DISSERTATION

Faithful representations
of minimal degree
for Lie algebras
with an abelian radical

Verfasser
Wolfgang Alexander Moens

angestrebter akademischer Grad
Doktor der Naturwissenschaften (Dr. rer. nat.)

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To all those who are dear to me.

Preface

The protagonists in this thesis are the representations of Lie algebras. The interesting ones among them are faithful and have a finite degree. That there even exist such representations is not at all obvious. It is a standard and non-trivial result in the theory of Lie algebras that every finite-dimensional Lie algebra admits a faithful representation of a finite degree. This existence-result is classically known as the theorem of Ado and Iwasawa.

The characterisation of these representations is a hopeless problem. By this, we mean the classification of the representations of a generic Lie algebra up to conjugation. This can already be seen in the case of the abelian Lie algebras. But not all is lost: many of the proofs of this Ado-Iwasawa theorem can be modified to give an explicit construction of such representations. The main disadvantage, however, is that these representations have a degree that is very large with respect to the dimension of the original Lie algebra. This is in stark contrast to the fact that it is very difficult to show that there are Lie algebras with only large faithful representations.

Precisely these algebras turned out to be relevant in the context of a well-known conjecture by J. Milnor. Formulated algebraically, he asked if every solvable Lie algebra $\mathfrak{r}$ admits a left-symmetric Lie algebra structure and hence a faithful representation of degree $\dim(\mathfrak{r})+1$. Any solvable Lie algebra $\mathfrak{r}$ that has only large faithful representations, i.e. representations of degree at least $\dim(\mathfrak{r}) + 2$, are counterexamples to Milnor's conjecture. Such examples were produced by Benoist and independently by Burde and Grunewald. At this point, the following invariant was introduced.

> **Definition** Let $\mathfrak{g}$ be a Lie algebra. Then we define $\mu(\mathfrak{g})$ to be the minimal degree of a faithful linear $\mathfrak{g}$-representation.

Ado's theorem then states that the μ-invariant of a Lie algebra is a natural number and Milnor's conjecture claims (incorrectly) that $\mu(\mathfrak{r}) \leq \dim(\mathfrak{r}) + 1$ for all solvable Lie algebras $\mathfrak{r}$.

What can be said about this invariant? Its history is closely related to that of the Ado-Iwasawa theorem, and there are typically four kinds of results. **(i)** Bounds were obtained for the μ-invariant as a function of other natural Lie algebra invariants such as the dimension (and the solvability or nilpotency class when applicable). **(ii)** Stronger versions of the theorem were obtained in terms of nilpotency conditions. **(iii)** The μ-invariant was computed explicitly for certain families such as the family of the generalised Heisenberg Lie algebras. And **(iv)** Lie algebras with a large μ-invariant were constructed.

This thesis aims to refine two of these four points.

In chapter one, we give a proof for Ado's theorem which will give us an upper bound for the μ-invariant as a function of only the dimension. We then introduce the μ-invariant and its elementary properties. A brief discussion of the different versions of Ado's theorem naturally leads to other invariants related to the original μ-invariant. For example: the μ-invariant is to Ado's theorem, as the μ_0-invariant is to a theorem by Block. For the Lie algebras that we will consider, the μ_0-invariant will have the advantage that it can be computed much more easily than the μ-invariant. We briefly touch the solvable and nilpotent case and interpret the theorems of Engel and Lie in this context. Finally, we illustrate that the notion of a μ-invariant also makes sense in other algebraic settings: there is one for color Lie algebras, generalised Lie algebras, groups and so on.

Chapter two is devoted to the reductive Lie algebras and their μ-invariants, cf. (iii). As any such Lie algebra decomposes into a semisimple and an abelian Lie algebra, the chapter is roughly divided into three parts. First, we illustrate how a theorem by Schur was reinterpreted in order to compute the μ-invariant for all abelian Lie algebras. Next the simple and semisimple Lie algebras are discussed. Just as every semisimple Lie algebra is the sum of its simple ideals, the μ-invariant of a simple Lie algebra turns out to be the sum of the μ-invariants of its simple ideals. Here, and in contrast to the general case, the faithful representations of minimal degree can be classified quite easily. In the third part, we compute the centraliser of a semisimple Lie algebra in a general linear Lie algebra. After this, it is not difficult to express the μ-invariant of a reductive Lie algebra in terms of that of a naturally associated semisimple and abelian ideal. We also obtain formulas for Lie algebras that have a decomposition very much like that of a reductive Lie algebra. Finally, we discuss how this is related to the classification of maximal reductive subalgebras.

In the third chapter we consider the class of all Lie algebras with an abelian radical. This class naturally contains the reductive Lie algebras. We illustrate how the μ_0-invariant, and hence an upper bound for μ, can be computed algorithmically (cf. (iii) resp. (i)). In order to do this, we generalise a construction for affine Lie algebras by introducing the so-called paired representations. These can be interpreted as a tensor product of two Lie algebra representations. The algorithm depends on the decomposition of tensor products into irreducible subrepresentations. We also obtain combinatorical bounds and bounds in terms of only the dimension. Finally, we apply these constructions to some examples in the family $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathbb{C}^t$. Additionally, we express the μ-invariant of a Lie algebra with a filiform radical in terms of the μ-invariant of its radical and Levi-complement.

We conclude with several remarks. There are other methods to construct representations: extensions of Lie algebras and holomorphs are two of them. The class of characteristically nilpotent Lie algebras, the family of finitely generated nilpotent Lie algebras and a certain family of filiform Lie algebras seem to be very interesting from the point of view of our invariants. We formulate some of the most obvious open problems in this theory (the growth of the invariant and the role played by the characteristic of the field) and include some extra material at the end.

Contents

Chapter 1

The theorem of Ado and Iwasawa

1.1 The theorem of Ado and Iwasawa

What does a Lie algebra look like? Lie algebras can be approached in several different ways. One way to present a Lie algebra, is to give a set of generators and a set of commutation relations. In a similar way, one can encode the Lie algebra as a set of structure constants with respect to a certain basis. Most well-known Lie algebras, however, occur naturally as matrix Lie algebras. We can for example think of the classical Lie algebras $\mathfrak{sl}_n, \mathfrak{so}_n$ and $\mathfrak{sp}_n$. One is tempted to ask the following question:

"Can every Lie algebra be realised as a matrix algebra?"

Or more precisely: "does every Lie algebra admit a faithful linear representation?". The answer is a definite "yes" since any Lie algebra acts naturally and faithfully on its universal enveloping algebra. Whether every Lie algebra admits a faithful representation of a *finite* degree is much less obvious: the universal enveloping algebra does not have a finite dimension. Other obvious examples of representations, such as the adjoint representation, need not be faithful. Fortunately, the answer to this question too, turns out to be pleasantly positive. In 1935, the Russian mathematician I. D. Ado published the paper *"The representation of Lie algebras by matrices"* [Ado] in which he formulated and proved the following result.

1.1 The theorem of Ado and Iwasawa

Theorem (Ado). Every finite-dimensional Lie algebra over an algebraically closed field of characteristic zero has a faithful linear representation of finite degree.

This illustrates that the matrix Lie algebras are not some very special examples of Lie algebras: every Lie algebra can be re-written as a Lie algebra of (finite square) matrices. Other mathematicians later showed that we need not assume the algebraic closedness of the field. In 1947, K. Iwasawa also removed the condition on the characteristic of the base field ([Iwas]). The theorem in prime characteristic is generally attributed to Iwasawa, while the theorem in characteristic zero is usually called Ado's theorem.

Theorem (Ado-Iwasawa). Every finite-dimensional Lie algebra has a faithful linear representation of finite degree.

1.1.1 Chronology

In the course of history, many respectable mathematicians have given a proof for the Ado-Iwasawa theorem or some variant of it. Ado's original proof of 1935 was quite involved and he presented an improved version in 1947. In 1937, Birkhoff gave a proof of the theorem for the class of all nilpotent Lie algebras. He introduced an associative algebra with unit, associated to a given Lie algebra [Bir]. In modern terminology, this is the *universal enveloping algebra* (also independently constructed by Witt and Artin). By taking the appropriate quotient in this algebra, Birkhoff constructed a faithful linear representation. In fact, he noted that the matrices in the representation are properly triangular.

A rigorous proof for Ado's theorem (for real and complex numbers) was given by Cartan in 1938. He used the more analytic theory of Lie groups. About a decade later, Iwasawa [Iwas] completely removed any restriction on the base field. He wanted to construct a proof that would work for both kinds of characteristics (and he succeeded). Inspired by abelian extensions in group theory, he first constructed a "universal splitting algebra". Then, he proved that there are finite-dimensional splitting Lie algebras in both kinds of characteristic. The theorems of Poincare-Birkhoff-Witt and Ado-Iwasawa then follow immediately.

About ten years later, Harish-Chandra [HC] gave another algebraic proof and sharpened the result slightly: "Every Lie algebra admits a faithful representation such that the elements of the maximal nilpotent ideal are mapped to nilpotent operators". In [Hoc], Hochschild proved that there is always a faithful representation such that all ad-nilpotent elements are mapped to nilpotent operators. In 1969, Reed [Ree] tried to find out how small the degrees of the faithful representations can be chosen in the nilpotent and solvable case. (See *subsection* 1.3.1). A few other contributors that should definitely be mentioned, are: Jacobson [Jac3], Block [Blo], de Graaf [deG] and Neretin [Ner].

1.1.2 A proof for Ado's theorem

Is it possible to present a constructive, compact and transparent proof for Ado's theorem? Let us try to do just that. The pioneering works that were mentioned above are neither known for their compactness nor for their transparency. We give a proof that is based on one given by Neretin [Ner] and Burde [Bur1] and many ideas go back to Birkhoff [Bir]. We will also use notation, results and terminology that will only be introduced later on. Consider a finite-dimensional Lie algebra $\mathfrak{g}$ over the complex numbers. We want to construct a $\mathfrak{g}$-module that is faithful and of finite dimension. We do this in two steps. First, we will use Neretin's embedding theorem. This will reduce the problem to the class of all nilpotent Lie algebras. Next, we use the enveloping algebra to construct faithful modules of finite dimension for these nilpotent Lie algebras. It then suffices to bring both results together.

Theorem [Neretin] The Lie algebra $\mathfrak{g}$ can be embedded into a semidirect product of the reductive Lie algebra $\frac{\mathfrak{g}}{\mathrm{nil}(\mathfrak{g})}$ and a nilpotent ideal of dimension $\dim(\mathrm{rad}(\mathfrak{g}))$.

Proof*:* <u>Step 0.</u> We first introduce the so-called elementary expansions. The theorem can then be obtained by applying successive elementary extensions. Let $\mathfrak{g}$ be any complex Lie algebra of finite dimension. Suppose it has an ideal I of co-dimension one. Let $\langle x\rangle$ be a complementary subspace. Consider the vector space

$$E(I,x) = \mathbb{C}y + \mathbb{C}z + I,$$

where y and z are formal vectors. The restriction of ad_x to the ideal I is a derivation d of this ideal. Let Let $d = d_s + d_n$ be a Jordan-Chevalley decomposition for d. Then $E(I, x)$ is a Lie algebra for the brackets

$$[y, z] = 0 \ , \ [y, u] = d_s(u) \ , \ [z, u] = d_n(u),$$

for all u in I and with the normal bracket on I. The Lie algebra $E(I, x)$ properly contains an isomorphic copy of $\mathfrak{g}$: $\mathbb{C}(y+z)+I$. We call it the *elementary expansion* of $\mathfrak{g}$ with respect to I and x. Note that the commutator of the expansion coincides with the commutator of the original Lie algebra.

<u>Step 1.</u> Fix a complex, finite-dimensional Lie algebra $\mathfrak{g}$. Let $\mathfrak{s}$ be a Levi-complement for $\mathfrak{g}$. Consider all triples $(\mathfrak{g}, \mathfrak{p}, \mathfrak{n})$ satisfying the following properties:

> (C_1) $\mathfrak{p} \cap \mathfrak{n} = 0$. (C_2) $\mathfrak{n}$ *is a nilpotent ideal of* $\mathfrak{g}$ *and it contains the ideal* $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$. (C_3) $\mathfrak{p}$ *is a reductive subalgebra of* $\mathfrak{g}$ *and it contains the Levi-complement* $\mathfrak{s}$ *of* $\mathfrak{g}$. (C_4) $\mathfrak{p}$ *acts completely reducibly on* $\mathfrak{g}$.

In this first part of the proof we will show that any triple $(\mathfrak{g}, \mathfrak{p}, \mathfrak{n})$ satisfying the conditions (1), (2), (3) and (4) can be used to define another triple $(\mathfrak{g}', \mathfrak{p}', \mathfrak{n}')$ satisfying the same conditions. Here, we assume that $\mathfrak{p} \ltimes \mathfrak{n}$ is a proper subalgebra of $\mathfrak{g}$.

Consider such a triple $(\mathfrak{g}, \mathfrak{p}, \mathfrak{n})$. Since $\mathfrak{p} \ltimes \mathfrak{n}$ is a proper subalgebra, there exists a subspace I of $\mathfrak{g}$ that contains $\mathfrak{p} \ltimes \mathfrak{n}$ and is of co-dimension one. Since $\mathfrak{p} \ltimes \mathfrak{n}$ contains $\mathfrak{s} \ltimes [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = [\mathfrak{g}, \mathfrak{g}]$, this I must be an ideal of $\mathfrak{g}$. It is hence also invariant under the action of $\mathfrak{p}$. Since the action of $\mathfrak{p}$ on $\mathfrak{g}$ is fully reducible, there exists a complementary subspace, necessarily of dimension one. Let it be generated by an element x: $\mathfrak{g} = \mathbb{C}x + I$ (as $\mathfrak{p}$-modules). Consider the elementary expansion $\mathfrak{g}' = E(I, x) = \mathbb{C}y + \mathbb{C}z + I$ and

$$\begin{aligned} \mathfrak{p}' &= \mathbb{C}y + \mathfrak{p} \\ \mathfrak{n}' &= \mathbb{C}z + \mathfrak{n}. \end{aligned}$$

We claim that the triple $(\mathfrak{g}', \mathfrak{p}', \mathfrak{n}')$ satisfies all four conditions. The spaces $\mathfrak{p}'$ and $\mathfrak{n}'$ have no intersection by construction. Since I contains all commutators, $\mathbb{C}x$ must be the trivial module and x commutes with $\mathfrak{p}$. The Jordan-Chevalley-decomposition $x = y+z$ then implies that also y commutes with $\mathfrak{p}$ so that $\mathfrak{p}' = \mathbb{C}y \oplus \mathfrak{p}$ and $\mathfrak{p}'$ is reductive. Since y is a semisimple operator and $\mathfrak{p}$ acts fully reducibly on $\mathfrak{g}$, the action of $\mathfrak{p}'$ on $\mathfrak{q}$ and hence on $\mathfrak{q}'$ will also be fully reducible. Since the commutator of an expansion is not bigger than the commutator of the original algebra, we have $\mathfrak{s}' = \mathfrak{s}$ and $[\mathfrak{g}', \mathrm{rad}(\mathfrak{g}')] = [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$. So $\mathfrak{n}' \geq [\mathfrak{g}', \mathrm{rad}(\mathfrak{g}')]$ and $\mathfrak{p}' \geq \mathfrak{s}'$. Since the semidirect product of a derivation of a nilpotent Lie algebra with that Lie algebra is nilpotent, $\mathfrak{n}'$ will be nilpotent.

Note that $\mathfrak{g}$ is proper subalgebra of $\mathfrak{g}'$ of co-dimension one. Similarly, $\mathfrak{n}$ and $\mathfrak{p}$ are subalgebras of $\mathfrak{p}'$ resp. $\mathfrak{n}'$ of co-dimension one. We thus have,

$$\dim(\frac{\mathfrak{g}'}{\mathfrak{p}' \ltimes \mathfrak{n}'}) = \dim(\frac{\mathfrak{g}}{\mathfrak{p} \ltimes \mathfrak{n}}) - 1.$$

<u>Step 2.</u> We can now prove the theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the complex numbers. We need to construct an embedding $\iota : \mathfrak{g} \longrightarrow \mathfrak{p} \ltimes \mathfrak{n}$ where $\mathfrak{p}$ is reductive and $\mathfrak{n}$ nilpotent. Consider the subalgebra $\mathfrak{p}_0 \ltimes \mathfrak{n}_0 = \mathfrak{s} \ltimes \mathrm{nil}(\mathfrak{g})$. If it is not a proper subalgebra, the identity $\mathbb{1} : \mathfrak{g} \longrightarrow \mathfrak{g}$ gives us the desired embedding. We may thus assume that the subalgebra of $\mathfrak{g} = \mathfrak{g}_0$ is proper.

Since the triple $(\mathfrak{g}, \mathfrak{s}, \mathrm{nil}(\mathfrak{g}))$ satisfies conditions one to four, we may apply the above construction to the data $(\mathfrak{g}_0, \mathfrak{p}_0, \mathfrak{n}_0)$ and repeat the procedure from step 1. Define α to be the co-dimension of $\mathfrak{p}_0 \ltimes \mathfrak{n}_0$ in $\mathfrak{g}$: $\dim(\frac{\mathrm{rad}(\mathfrak{g})}{\mathrm{nil}(\mathfrak{g})})$. We then obtain a sequence of Lie algebra embeddings

$$\mathfrak{g} = \mathfrak{g}_0 \hookrightarrow \mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \ldots \hookrightarrow \mathfrak{g}_\alpha.$$

The Lie algebra $\mathfrak{g}_\alpha$ is the semidirect product of a perfect subalgebra $\mathfrak{p}_\alpha$ and a nilpotent ideal $\mathfrak{n}_\alpha$. The Lie algebra $\mathfrak{p}_\alpha$ is of the form $\mathbb{C}^\alpha \oplus \mathfrak{p}_0$ and thus $\frac{\mathfrak{g}}{\mathrm{nil}(\mathfrak{g})}$. The ideal $\mathfrak{n}_\alpha$ has dimension $\dim(\mathfrak{n}) + \alpha = \dim(\mathrm{rad}(\mathfrak{g}))$.

□

Consider the semidirect product $\mathfrak{p} \ltimes \mathfrak{n}$ of a reductive Lie algebra $\mathfrak{p}$ and a nilpotent Lie algebra $\mathfrak{n}$. Then the action of $\mathfrak{p}$ on $\mathfrak{n}$ need not be faithful. However, since $\mathfrak{p}$ is reductive, it is possible to write the Lie algebra as $\mathfrak{p}_1 \oplus (\mathfrak{p}_2 \ltimes \mathfrak{n})$ where $\mathfrak{p}_1$ is the kernel of the action and $\mathfrak{p}_2$ is some complementary ideal. It follows that both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are reductive and that $\mathfrak{p}_2$ acts faithfully on $\mathfrak{n}$.

For natural numbers $c \leq n$, we define $p(n,c)$ to be $\sum_{0\leq j\leq c} \binom{n-j}{c-j} p(j)$. Here, $p(j)$ is the number of partitions of the number j. We define $p(0)$ to be 1.

Theorem Consider the semidirect product $\mathfrak{p} \ltimes \mathfrak{n}$ of a reductive Lie algebra $\mathfrak{p}$ and a nilpotent Lie algebra $\mathfrak{n}$ of dimension n and class c. Assume that the action of $\mathfrak{p}$ on $\mathfrak{n}$ is faithful. Then $\mathfrak{p} \ltimes \mathfrak{n}$ has some faithful module of dimension at most $p(n,c)$.

PROOF*:* Consider the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. Then $\mathfrak{p} \ltimes \mathfrak{n}$ acts on $U(\mathfrak{n})$ in the following way. For any element (z,y) of $\mathfrak{p} \ltimes \mathfrak{n}$ and any monomial $x_1 \cdots x_t$ in $U(\mathfrak{n})$, we define

$$\begin{aligned}(z,y) * (x_1\, x_2 \cdots x_t) &= y \cdot (x_1\, x_2 \cdots x_t) \\ &\quad + [z,x_1]\, x_2 \cdots x_t + x_1\, [z,x_2] \cdots x_t \\ &\quad + \ldots + x_1\, x_2 \cdots [z,x_t].\end{aligned}$$

We want to construct the faithful $(\mathfrak{p} \ltimes \mathfrak{n})$-module as a quotient of $U(\mathfrak{n})$ through some submodule. For this, we introduce an order on $U(\mathfrak{n})$. Consider the natural descending series for $\mathfrak{n}$:

$$\mathfrak{n} = \mathfrak{n}^1 > \mathfrak{n}^2 > \ldots > \mathfrak{n}^c > \mathfrak{n}^{c+1} = 0.$$

Choose a basis $x_1, \ldots, x_n$ for $\mathfrak{n}$ such that the first n_1 of them form a basis for the subspace $\mathfrak{n}^c$, the first n_2 form a basis for $\mathfrak{n}^{c-1}$, …and the first $n_c = n$ form a basis for $\mathfrak{n}^1 = \mathfrak{n}$. We now define the map $\mathrm{ord} : U(\mathfrak{n}) \longrightarrow \mathbb{N}$. On the monomials, it is given by the rules:

$$\begin{array}{rcl} \operatorname{ord}(1) & = & 0 \\ \operatorname{ord}(x_t) & = & \max\{ \ m \mid x_t \in \mathfrak{n}^m \ \} \\ \operatorname{ord}(x_1^{\alpha_1} \ \cdots \ x_n^{\alpha_n}) & = & \alpha_1 \operatorname{ord}(x_1) + \ldots + \alpha_n \operatorname{ord}(x_n) \\ \operatorname{ord}(0) & = & \infty. \end{array}$$

The theorem of Poincaré-Birkhoff-Witt states that the standard monomials $x^\alpha = x_1^{\alpha_1} \ \ldots \ x_n^{\alpha_n}$ form a basis for the universal enveloping algebra $U(\mathfrak{n})$. For a linear combination $\sum_\alpha c_\alpha x^\alpha$, with the scalars c_α all different from zero, we define

$$\operatorname{ord}(\sum_\alpha c_\alpha x^\alpha) = \min_\alpha \{ \ \operatorname{ord}(x^\alpha) \ \}.$$

For any natural number m, we can define a vector subspace U^m of $U(\mathfrak{n})$:

$$U^m = \{ \ u \in U(\mathfrak{n}) \mid \ \operatorname{ord}(u) \geq m \ \}.$$

We show that U^m is a $(\mathfrak{p} \ltimes \mathfrak{n})$-submodule of $U(\mathfrak{n})$. Let u be any element of $U(\mathfrak{n})$ and (z, y) any element in $\mathfrak{p} \ltimes \mathfrak{n}$. It suffices to show that $\operatorname{ord}((z, y) * u) \geq \operatorname{ord}(u)$. It follows from the definition that $\operatorname{ord}(y \cdot u) \geq \operatorname{ord}(y) + \operatorname{ord}(u) \geq \operatorname{ord}(u)$. Since $\mathfrak{p}$ acts on $\mathfrak{n}$ through derivations, it leaves every characteristic ideal invariant. In particular, $\operatorname{ord}([z, x_t]) \geq \operatorname{ord}(x_t)$ for every basis element x_t. So $\operatorname{ord}(z * u) \geq \operatorname{ord}(u)$. Finally, we conclude that

$$\begin{array}{rcl} \operatorname{ord}((z, y) * u) & = & \operatorname{ord}(z * u + y * u) \\ & = & \min\{ \ \operatorname{ord}(z * u) \ , \ \operatorname{ord}(y * u) \ \} \\ & \geq & \operatorname{ord}(u). \end{array}$$

Now assume that m is at least $c + 1$ and consider the quotient module $\overline{U} = \frac{U(\mathfrak{n})}{U^m}$. We show that it is faithful. Suppose an element (z, y) in $\mathfrak{p} \ltimes \mathfrak{n}$ acts on $\overline{U}$ as the zero-map. This means that $(z, y) * u$ has degree at least m for all u in $U(\mathfrak{n})$.

First, take u to be the unit element of $U(\mathfrak{n})$. Then $y = z * 1 + y \cdot 1 = (z, y) * 1$ so that y has order at least $c + 1$. But the only element in $\mathfrak{n}$ with order at least $c + 1$ is zero. We conclude that $y = 0$.

Now take u to be any element of $\mathfrak{n} \leq U(\mathfrak{n})$. Then the assumption tells us that $[z,u] = z * u$ has degree at least $c+1$. Since $[z,u]$ is an element of $\mathfrak{n}$ and the only element of $\mathfrak{n}$ of degree at least $c+1$ is zero, we conclude that $[z,u]$ is zero for every u in $\mathfrak{n}$. Since the action of $\mathfrak{p}$ on $\mathfrak{n}$ is assumed to be faithful, we conclude that also $z = 0$. This shows that $\overline{U}$ is a faithful $(\mathfrak{p} \ltimes \mathfrak{n})$-module.

What is the dimension of the module $\overline{U} = \frac{U(\mathfrak{n})}{U^{c+1}}$? Burde gave a combinatorical estimate for the dimension in [Bur1]: he obtains $\dim(\overline{U}) \leq p(n,c)$.

□

In [Bur5], he also proved the upper bound $p(n,k) \leq \frac{3}{\sqrt{n}} 2^n$ for all $k \leq n$ in $\mathbb{N}$. The Lie algebras with a trivial solvable radical, are precisely the semisimple Lie algebras. They have a trivial centre. For such a Lie algebra $\mathfrak{g}$, the adjoint-representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is faithful so that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})$. The other situation is described by the following corollary.

Corollary Let $\mathfrak{g}$ be any finite-dimensional complex Lie algebra with a non-trivial radical. Then $\mathfrak{g}$ has a faithful module of dimension at most

$$\mu(\frac{\mathfrak{g}}{\mathrm{nil}(\mathfrak{g})}) + 3\frac{2^{\dim(\mathrm{rad}(\mathfrak{g})))}}{\sqrt{\dim(\mathrm{rad}(\mathfrak{g}))}}.$$

Proof*:* According to Neretin's embedding theorem, we can embed the Lie algebra $\mathfrak{g}$ into a Lie algebra $\mathfrak{p} \ltimes \mathfrak{n}$ where $\mathfrak{p}$ is $\frac{\mathfrak{g}}{\mathrm{nil}(\mathfrak{g})}$ and $\mathfrak{n}$ is some nilpotent Lie algebra of dimension $\dim(\mathrm{rad}(\mathfrak{g}))$. The monotonicity of the μ-invariant gives us $\mu(\mathfrak{g}) \leq \mu(\mathfrak{p} \ltimes \mathfrak{n})$. The remark and the previous theorem then give us the upper bound $\mu(\mathfrak{p} \ltimes \mathfrak{n}) \leq \mu(\mathfrak{p}) + p(\dim(\mathrm{rad}(\mathfrak{g})), c_n(\mathfrak{n}))$. Since $p(n,c) \leq \frac{3}{\sqrt{n}} 2^n$ for $c \leq n$, we can combine all of these bounds to obtain,

$$\begin{aligned} \mu(\mathfrak{g}) &\leq \mu(\mathfrak{p} \ltimes \mathfrak{n}) \\ &\leq \mu(\mathfrak{p}) + p(\dim(\mathrm{rad}(\mathfrak{g})), c_{\mathrm{nil}}(\mathfrak{n})) \\ &\leq \mu(\frac{\mathfrak{g}}{\mathrm{nil}(\mathfrak{g})}) + 3\frac{2^{\dim(\mathrm{rad}(\mathfrak{g})))}}{\sqrt{\dim(\mathrm{rad}(\mathfrak{g}))}}. \end{aligned}$$

□

The Lie algebra $\frac{\mathfrak{g}}{\mathrm{nil}(\mathfrak{g})}$ is reductive. Since the μ-invariant of a reductive Lie algebra is bounded by its dimension (see section 2.3), we may conclude that $\mu(\mathfrak{g})$ is $O(2^{\dim(\mathfrak{g})})$. This finishes the proof of Ado's theorem.

