universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

Linear Stability for Self-Similar Wave Maps

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag. rer.nat.)

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Wien, im Februar 2010

Danksagungen

Ich möchte mich in erster Linie bei Prof. Peter C. Aichelburg für die hervorragende Betreuung bedanken, für die vielen hilfreichen Diskussionen und dafür, dass er mir bei allen Problemen stets mit Rat und Tat zur Seite stand.
Mein besonderer Dank gilt auch Roland Donninger, der mir das notwendige technische Rüstzeug vermittelte, der immer ein offenes Ohr für meine Fragen hatte und dessen geduldige Ausführungen mir zu vielen mathematischen Einsichten verhalfen. Ich danke auch meinen Kolleginnen und Kollegen der Gravitationsgruppe für ihre Hilfsbereitschaft, die vielen interessanten Gespräche und die unterhaltsamen Stunden im Büro.
Diese Arbeit ist meiner Familie gewidmet und all jenen geliebten Menschen, die mich während dieses Studiums begleitet und unterstützt haben.

Zusammenfassung

Diese Arbeit behandelt eine Klasse von Abbildungen, genannt *wave maps*, vom Minkowski-Raum auf die 3-Sphäre. Solche Abbildungen genügen einer nichtlinearen Wellengleichung, für die eine selbstähnliche Lösung, genannt der Grundzustand, in geschlossener Form bekannt ist. Diese Lösung entwickelt in endlicher Zeit eine Singularität (*blow-up).* Numerische Untersuchungen legen nahe, dass der Grundzustand einen Attraktor für generische Anfangsdaten darstellt. In dieser Arbeit werden lineare Störungen des Grundzustands untersucht, wobei das Ziel ist, die lineare Stabilität mit analytischen Methoden zu beweisen.

Die linearisierte Gleichung wird als Operatorgleichung formuliert und in zwei verschiedenen Funktionenräumen betrachtet - im Energieraum und in einem Raum, in dem die Norm mit einer höheren Energie assoziiert werden kann. Mit Methoden aus der Theorie starkstetiger, ein-parametriger Halbgruppen und durch Untersuchung des Spektralproblems kann eine Abschätzung für die zeitliche Entwicklung der Energie der Störung angegeben werden. Der Grundzustand ist linear stabil, wenn die Energie der Störung mit der Zeit abnimmt. Es wird gezeigt, dass nur eine Formulierung des Problems im höheren Energieraum zum gewünschten Ergebnis führt.

Abstract

This work studies a particular class of maps, called *wave maps*, from Minkowski space to the three-sphere. Such maps fulfill a nonlinear wave equation, for which a self-similar solution, called the ground state, is known in closed form. This solution develops a singularity in finite time (*blow-up*). Numerical investigations suggest that the ground state is an attractor for generic smooth initial data. In this work linear perturbations of the ground state solution are investigated. The aim is to prove linear stability with analytic methods.

We give an operator formulation of the linearized equation and consider it in two different functions spaces - in the energy space and in a *higher energy space*, where the norm can be associated with a higher energy. With methods from the theory of strongly continuous one-parameter semigroups and by investigation of the spectral problem an estimate for the temporal evolution of the energy of the perturbation can be found. The ground state solution is linearly stable if the energy of the perturbation decreases in time. It will be shown that only a formulation of the problem in the higher energy space leads to the intended result.

Contents

1. Introduction

The development of partial differential equations (PDEs) as a mathematical tool to describe the dynamics of systems by infinitesimal changes of physical quantities certainly marks one of the cornerstones for the enormous success of natural sciences. The theory of linear PDEs is well established with numerous applications in different scientific fields. However, as systems and the interactions within get more complex, nonlinearities arise naturally in the equations. Therefore, insights gained from the mathematical investigation of nonlinear PDEs are of tremendous importance and provide the key to a deeper understanding of many questions arising in physics, chemistry or biology. One of the central questions in the analysis of partial differential equations is known as *well-posedness of the Cauchy problem*. Given the initial state of a system one wants to ensure that the equation provides a unique solution. Furthermore, small changes of the initial state should only cause small changes of the solution. However, it is possible that these requirements are fulfilled only for a finite time interval after that break down of solutions occurs. Singularity formation in finite time from smooth initial data, also called *blow-up of solutions*, is a feature that many nonlinear PDEs seem to have in common and it is particularly interesting how such a break down occurs.

In this work a nonlinear wave equation, called the *wave maps equation*, will be considered and aspects of singularity formation will be investigated. Wave maps are defined on a pseudo-Riemannian manifold with values in a Riemannian manifold. The field equations, generally a system of semilinear wave equations, can be derived from an action principle. Wave maps can be considered as a generalization of the ordinary wave equation, which is a wave map from $(3+1)$-Minkowski space to the one-dimensional Euclidean space $\mathbb{R}$. For non-flat targets, which are often chosen to be spherical or hyperbolic, the resulting field equations turn out to be nonlinear. From a pure mathematical point of view, wave maps provide a rich source of interesting and challenging problems. In regard to singularity formation and global existence of solutions much progress has been made in the past years (see [22] for a survey).

In this work we focus on wave maps from (3 + 1)-Minkowski space to the three-sphere. In particle physics this model was introduced by Gell-Mann and Levy (see [17]) in 1960 and is known as SU(2)-σ model. By imposing additional symmetry, the wave maps equation turns out to be a single semilinear wave equation, which was intensively studied in the last two decades with analytic and numerical tools. In 1988 J. Shatah showed the existence of self-similar solutions that blow up in finite time (see [33]) and an explicit example, called the *ground state solution* was found by Turok and Spergel [39] in 1999. Numerical investigations of the wave maps equation started around 2000, mainly performed by P. Bizon and collaborators ([7], [6]). They showed the existence of a family of self-similar solutions, denoted by f_n. The ground state solution is given by f_0 and it is the only one that is known is closed form. It turned out that these solutions play an important role in the dynamics of the system. Evolution of initial data either disperses or blows up in finite time. Numerical investigations performed for large classes of initial data depending on a single parameter revealed the existence of a threshold between these two endstates. It was observed that the profile of the solutions near this threshold is given by the first excited state f_1, therefore called the critical solution, before either dispersion or singularity formation takes place.

In the early nineties, similar phenomena were found in gravitational physics and are known as *critical gravitational collapse*. Important discoveries in this field are due to numerical investigations performed by M. Choptuik [8]. He considered a massless scalar field coupled to gravity, where the only possible endstates of the system are dispersion to flat space or formation of a black hole. Among other remarkable features that characterize critical collapse (see [18] for a survey), the existence of a universal self-similar critical solution was demonstrated. Since the the Einstein equations are very difficult to handle, the wave maps equation serves as a toy model for critical phenomena in gravitational physics.

This work is dedicated to another suggestion that was made on the basis of numerical studies performed by P. Bizon, T. Chmaj and Z. Tabor in [7] and which concerns blow-up from smooth initial data. They investigated large classes of initial data that become singular in finite time and showed that the asymptotic shape near the singularity is locally given by the ground state solution f_0. Moreover, they were not able to find initial data, for which this was not the case. It is therefore believed that the ground state solution is a local attractor for singularity formation. If this is true it should be stable under small linear (and further nonlinear) perturbations. Thus, the investigation of stability of the ground state solution is rather mathematically than physically motivated. The wave maps equation, which is studied in this work, as well as the Einstein equations belong to the class of *super-critical equations*. The hope is that the techniques developed in the analysis of the super-critical wave maps equation may also shed some light on more involved problems.

In regard to the problem of linear stability of f_0 important numerical and analytic results have been obtained in the past years, mainly by P. Bizon (see for example [5],[4]) as well as by P. C. Aichelburg and R. Donninger ([1], [13], [11]). However, a rigorous proof turned out to be a tedious task and not even mode stability could be established so far. The aim of this work is to show linear stability of the ground state solution with methods from functional analysis, operator theory and the theory of strongly continuous one-parameter semigroups.

In chapter 2 wave maps on Minkowski space will be defined and we derive the field equations for co-rotational maps from $(3+1)$-Minkowski space to the three-sphere. We discuss the behavior of the wave maps equation under scaling and put this into the context of criticality classes and scaling heuristics. Then, well-posedness results for general wave maps will be briefly summarized. Finally we introduce self-similar solutions and discuss the ground state solution in more detail.

In the third chapter two different coordinate systems adapted to the problem of linear stability will be presented, *self-similar hyperbolic coordinates* and *CSS-coordinates*. For reasons that will be explained below, we choose the latter and derive the linearized equation. Due to the nature of the new time variable, linear stability of f_0 becomes a problem of *asymptotic stability*. We review the results obtained so far and also present new results in regard to mode stability and solutions of the eigenvalue equation.

In chapter 4 we give an operator formulation of the linearized equation in CSS-coordinates and show well-posedness of the system in the energy space. With

spectral analysis and semigroup theory we derive a growth estimates for the perturbation field, which turns out to be not sufficient to prove stability.
In chapter 5 we will overcome the hurdles of the previous chapter by formulating the problem in a different function space. We require higher differentiability of the solutions and operate on a space with a norm inspired by a higher energy. We will show that the system is well-posed and that we can derive an appropriate growth estimate, such that asymptotic stability of the ground state solution can be shown.

2. Self-similar wave maps from Minkowski space to S^3

In this section we derive the field equation for co-rotational wave maps from Minkowski space to the three-sphere. Then self-similar solutions and their importance for the time evolution of the system will be discussed. For the basic definitions related to the theory of smooth manifolds, which will be used in this section, we refer to an overview given in [11] as well as to textbooks on the subject (see for example [27]).

2.1. Derivation of the field equation

In general, wave maps are defined as maps on a pseudo-Riemannian manifold (M, η) taking values in a Riemannian manifold (N, g) (both of arbitrary dimension), where η and g denote the metrics on domain and target space, respectively. First we discuss wave maps from $(n+1)$-dimensional Minkowski space, denoted by $\mathbb{R}^{n+1}$, to an arbitrary n-dimensional Riemannian manifold. Then we focus on maps on $\mathbb{R}^{3+1}$ where the target is given by the three-sphere. Finally, by imposing additional symmetry, the field equation for co-rotational wave maps will be derived.

2.1.1. Wave maps on Minkowski space

Let $M := \mathbb{R}^{n+1}$ denote the $(n+1)$-dimensional Minkowski space with metric η. We define the signature of the metric to be $(-, +, ..., +)$. In standard coordinates the components of η are given by $\eta_{\mu\nu}$ =diag$(-1, 1, ..., 1)$, with indices running from 0 to n. Let $\Psi : M \to N$ be a smooth mapping from Minkowski space to an

n-dimensional Riemannian manifold (N, g). The components of the metric g in chosen coordinates are denoted by g_{AB} and we write Ψ^A for the components of the coordinate representation of Ψ, where $A = 1, ..., n$.
The *wave maps functional* S is defined as

$$S(\Psi) := \int_{\mathbb{R}^{n+1}} \eta^{\mu\nu}(\partial_\mu \Psi^A)(\partial_\nu \Psi^B) g_{AB}(\Psi) \tag{2.1}$$

with $\partial_\mu := \frac{\partial}{\partial x^\mu}$. A detailed discussion of this expression can be found for example in [11]. The map Ψ is called a *wave map* if it is a critical point of the action functional S. Such points can be formally calculated by considering compactly supported variations where the condition $\delta S = 0$ yields a system of nonlinear wave equations

$$\Box \Psi^A + \eta^{\mu\nu} \Gamma^A_{BC}(\Psi)(\partial_\mu \Psi^B)(\partial_\nu \Psi^C) = 0 \tag{2.2}$$

called the wave maps equation (see [34] for a derivation). The wave operator is defined by

$$\Box \Psi^A := \eta^{\mu\nu} \partial_\mu \partial_\nu \Psi^A$$

and

$$\Gamma^A_{BC} := \frac{1}{2} g^{AD}(\partial_B g_{CD} + \partial_C g_{BD} - \partial_D g_{BC})$$

denote the Christoffel symbols on N.
In regard to the above analysis it is common to define a Lagrange density $\mathcal{L}(\Psi, \partial\Psi)$, which is related to the action by

$$S(\Psi) = \int \mathcal{L}(\Psi, \partial\Psi).$$

It can be shown (see for example [16]) that the map, in order to be a critical point, must satisfy the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^A)} - \frac{\partial \mathcal{L}}{\partial \Psi^A} = 0.$$

Here the Lagrange density is given by

$$\mathcal{L}(\Psi, \partial_\mu \Psi) := \eta^{\mu\nu}(\partial_\mu \Psi^A)(\partial_\nu \Psi^B) g_{AB}(\Psi).$$

2.1.2. Co-rotational wave maps to the three-sphere

We consider wave maps on $M := \mathbb{R}^{3+1}$ and introduce spherical coordinates (t, r, θ, ϕ) on the Minkowski space, which are related to the standard coordinates by

$$\begin{aligned} x_0 &= t \\ x_1 &= r \sin\theta \cos\phi \\ x_2 &= r \sin\theta \sin\phi \\ x_3 &= r \cos\theta \end{aligned}$$

with $r > 0, 0 < \theta < \pi, 0 \leq \phi < 2\pi$. In these coordinates the components of the metric are given by

$$\eta_{\mu\nu}(t, r, \theta, \phi) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Let the target manifold N be the three-sphere $S^3 \subset \mathbb{R}^4$ defined by

$$S^3 := \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}.$$

The three-sphere with a metric induced by the ambient Euclidean metric is a three-dimensional Riemannian manifold (S^3, g). We choose hyperspherical coordinates (ψ, Θ, Φ), which are related to the Cartesian coordinates on $\mathbb{R}^4$ by

$$\begin{aligned} x_0 &= \sin\psi \sin\Theta \cos\Phi \\ x_1 &= \sin\psi \sin\Theta \sin\Phi \\ x_2 &= \sin\psi \cos\Theta \\ x_3 &= \cos\psi \end{aligned}$$

where $0 < \psi < \pi, 0 < \Theta < \pi, 0 \leq \Phi < 2\pi$ and

$$g_{AB}(\psi, \Theta, \Phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\psi & 0 \\ 0 & 0 & \sin^2\psi \sin^2\Theta \end{pmatrix}.$$

Let Ψ: $\mathbb{R}^{3+1} \to S^3$ be a smooth map from Minkowski space to the three-sphere. In coordinates it assigns 4-tuples (t, r, θ, ϕ) on Minkowski space to 3-tuples (ψ, Θ, Φ) on S^3.

Henceforth we only consider *co-rotational* maps, i.e. we require

$$\psi = \psi(t, r) \quad \Theta \equiv \theta \quad \Phi \equiv \phi. \tag{2.3}$$

With these assumptions it can be shown (cf. [11]) that the action functional given by (2.1) reduces to

$$S(\psi) = 4\pi \int_0^\infty \int_{-\infty}^\infty \left(-\psi_t^2(t,r) + \psi_r^2(t,r) + \frac{2\sin^2\psi(t,r)}{r^2} \right) r^2 dt dr \tag{2.4}$$

In the above expression we again abbreviate

$$\psi_i := \partial_i \psi = \frac{\partial \psi}{\partial x^i}.$$

A Lagrange density can be defined by

$$\mathcal{L} = -r^2\psi_t^2 + r^2\psi_r^2 + 2\sin^2\psi.$$

and the Euler-Lagrange equations then yield the semilinear wave equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0. \tag{2.5}$$

In the following, when we talk about the *wave map equation* without further specifying domain and target, we have equation (2.5) in mind.

With the Lagrange density an energy density can be assigned to the system by

$$\mathcal{E} = -\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\psi_t - \mathcal{L}\right).$$

Integration over the radial variable yields the conserved energy

$$E_\psi(t) = \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{2\sin^2(\psi)}{r^2}\right) r^2 dr. \tag{2.6}$$

2.1.3. The Cauchy problem

The main problem for evolution equations is known as the Cauchy problem. Consider for example the wave maps equation (2.5). Given data at $t = 0$

$$\psi|_{t=0} = \psi_0 \qquad \partial_t\psi|_{t=0} = \psi_1 \tag{2.7}$$

one wants to ensure that the problem is well-posed. Loosely speaking, this means that there exists a unique solution, which depends continuously on the initial data (the last requirement ensures that small changes in the data cause only small changes in the solution). If this is the case for a finite time interval, the problem is locally well-posed, whereas for global well-posedness the above properties hold at all times. If a solution ceases to exist after some time, one is naturally interested in the details of the break-down. The notion of well-posedness has of course to be made mathematically precise, and we will do this when we consider an operator formulation of an evolution equation in the following chapters.

In a proper definition it has to be specified what kind of solutions are admitted, i.e. the degree of regularity. In the above derivation of the wave maps equation we required the solution to be a smooth function (this is also called a *classical wave map*). From the view point of partial differential equations this is a rather restrictive assumption, which is usually dropped. In PDE theory it is common to only require a very low degree of regularity (at the start) and to consider weak solutions of the corresponding equation.

Criticality with respect to energy scaling

The wave maps equation is invariant under dilation. If $\psi(t,r)$ is a solution, then for $r \to r/\lambda$

$$\psi_\lambda(t,r) := \psi(t/\lambda, r/\lambda) \tag{2.8}$$

also solves eq. (2.5) for arbitrary $\lambda > 0$.
It turns out that invariance under scaling is a property that applies to many non-linear evolution equations, although the scaling of the solutions might be slightly different to (2.8). The scaling-behavior of conserved quantities associated with such equations gives rise to a classification known as *criticality* with respect to the considered quantity.
To illustrate this, we consider the energy associated with the wave maps equation given by (2.6). Under the above defined transformation the energy scales as

$$E_{\psi_\lambda}(t) = \lambda E_\psi(t/\lambda). \tag{2.9}$$

This is a special case of the more general scaling behavior

$$E_{\psi_\lambda}(t) = \lambda^\alpha E_\psi(t/\lambda).$$

Distinguishing between $\alpha < 0$, $\alpha = 0$, $\alpha > 0$, the corresponding field equations are then called *energy sub-critical*, *energy critical* or *energy super-critical* (see [38]).
It is widely believed that the criticality class is strongly connected with the possibility of singularity formation during the evolution. Sub-critical equations are supposed to be globally well-posed, whereas in the super-critical case blow-up of solutions is expected. Qualitatively the argument goes as follows: A solution $\psi(t,r)$ has a fine-scale counterpart for $\lambda \ll 1$, which is more concentrated and highly oscillating (hence less regular), whereas for $\lambda \gg 1$ the scaled solution is smoother than the original one. Given initial data with a certain amount of energy, then in the super-critical case, fine-scale solutions have smaller energy than the original one due to the scaling law (2.9). Thus, shrinking the solution decreases the energy locally, which favours the blow-up. In contrast, in the sub-critical case shrinking would require more energy.

