
- [Wu97] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2D. *Invent. Math.*, 130(1):39–72, 1997.
- [Yih77] C.-S. Yih. *Fluid mechanics: a concise introduction to the theory*. West River Press, Ann Arbor, Mich., corrected ed edition, 1977.
- [Yos82] H. Yosihara. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.*, 18:49–96, 1982.
- [Zab68] N. J. Zabusky. Solitons and bound states of the time-independent schrödinger equation. *Phys. Rev.*, 168(1):124–128, Apr 1968.



# Index

- Airy, G.B., 17
- Bernoulli's equation, 7
- bound states, 23
- Boundary conditions, 5
- Boussinesq equation, 16
- Burger's equation, 20
- continuity equation, 4
- dynamic boundary condition, 5
- equation of mass conservation, 4
- equation of state, 4
- Euler equations, 5
- Faddeev condition, 22
- Far-field region, 14
- far-field variables, 13
- fluid, 3
- free surface, 5
- Gel'fand-Levitan-Marchenko equation, 25
- Gerstner wave, 9
- inverse scattering theory, 21
- irrotational, 6
- J. Scott Russell, 17
- KdV balance, 40
- kinematic condition, 6
- Korteweg-de Vries equation, 14
- Lagrange's theorem, 6
- Lax pair, 26
- long-wave paradox, 19
- Mass conservation, 4
- Material derivative, 5
- Miura transformation, 22
- Nondimensionalization, 10
- Riemann-Hilbert problem, 27
- run up, 36
- Russell, J. Scott, 17
- Scaling, 11
- Schrödinger equation, 22
- Sokhotski-Plemelj formula, 27
- solitary wave, 19
- soliton, 20
- transfer of boundary conditions, 13
- transport equation, 19
- traveling wave, 19
- tsunami, 35
- tsunami period, 47
- unbound states, 23
- Ursell number, 39
- velocity potential, 7
- vorticity, 6
- Waimea Bay, 1
- wave, 2
- well-posed, 31





# Curriculum Vitae

Raphael Stuhlmeier

## PERSONAL

Born: Graz, Austria on July 5, 1985

Austrian Citizen

## EDUCATION

1991 - 1992: Grade 1 Longfellow Elementary School, Teaneck, NJ, USA

1992 - 2000: Grades 2-8 William H. Lincoln School, Brookline, MA, USA

2000 - 2003: Grades 9-12 GRg 23/VBS Draschestraße, Vienna, Austria

2005 - 2010: Diploma Studies in Mathematics, University of Vienna, Austria

## PUBLICATIONS

R. Stuhlmeier and K. M. Stuhlmeier, Fast, simultaneous, and sensitive detection of staphylococci, *Journal of Clinical Pathology*, **56** (2003), 782-785

R. Stuhlmeier, KdV theory and the Chilean tsunami of 1960, *Discrete and Continuous Dynamical Systems - Series B*, **12** (2009), 623-632