1.2 Faithful representations of minimal degree

The classification problem According to the Ado-Iwasawa theorem, every Lie algebra has a faithful linear representation of a finite degree. For the semisimple Lie algebras we can not only produce such a representation but we can even describe all of them. This is done in subsection 2.2.3 by using the standard theory for representations of semisimple Lie algebras. Classifying the representations of the one-dimensional Lie algebra can be reduced to a classical problem of linear algebra, as is shown in the following example.

Example 1.2.0.1. [Jordan canonical form] Consider $\mathbb{C}$, the field of the complex numbers. It is a one-dimensional (complex) Lie algebra that is in fact abelian. The representations $\rho : \mathbb{C} \longrightarrow \mathfrak{gl}(V)$ of $\mathbb{C}$ on a finite-dimensional vector space V correspond uniquely to the linear transformations of V: $\rho \mapsto \rho(1)$. Two representations σ and τ are equivalent if and only if $\sigma(1)$ and $\tau(1)$ are so that the classification of representations of $\mathbb{C}$ on V up to conjugation is reduced to the determination of the Jordan canonical form for a given matrix. Only the zero-transformation corresponds to a non-faithful representation.

$\triangle$

These two cases are extreme: it is in general much more difficult to classify the finite-dimensional (faithful) representations up to equivalence, than to produce one of them. The representations of the n-dimensional (abelian) algebra ($2 \leq n$) for example cannot be classified so easily. To see this, consider the following, closely related problem.

Problem Consider n-tuples $A = (A_1, \dots, A_n)$ of $(m \times m)$-matrices over the complex numbers. Then $GL_m(\mathbb{C})$ acts on A by simultaneous conjugation: $X \cdot A = (XA_1X^{-1}, \dots, XA_nX^{-1})$

for $X \in GL_n(\mathbb{C})$. Classify the orbits under this action, possibly under additional conditions such as commutativity and nilpotency.

This problem is known to be very *difficult*. For an exact formulation of this problem and its difficulty, we refer to [Nath], [GelPo] and [Frie]. But it is still possible to obtain some information about the faithful representations and we do this by introducing the μ-invariants.

1.2.1 The μ-invariant

Suppose that $\mathfrak{g}$ has a faithful representation of degree m. Then it is clear that it also has a faithful representation of degree $m+1, m+2, \ldots$ and that the degree of a faithful linear representation can be arbitrarily large. It cannot be arbitrarily small however. Consider for example an embedding of $\mathfrak{g}$ in $\mathfrak{gl}_m(\mathbb{C})$. Then $\dim(\mathfrak{g}) \leq \dim(\mathfrak{gl}_m(\mathbb{C})) = m^2$ and hence $\sqrt{\dim(\mathfrak{g})} \leq m$. So the degree of any faithful representation is bounded from below by the square root of the dimension of the algebra. Let us consider the minimal degree.

Definition [[Bur1]] Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of arbitrary characteristic. Then we define $\mu(\mathfrak{g}, K)$ to be the minimal dimension of a faithful $\mathfrak{g}$-module.

If it is clear which field we are working with, we drop K from the notation: $\mu(\mathfrak{g}, K) = \mu(\mathfrak{g})$. There is an alternative interpretation for the invariant. Suppose $\mathfrak{g}$ is a finite-dimensional Lie algebra and m a natural number. Then there exists a faithful representation of degree m, if and only if, $\mu(\mathfrak{g}) \leq m$. In this thesis we will only be interested in the complex situation, $K = \mathbb{C}$, unless stated otherwise. That the invariant does depend on the field, is suggested by the following lemma.

Lemma 1.2.1.1. *Let $\mathfrak{g}$ be a Lie algebra over a field K and let $\widehat{K}$ be a field extension of K. Then $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \widehat{K}$ is a $\widehat{K}$-Lie algebra and $\mu(\widehat{\mathfrak{g}}, \widehat{K}) \leq \mu(\mathfrak{g}, K)$.*

PROOF: Consider a faithful K-linear representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(K^m)$ of degree m. Let $\iota : \widehat{K} \longrightarrow \mathfrak{gl}(\widehat{K})$ be the natural embedding. Then the tensor product $\rho \otimes \iota : \mathfrak{g} \otimes \widehat{K} \longrightarrow \mathfrak{gl}(K^m \otimes \widehat{K})$ is a faithful $\widehat{K}$-representation of degree $m \times 1 = m$. If we take m to be $\mu(\mathfrak{g}, K)$, then we obtain $\mu(\widehat{\mathfrak{g}}, \widehat{K}) \leq \mu(\mathfrak{g}, K)$.

□

This μ has two obvious properties that are worth mentioning. The first one states that the invariant is *monotone*, i.e. that it is compatible with Lie algebra embeddings.

Lemma 1.2.1.2. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras. If $\mathfrak{g}_1 \leq \mathfrak{g}_2$, then $\mu(\mathfrak{g}_1) \leq \mu(\mathfrak{g}_2)$.*

PROOF*:* Consider an embedding $\iota : \mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$. Every embedding $\rho : \mathfrak{g}_2 \hookrightarrow \mathfrak{gl}(V)$ can be composed with this ι to obtain an embedding of $\mathfrak{g}_1$, $\rho \circ \iota : \mathfrak{g}_1 \hookrightarrow \mathfrak{gl}(V)$, of the same degree. In particular, we can do this for a faithful embedding of $\mathfrak{g}_2$ of degree $\mu(\mathfrak{g}_2)$ to obtain $\mu(\mathfrak{g}_1) \leq \mu(\mathfrak{g}_2)$.

□

In particular, this implies that the μ really is an invariant of Lie algebras. The second property, the *subadditivity*, states that the invariant is in some sense also compatible with the direct sums of Lie algebras.

Lemma 1.2.1.3. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras. Then $\mu(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \leq \mu(\mathfrak{g}_1) + \mu(\mathfrak{g}_2)$.*

PROOF*:* We consider two embeddings: $\rho_1 : \mathfrak{g}_1 \hookrightarrow \mathfrak{gl}(V_1)$ and $\rho_2 : \mathfrak{g}_2 \hookrightarrow \mathfrak{gl}(V_2)$. Then the direct sum $(\rho_1 \circ \pi_1) \oplus (\rho_2 \circ \pi_2) : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \hookrightarrow \mathfrak{gl}(V_1 \oplus V_2)$ gives a well-defined embedding of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of degree $\deg(\rho_1) + \deg(\rho_2)$. If we take ρ_1 and ρ_2 to be minimal, we obtain the desired inequality.

□

The three inequalities may, but need not be, strict.

Example 1.2.1.1. [Field extensions] For any field k and any natural number n, we have $\mu(\mathfrak{gl}_n(k)) = n$. Now consider the (strict) inclusion of fields. $\mathbb{Q} < \mathbb{R} < \mathbb{C}$. Then we have:

$$\mu(\mathfrak{gl}_n(\mathbb{C}), \mathbb{C}) = \mu(\mathfrak{gl}_n(\mathbb{R}), \mathbb{R}) = \mu(\mathfrak{gl}_n(\mathbb{Q}), \mathbb{Q}) = n.$$

△

Example 1.2.1.2. [Subalgebras] For any field k and any natural number n, we have $\mu(\mathfrak{sl}_n(k)) = n$. It is clear that $\mathfrak{sl}_n(k)$ admits an n-dimensional faithful representation so that $\mu(\mathfrak{sl}_n(k)) \leq n$. Suppose that $\mathfrak{sl}_n(k)$ can be embedded into $\mathfrak{gl}_m(k)$ for some m. Then $[\mathfrak{sl}_n(k), \mathfrak{sl}_n(k)] \leq [\mathfrak{gl}_m(k), \mathfrak{gl}_m(k)] = \mathfrak{sl}_m(k)$. Then it is clear that $n \leq m$ and we conclude that $\mu(\mathfrak{sl}_n(k)) = n$. Now consider the (strict) inclusion $\mathfrak{sl}_n(K) < \mathfrak{gl}_n(K)$. Then

$$\mu(\mathfrak{sl}_n(k), k) = \mu(\mathfrak{gl}_n(k), k).$$

△

Example 1.2.1.3. [Direct sums] For any field k and any natural number n, we consider the direct sum $\mathfrak{gl}_n(k) \oplus k$. Suppose $\mathfrak{gl}_n(k) \oplus k \leq \mathfrak{gl}_m(k)$. Then $n^2 + 1 = \dim(\mathfrak{gl}_n(k) \oplus k) \leq \dim(\mathfrak{gl}_m(k)) = m^2$. This implies that $m > n$ and $\mu(\mathfrak{gl}_n(k) \oplus k) \geq n + 1$. The subadditivity then shows that

$$\mu(\mathfrak{gl}_n(k) \oplus k) = \mu(\mathfrak{gl}_n(k)) + \mu(k).$$

△

1.2.2 The μ-invariants

Refinements of Engel's theorem Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a finite-dimensional representation of $\mathfrak{g}$. Then we say that $\mathfrak{g}$ acts nilpotently on V, or that ρ is a nilrepresentation, if the image $\rho(\mathfrak{g})$ consists of only nilpotent operators of V. That is, for every element A in the image of the representation, there exists some natural number ε depending on A, such that $A^\varepsilon = 0$. Representations into the Lie algebra of all strictly upper-triangular matrices of a finite-dimensional vector space are examples of nil-representations. Engel's theorem, which holds in any characteristic, then says that every nil-representation is of this form.

Theorem [Engel] Nil-representations are exactly the ones that can be strictly-upper-triangularised by conjugation.

Note that only nilpotent Lie algebras admit faithful nil-representations. Also note that not every (faithful) representation of a nilpotent Lie algebra is a nil-representation. The following lemma shows that the largest part of a representation of a nilpotent Lie algebra is a nilrepresentation.

> **Theorem** [Jordan-Chevalley] Consider a nilpotent Lie algebra $\mathfrak{n}$. Then every representation $\rho : \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{n}$ can be decomposed into a semisimple and nilpotent representation of $\mathfrak{n}$, $\rho = \rho_{\mathrm{s}} + \rho_{\mathrm{n}}$. These two representations commute.

The theorem can actually be stated in a stronger way. Consider a representation (ρ, V) of $\mathfrak{n}$. Then there is a direct sum decomposition $(\rho, V) = \bigoplus_c (\rho_c, V_c)$ that is indexed by characters of $\mathfrak{n}$, such that $\tilde{\rho}_c(x) = \rho_c(x) - c(x)\mathbb{1}_{V_c}$ is a nil-representation. Then $(\rho_n, V) = \bigoplus_c (\tilde{\rho}_c, V_c)$ is a nil-representation, $(\rho_s, V) = \bigoplus_c (c \cdot \mathbb{1}_{V_c}, V_c)$ is semisimple and we have the decomposition $\rho = \rho_s + \rho_n$. These two representations clearly commute.

From this we can immediately deduce a lower bound for μ in function of the nilpotency class. Nilpotent Lie algebras with a high class cannot have a low μ-invariant:

Corollary 1.2.2.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra, not 0 or $\mathbb{C}$. Then*

$$c_n(\mathfrak{n}) + 1 \leq \mu(\mathfrak{n}).$$

> PROOF*:* First suppose $\mathfrak{n}$ is not abelian. Consider an embedding $\rho : \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$. Let $\rho = \rho_s + \rho_n$ be the decomposition of the lemma. Then $[\rho(x), \rho(y)] = [\rho_n(x), \rho_n(y)]$ for all x and y in $\mathfrak{n}$. We see that the nilpotent Lie algebras $\rho(\mathfrak{n})$ and $\rho_n(\mathfrak{n}) \neq 0$ have the same class: $c_n(\mathfrak{n})$. According to Engel's theorem, we see that $\rho_n(\mathfrak{n})$ is contained in the nilpotent subalgebra B of $\mathfrak{gl}(V)$ that consists of all strictly upper triangular matrices. Since the nilpotency class is monotone, we have $c_n(\mathfrak{n}) = c_n(\rho_n(\mathfrak{n})) \leq c_n(B) = \dim(V)-1$. If we take V to be a minimal representation, we obtain $c_n(\mathfrak{n}) + 1 \leq \mu(\mathfrak{n})$. In case $\mathfrak{n}$ is abelian, then clearly only 0 and $\mathbb{C}^1$ satisfy $\mu(\mathfrak{n}) \leq c_n(\mathfrak{n}) = 1$.
>
> □

We have the following general property: for any Lie algebra $\mathfrak{g}$ and any $\mathfrak{g}$-module, the nilpotent ideal $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ acts nilpotently. Can we also find a representation of $\mathfrak{g}$ such that *every* nilpotent ideal acts nilpotently? The answer was shown to be positive by Harish-Chandra and Birkhoff (among others). We present a simplified version of their results.

> **Theorem** [Harish-Chandra, Birkhoff] Consider a finite-dimensional Lie algebra over an arbitrary field. Then there exists a faithful representation of the Lie algebra such that the restriction to the nilradical is a nil-representation.

Definition 1.2.2.1 (μ_∞)**.** Consider a Lie algebra $\mathfrak{g}$. Define μ_∞ to be the minimal dimension of a faithful representation for which the restriction to the nilradical is a nilrepresentation.

The theorem then proves that this is a well defined invariant with values in the natural numbers. It follows from the definition that $\mu(\mathfrak{g}) \leq \mu_\infty(\mathfrak{g})$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and let c be the nilpotency class of $\mathfrak{n}$. Then there exists a finite-dimensional faithful nil-representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$ and, by Engel's theorem, a natural number ε such that $\rho(\mathfrak{n})^\varepsilon = 0$. Note that ε must be at least $c+1$ because of the faithfulness condition. This leads us to the following question. Given a Lie algebra $\mathfrak{g}$ with a nilradical $\mathfrak{n}$ of class c, does there exist a faithful representation of $\mathfrak{g}$ such that $\rho(\mathfrak{n})^{c+1} = 0$? The answer is positive. It can be found, although not always very explicitly, in the works by Birkhoff [Bir], Reed [Ree], Burde [Bur1], De Graaf [deG] and Block [Blo].

> **Theorem** [Birkhoff, Reed, Burde] Let $\mathfrak{n}$ be a nilpotent Lie algebra of class c. Then there exists a finite-dimensional faithful representation $\rho : \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{n}$ such that $\rho(\mathfrak{n})^{c+1} = 0$.

> **Theorem** [Block] Consider a Lie algebra $\mathfrak{g}$ with nilradical $\mathfrak{n}$ of class c. Then there exists a faithful finite-dimensional representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of $\mathfrak{g}$ such that $\rho(\mathfrak{n})^{c+1} = 0$.

Definition 1.2.2.2 (μ_ε)**.** Let $\mathfrak{g}$ be a Lie algebra with nilradical $\mathfrak{n}$ of class c. Let ε be some natural number. Then $\mu_\varepsilon(\mathfrak{g})$ is the minimal dimension of a faithful representation ρ satisfying $\rho(\mathfrak{n})^{c+1+\varepsilon} = 0$. A representation is of type ε if it satisfies $\rho(\mathfrak{n})^{c+1+\varepsilon} = 0$ but $\rho(\mathfrak{n})^{c+\varepsilon} \neq 0$.

The theorem of Block then states that every Lie algebra $\mathfrak{g}$ admits a finite-dimensional faithful representation of type 0. We can summarise the information about the μ-invariants in the following proposition.

Proposition 1.2.2.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field. The sequence $(\mu_\varepsilon(\mathfrak{g}))_\varepsilon = (\mu_0(\mathfrak{g}), \mu_1(\mathfrak{g}), \ldots)$ is a descending sequence of natural numbers. It stabilises and thus converges to its minimum. This minimum is $\mu_\infty(\mathfrak{g})$. For every natural number ε, we have*

$$\mu(\mathfrak{g}) \leq \mu_\infty(\mathfrak{g}) \leq \mu_\varepsilon(\mathfrak{g}) < \infty.$$

Proof*:* Let $\mathfrak{n}$ be the nilradical of class c. If a (faithful) representation ρ satisfies $\rho(\mathfrak{n})^{c+1+\varepsilon_1} = 0$, then also $\rho(\mathfrak{n})^{c+1+\varepsilon_2} = 0$ for $\varepsilon_1 \leq \varepsilon_2$. This shows that the sequence $(\mu_\varepsilon(\mathfrak{g}))_\varepsilon$ is descending. Suppose ρ is faithful, nilpotent on the radical and of degree $\mu_\infty(\mathfrak{g})$. Then this ρ has a type, say ε, and $(\mu_\varepsilon(\mathfrak{g}))_\varepsilon$ stabilises in (at most) ε.

$\square$

Finally, let us remark that $\mu(\mathfrak{g}) = \mu_\infty(\mathfrak{g})$ if $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = \mathrm{nil}(\mathfrak{g})$.

Representing the class $\mathcal{Z}_{\leq}\mathcal{C}$ For convenience later, we now introduce a class $\mathcal{Z}_{\leq}\mathcal{C}$ of Lie algebras that is defined in terms of central elements and commutators. It consists of all Lie algebras $\mathfrak{g}$ that satisfy the condition $\mathcal{Z}(\mathfrak{g}) \leq [\mathfrak{g}, \mathfrak{g}]$. All centreless Lie algebras belong to this class. All perfect Lie algebras do too. Nilpotent (non-abelian) examples are: (*i.*) free nilpotent Lie algebras, (*ii.*) nilpotent Lie algebras with one-dimensional centre (e.g.: generalised Heisenberg Lie algebras and complex filiform nilpotent Lie algebras) and (*iii.*) characteristically nilpotent Lie algebras.

Proof*:* (*i.*) For a (non-abelian) free nilpotent Lie algebra $\mathfrak{g}$ of class c, $\mathcal{Z}(\mathfrak{g}) = \mathfrak{g}^{c-1} \leq \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$. (*ii.*) In a nilpotent Lie algebra, every (non-trivial) ideal intersects the centre non-trivially. Thus if the centre is one-dimensional, it is contained in every ideal. In particular, $\mathcal{Z}(\mathfrak{g}) \leq [\mathfrak{g}, \mathfrak{g}]$. (*iii.*) If the condition fails, semisimple derivations can be constructed, [LegTo].

$\square$

The following two results have also been obtained by Calgliero and Rojas, [CaRo], in essentially the same way (but independently). They show that we may restrict our attention to nil-representations if the Lie algebra is contained in the class $\mathcal{Z}_{\leq}\mathcal{C}$.

Proposition 1.2.2.2. *Consider a nilpotent Lie algebra $\mathfrak{n}$ satisfying $\mathcal{Z}(\mathfrak{n}) \leq [\mathfrak{n}, \mathfrak{n}]$. Consider a representation ρ of $\mathfrak{n}$ with decomposition $\rho = \rho_n + \rho_s$. Then ρ is faithful if and only if ρ_n is.*

> Proof: Consider the Jordan-Chevalley-decomposition. It suffices to show that ρ_n is faithful if ρ is. Now suppose that ρ is faithful but that ρ_n is not. By applying the appropriate conjugation, we may assume that ρ_s is semisimple and ρ_n is strictly-upper-triangular. Since $\mathfrak{n}$ is nilpotent, we may then select a non-zero central element z that is mapped to zero by ρ_n. But then for every character c in the decomposition, we have $0 = \rho_n(z) = \rho_c(z) - c(z)\mathbb{1}_{V_c}$, which implies that $\rho_c(z) = c(z)\mathbb{1}_{V_c}$ and $\rho(z) = \bigoplus_c c(z)\mathbb{1}_{V_c}$. The only element of this form that is a commutator of strictly-upper-triangular transformations, is the zero-transformation. We conclude that (ρ, V) is not faithful, which is a contradiction. This finishes the proof.
>
> □

Corollary 1.2.2.2. *Let $\mathfrak{n}$ be a nilpotent Lie algebra for which $\mathcal{Z}(\mathfrak{n}) \leq [\mathfrak{n}, \mathfrak{n}]$ holds. Then*

$$\mu(\mathfrak{n}) = \mu_\infty(\mathfrak{n}).$$

Remark 1. It is not at all obvious how to compute the μ-invariants for a given Lie algebra. The brute-force approach using computer programmes only works in very special cases. For some Lie algebras, there is an obvious way to proceed. For example, if the dimension of the Lie algebra is low (for nilpotent Lie algebras with dimension at most 5 [BNT], for Lie algebras of dimension at most 4 [KaBa] and related to this, [GST]), the answer is obtained by using the standard classification and explicit calculations. For other examples, we refer to [Han] and [RaTh]. The equations involved are in general too difficult to solve for the usual algorithms.

1.3 Nilpotent and solvable Lie algebras

1.3.1 Nilpotent Lie algebras

There is much historical background material available for the μ-invariants of nilpotent Lie algebras. More concretely, we will present a succession of upper bounds for $\mu(\mathfrak{g})$ in function of the dimension $\dim(\mathfrak{g})$ and the nilpotency class $c_n(\mathfrak{g})$ of $\mathfrak{g}$. First, we focus our attention to two important families of nilpotent Lie algebras: the generalised Heisenberg Lie algebras and the filiform nilpotent Lie algebras.

Heisenberg Lie algebras Consider the family of the (generalised) Heisenberg Lie algebras. They form a one-parameter family of two-step nilpotent Lie algebras and they can be defined by the presentation

$$\mathfrak{h}_m = \langle\, x_1, \ldots, x_m; y_1, \ldots, y_m; z \mid [x_i, y_j] = \delta_{i,j} z \,\rangle.$$

They occur frequently in quantum mechanics. So it seems quite reasonable to study their representations and the size of these representations. Note that the centre and the commutator algebra coincide so that $\mu(\mathfrak{h}) = \mu_\infty(\mathfrak{h})$ according to corollary 1.2.2.2.

> **Proposition** [[Bur1]] For the generalised Heisenberg Lie algebras we have
> $$2\mu(\mathfrak{h}_m) = \dim(\mathfrak{h}_m) + 3.$$

The standard representation of minimal degree is actually of type 0 so that μ, μ_∞ and μ_ε coincide for all $\varepsilon \geq 0$. Cagliero and Rojas gave a generalisation to *current Lie algebras.*

> **Proposition** [[CaRo]] For any polynomial $p(t)$ of degree r in $K[t]$, we have
> $$\mu(\mathfrak{h}_m \otimes \frac{K[t]}{p(t)}) = mr + \lceil 2\sqrt{r}\,\rceil.$$

These nilpotent Lie algebras are of dimension $r(2m+1)$ and the commutator is defined by $[v \otimes f, w \otimes g] = [v, w] \otimes (f \cdot g)$ for $v, w \in \mathfrak{h}_m$ and $f, g \in K[t]$. By considering the monomial $p(t) = t$ of degree one, we recover the previous proposition. The proof of Burde's proposition can be generalised to prove the following (slightly weaker) result.

Proposition 1.3.1.1. *Consider a Lie algebra* $\mathfrak{g}$ *with one-dimensional commutator algebra. Then we have the lower bound* $\dim(\mathfrak{g}) - \alpha_*(\mathfrak{g}) \leq \mu(\mathfrak{g})$.

For a Lie algebra $\mathfrak{g}$, $\alpha_*(\mathfrak{g})$ is the maximal dimension of an abelian subalgebra not containing any commutators. E.g.: for an abelian Lie algebra $\mathfrak{a}$, we have $\alpha_*(\mathfrak{a}) = \alpha(\mathfrak{a}) = \dim(\mathfrak{a})$. For the generalised Heisenberg Lie algebras, we have $\alpha_*(\mathfrak{h}_m) + 1 = \alpha(\mathfrak{h}_m) = m + 1$ and $\alpha_*(\mathfrak{g}) + 1 \leq \alpha(\mathfrak{g})$ for Lie algebras $\mathfrak{g}$ satisfying $0 < \mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$.

Proof*:* Let c be a generator for the commutator algebra and choose some faithful $\mathfrak{g}$-module (ρ, V) of minimal degree. Since the action is faithful, $\rho(c)$ is not the zero-operator on V and there exists a vector v in V such that $\rho(c)v \neq 0$. Fix this v. Consider the evaluation map with respect to v:

$$E : \mathfrak{g} \longrightarrow V : x \longmapsto \rho(x)v.$$

Define $\mathfrak{a}$ to be the kernel and $\mathfrak{b}$ to be the image of this map. The dimension theorem for vector spaces gives us the (in)equalities,

$$\begin{aligned} \mu(\mathfrak{g}) &= \dim(V) \\ &\geq \dim(\mathfrak{b}) \\ &= \dim(\mathfrak{g}) - \dim(\mathfrak{a}). \end{aligned}$$

It is clear that $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$. For, suppose that x and y are elements of $\mathfrak{a}$, the kernel of E. Then

$$\begin{aligned} E([x,y]) &= \rho([x,y])v \\ &= \rho(x)\rho(y)v - \rho(y)\rho(x)v \\ &= 0. \end{aligned}$$

In particular, $[\mathfrak{a}, \mathfrak{a}] \leq \mathfrak{a}$. Since the commutator algebra of $\mathfrak{g}$ is one-dimensional, such an element $[x, y]$ must be proportional to c. Since $E(c) \neq 0$ by definition, we conclude that $[x, y] = 0$ for all x and y in $\mathfrak{a}$. So $\mathfrak{a}$ is an *abelian* subalgebra of $\mathfrak{g}$, not intersecting the commutator. So we have,

$$\begin{aligned} \mu(\mathfrak{g}) &\geq \dim(\mathfrak{g}) - \dim(\mathfrak{a}) \\ &\geq \dim(\mathfrak{g}) - \alpha_*(\mathfrak{g}). \end{aligned}$$

□

Filiform Lie algebras and Milnor's conjecture A family that is of particular interest with respect to the μ-invariant consists of the filiform nilpotent Lie algebras. A complex Lie algebra $\mathfrak{g}$ is filiform if it is nilpotent of maximal class, i.e.: $c_n(\mathfrak{g}) + 1 = \dim(\mathfrak{g})$. Since the centre of a filiform Lie algebra $\mathfrak{g}$ is one-dimensional, we can apply corollaries 1.2.2.1 and 1.2.2.2 to conclude that $\dim(\mathfrak{g}) \leq \mu_\infty(\mathfrak{g}) = \mu(\mathfrak{g})$. (See [Ben] and [Bu6] for the original proofs.)

The μ-invariant was used to disprove a famous conjecture by Milnor. In his paper on the fundamental groups of complete affinely flat manifolds [Mil2], he enquired about the existence of left invariant affine structures on Lie groups. The question can be formulated in purely algebraic terms:

> **Problem** Which Lie algebras admit a left-symmetric structure?

Milnor claimed that all nilpotent, and more generally, all solvable Lie algebras (groups) admit such a structure. It turns out that this claim is false: Benoist constructed an 11-dimensional nilpotent counterexample. This was followed by the construction of a family of nilpotent counterexamples by Burde and Grunewald. Burde later found simpler counter-examples in dimension 10. All of the mentioned Lie algebras $\mathfrak{g}$ have the following crucial property in common: they have a μ-invariant that is in a certain sense big compared to their dimension: $\dim(\mathfrak{g}) + 1 < \mu(\mathfrak{g})$. For this reason, they cannot admit left symmetric structures:

> **Proposition** If a Lie algebra $\mathfrak{g}$ admits a left symmetric algebra structure, then $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$.

More detailed results can be found in the papers of Burde, Grunewald and Benoist: [Ben], [BuGr] and [Bu6]. Propositions 3.0.4.1 and 3.0.4.2 on page 55 express the μ-invariant of a Lie algebra $\mathfrak{g}$ with a filiform radical $\mathfrak{f}$ in terms of $\mu(\frac{\mathfrak{g}}{\mathfrak{f}})$ and $\mu(\mathfrak{f})$.