Global and local well-posedness for wave maps

The Cauchy-Problem for wave maps on Minkowski space has been studied extensively in the past years. Since the technical requirements, which would be necessary to discuss the results in detail, go beyond the scope of this work, we refer to a survey by Krieger an the references therein [22].
For $n = 1$ the wave maps equation is sub-critical and global well-posedness has been established. For $n \geq 2$ a sharp local well-posedness result (independent of the target manifold) has been obtained in [20],[21] for data in suitable Sobolev spaces.
In two spatial dimensions the wave maps equation is critical, which means that the energy is invariant under scaling. In regard to global existence of solutions substantial progress has been made in the last years and we refer again to [22] and to [23], [30], [36] for more recent results.
For the super-critical case $n \geq 3$ many questions remain unresolved. For wave maps from $\mathbb{R}^{n+1}$ to S^{m-1} (where $m, n \geq 2$) it was shown by Tao in [37] that global existence is provided if the initial data are small enough in a particular norm (the energy norm). The Cauchy problem and the question of global well-posedness for wave maps from $(3+1)$-Minkowski space to the three-sphere was studied earlier in [24] and [35]. For the particular case of co-rotational wave maps that is considered here, we refer to a result obtained by Shatah and Tahvildar-Zadeh in [32]. Again, global existence of solutions can only be shown if the data are small. For eq. (2.5) an explicit example of singularity formation in finite time is known and will be presented in the next section.

2.2. Self-similar solutions

Since eq. (2.5) is scale invariant it is reasonable to look for solutions that share this property, as it is provided by self-similar solutions.

2.2.1. The ground state solution

In [33] Shatah considered solutions of the form

$$\psi(t,r) = f\left(\tfrac{r}{T-t}\right)$$

for an arbitrary constant $T > 0$.
Defining a radial coordinate adapted to self-similarity

$$\rho := \frac{r}{T-t}$$

the wave maps equation reads

$$f'' + \frac{2}{\rho}f' - \frac{\sin(2f)}{\rho^2(1-\rho^2)} = 0 \tag{2.10}$$

where $' := \frac{d}{d\rho}$, $t \leq T$ and $0 \leq \rho < \infty$. At $\rho = 0$ and $\rho = 1$ one has regularity conditions

$$f(0) = 0 \qquad f(1) = \frac{\pi}{2}. \tag{2.11}$$

Shatah proved that such solutions exist and later Turok and Spergel [39] found an example in closed form, which is given by

$$f_0(\rho) = 2\arctan(\rho) \tag{2.12}$$

In the following we will refer to (2.12) as *the ground state solution*. It turns out that it is sufficient to consider eq. (2.10) only for $\rho \in [0,1]$ (see [6]). This can be seen by the argument of *finite speed of propagation*: Calculating the characteristics of eq. (2.5) (this was done for example in [11]) one observes that information propagates with velocity equal to one along straight lines in the spacetime diagram with slope -1 and 1. This means that the point $(T,0)$ can only be influenced by information contained in its past lightcone, the boundary of which is given by $r = T - t$ corresponding to $\rho = 1$.
Set $\psi_0(t,r) := f_0(\frac{r}{T-t})$ and define smooth initial data

$$\psi(0,r) := \psi_0(0,r) \qquad \partial_t\psi(0,r) = \psi_{0,t}(0,r)$$

then the ground state solution provides a solution of the wave maps equation, which is smooth for $t < T$. However, the spatial derivative at the origin

$$\partial_r \psi_0(t,0) \sim (T-t)^{-1}$$

becomes singular for $t \to T$. Thus, it is an explicit example for a blow-up solution. Numerical studies of solutions of eq. (2.10), first performed by Ammineborg and Bergstrom in [3] and later by Bizon, Chmaj and Tabor in [7], suggested that f_0 is a member of a family of self-similar solutions. This is given in the next theorem, obtained by Bizon in [6].

Theorem 1. There exists a countable family of smooth solutions f_n of eq. (2.10) satisfying the boundary conditions (2.11). The index $n = 0, 1, 2, ...$ denotes the number of intersections of $f_n(\rho)$ with the line $f = \frac{\pi}{2}$ on $\rho \in [0, 1)$.

The only member of this family, which is known is closed form, is the ground state solution. The others have to be constructed numerically.

2.2.2. The role of self-similar solutions in the time evolution

For given initial data, there are two possible endstates for the dynamics governed by the wave maps equation: dispersion, that is convergence to the vacuum solution, or formation of a singularity. In regard to this, the ground state solution as well as the first excited state seem to play an important role. Bizon et al. [7] evolved families of initial data depending on one parameter p. By adjusting the parameter they were able to identify the critical value p^*, which marks the threshold between the two endstates. It was shown that for large classes of initial data with values p close to p^*, the solution f_1 is approached locally before finally dispersion or singularity formation takes place.
Another conjecture that was made in [7], and which is of major importance here, concerns the ground state solution f_0. It was observed that for large sets of data, which become singular after a finite time, the asymptotic shape near the singularity is given by f_0. It is therefore believed that the blow-up is universal and that the ground state solution acts as a local attractor for singularity formation (local in the sense that it is approached near the centre of spherical symmetry). To prove this conjecture it has to be shown that the ground state solution is stable under small

linear (and further nonlinear) perturbations. Although in the last years numerical and analytic arguments have been obtained that point in this direction (they will be presented in the next chapter), there was no rigorous proof of linear stability.

3. Linear stability of the ground state solution

The basic idea of linear stability analysis is to linearize the evolution equation for small perturbations around the ground state solution. Then it has to be shown that the perturbing field (measured by a suitable norm) converges to zero as the time approaches the blow-up time. In this section we first introduce coordinate systems adapted to the problem and review the results obtained so far. In regard to mode stability in CSS-coordinates (see below) new results for solutions of the eigenvalue equation will be presented.

3.1. Adapted coordinates

We have already defined a radial coordinate adapted to self-similarity

$$\rho = \frac{r}{T-t}.$$

3.1.1. Self-similar hyperbolic coordinates

One can define a new time variable σ by

$$\sigma := -\log\sqrt{(T-t)^2 - r^2}$$

for $r < T - t$, which corresponds to the interior of the backward lightcone of the blow-up point $(t, r) = (T, 0)$. Since the lines $\sigma = const.$ are hyperbolae in the spacetime diagram, the coordinates (σ, ρ) are called *self-similar hyperbolic coordinates.* The system is orthogonal by construction (see [11] for a derivation). The wave maps equation (2.5) in (σ, ρ) reads

$$\psi_{\sigma\sigma} - 2\psi_\sigma - (1-\rho^2)^2\psi_{\rho\rho} - \frac{2(1-\rho^2)^2}{\rho}\psi_\rho + \frac{(1-\rho^2)\sin(2\psi)}{\rho^2} = 0. \tag{3.1}$$

Linear stability of the ground state solution in hyperbolic coordinates was rigorously studied by Donninger in [11] and by Aichelburg and Donninger in [13]. With an appropriate operator formulation of the linearized equation well-posedness of the Cauchy problem in a particular Sobolev space (the energy space) was shown and an upper bound for the growth rate of the perturbation was obtained (given by the growth rate of the gauge mode, see section 3.2.1). An explanation, why this result is the best that can be achieved in these coordinates can be found in [11]: Generic inital data leads to outgoing wave packets that leave the backward lightcone after some time. However, in hyperbolic coordinates this is troublesome because the time coordinate σ breaks down at the boundary of the lightcone, hence outgoing wave packets will never leave this spacetime region and eventually cumulate near $\rho = 1$. This leads to exponential growth of solutions.

3.1.2. CSS-coordinates

Another possibility is to define

$$\tau := -\log(T-t)$$

for $t < T$. This coordinate exists not only in the interior of the backward lightcone of the blow-up point, but also on the boundary and outside of it. The coordinates (τ, ρ) are called *continuously self-similar* and in the following we refer to them as *CSS-coordinates* or simply *similarity coordinates.*
The inverse transformation is given by

$$t = T - e^{-\tau} \qquad r = \rho e^{-\tau}.$$

Derivatives transform to

$$\partial_r = e^\tau \partial_\rho \qquad \partial_t = e^\tau(\partial_\tau + \rho\partial_\rho).$$

The wave maps equation in (τ, ρ) reads

$$\psi_{\tau\tau} - (1-\rho^2)\psi_{\rho\rho} + 2\rho\psi_{\tau\rho} + \psi_\tau - 2\frac{1-\rho^2}{\rho}\psi_\rho + \frac{\sin(2\psi)}{\rho^2} = 0 \tag{3.2}$$

Note that there are mixed derivatives, which is due to the fact that the coordinate system is not orthogonal.

3.2. Mode stability in CSS-coordinates

The wave maps equation in similarity coordinates can be linearized around the ground state solution by making the ansatz

$$\psi(\tau,\rho) = f_0(\rho) + \tilde{\psi}(\tau,\rho)$$

where $\tilde{\psi}(\tau,\rho)$ denotes a small perturbation.
Neglecting quadratic and higher order terms, the resulting linear time evolution equation for the perturbation field reads

$$\tilde{\psi}_{\tau\tau} - (1-\rho^2)\tilde{\psi}_{\rho\rho} + 2\rho\tilde{\psi}_{\tau\rho} + \tilde{\psi}_\tau - 2\frac{1-\rho^2}{\rho}\tilde{\psi}_\rho + \frac{2\cos(2f_0)}{\rho^2}\tilde{\psi} = 0 \tag{3.3}$$

with regularity conditions (cf. [10])

$$\tilde{\psi}(\tau,0) = \tilde{\psi}_\tau(\tau,0) = 0$$

and initial data $\tilde{\psi}(0,\rho), \tilde{\psi}_\tau(0,\rho)$.
With the above argument of finite speed of propagation it is sufficient to consider the equation only in the backward lightcone of $(r,t) = (0,T)$, that is for

$$\tau \geq -\log T \qquad 0 \leq \rho \leq 1$$

As solutions we admit smooth complex-valued functions

$$\tilde{\psi} : [-\log T, \infty) \times [0,1] \to \mathbb{C}.$$

Since $t \to T$ corresponds to $\tau \to \infty$, *asymptotic stability* of f_0 has to be shown. By this we mean that the perturbation converges to zero as τ approaches infinity.

3.2.1. The eigenvalue equation - known results

We consider mode solutions

$$\tilde{\psi}(\tau,\rho) = e^{\lambda\tau} u(\rho)$$

for $\lambda \in \mathbb{C}$ and u:$[0,1] \to \mathbb{C}$ a smooth function. A necessary condition for asymptotic stability is the nonexistence of such solutions with $Re\lambda > 0$. Inserting the above ansatz in the linearized equation yields

$$u'' + \left(\frac{2}{\rho} - \frac{2\lambda\rho}{1-\rho^2}\right) u' - \left(\frac{2\cos(2f_0)}{\rho^2(1-\rho^2)} + \frac{\lambda(\lambda+1)}{1-\rho^2}\right) u = 0 \tag{3.4}$$

Although this equation is not a standard eigenvalue problem, those $\lambda \in \mathbb{C}$ for which it admits smooth solutions are called eigenvalues.

The gauge mode

A particular solution of eq. (3.4) corresponding to $\lambda = 1$ can be found in closed form and we refer to it as the *gauge mode*. It is given by

$$u^g(\rho) = \frac{2\rho}{1+\rho^2}.$$

Obviously, the gauge mode is a growing mode solution. However, in the dynamics of the system it does not have a physical meaning. Its existence is related to the time translation symmetry of the problem, since the ground state solution f_0 denotes in fact a family of solutions depending on the blow-up time T.
We illustrate this with the following abstract argument given by Donninger in [12]. Consider a nonlinear equation

$$F(u) = 0$$

where F is nonlinear partial differential operator, which maps elements of an open subset in a Banach space X to elements in another Banach space Y.

$$u \mapsto F(u) : U \subset X \to Y$$

Further we assume that F is Fréchet differentiable (see A.1). Suppose there exists

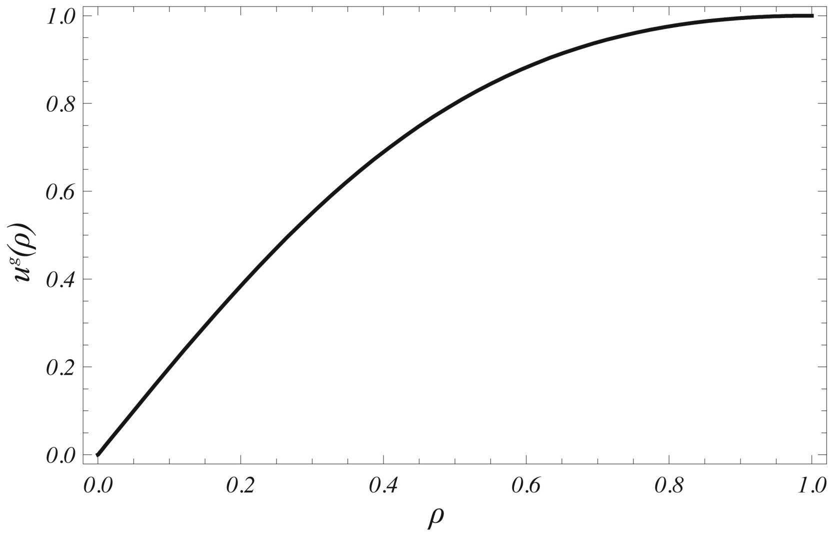

Figure 3.1.: The gauge mode

a one-parameter family of solutions $\{u_s \in U : s \in (a,b)\}$ such that the mapping

$$s \mapsto u_s : (a,b) \to X$$

is differentiable at $s_0 \in (a,b)$. Then

$$0 = \frac{d}{ds}\bigg|_{s=s_0} F(u_s) = DF(u_{s_0}) \frac{d}{ds}\bigg|_{s=s_0} u_s$$

Hence $\frac{d}{ds}|_{s=s_0} u_s \in X$ is a solution of the linearized problem

$$DF(u_{s_0})u = 0$$

where DF denotes the Fréchet derivative of F.

With this argument we can give a qualitative explanation of the gauge mode. Here, the one-parameter family of solutions for the nonlinear problem corresponds to the ground state solution (2.12), which depends on the blow-up time T. Calculating the derivative with respect to T yields

$$\frac{d}{dT} f_0\left(\tfrac{r}{T-t}\right) = \frac{\frac{2r}{(T-t)^2}}{1 + (\frac{r}{T-t})^2}.$$

Switching to similarity coordinates and defining

$$\tilde{\psi}(\tau,\rho) := \frac{2\rho}{1+\rho^2}e^{\tau} = u^g(\rho)e^{\tau}$$

provides a solution of the linearized equation (3.2). Thinking in terms of mode solutions, the eigenvalue corresponding to the gauge mode is given by $\lambda = 1$. Due to its nature, we are only interested in asymptotic stability of the ground state solution modulo this instability.

Analytic results

The eigenvalue equation (3.4) has singularities in the complex plane at $\rho = 0$, $\rho = \pm 1$ and $\rho = \infty$. By rewriting the term

$$\cos\left(4\arctan(\rho)\right) = \frac{1-6\rho^2+\rho^4}{(1+\rho^2)^2} \tag{3.5}$$

it becomes obvious that there are additional singular points at $\rho = \pm i$. From the view point of ODE theory the high number of singularities makes the equation extremely difficult to handle. Neverthless one can obtain information about the solutions in the interval $[0,1]$ by applying Frobenius method (see [2]) at $\rho = 0$ and $\rho = 1$.

Given an ordinary differential equation in $\mathbb{C}$ of the form

$$u''(z) + p(z)u'(z) + q(z)u(z) = 0$$

and assume that p and/or q have a singular point at z_0. Then z_0 is called a regular singular point if it is a pole of order 1 for p and a pole of order 2 for q (at most). If this is the case, one can obtain a power series expansion of the solution. The series converges within a circle of radius r, which is given by the distance to the next singular point in the complex plane. For the eigenvalue equation the points $\rho = 0$ and $\rho = 1$ are regular singular points and at each endpoint Frobenius method provides a pair of linearly independent solutions that converge in $[0,1)$ and $(0,1]$, respectively. A detailed derivation of the power series expansions can be found in [10].

The behavior of the solutions near the endpoints of $[0,1]$ is given in table (3.1) (cf. [1]). At $\rho = 0$ there is an analytic solution $u_0^a(\cdot,\lambda)$ and a non-analytic solution $u_0^n(\cdot,\lambda)$. At $\rho = 1$ the situation is more complicated: $u_1^a(\cdot,\lambda)$ is analytic, but the regularity of $u_1^n(\cdot,\lambda)$ depends on the value of λ. Since these solutions are linearly independent it follows that for $\rho \in (0,1)$ the solutions are connected by the relation

$$u_0^a(\rho,\lambda) = a(\lambda)u_1^a(\rho,\lambda) + b(\lambda)u_1^n(\rho,\lambda).$$

Unfortunately the explicit expressions for the coefficients are not known for ordinary differential equations with more than three singularities (cf. [4]).

Table 3.1.: Asymptotic estimates for (3.4)

ρ	λ	Analytic solution	Non-analytic solution
$\rho \to 0$	any	$u_0^a \sim \rho$	$u_0^n \sim \rho^{-2}$
$\rho \to 1$	$\lambda \notin \mathbb{Z}$	$u_1^a \sim 1$	$u_1^n \sim (1-\rho)^{1-\lambda}$
	$\lambda \in \mathbb{Z},\ \lambda > 1$	$u_1^a \sim 1$	$u_1^n \sim c\log(1-\rho) + (1-\rho)^{1-\lambda}$
	$\lambda \in \mathbb{Z},\ \lambda \leq 1$	$u_1^a \sim (1-\rho)^{1-\lambda}$	$u_1^n \sim c\log(1-\rho)(1-\rho)^{1-\lambda} + 1$

The above asymptotic estimates for the solutions of the eigenvalue equation provide one of the key ingredients for the proofs of the results, which have been obtained so far. First, we cite a result given by Bizon et al. in [7] and later by Aichelburg and Donninger in [13].

Theorem 2. The eigenvalue problem given by equation (3.4) has no solutions that are analytic at both endpoints $\rho = 0$ and $\rho = 1$ for $\lambda \in \mathbb{C}$ with $Re\lambda > 1$.

Another theorem, also obtained by Aichelburg and Donninger, can be found in [1].

Theorem 3. Eq. (3.4) does not have regular solutions for $\lambda \in (0,1)$.

Here, a solution is called regular when it is continuous on $[0,1]$, an element of $C^2(0,1)$ and when the first derivative can be continuously extended to the whole interval $[0,1]$. Finally we present another result recently obtained by Donninger [14]. The underlying idea of the proof can be found in [9].

Theorem 4. The eigenvalue equation (3.4) does not have analytic solutions for $\lambda \in \mathbb{C}$ with $Re\lambda = 1$ and $Im\lambda \neq 0$.

Proof. Set

$$v(\rho) := \rho(1-\rho)^{\frac{\lambda}{2}} u(\rho).$$

Then eq. (3.4) reads

$$v'' - \frac{2\cos(2f_0)}{\rho^2(1-\rho^2)} v = \frac{\lambda(\lambda-2)}{(1-\rho^2)^2} v. \tag{3.6}$$

The asymptotic estimates for the solutions of the eigenvalue equation given in table (3.1) show that there is a solution $u_1^a(\cdot,\lambda)$, which is analytic around $\rho = 1$. We denote the corresponding solution of eq. (3.6) by $v^1(\cdot,\lambda)$, where $v^1 \sim (1-\rho)^{\frac{\lambda}{2}}$. The solution, which is analytic around $\rho = 0$ is given by $u_0^a(\cdot,\lambda)$ and the corresponding solution of eq. (3.6) will be denoted by $v^0(\cdot,\lambda)$. In the following we will sometimes omit the argument and write for example v^1 instead of $v^1(\cdot,\lambda)$.