Bounds The Ado-Iwasawa theorem states that any finite-dimensional Lie algebra has a finite μ-invariant and Iwasawa asked how the μ-invariant can be bounded by the dimension. This question too, has a long history. Already in 1937, Birkhoff gave attention to this problem for *nilpotent* Lie algebras.

Proposition [[Bir]] "Any c-step-nilpotent Lie algebra of dimension d is isomorphic with a Lie algebra of finite matrices of degree at most $\frac{d^{c+1}-1}{d-1}$, with coefficients in the same field."

In 1969, Reed sharpened the result for nilpotent Lie algebras and generalised the argument to deal with the more general class of *solvable* Lie algebras.

Proposition [[Ree]] Every nilpotent Lie algebra of dimension d and nilpotency class c over an algebraically closed field of characteristic zero has a faithful representation of degree at most $1 + d + d^c$.

Other bounds for nilpotent Lie algebras were constructed by Burde and de Graaf, where Burde's bound is sharper than the one of de Graaf.

Proposition [[Bur1]] "Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension d and nilpotency class c. Then $\mu(\mathfrak{g}) \leq p(d,c)$ with $c < d$. Here $\nu(d,c) = \Sigma_{j=0}^{c}\binom{d-j}{c-j}p(j)$ and $p(j)$ is the number of partitions of j. [Also,]

$$\mu(\mathfrak{g}) < \frac{\alpha}{\sqrt{d}}2^d."$$

Proposition [[deG]] "Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension d and nilpotency class c. Then the degree of [the faithful representation] σ is bounded above by $\binom{d+c}{c}$."

1.3.2 Solvable Lie algebras

Lie's theorem We have seen that the nil-representations can be conjugated to a subalgebra of all strictly upper triangular matrices. Similarly, we have a theorem that describes the (finite-dimensional) representations of a solvable Lie algebra. Unlike in Engel's theorem, there are restrictions on the underlying base field.

Theorem [Lie] Consider any finite-dimensional linear representation of a finite-dimensional solvable Lie algebra. If the base field is algebraically closed and of characteristic zero, then the representation can be triangularised.

The corresponding statement over the real numbers does *not* hold. To each pair of non-zero real numbers α and β, we associate a (real) Lie algebra that is two-step solvable (but not nilpotent):

$$L_{\alpha,\beta} = \langle\, x, y, z \mid [z,x] = \alpha y, [z,y] = \beta x \,\rangle.$$

We will show that if $\mathfrak{g}$ is a (real) Lie algebra containing some $L_{\alpha,\beta}$ for $\alpha\beta < 0$, then $\mathfrak{g}$ cannot be embedded into (real) upper-triangular matrices of any degree.

Proof*:* Suppose there is an embedding of $\mathfrak{g}$ into strictly upper-triangular matrices of a finite degree. We may then identify elements of $L_{\alpha,\beta}$ with their images. Since $x = \frac{1}{\beta}[z,y]$ and $y = \frac{1}{\alpha}[z,x]$, the elements x and y are strictly upper triangular. Let $x_{i,j}, y_{i,j}$ be their coefficients with respect to the chosen basis. The two commutator-relations give the following system in the variables $x_{t+1,t}$ and $y_{t+1,t}$:

$$\begin{aligned} \alpha y_{t+1,t} &= (z_{t,t} - z_{t+1,t+1})x_{t+1,t} \\ \beta x_{t+1,t} &= (z_{t,t} - z_{t+1,t+1})y_{t+1,t}. \end{aligned}$$

This system is regular over the real numbers since $\alpha\beta < 0$. Hence, there are only trivial solutions for $x_{t+1,t}$ and $y_{t+1,t}$. By repeating this argument, one can show that $x_{t+u,t}$ and $y_{t+u,t}$ are zero for all u. This implies that x and y are the zero-operators. This contradicts the faithfulness of the embedding and it finishes the proof.

□

All real solvable Lie algebras of dimension at most 2 can be embedded into upper-triangular matrices. Thus the three-dimensional Lie algebras above are counterexamples to Lie's theorem over the real numbers of lowest possible dimension. Note that for $\alpha, \beta, A, B \neq 0$, we have $L_{\alpha,\beta} \cong L_{A,B}$. Also note that $\mu(L_{\alpha,\beta}) = 3$.

Problem Consider a field K. Which solvable K-Lie algebras can be brought in upper triangular form?

Bounds Also for solvable Lie algebras, we have bounds for the μ-invariant in terms of its dimension and solvability class. For nilpotent Lie algebras, Engel's theorem gave us a lower bound for the μ-invariant in terms of the nilpotency class. Similarly, in the case of solvable Lie algebras, we can deduce a lower bound for the μ-invariant in terms of the solvability class. Where Engel's theorem gave a linear lower bound, Lie's theorem gives us an exponential bound:

Corollary 1.3.2.1. *For a complex solvable Lie algebra $\mathfrak{g}$, we have*

$$2^{c_s(\mathfrak{g})-2} \leq \mu(\mathfrak{g}).$$

> Proof*:* Suppose that $\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{C})$. Then after conjugation, we may assume that $\mathfrak{g}$ is contained in the solvable Lie algebra of all upper-triangular matrices, $B(n)$. Then $c_{\mathrm{s}}(\mathfrak{g}) \leq c_{\mathrm{s}}(B(n))$. The latter solvability class is $\lceil \log_2(n) \rceil + 1$. This implies that $2^{c_s-2} \leq n$. The desired inequality is obtained for $n = \mu(\mathfrak{g})$.
>
> $\square$

The following result was claimed by Reed. Note that we have obtained a better upper bound in subsection 1.1.2.

> **Proposition** [[Ree]] Every solvable Lie algebra of dimension d over an algebraically closed field of characteristic zero has a faithful representation of degree at most $1 + d + d^d$.

1.4 Analogues and generalisations

An analogue to Ado's theorem can also be formulated for other algebraic structures such as algebras and groups, where the definitions also make sense. In this section we mention a few parallel results.

1.4.1 Generalisations of Lie algebras

Color Lie algebras Super Lie algebras are natural generalisations of Lie algebras. Scheunert and later Kac proved that Ado's theorem also holds in this setting. In his proof [Kac], Kac even uses the classical theorem of Ado to obtain that every finite-dimensional Lie superalgebra admits a faithful finite-dimensional representation. In particular, it is possible to extend the definition of μ to the larger class of Super Lie algebras. There is the even broader class of *generalised* Lie algebras, also called *color* Lie algebras. The following theorem is due to Scheunert [Scht].

> **Theorem** [Scheunert] Let Γ be an abelian group, let ε be a commutation factor on Γ [over an arbitrary field], and let L be a finite-dimensional ε Lie algebra. Then L has a faithful finite-dimensional Γ-graded representation.

Lie algebras over a ring Lie algebras are usually defined over a field but it is also possible to consider Lie algebras over a ring. Churkin, Kuz'min and Weigel generalised the theorem of Ado and Iwasawa in the following ways (See [Chur, Wei] and [Iwas]).

> **Theorem** [Churkin,Kuz'min,Weigel] Let D be a ring and let $\mathfrak{g}$ be a Lie algebra over D that is finitely generated and free. Then $\mathfrak{g}$ admits a faithful representation by matrices in $M_n(D)$ for some $n \in \mathbb{N}$ if D is a
>
> ($i.$) principle ideal domain of characteristic zero,
>
> ($ii.$) commutative ring of prime characteristic, or
>
> ($iii.$) noetherian ring of prime characteristic.

Note that the theorem of Ado is recovered from ($i.$) and Iwasawa's from ($ii.$), ($iii.$) by taking D to be a field.

1.4.2 Linear groups

Lie groups Ado's theorem does not hold in general for the class of all Lie groups. Consider for example the simply connected universal covering group of $SL_2(\mathbb{R})$. Then it is well-known that this Lie group does not admit any faithful linear representation of finite dimension. For a complete characterisation of the (finite-dimensional) connected Lie groups that admit a faithful representation of finite degree, see the work [BeNe]. If we pass to matrices of countable size, $\mathcal{M}_\infty(k)$, and to $\mathcal{M}_\infty^*(k)$, the corresponding subset of invertible matrices, we obtain the following theorem by Bourgin and Robart. (See [BoRo1, BoRo2] for exact definitions and notation.)

Theorem [Bourgin-Robart] Each finite-dimensional Lie group can be faithfully represented in $\mathcal{M}_{\infty}^{*}(k)$.

If we define $\mu(G)$ to be the minimal degree of a faithful Lie-representation for the Lie group G, then the theorem says that the invariant $\mu(G)$ is either finite or countably infinite. It is possible to compare the μ-invariant of a Lie group with that of its Lie algebra.

Proposition 1.4.2.1. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then we have,*

$$\mu(\mathfrak{g}) \leq \mu(G).$$

Proof*:* Suppose that $\rho : G \longrightarrow GL(V)$ is a faithful linear representation of the Lie group G in the vectorspace V. Then the projection $d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ with the same degree. Since the kernel of $d\rho$ is equal to the projection of the kernel of ρ, it is the projection of $\{0\}$ and so $d\rho$ is faithful. So $\mu(\mathfrak{g}) \leq \deg(d\rho) = \deg(\rho)$ and since this is true for all faithful embeddings ρ, we obtain $\mu(\mathfrak{g}) \leq \mu(G)$.

□

Linear groups The groups that admit a finite-dimensional faithful representation are also called *linear*. Finite groups are linear by Cayley's theorem of permutation groups. The free groups and the virtually-polycyclic groups ([Aus, Hall] and [Swa]) are linear too. There are many results available in the literature, mainly concerned with the following two questions.

Problem Which groups are linear? And what are the properties of linear groups?

One of the first to attack these problems was Mal'cev. He studied "*the representation by matrices of some finite degree with elements from some field*" ([Mal1]). Merzlyakov devoted his thesis to the subject and many have worked on the problems since ([Mer]): Mal'cev, Hall, Auslander, Zassenhaus, Kolchin, Suprunenko and Tits. Mal'cev proved that linearity is a *local* property. Let n be a natural number.

Theorem [Mal'cev] A group is linear of degree n if and only if it is locally linear of degree n.

An interesting overview of the subject can be found in *"Matrix groups"* by Cameron ([Came]): linear groups that are finite, finitely generated, periodic, solvable or nilpotent are discussed. Since Ado's theorem fails for groups, the μ-invariant may not be finite.

Definition The μ-invariant of a group G over a field k is,

$$\mu(G,k) = \min\{ \dim_k(V) \mid G \leq GL(V) \}.$$

Note that $\mu(G,k) \leq |G|$ since the left-regular representation is faithful and of degree $|G|$. As in the Lie-algebra-case, the invariant is monotone and subadditive. It is also known as the *representation-dimension* of G, $\mathrm{rdim}_k(G)$. For p-groups, it generically coincides with the *essential-dimension* ([KaMe, Flo]). Note that this invariant strongly depends on the properties of the field. For finitely generated abelian groups, the μ-invariant is finite and it can be expressed in terms of an other natural invariant of the group (cf. [KaMe, Flo] and [Kar]).

Proposition Consider a finitely generated abelian group A with torsion subgroup T. Let t be the number of invariant factors of T. Then $\mu(A,\mathbb{C}) = \mu(T,\mathbb{C}) = t$.

An elementary proof can be found in the appendix. Note the importance of the field in the above proposition. For natural numbers $n > 2$, $\mu(\mathbb{Z}_n,\mathbb{C}) < \mu(\mathbb{Z}_n,\mathbb{Q})$. Finally, we should note that there is an other representation-invariant associated to (finite) groups. Unfortunately, the same notation is used in that context. According to Cayley's theorem, every finite group G can be embedded into some permutation group on $|G|$ many elements. The *other* μ-invariant for a finite group G is then defined as the minimal natural number n for which there exist an embedding of G into S_n,

$$\mu(G) = \min\{ n \in \mathbb{N} \mid G \leq S_n \} \leq |G|.$$

This invariant too is monotone and subadditive. In this field, there is the following important question ([Kar],[Joh] and [Wri]). For a nice and compact overview of the subject, we refer to [Sau].

Problem For which groups G_1 and G_2 do we have the equality $\mu(G_1 \times G_2) = \mu(G_1) + \mu(G_2)$?

Chapter 2

Reductive Lie algebras

In this chapter we study the faithful representations of minimal degree for Lie algebras that are reductive. A Lie algebra $\mathfrak{g}$ is called reductive if every ideal $\mathfrak{h}$ of $\mathfrak{g}$ admits a complementary ideal, i.e., an ideal $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$ as a direct sum of ideals. Every Lie algebra that does not contain any proper ideals is reductive and it cannot be decomposed further. In this case, there are two possibilities: the algebra is either abelian and of dimension one, or, it is not. In the latter case, we call the Lie algebra *simple*. In the classical theory of Lie algebras, there is a standard classification of the simple Lie algebras, say over the complex numbers. It turns out that every reductive Lie algebra can be decomposed into a sum of ideals that are either simple or abelian and that this decomposition is essentially unique.

We will first focus on the *abelian* Lie algebras, which are in a way the easiest examples of reductive Lie algebras. A theorem due to Schur can be transformed to obtain the μ-invariant for abelian Lie agebras as a function of their dimension. Then we will consider the *simple* and *semisimple* Lie algebras, which behave quite differently. Theorems by Weyl and Iwahori are used to decompose representations of semisimple Lie algebras. The faithful representations can then be characterised in terms of these decompositions. It turns out that the μ-invariant is additive for semisimple Lie algebras. In general, a reductive Lie algebra contains both simple and abelian ideals. To tackle this case, we first compute the centraliser of a semisimple Lie algebra in a general linear algebra $\mathfrak{gl}(V)$. From this, we can draw a number of conclusions about the μ-invariant, including:

Theorem Consider a reductive Lie algebra $\mathfrak{g}$. Let $l_0 = \max\{\ 0, z - l\ \}$ where l is the length of the commutator and z is the dimension of the centre. Then

$$\mu(\mathfrak{g}) = \mu([\mathfrak{g}, \mathfrak{g}]) + \mu(\frac{\mathfrak{z}(\mathfrak{g})}{\mathbb{C}^{l_0}}).$$

After this, it is not difficult to produce a list of all reductive subalgebras of a general linear algebra $\mathfrak{gl}(V)$ up to isomorphism, where V is an arbitrary finite-dimensional vector space. Finally, we discuss how this is related to the classification of maximal reductive subalgebras of a given simple Lie algebra.

2.1 Abelian Lie algebras

Abelian subalgebras In the first case, we consider abelian Lie algebras. These Lie algebras are, up to isomorphism, completely determined by their dimension. Note that if a Lie algebra $\mathfrak{g}$ contains an abelian subalgebra, then it also contains the abelian Lie algebras of lower dimensions. For convenience later, we introduce the following invariant which respects inclusions and direct sums. More precisely, it satisfies the monotonicity and the additivity properties.

Definition 2.1.0.1. For any finite-dimensional Lie algebra $\mathfrak{g}$, we define the α-invariant $\alpha(\mathfrak{g})$ to be the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$.

Lemma 2.1.0.1. *Suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two Lie algebras. Then we have:*

(*i.*) *If $\mathfrak{g}_1 \leq \mathfrak{g}_2$, then $\alpha(\mathfrak{g}_1) \leq \alpha(\mathfrak{g}_2)$.*

(*ii.*) *$\alpha(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \alpha(\mathfrak{g}_1) + \alpha(\mathfrak{g}_2)$.*

Proof*:* (*i.*) Any abelian subalgebra of $\mathfrak{g}_1$ is an abelian subalgebra of $\mathfrak{g}_2$. (*ii.*) Let $\pi_{\mathfrak{g}_1}$ and $\pi_{\mathfrak{g}_2}$ be the natural projections of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ onto $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$. If $\mathfrak{a}$ is any abelian subalgebra of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, then so is $\overline{\mathfrak{a}} = \pi_{\mathfrak{g}_1}(\mathfrak{a}) \oplus \pi_{\mathfrak{g}_2}(\mathfrak{a})$. Since $\mathfrak{a} \leq \overline{\mathfrak{a}}$, we may conclude that $\alpha(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \leq \alpha(\mathfrak{g}_1) + \alpha(\mathfrak{g}_2)$. The converse inequality can be obtained from the direct sum of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ where $\mathfrak{a}_1$, resp. $\mathfrak{a}_2$ is an abelian subalgebra of maximal dimension of $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$.

$\square$

In particular, $\alpha(\mathfrak{g})$ is an invariant of $\mathfrak{g}$. It can be interpreted in the following way. An abelian Lie algebra $\mathfrak{a}$ can be embedded into a Lie algebra $\mathfrak{g}$ if and only if $\dim(\mathfrak{a}) \leq \alpha(\mathfrak{g})$. A Lie algebra is abelian if and only if its α-invariant and dimension coincide. An abelian subalgebra of dimension $\alpha(\mathfrak{g})$ is a maximal abelian subalgebra of $\mathfrak{g}$ but the converse is generally not true. To see this, consider for example a Cartan subalgebra of the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$: it is maximal abelian but not of maximal abelian dimension. Over one century ago, Schur computed what we would now call the α-invariant of the general linear matrix algebra.

Lemma [[Schu]] The maximal dimension, $\alpha(\mathfrak{gl}_n(\mathbb{C}))$, of an abelian subalgebra of $\mathfrak{gl}(\mathbb{C}^n)$ is given by $\lfloor (\frac{n}{2})^2 \rfloor + 1$.

This result can be used to compute the μ-invariant for the abelian Lie algebras. It says that an abelian Lie algebra $\mathfrak{a}$ can be embedded into $\mathfrak{gl}(\mathbb{C}^n)$ if and only if $\dim(\mathfrak{a}) \leq \lfloor (\frac{n}{2})^2 \rfloor + 1$.

Proposition [[Bur1]] *For a d-dimensional abelian Lie algebra $\mathbb{C}^d$, we have*

$$\mu(\mathbb{C}^d) = \begin{cases} 0 & \text{if} \quad d \leq 0 \\ 1 & \text{if} \quad d = 1. \\ \lceil 2\sqrt{d-1} \rceil & \text{if} \quad d \geq 2 \end{cases}$$

Here, the first case $d \leq 0$ is really a definition that will be useful later.

The history of α Schur proved in fact much more than a formula for $\alpha(\mathfrak{gl}(V))$. He even classified the abelian subalgebras of maximal dimension (See [Schu]). Using some explicit calculations, he obtained the following result in 1905. Let n be a natural number.

Theorem [Schur] We have $\alpha(\mathfrak{gl}_n(\mathbb{C})) = \lfloor (\frac{n}{2})^2 \rfloor + 1$. If $n > 3$, then any abelian subalgebra of this dimension is the direct sum of the scalar subalgebra $\mathbb{C}\mathbb{1}_n$ and a Lie subalgebra N consisting of only nilpotent operators such that the product of any two such operators vanishes.

Consider a vector space V of dimension n. First suppose $n = 2m$ is even. Let $V = V_1 \oplus V_2$ be a vector space decomposition of V such that both terms have dimension m. The linear transformations A of V satisfying $A(V_1) = \{0\}$ and $A(V_2) \leq V_1$, form a linear subspace of $\mathrm{End}(V)$ and even an abelian Lie subalgebra of $\mathfrak{gl}(V)$. Denote this space by $\mathcal{A}_n$. Note that $\mathcal{A}_n \circ \mathcal{A}_n = 0$ so that $[\mathcal{A}_n, \mathcal{A}_n] = 0$. Now suppose $n = 2m+1$ is odd. Consider a vector space decomposition $V = V_1 \oplus V_2$ with $(\dim(V_1), \dim(V_2)) = (m, m+1)$. Define $\mathcal{A}_{n,1}$ as the space of linear transformations A satisfying $A(V_1) = \{0\}$ and $A(V_2) \leq V_1$. Consider another vector space decomposition $V = W_1 \oplus W_2$ with $(\dim(W_1), \dim(W_2)) = (m+1, m)$. Define $\mathcal{A}_{n,2}$ to be the space of linear transformations A satisfying $A(W_1) = \{0\}$ and $A(W_2) \leq W_1$. Then both $\mathcal{A}_{n,1}$ and $\mathcal{A}_{n,2}$ are abelian subalgebras of $\mathfrak{gl}(V)$. Note that $\mathcal{A}_{n,1} \circ \mathcal{A}_{n,1} = 0$ so that $[\mathcal{A}_{n,1}, \mathcal{A}_{n,1}] = 0$. The same goes for $\mathcal{A}_{n,2}$.

Theorem [ctd.] Such a Lie algebra N is equivalent to:

- For $n \neq 3$: $\mathcal{A}_n$, $\mathcal{A}_{n,1}$ or $\mathcal{A}_{n,2}$ depending on the parity of n.
- For $n = 3$: $\mathcal{A}_{3,1}$, $\mathcal{A}_{3,2}$ or to the algebra of transformations with a matrix of the form,

$$\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & a & 0 \end{pmatrix}.$$

Corollary 2.1.0.1. *For every abelian Lie algebra $\mathfrak{a} > 0$ and every $\varepsilon \in \mathbb{N}$, we have*

$$\mu_\varepsilon(\mathfrak{a}) = \mu_\infty(\mathfrak{a}) = \mu(\mathfrak{a} \oplus \mathbb{C}).$$

Later, in 1944, Jacobson [Jac] simplified the techniques and generalised the result to characteristic zero. It should be noted however, that he still makes explicit matrix manipulations. Then, in 1975, there was a completely new proof given by Gustafson. He noted that a maximal abelian Lie subalgebra is also a finite-dimensional commutative algebra with unit and so in [Gus] he applied the standard structure theory of commutative algebra in a very elegant way to classify abelian subalgebras of maximal dimension. In 1945, Malcev [Mal3, Mal4] generalised Schur's result by essentially classifying the abelian subalgebras of maximal dimension for the semisimple Lie algebras. He did this by classifying maximal commutative systems of roots. Finally, Suter [Sut] computed the maximal dimension of abelian ideals in the semisimple Lie algebras. He did not do it case-by-case as Malcev did, but using a general approach with root systems.

2.2 Semisimple Lie algebras

Let $\mathfrak{s}$ be a semisimple Lie algebra. Then it is well known that this Lie algebra decomposes into simple ideals $\mathfrak{s}_1, \ldots, \mathfrak{s}_l$: $\mathfrak{s} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_l$. The number of simple ideals into which $\mathfrak{s}$ decomposes and the simple ideals themselves are unique and this decomposition is unique up to order of the terms. The *length* $l = l(\mathfrak{s})$ of $\mathfrak{s}$ is defined as the number of simple ideals in such a decomposition. More generally, the length of a (finite-dimensional) Lie algebra can be defined as the maximal number of terms in a reduced direct sum decomposition of ideals. The length and dimension coincide exactly for the abelian Lie algebras.

2.2.1 Decomposition theorems

Just as every semisimple Lie algebra decomposes into elementary pieces, so do their representations. These elementary pieces are the *irreducible* representations. There are two theorems that describe the decomposition.

> **Theorem** [Weyl] Every complex representation $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ of a semisimple Lie algebra $\mathfrak{s}$ is equivalent to a representation that is the direct sum of irreducible representations $\rho_1, \ldots, \rho_k$: $\rho = \rho_1 \oplus \ldots \oplus \rho_k$. Two representations $\rho \sim \oplus_{i=1}^{k} \rho_i$ and $\rho' \sim \oplus_{i=1}^{k'} \rho'_i$ are equivalent if and only if $k = k'$ and there exists a permutation $\sigma \in \mathcal{S}_k$ such that $\rho_i \sim \rho'_{\sigma(i)}$ for all i.

The irreducible representations of a semisimple Lie algebra can be broken up into irreducible representations of the *simple* ideals. Let π_j be the natural projection of $\mathfrak{s}$ onto $\mathfrak{s}_j$ with respect to the above decomposition.

> **Theorem** [Schur-Iwahori] Every irreducible representation $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ is equivalent to a representation that is the tensor product $(\rho_1 \circ \pi_1) \otimes \ldots \otimes (\rho_l \circ \pi_l)$ of irreducible $\mathfrak{s}_j$-representations $\rho_j : \mathfrak{s}_j \longrightarrow \mathfrak{gl}(V_j)$. Two such (irreducible)

representations $\rho \sim \otimes_{j=1}^{l} \rho_j$ and $\rho' \sim \otimes_{j=1}^{l} \rho'_j$ are equivalent if and only if $\rho_j \sim \rho'_j$ for all j.

A representation of a simple Lie algebra is either trivially zero or faithful since the kernel is an ideal of the Lie algebra. In particular, a minimal faithful representation of a simple Lie algebra is automatically irreducible. Note that these minimal representations are not necessarily unique up to isomorphism. Consider the list of invariants for the simple Lie algebras in Table 2.1. Here, $\nu(\mathfrak{s})$ is the number of minimal faithful representations up to equivalence, i.e.: linear conjugation.

$\mathfrak{s}$	$d(\mathfrak{s})$	$\alpha(\mathfrak{s})$	$\mu(\mathfrak{s})$	$\nu(\mathfrak{s})$		n	Type
A_n	$n(n+2)$	$\lfloor (\frac{n+1}{2})^2 \rfloor$	$n+1$	$\frac{1}{2}$		$\frac{1=n}{2 \leq n}$	$\mathfrak{sl}_{n+1}$
B_n	$n(2n+1)$	$\frac{n(n-1)}{2}+1$	$2n+1$	1		$3 \leq n$	$\mathfrak{o}_{2n+1}$
C_n	$n(2n+1)$	$\frac{n(n+1)}{2}$	$2n$	1		$2 \leq n$	$\mathfrak{sp}_{2n}$
D_n	$n(2n-1)$	$\frac{n(n-1)}{2}$	$2n$	1		$4 \leq n$	$\mathfrak{o}_{2n}$
G_2	14	3	7	1			$\mathfrak{g}_2$
F_4	52	9	26	1			$\mathfrak{f}_4$
E_6	78	16	27	2			$\mathfrak{e}_6$
E_7	133	27	56	1			$\mathfrak{e}_7$
E_8	248	36	248	1			$\mathfrak{e}_8$

Table 2.1: Invariants of the Semisimple Lie Algebras

So consider an $\mathfrak{s}$-representation $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$. The theorems above give an equivalent representation that decomposes as $\oplus_i^k \otimes_j^l \rho_{i,j}$. This induces a $(k \times l)$-matrix with (i,j)'th entry equal to $[\rho_{i,j}]$, the equivalence class of $\rho_{i,j}$ as an $\mathfrak{s}_j$-representation:

$$\begin{array}{|c|c|c|} \hline [\rho_{1,1}] & \cdots & [\rho_{1,l}] \\ \hline \vdots & \ddots & \vdots \\ \hline [\rho_{k,1}] & \cdots & [\rho_{k,l}] \\ \hline \end{array} .$$

Remark 2. Define an equivalence relation on the set $\mathcal{R} = \mathcal{R}(\mathfrak{s})$ of these matrices as follows. Two elements P and Q of $\mathcal{R}$ are equivalent, $P \sim Q$, if and only if they are of the same size ($k = k'$) and there exists a permutation $\sigma \in \mathcal{S}_k$ such that $P_{i,j} = Q_{\sigma(i),j}$ for all (i,j). So we have a natural one-to-one correspondence between the equivalence classes of $\mathfrak{s}$-representations and the equivalence classes of $\mathcal{R}(\mathfrak{s})$. From now on, we will identify these two sets. The degree of $[R] \in \mathcal{R}/\sim$ is just $\sum_i^k \prod_j^l \deg(R_{i,j})$, which (of course), does not depend upon the choice of the representative. Note that for an equivalence class there are only two possibilities: either all representatives are faithful, or none of them are.

Lemma 2.2.1.1. *For $\rho_{i,j}$ as before, we have:* $\ker(\otimes_t^l \rho_{i,t} \circ \pi_t) = \oplus_t^l \ker(\rho_{i,t} \circ \pi_t)$.

> Proof*:* We can drop the subscript i. First, assume that $l = 2$, i.e.: we work with the irreducible representations $\rho_t : \mathfrak{s}_t \longrightarrow \mathfrak{gl}(V_t)$. Then $\ker(\rho_1 \circ \pi_1) \oplus \ker(\rho_2 \circ \pi_2)$ is contained in the kernel of $(\rho_1 \circ \pi_1) \otimes (\rho_2 \circ \pi_2)$. Conversely, assume that $z \in \ker((\rho_1 \circ \pi_1) \otimes (\rho_2 \circ \pi_2))$, i.e.: $(\rho_1(\pi_1(z))v_1) \otimes v_2 + v_1 \otimes (\rho_2(\pi_2(z))v_2) = \mathbb{O}$ for all v_1 in V_1 and all v_2 in V_2. Then, using explicit bases, we see that there is a constant $\alpha \in \mathbb{C}$ such that
>
> $$\rho_1(\pi_1(z)) = \alpha \mathbb{1}_{V_1} \text{ and } \rho_2(\pi_2(z)) = -\alpha \mathbb{1}_{V_2}.$$
>
> Since $\rho_t(\pi_t(z))$ is traceless, the constant is zero and in particular we have that $\pi_1(z) \in \ker(\rho_1)$ and $\pi_2(z) \in \ker(\rho_2)$. For $l > 2$, we can use the same argument inductively by replacing $\mathfrak{s}_1$ by the first $l-1$ ideals and $\mathfrak{s}_2$ by $\mathfrak{s}_l$. This finishes the proof.
>
> $\square$

Lemma 2.2.1.2. *An equivalence class $[R]$ is faithful if and only if R has no columns that contain only trivial elements. This does not depend on the choice of the representative.*

> Proof*:* Suppose there is a column with all elements equal to the trivial class. Then the corresponding ideal is contained in the kernel. Conversely, suppose that there is no trivial column for $R = ([\rho_{i,j}])_{i,j}$ and let z be an element of the kernel. Fix an index j. Then there exists an i such that $[\rho_{i,j}]$ is faithful. In particular, we have
>
> $$0 = \rho_i(z) = \otimes_t^l \rho_{i,t}(z_t).$$

Since $\ker(\otimes_t^l \rho_{i,t}) = \oplus_t^l \ker(\rho_{i,t})$, we have $z_j = \pi_j(z) \in \ker(\rho_{i,j}) = \{0\}$. Similarly, we can prove that all the other projections of z are zero. This means that z itself is zero and this finishes the proof.

□

2.2.2 The additivity of the μ-invariant

Now we are ready to compute the minimal dimension of a faithful representation for a semisimple Lie algebra and to classify the faithful representations of this minimal degree.

Theorem 2.2.2.1. *Let $\mathfrak{s} = \oplus_i^l \mathfrak{s}_i$ as above. Then the minimal dimension of a faithful representation is*

$$\mu(\mathfrak{s}) = \mu(\mathfrak{s}_1) + \ldots + \mu(\mathfrak{s}_l).$$

Proof*:* First, consider the classes of the form $[R_0]$ for

$$R_0 = \begin{array}{|c|c|c|c|} \hline [\rho_{1,1}] & [1] & \cdots & [1] \\ \hline [1] & [\rho_{2,2}] & \ddots & [1] \\ \hline \vdots & \ddots & \ddots & [1] \\ \hline [1] & [1] & [1] & [\rho_{l,l}] \\ \hline \end{array},$$

where $\rho_{i,i}$ is some minimal, faithful representation for $\mathfrak{s}_i$ (in particular of degree $\mu(\mathfrak{s}_i)$) and $[1]$ is the trivial class. Then $[R_0]$ is faithful, $\deg([R_0]) = \sum_i^l \mu(\mathfrak{s}_i)$ and we have the upper bound $\mu(\mathfrak{s}) \leq \sum_i^l \mu(\mathfrak{s}_i)$. We will prove that this inequality is an equality. Let $[R]$ be a minimal faithful class with $R = ([\rho_{i,j}])_{i,j}$.

- Note that for any two natural numbers p and q that are at least two, we have the inequality $p + q \leq p \cdot q$. Suppose that R has a row, say row i, that contains more than one non-trivial element, say elements at the position (i, x) and (i, y). Then we can remove this row and add two new rows:

$$\begin{array}{|c|c|c|c|c|} \hline \cdots & [\rho_{i,j_x}] & \cdots & [\rho_{i,j_y}] & \cdots \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|} \hline \cdots & [\rho_{i,j_x}] & \cdots & [1] & \cdots \\ \hline \cdots & [1] & \cdots & [\rho_{i,j_y}] & \cdots \\ \hline \end{array},$$

  so that we obtain a new matrix R'. Note that this matrix is faithful and that $\deg(R') \leq \deg(R)$. Since R was minimal, we have equality. In particular, R' is a minimal faithful representation. By applying this transformation repeatedly, we finally obtain a minimal, faithful matrix $\overline{R}$ such that every row of $\overline{R}$ contains at most one non-trivial element.

- There cannot be a row that contains only trivial elements, since removing this row would give a faithful matrix of strictly smaller degree. So every row of $\overline{R}$ contains exactly one element unequal to the trivial one.

- Suppose there is a column, say column j, that contains more than one non-trivial element, say elements at position (x, j) and (y, j). Then we can remove row x to obtain a faithful matrix of dimension strictly smaller. This contradicts the minimality of $\overline{R}$. So every column contains exactly one element different from the trivial one.

- Consider the j'th column of $\overline{R}$. Then the non-trivial element is a faithful representation of the ideal $\mathfrak{s}_j$. If it is not of minimal degree, we can replace it by an other faithful class of strictly lower degree to obtain a faithful matrix $\overline{R}'$ of strictly lower degree. This contradicts the minimality of $\overline{R}$. So, up to equivalence, $\overline{R}$ is of the form R_0 from the beginning of the proof.

This gives us $\mu(\mathfrak{s}) = \deg([R]) = \deg([R_0]) = \sum_i^l \mu(\mathfrak{s}_i)$, which finishes the proof.

□

Remark 3. In the proof we have worked with complex Lie algebras and their complex representations. The proof holds if we work with real Lie algebras and their complex representations because the decomposition theorems of Weyl and Iwahori-Schur can be used in this case. Note however, that the additivity fails for real representations of real Lie algebras. Consider for example the semisimple Lie algebra $\mathfrak{so}_4(\mathbb{R}) = \mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$. Then $\mu(\mathfrak{so}_4(\mathbb{R}), \mathbb{R}) = 4$ and $\mu(\mathfrak{so}_3(\mathbb{R}), \mathbb{R}) = 3$. The natural, irreducible representation of $\mathfrak{so}_4(\mathbb{R})$ is an embedding of minimal degree. Hence $\mu(\mathfrak{so}_3(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})) \neq \mu(\mathfrak{so}_3(\mathbb{R})) + \mu(\mathfrak{so}_3(\mathbb{R}))$.

2.2.3 Classification

Recall that two representations ρ_1 and ρ_2 of a Lie algebra $\mathfrak{g}$ in V are called equivalent if they are conjugated, i.e., if there exists an isomorphism T of V such that $\rho_1(x) \circ T \equiv T \circ \rho_2(x)$ for all x in $\mathfrak{g}$. We now describe the minimal faithful representations of the semisimple Lie algebras and count them - up to equivalence.

Definition 2.2.3.1. For a Lie algebra $\mathfrak{g}$ we define $\nu(\mathfrak{g})$ to be the number of faithful representations of minimal degree $\mu(\mathfrak{g})$, up to equivalence.

In Table 2.1, the ν-invariant for (complex) semisimple Lie algebras is presented. For $\mathfrak{sl}_n(\mathbb{C})$ $(n \geq 2)$ and E_6 it is two. For all other simple Lie algebras it is one. The proof of theorem 2.2.2.1 already gave explicit examples of minimal faithful representations for semisimple Lie algebras but it did not give all of them.

Example 2.2.3.1. Let $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are copies of $\mathfrak{sl}_2(\mathbb{C})$. Then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ each have a unique faithful representation of minimal degree: ρ_1 resp. ρ_2. They are of degree two. Consider the following representations of $\mathfrak{s}_1 \oplus \mathfrak{s}_2$,

$$\begin{aligned} \rho_1 \oplus \rho_2 : \mathfrak{s}_1 \oplus \mathfrak{s}_2 &\longrightarrow \mathfrak{gl}_4(\mathbb{C}) \\ \rho_1 \otimes \rho_2 : \mathfrak{s}_1 \oplus \mathfrak{s}_2 &\longrightarrow \mathfrak{gl}_4(\mathbb{C}), \end{aligned}$$

with associated tableaux,

$$\begin{array}{|c|c|} \hline [\rho_1] & [\mathbb{1}] \\ \hline [\mathbb{1}] & [\rho_2] \\ \hline \end{array} \text{ resp. } \begin{array}{|c|c|} \hline [\rho_1] & [\rho_2] \\ \hline \end{array}.$$

From this it is immediately clear that both representations are faithful and of degree $\mu(\mathfrak{s}_1 \oplus \mathfrak{s}_2) = 4$. They are also inequivalent since $\rho_1 \otimes \rho_2$ is irreducible and $\rho_1 \oplus \rho_2$ is not.

$\triangle$

First case Now consider the special case: $\mathfrak{s}' = \mathfrak{s}'_1 \oplus \ldots \oplus \mathfrak{s}'_{l'}$ where every $\mathfrak{s}'_i$ is a copy of $\mathfrak{sl}_2(\mathbb{C})$. Let ρ' be a faithful representation of $\mathfrak{s}'$ of *minimal* degree: $\mu(\mathfrak{s}') = 2l'$. Fix a tableau for ρ. Since, $p + q \leq p \cdot q$ for natural numbers $2 \leq p, q$ with equality for only $p = q = 2$, we can make the following observations:

(*i.*) Any column contains exactly one non-trivial element. (*ii.*) All non-trivial elements are the same, of degree two. (*iii.*) Any row contains either one or two non-trivial elements.

For otherwise, it would be easy to construct an other faithful representation of strictly smaller degree (as was shown in the proof of theorem 2.2.2.1). Iwahori's theorem says that two such representations are equivalent if and only if the associated tableaux are, i.e.: if they are the same up to a permutation of the rows. These observations prove the following lemma.

Lemma 2.2.3.1. *The equivalence classes of minimal faithful representations of the Lie algebra $\mathfrak{s}'$ are in a natural one-to-one correspondence to the partitions of the set of the l' simple ideals of $\mathfrak{s}'$ into subsets of size one and two.*

In the example above, the partitions of ideals into subsets of size one and two, are: $\{ \{ \mathfrak{s}'_1 \}, \{ \mathfrak{s}'_2 \} \}$ and $\{ \{ \mathfrak{s}'_1, \mathfrak{s}'_2 \} \}$. This implies that $\nu(\mathfrak{s}') = 2$. In the general case, we obtain the formula, $\nu(\mathfrak{s}') = c_{l'}$ where $(c_n)_n$ is the unique solution to the linear recurrence relation

$$c_{n+2} - c_{n+1} = (n+1)c_n$$

with initial conditions $c_0 = c_1 = 1$.

Second case Now consider another special case: $\mathfrak{s}'' = \mathfrak{s}''_1 \oplus \ldots \oplus \mathfrak{s}''_{l''}$ where each $\mathfrak{s}''_i$ is simple and *not* isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let ρ'' be a faithful representation of minimal degree $\mu(\mathfrak{s}'') = \sum_i \mu(\mathfrak{s}''_i)$ and fix a tableau for it. Since $\mu(\mathfrak{s}''_i) > 2$ for all i, the $(p+q), (p \cdot q)$-argument shows that every row and every column contains exactly one non-trivial element. This implies that the tableau is square and, up to permutation, of the form,

$$\begin{array}{|c|c|c|c|}\hline [\rho''_1] & \mathbb{1} & \mathbb{1} & \mathbb{1} \\ \hline \mathbb{1} & [\rho''_2] & \mathbb{1} & \mathbb{1} \\ \hline \mathbb{1} & \mathbb{1} & \ddots & \mathbb{1} \\ \hline \mathbb{1} & \mathbb{1} & \mathbb{1} & [\rho''_{l''}] \\ \hline \end{array},$$

where each ρ''_i is a faithful representation of $\mathfrak{s}''_i$ of minimal degree. Iwahori's theorem then gives the following lemma.

Lemma 2.2.3.2. *The minimal faithful representations of the Lie algebra $\mathfrak{s}''$ are of the form $\rho''_1 \oplus \ldots \oplus \rho''_{l''}$ where each ρ''_i is a faithful representation for the simple ideal $\mathfrak{s}''_i$ of minimal degree. Two representations are equivalent if and only if all corresponding terms are.*

This implies that $\nu(\mathfrak{s}'') = \nu(\mathfrak{s}''_1) \cdot \ldots \cdot \nu(\mathfrak{s}''_{l''})$.

General case Fix a semisimple Lie algebra $\mathfrak{s}$ of length l and decompose it into simple ideals. We may assume that the first l' of them are copies of $\mathfrak{sl}_2(\mathbb{C})$ and that the l'' rest of them are not:

$$\mathfrak{s} = (\mathfrak{s}'_1 \oplus \cdots \oplus \mathfrak{s}'_{l'}) \bigoplus (\mathfrak{s}''_1 \oplus \cdots \oplus \mathfrak{s}''_{l''}) = \mathfrak{s}' \oplus \mathfrak{s}''.$$

The length of $\mathfrak{s}'$ and $\mathfrak{s}''$ is l' resp. l'' so that in particular, $l = l' + l''$. Fix a faithful representation ρ of minimal degree for $\mathfrak{s}$ and fix a tableau. Then the usual $(p+q), (p \cdot q)$-argument shows that each row contains either one non-trivial element or two non-trivial elements. The latter can only occur if both entries correspond to ideals of $\mathfrak{s}'$. Up to equivalence, we obtain a representation with a tableau of the form,

$$\begin{array}{|c|c|}\hline [\rho'] & [\mathbb{1}] \\ \hline [\mathbb{1}] & [\rho''] \\ \hline \end{array},$$

where ρ' resp: ρ'' is a representation of $\mathfrak{s}'$ resp. $\mathfrak{s}''$. These are necessarily faithful and of minimal degree. The decomposition of ρ into ρ' and ρ'' is unique (up to equivalence) according to Iwahori's theorem. These observations prove the following lemma.

Lemma 2.2.3.3. *The faithful representations ρ of minimal degree for $\mathfrak{s}$ are the ones equivalent to those of the form $\rho' \oplus \rho'' = (\rho' \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes \rho'')$ where ρ' resp. ρ'' is a faithful representation of $\mathfrak{s}'$ resp. $\mathfrak{s}''$ of minimal degree. Two such representations are equivalent if and only if the corresponding terms are.*

This allows us to conclude that $\nu(\mathfrak{s}) = \nu(\mathfrak{s}') \cdot \nu(\mathfrak{s}'')$, where the factors were computed in the two previous cases.

2.3 Reductive Lie algebras

A reductive Lie algebra $\mathfrak{g}$ decomposes into a direct sum of simple and abelian ideals. We already know how to compute the μ-invariant for each of these ideals. We will reduce the computation of $\mu(\mathfrak{g})$ to that of its ideals.

Lemma 2.3.0.4. *For any Lie algebra* $\mathfrak{g}$ *satisfying* $\mathcal{Z}(\mathfrak{g}) \leq [\mathfrak{g}, \mathfrak{g}]$*, we have* $\mu(\mathfrak{g} \oplus \mathbb{C}) = \mu(\mathfrak{g})$.

Proof*:* Consider a faithful representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Then the representation $\tilde{\rho} : \mathfrak{g} \oplus \mathbb{C} \longrightarrow \mathfrak{gl}(V) : (x, t) \longmapsto \rho(x) + t \cdot \mathbb{1}_V$ is faithful if $\mathfrak{g}$ satisfies $\mathcal{Z}(\mathfrak{g}) \leq [\mathfrak{g}, \mathfrak{g}]$. For such algebras $\mathfrak{g}$, we conclude $\mu(\mathfrak{g} \oplus \mathbb{C}) \leq \mu(\mathfrak{g})$. The monotonicity of μ implies the converse inequality.

□

Corollary 2.3.0.1. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_l$ *be Lie algebras as above let and* k *be a natural number. Then*

$$\mu(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^k) \leq \mu(\mathfrak{g}_1) + \ldots + \mu(\mathfrak{g}_l) + \mu(\mathbb{C}^{l-k}).$$

Proof*:* First suppose that $l \leq k$. By using the monotonicity and lemma 2.3.0.4 we obtain,

$$\begin{aligned} \mu(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^k) &\leq \mu(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^l) + \mu(\mathbb{C}^{k-l}) \\ &= \mu(\mathfrak{g}_1 \oplus \mathbb{C}) + \ldots + \mu(\mathfrak{g}_l \oplus \mathbb{C}) + \mu(\mathbb{C}^{k-l}) \\ &= \mu(\mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_l) + \mu(\mathbb{C}^{k-l}). \end{aligned}$$

If $k \leq l$, then we can embed $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^k$ into $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^l$. Then the monotonicity and the previous case give,

$$\begin{aligned} \mu(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^k) &\leq \mu(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l \oplus \mathbb{C}^l) \\ &\leq \mu(\mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_l) + \mu(\mathbb{C}^{l-l}) \\ &= \mu(\mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_l) + \mu(\mathbb{C}^{k-l}). \end{aligned}$$

□

Let $\mathfrak{g}$ be a reductive Lie algebra. We already know that it decomposes into a direct sum of ideals that are simple or abelian. The sum of all simple ideals of $\mathfrak{g}$, which is a semisimple Lie algebra, coincides with the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. The sum of all abelian ideals gives us exactly the centre $\mathcal{Z}(\mathfrak{g})$ of $\mathfrak{g}$. The reductive Lie algebras are then exactly the direct sums of a semisimple Lie algebra $\mathfrak{s}$ and an abelian Lie algebra $\mathfrak{a}$: $\mathfrak{s} \oplus \mathfrak{a}$.

Corollary 2.3.0.2. *For any reductive Lie algebra $\mathfrak{g}$ with commutator $\mathfrak{s}$, centre $\mathfrak{a}$ and commutator length $l(\mathfrak{s})$, we have*

$$\mu(\mathfrak{g}) \leq \mu(\mathfrak{s}) + \mu(\mathbb{C}^{\dim(\mathfrak{a})-l(\mathfrak{s})}).$$

We can already note that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})$ for all reductive Lie algebras $\mathfrak{g}$.

2.3.1 Centralisers of semisimple Lie algebras

In order to prove equality in corollary 2.3.0.2, the following lemma is crucial.

Lemma 2.3.1.1. *Let $\mathfrak{s}$ be semisimple, $\mathfrak{h}$ be some Lie algebra and choose a natural number n, at least $\mu(\mathfrak{s})$. Then $\mathfrak{s} \oplus \mathfrak{h}$ can be embedded into $\mathfrak{gl}(\mathbb{C}^n)$ if and only if $\mathfrak{h}$ can be embedded into $\mathfrak{gl}(\mathbb{C}^{n-\mu(\mathfrak{s})}) \oplus \mathbb{C}^{l(s)}$.*

We give the proof in several steps, essentially by breaking down the representations of reductive Lie algebras into smaller pieces. Two representations $\rho_1 : \mathfrak{g}_1 \longrightarrow \mathfrak{gl}(V)$ and $\rho_2 : \mathfrak{g}_2 \longrightarrow \mathfrak{gl}(V)$ are said to commute, if $[\rho_1(x_1), \rho_2(x_2)] = 0$ for every $x_1 \in \mathfrak{g}_1$ and every $x_2 \in \mathfrak{g}_2$.

Lemma 2.3.1.2 (Cut-and-Paste)**.** *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras and suppose that one of them has a trivial centre. Then there is a bijective correspondence between representations as follows:*

(1) A faithful representation $\rho : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \longrightarrow \mathfrak{gl}_n(\mathbb{C})$ induces a pair of commuting representations (ρ_1, ρ_2) by inclusion, given by $\rho_j = \rho \circ \iota_j : \mathfrak{g}_j \longrightarrow \mathfrak{gl}_n(\mathbb{C})$ for $j = 1, 2$, where ι_j is the natural inclusion of $\mathfrak{g}_j$ into $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

(2) Conversely a pair of commuting faithful representations $\rho_j : \mathfrak{g}_j \longrightarrow \mathfrak{gl}_n(\mathbb{C})$ induces a faithful representation $\rho : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \longrightarrow \mathfrak{gl}_n(\mathbb{C})$ by $\rho = \rho_1 \circ \pi_1 + \rho_2 \circ \pi_2$, where π_j is the natural projection of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ onto $\mathfrak{g}_j$.

Proof. It is clear that ρ_1, ρ_2 are faithful representations. We have

$$[\rho_1(x), \phi_2(y)] = [\rho(x, 0), \phi(0, y)] = \rho([(x, 0), (0, y)]) = 0.$$

This shows (1). For (2) we may assume that $\mathfrak{g}_1$ has no centre. Choose any $(x, y) \in \ker(\phi)$. This means $\rho_1(x) + \rho_2(y) = 0$, so that

$$\rho_1(x) = -\rho_2(y) \in \rho_1(\mathfrak{g}_1) \cap \rho_2(\mathfrak{g}_2) \subseteq Z(\rho_1(\mathfrak{g}_1)) = Z(\mathfrak{g}_1) = 0.$$

In particular, x and y are mapped to zero. Since both ρ_1 and ρ_2 are faithful, we have $(x, y) = (0, 0)$. This implies that ρ is faithful. $\square$

Consider a Lie algebra embedding $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Then we define the centraliser of this representation to be,

$$\begin{aligned} \mathcal{C}_{\mathfrak{gl}(V)}(\rho) &= \mathcal{C}(\rho) \\ &= \{ M \in \mathfrak{gl}(V) \mid [\rho(x), M] = 0 \text{ for every } x \in \mathfrak{g} \}. \end{aligned}$$

Note that $\mathcal{C}(\rho)$ is a subalgebra of $\mathfrak{gl}(V)$. The centraliser of a representation of a semisimple Lie algebra can easily be described in terms of it's Weyl decomposition. Let $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(\mathbb{C}^n)$ be a representation that decomposes into irreducibles $\oplus_{i=1}^k \rho_i$ according to theorem 2.2.1. It is possible that there are equivalent terms in this decomposition. So, up to equivalence, we may assume that ρ is of the form $\rho = \bigoplus_{i=1}^k m_i\rho_i$ where $\rho_i \sim \rho_j$ if and only if $i = j$. The number m_i is the multiplicity of the irreducible representation ρ_i in ρ. We may also assume that none of the multiplicities is zero. The following is a generalisation of Schur's lemma for irreducible representations.

Lemma 2.3.1.3. *Consider the representation $\oplus_i^k m_i\rho_i$ for inequivalent, irreducible representations ρ_i. Then we have the isomorphisms of Lie algebras*

$$\mathcal{C}(\bigoplus_i^k m_i\rho_i) \cong \bigoplus_i^k \mathcal{C}(m_i\rho_i) \cong \bigoplus_i^k \mathfrak{gl}(\mathbb{C}^{m_i}).$$

In particular, we see that the centraliser is automatically a reductive Lie algebra. Note also that the lemma reduces to the traditional lemma for $k = 1$ and $m_1 = 1$.

Definition 2.3.1.1. Fix a semisimple Lie algebra decomposition $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_l$ of length l, and an integer $n \geq \mu(\mathfrak{s})$. Consider the following $\mathfrak{s}$-representation, $\sigma_n = \bigoplus_{i=1}^l (\rho_i \circ \iota_i) \bigoplus (n - \mu(\mathfrak{s}))\tau$, where the ρ_i are faithful $\mathfrak{s}_i$-representations of degree $\mu(\mathfrak{s}_i)$ and where τ is the trivial representation of dimension one, as usual. A representation of this form (or, a representation equivalent to one of this form) is called a standard-block representation of degree n for $\mathfrak{s} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_l$.

Here, the associated tableaux is of the form of Figure 2.1. In particular, it is immediately clear that the σ_n are faithful and that $\sigma_n = \sigma_{\mu(\mathfrak{s})} \oplus (n - \mu(\mathfrak{s}))\tau$ for some standard block representation $\sigma_{\mu(\mathfrak{s})}$ of degree $\mu(\mathfrak{s})$. These representations turn out to have the biggest centralisers: For every standard block representation σ_n of degree n, we have the isomorphism of Lie algebras,

$$\mathcal{C}(\sigma_n) \cong \mathfrak{gl}(\mathbb{C}^{n-\mu(\mathfrak{s})}) \oplus \mathbb{C}^{l(\mathfrak{s})}.$$

$$[\sigma_n] = \begin{array}{|c|c|c|c|}
\hline
[\rho_1] & [\tau] & \cdots & [\tau] \\
\hline
[\tau] & [\rho_2] & \ddots & [\tau] \\
\hline
\vdots & \ddots & \ddots & [\tau] \\
\hline
[\tau] & [\tau] & [\tau] & [\rho_l] \\
\hline
\hline
[\tau] & [\tau] & [\tau] & [\tau] \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
[\tau] & [\tau] & [\tau] & [\tau] \\
\hline
\end{array}.$$

Figure 2.1: Standard Block Representation σ_n

Lemma 2.3.1.4. *The centraliser of any faithful representation $\rho_n : \mathfrak{s} \longrightarrow \mathfrak{gl}(\mathbb{C}^n)$ of degree n can be embedded into the centraliser of a standard block representation σ_n of degree n:*

$$f : \mathcal{C}(\rho_n) \hookrightarrow \mathcal{C}(\sigma_n).$$

Proof*:* Let $\rho = \bigoplus_{i=0}^{k} m_i \rho_i$ be a decomposition into the (inequivalent) irreducible representations $\rho_i : \mathfrak{s} \longrightarrow \mathfrak{gl}(\mathbb{C}^{d_i})$ of degree d_i with multiplicities m_i, where $\rho_0 = \tau$, the trivial representation.

- We may assume that $m_1, \ldots, m_r = 1$ and $m_{r+1}, \ldots, m_k > 1$ after a permutation of the terms in the decomposition. Consider the associated representation $\widetilde{\rho} = \bigoplus_{i=0}^{r} \widetilde{m}_i \rho_i$ of the same degree, where

  $$\begin{cases} \widetilde{m}_0 &= m_0 + \sum_{i=r+1}^{k} (m_i - 1) d_i, \\ \widetilde{m}_i &= 1 \qquad \text{for } 0 < i. \end{cases}$$

  It is obtained from the decomposition of ρ by lowering the multiplicities of the non-trivial terms $\rho_{r+1}, \ldots, \rho_k$ to one. Note that lemma 2.2.1.2 implies that this new representation is faithful. Then we have the following isomorphisms of Lie algebras,

  $$\begin{cases} \mathcal{C}(\rho) &\cong \mathfrak{gl}(\mathbb{C}^{m_0}) \oplus \mathbb{C}^r \bigoplus_{i=r+1}^{k} \mathfrak{gl}(\mathbb{C}^{m_i}), \\ \mathcal{C}(\widetilde{\rho}) &\cong \mathfrak{gl}(\mathbb{C}^{\widetilde{m}_0}) \oplus \mathbb{C}^r \oplus \mathbb{C}^{k-r}. \end{cases}$$

We claim that the first centraliser can be embedded into the second one. First note that we have the natural embeddings

$$\begin{cases} \mathfrak{gl}(\mathbb{C}^{\alpha_0}) & \hookrightarrow & \mathfrak{gl}(\mathbb{C}^{\alpha}) \\ \mathfrak{gl}(\mathbb{C}^{\alpha}) \oplus \mathfrak{gl}(\mathbb{C}^{\beta}) & \hookrightarrow & \mathfrak{gl}(\mathbb{C}^{\alpha+\beta}) \\ \mathbb{C}^{\alpha} & \hookrightarrow & \mathfrak{gl}(\mathbb{C}^{\alpha}) \end{cases}.$$

for all $\alpha \leq \alpha_0$ and all β in $\mathbb{N}$. It then suffices to prove that $m_0 + \sum_{i=r+1}^{k} m_i \leq \widetilde{m}_0$, which is equivalent to $\sum_{i=r+1}^{k} m_i \leq \sum_{i=r+1}^{k} (m_i - 1)d_i$. Since $m_i \geq 2$ for $1 \leq i \leq r$ and $d_i \geq 2$ for $i \neq 0$, we have $\frac{m_i}{m_i - 1} \leq 2 \leq d_i$, which implies the desired inequality.