If the two solutions u_0^a and u_1^a (v^0 and v^1, respectively) are linearly dependent for a given λ and $\rho \in (0,1)$, i.e. their Wronskian vanishes, then λ is an eigenvalue.

Note that the expression on the right hand side of equation (3.6) is real for $Re\lambda = 1$.

$$\lambda(\lambda-2) = -(1+(Im\lambda)^2)$$

Thus v^1 amd $\overline{v^1}$ are both solutions of the same equation. In general, for a second order ODE of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

with solutions y^1, y^2 one can find an expression for the Wronskian given by

$$W(y^1,y^2)(x) = W(y^1,y^2)(x_0)e^{\int_{x_0}^{x} p(s)ds}$$

Now consider the Wronskian of $v^1$and $\overline{v^1}$. Since eq. (3.6) does not depend on the first derivative, the Wronskian must be constant and for $Re\lambda = 1$ and $Im\lambda \neq 0$ one finds

$$W(v^1,\overline{v^1})(\rho) = W(v^1,\overline{v^1})(1) = iIm\lambda.$$

Thus v^1 and $\overline{v^1}$ are linearly independent and v^0 must be of the form

$$v^0(\cdot,\lambda) = a(\lambda)v^1(\cdot,\lambda) + b(\lambda)\overline{v^1(\cdot,\lambda)} \tag{3.7}$$

for $\rho \in (0,1)$ and complex constants a, b depending on λ.
Since eq. (3.6) is real for $Re\lambda = 1$ we assume without loss of generality that v^0 is a real valued function. Then we get

$$\begin{aligned} 0 = W(v^0, \overline{v^0}) &= W(a(\lambda)v^1 + b(\lambda)\overline{v^1}, \overline{a(\lambda)v^1} + \overline{b(\lambda)}v^1) = \\ &= |a|^2 W(v^1, \overline{v^1}) + |b|^2 W(\overline{v^1}, v^1) = (|a|^2 - |b|^2)W(v^1, \overline{v^1}). \end{aligned}$$

Since $W(v^1, \overline{v^1})$ does not vanish it follows that

$$|a|^2 - |b|^2 = 0. \tag{3.8}$$

Now we turn back to the original equation. Suppose eq. (3.4) has a nontrivial analytic solution for a $\lambda \in \mathbb{C}$ with $Re\lambda = 1$ and $Im\lambda \neq 0$. This means that $W(u_0^a, u_1^a) = 0$. The transformed solution is a solution of eq. (3.6) and we have $W(v^0, v^1) = 0$. Inserting the above relation for v^0 yields

$$0 = W(v^0, v^1) = W(a(\lambda)v^1 + b(\lambda)\overline{v^1}, v^1) = b(\lambda)W(\overline{v^1}, v^1).$$

$W(\overline{v^1}, v^1)$ does not vanish and it follows that $b(\lambda) = 0$. From eq. (3.8) we conclude that $a(\lambda) = 0$. Eq. (3.7) implies that the eigenfunction corresponding to the eigenvalue λ is identically zero and the proof follows by contradiction.

□

Numerical results

In [7] Bizon et al. numerically studied solutions of the linearized equation around representatives of the family of self-similar solutions $\{f_n\}$. Their results imply that every f_n has exactly $n+1$ positive eigenvalues

$$\lambda_1^{(n)} > \lambda_2^{(n)} > ... > \lambda_{n+1}^{(n)} = 1.$$

In [5] Bizon focused on equation (3.4). The eigenvalues that were found are real and the only positive one corresponds to $\lambda = 1$ (see table (3.2)).

Table 3.2.: First 5 analytic eigenvalues

λ_1	λ_2	λ_3	λ_4	λ_5
1.00	-0.54	-2.00	-3.40	-4.77

3.2.2. The nonexistence of eigenvalues for Re $\lambda \geq 1/2$

In this section a new result for solutions of eq. (3.4) obtained by R. Donninger and the author will be presented.

Theorem 5. Eq. (3.4) does not have nontrivial smooth solutions for $\lambda \in \mathbb{C}$ with $Re\lambda \geq 1/2$ except for $\lambda = 1$.

To prove this assertion, we first have to study another equation, which looks very similar to the linearized equation (3.3) written in the original coordinates (t, r), but which has some nice properties.
Transformed back to original coordinates (t, r) eq. (3.3) reads

$$\tilde{\psi}_{tt} - \tilde{\psi}_{rr} - \frac{2}{r}\tilde{\psi}_r + \frac{V_0}{r^2}\tilde{\psi} = 0 \tag{3.9}$$

for smooth $\tilde{\psi} : [0, T) \times [0, T - t] \to \mathbb{C}$ and arbitrary $T = const$. The potential V_0 depending on t and r is given by

$$V_0\left(\tfrac{r}{T-t}\right) := 2\cos\left(4\arctan\left(\tfrac{r}{T-t}\right)\right) = \frac{2(1 - 6\left(\frac{r}{T-t}\right)^2 + \left(\frac{r}{T-t}\right)^4)}{\left(1 + \left(\frac{r}{T-t}\right)^2\right)^2}.$$

Now consider a similar evolution equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{V_1}{r^2}\psi = 0 \tag{3.10}$$

for smooth $\psi : [0,T) \times [0, T-t] \to \mathbb{C}$ and a potential V_1 defined by

$$V_1\left(\tfrac{r}{T-t}\right) := \frac{6 - 2(\frac{r}{T-t})^2}{1 + (\frac{r}{T-t})^2}.$$

This potential is a strictly positive function, whereas V_0 changes sign. This can be seen in fig. (3.2), where the potentials are plotted as functions of $\rho := \frac{r}{T-t}$ for $\rho \in [0,1]$.

Lemma 1. Equation (3.10) does not have nontrivial mode solutions for $Re\lambda \geq \frac{1}{2}$.

Proof. We consider the local energy in the backward lightcone associated with eq. (3.10) given by

$$E^{loc}(t) = \frac{1}{2}\int_0^{T-t} \left(r^2|\psi_t(t,r)|^2 + r^2|\psi_r(t,r)|^2 + V_1\left(\tfrac{r}{T-t}\right)|\psi(t,r)|^2\right)dr. \qquad (3.11)$$

We calculate

$$\begin{aligned}
&\frac{dE^{loc}(t)}{dt} = \frac{1}{2}\frac{d}{dt}\left(\int_0^{T-t}\left(r^2|\psi_t(t,r)|^2 + r^2|\psi_r(t,r)|^2 + V_1\left(\tfrac{r}{T-t}\right)|\psi(t,r)|^2\right)dr\right) = \\
= \quad & -\frac{1}{2}\left[r^2|\psi_t(t,r)|^2 + r^2|\psi_r(t,r)|^2 + V_1\left(\tfrac{r}{T-t}\right)|\psi(t,r)|^2\right]_{r=T-t} + \\
+ \quad & Re\left(\int_0^{T-t}\left(r^2\overline{\psi_t(t,r)}\psi_{tt}(t,r) + r^2\overline{\psi_r(t,r)}\psi_{rt}(t,r) + V_1\left(\tfrac{r}{T-t}\right)\overline{\psi(t,r)}\psi_t(t,r)\right)dr\right) + \\
+ \quad & \frac{1}{2}\int_0^{T-t}\frac{r}{(T-t)^2}V_1'\left(\tfrac{r}{T-t}\right)|\psi(t,r)|^2 dr.
\end{aligned}$$

Inserting eq. (3.10 and integrating per parts yields

$$\begin{aligned}
& Re\left(\int_0^{T-t}\left(r^2\overline{\psi_t(t,r)}\psi_{tt}(t,r) + r^2\overline{\psi_r(t,r)}\psi_{rt}(t,r) + V_1\left(\tfrac{r}{T-t}\right)\overline{\psi(t,r)}\psi_t(t,r)\right)dr\right) = \\
= \quad & Re\left(\left[r^2\overline{\psi_t(t,r)}\psi_r(t,r)\right]_{r=T-t}\right).
\end{aligned}$$

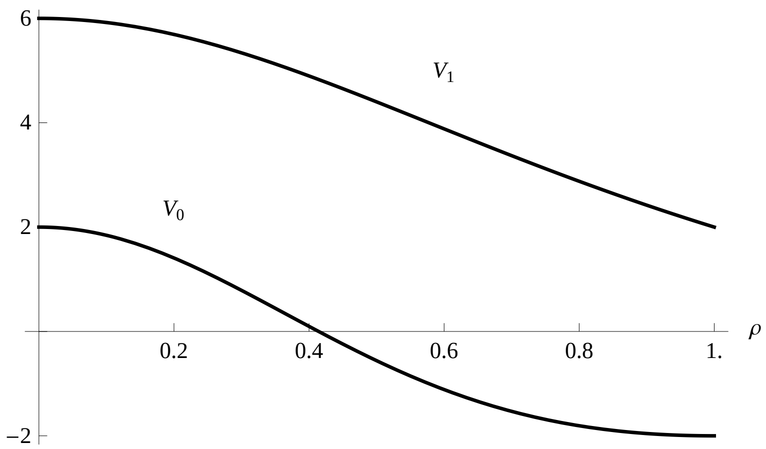

Figure 3.2.: Potential functions V_0 and V_1

Hence

$$\begin{aligned}\frac{dE^{loc}(t)}{dt} &= -\frac{1}{2}(T-t)^2\left[|\psi_t(t,r)-\psi_r(t,r)|^2\right]_{r=T-t} - \\ &- \frac{1}{2}V_1(1)\left[|\psi(t,r)|^2\right]_{r=T-t} + \frac{1}{2}\int_0^{T-t}\frac{r}{(T-t)^2}V_1'\left(\tfrac{r}{T-t}\right)|\psi(t,r)|^2 dr.\end{aligned}$$

Observe that

$$V_1'\left(\tfrac{r}{T-t}\right) = \frac{-16\left(\frac{r}{T-t}\right)}{\left(1+(\frac{r}{T-t})^2\right)^2}.$$

Now we consider mode solutions

$$\psi(t,r) = (T-t)^{-\lambda}u(\tfrac{r}{T-t})$$

with $\lambda \in \mathbb{C}$ and smooth functions $u : [0,1] \to \mathbb{C}$ We get

$$\begin{aligned}\psi_t(t,r) &= (T-t)^{-\lambda-1}\left(\lambda u(\tfrac{r}{T-t}) + \frac{r}{T-t}u'(\tfrac{r}{T-t})\right) \\ \psi_r(t,r) &= (T-t)^{-\lambda-1}u'(\tfrac{r}{T-t})\end{aligned}$$

and

$$\begin{aligned}
\frac{dE^{loc}(t)}{dt} &= -\frac{1}{2}(T-t)^{-2\lambda}\left[\left|\lambda u(\tfrac{r}{T-t}) + \frac{r}{T-t}u'(\tfrac{r}{T-t}) - u'(\tfrac{r}{T-t})\right|^2\right]_{r=T-t} - \\
&- (T-t)^{-2\lambda}|u(1)|^2 - \frac{1}{2}(T-t)^{-2\lambda}\int_0^1 \frac{16\rho^2}{(1+\rho^2)^2}|u(\rho)|^2 d\rho = \\
&= -\frac{1}{2}(T-t)^{-2\lambda}\left(|u(1)|^2(|\lambda|^2+2) + \int_0^1 \frac{16\rho^2}{(1+\rho^2)^2}|u(\rho)|^2 d\rho\right)
\end{aligned}$$

hence $\frac{dE^{loc}(t)}{dt} < 0$ for all $t \leq T$. We conclude that the energy of mode solutions is a positive, monotone decreasing function in the backward lightcone. Inserting the ansatz for mode solutions into the energy (3.11) implies

$$E(t) \propto (T-t)^{-2\lambda+1}.$$

Since the energy is decaying it follows that

$$Re\lambda < \frac{1}{2}.$$

□

We consider eq. (3.10) in similarity coordinates (τ, ρ).

$$\psi_{\tau\tau} - (1-\rho^2)\psi_{\rho\rho} + 2\rho\psi_{\tau\rho} + \psi_\tau - 2\frac{1-\rho^2}{\rho}\psi_\rho + \frac{V_1(\rho)}{\rho^2}\psi = 0$$

where the potential is given by

$$V_1(\rho) = \frac{6-2\rho^2}{1+\rho^2}.$$

In these coordinates mode solutions are of the form $\psi(\tau,\rho) = e^{\lambda\tau}u(\rho)$, for complex valued λ and smooth $u : [0,1] \to \mathbb{C}$. Plugging this ansatz into the above equation yields an equation for u, where the only difference to the original eigenvalue equation is the different potential function. Applying the transformation

$$\tilde{u} := (1-\rho^2)^{\frac{\lambda}{2}}u$$

we get

$$-(1-\rho^2)^2\tilde{u}'' - \frac{2(1-\rho^2)^2}{\rho}\tilde{u}' + \left(\frac{(1-\rho^2)V_1(\rho) + \rho^2\lambda(\lambda-2)}{\rho^2}\right)\tilde{u} = 0 \qquad (3.12)$$

Proposition 1. Equation (3.12) does not have nontrivial, smooth solutions for $Re\lambda \geq 1/2$.

This follows immediately from Lemma 1.

Implications on the eigenvalue equation

Now we are ready to prove the main theorem

Proof (of Theorem 5). Suppose there exists an eigenvalue with $Re\lambda \geq 1/2$, $\lambda \neq 1$ and an associated eigenfunction $u \neq u^g$ satisfying the eigenvalue equation

$$u'' + \left(\frac{2}{\rho} - \frac{2\lambda\rho}{1-\rho^2}\right)u' - \left(\frac{2\cos(2f_0(\rho))}{\rho^2(1-\rho^2)} + \frac{\lambda(\lambda+1)}{1-\rho^2}\right)u = 0.$$

Then with

$$\tilde{u} = (1-\rho^2)^{\frac{\lambda}{2}}u$$

the transformed eigenfunction is a solution of

$$a\tilde{u} := \frac{1}{\omega}(-(p\tilde{u}')' + q\tilde{u}) = -(\lambda-1)^2\tilde{u}$$

where

$$\omega(\rho) := \frac{\rho^2}{(1-\rho^2)^2}, \quad p(\rho) := \rho^2, \text{ and } \quad q(\rho) := \frac{2(1-\rho^2)\cos(2f_0) - \rho^2}{(1-\rho^2)^2}.$$

The above equation was studied by Donninger and Aichelburg in [13]. They showed that there exists a factorization

$$a\tilde{u} = \hat{b}b\tilde{u}$$

where the formal differential expression b and $\hat{b}$ are defined by

$$b\tilde{u}(\rho) := (1-\rho^2)\tilde{u}'(\rho) - \frac{1-3\rho^2}{\rho(1+\rho^2)}\tilde{u}(\rho)$$

and

$$\hat{b}\tilde{u}(\rho) := -(1-\rho^2)\tilde{u}'(\rho) - \frac{3-\rho^2}{\rho(1+\rho^2)}\tilde{u}(\rho).$$

Consider the equation

$$b\hat{b}b\tilde{u} = -(\lambda-1)^2 b\tilde{u}.$$

We define $\tilde{v} := b\tilde{u}$, which is again a transformed eigenfunction and a solution of

$$b\hat{b}\tilde{v} = -(\lambda-1)^2\tilde{v}.$$

Calculating the above expression yields

$$-(1-\rho^2)^2\tilde{v}'' - \frac{2(1-\rho^2)^2}{\rho}\tilde{v}' + \left(\frac{(1-\rho^2)V_1(\rho) + \rho^2\lambda(\lambda-2)}{\rho^2}\right)\tilde{v} = 0$$

with $V_1(\rho) = \frac{6-2\rho^2}{1+\rho^2}$. This equation corresponds to eq. (3.12). It was shown above that this equation does not have solutions for $Re\lambda \geq \frac{1}{2}$, what is a contradiction to the initial assumption. The case $\lambda = 1$ and $u = u^g$ is excluded because the differential expression b annihilates the transformed gauge mode, i.e.

$$b\tilde{u}^g = 0.$$

□

At this point one is far from a rigorous result on asymptotic stability. First, not even mode stability of the ground state solution can be shown. Even if the gauge mode is not taken into account, growing mode solution can't be excluded. Moreover, it has to be shown that the linearized equation is well-posed. Thus it is reasonable to consider an operator formulation of the linearized equation and to study both the Cauchy problem and the problem of asymptotic stability in a suitable function space. In hyperbolic coordinates there is a self-adjoint formulation of the problem, which is not the case for CSS-coordinates. This further complicates the situation because a lot of results, which are available for this class of operators, do not apply.

4. Well-posedness and growth estimates in the energy space

In this section we give an operator formulation of the linearized equation in similarity coordinates. By applying results from semigroup theory (see A.2) we show that the problem is well-posed in the energy space, which is in this formulation a weighted L^2-space. Then we analyze the spectrum of the generator and derive a growth estimate for the energy of the perturbation. The work of Donninger [12], concerning the problem of linear stability for the wave equation with power non-linearity, provides a base frame for the following program.
Henceforth smoothness assumptions for solutions of the linearized equation will be dropped.

4.1. First order system - similarity coordinates

We start with the linearized equation (3.9) transformed to the original coordinates (t, r), which is given by

$$\tilde{\psi}_{tt} - \tilde{\psi}_{rr} - \frac{2}{r}\tilde{\psi}_r + \frac{V_0}{r^2}\tilde{\psi} = 0 \tag{4.1}$$

and $V_0(t,r) = 2\cos(2f_0(\frac{r}{T-t}))$. Recall that this equation is considered in the the backward lightcone of the blow-up point $(T, 0)$, that is for $0 \leq t \leq T$ and $0 \leq r \leq T - t$ and in general solutions are complex-valued functions. The last term in the above equation is singular for $r = 0$. We expand the potential and split off the part that causes the singular behavior.