- So consider the representation $\widetilde{\rho} = \widetilde{m}_0\rho_0 \bigoplus_{i=0}^{k} \rho_i$, for inequivalent ρ_i. Lemma 2.2.1.2 says that for every ideal $\mathfrak{s}_j$, we can choose some ρ_{i_j} such that the restriction to $\mathfrak{s}_j$ is faithful. Even if $\mathfrak{s}_j \neq \mathfrak{s}_{j'}$, we can have $\rho_{i_j} = \rho_{i_{j'}}$ so that we end up with $t \leq k$ distinct representations. After a permutation of the terms, we may assume that they are $\rho_1, \ldots, \rho_t$. Note that $\overline{\rho} = \overline{m}_0\rho_0 \bigoplus_{i=1}^{t} \rho_i$, with $\overline{m}_0 = \widetilde{m}_0 + \sum_{i=t+1}^{k} d_i$, is a faithful representation of degree $\deg(\overline{\rho}) = \deg(\widetilde{\rho}) = \deg(\rho)$. According to lemma 2.3.1.3, we have the isomorphisms of Lie algebras,

$$\begin{cases} \mathcal{C}(\widetilde{\rho}) & \cong & \mathfrak{gl}(\mathbb{C}^{\widetilde{m}_0}) \oplus \mathbb{C}^k, \\ & \cong & \mathfrak{gl}(\mathbb{C}^{\widetilde{m}_0}) \oplus \mathbb{C}^t \oplus \mathbb{C}^{k-t}, \\ \mathcal{C}(\overline{\rho}) & \cong & \mathfrak{gl}(\mathbb{C}^{\overline{m}_0}) \oplus \mathbb{C}^t. \end{cases}$$

  We claim that the first centraliser can be embedded into the second one. For this, it suffices to prove that $\widetilde{m}_0 + (k-t) \leq \overline{m}_0$, or equivalently, that $k - t \leq \sum_{i=t+1}^{k} d_i$. But this is true since $d_i \geq 1$.

- Now consider a standard block representation σ_n of degree $n = \deg(\overline{\rho}) = \deg(\rho)$. Then we have the isomorphisms of Lie algebras,

$$\begin{cases} \mathcal{C}(\overline{\rho}) & \cong & \mathfrak{gl}(\mathbb{C}^{\overline{m}_0}) \oplus \mathbb{C}^t, \\ \mathcal{C}(\sigma_n) & \cong & \mathfrak{gl}(\mathbb{C}^{n-\mu(\mathfrak{s})}) \oplus \mathbb{C}^{l(\mathfrak{s})}. \end{cases}$$

  We claim that the first centraliser can be embedded into the second one. We already noted that $t \leq l(\mathfrak{s})$, so it suffices to prove that $\overline{m}_0 \leq n - \mu(\mathfrak{s})$, or equivalently $\mu(\mathfrak{s}) + \overline{m}_0 \leq n$. Note that the $\mathfrak{s}$-representation $\bigoplus_{i=1}^t \rho_i$ of degree $\sum_{i=1}^t d_i$ is faithful so that $\mu(\mathfrak{s}) \leq \sum_{i=1}^t d_i$. But then $\mu(\mathfrak{s}) + \overline{m}_0 \leq \sum_{i=1}^t d_i + \overline{m}_0 = n$.

We finish the proof by combining the inclusions

$$\mathcal{C}(\rho) \hookrightarrow \mathcal{C}(\widetilde{\rho}) \hookrightarrow \mathcal{C}(\overline{\rho}) \hookrightarrow \mathcal{C}(\sigma_n).$$

$\square$

Proof*:* [of Lemma 2.3.1.1] Consider the Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$.

- Let $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathbb{C}^n)$ be a faithful representation of $\mathfrak{g}$. Then by lemma 2.3.1.2 we have that $\rho_{\mathfrak{h}}$, the restriction of ρ to $\mathfrak{h}$, is contained in the centraliser of $\rho_{\mathfrak{s}}$, the restriction of ρ to $\mathfrak{s}$. Lemma 2.3.1.4 then gives the embedding of $\mathfrak{h}$ into the centraliser of some σ_n, which is isomorphic to $\mathfrak{gl}(\mathbb{C}^{n-\mu(\mathfrak{s})}) \oplus \mathbb{C}^{l(\mathfrak{s})}$.
- If $\rho_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{gl}(\mathbb{C}^{n-\mu(\mathfrak{s})}) \oplus \mathbb{C}^{l(\mathfrak{s})}$ is an embedding, then $\mathfrak{h}$ embeds into the centraliser of any standard block $\mathfrak{s}$-representation σ_n of degree n. According to lemma 2.3.1.2, there is an embedding of $\mathfrak{s} \oplus \mathfrak{h}$ into $\mathfrak{gl}(\mathbb{C}^n)$.

This finishes the proof of lemma 2.3.1.1.

$\square$

2.3.2 Corollaries of the centraliser lemma

Corollary 2.3.2.1. *For a reductive Lie algebra $\mathfrak{s} \oplus \mathfrak{a}$, we have*

$$\mu(\mathfrak{s} \oplus \mathfrak{a}) = \mu(\mathfrak{s}) + \mu(\mathbb{C}^{\dim(\mathfrak{a}) - l(\mathfrak{s})}).$$

Proof*:* First note that we have already derived the inequality $\mu(\mathfrak{s} \oplus \mathfrak{a}) \leq \mu(\mathfrak{s}) + \mu(\mathbb{C}^{\dim(\mathfrak{a}) - l(\mathfrak{s})})$. Let $\rho : \mathfrak{s} \oplus \mathfrak{a} \hookrightarrow \mathfrak{gl}(V)$ be a faithful representation of degree $\mu(\mathfrak{s} \oplus \mathfrak{a})$. Then, after identifying $\mathfrak{s}$ and $\mathfrak{a}$ with their images under ρ, we may assume that $\mathfrak{a}$ is contained in the centraliser of $\mathfrak{s}$ in $\mathfrak{gl}(V)$. This centraliser can be embedded into $\mathfrak{c} = \mathfrak{gl}(\mathbb{C}^{\mu(\mathfrak{s} \oplus \mathfrak{a}) - \mu(\mathfrak{s})}) \oplus \mathbb{C}^{l(\mathfrak{s})}$. In particular, there exists an embedding of $\mathfrak{a}$ into $\mathfrak{c}$ and this is equivalent with $\dim(\mathfrak{a}) \leq \alpha(\mathfrak{c}) = \alpha(\mathfrak{gl}(\mathbb{C}^{\mu(\mathfrak{s}\oplus\mathfrak{a})-\mu(\mathfrak{s})})) + \alpha(\mathbb{C}^{l(\mathfrak{s})}) = \alpha(\mathfrak{gl}(\mathbb{C}^{\mu(\mathfrak{s}\oplus\mathfrak{a})-\mu(\mathfrak{s})})) + l(\mathfrak{s})$. This means $\dim(\mathbb{C}^{\dim(\mathfrak{a})-l(\mathfrak{s})}) = \dim(\mathfrak{a}) - l(\mathfrak{s}) \leq \alpha(\mathfrak{gl}(\mathbb{C}^{\mu(\mathfrak{s}\oplus\mathfrak{a})-\mu(\mathfrak{s})}))$ and it implies $\mu(\mathbb{C}^{\dim(\mathfrak{a})-l(\mathfrak{s})}) \leq \mu(\mathfrak{s}\oplus\mathfrak{a}) - \mu(\mathfrak{s})$. This finishes the proof.

□

Corollary 2.3.2.2. *For a Lie algebra of the form $\mathfrak{s}\oplus\mathfrak{r}$, with $\mathfrak{s}$ semisimple and $\mathfrak{r}$ arbitrary, we have*

$$\mu([\mathfrak{r},\mathfrak{r}]) \leq \mu(\mathfrak{s}\oplus\mathfrak{r}) - \mu(\mathfrak{s}) \leq \mu(\mathfrak{r}).$$

Proof*:* The upper bound $\mu(\mathfrak{s}\oplus\mathfrak{r}) \leq \mu(\mathfrak{s}) + \mu(\mathfrak{r})$ follows from the subadditivity of μ. To prove the remaining inequality, we now consider an embedding $\rho : \mathfrak{s}\oplus\mathfrak{r} \longrightarrow \mathfrak{gl}_{\mu(\mathfrak{s}\oplus\mathfrak{r})}(\mathbb{C})$ of minimal degree. Lemma 2.3.1.1 implies that $\mathfrak{r}$ can be embedded into $\mathfrak{c} = \mathfrak{gl}_{\mu(\mathfrak{s}\oplus\mathfrak{r})-\mu(\mathfrak{s})}(\mathbb{C}) \oplus \mathbb{C}^{l(\mathfrak{s})}$. In particular, we have an embedding of $[\mathfrak{r},\mathfrak{r}]$ into $\mathfrak{sl}_{\mu(\mathfrak{s}\oplus\mathfrak{r})-\mu(\mathfrak{s})} \leq \mathfrak{gl}_{\mu(\mathfrak{s}\oplus\mathfrak{r})-\mu(\mathfrak{s})}$. We conclude that $\mu([\mathfrak{r},\mathfrak{r}]) \leq \mu(\mathfrak{s}\oplus\mathfrak{r}) - \mu(\mathfrak{s})$. This finishes the proof.

□

Corollary 2.3.2.3. *For a Lie algebra of the form $\mathfrak{s}\oplus\mathfrak{p}$, with $\mathfrak{s}$ semisimple and $\mathfrak{p}$ perfect, we have*

$$\mu(\mathfrak{s}\oplus\mathfrak{p}) = \mu(\mathfrak{s}) + \mu(\mathfrak{p}).$$

In general, for two perfect Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$, we do *not* have the identity $\mu(\mathfrak{p}_1\oplus\mathfrak{p}_2) = \mu(\mathfrak{p}_1) + \mu(\mathfrak{p}_2)$. We will illustrate this with some examples later (Example 3.1 and corollary 3.1.0.2 on page 60). Note that both perfect and centreless Lie algebras $\mathfrak{g}$ satisfy the condition $\mathfrak{z}(\mathfrak{g}) \leq [\mathfrak{g},\mathfrak{g}]$. Other Lie algebras satisfying this condition, are: nilpotent Lie algebras with one-dimensional centre (e.g.: generalised Heisenberg Lie algebras, filiform nilpotent Lie algebras), free nilpotent Lie algebras. This leads us to the following generalisation:

Proposition 2.3.2.1. *Let $\mathfrak{h}$ be any Lie algebra satisfying $\mathcal{Z}(\mathfrak{h}) \leq [\mathfrak{h},\mathfrak{h}]$. Then, for any semisimple Lie algebra $\mathfrak{s}$, we have:*

$$\mu(\mathfrak{s}\oplus\mathfrak{h}) = \mu(\mathfrak{s}) + \mu(\mathfrak{h}).$$

Proof*:* The subadditivity of μ provides the upper bound. To prove the lower bound, we use the centralising lemma. Suppose that $\mathfrak{s} \oplus \mathfrak{h}$ can be embedded into $\mathfrak{gl}_n(\mathbb{C})$ for some $n \geq \mu(\mathfrak{s})$. Then $\mathfrak{h}$ can be embedded into $\mathfrak{gl}_{n-\mu(\mathfrak{s})} \oplus \mathbb{C}^{l(\mathfrak{s})}$. Consider the projection $\pi : \mathfrak{gl}_{n-\mu(\mathfrak{s})}(\mathbb{C}) \oplus \mathbb{C}^{l(\mathfrak{s})} \longrightarrow \mathfrak{gl}_{n-\mu(\mathfrak{s})}(\mathbb{C})$. It is a morphism and we claim that $\mathfrak{h} \cong \pi(\mathfrak{h}) \leq \mathfrak{gl}_{n-\mu(\mathfrak{s})}(\mathbb{C})$ so that $\mu(\mathfrak{s}) + \mu(\mathfrak{h}) \leq n$. The lower bound is then obtained for $n = \mu(\mathfrak{s} \oplus \mathfrak{h})$.

Note that the commutator of $\mathfrak{h}$ is contained in $\mathfrak{gl}_{n-\mu(\mathfrak{s})}(\mathbb{C})$. The restriction of π to commutators of $\mathfrak{h}$ is then nothing but the identity. To prove the faithfulness of π, we choose an element x in the kernel of π. Then, for any $y \in \mathfrak{h}$, we have:

$$\begin{aligned} 0 &= [\pi(x), \pi(y)] \\ &= \pi([x, y]) \\ &= [x, y]. \end{aligned}$$

So x is a central element. This implies that $x = \pi(x) = 0$. So π is faithful and this suffices to finish the proof.

□

With some minor adjustments to the proof of the centraliser-lemma, we can obtain a formula for the μ_ε-invariants for Lie algebras that are the direct sum of a semisimple and nilpotent Lie algebra, $\mathfrak{s} \oplus \mathfrak{n}$. Unlike for the μ-invariant, we will have additivity. For this we only need to remark that the $\mathbb{C}^{l(\mathfrak{s})}$-term in the centraliser-lemma gives us semisimple endomorphisms. Since the representations we consider here are of finite type, we obtain the following result.

Corollary 2.3.2.4. *Let $\mathfrak{s}$ and $\mathfrak{n}$ be a semisimple resp. nilpotent Lie algebra. Then for every $\varepsilon \in \overline{\mathbb{N}}$, $\mu_\varepsilon(\mathfrak{s} \oplus \mathfrak{n}) = \mu_\varepsilon(\mathfrak{s}) + \mu_\varepsilon(\mathfrak{n})$. For $\mathfrak{a}$ abelian, we obtain $\mu_\varepsilon(\mathfrak{s} \oplus \mathfrak{a}) = \mu_\varepsilon(\mathfrak{s}) + \mu_\varepsilon(\mathfrak{a}) = \mu(\mathfrak{s}) + \mu(\mathfrak{a} \oplus \mathbb{C})$.*

2.3.3 Reductive subalgebras of $\mathfrak{gl}_n(\mathbb{C})$

We can use the formula above to obtain a list of all reductive subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ up to isomorphism. We do this explicitly for $\mathfrak{gl}(\mathbb{C}^1), \ldots, \mathfrak{gl}(\mathbb{C}^8)$ in Table 2.2. For each $\mathfrak{gl}_n(\mathbb{C})$, we present a set S_n of reductive subalgebras.

The set of all subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ then consists of all quotients of all Lie algebras in S_n. For example, $S_2 = \{\,\mathbb{C} \oplus \mathbb{C}, A_1 \oplus \mathbb{C}\,\}$ so that the reductive subalgebras of $\mathfrak{gl}_2(\mathbb{C})$ are $0, \mathbb{C}, \mathbb{C}^2, A_1$ and $A_1 \oplus \mathbb{C}$. Note that we consider the Lie algebras up to isomorphism and not up to linear conjugation in $\mathfrak{gl}_n(\mathbb{C})$.

In general, the Lie subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ can be obtained in the following algorithmic way. First list all simple Lie algebras $\mathfrak{s}$ with $\mu(\mathfrak{s}) \leq n$. From this, we obtain a list of all semisimple subalgebras of $\mathfrak{gl}_n(\mathbb{C})$. To such an $\mathfrak{s}$, we can add an abelian Lie algebra of dimension $0, 1, \ldots$ up to $l(\mathfrak{s}) + \alpha(\mathfrak{gl}_{n-\mu(\mathfrak{s})}(\mathbb{C}))$. In this way we obtain all reductive subalgebras of $\mathfrak{gl}_n(\mathbb{C})$.

2.3.4 Maximal reductive subalgebras

Suppose we have a reductive Lie algebra $\mathfrak{g}$. A faithful representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_n$ identifies $\mathfrak{g}$ with a subalgebra of a proper maximal reductive subalgebra of $\mathfrak{gl}_n$, unless $\mathfrak{g} \cong \mathfrak{gl}_n$. There is a complete classification of all maximal reductive Lie algebras in semisimple Lie algebras, due to Malcev [Mal3, Mal4], Dynkin [Dyn1], [Dyn2], and Borel. These lists allow us to give an alternative way to compute the μ-invariant of reductive Lie algebras, at least in certain cases. Consider the following example.

Example 2.3.4.1. [We have $\mu(A_1 \oplus \mathbb{C}^4) = 5$.] First, we note that there is an obvious five-dimensional faithful representation for $\mathfrak{g}$: $\mathfrak{g} \cong \mathfrak{gl}_2 \oplus \mathbb{C}^3$ is a decomposition so that we can take the direct sum of the natural representations of the terms. In particular, we have $\mu(\mathfrak{g}) \leq 5$.

For the converse inequality, we proceed as follows. Supposing we can embed $\mathfrak{g}$ into $\mathfrak{gl}_4$, we derive a contradiction. So suppose $\mathfrak{g}$ is identified with a subalgebra of $\mathfrak{gl}_4 = A_3 \oplus \mathbb{C}$ and let $\pi : A_3 \oplus \mathbb{C} \longrightarrow A_3$ be the natural projection through the centre. Then $\pi(\mathfrak{g})$ is a subalgebra of A_3 isomorphic to $\frac{\mathfrak{g}}{\ker(\pi)}$. There are two possibilities: either $\pi(\mathfrak{g}) \cong \mathfrak{g}$ or $\pi(\mathfrak{g}) \cong A_1 \oplus \mathbb{C}^3$ since the kernel is a central ideal of dimension at most one. So in either case, we may assume that $\mathfrak{g}_0 = A_1 \oplus \mathbb{C}^3$ is contained in A_3. This $\mathfrak{g}_0$ is reductive and thus contained in a (proper) maximal reductive subalgebra of A_3, say $\mathfrak{h}$. But the maximal reductive

$\mathfrak{gl}_n$	S_n
$\mathfrak{gl}_1$	$\mathbb{C}^1$
$\mathfrak{gl}_2$	$\mathbb{C}^2, A_1 \oplus \mathbb{C}^1$
$\mathfrak{gl}_3$	$\mathbb{C}^3, A_1 \oplus \mathbb{C}^2, A_2 \oplus \mathbb{C}$
$\mathfrak{gl}_4$	$\mathbb{C}^5, A_1 \oplus \mathbb{C}^3, A_2 \oplus \mathbb{C}^2, A_3 \oplus \mathbb{C}^1, C_2 \oplus \mathbb{C}^1, A_1 \oplus A_1 \oplus \mathbb{C}^2$
$\mathfrak{gl}_5$	$\mathbb{C}^7, A_1 \oplus \mathbb{C}^4, A_2 \oplus \mathbb{C}^3, A_3 \oplus \mathbb{C}^2, A_4 \oplus \mathbb{C}^1,$ $C_2 \oplus \mathbb{C}^2, A_1 \oplus A_1 \oplus \mathbb{C}^3, A_1 \oplus A_2 \oplus \mathbb{C}^2$
$\mathfrak{gl}_6$	$\mathbb{C}^{10}, A_1 \oplus \mathbb{C}^6, A_2 \oplus \mathbb{C}^4, A_3 \oplus \mathbb{C}^3, A_4 \oplus \mathbb{C}^2,$ $A_5 \oplus \mathbb{C}^1, C_2 \oplus \mathbb{C}^3, C_3 \oplus \mathbb{C}^1, A_2 \oplus A_2 \oplus \mathbb{C}^2,$ $A_1 \oplus A_2 \oplus \mathbb{C}^3, A_1 \oplus A_3 \oplus \mathbb{C}^2, C_2 \oplus A_1 \oplus \mathbb{C}^2,$ $A_1 \oplus A_1 \oplus \mathbb{C}^4, A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^3$
$\mathfrak{gl}_7$	$\mathbb{C}^{13}, G_2 \oplus \mathbb{C}, B_3 \oplus \mathbb{C}^1, A_6 \oplus \mathbb{C}, C_3 \oplus \mathbb{C}^2, A_5 \oplus \mathbb{C}^2, A_4 \oplus \mathbb{C}^3,$ $C_2 \oplus \mathbb{C}^4, A_3 \oplus \mathbb{C}^4, A_2 \oplus \mathbb{C}^6, A_1 \oplus \mathbb{C}^8, A_4 \oplus A_1 \oplus \mathbb{C}^2,$ $C_2 \oplus A_2 \oplus \mathbb{C}^2, C_2 \oplus A_1 \oplus \mathbb{C}^3, A_3 \oplus A_2 \oplus \mathbb{C}^2, A_3 \oplus A_2 \oplus \mathbb{C}^2,$ $A_3 \oplus A_1 \oplus \mathbb{C}^3, A_2 \oplus A_2 \oplus \mathbb{C}^3, A_2 \oplus A_1 \oplus \mathbb{C}^4,$ $A_1 \oplus A_1 \oplus \mathbb{C}^5, A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^3, A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^4$
$\mathfrak{gl}_8$	$\mathbb{C}^{17}, D_4 \oplus \mathbb{C}^1, C_4 \oplus \mathbb{C}^1, A_7 \oplus \mathbb{C}^1, G_2 \oplus \mathbb{C}^2, B_3 \oplus \mathbb{C}^2,$ $A_6 \oplus \mathbb{C}^2, C_3 \oplus \mathbb{C}^3, A_5 \oplus \mathbb{C}^3, A_4 \oplus \mathbb{C}^4, C_2 \oplus \mathbb{C}^6,$ $A_3 \oplus \mathbb{C}^6, A_2 \oplus \mathbb{C}^8, A_1 \oplus \mathbb{C}^{11}, C_3 \oplus A_1 \oplus \mathbb{C}^2, A_5 \oplus A_1 \oplus \mathbb{C}^2,$ $A_4 \oplus A_2 \oplus \mathbb{C}^2, A_4 \oplus A_1 \oplus \mathbb{C}^3, C_2 \oplus C_2 \oplus \mathbb{C}^2, C_2 \oplus A_3 \oplus \mathbb{C}^2,$ $C_2 \oplus A_2 \oplus \mathbb{C}^2, C_2 \oplus A_1 \oplus \mathbb{C}^4, A_3 \oplus A_3 \oplus \mathbb{C}^2, A_3 \oplus A_2 \oplus \mathbb{C}^3,$ $A_3 \oplus A_1 \oplus \mathbb{C}^4, A_2 \oplus A_2 \oplus \mathbb{C}^4, A_2 \oplus A_1 \oplus \mathbb{C}^5,$ $A_1 \oplus A_1 \oplus \mathbb{C}^7, C_2 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^3, A_3 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^3,$ $A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{C}^3, A_2 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^4,$ $A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^5, A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{C}^4$

Table 2.2: The sets S_n

subalgebras are summed up in the following list:

$$C_2, A_2 \oplus \mathbb{C}, A_1 \oplus A_1, A_1 \oplus A_1 \oplus \mathbb{C}.$$

Since $\mathfrak{g}_0 \leq \mathfrak{h}$ for some element $\mathfrak{h}$ in the list above, and because of the monotonicity of the α-invariant, we have $\alpha(\mathfrak{g}_0) \leq \alpha(\mathfrak{h})$. Since α is additive, $\alpha(\mathfrak{g}_0) = \alpha(A_1) + \alpha(\mathbb{C}^3) = 4$. But every Lie algebra from the list above will have an invariant at most three. Malcev computed $\alpha(\mathfrak{r})$ for all reductive Lie algebras $\mathfrak{r}$. In a more recent article, Suter [Sut] obtained the same results. For example,

$\mathfrak{h}$	C_2	$A_2 \oplus \mathbb{C}$	$A_1 \oplus A_1$	$A_1 \oplus A_1 \oplus \mathbb{C}$
$\alpha(\mathfrak{h})$	3	3	2	3

.

This produces a contradiction and finishes the proof.

△

Using similar techniques, one can again obtain a formula for $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^k$. The following result, however, illustrates that the use of Dynkin's results can lead to very complicated situations.

Example 2.3.4.2. [$\mu(A_1 \oplus C_3 \oplus \mathbb{C}^6) = 12$.] There is an obvious twelve-dimensional faithful representation suggested by the decomposition $\mathfrak{g} = (A_1 \oplus \mathbb{C}) \oplus (C_3 \oplus \mathbb{C}) \oplus \mathbb{C}^4$: simply take direct sum of the natural embeddings of each of the terms $A_1 \oplus \mathbb{C}, C_3 \oplus \mathbb{C}$ and $\mathbb{C}^4$. We may conclude that $\mu(\mathfrak{g}) \leq 12$.

Now suppose that we can embed $\mathfrak{g}$ into $\mathfrak{gl}_{11} = A_{10} \oplus \mathbb{C}$ in an attempt to obtain a contradiction. Using arguments that are similar to those from the previous example, we may assume that $\mathfrak{g}_0 = A_1 \oplus C_2 \oplus \mathbb{C}^5$ is a reductive subalgebra of A_{10}. In particular, $\mathfrak{g}_0$ is contained in a maximal (proper) reductive subalgebra of A_{10}, say $\mathfrak{h}$. The following list describes those subalgebras with their corresponding α-invariant:

$\mathfrak{h}$	$A_9 \oplus \mathbb{C}$	$A_1 \oplus A_8 \oplus \mathbb{C}$	$A_2 \oplus A_7 \oplus \mathbb{C}$
$\alpha(\mathfrak{h})$	26	22	19

$\mathfrak{h}$	$A_3 \oplus A_6 \oplus \mathbb{C}$	$A_4 \oplus A_5 \oplus C$	B_5
$\alpha(\mathfrak{h})$	17	16	11

.

Since $\mathfrak{g}_0 \leq \mathfrak{h}$, we have $\alpha(\mathfrak{g}_0) \leq \alpha(\mathfrak{h})$. Now, $\alpha(\mathfrak{g}_0) = \alpha(A_1) + \alpha(C_3) + \alpha(\mathbb{C}^5) = 12$. Unfortunately, the only Lie algebra that can be excluded immediately, is B_5. All remaining Lie algebras need to be treated in a similar way and these ramify to even more cases in the next step. Moreover, the maximal reductive subalgebras of types other than "A_n" play a role.

$\triangle$

Finally we note that the knowledge of $\mu(\mathfrak{g})$ for reductive Lie algebras $\mathfrak{g}$ does not allow us to recover the results of Dynkin and others. They are specifically working with representations up to linear conjugation, which is a special kind of isomorphism.

Chapter 3

Lie algebras with an abelian radical

In the previous chapter we studied the faithful representations for reductive Lie algebras. These Lie algebras are exactly the ones for which the solvable radical and the centre coincide. In general, the radical need not be central. We now consider the larger class of all Lie algebras with a solvable radical that is in fact *abelian*. Ideally, we would like to construct faithful linear representations of small degree for Lie algebras of this type.