We define a bounded potential function

$$V(t,r) := \frac{2\cos\left(2f_0\left(\frac{r}{T-t}\right)\right)-2}{r^2}$$

and rewrite eq. (4.1). Then we get

$$\tilde{\psi}_{tt} - \tilde{\psi}_{rr} - \frac{2}{r}\tilde{\psi}_r + \frac{2}{r^2}\tilde{\psi} + V\tilde{\psi} = 0.$$

We give initial data $\tilde{\psi}(0,r), \tilde{\psi}_t(0,r)$, which has to satisfy the regularity condition $\tilde{\psi}(t,0) = 0$ for all t. To simplify the system we apply the transformation

$$\psi(t,r) := r^2\tilde{\psi}(t,r).$$

This yields

$$\psi_{tt} - \psi_{rr} + \frac{2}{r}\psi_r + V\psi = 0 \tag{4.2}$$

with initial data $\psi(0,r), \psi_t(0,r)$ and

$$\psi(t,0) = \psi_r(t,0) = \psi_{rr}(t,0) = 0, \forall t.$$

We introduce new variables

$$\Psi(t,r) := (\psi_1(t,r), \psi_2(t,r))^T$$

where

$$\psi_1 := \frac{\psi_t}{T-t} \qquad \psi_2 := \frac{\psi_r}{T-t}.$$

Since ψ vanishes at $r=0$, it can be obtained by integration.

$$\psi(t,r) = (T-t)\int_0^r \psi_2(t,s)ds$$

A first order formulation of eq. (4.2) then reads

$$\partial_t \Psi(t,r) = \begin{pmatrix} \frac{1}{T-t} & \partial_r - \frac{2}{r} \\ \partial_r & \frac{1}{T-t} \end{pmatrix} \Psi(t,r) + \begin{pmatrix} -V(t,r) \int\limits_0^r \psi_2(t,s)ds \\ 0 \end{pmatrix}.$$

Our aim is to study the system in similarity coordinates (τ, ρ). In these coordinates the above system transforms to

$$\partial_\tau \Phi(\tau,\rho) = \begin{pmatrix} 1-\partial_\rho & \partial_\rho - \frac{2}{\rho} \\ \partial_\rho & 1-\partial_\rho \end{pmatrix} \Phi(\tau,\rho) + \begin{pmatrix} -V(\rho) \int\limits_0^\rho \phi_2(\tau,\xi)d\xi \\ 0 \end{pmatrix}$$

where $\Phi(\tau,\rho) := \Psi(T-e^{-\tau}, \rho e^{-\tau})$. The original field in similarity coordinates is given by

$$\phi(\tau,\rho) = e^{-2\tau} \int\limits_0^\rho \phi_2(\tau,\xi)d\xi.$$

Note that the potential $V(\rho)$ can be simplified to

$$V(\rho) = \frac{2\cos\left(4\arctan(\rho)\right) - 2}{\rho^2} = -\frac{16}{(1+\rho^2)^2}.$$

4.2. Energy space - Operator formulation

In the following we want to define a suitable function space, such that the norm corresponds to the energy of the system. The energy of eq. (4.2) reads

$$E^{full}(t) = \int\limits_0^\infty r^{-2}\left(|\psi_t(t,r)|^2 + |\psi_r(t,r)|^2 + V\left(\tfrac{r}{T-t}\right)|\psi(t,r)|^2\right)dr.$$

This quantity is not well-suited because the potential has negative sign. Moreover it is time dependent, thus the energy is not conserved

$$\begin{aligned}\frac{dE^{full}(t)}{dt} &= (T-t)^{-2}\int_0^\infty \frac{V'(\frac{r}{T-t})}{r}|\psi(t,r)|^2 dr = \\ &= \frac{64}{(T-t)^3}\int_0^\infty \left(1+\left(\tfrac{r}{T-t}\right)^2\right)^{-3}|\psi(t,r)|^2 dr.\end{aligned}$$

A common approach is to consider a norm associated with the (globally) conserved energy of the free equation

$$\psi_{tt} - \psi_{rr} + \frac{2}{r}\psi_r = 0, \tag{4.3}$$

which is given by

$$E(t) = \int_0^\infty \left(r^{-2}|\psi_t(t,r)|^2 + r^{-2}|\psi_r(t,r)|^2\right) dr$$

with $\frac{dE(t)}{dt} = 0$. We are only interested in the behavior of solutions in the backward lightcone of the blow-up point, this means we consider the energy only for $0 \leq r \leq T-t$. In the new variables and transformed to similarity coordinates we get the following expression for the local energy associated with the free equation

$$E^{loc}(\tau) := e^{-\tau}\int_0^1 \left(\frac{1}{\rho^2}|\phi_1(\tau,\rho)|^2 + \frac{1}{\rho^2}|\phi_2(\tau,\rho)|^2\right) d\rho.$$

The hope is that the solution can still be controlled by this quantity when the potential term is added to the free equation as a perturbation. We define the weighted Lebesgue space $L^2_w := L^2\left((0,1),\rho^{-2}d\rho\right)$ as the completion of $C^\infty_c(0,1)$ with inner product

$$(f|g)_{L^2_w} = \int_0^1 \frac{1}{\rho^2} f(\rho)\overline{g(\rho)} d\rho.$$

Let $\mathcal{H}$ denote the productspace $L^2_w(0,1)^2$ where the norm is given by

$$||\mathbf{f}||^2_{\mathcal{H}} := \int_0^1 \frac{1}{\rho^2}|f_1(\rho)|^2 d\rho + \int_0^1 \frac{1}{\rho^2}|f_2(\rho)|^2 d\rho$$

for $\mathbf{f} = (f_1, f_2)^T \in \mathcal{H}$. We define an operator $(\tilde{A}_0, \mathcal{D}(\tilde{A}_0))$ by

$$\mathcal{D}(\tilde{A}_0) := \{\mathbf{u} \in C^1[0,1]^2 : u_1(0) = u_2(0) = 0\}$$

where

$$\tilde{A}_0\mathbf{u} := \begin{pmatrix} u_1 + u_2' - \rho u_1' - \frac{2}{\rho}u_2 \\ u_2 + u_1' - \rho u_2' \end{pmatrix}.$$

The operator $\tilde{A}_0$ is densely defined and we get an operator formulation of the free equation for $\Phi(\tau, \rho) = \Phi(\tau)(\rho)$

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= \tilde{A}_0\Phi(\tau) \\ \Phi(\tau_0) &= \Phi_0 \end{aligned}$$

with $\Phi : [\tau_0, \infty) \to \mathcal{H}$ and $\tau_0 := -\log T$. The local energy can be obtained by

$$E^{loc}(\tau) = e^{-\tau} \left|\left|\Phi(\tau, \cdot)\right|\right|_{\mathcal{H}}^2. \tag{4.4}$$

This function space is therefore called *the energy space.* Our aim is to derive a growth estimate for solutions of the full equation of the form $||\Phi(\tau,\cdot)||_{\mathcal{H}} \leq Ce^{\mu\tau}$. This implies

$$E^{loc}(\tau) = e^{-\tau} \left|\left|\Phi(\tau, \cdot)\right|\right|_{\mathcal{H}}^2 \leq Ce^{2(\mu-\frac{1}{2})\tau}.$$

Definition. The ground state solution is asymptotically stable in the energy space, if the local energy of the perturbation field in the backward lightcone given by (4.4) is decreasing for $\tau \to \infty$.

This implies that we have to find a growth estimate with $\mu < \frac{1}{2}$.

4.3. Well-posedness

4.3.1. The free equation

We now want to show that the above defined operator (its closure, respectively) generates a one-parameter semigroup. Therefore we need the following Lemma.

Lemma 2. $\tilde{A}_0$ satisfies $Re(\tilde{A}_0\mathbf{u}|\mathbf{u}) \leq \frac{1}{2}\,||\mathbf{u}||^2_{\mathcal{H}}$ for all $\mathbf{u} \in \mathcal{D}(\tilde{A}_0)$.

Proof. A straightforward calculation yields

$$\begin{aligned}
Re(\tilde{A}_0\mathbf{u}|\mathbf{u})_{\mathcal{H}} &= Re\left(\int_0^1 \frac{1}{\rho^2}\left(u_1(\rho) - \rho u_1'(\rho) + u_2'(\rho) - \frac{2}{\rho}u_2(\rho)\right)\overline{u_1(\rho)}d\rho\right) + \\
&+ Re\left(\int_0^1 \frac{1}{\rho^2}\left(u_2(\rho) - \rho u_2'(\rho) + u_1'(\rho)\right)\overline{u_2(\rho)}d\rho\right) \\
&= \frac{1}{2}\,||\mathbf{u}||^2_{\mathcal{H}} - \frac{1}{2}|u_1(1)|^2 - \frac{1}{2}|u_2(1)|^2 + \\
&+ Re\Big(\int_0^1 \frac{1}{\rho^2}u_2'(\rho)\overline{u_1(\rho)}d\rho - \int_0^1 \frac{2}{\rho^3}u_2(\rho)\overline{u_1(\rho)}d\rho + \int_0^1 \frac{1}{\rho^2}u_1'(\rho)\overline{u_2(\rho)}d\rho\Big) = \\
&= \frac{1}{2}\,||\mathbf{u}||^2_{\mathcal{H}} - \frac{1}{2}|u_1(1) - u_2(1)|^2 + Re\left(2iIm\left(\int_0^1 \frac{1}{\rho^2}u_1'(\rho)\overline{u_2(\rho)}d\rho\right)\right) \\
&\leq \frac{1}{2}\,||\mathbf{u}||^2_{\mathcal{H}}
\end{aligned}$$

□

Lemma 3. The range of $1 - \tilde{A}_0$ is dense in $\mathcal{H}$.

Proof. Let $\mathbf{f} \in C_c^\infty(0,1)^2$. We define $\mathbf{u} := (u_1, u_2)^T$ by

$$u_2(\rho) = \frac{\rho^2}{1-\rho^2}\int_\rho^1 \frac{1}{\rho^2}F(\xi)d\xi$$

with $F(\rho) = f_1(\rho) + \rho f_2(\rho)$ and

$$u_1(\rho) = \rho u_2(\rho) - \int_0^\rho u_2(\xi)d\xi - \int_0^\rho f_2(\xi)d\xi$$

Obviously $u_1(0) = u_2(0) = 0$ and $\mathbf{u} \in C^1[0,1]^2$. Thus $\mathbf{u} \in \mathcal{D}(\tilde{A}_0)$ and $(1-\tilde{A}_0)\mathbf{u} = \mathbf{f}$. Lemma 3 follows from the density of $C_c^\infty(0,1)^2$ in $\mathcal{H}$.

□

Proposition 2. The operator $\tilde{A}_0$ is closable and the closure A_0 generates a strongly continuous one-parameter semigroup $S_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ with

$$||S_0(\tau)|| \le e^{\frac{1}{2}\tau}.$$

Proof. Apply Lemma 2, Lemma 3 and the Lumer-Phillips Theorem (see A.2).

4.3.2. The full system

We define an operator on $\mathcal{H}$ by

$$A'\mathbf{u}(\rho) := \begin{pmatrix} -V(\rho)\int_0^\rho u_2(\xi)d\xi \\ 0 \end{pmatrix}.$$

Lemma 4. $A' : \mathcal{H} \to \mathcal{H}$ is bounded.

Proof. Applying Hardy's inequality (see [26]) yields

$$\begin{aligned} ||A'\mathbf{u}||_{\mathcal{H}}^2 &= \int_0^1 \rho^{-2}\left|V(\rho)\int_0^\rho u_2(\xi)d\xi\right|^2 d\rho \le \sup_{\rho\in[0,1]} |V(\rho)|^2 \int_0^1 \rho^{-2}\left(\int_0^\rho |u_2(\xi)|^2 d\xi\right) d\rho \\ &\lesssim \int_0^1 |u_2(\rho)|^2 d\rho \le ||\mathbf{u}||_{\mathcal{H}}^2 . \end{aligned}$$

□

We define $A := A_0 + A'$ with domain $\mathcal{D}(A) = \mathcal{D}(A_0)$ and get an operator formulation of eq. (4.2).

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= A\Phi(\tau) \\ \Phi(\tau_0) &= \Phi_0 \end{aligned}$$

$\Phi : [\tau_0, \infty) \to \mathcal{H}$ and $\tau_0 := -\log T$.

Proposition 3. The operator $A = A_0 + A'$ generates a strongly continuous one-parameter semigroup $S : [0, \infty) \to \mathcal{B}(\mathcal{H})$ with

$$||S(\tau)|| \leq e^{(\frac{1}{2}+||A'||)\tau}.$$

Proof. The assertion follows from Proposition 2, Lemma 4 and the Bounded Perturbation Theorem (see A.2).

□

We conclude that the full system is well-posed in the energy space (see A.2). However, the growth estimate is not very satisfactory because it does not imply asymptotic stability. In order to improve this result we analyze the spectral properties of the generator.

4.4. The spectrum of A_0

Lemma 5. The domain of A_0 is given by

$$\mathcal{D}(A_0) = \{\mathbf{u} \in \mathcal{H} : \mathbf{u} \in H^1_{loc}(0,1)^2, a_0\mathbf{u} \in \mathcal{H}, u_1(0) = u_2(0) = 0\}$$

where the formal differential expression a_0 is defined by

$$a_0\mathbf{u} := \begin{pmatrix} u_1 + u_2' - \rho u_1' - \frac{2}{\rho}u_2 \\ u_2 + u_1' - \rho u_2' \end{pmatrix}$$

and $A_0\mathbf{u} = a_0\mathbf{u}$.

Proof. Let $\mathbf{u} \in \mathcal{D}(A_0)$. This implies that there exists a sequence $(\mathbf{u}_j) \subset \mathcal{D}(A_0)$ such that

$$\begin{aligned} (\mathbf{u}_j) &\to \mathbf{u} && (4.5) \\ \tilde{A}_0 \mathbf{u}_j &\to A_0 \mathbf{u} && (4.6) \end{aligned}$$

in $\mathcal{H}$.
It follows that $((1-\rho^2)u'_{1j})$ and $((1-\rho^2)u'_{2j})$ are Cauchy sequences in $L^2(0,1)$. Thus, u_1 and u_2 are elements of $H^1(0,1-\delta)$. With the Sobolev embedding theorem (see for example [31]) we conclude that $H^1(0,1-\delta) \hookrightarrow C[0,1-\delta]$, which yields the boundary conditions $u_1(0) = u_2(0) = 0$. Conversely consider an element $\mathbf{u} \in \mathcal{H}$ with $\mathbf{u} \in H^1_{loc}(0,1)^2, a_0\mathbf{u} \in \mathcal{H}, u_1(0) = u_2(0) = 0$ then $(1-a_0)\mathbf{u}$ is an element of the Hilbert space. From semigroup theory we know that the spectrum of A_0 is contained in some left half plane. The growth estimate implies that the resolvent map $R_{A_0}(\lambda) : \mathcal{H} \to \mathcal{D}(A_0)$ is a well defined object for $\lambda = 1$. We set $\mathbf{v} := R_{A_0}(1)(1-a_0)\mathbf{u}$, which is an element of $\mathcal{D}(A_0)$. It follows that $(1-a_0)\mathbf{v} = (1-a_0)\mathbf{u}$ hence $(1-a_0)(\mathbf{v}-\mathbf{u}) = 0$. The only solution of this equation in $\mathcal{H}$ is trivial and therefore $\mathbf{v} = \mathbf{u}$.

□

Now we show that the spectral analysis of the operator A_0 can be reduced to the invertibility of an operator-valued function. First we introduce another function space. Let H denote Sobolev space $H^{1,2}\big((0,1),\rho,-2\big)$ (for a definition see [25], p. 67) with singular power weights and a norm given by

$$||u||_H^2 := \int_0^1 \frac{1}{\rho^4}|u(\rho)|^2 d\rho + \int_0^1 \frac{1}{\rho^2}|u'(\rho)|^2 d\rho.$$

For $\lambda \in \mathbb{C}$ we define the formal differential expression

$$t_0(\lambda)u(\rho) := -(1-\rho^2)u''(\rho) + \left(2(\lambda-1)\rho + \frac{2}{\rho}\right)u'(\rho) + (\lambda-1)(\lambda-2)u(\rho)$$

and set

$$D(T_0(\lambda)) := \{u \in H : u \in H^2_{loc}(0,1), t_0(\lambda)u \in L^2_w(0,1), u(0) = u'(0) = 0\}$$

where $T_0(\lambda)u := t_0(\lambda)u$.

Proposition 4. The operator $\lambda - A_0$ is bounded invertible for $\lambda \in \mathbb{C}$ if and only if $T_0(\lambda)$ is invertible, further $\lambda \in \sigma_p(A_0) \iff \dim\ker T_0(\lambda) = 0$. If λ is an eigenvalue of A_0 then the eigenfunction is given by $\mathbf{u} = (u_1, u_2)^T$ with

$$u_1(\rho) = (\lambda - 2)u(\rho) + \rho u'(\rho) \qquad u_2(\rho) = u'(\rho)$$

for $u \in \ker T_0(\lambda)$.

Proof. Suppose $\lambda \in \sigma_p(A_0)$ and $\mathbf{u}$ is the associated eigenfunction. Then the $(\lambda - A_0)\mathbf{u} = 0$ yields

$$\begin{aligned} u_1'(\rho) &= (\lambda - 1)u_2(\rho) + \rho u_2'(\rho) \\ \Rightarrow u_1(\rho) &= (\lambda - 2)\int_0^\rho u_2(\xi)d\xi + \rho u_2(\rho). \end{aligned}$$

Inserting in

$$(\lambda - 1)u_1(\rho) + \rho u_1'(\rho) - u_2'(\rho) + \frac{2}{\rho}u_2(\rho) = 0$$

implies

$$-(1-\rho^2)u_2'(\rho) + \left(2(\lambda - 1)\rho - \frac{2}{\rho}\right)u_2(\rho) + (\lambda - 1)(\lambda - 2)\int_0^\rho u_2(\xi)d\xi = 0.$$

Set $u(\rho) := \int_0^\rho u_2(\xi)d\xi$, then it has to be shown that $u \in \ker T_0(\lambda)$. First we observe that we can apply Hardy's inequality for functions $v \in C_c^\infty(0,1)$ to get the following estimate

$$\int_0^1 \frac{1}{\rho^4}\left|\int_0^\rho v(\xi)d\xi\right|^2 d\rho \lesssim \int_0^1 \frac{1}{\rho^2}|v(\rho)|^2 d\rho. \tag{4.7}$$

Smooth functions with compact support are dense in $L^2_w(0,1)$ and thus the inequality holds for elements of $L^2_w(0,1)$. From the properties of u_2 we obtain the

boundary conditions $u(0) = u'(0) = 0$. Since $u' = u_2 \in L^2_w(0,1)$ we use (4.7) to show that $u \in H$. Further we have $u'' = u_2' \in L^2_{loc}(0,1)$ from what follows that $u \in H \cap H^2_{loc}(0,1)$. The above equation yields $t_0(\lambda)u = 0$ thus $u \in \ker T_0(\lambda)$.
Conversely let $u \in \ker T_0(\lambda)$, $u \neq 0$. Define $u_1(\rho) = (\lambda - 2)u(\rho) + \rho u'(\rho)$ and $u_2(\rho) = u'(\rho)$, then $\mathbf{u} = (u_1, u_2)^T \in \mathcal{H} \cap H^1_{loc}(0,1)^2, u_1(0) = u_2(0) = 0$ and $(\lambda - a_0)\mathbf{u} = 0$. Thus $\mathbf{u} \in \ker(\lambda - A_0)$ and $\lambda \in \sigma_p(A_0)$.
Suppose $\lambda - A_0$ is surjective and set $\mathbf{f} = (f, 0)^T \in \mathcal{H}$. Then there exists a $\mathbf{u} \in \mathcal{D}(A_0)$ such that $(\lambda - A_0)\mathbf{u} = \mathbf{f}$. This implies that $u(\rho) := \int_0^\rho u_2(\xi)d\xi \in \mathcal{D}(T_0(\lambda))$ and $T_0(\lambda)u = f$. Thus $T_0(\lambda)$ is surjective.
Conversely, if $T_0(\lambda)$ is surjective we can find a $u \in \mathcal{D}(T_0(\lambda))$ satisfying

$$T_0(\lambda)u(\rho) = f_1(\rho) + \rho f_2(\rho) + (\lambda - 1)\int_0^\rho f_2(\xi)d\xi$$

for any $\mathbf{f} = (f_1, f_2)^T \in \mathcal{H}$.
Defining $\mathbf{u}$ by $u_1(\rho) = (\lambda - 2)u(\rho) + \rho u'(\rho)$ and $u_2(\rho) = u'(\rho)$ we observe that $\mathbf{u} \in \mathcal{D}(A_0)$ and $(\lambda - A_0)\mathbf{u} = \mathbf{f}$, which shows the surjectivity of $\lambda - A_0$.
Thus we have shown that $\lambda - A_0$ is bijective if and only if $T_0(\lambda)$ is bijective. The closed graph theorem states that $(\lambda - A_0)^{-1}$ is bounded if it exists. From ODE theory it follows that $\ker T_0(\lambda)$ is at most one-dimensional.

□

Lemma 6. The spectrum of A_0 is given by $\sigma(A_0) = \{\lambda \in \mathbb{C} : Re\lambda \leq \frac{1}{2}\}$ where

$$\begin{aligned} \sigma_p(A_0) &= \left\{\lambda \in \mathbb{C} : Re\lambda < \frac{1}{2}\right\} \\ \sigma_c(A_0) &= \left\{\lambda \in \mathbb{C} : Re\lambda = \frac{1}{2}\right\} \\ \sigma_r(A_0) &= \{\emptyset\} \end{aligned}$$

Proof. From semigroup theory and the growth estimate we know that for $\lambda \in \mathbb{C}$ with $Re\lambda > 1/2 \Rightarrow \lambda \in \rho(A_0)$. To determine the point spectrum we consider the equation $t_0(\lambda)u = 0$, which can be solved explicitly. The solution satisfying the boundary conditions $u(0,\lambda) = u'(0,\lambda) = 0$ for $\lambda = 0$ reads

$$u^0(\rho) = (1-\rho^2)\,\text{arctanh}\, x - x \tag{4.8}$$

and for $\lambda \leq \frac{1}{2}, \lambda \neq 0$ we have

$$u^\lambda(\rho) = (1-\rho)^{1-\lambda}(\rho(\lambda-1)-1) + (1+\rho)^{1-\lambda}(\rho(\lambda-1)+1) \tag{4.9}$$

$u^0(\rho) := u(\rho, 0)$ and $u^\lambda(\rho) := u(\rho,\lambda)$ for $\lambda \neq 0$. u^0 is an element of $\mathcal{D}(T_0(\lambda))$. Near $\rho = 0$ we have $u^\lambda \sim \rho^3$ and near $\rho = 1$ the solution behaves like $u^\lambda \sim (1-\rho)^{1-\lambda}$. The condition

$$u^\lambda \in \mathcal{D}(T_0(\lambda)) \Longleftrightarrow Re\lambda < \frac{1}{2}$$

determines the point spectrum of A_0. The spectrum of an operator is a closed set and it is known (c.f. [15], p. 55) that the topological boundary $\frac{1}{2} + i\mathbb{R}$ of the spectrum is contained in the approximate point spectrum given by $\sigma_p(A_0) \cup \sigma_c(A_0)$. We conclude that $\sigma_c(A_0) = \frac{1}{2} + i\mathbb{R}$.

□

The spectrum of A_0 reveals a very interesting structure. Every point in the complex plane with $Re\lambda < 1/2$ is an eigenvalue and for $\lambda = \{-1, -2, -3, ...\}$ the corresponding eigenfunctions are analytic.

4.5. The spectral problem for the operator A

For the analysis of the the spectral properties of $A = A_0 + A'$ we need the following result.

Lemma 7. $A' : \mathcal{H} \to \mathcal{H}$ is compact

Proof. First we define an operator $U : L^2_w(0,1) \to L^2(0,1)$ by

$$u \to Uu(\rho) = \frac{u(\rho)}{\rho} := \tilde{u}(\rho).$$

This is a unitary transformation, which is in particular bounded, since $(Uf|Ug)_{L^2} = (f|g)_{L^2_w}$. The inverse transformation $U^{-1} : L^2(0,1) \to L^2_w(0,1)$ is given by

$$U^{-1}\tilde{u}(\rho) = \rho\tilde{u}(\rho).$$

Consider an integral operator $K : L^2_w(0,1) \to L^2_w(0,1)$ where

$$Ku(\rho) = \int\limits_0^\rho u(\xi)d\xi$$

We define $\tilde{K} : L^2(0,1) \to L^2(0,1)$ by

$$\tilde{K} = UKU^{-1}$$

and

$$\tilde{K}\tilde{u}(\rho) = \frac{1}{\rho}\int\limits_0^\rho \xi\tilde{u}(\xi)d\xi = \frac{1}{\rho}\int\limits_0^1 \Theta(\rho-\xi)\xi\tilde{u}(\xi)d\xi.$$

The integral kernel $\tilde{k}(\rho,\xi) : (0,1)\times(0,1) \to \mathbb{C}$ defined by

$$\tilde{k}(\rho,\xi) = \Theta(\rho-\xi)\frac{\xi}{\rho}$$

is Hilbert-Schmidt, since

$$\int_0^1\int_0^1 |\tilde{k}(\rho,\xi)|^2 d\rho d\xi < \infty.$$

Thus, $\tilde{K}$ is compact and, as a product of compact and continuous operators, K is compact as well. To complete the proof we define a few more operators:

$$\begin{array}{llr} T_1 : & \mathcal{H} \to L^2_w(0,1) & T_1\mathbf{u}(\rho) = u_2(\rho) \\ M : & L^2_w(0,1) \to L^2_w(0,1) & Mu(\rho) = -V(\rho)u(\rho) \\ T_2 : & L^2(0,1) \to \mathcal{H} & T_2 u(\rho) = (u,0)^T. \end{array}$$

Now A' can be constructed in the following manner

$$A' = T_2 \cdot M \cdot U^{-1} \cdot \tilde{K} \cdot U \cdot T_1.$$

Since A' is a product of compact and continuous operators, it is compact as well.

□

A priori we don't know, how the spectrum of A_0 changes under the perturbation A'. However, we will show that for $Re\lambda > \frac{1}{2}$ we only get additional point spectrum, i.e. in this case the investigation of the eigenvalue problem is sufficient to fully determine the spectrum. Suppose that λ is a spectral point of A, such that it belongs to the resolvent set of A_0, then the followung Lemma holds.

Lemma 8. $\lambda \in \mathbb{C} : \lambda \in \sigma(A) \setminus \sigma(A_0) \Rightarrow \lambda \in \sigma_p(A)$

Proof. With the definition of the resolvent $R_{A_0}(\lambda) : \mathcal{H} \to \mathcal{D}(A_0)$,

$$R_{A_0}(\lambda)u = (\lambda - A_0)^{-1}u.$$

for $\lambda \in \rho(A_0)$ we get the following identity:

$$(\lambda - L) = (1 - A'R_{A_0}(\lambda))(\lambda - A_0)$$

For every $\lambda \in \sigma(A) \setminus \sigma(A_0)$ we know that $(\lambda - A)$ is not bounded invertible. Since the inverse of $(\lambda - A_0)$ for $\lambda \in \rho(A_0)$ exists per definition, it follows that $(1 - A'R_{A_0}(\lambda))$ is not bounded invertible. Now we use compactness of the A' to define the compact operator

$$B(\lambda) := A'R_{A_0}(\lambda).$$

With the above observation it is clear that $(1 - B(\lambda))$ is not invertible and with the result for the spectral properties of compact operators (see A.1) we conclude that 1 must be an eigenvalue. Thus one can find an eigenfunction $\mathbf{f} \in \mathcal{H}$ with

$$(1 - B(\lambda))\mathbf{f} = (1 - A'R_{A_0}(\lambda))\mathbf{f} = 0.$$

Defining $\mathbf{u} := (\lambda - A_0)^{-1}\mathbf{f}$, which is an element of $\mathcal{D}(A_0)$, it follows that

$$(\lambda - A)\mathbf{u} = (1 - A'R_{A_0}(\lambda))(\lambda - A_0)\mathbf{u} = (1 - B(\lambda))\mathbf{f} = 0.$$

We conclude that λ must be an eigenvalue of the operator A.

4.5.1. The eigenvalue equation

We define

$$D(T(\lambda)) := \{u \in H : u \in H^2_{loc}(0,1), t_0(\lambda)u \in L^2_w(0,1), u(0) = u'(0) = 0\}$$

and

$$t(\lambda)u(\rho) := -(1-\rho^2)u''(\rho) + \left(2(\lambda-1)\rho + \frac{2}{\rho}\right)u'(\rho) + ((\lambda-1)(\lambda-2) + V(\rho))\,u(\rho)$$

where $T(\lambda)u := t(\lambda)u$. The potential was defined above and reads

$$V(\rho) = -\frac{16}{(1+\rho^2)^2}.$$

Proposition 5. The operator $\lambda - A$ is bounded invertible for $\lambda \in \mathbb{C}$ if and only if $T(\lambda)$ is invertible, further $\lambda \in \sigma_p(A) \Longleftrightarrow \dim\ker T(\lambda) = 0$. If λ is an eigenvalue of A then the eigenfunction is given by $\mathbf{u} = (u_1, u_2)^T$ with

$$u_1(\rho) = (\lambda-2)u(\rho) + \rho u'(\rho) \qquad u_2(\rho) = u'(\rho)$$

for $u \in \ker T(\lambda)$

The proof of the above Proposition will be omitted since it consists of obvious modifications to the proof of Proposition 4. For $Re\lambda > \frac{1}{2}$ the spectrum of A is fully determined by solutions of the eigenvalue equation $t(\lambda)u = 0$, which reads

$$u''(\rho) - \left(\frac{2}{\rho} + \frac{2\lambda\rho}{(1-\rho^2)}\right)u' - \left(\frac{(\lambda-1)(\lambda-2) + V(\rho)}{(1-\rho^2)}\right)u(\rho) = 0. \qquad (4.10)$$

Set $\tilde{u} = \frac{u}{\rho^2}$, then the above equation transformes to eq. (3.4). It is not surprising that there is a solution of eq. (5.7) for $\lambda = 1$, which corresponds to the the gauge mode.

$$u^g(\rho) = \frac{2\rho^3}{1+\rho^2}. \qquad (4.11)$$

The transformation only influences the behavior of solutions near $\rho = 0$, thus we can adapt the asymptotic estimates (see table (3.2)) to the above eigenvalue equation. The behavior near the endpoints $\rho = 0$ and $\rho = 1$ is given in table (4.1).

Table 4.1.: Asymptotic estimates for (5.7)

$\rho \to 0$	all λ	$u_0^a \sim \rho^3$	$u_0^n \sim 1$
$\rho \to 1$	$\lambda \notin \mathbb{Z}$	$u_1^a \sim 1$	$u_1^n \sim (1-\rho)^{1-\lambda}$
	$\lambda \in \mathbb{Z},\, \lambda > 1$	$u_1^a \sim 1$	$u_1^n \sim c\log(1-\rho) + (1-\rho)^{1-\lambda}$
	$\lambda \in \mathbb{Z},\, \lambda \leq 1$	$u_1^a \sim (1-\rho)^{1-\lambda}$	$u_1^n \sim c\log(1-\rho)(1-\rho)^{1-\lambda} + 1$

We only consider the case $Re\lambda > \frac{1}{2}$. We see that admissible solutions, i.e. solutions belonging to $D(T(\lambda))$, are analytic at both endpoints. With the results derived in chapter 3 we obtain the following Lemma.

Lemma 9. Equation (5.7) does not have nontrivial analytic solutions for

$$Re\lambda > \frac{1}{2},$$

except for $\lambda = 1$.

Proof. Suppose there is an analytic solution u for $\lambda \in \mathbb{C}$ with $Re\lambda > \frac{1}{2}$. Then $\tilde{u} = \frac{u}{\rho^2}$ is a solution of eq. (3.4) and by applying theorem 5 the assertion follows by contradiction.

Proposition 6. The spectrum of A consists of the single eigenvalue $\lambda = 1$ for $Re\lambda > \frac{1}{2}$.

4.5.2. Implications on asymptotic stability

In the following we will qualitatively explain that is not possible to prove asymptotic stability in the energy space. The growth estimate for the perturbation field was given by

$$||S(\tau)|| \leq e^{(\frac{1}{2}+||A'||)\tau}.$$

Suppose we can remove the eigenvalue $\lambda = 1$ in order to operate on a subspace $N \subset \mathcal{H}$, where A_N is the restriction of A to N with spectral bound $s(A_N) = \frac{1}{2}$

(see (A.2) for a definition). This can be achieved by defining a spectral projection as will be described in the next chapter. It can be shown that A_N generates a semigroup S_N with growth estimate $||S_N(\tau)|| \leq e^{(\frac{1}{2}+||A'||)\tau}$. This can be optimized by considering the spectral properties of the generator, cf. (A.2). At best one gets the result

$$s(A_N) = \omega_0 = \frac{1}{2}$$

where ω_0 denotes the growth bound of the subspace semigroup. However, ω_0 is defined as an infimum, which is generally not attained. However, for every $\epsilon > 0$ one can find an $\omega > s(A)$ such that $\omega < \omega_0 + \epsilon$.
We formulate the Cauchy problem on the subset N

$$\frac{d}{d\tau}\Phi(\tau) = A_N \Phi(\tau) \tag{4.12}$$

$$\Phi(\tau_0) = \Phi_0 \tag{4.13}$$

with $\Phi : [\tau_0, \infty) \to N \subset \mathcal{H}$, $\Phi_0 \in N$ and $\tau_0 := -\log T$. The time evolution is then determined by $S_N(\tau)$.

$$\Phi(\tau) = S_N(\tau - \tau_0)\Phi(\tau_0)$$

Set $\omega = \frac{1}{2} + \frac{\epsilon}{2}$, then this yields

$$||\Phi(\tau)||_{\mathcal{H}} = ||S(\tau - \tau_0)\Phi(\tau_0)||_{\mathcal{H}} \lesssim e^{\frac{1}{2}(1+\epsilon)\tau} ||\Phi(\tau_0)||_{\mathcal{H}}.$$

Finally we get the energy estimate

$$E_\Phi(\tau) \leq e^{-\tau} ||\Phi(\tau)||^2_{\mathcal{H}} \lesssim e^{\epsilon\tau}.$$

To summarize, it was shown above that for $Re\lambda > \frac{1}{2}$ the spectral problem for the generator of the full system can be reduced to the investigation of the eigenvalue equation by using the spectral properties of A_0 and compactness of A'. Further it was demonstrated that in this case there are no eigenvalues except for $\lambda = 1$, which is not a problem because it can be removed by a spectral projection. Despite this, it is not possible to derive an appropriate growth estimate that leads to decreasing energy. Thus, the main problems are given by the spectral bound of A_0 and by the eigenvalue equation as an ODE problem. In the following it will be shown that by formulating the problem in another function space these obstacles can be overcome.

5. Higher energy space - Asymptotic stability

We formulate the main result of the following analysis.

Theorem 6. The ground state solution (2.12) of the wave maps equation (2.5) is asymptotically stable.

5.1. First order formulation

For convenience we define variables slightly different to the ones we used above and therefore get a different first order formulation of the problem. We start with the evolution equation for the (transformed) pertubation field (cf. eq. (4.2)), which reads

$$\psi_{tt} - \psi_{rr} + \frac{2}{r}\psi_r + V\psi = 0 \tag{5.1}$$

with initial data $\psi(0,r), \psi_t(0,r)$ and

$$\psi(t,0) = \psi_r(t,0) = \psi_{rr}(t,0) = 0, \forall t.$$

The potential is given by $V(t,r) := \frac{2\cos\left(4\arctan(\frac{r}{T-t})\right)-2}{r^2}$.
We define variables $\Psi := (\psi_1, \psi_2)^T$ by

$$\psi_1 := \frac{\psi_t}{(T-t)} \qquad \psi_2 := \frac{\psi_r}{r}.$$

The field can again be obtained by integration

$$\psi(t,r) = \int_0^r s\psi_2(t,s)ds.$$

The first order formulation in CSS coordinates (τ,ρ) then reads

$$\begin{pmatrix} \partial_\tau\phi_1 \\ \partial_\tau\phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 - \rho\partial_\rho\phi_1 + \rho\partial_\rho\phi_2 - \phi_2 - V(\rho)\int\limits_0^\rho \xi\phi_2(\xi)d\xi \\ \frac{1}{\rho}\partial_\rho\phi_1 - \rho\partial_\rho\phi_2 \end{pmatrix} \tag{5.2}$$

where $\phi_j(\tau,\rho) := \psi_j(T-e^{-\tau}, \rho e^{-\tau})$ for $j=1,2$ and $\Phi = (\phi_1,\phi_2)^T$. The potential is given by

$$V(\rho) = -\frac{16}{(1+\rho^2)^2}$$

and regularity conditions are

$$\phi_1(\tau,0) = \partial_\rho\phi_1(\tau,0) = \partial_{\rho\rho}\phi_1(\tau,0) = 0 \quad \text{and} \quad \phi_2(\tau,0) = 0$$

for all τ. The field can be calculated by

$$\phi(\tau,\rho) = e^{-2\tau}\int\limits_0^\rho \xi\phi_2(\tau,\xi)d\xi.$$

5.1.1. Higher energy

We consider the energy in the backward lightcone of $(t,r)=(T,0)$ associated with the free equation

$$\psi_{tt} - \psi_{rr} + \frac{2}{r}\psi_r = 0. \tag{5.3}$$

Transformed to the above defined variables and written in CSS-coordinates it reads

$$E_\phi^{loc}(\tau) := e^{-\tau}\int_0^1 \left(\rho^{-2}|\phi_1(\tau,\rho)|^2 + |\phi_2(\tau,\rho)|^2\right) d\rho. \tag{5.4}$$

Let $\mathcal{H}^E$ denote the energy space defined as the productspace

$$\mathcal{H}^E := L^2\left((0,1),\rho^{-2}d\rho\right) \times L^2(0,1).$$

An estimate for the field of the form $||\Phi(\tau,\cdot)||^2_{\mathcal{H}^E} \leq Ce^{\mu\tau}$ implies the energy estimate

$$E_\phi(\tau) = e^{-\tau}\, ||\Phi(\tau,\cdot)||^2_{\mathcal{H}^E} \leq Ce^{2(\mu-\frac{1}{2})\tau}.$$

It was shown above that an operator formulation in the energy space is not very fruitful in the sense that we cannot derive an appropriate growth bound. In order to improve this result we require higher differentiablity of the solutions. First we motivate the choice of the function space and the inner product we will use. The derivative of eq. (5.3) with respect to r yields

$$\psi_{rtt} - \psi_{rrr} + \frac{2}{r}\psi_{rr} - \frac{2}{r^2}\psi_r = 0.$$

With $\tilde{\phi} = \frac{\psi_r}{r}$ the above equation transforms to the 1+1 wave equation with energy density $\mathcal{E}_{\tilde{\phi}} = |\tilde{\phi}_t|^2 + |\tilde{\phi}_r|^2$. This yields a conserved quantity for eq. (5.3) given by

$$E^{diff}(t) := \int_0^\infty \left(\left| \tfrac{\psi_{tr}(t,r)}{r} \right|^2 + \left| \partial_r \left(\tfrac{\psi_r(t,r)}{r} \right) \right|^2 \right) dr. \tag{5.5}$$

We will show that by introducing a norm inspired by this quantity the spectrum of the generator for the free equation can be shifted towards the left. The aim is to derive a growth estimate in the *higher energy space* and to show that this yields an estimate for the original energy given by (5.4).