Since the representations of semisimple Lie algebras are well-understood, we would like to reduce this problem to the semisimple case. In order to do this, we introduce the so-called *paired* modules. Suppose a semisimple Lie algebra $\mathfrak{s}$ acts on the vector space V through a morphism $\sigma : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. Then $\mathfrak{s}$ also acts on $\mathrm{Hom}(V, V)$ by conjugation:

$$(s \cdot M)(v) = (\sigma(s) \circ M - M \circ \sigma(s))(v)$$

for all $s \in \mathfrak{s}, M \in \mathrm{Hom}(V, V)$ and $v \in V$. According to Weyl's theorem, the module $\mathrm{Hom}(V, V)$ decomposes completely into (irreducible) submodules. If we identify $\mathrm{Hom}(V, V)$ with $\mathfrak{gl}(V)$, we notice that some of these submodules are at the same time Lie subalgebras. This leads us to the following problem.

Problem Let $\mathfrak{s}$ be a semisimple subalgebra of $\mathfrak{gl}(V)$. Describe the abelian subalgebras of $\mathfrak{gl}(V)$ that are invariant under the conjugation-action of $\mathfrak{s}$.

The decomposition theorem of Levi will be used to translate the first problem into the second. As an application we present a method that is algorithmic and combinatorical in nature, to compute the μ_0-invariant for all Lie algebras with an abelian radical. This algorithm relies on the (abstract) decomposition of a tensor product of two irreducible $\mathfrak{s}$-modules into irreducible submodules. We also obtain upper-bounds for the μ-invariant in terms of other invariants associated to the Levi-decomposition. As a third example, we have:

> **Proposition** Let $\mathfrak{g}$ be a complex, finite-dimensional Lie algebra with an abelian radical. Then we have the sharp bounds
>
> $$\sqrt{\dim(\mathfrak{g})} \leq \mu(\mathfrak{g}) \leq \dim(\mathfrak{g}).$$

Finally, we consider the family of Lie algebras that have an abelian radical and a Levi-complement isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. We obtain some results about the μ-invariant.

The Levi-decomposition

Finite-dimensional Lie algebras $\mathfrak{g}$ over a field of characteristic zero can be decomposed as a semidirect product of an ideal and a complementary subalgebra. These two pieces come from special classes of Lie algebras that are very far away from each other. The ideal is in fact the solvable radical $\mathrm{rad}(\mathfrak{g})$ of $\mathfrak{g}$, i.e., the (unique) maximal solvable ideal of $\mathfrak{g}$. The complementary algebras are in this case semisimple and there is a lot of theory available for semisimple Lie algebras.

> **Theorem** [Levi] For every Lie algebra $\mathfrak{g}$ of finite dimension over a field of characteristic zero there exists a semisimple Lie algebra complement $\mathfrak{s}$ to the solvable radical $\mathfrak{r}$, i.e.: $\mathfrak{g} \cong \mathfrak{s} \ltimes \mathfrak{r}$. Such an algebra $\mathfrak{s}$ is called a Levi-complement for $\mathfrak{g}$.

The action of $\mathfrak{s}$ on $\mathfrak{r}$ is described by a homomorphism $\delta : \mathfrak{s} \longrightarrow \mathrm{Der}(\mathfrak{r})$. We will say that δ *describes* this Levi-decomposition. We will also write $\mathfrak{g} \cong \mathfrak{s} \ltimes_\delta \mathfrak{r}$. Although the radical of a Lie algebra is unique, it Levi-complements are not:

Theorem [Mal'cev] The Levi-complements are unique up to conjugation by an inner automorphism of the form, $\exp(\mathrm{Ad}(z))$, for some z in the nilradical.

So, even though Levi-complements need not be unique, they are unique up to conjugation and in particular up to isomorphism. Let us mention the following result without proof. For a Lie algebra $\mathfrak{g}$ with abelian radical $\mathfrak{a}$, Levi-complement $\mathfrak{s}$ and defining representation δ, the following conditions are equivalent: (a) $\mathfrak{z}(\mathfrak{g}) = 0$. (b) $\mathrm{mult}_{\delta_1}(\delta) = 0$. (c) $[\mathfrak{s}, \mathfrak{a}] = \mathfrak{a}$.(c') $\mathfrak{g}$ is perfect. (d) $H^0(\mathfrak{s}, \mathfrak{a}) = 0$. (e) $H^0(\mathfrak{g}, \mathfrak{g}) = 0$.

Corollary 3.0.4.1. *Let $\mathfrak{g}$ be a Lie algebra with an abelian radical. If it is perfect, then*

$$\mu(\mathfrak{g}) = \mu_\infty(\mathfrak{g}).$$

Lie algebras with a filiform radical We illustrate the use of Levi's theorem by applying it to the family of Lie algebras with a radical that is filiform nilpotent. In his paper ([Camp]), the author writes *"Any complex perfect Lie algebra [...] with a Heisenberg radical $\mathfrak{h}$ admits a faithful representation of degree* $\dim(\mathfrak{h}) + 1$ *[...]"* - which is incorrect. He assumes that the action of the Levi-complement on the radical is faithful. We have the following result:

Proposition 3.0.4.1. *Let $\mathfrak{g}$ be a Lie algebra such that* $\mathrm{rad}(\mathfrak{g}) \cong \mathfrak{h}_1$. *Then we have the formula,*

$$\mu(\mathfrak{g}) = \mu\Big(\frac{\mathfrak{g}}{\mathrm{rad}(\mathfrak{g})}\Big) + \mu(\mathrm{rad}(\mathfrak{g})) - \begin{cases} 1 & \text{if} \quad [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF*:* Let $\mathfrak{g} = \mathfrak{s} \ltimes_\delta \mathfrak{h}_1$ be a Levi-decomposition for $\mathfrak{g}$. Consider the characteristic filtration $\mathfrak{h}_1 > [\mathfrak{h}_1, \mathfrak{h}_1] > 0$. Since $\mathfrak{s}$ acts through derivations of $\mathfrak{h}_1$, it leaves all characteristic ideals invariant. Since $\mathfrak{s}$ is semisimple, each invariant subspace has an invariant complement. This implies that δ is either of the form $\delta_1 \oplus \delta_1 \oplus \delta_1$ or $\alpha \oplus \beta$ with α and β irreducible of degree 2 resp. 1. In the first case, $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{s} \oplus \mathfrak{h}_1$ and it is clearly not perfect. Since the centre of $\mathfrak{h}_1$ consists of commutators, we can apply proposition 2.3.2.1 to $\mathfrak{s}$ and $\mathfrak{h}_1$. This finishes the proof in the first case.

So assume that δ is not the trivial representation. It can be verified easily that $\mathfrak{g}$ is perfect in this case. First assume that the representation δ is faithful. As we have noted before, δ must be of the form $\delta_2 \oplus \delta_1$. This δ is faithful if and only of δ_2 is. This implies that $\mathfrak{s}$ is either 0 or $\mathfrak{sl}_2(\mathbb{C})$. We may assume that $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C})$ since δ is assumed to be non-trivial. This leaves us with exactly one Lie algebra: $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{C}) \ltimes_{\delta_2 \oplus \delta_1} \mathfrak{h}_1$. It can be verified that $\mu(\mathfrak{g}_0) = 4$. Now we drop the faithfulness condition on δ. Write $\mathfrak{g}$ as $\ker \delta \oplus (\frac{\mathfrak{s}}{\ker \delta} \ltimes_{\delta_0} \mathfrak{h}_1)$. Note that the second term is perfect, that its radical is $\mathfrak{h}_1$ and that the defining representation is faithful. So this term must be $\mathfrak{g}_0$. Lemma 2.3.2.3 gives $\mu(\mathfrak{g}) = \mu(\ker \delta) + \mu(\mathfrak{g}_0) = \mu(\mathfrak{s}) + \mu(\mathfrak{g}_0) - 2 = \mu(\mathfrak{s}) + 2$. This finishes the proof.

□

The Lie algebra $\mathfrak{h}_1$ is the complex filiform Lie algebra of dimension three. Any other filiform Lie algebra can also occur as the radical of a Lie algebra, but only in a trivial way. This was shown by Bermudez, Campoamor and Vergnolle using deformation theory: see [BeCaVe]. We give an alternative proof and an immediate corollary.

Proposition 3.0.4.2. *Let $\mathfrak{g}$ be a Lie algebra with filiform radical. If* $\mathrm{rad}(\mathfrak{g}) \neq \mathfrak{h}_1$*, then*

$$\mu(\mathfrak{g}) = \mu(\frac{\mathfrak{g}}{rad(\mathfrak{g})}) + \mu(rad(\mathfrak{g})).$$

Proof*:* Consider the Levi-decomposition $\mathfrak{g} \cong \mathfrak{s} \ltimes_\delta \mathfrak{r}$ and assume that $\mathfrak{r} > \mathfrak{h}_1$. It suffices to show that the defining representation is the zero-map. Then the Levi-decomposition $\mathfrak{s} \ltimes_\delta \mathfrak{r}$ splits trivially and we can apply proposition 2.3.2.1 to finish the proof. We suppose the converse and try to deduce a contradiction.

Suppose that the defining representation is non-trivial. We may then replace $\mathfrak{g}$ by a subalgebra with the same radical such that the defining representation is non-trivial and faithful. Consider the natural characteristic filtration of $\mathfrak{r}$ with co-dimensions $(2, 1, 1, \ldots, 1)$:

$$\mathfrak{r} = \mathfrak{r}^1 > \mathfrak{r}^2 > \ldots > \mathfrak{r}^{d-2} > \mathfrak{r}^{d-1} > \mathfrak{r}^d = 0.$$

Note that $\mathfrak{s}$ acts trivially on $\mathfrak{r}^2$. Let E be an $\mathfrak{s}$-complement to $\mathfrak{r}^2$ in $\mathfrak{r}$. Let y be a generator of $\mathfrak{r}^{d-1}$ and let $\{y, z\}$ be a set of generators for $\mathfrak{r}^{d-2}$. Take an element x in E such that $[x, y] = z$. Any derivation f in $\delta(\mathfrak{s}) \leq \mathcal{D}er(\mathfrak{r})$ then satisfies $[f(x), y] = 0$ since $d > 3$. This means that $[\delta(\mathfrak{s})x, y] = 0$. Since δ is non-trivial, $\delta(\mathfrak{s})x = E$. We then have,

$$0 = [E, y] = [E, y] + [\mathfrak{r}^2, y] = [\mathfrak{r}, y] = [\mathfrak{r}, y] + [\mathfrak{r}, z] = [\mathfrak{r}, \mathfrak{r}^{d-2}],$$

which contradicts the fact that $\mathfrak{r}$ is nilpotent of class $d - 1$.

□

Remark 4. We can obtain an other proof using weight theory. If a semisimple Lie algebra acts on a filiform nilpotent Lie algebra $\mathfrak{g}$, it is faithful if and only if the action on the abelianisation is. So we may assume that the semisimple Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$. The action is fully reducible. Using the descending central series, we obtain a chain of modules with alternating parity of the weights. We then have $\dim(\mathfrak{g}) \geq 3\lceil c(\mathfrak{g})/2 \rceil$. The only two solutions are $\mathfrak{g} = \mathfrak{h}_1$ and $\mathfrak{g} = \mathbb{C}^2$.

3.1 Paired representations

Let us start from an easy example and consider the affine Lie algebra $\mathfrak{aff}_n(k) = \mathfrak{gl}_n(k) \ltimes k$ over a field k. It is quite obvious how to compute the μ-invariant from what we know already: it is $n + 1$. Suppose we can embed $\mathfrak{aff}_n(k)$ into $\mathfrak{gl}_m(k)$ for some m. Since the dimension is monotone, we have $n^2 + 1 = \dim(\mathfrak{aff}_n(k)) \leq \dim(\mathfrak{gl}_m(k)) = m^2$ and hence also $n < m$. This implies $n + 1 \leq \mu(\mathfrak{aff}_n(k))$. It will be equality since we can construct a faithful representation of this degree:

$$\rho : \mathfrak{aff}_n(k) \longrightarrow \mathfrak{gl}_{n+1}(k) : (A, b) \longmapsto \begin{array}{|c|c|} \hline A & b \\ \hline 0 & 0 \\ \hline \end{array}.$$

Note that the blocks on the diagonal define a representation for $\mathfrak{gl}_n(k) \leq \mathfrak{aff}_n(k)$ and that the blocks off the diagonal are the natural $\mathfrak{aff}_n(k)$-module

k and its dual. Both modules are even abelian as Lie algebras (cf. Schur's theorem). We would like to generalise this construction in order to obtain many faithful representations for a given Lie algebra. All that we require, is a semidirect product decomposition as for the affine Lie algebras. In view of Levi's theorem, every Lie algebra has such a decomposition (or indeed one that is quite similar).

Paired representations Consider a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module (ρ, V). Then the vector space $\mathrm{Hom}(V, V)$ of all V-endomorphisms is also a $\mathfrak{g}$-module for the following action. For $x \in \mathfrak{g}$ and $M \in \mathrm{Hom}(V, V)$ we define $x \cdot M$ to be $\rho(x) \circ M - M \circ \rho(x)$. More generally, we have the following definition.

Definition 3.1.0.1 (Pairing)**.** Let $\mathfrak{g}$ be a Lie algebra. Consider two finite-dimensional $\mathfrak{g}$-representations (α, V) and (β, W). Then $\mathfrak{g}$ defines a module structure on $\mathrm{Hom}(W, V)$ by $\langle \alpha, \beta \rangle : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathrm{End}(W, V)) : x \longmapsto L_{\alpha(x)} - R_{\beta(x)}$: I.e., for $x \in \mathfrak{g}$, $f \in \mathrm{End}(W, V)$ and $w \in W$:

$$((\langle \alpha, \beta \rangle(x))(f))(w) = \alpha(x)f(w) - f(\beta(x)(w)).$$

On the other hand, for the given representations α and β, $\beta^* \otimes \alpha$ defines a $\mathfrak{g}$-module structure on the vector space $W^* \otimes V$. For $x \in \mathfrak{g}$, $f \in W^*$ and $v \in V$,

$$((\beta^* \otimes \alpha)(x))(f(\cdot) \otimes v) = -f(\beta(x)\cdot) \otimes v + f(\cdot) \otimes ((\alpha(x))(v)).$$

Lemma 3.1.0.1. *For $\mathfrak{g}, \alpha, \beta, V$ and W as above, we have the isomorphism of modules*

$$(\mathit{Hom}(W, V), \langle \alpha, \beta \rangle) \cong (W^* \otimes V, \beta^* \otimes \alpha).$$

PROOF*:* Consider the natural map $\theta : W^* \otimes V \longrightarrow \mathrm{End}(W, V)$: $f \otimes v \longmapsto f(\cdot)v$. By applying the definitions, we can see that this map defines an isomorphism of modules.

□

Remark 5. Fix a Lie algebra $\mathfrak{g}$ and consider its associated pairing $\langle \cdot, \cdot\cdot \rangle$: $\mathrm{Rep}(\mathfrak{g}) \times \mathrm{Rep}(\mathfrak{g}) \longrightarrow \mathrm{Rep}(\mathfrak{g}) : (\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle$. Here, $\mathrm{Rep}(\mathfrak{g})$ is the set of all finite-dimensional $\mathfrak{g}$-modules up to isomorphism. For $\mathfrak{g}$-representations α, β, γ and the trivial representation ε, we have the following identities:

- $\deg(\langle \alpha, \beta \rangle) = \deg(\alpha) \deg(\beta)$.

- $\langle \alpha \oplus \beta, \gamma \rangle = \langle \alpha, \gamma \rangle \oplus \langle \beta, \gamma \rangle$ and $\langle \alpha, \beta \oplus \gamma \rangle = \langle \alpha, \beta \rangle \oplus \langle \alpha, \gamma \rangle$.
- $\langle \alpha, \varepsilon \rangle = \alpha$ and $\langle \varepsilon, \alpha \rangle = \alpha^*$.

Block decomposition Suppose we have a representation $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ of a semisimple Lie algebra $\mathfrak{s}$. Then there are some obvious invariant subspaces of $(\langle \rho, \rho \rangle, \mathfrak{gl}(V))$: we have the decomposition $\mathfrak{gl}(V) = \mathfrak{sl}(V) \oplus \mathbb{C}\mathbb{1}_V$. Clearly, $\rho(\mathfrak{s})$ is an invariant subspace of $\mathfrak{sl}(V)$ and every ideal of $\mathfrak{s}$ is mapped by ρ to an invariant subspace of $\rho(\mathfrak{s})$. But we can go even further. Decompose (ρ, V) into irreducibles: $(\oplus_r \rho_r, \oplus_r V_r)$. Then the $\mathfrak{s}$-module $\mathrm{Hom}(V, V)$ decomposes (as direct sum of modules) into the invariant block-submodules $(\langle \rho_p, \rho_q \rangle, \mathrm{Hom}(V_q, V_p))_{p,q}$ of $(\dim(\rho_q) \times \dim(\rho_p))$-matrices:

$$\mathfrak{gl}(V) = \begin{array}{|c|c|c|c|c|} \hline \langle \rho_1, \rho_1 \rangle & \langle \rho_1, \rho_2 \rangle & & & \\ \hline \langle \rho_2, \rho_1 \rangle & \ddots & & & \\ \hline & & \langle \rho_q, \rho_p \rangle & & \\ \hline & & & \ddots & \\ \hline & & & & \langle \rho_r, \rho_r \rangle \\ \hline \end{array}.$$

These submodules need not be irreducible. In particular, the study of the submodules of $\mathrm{Hom}(V, V)$ is reduced to the study of the submodules of $\langle \rho_p, \rho_q \rangle$ (equivalently, of $\rho_q^* \otimes \rho_p$) with ρ_p and ρ_q irreducible. Note, however, that these submodules need not be subalgebras.

Example 3.1.0.3. Consider the simple Lie algebra $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C})$.

- Let δ_2 be the irreducible representation of degree 2 on $\mathbb{C}^2$. Then the induced module $(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2), \langle \delta_2, \delta_2 \rangle)$ decomposes as an isomorphic copy of $\delta_1 \oplus \delta_3$, or more explicitly:

$$\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) = \begin{array}{|c|c|} \hline r & \\ \hline & r \\ \hline \end{array} \bigoplus \begin{array}{|c|c|} \hline z & x \\ \hline y & -z \\ \hline \end{array}.$$

  Both modules are Lie algebras with the bracket induced by $\mathfrak{gl}_2(\mathbb{C})$: the first one is the scalar subalgebra, hence abelian, and the second one is $\mathfrak{sl}_2(\mathbb{C})$, hence simple.
- Let δ_3 be the irreducible representation of degree 3. Then the induced module $(\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3), \langle \delta_3, \delta_3 \rangle)$ decomposes as

$\delta_1 \oplus \delta_3 \oplus \delta_5$, or more explicitly:

$$\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3) = \begin{array}{|c|c|c|}\hline t & & \\ \hline & t & \\ \hline & & t \\ \hline\end{array} \bigoplus \begin{array}{|c|c|c|}\hline 2z & x & \\ \hline 2y & & 2x \\ \hline & y & -2z \\ \hline\end{array} \bigoplus \begin{array}{|c|c|c|}\hline a & b & d \\ \hline -2c & -2a & -2b \\ \hline e & c & a \\ \hline\end{array}.$$

We note that even though the first two modules are Lie subalgebras of $\mathfrak{gl}_3(\mathbb{C})$, the third one is not. The subalgebras of $\mathfrak{gl}_2(\mathbb{C})$ that are invariant under the action of $\mathfrak{s}$ are : 0, δ_1, δ_3, $\delta_1 \oplus \delta_3$ and $\mathfrak{gl}_3(\mathbb{C})$ itself.

- The module $\langle \delta_2, \delta_3 \rangle$ decomposes as $\delta_2 \oplus \delta_4$, or more explicitly:

$$\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^2) = \begin{array}{|c|c|c|}\hline 2q & p & \\ \hline & q & 2p \\ \hline\end{array} \bigoplus \begin{array}{|c|c|c|}\hline c & -b & a \\ \hline d & -c & b \\ \hline\end{array}.$$

Note that the only abelian submodules are the trivial ones. The decompositions can be obtained by using arguments from linear algebra.

$\triangle$

Note that for $\mathfrak{sl}_2$-representations α, β, γ (not necessarily irreducible) and the trivial representation ε we have the following isomorphisms of modules:

$$\langle \alpha \oplus \beta, \gamma \rangle \cong \langle \alpha, \gamma \rangle \oplus \langle \beta, \gamma \rangle \text{ and } \langle \alpha, \beta \rangle \cong \langle \beta, \alpha \rangle \text{ and } \langle \varepsilon, \alpha \rangle \cong \alpha.$$

Example In subsection 2.3.2 we have seen examples of pairs of perfect Lie algebras $(\mathfrak{p}_1, \mathfrak{p}_2)$ for which the equality $\mu(\mathfrak{p}_1 \oplus \mathfrak{p}_2) = \mu(\mathfrak{p}_1) + \mu(\mathfrak{p}_2)$ holds. A sufficient condition would be that one of the terms is semisimple. We now wish to illustrate that also $\mu(\mathfrak{p}_1 \oplus \mathfrak{p}_2) < \mu(\mathfrak{p}_1) + \mu(\mathfrak{p}_2)$ can occur. For this, we consider the family of perfect Lie algebras $L_n = \mathfrak{sl}_n \ltimes_{\mathbb{1}} \mathbb{C}^n$, for every $n \in \mathbb{N} \setminus \{0, 1\}$. Then it is clear that $\mu(L_n) = n + 1$. We will show that

$$\begin{aligned} \mu(L_{m_1} \oplus \ldots \oplus L_{m_t}) &= \mu(L_{m_1}) + \ldots + \mu(L_{m_t}) + 1 - t \\ &= m_1 + \ldots + m_t + 1. \end{aligned}$$

PROOF*:* Consider such a Lie algebra $L = L_{m_1} \oplus \ldots \oplus L_{m_t}$. First we construct a faithful representation of degree $m = m_1 + \ldots m_t +$ 1. Let $\iota_j : \mathfrak{sl}_{m_j}(\mathbb{C}) \longrightarrow \mathfrak{gl}_{m_j}(\mathbb{C})$ be the natural embeddings and let τ be the trivial representation of $\mathfrak{sl}_{m_1} \oplus \cdots \oplus \mathfrak{sl}_{m_t}(\mathbb{C})$. Then

$\rho_{m_1} \oplus \ldots \oplus \rho_{m_t} \oplus \tau$ is a faithful representation of $\mathfrak{sl}_{m_1}(\mathbb{C}) \oplus \ldots \oplus \mathfrak{sl}_{m_t}(\mathbb{C}) \leq L$ of degree $m+1$. Then

$$\rho : L \longrightarrow \mathfrak{gl}_{m+1}(\mathbb{C}) : ((M_j; x_j))_j \longmapsto \begin{array}{|c|c|c|c|}\hline \iota_1(M_1) & & & x_1 \\ \hline & \ddots & & \vdots \\ \hline & & \iota_t(M_t) & x_t \\ \hline & & & \tau \\ \hline \end{array}$$

is a faithful representation of L of the desired degree. So we conclude that $\mu(L) \leq m+1$.

We will prove the converse inequality by induction on the number t of terms of L. If there is only one term, then there is nothing to prove. So assume that $\mu(L_{m_1} \oplus \ldots \oplus L_{m_t}) = m_1 + \ldots + m_t + 1$ and consider the Lie algebra $L = L_{m_0} \oplus (L_{m_1} \oplus \ldots \oplus L_{m_t})$. Let $\overline{L}$ be $\frac{L}{L_{m_0}}$. Suppose L is embedded into $\mathfrak{gl}_s(\mathbb{C})$. Then $\overline{L}$ is contained in the centraliser of $\mathfrak{sl}_{m_0}(\mathbb{C})$ in $\mathfrak{gl}_s(\mathbb{C})$. The centraliser-lemma then gives us an embedding of $\overline{L}$ into $\mathfrak{gl}_{s-\mu(\overline{L})}(\mathbb{C}) \oplus \mathbb{C}$. Since $\overline{L}$ is perfect, it can even be embedded into $\mathfrak{gl}_{s-\mu(\overline{L})}(\mathbb{C})$. In particular, we have $m_0 = \mu(\mathfrak{sl}_{m_0}(\mathbb{C})) \leq s - \mu(\overline{L})$ and thus $m_0 + \mu(\overline{L}) \leq s$. For $s = \mu(L)$, we obtain $m_0 + \mu(\overline{L}) \leq \mu(L)$. The induction hypothesis can then be applied to $\overline{L}$ to finish the proof.

$\square$

Corollary 3.1.0.2. *In particular, there are many pairs of perfect Lie algebras* $(\mathfrak{p}_1, \mathfrak{p}_2)$ *satisfying*

$$\mu(\mathfrak{p}_1 \oplus \mathfrak{p}_2) < \mu(\mathfrak{p}_1) + \mu(\mathfrak{p}_2).$$

3.2 Constructing representations

We have seen that the pairing of modules gives us a rich family of modules and submodules. We now want to use these pairing to compute μ-invariants (or bounds for them). As in the reductive case, we can cut embeddings into pieces and glue them together (see lemma 2.3.1.2). These two processes are each others' inverse. This gluing and cutting process is described in the following two lemmas.

Lemma 3.2.0.2. *Let $\mathfrak{g}$ be a complex, finite-dimensional Lie algebra with a Levi-decomposition $\mathfrak{s} \ltimes_\delta \mathfrak{r}$. Consider a finite-dimensional vector space V and a pair of Lie algebra morphisms $\sigma : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ and $\rho : \mathfrak{r} \longrightarrow \mathfrak{gl}(V)$. Define the representation $\theta : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ by $\theta|_\mathfrak{s} = \sigma$ and $\theta|_\mathfrak{r} = \rho$. Then θ is a well-defined embedding of $\mathfrak{g}$ if*

1. *$\sigma : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ resp. $\rho : \mathfrak{r} \longrightarrow \mathfrak{gl}(V)$ are embeddings of Lie algebras.*
2. *$[\sigma(s), \rho(r)] = \rho([s,r])$ for all $s \in \mathfrak{s}$ and $r \in \mathfrak{r}$.*

PROOF*:* It is clear that the map θ is a Lie algebra morphism. We only need to check the faithfulness. Define M to be the set of elements s in $\mathfrak{s}$ for which $\sigma(s) \in \rho(\mathfrak{r})$. We will first prove that M is a solvable ideal of $\mathfrak{s}$. Suppose $s \in \mathfrak{s}$ and $m \in M$. Then there exists an r in $\mathfrak{r}$ such that $\rho(r) = \sigma(m)$. Then we have,

$$\begin{aligned} \sigma([s,m]) &= [\sigma(s), \sigma(m)] \\ &= [\sigma(s), \rho(r)] \\ &\leq \rho([\mathfrak{s}, \mathfrak{r}]) \leq \rho(\mathfrak{r}). \end{aligned}$$

This implies that $[\mathfrak{s}, M] \leq M$ and that M is an ideal of $\mathfrak{s}$. Since M is the image of the morphism $\sigma^{-1} \circ \rho$, it is also solvable. But in a semisimple Lie algebra, there are no non-zero solvable ideals. This implies that M is zero. So $\rho(\mathfrak{r}) \cap \sigma(\mathfrak{s}) = 0$ and we conclude that θ is faithful.

$\square$

Lemma 3.2.0.3. *Let $\mathfrak{g}$ be a complex, finite-dimensional Lie algebra with a Levi-decomposition $\mathfrak{s} \ltimes_\delta \mathfrak{r}$. Consider a faithful representation $\theta : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a finite-dimensional vector space V. Define the representations σ and ρ by restricting θ to $\mathfrak{s}$ resp. $\mathfrak{r}$. Then,*

1. *$\sigma : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ resp. $\rho : \mathfrak{r} \longrightarrow \mathfrak{gl}(V)$ are embeddings of Lie algebras.*
2. *$[\sigma(s), \rho(r)] = \rho([s,r])$ for all $s \in \mathfrak{s}$ and $r \in \mathfrak{r}$.*

These two lemmas allow us to shift our attention to the following problem.