5.2. Higher energy space

We define two Hilbert spaces, denoted by $\dot{X}(0,1)$ and $\dot{X}_2(0,1)$, as the completion of $C^\infty_{\{0\}}(0,1)$ with inner products

$$(f|g)_{\dot{X}} := \int_0^1 f'(\rho)\overline{g'(\rho)} d\rho,$$

$$(f|g)_{\dot{X}_2} := \int_0^1 \frac{1}{\rho^2} f'(\rho)\overline{g'(\rho)} d\rho.$$

The index in $\dot{X}_2(0,1)$ indicates the exponent of the weight function ρ^{-e}.

We define a product space $\mathcal{H} := \dot{X}_2(0,1) \times \dot{X}(0,1)$ with norm

$$||\mathbf{u}||^2_{\mathcal{H}} = \int_0^1 \frac{1}{\rho^2} |u_1'(\rho)|^2 d\rho + \int_0^1 |u_2'(\rho)|^2 d\rho$$

for $\mathbf{u} = (u_1, u_2)^T \in \mathcal{H}$.
Next we study the properties of the above defined spaces and prove some embedding thoerems, which turn out to be very useful for further application.

Lemma 10. $\dot{X}(0,1)$ and $\dot{X}_2(0,1)$ are continuously embedded in $H^1(0,1)$.

Proof. Let $H^1(0,1)$ denote a Sobolev space with norm

$$||f||^2_{H^1} := \int_0^1 |f(\rho)|^2 d\rho + \int_0^1 |f'(\rho)|^2 d\rho.$$

First we prove the assertion for $\dot{X}(0,1)$. Consider the following inequality on the dense subset $C^\infty_{\{0\}}(0,1)$:

$$|u(\rho)| = \left| \int_0^\rho u'(\xi) d\xi \right| \leq \int_0^\rho |u'(\xi)| d\xi \leq \int_0^1 |u'(\rho)| d\rho \leq \left(\int_0^1 |u'(\rho)|^2 d\rho \right)^{\frac{1}{2}} = ||u||_{\dot{X}} \,,$$

where Cauchy-Schwarz inequalitity was applied. Integrating $|u(\rho)|^2$ then yields

$$||u||^2_{L^2} \leq ||u||^2_{\dot{X}} \,.$$

Thus

$$||u||^2_{H^1} = \int_0^1 |u(\rho)|^2 d\rho + \int_0^1 |u'(\rho)|^2 \leq 2\, ||u||^2_{\dot{X}}$$

for all $u \in \dot{X}(0,1)$. The continuous embedding of $\dot{X}_2(0,1)$ in $H^1(0,1)$ follows from

$$||u||^2_{\dot{X}} \leq ||u||^2_{\dot{X}_2}$$

for all $u \in \dot{X}_2(0,1)$.

□

Lemma 11. Let $\mathbf{u} \in \mathcal{H}$. Then $u_j \in C[0,1]$ and $u_j(0) = 0$ for $j = 1, 2$.

Proof. The claim follows from Lemma 10, the continuous embedding of $H^1(0,1)$ in $C[0,1]$ and the composition of continuous mappings.

□

We denote the weighted Lebesgue spaces $L^2((0,1), \rho^{-4} d\rho)$ and $L^2((0,1), \rho^{-2} d\rho)$ by $L^2_4(0,1)$ and $L^2_2(0,1)$.

Lemma 12. $\dot{X}(0,1)$ and $\dot{X}_2(0,1)$ are continuously embedded in $L^2_2(0,1)$ and $L^2_4(0,1)$, respectively.

Proof. Operating on the dense subset $C^\infty_{\{0\}}(0,1)$ and applying Hardy's inequality yields

$$||u||^2_{L^2_2} = \int_0^1 \frac{1}{\rho^2} |u(\rho)|^2 d\rho \lesssim ||u||^2_{\dot{X}}$$
$$||u||^2_{L^2_4} = \int_0^1 \frac{1}{\rho^4} |u(\rho)|^2 d\rho \lesssim ||u||^2_{\dot{X}_2}$$

for all $u \in \dot{X}(0,1)$ and $\dot{X}_2(0,1)$, respectively.

□

Lemma 13. The space $\dot{X}(0,1)$ is compactly embedded in $L^2(0,1)$ and

$$||u||^2_{L^2} \leq ||u||^2_{\dot{X}}$$

for all u in $\dot{X}(0,1)$.

Proof. From Lemma 10 we know that $\dot{X}(0,1) \hookrightarrow H^1(0,1)$ holds. Since $H^1(0,1)$ is compactly embedded in $L^2(0,1)$ (see e.g. [16]), it follows that the inclusion $I : \dot{X}(0,1) \to L^2(0,1), Iu = u$ can be constructed as a product of compact and continuous operators and is therefore compact as well. The inequality was already shown in the proof of Lemma 10.

□

Lemma 14. $\mathcal{H}$ is continuously embedded in the energy space, i.e.

$$||\mathbf{u}||_{\mathcal{H}^E} \leq C\,||\mathbf{u}||_{\mathcal{H}}$$

for all $\mathbf{u} \in \mathcal{H}$.

Proof. The assertion follows from Lemma 13, the inequalitity

$$||u||^2_{L^2_2} \lesssim ||u||^2_{\dot{X}} \leq ||u||^2_{\dot{X}_2}$$

for all $u \in \dot{X}_2(0,1)$ and the construction of $\mathcal{H}$ as a product space.

□

At the end of this list of properties another useful relation should be mentioned

Lemma 15. The multiplication operator M_1 defined by $M_1 u := \rho u$ is a bounded operator from $\dot{X}(0,1) \to \dot{X}_2(0,1)$.

Proof. With Lemma 12 we get

$$\begin{aligned} ||M_1 u||_{\dot{X}_2} &= ||\rho u||_{\dot{X}_2} = \left|\left|u' + \rho^{-1}u\right|\right|_{L^2} \\ &\leq ||u'||_{L^2} + \left|\left|\rho^{-1}u\right|\right|_{L^2} = ||u||_{\dot{X}} + ||u||_{L^2_2} \lesssim ||u||_{\dot{X}}\,. \end{aligned}$$

□

5.3. Well-posedness

5.3.1. The free equation

We set

$$\mathcal{D}(\tilde{L}_0) := \{\mathbf{u} \in \mathcal{H} : u_1, u_2 \in C^1(0,1), \ell_0\mathbf{u} \in \mathcal{H}, u_1'(0) = 0\}$$

and $\tilde{L}_0\mathbf{u} := l_0\mathbf{u}$, where the formal differential expression l_0 is given by

$$l_0\mathbf{u}(\rho) := \begin{pmatrix} u_1(\rho) - \rho u_1'(\rho) + \rho u_2'(\rho) - u_2(\rho) \\ \frac{u_1'(\rho)}{\rho} - \rho u_2'(\rho) \end{pmatrix}.$$

Then the operator $(\tilde{L}_0, \mathcal{D}(\tilde{L}_0))$ is densely defined and we get an operator formulation of the free equation by

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= \tilde{L}_0\Phi(\tau) \\ \Phi(\tau_0) &= \Phi_0 \end{aligned}$$

where $\Phi : [\tau_0, \infty) \to \mathcal{H}$ and $\tau_0 := -\log T$.
We will show that the operator is closable and that the closure generates a one parameter semigroup. The next Lemma follows from straightforward calculation (see proof of Lemma 2).

Lemma 16. $\tilde{L}_0$ satisfies $Re(\tilde{L}_0\mathbf{u}|\mathbf{u}) \leq -\frac{1}{2}\,||\mathbf{u}||^2_{\mathcal{H}}$ for all $\mathbf{u} \in \mathcal{D}(\tilde{L}_0)$

Lemma 17. The range of $1 - \tilde{L}_0$ is dense in $\mathcal{H}$.

Proof. Let $\mathbf{f} \in C^\infty_{\{0\}}(0,1)^2$. We define $\mathbf{u} := (u_1, u_2)$ by

$$u_2(\rho) = \frac{\rho}{1-\rho^2}\int_\rho^1 \frac{F(\xi)}{\xi^2}d\xi$$

with $F(\rho) = f_1(\rho) + \rho^2 f_2(\rho)$ and

$$u_1(\rho) = \rho^2 u_2(\rho) - \int_0^\rho \xi u_2(\xi)d\xi - \int_0^\rho \xi f_2(\xi)d\xi$$

Then $u_1(0) = u_2(0) = 0$ and $\mathbf{u} \in \mathcal{H}$. Furthermore, $\mathbf{u} \in C^1(0,1)^2$ and $u_1'(0) = 0$. We conclude that $\mathbf{u} \in \mathcal{D}(\tilde{L}_0)$. The claim follows from the density of $C^\infty_{\{0\}}(0,1)^2$ in $\mathcal{H}$.

□

Proposition 7. The operator $\tilde{L}_0$ is closable and the closure L_0 generates a strongly continuous one-parameter semigroup $S_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ with

$$||S_0(\tau)|| \leq e^{-\frac{1}{2}\tau}.$$

Proof. Lemma 16, Lemma 17 and Lumer-Phillips Theorem.

∎

5.3.2. Well-posedness of the full system

Consider the perturbation operator

$$(L'\mathbf{u})(\rho) := \begin{pmatrix} -V(\rho)\int\limits_0^\rho \xi u_2(\xi)d\xi \\ 0 \end{pmatrix}.$$

Lemma 18. The operator $L' : \mathcal{H} \to \mathcal{H}$ is compact and in particular bounded.

Proof. First we define an integral operator $K : \dot{X}(0,1) \to \dot{X}_2(0,1)$ by

$$(Ku)(\rho) = \int\limits_0^\rho \xi u(\xi)d\xi$$

The operator K is compact if and only if given a bounded sequence (u_j) in $\dot{X}(0,1)$ it follows that (Ku_j) has a convergent subsequence (Ku_{j_k}) in $\dot{X}_2(0,1)$.
From Lemma 13 we know that any bounded sequence (u_j) in $\dot{X}(0,1)$ has a subsequence (u_{j_k}), which converges in $L^2(0,1)$, i.e. $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ such that $\forall j_k \geq N(\epsilon) : ||u_{j_k} - u^*||_{L^2} < \epsilon$.
This implies

$$\begin{aligned} ||u_{j_k} - u^*||_{L^2} &= \int_0^1 |u_{j_k}(\rho) - u^*(\rho)|^2 d\rho = \int_0^1 \frac{1}{\rho^2}|\rho u_{j_k}(\rho) - \rho u^*(\rho)|^2 d\rho = \\ &= \int_0^1 \frac{1}{\rho^2}\left|\left(\int\limits_0^\rho \xi u_{j_k}(\xi)d\xi - \int\limits_0^\rho \xi u^*(\xi)d\xi\right)'\right|^2 d\rho = \\ &= \int_0^1 \frac{1}{\rho^2}\left|(Ku_{j_k}(\rho) - \hat{u}(\rho))'\right|^2 d\rho = \\ &= ||Ku_{j_k} - \hat{u}||_{\dot{X}_2} < \epsilon \end{aligned}$$

where $\hat{u}(\rho) := \int\limits_0^\rho \xi u^*(\xi)d\xi$ is the limit of (Ku_{j_k}) in $\dot{X}_2(0,1)$.

Next we construct L' as a product of continuous and compact operators.

$$\begin{array}{rcl} T_1: & \mathcal{H} \to \dot{X}(0,1) & T_1\mathbf{u}(\rho) = u_2(\rho) \\ M_2: & \dot{X}_2(0,1) \to \dot{X}_2(0,1) & M_2 u(\rho) = -V(\rho)u(\rho) \\ T_2: & \dot{X}_2(0,1) \to \mathcal{H} & T_2 u(\rho) = (u,0)^T \end{array}$$

$$L' = T_2 \cdot M_2 \cdot K \cdot T_1$$

□

Defining $L := L_0 + L'$ with domain $\mathcal{D}(L) = \mathcal{D}(L_0)$ then the operator formulation of eq. (5.1) reads

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= L\Phi(\tau) \\ \Phi(\tau_0) &= \Phi_0 \end{aligned}$$

with $\Phi : [\tau_0, \infty) \to \mathcal{H}$ and $\tau_0 := -\log T$.

Proposition 8. The operator L generates a strongly continuous one-parameter semigroup $S : [0,\infty) \to \mathcal{B}(\mathcal{H})$ with

$$||S(\tau)|| \leq e^{(-\frac{1}{2}+||L'||)\tau}.$$

Proof. Proposition 7, Lemma 18 and the Bounded Perturbation Theorem.

□

5.4. The spectrum of L_0

We describe the operator L_0 in more detail and analyze its spectrum.

Lemma 19. The domain of the operator $(L_0, \mathcal{D}(L_0))$ generated by l_0 is given by

$$\mathcal{D}(L_0) = \{\mathbf{u} \in \mathcal{H} : \mathbf{u}' \in H^1_{loc}(0,1)^2, l_0\mathbf{u} \in \mathcal{H}, u_1'(0) = 0\}$$

Proof. Following the definition, for every $\mathbf{u} \in \mathcal{D}(L_0)$ there exists a sequence $(\mathbf{u_j}) \in \mathcal{D}(\tilde{L_0})$ with

$$\begin{aligned} \mathbf{u}_j &\to \mathbf{u} \\ \tilde{L_0}\mathbf{u}_j &\to L_0\mathbf{u} \end{aligned}$$

in $\mathcal{H}$.

In the norm we have to consider derivatives and the first condition yields

$$\begin{aligned} \frac{u'_{1_j}}{\rho} &\to \frac{u'_1}{\rho} \\ u'_{2_j} &\to u'_2 \end{aligned}$$

in $L^2(0,1)$.

The second condition implies

$$\begin{aligned} u''_{2_j} - u''_{1_j} &\to u''_2 - u''_1 \\ \frac{u''_{1_j}}{\rho} - \frac{u'_{1_j}}{\rho^2} - \rho u''_{2_j} - u'_{2_j} &\to \frac{u''_1}{\rho} - \frac{u'_1}{\rho^2} - \rho u''_2 - u'_2 \end{aligned}$$

in $L^2(0,1)$.

A suitable combination ob the above statements shows that $((1-\rho^2)u''_{1_j})$ and $((1-\rho^2)u''_{2_j})$ are Cauchy sequences in $L^2(0,1)$. Therefore $\mathbf{u}'$ is an element of $H^1(0,1-\delta)^2$ for any $\delta \in (0,1)$. The Sobolev embedding $H^1(0,1-\delta) \hookrightarrow C[0,1-\delta]$ then yields the boundary condition $u'_1(0) = 0$.

Conversely, if $\mathbf{u} \in \mathcal{H}, \mathbf{u}' \in H^1_{loc}(0,1)^2$, $l_0\mathbf{u} \in \mathcal{H}$ and $u'_1(0) = 0$ then $(1-l_0)\mathbf{u} \in \mathcal{H}$. The above growth estimate for the semigroup shows that the resolvent map $R_{L_0}(\lambda) : \mathcal{H} \to \mathcal{D}(L_0)$ is a well defined object for $\lambda = 1$.

We set $\mathbf{v} := R_{L_0}(1)(1-l_0)\mathbf{u}$, which is an element of $\mathcal{D}(L_0)$. It follows that $(1-l_0)\mathbf{v} = (1-l_0)\mathbf{u}$ hence $(1-l_0)(\mathbf{v}-\mathbf{u}) = 0$. The only solution for this equation in $\mathcal{H}$ is the trivial solution and $\mathbf{v} = \mathbf{u}$.

□

Let H denote the Sobolev space $H^{2,2}((0,1);\rho;-2)$ (cf. [25], p. 67) with norm

$$||u||_H^2 := \int_0^1 \frac{1}{\rho^6}|u(\rho)|^2 d\rho + \int_0^1 \frac{1}{\rho^4}|u'(\rho)|^2 d\rho + \int_0^1 \frac{1}{\rho^2}|u''(\rho)|^2 d\rho$$

Lemma 20. Let $u \in H$. Then $u \in C^1[0,1]$ and

$$u(0) = u'(0) = 0.$$

Proof. It is known that $H^{2,2}((0,1);\rho;-2)$=$W^{2,2}((0,1);\rho;-2)$ (see [25], p. 73), where $W^{2,2}((0,1);\rho;-2)$ is a weighted Sobolev space with norm

$$||u||_{W^{2,2}_{-2}}^2 := \int_0^1 \frac{1}{\rho^2}|u(\rho)|^2 d\rho + \int_0^1 \frac{1}{\rho^2}|u'(\rho)|^2 d\rho + \int_0^1 \frac{1}{\rho^2}|u''(\rho)|^2 d\rho$$

Since $||u||_{H^2} \leq ||u||_{W^{2,2}_{-2}}$ for $u \in W^{2,2}((0,1);\rho;-2)$ we have

$$W^{2,2}((0,1);\rho;-2) \hookrightarrow H^2(0,1).$$

The assertion follows from the continuous embedding $H^2(0,1) \hookrightarrow C^1[0,1]$ and the density of $C^\infty_{\{0\}}(0,1)$ in H ([25] p. 73).

□

We define an operator valued function $(T_0(\lambda), \mathcal{D}(T_0(\lambda)))$ generated by the formal differential expression

$$t_0(\lambda)u(\rho) := -(1-\rho^2)u''(\rho) + \left(2(\lambda-1)\rho + \frac{2}{\rho}\right)u'(\rho) + (\lambda-1)(\lambda-2)u(\rho)$$

where $T_0(\lambda)u := t_0(\lambda)u$. We set

$$\mathcal{D}(T_0(\lambda)) := \{u \in H : u \in H^3_{loc}(0,1), t_0(\lambda)u \in \dot{X}_2(0,1), u''(0) = 0\}.$$

Lemma 21. The operator $\lambda - L_0$ for a $\lambda \in \mathbb{C}$ is bounded invertible if and only if $T_0(\lambda)$ is invertible. Furthermore $\lambda \in \sigma_p(L_0) \iff \dim\ker T_0(\lambda) = 1$ and for $\lambda \in \sigma_p(L_0)$ the vector $\mathbf{u} = (u_1, u_2)^T$ defined by

$$u_1(\rho) = (\lambda - 2)u(\rho) + \rho u'(\rho) \qquad u_2(\rho) = \frac{u'(\rho)}{\rho}$$

for $u \in \ker T_0(\lambda)$ is an eigenfunction of L_0.

To prove the above Proposition we show that
(1) $\mathbf{u} \in \ker(\lambda - L_0) \iff u \in \ker T_0(\lambda)$.
(2) $(\lambda - L_0)$ is surjective $\iff T_0(\lambda)$ is surjective.