Problem Let $\mathfrak{s}$ be a semisimple subalgebra of $\mathfrak{gl}(V)$. Describe the abelian subalgebras of $\mathfrak{gl}(V)$ that are invariant under the conjugation-action of $\mathfrak{s}$.

We can put even more structure on the submodules. Suppose $\mathfrak{a}$ is such an abelian submodule of $\mathfrak{gl}(V)$. Consider the commutative algebra A in $\mathfrak{gl}(V)$ that is generated by $\mathfrak{a}$ and $\mathbb{1}_V$. Then this A and various natural subspaces, such as the space $\langle A_1 \circ A_2 \circ \cdots \circ A_t \mid A_1, A_2, \ldots, A_t \in \mathfrak{a}\rangle$ for some $t \in \mathbb{N}$, are also invariant under the action of $\mathfrak{s}$. Conversely, every commutative subalgebra that is invariant under conjugation by $\mathfrak{s}$ defines an invariant abelian Lie algebra. It seems reasonable that the standard structure theory of commutative algebra could be used to obtain more information about the abelian submodules of $\mathfrak{gl}(V)$.

Problem Let $\mathfrak{s}$ be a semisimple subalgebra of $\mathfrak{gl}(V)$. Describe the commutative subalgebras of $\mathfrak{gl}(V)$ that are invariant under the conjugation-action of $\mathfrak{s}$.

3.2.1 Representations of type zero

Invariant subspaces We will now characterise the faithful representations (ρ, V) of type zero for the Lie algebras $\mathfrak{s} \ltimes \mathfrak{a}$ with a solvable radical $\mathfrak{a}$ that is abelian. In this context we should consider $\mathfrak{s}$-invariant subspaces of V. We can obtain some of them from images and invariants of characteristic ideals. Recall that an ideal of a Lie algebra $\mathfrak{g}$ is *characteristic* if it is invariant under the action of the derivation algebra $\mathrm{Der}(\mathfrak{g})$ of $\mathfrak{g}$. If M is a $\mathfrak{g}$-module, then $M^{\mathfrak{g}}$ is the submodule of all *invariants*, i.e. the set of vectors of M that are annihilated by the Lie algebra $\mathfrak{g}$.

Lemma 3.2.1.1. *Consider the linear Lie algebra $\mathfrak{s} \ltimes \mathfrak{r} \leq \mathfrak{gl}(V)$. For any characteristic ideal $\mathfrak{c}$ of $\mathfrak{r}$, $\langle \mathfrak{c}(V)\rangle$ and $V^{\mathfrak{c}}$ are invariant subspaces of V as an $\mathfrak{s}$-module.*

PROOF*:* Let $\mathfrak{c}$ be a characteristic ideal of $\mathfrak{r}$. Then for every element c in $\mathfrak{c}$, every s in $\mathfrak{s}$ and every v in V, we have:

$$\begin{aligned} s(c(v)) &= ([s,c] + c \circ s)(v) \\ &= [s,c](v) + c(s(v)) \\ &\leq \mathfrak{c}(v) + \mathfrak{c}(V) \\ &\leq \langle \mathfrak{c}(V)\rangle. \end{aligned}$$

Hence the vector space generated by $\mathfrak{c}(V)$ is invariant under the action of $\mathfrak{s}$. Similarly, for every $c \in \mathfrak{c}$, $s \in \mathfrak{s}$ and $v \in V^{\mathfrak{c}}$, we have:

$$\begin{aligned} c(s(v)) &= ([c,s] + s \circ c)(v) \\ &= [c,s](v) + s(c(v)) \\ &\leq \mathfrak{c}(v) + s(0) \\ &= 0. \end{aligned}$$

This means that $V^{\mathfrak{c}} \leq V$ is invariant under the action of $\mathfrak{s}$.

□

Corollary 3.2.1.1. *If the Levi-complement of a linear Lie algebra is irreducible, then the radical is abelian.*

Proof*:* Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r} \leq \mathfrak{gl}(V)$ be a Levi-decomposition. Since $\mathfrak{r}$ is solvable, it can be triangularised by Lie's theorem. The derived ideal $\mathfrak{c} = [\mathfrak{r}, \mathfrak{r}]$ is characteristic and is strictly upper triangular. In particular, there exists a non-trivial $\mathfrak{c}$-invariant in V. The subspace $V^{\mathfrak{c}}$ of V is then also invariant under the action of $\mathfrak{s}$ by the previous lemma. Since $\mathfrak{s}$ is assumed to be irreducible, $V^{\mathfrak{c}}$ must be V. This is equivalent to $[\mathfrak{r}, \mathfrak{r}] = 0$ and it finishes the proof.

□

A flag Let $\mathfrak{n}$ be any nilpotent Lie algebra and let (ρ, V) be any $\mathfrak{n}$-module. Then we can associate to $\mathfrak{n}$ and V a flag of V in a very natural way. Let V_0 be V itself and define V_{t+1} recursively as $\langle \mathfrak{n} \cdot V_t \rangle$. Equivalently, V_t is the space generated by all elements of the form $\rho(n_1) \circ \cdots \circ \rho(n_t)(v)$ for $n_1, \ldots, n_t \in \mathfrak{n}$ and v in V. Note that this gives us a descending chain of subspaces and it satisfies the following property: for every $t \geq 0$, we have $\rho(\mathfrak{n})V_t \leq V_{t+1} \leq V_t$. This chain will not terminate in the zero-vector space unless the representation is a nilrepresentation, i.e. if it has a finite type. If the representation is of type zero, the chain will terminate after exactly $c+1$ steps:

$$0 = V_{c+1} < V_c < \ldots < V_1 < V_0 = V.$$

Lemma 3.2.1.2. *Consider a Lie algebra $\mathfrak{g}$ with nilradical $\mathfrak{n}$ of class c. Let V be a finite-dimensional $\mathfrak{g}$-module of type zero. Then there exists a flag*

$(V_i)_i$ of V of length $c+1$ such that for all $t \in \mathbb{N}$:

$$\mathfrak{s} \cdot \frac{V_t}{V_{t+1}} \leq \frac{V_t}{V_{t+1}} \text{ and } \mathfrak{n} \cdot V_t \leq V_{t+1}.$$

Here, we identify $\frac{V_t}{V_{t+1}}$ with a suitable $\mathfrak{s}$-complement W_{t+1} of V_{t+1} in V_t.

PROOF*:* Let $0 < V_c < \ldots < V_1 < V_0 = V$ be the flag associated to $\mathfrak{n}$ and V. Then it is not only invariant under $\mathfrak{n}$ but also under $\mathfrak{s}$. We use induction to show this. It is clear that $\mathfrak{s} \cdot V_0 \leq V_0$. Now assume that $\mathfrak{s} \cdot V_t \leq V_t$. Consider elements $s \in \mathfrak{s}, n \in \mathfrak{n}$ and $v \in V_t$. Then,

$$\begin{aligned} s \cdot (m \cdot v) &= [s, m] \cdot v + (m \cdot s) \cdot v \\ &\leq \mathfrak{n} \cdot v + m \cdot (s \cdot V_t) \\ &\leq \mathfrak{n} \cdot V_t + \mathfrak{n} \cdot V_t \\ &\leq V_{t+1}. \end{aligned}$$

This implies that $\mathfrak{s} \cdot V_{t+1} \leq V_{t+1}$, which completes the induction. We will construct the W_t from the V_t. For every $t \geq 0$, we can find an $\mathfrak{s}$-module complement W_{t+1} for V_{t+1} in V_t. We can then identify W_{t+1} with $\frac{V_t}{V_{t+1}}$ and we have the desired identities.

□

Characterisation We now assume that the radical of the Lie algebra $\mathfrak{g}$ is in fact abelian. Let $\mathfrak{s} \ltimes \mathfrak{a}$ be a Levi-decomposition. Then any finite-dimensional module $(\mathfrak{g}, V)$ of type 0 decomposes as a direct sum of two proper $\mathfrak{s}$-invariant subspaces V_1 and W_1 such that $\mathfrak{a} \cdot W_1 \leq V_1$ and $\mathfrak{a} \cdot V_1 = 0$. In particular, the type-zero faithful representations of $\mathfrak{s} \ltimes \mathfrak{a}$ are of the form,

$$\rho(\mathfrak{s} \ltimes \mathfrak{a}) = \begin{array}{|c|c|} \hline \rho_1(\mathfrak{s}) & \rho(\mathfrak{a}) \\ \hline \mathbb{O} & \rho_2(\mathfrak{s}) \\ \hline \end{array}.$$

Note that $\rho(\mathfrak{a})$ can be identified with a vector subspace of $\mathrm{Hom}(W_1, V_1)$. In this way, it is even a submodule of $(\langle \rho_1(\mathfrak{s}), \rho_2(\mathfrak{s}) \rangle, \mathrm{Hom}(W_1, V_1))$. Conversely, every representation of this form has type zero.

Corollary 3.2.1.2. *The type-zero representations of a Lie algebra $\mathfrak{s} \ltimes \mathfrak{a}$ correspond to the triples $(\alpha, \beta, \mathfrak{b})$ where α and β are $\mathfrak{s}$-representations and $\mathfrak{b}$ is a submodule of $\langle \alpha, \beta \rangle$, isomorphic to a quotient of $\mathfrak{a}$. Such a representation $(\alpha, \beta, \mathfrak{b})$ is a* faithful *representation of $\mathfrak{g}$ if and only if: $\alpha \oplus \beta$ is a faithful representation of $\mathfrak{s}$ and $\mathfrak{b} \leq Hom(V_\beta, V_\alpha)$ is isomorphic to $\mathfrak{a}$ as an $\mathfrak{s}$-module.*

Fix a Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{a}$ and a natural number n. We are now ready, at least in theory, to check if there is a faithful type-zero-representation of $\mathfrak{g}$ of degree n. If there is one, the following procedure will produce all such representations of degree n. If there is no such representation, the output is empty.

[**Algorithm**] Decompose $\mathfrak{s}$ into simple ideals $\mathfrak{s}_1, \cdots, \mathfrak{s}_l$. Let $\Sigma(\mathfrak{s}_i)$ be the set of irreducible $\mathfrak{s}_i$-representations. Then according to theorem 2.2.1, $\Sigma(\mathfrak{s}) = \Sigma(\mathfrak{s}_1) \times \cdots \times \Sigma(\mathfrak{s}_l)$ is the set of all irreducible $\mathfrak{s}$-representations. According to Weyl's theorem, the set Σ of all $\mathfrak{s}$-representations consists of the (finite) direct sums of elements in $\Sigma(\mathfrak{s})$. Construct the *finite* set

$$\overline{P}_n(\mathfrak{s}) = \{\ (\sigma, \tau) \in \Sigma \times \Sigma \mid\ \deg(\sigma) + \deg(\tau) = n\ \}\,.$$

Lemma 2.2.1.2 describes faithful representations of $\mathfrak{s}$ in terms of the mentioned decomposition. Let $P_n(\mathfrak{s})$ be the set elements (σ, τ) in $\overline{P}_n(\mathfrak{s})$ such that $\sigma \oplus \tau$ is faithful. For each element (σ, τ) in $P_n(\mathfrak{s})$, one can decompose the $\mathfrak{s}$-module $\langle \sigma, \tau \rangle$ into irreducibles and check if $\mathfrak{a} \leq \langle \sigma, \tau \rangle$ as an $\mathfrak{s}$-module. Define the set

$$Q_n(\mathfrak{s}, \mathfrak{a}) = \{\ (\sigma, \tau) \in P_n(\mathfrak{s}) \mid \mathfrak{a} \leq \langle \sigma, \tau \rangle\ \}\,.$$

If $Q_n(\mathfrak{s}, \mathfrak{a})$ is empty, there is no n-dimensional faithful representation of $\mathfrak{s} \ltimes \mathfrak{a}$ of the desired form. If $Q_n(\mathfrak{s}, \mathfrak{a})$ is not empty, then every element $(\sigma, \tau) \in Q_n(\mathfrak{s}, \mathfrak{a})$ induces an n-dimensional faithful representation of $\mathfrak{s} \ltimes \mathfrak{a}$:

$$\rho(\mathfrak{s} \ltimes \mathfrak{a}) = \begin{array}{|c|c|} \hline \sigma(\mathfrak{s}) & f(\mathfrak{a}) \\ \hline \mathbb{O} & \tau(\mathfrak{s}) \\ \hline \end{array},$$

where $f : \mathfrak{a} \longrightarrow \langle \sigma, \tau \rangle$ is an embedding of $\mathfrak{s}$-modules.

Finally, fix a natural number n. Applying the above algorithm to n, we conclude that either $n < \mu_0(\mathfrak{s} \ltimes \mathfrak{a})$ or $\mu_0(\mathfrak{s} \ltimes \mathfrak{a}) \leq n$. In the former case, we increase n by one and repeat the algorithm until it gives a positive answer. Then $\mu_0(\mathfrak{s} \ltimes \mathfrak{a}) = n$. In the latter case, we decrease n by one until the algorithm produces a negative answer. Then $\mu_0(\mathfrak{s} \ltimes \mathfrak{a}) = n + 1$. For practical applications, we refer to subsection 3.3.1 and to section 4.4.

3.2.2 Combinatorical bounds

In the above it was explained how the μ_0-invariant can be computed for an arbitrary Lie algebra with abelian radical. This computation uses the explicit decomposition of tensor products of representations into irreducibles, which can be quite involved. Here, we present an upper bound for $\mu_0(\mathfrak{g})$, and hence for $\mu(\mathfrak{g})$, in terms of the dimensions and multiplicities of the irreducible components of the defining representation. For convenience, we also assume that this latter representation is faithful.

So suppose we are working with a Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes_\delta \mathfrak{a}$ and assume that the defining representation δ is faithful. In particular, the faithful representation $\rho_\mathfrak{s} = \delta \oplus \tau$ of the semisimple part $\mathfrak{s}$ induces a faithful representation of the entire Lie algebra (that squares to zero on the radical). This representation has degree $\deg(\delta) + 1$. But it is possible to refine this upper bound.

Definition 3.2.2.1. For any natural number k, we define the function $p = p_k : \mathbb{N}_0^k \times \mathbb{N}_0^k \longrightarrow \mathbb{N}_0$ as follows. Let $(m; d) = ((m_1, \ldots, m_k); (d_1, \ldots, d_k)) \in \mathbb{N}_0^k \times \mathbb{N}_0^k$, then

$$p_k((m; d)) = \min \left\{ M_0 + \sum_{i=1}^{k} M_i d_i \mid M_i \in \mathbb{N}_0 \text{ and } M_0 M_i \geq m_i \right\}.$$

Let k be the number of inequivalent irreducible representations in the decomposition of δ. If we take the m_i to be the multiplicities of these representations and d_i to be their degrees, we obtain the following result.

Proposition 3.2.2.1. *For $\mathfrak{s} \ltimes_\delta \mathfrak{a}$ as above, $\mu(\mathfrak{s} \ltimes_\delta \mathfrak{a}) \leq \mu_0(\mathfrak{s} \ltimes_\delta \mathfrak{a}) \leq p_k(m; d)$. And in particular, an upper bound for $p_k(m; d)$ gives,*

$$\lceil 2\sqrt{\deg(\delta)} \rceil \leq \mu(\mathfrak{s} \ltimes_\delta \mathfrak{a}) \leq \lceil \sqrt{\deg(\delta)} \rceil + \sum_i d_i \lceil \frac{m_i}{\sqrt{\deg(\delta)}} \rceil.$$

This upper bound for $p_k(m; d)$ is in general a very rough one so it is worth it to compute the value for $p_k(m; d)$ explicitly. In the appendix we give algorithms for this computation.

Proof*:* Since the algebra $\mathfrak{g}$ is perfect, every embedding will be traceless. In particular, we have an embedding $\rho : \mathfrak{g} \longrightarrow \mathfrak{sl}(\mathbb{C}^{\mu(\mathfrak{g})})$. This induces an embedding of the radical $\mathfrak{a}$ in $\mathfrak{sl}(\mathbb{C}^{\mu(\mathfrak{g})})$

and the lower bound $\mu(\mathbb{C}^{\deg(\delta)+1}) \leq \mu(\mathfrak{g})$. For the upper bound, it suffices to check that

$$(M_0; M_1, \dots, M_k) = (\lceil \sqrt{\deg(\delta)} \rceil; \lceil \frac{m_1}{\sqrt{\deg(\delta)}} \rceil, \dots, \lceil \frac{m_k}{\sqrt{\deg(\delta)}} \rceil)$$

satisfies the condition $m_i \leq M_0 M_i$ for all i. And this is the case since,

$$m_i = \sqrt{\deg(\delta)} \frac{m_i}{\sqrt{\deg(\delta)}} \leq \lceil \sqrt{\deg(\delta)} \rceil \lceil \frac{m_i}{\sqrt{\deg(\delta)}} \rceil = M_0 M_i.$$

□

3.2.3 Dimensional bounds

What is known about the μ-invariant of a Lie algebra if only the dimension of that Lie algebra is given? In this subsection we present some lower and upper bounds which turn out to be sharp bounds.

Proposition 3.2.3.1. *The μ-invariant of a Lie algebra $\mathfrak{g}$ with abelian radical satisfies,*

$$\sqrt{\dim(\mathfrak{g})} \leq \mu(\mathfrak{g}) \leq \dim(\mathfrak{g}).$$

In particular, we have that any Lie algebra $\mathfrak{h}$ with $\dim(\mathfrak{h}) < \mu(\mathfrak{h})$ must have a non-abelian radical.

Proof*:* The lower bound is obvious. For suppose that $\mathfrak{g}$ can be embedded into $\mathfrak{gl}_n$. Then $\dim(\mathfrak{g}) \leq \dim(\mathfrak{gl}_n) = n^2$ and $\sqrt{\dim(\mathfrak{g})} \leq n$. In particular this is true for $n = \mu(\mathfrak{g})$. For reductive Lie algebras, the upper bound is obtained from the subadditivity of μ and the fact that it holds for the simple and the abelian ones.

So suppose that $\mathfrak{g} = \mathfrak{s} \ltimes_\delta \mathfrak{a}$ is *not* reductive. Note that $\mathfrak{g} \cong \ker(\delta) \oplus (\overline{\mathfrak{s}} \ltimes_{\overline{\delta}} \mathfrak{a})$ with $\overline{\mathfrak{s}} = \frac{\mathfrak{s}}{\ker(\delta)}$ and $\overline{\delta} : \overline{\mathfrak{s}} \longrightarrow \mathfrak{gl}(\mathfrak{a})$ the induced, faithful representation. Since $\mathfrak{g}$ is not reductive, $\overline{\mathfrak{s}}$ is a proper ideal of $\mathfrak{s}$. Then, $\dim(\ker(\delta)) = \dim(\mathfrak{s}) - \dim(\overline{\mathfrak{s}}) \leq \dim(\mathfrak{s}) - 3$.

This gives us the following estimates,

$$\begin{aligned}\mu(\mathfrak{g}) &\leq \mu(\ker(\delta)) + \mu(\overline{\mathfrak{s}} \ltimes_{\overline{\delta}} \mathfrak{a}) \\ &\leq \dim(\ker(\delta)) + \deg(\overline{\delta}) + 1 \\ &= \dim(\ker(\delta)) + \dim(\overline{\mathfrak{s}}) - \dim(\overline{\mathfrak{s}}) + \dim(\mathfrak{a}) + 1 \\ &= \dim(\mathfrak{g}) - \dim(\overline{\mathfrak{s}}) + 1 \\ &\leq \dim(\mathfrak{g}) - 2.\end{aligned}$$

This finishes the proof.

□

These bounds are sharp. The Lie algebras that satisfy $\sqrt{\dim(\mathfrak{g})} = \mu(\mathfrak{g})$ are of course full linear algebras $\mathfrak{gl}_n(\mathbb{C})$ for $n \in \mathbb{N}$. The ones that reach the upper bound are described in the following lemma.

Proposition 3.2.3.2. *The Lie algebras* $\mathfrak{g}$ *with abelian radical that satisfy* $\mu(\mathfrak{g}) = \dim(\mathfrak{g})$*, are precisely*

$$0, \mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \mathbb{C}^4 \text{ and } n\mathfrak{e}_8 \text{ for } n \in \mathbb{N}.$$

PROOF*:* The proof of the previous lemma shows that such a Lie algebra is necessarily reductive. The abelian Lie algebras $\mathfrak{a}$ that satisfy $\dim(\mathfrak{a}) = \mu(\mathfrak{a})$ are $\mathbb{C}^0, \mathbb{C}^1, \mathbb{C}^2, \mathbb{C}^3$ and $\mathbb{C}^4$. For the semisimple case $\mathfrak{s} = \oplus_j \mathfrak{s}_j$, we have $\mu(\mathfrak{s}) \leq \sum_j \mu(\mathfrak{s}_j) \leq \sum_j \dim(\mathfrak{s}_j) = \dim(\mathfrak{s})$. So the μ-invariant and dimension of $\mathfrak{s}$ coincide, precisely when the same is true for all the simple ideals $\mathfrak{s}_j$. The only simple, (non-zero) Lie algebra $\mathfrak{s}_j$ for which $\mu(\mathfrak{s}_j) = \dim(\mathfrak{s}_j)$, is $\mathfrak{e}_8$.

Now suppose that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$. Then we have

$$\begin{aligned}\mu(\mathfrak{g}) &\leq \mu(\mathfrak{s}) + \mu(\mathbb{C}^{\dim(\mathfrak{a}) - l(\mathfrak{s})}) \\ &\leq \dim(\mathfrak{s}) + \max\{0, \dim(\mathfrak{a}) - l(\mathfrak{s})\} \\ &\leq \dim(\mathfrak{s}) + \dim(\mathfrak{a}) \\ &= \dim(\mathfrak{g}).\end{aligned}$$

If $\mu(\mathfrak{g}) = \dim(\mathfrak{g})$, then $\mu(\mathfrak{s}) = \dim(\mathfrak{s})$ and $\max\{0, \dim(\mathfrak{a}) - l(\mathfrak{s})\} = \dim(\mathfrak{a})$. The latter equality implies that either $\mathfrak{a} = 0$

or $\mathfrak{s} = 0$. This reduces the classification to the semisimple, resp. abelian case. This finishes the proof.

□

Note that this only covers the algebras with *abelian* radical. Take for example the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_1$. Then this two-step nilpotent Lie algebra satisfies $\mu(\mathfrak{h}_1) = \dim(\mathfrak{h}_1)$, but the (non-abelian) Lie algebra equals its own radical.

Corollary 3.2.3.1. *The Lie algebras $\mathfrak{g}$ with abelian radical satisfying $\mu(\mathfrak{g}) + 1 = \dim(\mathfrak{g})$ are reductive and they are given by $\mathfrak{sl}_2(\mathbb{C}), \mathbb{C}^5, \mathbb{C}^6$, $n\mathfrak{e}_8 \oplus \mathbb{C}$ for $n > 0$ and $\mathfrak{e}_8 \oplus \mathbb{C}$, $\ldots \mathfrak{e}_8 \oplus \mathbb{C}^5$.*

PROOF*:* Because of 3.2.3.1, we see that $\mathfrak{g}$ must be reductive. The Lie algebras in the list clearly satisfy the condition. Using an argument as before, we can exclude all other possibilities.

□

One can see that most Lie algebras $\mathfrak{g}$ with an abelian radical have a μ-invariant that is much smaller than this sharp, but rough bound $\dim(\mathfrak{g})$ (see 3.2.1). The above procedure can be generalised to obtain the following results.

Proposition 3.2.3.3. *For any Lie algebra of the form $\mathfrak{h} \ltimes_\delta \mathfrak{a}$ with $\mathfrak{h}$ reductive, $\mathfrak{a}$ abelian and δ faithful, we have $\mu(\mathfrak{h} \ltimes \mathfrak{a}) \leq \dim(\mathfrak{a})$. In particular, for $\mathfrak{h} = \mathfrak{b}$ abelian, we have $\mu(\mathfrak{b} \ltimes \mathfrak{a}) \leq \dim(\mathfrak{a})$.*

3.3 The family $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathbb{C}^t$

3.3.1 The μ_0-invariant

In the previous section we have sketched how one can compute the μ_0-invariant for Lie algebras with an abelian radical in a combinatorical way. We will now illustrate this result by applying it to a family of Lie algebras for which the combinatorics is not too complicated. So consider the Lie algebras with an abelian radical $\mathfrak{a}$ and a Levi-complement $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$: $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{a}$. Note that this family is parametrised by the finite-dimensional $\mathfrak{s}$-representations. These representations are either faithful or identically zero. The trivial representations correspond precisely to the reductive Lie algebras with commutator $\mathfrak{sl}_2(\mathbb{C})$. From corollary 2.3.2.4, we get

$$\mu_0(\mathfrak{s} \oplus \mathfrak{a}) = \mu_0(\mathfrak{s}) + \mu_0(\mathfrak{a}) = \mu(\mathfrak{s}) + \mu(\mathfrak{a} \oplus \mathbb{C}).$$

A finite problem So we may assume that the defining representation ρ is faithful. Remember that $\mathfrak{sl}_2(\mathbb{C})$ has, up to conjugation, a unique irreducible representation δ_n for every natural number $n \geq 1$. So the representations ρ correspond to sequences of natural numbers $(\rho_t)_t = (\rho_1, \rho_2, \ldots)$ (the multiplicities of the $\delta_1, \delta_2, \ldots$ in ρ) which have only finitely many non-zero terms: $l^1(\mathbb{N})$. The faithful representations are those for which at least one ρ_α is non-zero for some $\alpha \geq 2$. From now on we identify the representations with their sequence of multiplicities.

This identification is compatible with addition in the natural way: if ρ and σ are two representations, then $(\rho \oplus \sigma)_t = \rho_t + \sigma_t$ for all t. It is also compatible with inclusion: the representation $\alpha = (\alpha_t)_t$ is a subrepresentation of the representation $\beta = (\beta_t)_t$ if and only if $\alpha_t \leq \beta_t$ for all $t \geq 1$. The degree of α is simply $\sum_i i\alpha_i$. Consider the pairing of two representations:

$$\langle \cdot, \cdot\cdot \rangle : l^1(\mathbb{N}) \times l^1(\mathbb{N}) \longrightarrow l^1(\mathbb{N}) : (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle.$$

What is $\langle \alpha, \beta \rangle_u$ in terms of the α_s and β_t? Since the pairing is additive in each component, we have $\langle \alpha, \beta \rangle = \bigoplus_{i,j} \alpha_i \beta_j \langle \delta_i, \delta_j \rangle$ and thus also $\langle \alpha, \beta \rangle_u = \sum_{i,j} \alpha_i \beta_j \langle \delta_i, \delta_j \rangle_u$. So we only need to determine the multiplicities of $\langle \delta_i, \delta_j \rangle$. Let $P(n)$ be the parity of the integer n. The formula of Clebsch-Gordan tells us that $\langle \delta_i, \delta_j \rangle_u$ is either 1 or 0. More precisely:

$$\langle \delta_i, \delta_j \rangle_u = \begin{cases} 1 & \text{if} \quad |i-j| + 1 \leq u \leq i + j - 1 \\ & \qquad \text{and } P(u) \neq P(i-j) \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\langle \alpha, \beta \rangle_1 = \sum_i \alpha_i \beta_i$. Similarly, we have $\langle \alpha, \beta \rangle_2 = \sum_i (\alpha_i \beta_{i+1} + \alpha_{i+1} \beta_i)$ and so on. We now fix the Lie algebra $\mathfrak{g} = \mathfrak{s} \ltimes_\theta \mathfrak{a}$, or equivalently, the representation θ. We wish to determine the minimal degree of a faithful type-zero representation of $\mathfrak{g}$. Let c be an upper bound for $\mu_0(\mathfrak{g})$, for example $\deg(\theta) + 1$. Consider the *finite*, non-empty class,

$$\mathcal{C}_\theta^c = \{\ \alpha \oplus \beta \in l^1(\mathbb{N}) \oplus l^1(\mathbb{N}) \mid \theta \leq \langle \alpha, \beta \rangle \text{ and } \deg(\alpha \oplus \beta) \leq c\ \}.$$

Note that θ is not the zero-map by assumption. So every $(\alpha, \beta) \in \mathcal{C}$ defines a representation of $\mathfrak{g}$ that is of of type zero, of degree at most c and faithful. Up to irrelevant identifications, $\mathcal{C}_\theta^c$ contains all such faithful $\mathfrak{g}$-representations that are of type zero and have a degree at most c. Then we have by definition $\mu_0(\mathfrak{g}) = \min\{ \deg(\alpha \oplus \beta) \mid (\alpha, \beta) \in \mathcal{C}_\theta^c \}$. So we are minimising $\sum_k k(\alpha_k + \beta_k) \leq c$ under the conditions $\theta_u \leq \sum_{i,j} \alpha_i \beta_j \langle \delta_i, \delta_j \rangle_u$ for all $u \geq 1$. For special choices of θ, the combinatorics can be reduced even more using the symmetry of the problem.