Proof. (1) For $\lambda \in \sigma_p(L_0)$ and $\mathbf{u}$ being the associated eigenfunction, $(\lambda - L_0)\mathbf{u} = 0$ yields

$$u_1'(\rho) = \lambda\rho u_2(\rho) + \rho^2 u_2'(\rho),$$
$$\Rightarrow u_1(\rho) = (\lambda - 2)\int_0^\rho \xi u_2(\xi)d\xi + \rho^2 u_2(\rho).$$

Inserting this in

$$(\lambda - 1)u_1(\rho) + \rho u_1'(\rho) - \rho u_2'(\rho) + u_2(\rho) = 0$$

we get

$$-\rho(1-\rho^2)u_2'(\rho) + ((2\lambda - 1)\rho^2 + 1)u_2(\rho) + (\lambda - 1)(\lambda - 2)\int_0^\rho \xi u_2(\xi)d\xi = 0.$$

Define $u(\rho) := \int_0^\rho \xi u_2(\xi)d\xi$. Then the above equation reads

$$-(1-\rho^2)u''(\rho) + \left(2(\lambda - 1)\rho + \frac{2}{\rho}\right)u'(\rho) + (\lambda - 1)(\lambda - 2)u(\rho) = 0.$$

From Lemma 15 we know that $u' = \rho u_2$ is an element of $\dot{X}_2(0,1)$ and with Lemma 12 we have $u' \in L^2_4(0,1)$ and $u'' \in L^2_2(0,1)$. With Hardy's inequaltity we get

$$\int_0^1 \frac{1}{\rho^6}\left|\int\limits_0^\rho \xi u_2(\xi)d\xi\right|^2 d\rho \lesssim \int_0^1 \frac{1}{\rho^2}|u_2(\rho)|^2 d\rho \lesssim \int_0^1 |u_2'(\rho)|^2 d\rho < \infty$$

Hence u is an element of H. Further we have $u''(0) = 0$ and

$$u'''(\rho) = \rho u_2''(\rho) + 2u_2'(\rho) \in L^2_{loc}(0,1)$$

It follows that $u \in \mathcal{D}(T_0(\lambda))$ and $t_0(\lambda)u(\rho) = 0$, thus $u \in \ker T_0(\lambda)$.
Conversely let $u \in \ker T_0(\lambda)$, $u \neq 0$. Define

$$u_1(\rho) = (\lambda - 2)u(\rho) + \rho u'(\rho)$$
$$u_2(\rho) = \frac{u'(\rho)}{\rho}$$

then u_1 and u_2 are elements of $\dot{X}_2(0,1)$ and $\dot{X}(0,1)$, respectively. The boundary conditions yield $u_1(0) = u_2(0) = u_1'(0) = 0$. Thus $\mathbf{u} = (u_1, u_2)^T \in \mathcal{H}$ and $l_0\mathbf{u} = \lambda\mathbf{u}$. Since $\mathbf{u}' \in H^1_{loc}$ and we conclude that $\mathbf{u} \in \mathcal{D}(L_0)$, $\mathbf{u} \in \ker(\lambda - L_0)$ and $\lambda \in \sigma_p(L_0)$.

□

Proof. (2) Suppose $\lambda - L_0$ is surjective and set $\mathbf{f} = (f, 0)^T \in \mathcal{H}$. Then one can find a $\mathbf{u} \in \mathcal{D}(L_0)$ such that $(\lambda - L_0)\mathbf{u} = \mathbf{f}$. Again define $u(\rho) := \int\limits_0^\rho \xi u_2(\xi)d\xi \in \mathcal{D}(T_0(\lambda))$, then the above equation yields $T_0(\lambda)u = f$. Thus $T_0(\lambda)$ is surjective.
Conversely, if $T_0(\lambda)$ is surjective we can find a $u \in \mathcal{D}(T_0(\lambda))$ satisfying

$$T_0(\lambda)u(\rho) = f_1(\rho) + \rho^2 f_2(\rho) + (\lambda - 1)\int\limits_0^\rho \xi f_2(\xi)d\xi$$

for any $\mathbf{f} = (f_1, f_2)^T \in \mathcal{H}$. Then for $\mathbf{u} := (u_1, u_2)$ with $u_1(\rho) = (\lambda - 2)u(\rho) + \rho u'(\rho)$ and $u_2(\rho) = \frac{u'(\rho)}{\rho}$ it follows that $\mathbf{u} \in \mathcal{D}(L_0)$ and $(\lambda - L_0)\mathbf{u} = \mathbf{f}$ which shows surjectivity of $\lambda - L_0$.
We complete the proof with the closed graph theorem, which states that $(\lambda - L_0)^{-1}$ is bounded if it exists. Furthermore ODE theory states that the kernel of $T_0(\lambda)$ is at most one-dimensional.

□

Solving the above ordinary differential equation $t_0(\lambda)u = 0$ we derive the following result for the spectrum of L_0.

Lemma 22. The spectrum of L_0 is given by $\sigma(L_0) = \{\lambda \in \mathbb{C} : Re\lambda \leq -\frac{1}{2}\}$ where

$$\begin{aligned}\sigma_p(L_0) &= \left\{\lambda \in \mathbb{C} : Re\lambda < -\frac{1}{2}\right\} \\ \sigma_c(L_0) &= \left\{\lambda \in \mathbb{C} : Re\lambda = \frac{1}{2}\right\} \\ \sigma_r(L_0) &= \{\emptyset\}\end{aligned}$$

Proof. From the growth estimate for $S_0(\tau)$ we know that $\lambda \in \mathbb{C}$ with $Re\lambda > -\frac{1}{2}$ belong to the resolvent set. The solutions of the eigenvalue equation for $Re\lambda < -\frac{1}{2}$ are given by

$$u(\cdot, \lambda) = (1-\rho)^{1-\lambda}(\rho(\lambda-1)-1) + (1+\rho)^{1-\lambda}(1+\rho(\lambda-1))$$

and $u(\cdot, \lambda) \in \mathcal{D}(T_0(\lambda)) \iff Re\lambda < -\frac{1}{2}$. For the continuous spectrum we refer to the proof of Lemma 6.

□

5.4.1. Resolvent estimates

We calculate the resolvent $R_{L_0}(\lambda)$ of the operator L_0 using the inverse of the above defined $T_0(\lambda)$. We define an operator $B(\lambda) : \mathcal{H} \to \dot{X}_2(0,1)$ by

$$B(\lambda)\mathbf{f}(\rho) := f_1(\rho) + \rho^2 f_2(\rho) + (\lambda - 1)\int_0^\rho \xi f_2(\xi) d\xi.$$

The resolvent $R_{L_0}(\lambda)$ then reads

$$R_{L_0}(\lambda)\mathbf{f}(\rho) = \begin{pmatrix} (\lambda-2)(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})(\rho) + \rho(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'(\rho) - \int_0^\rho \xi f_2(\xi)d\xi \\ \frac{1}{\rho}(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'(\rho) \end{pmatrix}.$$

The growth bound for the semigroup provides an upper bound for the resolvent (see A.2),

$$||R_{L_0}(\lambda)|| \leq \frac{1}{Re\lambda + \frac{1}{2}}$$

for all $\lambda \in \mathbb{C}$ with $Re\lambda > -\frac{1}{2}$.

This estimates implies

$$\left|\left|(\lambda-2)(T_0^{-1}(\lambda)B(\lambda)\mathbf{f}) + \rho(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})' - \int_0^\rho \xi f_2(\xi)d\xi\right|\right|_{\dot{X}_2} \leq \frac{1}{Re\lambda + \frac{1}{2}}\,||\mathbf{f}||_{\mathcal{H}}$$

for the first component of $R_{L_0}(\lambda)$. It follows that

$$\begin{aligned}
&\big|\big|(\lambda-2)(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})\big|\big|_{\dot{X}_2} - \big|\big|\rho(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'\big|\big|_{\dot{X}_2} \leq \\
&\quad \leq \big|\big|(\lambda-2)(T_0^{-1}(\lambda)B(\lambda)\mathbf{f}) + \rho(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'\big|\big|_{\dot{X}_2} \leq \\
&\quad \leq \left|\left|(\lambda-2)(T_0^{-1}(\lambda)B(\lambda)\mathbf{f}) + \rho(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})' - \int_0^\rho \xi f_2(\xi)d\xi\right|\right|_{\dot{X}_2} + \left|\left|\int_0^\rho \xi f_2(\xi)d\xi\right|\right|_{\dot{X}_2} \leq \\
&\quad \leq \left(\frac{1}{Re\lambda + \frac{1}{2}} + 1\right)||\mathbf{f}||_{\mathcal{H}}
\end{aligned}$$

where Lemma 13 was used in the last step. The second component of $R_{L_0}(\lambda)$ and Lemma 15 yield

$$\begin{aligned}
\big|\big|\rho(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'\big|\big|_{\dot{X}_2} &\leq \big|\big|(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'\big|\big|_{\dot{X}_2} \leq \\
&\leq c\big|\big|\rho^{-1}(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})'\big|\big|_{\dot{X}} \leq \frac{c}{Re\lambda + \frac{1}{2}}\,||\mathbf{f}||_{\mathcal{H}}
\end{aligned}$$

with constant $c > 0$.

Finally we get an estimate which will turn out to be very useful

$$\big|\big|(T_0^{-1}(\lambda)B(\lambda)\mathbf{f})\big|\big|_{\dot{X}_2} \leq \frac{1}{|\lambda-2|}\left(\frac{C}{Re\lambda + \frac{1}{2}} + 1\right)||\mathbf{f}||_{\mathcal{H}} \tag{5.6}$$

for $C > 0$.

5.5. The spectrum of the generator for the full system

Adding the perturbation L' we have the following important result (for a proof see Lemma 8).

Proposition 9. $\lambda \in \mathbb{C} : \lambda \in \sigma(L) \setminus \sigma(L_0) \Rightarrow \lambda \in \sigma_p(L)$.

We define $(T(\lambda), \mathcal{D}(T(\lambda))$ by

$$\mathcal{D}(T(\lambda)) := \{u \in H : u \in H^3_{loc}(0,1), t_0(\lambda)u \in \dot{X}_2(0,1), u''(0) = 0\},$$

$$t(\lambda)u(\rho) := -(1-\rho^2)u''(\rho) + \left(2(\lambda-1)\rho + \frac{2}{\rho}\right)u'(\rho) + ((\lambda-1)(\lambda-2) + V(\rho))\,u(\rho)$$

and $T(\lambda)u := t(\lambda)u$.

Proposition 10. The operator $\lambda - L$ for $\lambda \in \mathbb{C}$ is bounded invertible if and only if $T(\lambda)$ is invertible. $\lambda \in \sigma_p(L) \iff \dim\ker T(\lambda) = 1$ and for $\lambda \in \sigma_p(L)$ $\mathbf{u} = (u_1, u_2)^T$ defined by

$$u_1(\rho) = (\lambda-2)u(\rho) + \rho u'(\rho) \qquad u_2(\rho) = \frac{u'(\rho)}{\rho}$$

for $u \in \ker(T(\lambda))$ is an eigenfunction of L.

For a proof see Lemma 21 with obvious modifications.

5.5.1. The eigenvalue equation

For $Re\lambda > -\frac{1}{2}$ the spectrum of L is fully determined by solutions of the eigenvalue equation.

$$u'' - \left(\frac{2}{\rho} + \frac{2\lambda\rho}{(1-\rho^2)}\right)u' - \left(\frac{(\lambda-1)(\lambda-2) + V(\rho)}{(1-\rho^2)}\right)u = 0 \tag{5.7}$$

This equation has already been discussed in the previous chapter.

Note that for $Re\lambda > -\frac{1}{2}$ all solutions that belong to $\mathcal{D}(T(\lambda))$ are analytic solutions (cf. table (4.1)). $\lambda = 1$ is an eigenvalue with the gauge mode u^g (see (4.11)) as the corresponding eigenfunction.
The proof of the next result is analog to the proofs of Lemma 9 and Proposition 6, respectively. The only difference is that now we also consider the case $Re\lambda = \frac{1}{2}$.

Proposition 11. The spectrum of L consists of the single eigenvalue $\lambda = 1$ for $Re\lambda \geq \frac{1}{2}$.

Note that the resolvent set $\rho(L)$ is always an open set in the complex plane (see [19], p. 174). We define two sets $\Sigma(L)$ and $P(L)$ by

$$\Sigma(L) := \{\lambda \in \mathbb{C} : Re\lambda \leq \frac{1}{2} - \epsilon\} \cup \{1\}$$
$$P(L) := \{\lambda \in \mathbb{C} : Re\lambda > \frac{1}{2} - \epsilon, \lambda \neq 1\}$$

for $\epsilon > 0$. Then we know that

$$\sigma(L) \subseteq \Sigma(L)$$
$$P(L) \subseteq \rho(L)$$

5.6. Asymptotic stability

In the third chapter we discussed the gauge mode and showed that we do not consider it as a physical instability due to its origin in the freedom of choosing the blow up time T. $\lambda = 1$ is an isolated eigenvalue according to Proposition 11. Thus, it is possible to find a circle γ in the complex plane with center $\lambda = 1$ and suitable radius, such that the rest of the spectrum lies in the exterior. Then the decomposition theorem ([19] p. 178) holds and the operator L can be decomposed via spectral projection. To this aim one defines the bounded operator

$$P := \frac{1}{2\pi i}\int_\gamma R_L(\lambda)d\lambda. \tag{5.8}$$

P defines a projection on $M = P\mathcal{H}$ along $N = (1-P)\mathcal{H}$ and the Hilbert space can be decomposed according to $\mathcal{H} = M \oplus N$. The parts of the operator L on the closed subspaces N and M are denoted by $(L_N, \mathcal{D}(L) \cap N)$ and $(L_M, \mathcal{D}(L) \cap M)$, respectively.

The spectrum of L_M is given by $\sigma(L_M) = \{1\}$. Since we are not able to determine the spectrum of L completely we have $\Sigma(L_N) := \{\lambda \in \mathbb{C} : Re\lambda \leq \frac{1}{2} - \epsilon\}$ and $P(L_N) := \{\lambda \in \mathbb{C} : Re\lambda > \frac{1}{2} - \epsilon\}$. Again

$$\begin{aligned} \sigma(L_N) &\subseteq \Sigma(L_N) \\ P(L_N) &\subseteq \rho(L_N) \end{aligned}$$

The resolvent of L_N is given by $R_L(\lambda)|_N$. L_N is densely defined and since the operator L is closed, the same is true for L_N (see [19]). It follows that L_N generates a strongly continuous semigroup S_N on the subspace N with

$$||S_N(\tau)|| \leq e^{(-\frac{1}{2}+||L'||)\tau}.$$

We can improve the above growth estimate for the subspace semigroup by using the spectral properties of the generator and applying the formula for the growth bound ω_0 (see A.2) which is given by

$$\omega_0 = \inf\{\kappa > s(L_N) : \sup_{\omega \in \mathbb{R}} ||R_L(\kappa + i\omega)|_N|| < \infty\} \tag{5.9}$$

We know that

$$s(L_N) \leq \frac{1}{2} - \epsilon.$$

For our purpose it is sufficient to consider the above formula for $\kappa > \frac{1}{2} - \epsilon$. This yields a growth bound $\Omega_0 \geq \omega_0$. This, however, does not yield a sharp growth estimate as was discussed in the previous chapter.

Lemma 23. For the growth bound ω_0 of the semigroup $S_N(\tau)$ generated by the operator L_N,

$$\omega_0 \leq \Omega_0 = \frac{1}{2} - \epsilon$$

holds.

Proof. For $\lambda \in \mathbb{C}$ with $Re\lambda > \frac{1}{2} - \epsilon$ and $\lambda \neq 1$ we use the identity

$$\lambda - L = (I - L' R_{L_0}(\lambda))(\lambda - L_0).$$

It follows that

$$R_L(\lambda) = R_{L_0}(\lambda)(I - L' R_{L_0}(\lambda))^{-1}.$$

The resolvent of the generator of the free equation is given in section (5.4.1) and we have

$$L' R_{L_0}(\lambda)\mathbf{f} = \begin{pmatrix} -VT_0(\lambda)^{-1}B(\lambda)\mathbf{f} \\ 0 \end{pmatrix} \tag{5.10}$$

Now estimate (5.6) implies

$$||L' R_{L_0}(\lambda)|| \leq \frac{\sup |V(\rho)|^2}{|\lambda - 2|} \left(\frac{C}{Re\lambda + \frac{1}{2}} + 1 \right) \tag{5.11}$$

with $C > 0$.
For $|Im\lambda| \to \infty$ we observe that $||L' R_{L_0}(\lambda)|| \to 0$. Then $(I - L' R_{L_0}(\lambda)) \to I$ and $(I - L' R_{L_0}(\lambda))^{-1} \to I$. The claim then follows from the boundedness of $||R_{L_0}(\lambda)||$ for $Re\lambda > -\frac{1}{2}$ (cf. section (5.4.1)).

□

Now we are ready to prove the main theorem.

Proof (of Theorem 6). We formulate the Cauchy problem for equation (5.1) in CSS-coordinates on the subset $N \subset \mathcal{H}$.

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= L_N \Phi(\tau) \\ \Phi(\tau_0) &= \Phi_0 \end{aligned}$$

with $\Phi : [\tau_0, \infty) \to N \subset \mathcal{H}$, $\Phi_0 \in N$ and $\tau_0 := -\log T$.
Then the solution is given by

$$\Phi(\tau) = S_N(\tau - \tau_0)\Phi(\tau_0).$$

It was shown that the energy $E^{loc}(\tau)$ of the perturbation field in the backward lightcone of the blow-up point $(T, 0)$ is given by the norm of $\Phi(\tau, \cdot)$ in the energy space. Using the embedding given in Lemma 14 we get

$$E^{loc}(\tau) \leq e^{-\tau} \left|\left|\Phi(\tau)\right|\right|^2_{\mathcal{H}^E} \lesssim e^{-\tau} \left|\left|\Phi(\tau)\right|\right|^2_{\mathcal{H}}$$

The above derived growth bound for the semigroup $S_N(\tau)$ yields

$$\left|\left|\Phi(\tau)\right|\right|_{\mathcal{H}} = \left|\left|S_N(\tau - \tau_0)\Phi(\tau_0)\right|\right|_{\mathcal{H}} \lesssim e^{(\omega_0 + \frac{\epsilon}{2})\tau} \left|\left|\Phi(\tau_0)\right|\right|_{\mathcal{H}} \lesssim e^{(\frac{1}{2} - \epsilon + \frac{\epsilon}{2})\tau} = e^{\frac{1}{2}(1-\epsilon)\tau}$$

and finally

$$E^{loc}(\tau) \lesssim e^{-\epsilon\tau}. \tag{5.12}$$

The energy of the perturbation field converges to zero as $\tau \to \infty$ and we conclude that the ground state solution is asymptotically stable.

□

6. Discussion and outlook

We studied the linearization of the wave maps equation around the ground state solution f_0 and considered an operator formulation in CSS-coordinates in two different function spaces. Applying semigroup theory we showed that the equation is well-posed. In regard to asymptotic stability we have seen that in the energy space formulation the spectral properties of the generator of the free equation spoil the result. By formulating the problem in a weighted Sobolev space, where first derivatives appear in the norm, it has been demonstrated that the spectrum of the generator can be shifted towards the left in the complex plane, such that the spectral bound is negative. Applying the new results for solutions of the eigenvalue equation we were able to show that the generator of the full system does not have eigenvalues for $\mathrm{Re}\lambda \geq \frac{1}{2}$ except for the gauge mode. We removed this instability by spectral projection and derived a growth estimate for the semigroup in the higher energy space. Continuous embedding finally implied that the physical energy of the perturbation field decreases in the backward lightcone of the blow-up point. This shows that the ground state solution is asymptotically stable.

Although this is what we wanted to prove, the result is not optimal. It is known (see [7]) that the energy of the ground state solution decreases linearly in the backward lightcone with $T - t$. For the energy of the perturbation field we get a decay

$$E^{loc}(t) \lesssim (T - t)^{\epsilon}$$

for $\epsilon \ll 1$. This means that the energy of f_0 decreases faster for $t \to T$ than the energy of the perturbation. Actually, it would be desirable to show the converse. This, however hinges on the spectral bound of the operator L_N. If we could prove that the eigenvalue equation has no solutions for $Re\lambda \geq 0$, then this would imply that the perturbation field decays with $(T - t)^{1+\epsilon}$, hence faster than then ground state solution.

The analysis presented in this work can easily be extended to more general cases. Co-rotational wave maps are a special case of *equivariant wave maps* (see [32]), for which the wave maps equation is given by

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{m(m+1)}{2}\frac{\sin(2\psi)}{r^2} = 0. \tag{6.1}$$

The number $m \in \mathbb{N}$ is called the *equivariance index* and in the co-rotational case we have $m = 1$. It was shown by Bizon in [6] that for each m there exists a countable family of self-similar solutions $f_{n,m}$ with analogous properties for all m. Thus, the analysis of linear stability for the ground state solution in the co-rotational case can be generalized to the investigation of the linearization of eq. (6.1) around $f_{0,m}$. The resulting equation can be written as

$$\tilde{\psi}_{tt} - \tilde{\psi}_{rr} - \frac{2}{r}\tilde{\psi}_r + \frac{m(m+1)}{r^2}\tilde{\psi} + V_m\tilde{\psi} = 0$$

where $\tilde{\psi}$ denotes the perturbation and the (bounded) potential is given by

$$V_m(t,r) := \frac{m(m+1)\cos\left(2f_{0,m}(\frac{r}{T-t})\right) - m(m+1)}{r^2}.$$

With a formulation of this problem in a generalized (higher) energy space and sufficient knowledge of solutions of the corresponding eigenvalue equation it should be possible to derive an analogous result.

A. Mathematics

A.1. Operator theory

In this section we collect the basic definitions and results from operator theory, which are used in this work. The proofs of the results can be found for example in [31] or [40].

Definition (Linear operators on Banach spaces). Let X and Y be Banach spaces. A linear operator $(A, \mathcal{D}(A))$ is a linear transformation A defined on its domain $\mathcal{D}(A) \subset X$ with

$$A : \mathcal{D}(A) \to Y.$$

The range of $(A, \mathcal{D}(A))$ is a subspace $\mathrm{rg}(A) \subset Y$ defined by

$$\mathrm{rg}(A) := \{y \in Y : y = Ax, \text{ for some } x \in \mathcal{D}(A)\}.$$

The subspace $\ker(A) \subset X$ is given by

$$\ker(A) := \{x \in X : Ax = 0\}.$$

An operator is called *densely defined* if its domain is a dense subset of X, i.e. for every $x \in X$ there exists a sequence $(x_n) \in \mathcal{D}(A)$ such that $x_n \to x$.

Definition (Extension). An operator $(\tilde{A}, \mathcal{D}(\tilde{A}))$ is called an extension of $(A, \mathcal{D}(A))$ if $\mathcal{D}(A) \subseteq \mathcal{D}(\tilde{A})$ and

$$Ax = \tilde{A}x$$

for every $x \in \mathcal{D}(A)$.

Definition (Bounded operators). A linear operator $(A, \mathcal{D}(A))$ is *bounded* if there exists a constant C such that

$$||Ax||_Y \leq C\,||x||_X$$

for every $x \in \mathcal{D}(A)$. The operator is *unbounded* if no such constant exists. The *operator norm* is the smallest C for which the above inequality holds, i.e.

$$||A|| := \sup_{x \in \mathcal{D}(A), ||x||_X \neq 0} \frac{||Ax||_Y}{||x||_X}.$$

Theorem (Domain of bounded operators). If X is a Hilbert space then any bounded operator $(A, \mathcal{D}(A))$ can be extended such that $\mathcal{D}(\tilde{A}) = X$ without changing its norm. Therefore, the domain of a bounded operator is in most cases assumed to be the entire Hilbert space X.

Definition (Closed operators, closure). A linear operator $(A, \mathcal{D}(A))$ is closed if for every sequence $(x_n) \in \mathcal{D}(A)$ with

$$x_n \to x \quad \text{and} \quad Ax_n \to y$$

it follows that

$$x \in \mathcal{D}(A) \quad \text{and} \quad Ax = y.$$

An operator is said to be *closable* if for every sequence $(x_n) \in \mathcal{D}(A)$ such that $x_n \to 0$, either

$$Ax_n \to 0 \quad \text{or} \quad \lim_{n\to\infty} Ax_n \text{ does not exist.}$$

The smallest closed extension of a closable operator is called *the closure*.

Definition (**Resolvent and spectrum of closed operators**)**.** The set

$$\rho(A) := \{\lambda \in \mathbb{C} \text{ for which } \lambda - A : \mathcal{D}(A) \to X \text{ is bijective}\}$$

is called the *resolvent set* of the operator $(A, \mathcal{D}(A))$ and for $\lambda \in \rho(A)$ the *resolvent* is given by

$$R_A(\lambda) := (\lambda - A)^{-1}.$$

The *spectrum* of the operator $(A, \mathcal{D}(A))$ is defined as the complement of the resolvent set,

$$\sigma(A) := \mathbb{C} \backslash \rho(A).$$

One distinguishes between

- the *point spectrum*

$$\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective}\},$$

- the *continuous spectrum*

$$\sigma_c(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ injective, not surjective and } \operatorname{rg}(\lambda - A) \text{ dense}\},$$

- and the *residual spectrum*

$$\sigma_r(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ injective }, \operatorname{rg}(\lambda - A) \text{ not dense}\}.$$

Each $\lambda \in \sigma_p(A)$ is called an *eigenvalue* and each non-trivial $u \in \mathcal{D}(A)$ satisfying $(\lambda - A)u = 0$ is an *eigenvector* of A. By the *approximate point spectrum* one understands $\sigma_p(A) \cup \sigma_c(A)$.

Definition (**Compact operators**)**.** An operator $(A, \mathcal{D}(A))$ is compact if given any bounded sequence $(x_n) \in \mathcal{D}(A)$ it follows that $A(x_n)$ has a convergent subsequence. It can be shown that every compact operator is bounded.

Theorem (**Spectrum of compact operators**)**.** Consider a compact operator on a Hilbert space $A : H \to H$. In regard to its spectrum the following properties, known as the *Fredholm alternative theorem*, hold:

- $\sigma(A)$ is a compact set having no limit point except perhaps $\lambda = 0$,
- for $\lambda \in \sigma(A)\backslash\{0\}$, either
  $\lambda \in \rho(A)$, or
  $\lambda \in \sigma_p(A)$ is an eigenvalue of finite multiplicity.

Definition (**Integral operators**). Let $L^2(\Omega)$ denote the space of square integrable functions on Ω and consider an integral operator $K : L^2(\Omega) \to L^2(\Omega)$ defined by

$$Ku := \int_\Omega k(\mathbf{x}, \mathbf{y})u(\mathbf{y})d\mathbf{y}$$

where $k : \Omega \times \Omega \to \mathbb{C}$ is called the *kernel*. If

$$\int_\Omega \int_\Omega |k(\mathbf{x}, \mathbf{y})|^2 d\mathbf{x}d\mathbf{y} < \infty$$

holds, the kernel is called *Hilbert-Schmidt*.

Theorem (**Compactness of integral operators**). Let $k : \Omega \times \Omega \to \mathbb{R}$ be Hilbert-Schmidt. Then the above defined integral operator K is compact and in particular bounded.

Definition (**Fréchet derivative**). Let X and Y be Banach spaces, $U \subset X$ an open subset and $F : U \to Y$ a non-linear mapping. F is called Fréchet-differentiable at $x_0 \in U$ if there exists a bounded linear operator $T : X \to Y$ such that

$$\lim_{||h||_X \to 0} \frac{||F(x_0 + h) - Fx_0 - Th||_Y}{||h||_X} = 0$$

By the limit we mean that given an $\epsilon > 0$ there exists a $\delta > 0$ such that for $||h||_X < \delta$

$$||F(x_0 + h) - Fx_0 - Th||_Y \le \epsilon \, ||h||_X \, .$$

$DF(x_0) := T$ is called the Fréchet derivative of F at x_0.

A.2. Strongly continuous one-parameter semigroups

In this section we discuss the basic ideas from the theory of *strongly continuous one-parameter semigroups of bounded linear operators*, also called C_0 *-semigroups*, and the application to partial differential equations. For a detailed analysis and the proofs of the results, which will be presented here, we refer to textbooks such as [15] or [29].

As a motivation consider a linear ordinary differential equation with constant coefficients of the form

$$\begin{aligned} y'(t) &= Au(t) \\ y(0) &= y_0 \end{aligned}$$

where A is $(n \times n)$-matrix on $\mathbb{R}^n$ (or $\mathbb{C}^n$, respectively) and y_0 is the initial value. The solution is given by

$$y(t) = e^{At} y_0.$$

The matrix exponential is well-defined by

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

Linear partial differential equations can also be written in such a form, but then A is usually an unbounded linear operator on an infinite dimensional function space. To make this more precise we consider the *abstract Cauchy problem* on a Banach space X.

$$\frac{d}{dt} u(t) = Au(t) \tag{A.1}$$

with initial value $u(0) = u_0 \in X$, $u : [0, \infty) \to X$ a Banach space valued function and $A : \mathcal{D}(A) \subset X \to X$ a linear operator.

The core of the theory of one-parameter semigroups is to give a precise meaning to the intuitive notion of the solution "$u(t) = e^{At} u_0$" for the above stated problem. This is a subtle matter, especially when the operator A is unbounded, as will be assumed in the following.

Semigroups and generators

Definition (**C_0-semigroup**). Let $S(t)$ be family of bounded linear operators on a Banach space X depending on one parameter $t \in \mathbb{R}, t \geq 0$. Then $S(t)$ is called a strongly-continuous one-parameter semigroup iff

$$\begin{aligned} S(0) &= I, \\ S(t+s) &= S(t)S(s), \end{aligned}$$

and

$$||S(t)x_0 - x_0||_X \to 0$$

for all $x_0 \in X$ and $t \to 0$.

Definition (**Generators of a semigroups**). The infinitesimal generator of a C_0-semigroup $S(t)$ is an operator $(A, \mathcal{D}(A))$ on a Banach space X defined by

$$\begin{aligned} Ax &:= \lim_{t \to 0+} \frac{S(t)x - x}{t}, \\ D(A) &:= \{x \in X : \lim_{t \to 0+} \frac{S(t)x - x}{t} \text{ exists}\}. \end{aligned}$$

Theorem (**Properties of the generator**). For the generator $(A, \mathcal{D}(A))$ of a strongly continuous one-parameter semigroup $S(t)$, $t \geq 0$ the following properties hold:

- $A : \mathcal{D}(A) \subset X \to X$ is a linear operator,
- if $u \in \mathcal{D}(A)$, then $S(t)u \in \mathcal{D}(A)$ and

  $$\frac{d}{dt} S(t)u = S(t)Au = AS(t)u$$

  for all $t \geq 0$.

Theorem (**Growth estimates**). For every C_0-semigroup $S(t)$ there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$||S(t)|| \leq Me^{\omega t}, \forall t \geq 0$$

We have seen that given a C_0-semigroup one can find an infinitesimal generator. For application to partial differential equations the following question is even more interesting: Given an unbounded, linear operator $(A, \mathcal{D}(A))$ on a Banach space X. Is it possible to find a C_0-semigroup with A as infinitesimal generator?
In the literature there are various theorems to answer this question, where the most general one is the Hille-Yosida generation theorem. Here, another well-known theorem should be cited which does apply for semigroups on Hilbert spaces.

Theorem (Lumer-Phillips Theorem). Let H be a Hilbert space and let $(A, \mathcal{D}(A))$ be a linear operator satisfying the following conditions:

1. A is densely defined.
2. $Re(Au|u) \leq \omega(u|u)$ for every $u \in \mathcal{D}(A)$.
3. $\mathrm{rg}(\lambda - A)$ is dense in H for some $\lambda > \omega$.

Then it follows that A is closable and the closure of A generates a strongly continuous one-parameter semigroup with

$$||S(t)|| \leq e^{\omega t}$$

for all $t \geq 0$.

Another useful result concerns bounded perturbations of infinitesimal generators.

Theorem (Bounded Perturbation Theorem). Let $(A, \mathcal{D}(A))$ be the generator of a C_0-semigroup $S_0(t)$ satisfying

$$||S_0(t)|| \leq e^{\omega t}$$

for all $t \geq 0$. If B is a bounded operator then

$$C := A + B \text{ with } \mathcal{D}(C) = \mathcal{D}(A)$$

generates a C_0-semigroup $S(t)$ with

$$||S(t)|| \leq e^{(\omega + ||B||)t}$$

for all $t \geq 0$.

Properties of a semigroup are related to spectral properties of its generator.

Definition (**Spectral bound**). The spectral bound of a closed linear operator $(A, \mathcal{D}(A))$ on a Banach space X is defined as

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\}.$$

Theorem (**Spectral properties and resolvent estimates**). In regard to the resolvent of the generator $(A, \mathcal{D}(A))$ the following properties hold:

- If $Re\lambda > \omega$, then λ is an element of the resolvent set $\rho(A)$,
- $||R_A(\lambda)|| \leq \frac{M}{Re\lambda - \omega}$.

Definition (**Growth Bound**). For a strongly continuous, one parameter semigroup the growth bound is defined as

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \text{ there } \exists M_\omega \geq 1 \text{ such that } ||S(t)|| \leq M_\omega e^{\omega t}\}.$$

For the spectral bound of the generator and the growth bound of the semigroup we have

$$-\infty \leq s(A) \leq \omega_0 < \infty$$

Theorem (**Gearhart-Prüss**). According to [28] (or see [15]) the growth bound can be calculated by

$$\omega_0 = \inf\{\kappa > s(A) : \sup_{\omega \in \mathbb{R}} ||R(\kappa + i\omega)|| < \infty\}.$$

Well-Posedness of evolution equations

Now we turn back to the abstract Cauchy problem (A.1) and define what we mean by solutions.

Definition (**Classical solution**). Assume that $u_0 \in \mathcal{D}(A)$. A function $u : [0, \infty) \to X$ is a classical solution of (A.1) if u is continuously differentiable with respect to X and $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$.

By well-posedness of the Cauchy problem we mean that there exists a unique solution, which depends continuously on the initial data. With the above stated properties of the generator the following result can be derived.

Theorem (**Well-posedness**). Let X be a Banach space and let $(A, \mathcal{D}(A)$ be the generator of a strongly continuous one-parameter semigroup $S(t)$, $t \geq 0$ with $||S(t)|| \leq Me^{\omega t}$. Then, for every $u_0 \in \mathcal{D}(A))$, the function

$$u : t \mapsto u(t) := S(t)u_0$$

is the unique classical solution of (A.1) and

$$||u(t)||_X = ||S(t)u_0||_X \leq Me^{\omega t}\, ||u_0||_X$$

Furthermore, for every sequence $(u_0^n) \subset \mathcal{D}(A)$ with $n \in \mathbb{N}$ satisfying $\lim_{n\to\infty} u_0^n = 0$ one has $\lim_{n\to\infty} u(t, u_0^n) = 0$ uniformly in compact intervals $[0, t_0]$.

A.3. Notation and conventions

$\mathbb{N}$	The natural numbers $\{1, 2, ...\}$.
$\mathbb{Z}$	The integer numbers $\{..., -2, -1, 0, 1, 2, ...\}$.
$\mathbb{R}$; $\mathbb{C}$; $\mathbb{K}$	The real numbers; the complex numbers; $\mathbb{R}$ or $\mathbb{C}$.
Rez; Imz	The real part of $z \in \mathbb{C}$; the imaginary part of $z \in \mathbb{C}$.
$\overline{z}$	The complex conjugate of $z \in \mathbb{C}$.
$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$.
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$.
$[a, b)$	$\{x \in \mathbb{R} : a \leq x < b\}$.
$(a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$.
$(\cdot\|\cdot)_X$; $\|\|\cdot\|\|_X$; $\|\|\cdot\|\|$	Scalar product in X; norm in X; operator norm (see A.1).
$\lesssim$	$A \lesssim B \Leftrightarrow A \leq cB$ with $c = const.$
$\text{supp}(f)$	Support of a function $f : \Omega \to \mathbb{K}$ defined as $\text{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}$.
$C[0, 1]$	$\{f : [0, 1] \to \mathbb{C} : f \text{ is continuous}\}$.
$C^k(0, 1)$	$\{f : (0, 1) \to \mathbb{C} : f \text{ is k-times continuously differentiable}\}$.
$C^\infty(0, 1)$	The set of smooth functions defined as $\{f : (0, 1) \to \mathbb{C} : f \text{ has continuous derivatives of all orders}\}$.
$C_c^\infty(0, 1)$	The set of smooth functions f with $\text{supp}(f) \subset (0, 1)$ compact.
$C_{\{0\}}^\infty(0, 1)$	The set of smooth functions with $\text{supp}(f) \cap \{0\} = \emptyset$.
$L^2(0, 1)$	The space of square-integrable functions f with norm $\|\|f\|\|_{L^2} := \left(\int_0^1 \|f(x)\|^2 dx\right)^{1/2}$.
$L^2((0, 1), \omega(x)dx)$	Weighted L^2-space with norm $\|\|f\|\|_{L^2_\omega} := \left(\int_0^1 \omega(x)\|f(x)\|^2 dx\right)^{1/2}$ and weight function $\omega : (0, 1) \to \mathbb{R}$ continuous and positive.
$H^k(0, 1)$	The set of all functions with k weak derivatives in $L^2(0, 1)$, $\|\|f\|\|_{H^k} := \left(\sum_{n=0}^{k} \int_0^1 \|f^{(n)}(x)\|^2 dx\right)^{1/2}$.
$L^2_{loc}(0, 1)$	The set of locally square integrable functions defined as $\{f : (0, 1) \to \mathbb{C} : f \in L^2(0, 1) \text{ for all } [a, b] \subset (0, 1)\}$.
$H^k_{loc}(0, 1)$	The set of all functions with k weak derivatives in $L^2_{loc}(0, 1)$.

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