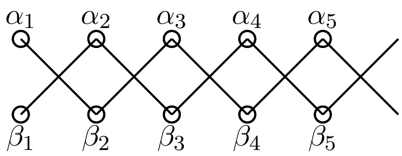
Multiplicity graphs Let $\alpha = (\alpha_i)_i$ and $\beta = (\beta_j)_j$ be two representations and let u be a non-zero natural number. Let us define a bipartite graph $G_u = G_u(\alpha, \beta) = (V_u, E_u)$ for α and β and u. The set V_u of vertices consists of $\{ (1, d) \mid d \in \mathbb{N}_0 \} \cup \{ (0, d) \mid d \in \mathbb{N}_0 \}$. Let us label the vertices: $L : V \longrightarrow \mathbb{N}$ such that $L(1, d) = \alpha_d$ and $L(0, d) = \beta_d$. Connect two vertices (ε, d) and (ε', d') if $1 = \langle \delta_d, \delta_{d'} \rangle_u$. Note that such a labelling is just a re-writing of the pair of representations and we may identify (α, β) with $G_u(\alpha, \beta)$. The degree of $\alpha \oplus \beta$ and $\langle \alpha, \beta \rangle_u$ can then be seen in the graph: $\deg(\alpha \oplus \beta) = \sum_d (d(\alpha_d + \beta_d))$ and

$$\langle \alpha, \beta \rangle_u = \sum_{((\varepsilon,d),(\varepsilon',d')) \in E_u} L(\varepsilon, d) L(\varepsilon', d').$$

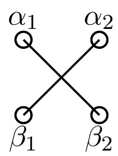
There is a natural order associated to this graph: $(\varepsilon, d) \leq (\varepsilon', d')$ if and only if $d \leq d'$. There are also natural transformations associated to such a graph. Any morphism $f : G_u \longrightarrow G_u$ satisfying $f(\varepsilon, d) \leq (\varepsilon, d)$ for all $(\varepsilon, d) \in V$ is called a *acceptable* morphism for the u'th component. It is acceptable if the property holds for all u. It defines a new labelling $L_f : G \longrightarrow G$ (i.e. a pair of representations) as follows: if $(1, d)$ is mapped to $(1, d')$, then $L(1, d)$ is decreased by $L(1, d)$ and $L(1, d')$ is increased by $L(1, d)$. Similarly for $(0, d)$. The pair (L, L_f) satisfies the following two properties: if (α, β) corresponds to the labelling L and (α', β') to the labelling L_f for some acceptable morphism f, then $\deg(\alpha' \oplus \beta') \leq \deg(\alpha \oplus \beta)$ and $\langle \alpha, \beta \rangle_u \leq \langle \alpha', \beta' \rangle_u$ for all u. In particular: the degree will not increase and the pairing will not decrease. Acceptable morphisms can often be realised as reflections or other elementary symmetries.

Example 3.3.1.1. Consider the Lie algebra associated to $\theta = m\delta_2 = (0, m, 0, \ldots)$ for some $m \in \mathbb{N}_0$. Then the pairing $\langle \alpha, \beta \rangle$

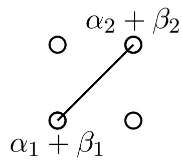
contains θ if and only if $m \leq \langle \alpha, \beta \rangle_2 = \sum_i (\alpha_i \beta_{i+1} + \alpha_{i+1} \beta_i)$. So we only need to consider the graphs $G_2(\alpha, \beta)$:



So suppose we have a pair (α, β) with $m \leq \langle \alpha, \beta \rangle_2$. Folding from right to left then clearly defines an acceptable morphism. We may apply such folding morphisms successively to obtain a reduced graph of the form,



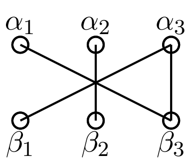
Then the transformation $(\alpha_1, \beta_1, \alpha_2, \beta_2) \mapsto (\beta_1, \beta_1, \alpha_2, \alpha_2)$ is also an acceptable morphism. In particular, we obtain the graph



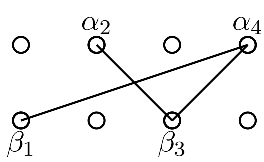
So we conclude that $\mu_0(\mathfrak{sl}_2(\mathbb{C}) \ltimes_{m\delta_2}) = p(m; 2)$ from subsection 3.2.2.

△

Example 3.3.1.2. Similarly, we can use acceptable morphisms to reduce the graph G_3 for $\theta = m\delta_3$ and G_4 for $\theta = m\delta_4$ to one of the form



resp.



and so on. Note that the graphs do not depend, of course, on the parameter $m \geq 1$.

△

Low dimensions Let us write L_δ for the Lie algebra with defining representation δ, i.e.: $L_\delta = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\delta \mathbb{C}^{\deg(\delta)}$. We include a table (3.1) of $\mu_0(L_\delta)$ up to dimension 9. We identify the defining representation $\delta = \oplus_\alpha m_\alpha \delta_\alpha$ with the sequence $(m_\alpha)_\alpha$. The representations that correspond to a reductive Lie algebra are marked with an asterisk.

$\deg(\delta)$	δ	$\mu_0(L_\delta)$
1	(1)	2^*
2	$(0,1)$	3
	(2)	3^*
3	$(0,0,1)$	4
	$(1,1)$	4
	(3)	4^*
4	$(0,0,0,1)$	5
	$(1,0,1)$	4
	$(0,2)$	4
	$(2,1)$	5
	(4)	5^*
5	$(0,0,0,0,1)$	6
	$(1,0,0,1)$	6
	$(0,1,1)$	5
	$(2,0,1)$	6
	$(1,2)$	5
	$(3,1)$	6
	(5)	6^*
6	$(0,0,0,0,0,1)$	7
	$(1,0,0,0,1)$	6
	$(0,1,0,1)$	5
	$(2,0,0,1)$	7
	$(0,0,2)$	5
	$(1,1,1)$	7
	$(3,0,1)$	7
	$(0,3)$	5
	$(2,2)$	4
	$(4,1)$	6
	(6)	6^*

Table 3.1: $\mu_0(L_\delta)$ up to dimension 9.

3.3.2 The μ-invariant

Irreducible representations

Lemma 3.3.2.1. *Consider an irreducible representation* (ρ, V) *of* $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C})$. *Then there is only one abelian submodule of* $(\mathrm{End}(V), \langle \rho, \rho \rangle)$*: the scalar one.*

Proof*:* Consider such a representation $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$ of degree x. Then according to the Clebsch-Gordan formula, we have the following isomorphism,

$$\langle \rho, \rho \rangle \cong \bigoplus_{j=0}^{x-1} \delta_{1+2j}.$$

Note that the multiplicities of all components are one and that all occurring irreducible representations have an even highest weight. After conjugation, we may assume that ρ has the standard form. Then the zero-weight space for ρ is exactly the centraliser of $\rho(H)$ in $\mathfrak{gl}(V)$ and consists of the diagonal matrices.

Suppose that $W \leq \mathfrak{gl}(V)$ is an abelian submodule. We will prove that any irreducible submodule of W (which is necessarily abelian) consists of only scalar matrices. This implies that W is scalar and this proves the proposition.

So take any irreducible, abelian submodule $U \leq \mathfrak{gl}(V)$. As noted above, there is a non-trivial zero-weight vector $M \in \mathfrak{gl}(V)$. (This matrix is diagonal.) Define the two elements

$$\begin{aligned} A^+ &= (\langle \rho, \rho \rangle(E))\,(M) = [\rho(E), M] \\ A^- &= (\langle \rho, \rho \rangle(F))\,(M) = [\rho(F), M]. \end{aligned}$$

The matrices A^+ and A^- belong to the module U since M does. Since U is abelian, A^+ and A^- commute. A straightforward calculation shows that this is only possible if M is scalar. This finishes the proof.

□

Proposition 3.3.2.1. *For any* $x \in \mathbb{N}$, *we have* $\mu(L_{\delta_x}) + 2 = \dim(L_{\delta_x})$.

Proof*:* Suppose first that $x = 1$. Then the Lie algebra is decomposable and isomorphic to $\mathfrak{gl}_2(\mathbb{C})$. The μ-invariant is then of course 2.

So we may now assume that $x > 1$. Let $\widetilde{\rho} : \mathfrak{s} \ltimes_{\delta_x} \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ be any faithful representation with restriction $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$. Since ρ induces an (abelian) δ_x-submodule of $\mathfrak{gl}(V)$, there must exist irreducible representations δ and δ' of degree α resp. β in a fixed decomposition of ρ (where δ and δ' can be the same representation) such that $P(x) \neq P(\alpha - \beta)$ and

$$\alpha - \beta + 1 \leq x \leq \alpha + \beta - 1.$$

We use the previous lemma to show that δ and δ' can be chosen distinct. For suppose the converse. Then the δ_x-module induced by ρ is exactly the one induced by $\delta = \delta'$. The lemma says that the module can only be abelian if $x = 1$ which contradicts our assumption.

So we may assume that δ and δ' are distinct terms in the decomposition of ρ. In particular, we have

$$\begin{aligned} x &\leq \alpha + \beta - 1 \\ &\leq \deg(\rho) - 1 = \deg(\widetilde{\rho}) - 1. \end{aligned}$$

This holds for any faithful representation $\widetilde{\rho}$ of $\mathfrak{s} \ltimes_{\delta_x} \mathfrak{a}$ so that $x + 1 \leq \mu(\mathfrak{s} \ltimes_{\delta_x} \mathfrak{a})$. The converse inequality is induced by the representation $\rho = \delta_x \oplus \delta_1$. This finishes the proof.

□

Corollary 3.3.2.1. *The Lie algebras $L = n\mathfrak{e}_8 \oplus L_{\delta_x}$ satisfy $\mu(L) + 2 = \dim(L)$.*

Proof: We may suppose that x is at least two. Then L_{δ_x} is perfect and lemma 2.3.2.3 tells us that $\mu(L) = \mu(n\mathfrak{e}_8) + \mu(L_{\delta_x})$. The previous proposition then implies $\mu(L) = \dim(n\mathfrak{e}_8) + \dim(L_{\delta_x}) - 2 = \dim(L) - 2$.

□

Reducible representations

Example 3.3.2.1. We have the following formula's for μ:

$$\mu(L_\delta) = \begin{cases} x+3 & \text{for} \quad \delta_x \oplus \delta_{x+1} \\ \max(2r, 2s+1)+2 & \text{for} \quad \delta_{2r} \oplus \delta_{2s+1} \\ x+3 & \text{for} \quad \delta_{x-1} \oplus \delta_x \oplus \delta_{x+1} \\ x+2k+1 & \text{for} \quad \delta_x \oplus \delta_{x+2} \oplus \ldots \oplus \delta_{x+2k} \end{cases}$$

PROOF: **(i)** Consider any faithful representation $\widetilde{\rho} : L_\delta \longrightarrow \mathfrak{gl}(V)$ with restriction $\rho : \mathfrak{s} \longrightarrow \mathfrak{gl}(V)$. Fix a decomposition of δ into irreducibles. Then ρ induces abelian δ_x and δ_{x+1}-submodules. Just as in the proof of the previous proposition, we can show that there exist terms $\delta \neq \delta'$ and $\eta \neq \eta'$ in the decomposition of ρ such that δ_x is a submodule of $\langle \delta, \delta' \rangle$ and δ_{x+1} is a submodule of $\langle \eta, \eta' \rangle$. Since the parities of x and $x+1$ differ, at least one of the sets $\{\delta, \delta', \eta'\}$, $\{\delta, \delta', \eta\}$, $\{\eta, \eta', \delta'\}$ or $\{\eta, \eta', \delta\}$ consists of three distinct elements. Without loss of generality, we assume it is true for the latter. Then we have

$$\begin{aligned} x+1 &\leq \deg(\eta) + \deg(\eta') - 1 \\ &\leq \deg(\eta) + \deg(\eta') + \deg(\delta) - 2 \\ &\leq \deg(\rho) - 2 \\ &= \deg(\widetilde{\rho}) - 2. \end{aligned}$$

Since this holds for all faithful representations $\widetilde{\rho}$ of L_δ, we have the bound $x+3 \leq \mu(L_\delta)$. The representation $\rho = \delta_x \oplus \delta_2 \oplus \delta_1$ induces the other inequality. This finishes the proof of the first point. **(ii)** Note that the proof above can be generalised to prove $\mu(\delta_x \oplus \delta_y) = \max(x, y) + 2$ for x and y of different parity. The lower bound is obtained in the same way. The upper bound is induced by $\rho = \delta_{\min(x,y)} \oplus \delta_{(\max - \min)(x,y)+1} \oplus \delta_1$. **(iii)** Using the same ρ as above, we obtain the upper bound $\mu(L_\delta) \leq x+3$. Using the monotonicity and the previous case, we obtain equality. **(iv)** Using the monotonicity and proposition 3.3.2.1, we obtain $x+2k+1$ as a lower bound. The representation $\rho = \delta_{x+k} \oplus \delta_{k+1}$ gives the other inequality. □

Example 3.3.2.2. For $t \geq 1$, we have the fowllowing formula's:

$$\mu(L_\delta) = \mu_0(L_\delta) = \begin{cases} \lceil 2\sqrt{2t} \rceil & \text{for} \quad t\delta_2 \\ \lceil 2\sqrt{3t} \rceil & \text{for} \quad t\delta_3. \end{cases}$$

Proof*:* **(i)** Note that $L = L_\delta$ is perfect for all t, so that $\lceil 2\sqrt{2t} \rceil = \mu(\mathfrak{a} \oplus \mathbb{C}) \leq \mu(L \oplus \mathbb{C}) = \mu(L) \leq \mu_0(L)$. This gives us the lower bound. This lower bound B_t is reached by $\mu_0(L)$ so that the inequalities are actually equalities. We can even explicitly construct a a faithful representation of type zero of this degree. For this, it is sufficient to find $a, b \in \mathbb{N}_0$ such that the following conditions hold:

$$\begin{array}{rcl} 2a + b & \leq & B_t \\ ab & \geq & 2t, \end{array}$$

We define a to be $\lceil \frac{\lfloor \frac{B_t}{2} \rfloor}{2} \rceil$ and b to be $B_t - 2a$. Since $t \geq 1$, B_t will be at least 3 and a is at least 1. It can then be checked that the representation $a\delta_2 \oplus b\delta_1$ induces a faithful representation for L. **(ii)** Similarly, for $\delta = t\delta_3$, we can consider the bound $B_t = \lceil 2\sqrt{3t} \rceil$ and the representation $a\delta_3 \oplus b\delta_1$. We define b as $B_t - 3a$ and a to be $\lceil \frac{\lfloor \frac{B_t}{2} \rfloor}{3} \rceil$.

□

Proposition 3.3.2.2. *Consider a non-reductive Lie algebra $L = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\delta \mathfrak{a}$. If δ is reducible, then $\mu(L) < \dim(L) - 2$.*

As noted in proposition 3.2.3.1, the Lie algebras $\mathfrak{g}$ with abelian radical satisfy $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})$. In case $\mathfrak{g}$ is reductive, we have obtained an explicit formula for μ and we proved that the bound is sharp. In the other case, the non-reductive one, we have the upper bound $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) - 2$. It follows from proposition 3.3.2.1 that this bound is sharp.

Corollary 3.3.2.2. *The non-reductive Lie algebras $\mathfrak{g}$ with abelian radical satisfying $\mu(\mathfrak{g}) + 2 = \dim(\mathfrak{g})$ are precisely the Lie algebras of the form $n\mathfrak{e}_8 \oplus \mathfrak{sl}_2(\mathbb{C}) \ltimes_{\delta_x} \mathfrak{a}$ with $n \in \mathbb{N}$ and $x \in \mathbb{N}_0$.*

Remark 6. Consider a Lie algebra with an abelian radical. If the Lie algebra happens to be reductive, then the centre is maximal in the following sense: the centre coincides with the radical and we have used this to obtain a formula for the μ-invariant. In the other extreme, the Lie algebra is perfect. Experimental evidence in low dimensions seems to suggest a very close relationship between μ_0, μ_∞ and μ:

$\deg(\delta)$	δ	$\mu_0(L_\delta) = \mu(L_\delta)$
2	$(0,1)$	3
3	$(0,0,1)$	4
4	$(0,0,0,1)$	5
	$(0,2)$	4
5	$(0,0,0,0,1)$	6
	$(0,1,1)$	5
6	$(0,0,0,0,0,1)$	7
	$(0,1,0,1)$	5
	$(0,0,2)$	5
7	$(0,0,0,0,0,0,1)$	8
	$(0,1,0,0,1)$	7
	$(0,0,1,1)$	6
	$(0,2,1)$	6
8	$(0,0,0,0,0,0,0,1)$	9
	$(0,1,0,0,0,1)$	7
	$(0,0,1,0,1)$	6
	$(0,0,0,2)$	6
	$(0,2,0,1)$	6
	$(0,1,2)$	7
	$(0,4)$	6

Table 3.2: $\frac{\mathfrak{g}}{\mathrm{rad}(\mathfrak{g})} \cong \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{z}(\mathfrak{g}) = 0$.

Problem Do $\mu(\mathfrak{g})$, $\mu_\infty(\mathfrak{g})$ and $\mu_0(\mathfrak{g})$ coincide for perfect Lie algebras with an abelian radical?

A positive answer is desirable since $\mu(\mathfrak{g})$ is difficult to compute, whereas $\mu_0(\mathfrak{g})$ can be computed easily.

Chapter 4

Remarks

In this final part we would like to discuss some other methods to construct faithful representations of low degree, classes of Lie algebras that are interesting in this field and some of the most evident open problems.

4.1 Construction of modules

In the first chapter we gave a proof of Ado's theorem by explicitly constructing faithful modules of finite dimension. Neretin's embedding theorem reduced the problem to the essential case, that of the nilpotent Lie algebras. For these nilpotent Lie algebras $\mathfrak{n}$ there is a standard way to proceed. Simply consider the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$. This is an infinite-dimensional faithful representation of $\mathfrak{n}$. By considering the appropriate quotients we obtain faithful representations, although not necessarily of low dimension.

4.1.1 Extensions

We can consider extensions of Lie algebras in order to construct modules. Consider for example the following proposition on central extensions.[Bu6]

> **Proposition** [Burde] Suppose the Lie algebra $\mathfrak{g}$ is the quotient of a Lie algebra $\mathfrak{h}$ through its centre $\mathfrak{z}(\mathfrak{h})$, i.e., we have a short exact sequence, $0 \longrightarrow \mathfrak{z}(\mathfrak{h}) \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow 0$. Then $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + \dim(\mathfrak{z}(\mathfrak{h}))$.

Derivations One can use derivations to construct split extensions. Consider a Lie algebra $\mathfrak{g}$ and the Lie algebra $\mathrm{Der}(\mathfrak{g})$ of all derivations. Then $\mathrm{Der}(\mathfrak{g})$ acts on $\mathfrak{g}$ and more generally, every subalgebra $\mathcal{D}$ of $\mathrm{Der}(\mathfrak{g})$ acts on every characteristic ideal $\mathfrak{g}_{\mathrm{char}}$ of $\mathfrak{g}$. We may then consider the associated semidirect product (split extension): $\mathrm{Der}(\mathfrak{g}) \ltimes \mathfrak{g}_{\mathrm{char}}$. Clearly, every derivation δ generates a subalgebra of $\mathrm{Der}(\mathfrak{g})$: $\widehat{\mathfrak{g}} = \langle \delta \rangle \ltimes_{\mathbb{1}} \mathfrak{g}$.

Lemma 4.1.1.1. *Let $\mathfrak{g}$ be a Lie algebra and suppose δ is an* outer *derivation. Then the one-dimensional extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by δ satisfies $\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathfrak{z}(\mathfrak{g}) \cap \ker \delta$.*

PROOF*:* In the Lie algebra $\widehat{\mathfrak{g}} = \langle \delta \rangle \ltimes_{\mathbb{1}} \mathfrak{g}$, the Lie bracket is given by

$$[(\lambda\delta, x), (\nu\delta, y)] \quad = \quad (0, \lambda\delta(y) - \nu\delta(x) + [x, y]),$$

for all $(\lambda\delta, x)$ and $(\nu\delta, y)$ in $\widehat{\mathfrak{g}}$. An element $z = (\lambda\delta, x)$ of $\widehat{\mathfrak{g}}$ is in the centre of $\widehat{\mathfrak{g}}$ if and only if, for every $(\nu\delta, y) \in \widehat{\mathfrak{g}}$, $0 = \lambda\delta(y) - \nu\delta(x) + [x, y]$. Suppose $(\lambda\delta, x)$ is central. For $\nu = 1$ and $y = 0$, we obtain $\delta(x) = 0$ so that x is contained in the kernel of δ. The equation then reduces to $\lambda\delta(y) = [-x, y]$ for all y in $\mathfrak{g}$. Since δ is an outer derivation, λ must be zero. This implies that x is central so that $(\lambda\delta, x) \in \ker(\delta) \cap \mathfrak{z}(\mathfrak{g})$. It is also clear that $\mathfrak{z}(\widehat{\mathfrak{g}}) \geq \mathfrak{z}(\mathfrak{g}) \cap \ker \delta$. This finishes the proof.

□

Corollary 4.1.1.1. *Let $\mathfrak{g}$ be a Lie algebra and f a derivation that induces an isomorphism on the centre. Then $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$.*

PROOF*:* Consider the extension $\widehat{\mathfrak{g}}$ from the previous lemma. Then $\widehat{\mathfrak{g}}$ has no centre and the restriction of the adjoint representation of $\widehat{\mathfrak{g}}$ to $\mathfrak{g}$ defines a faithful linear representation of degree $\dim(\widehat{\mathfrak{g}}) = \dim(\mathfrak{g}) + 1$.

□

Example 4.1.1.1. Consider a Levi-decomposition $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{a}$ for a Lie algebra $\mathfrak{g}$ with an abelian radical. Consider the natural projection onto the radical $\pi : \mathfrak{g} \longrightarrow \mathfrak{g} : (x; y) \longmapsto (0; y)$. Then this map is a derivation of $\mathfrak{g}$ that is non-singular on the centre. We conclude that $\mu(\mathfrak{g}) \leq \mu(\widehat{\mathfrak{g}}) \leq \dim(\mathfrak{g}) + 1$. Note however that we have already obtained the strictly better result $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})$.

$\triangle$

Is it possible to lift derivations of the radical to the entire Lie algebra in the obvious way? The following proposition gives us a necessary and sufficient condition.

Proposition 4.1.1.1. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi-decomposition for the Lie algebra $\mathfrak{g}$, with $\mathfrak{s} \leq \mathrm{Der}(\mathfrak{r})$. Suppose δ is a derivation of the radical. Then the extended map $\pi : \mathfrak{s} \ltimes \mathfrak{r} \longrightarrow \mathfrak{s} \ltimes \mathfrak{r} : (x;t) \longmapsto (0;\delta(t))$ is a derivation of $\mathfrak{g}$ if and only if δ commutes with $\mathfrak{s}$.*

Proof*:* Consider any pair $a = (x,t)$ and $b = (y,s)$ of elements in $\mathfrak{g}$. We need to show that $\pi([a,b]) = [\pi(a),b] + [a,\pi(b)]$. The commutator of a and b is given by $[(x,t),(y,s)] = ([x,y], x(s) - y(t) + [t,s])$ so that

$$\begin{aligned} \pi([(x,t),(y,s)]) &= (0, \delta([t,s]) + (\delta \circ x)(s) - (\delta \circ y)(t)) \\ &= (0, [\delta(t),s] + [s,\delta(t)] + (\delta \circ x)(s) - (\delta \circ y)(t)). \end{aligned}$$

We have $\pi(x,t) = (0,\delta(t))$ and $\pi(y,s) = (0,\delta(s))$ so that

$$\begin{aligned} [\pi(x,t),(y,s)] &+ [(x,t),\pi(y,s)] = [(0,\delta(t)),(y,s)] + [(x,t),\delta(s)] \\ &= (0, [\delta(t),s] + [t,\delta(s)] + (x \circ \delta)(s) - (y \circ \delta)(t)). \end{aligned}$$

We see that π is a derivation of $\mathfrak{g}$ if and only if these two expressions coincide for all values of $x, y \in \mathfrak{s}$ and $s, t \in \mathfrak{n}$. This condition reduces to $[\delta, x](s) = 0$ for all $x \in \mathfrak{s}$ and $s \in \mathfrak{n}$. This finishes the proof.

$\square$

Corollary 4.1.1.2. *If the radical of $\mathfrak{g}$ is at most two-step nilpotent, we have*

$$\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1.$$

Proof*:* Let $\mathfrak{s} \ltimes \mathfrak{n}$ be a Levi-decomposition. We wish to show that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g})+1$. We may assume that $\mathfrak{n}$ is of class two and that $\mathfrak{s} \leq \mathrm{Der}(\mathfrak{n})$. Then $\mathfrak{s}$ acts on $\mathfrak{n}$ and the subspace $\mathfrak{n}_2 = [\mathfrak{n},\mathfrak{n}]$ is invariant. So there exists an $\mathfrak{s}$-invariant complement $\mathfrak{n}_1$ to

$[\mathfrak{n}, \mathfrak{n}]$. The decomposition $\mathfrak{n}_1 + \mathfrak{n}_2$ of $\mathfrak{n}$ defines a derivation δ of $\mathfrak{n}$: $\delta|_{\mathfrak{n}_1} = \mathbb{1}_{\mathfrak{n}_1}$ and $\delta|_{\mathfrak{n}_2} = 2\mathbb{1}_{\mathfrak{n}_2}$. Then it also commutes with $\mathfrak{s}$ in $\mathrm{Der}(\mathfrak{n})$. The proposition allows us to lift δ to a derivation of $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{n}$. It is an isomorphism when restricted to the centre of $\mathfrak{g}$ so that $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$. Alternatively, we can construct an invertible derivation of the radical using the upper central series of the radical and lift this derivation.

□

More generally, we can consider the class of Lie algebras that are $\mathbb{Z}$-graded. Such a gradation $\oplus_\alpha \mathfrak{g}_\alpha$ defines a derivation δ by $\delta|_{\mathfrak{g}_\alpha} = \alpha \mathbb{1}_{\mathfrak{g}_\alpha}$. The derivation is non-singular if the gradation is strictly positive.

Cohomology Instead of taking a one-dimensional subalgebra of $\mathrm{Der}(\mathfrak{g})$, let us consider any subalgebra $\mathcal{D}$. Then $\mathfrak{g}$ and any characteristic ideal are $\mathcal{D}$-modules and we can consider the associated cohomology.

Proposition 4.1.1.2. *Consider the inclusion of subalgebras* $\mathrm{Inn}(\mathfrak{g}) \leq \mathcal{D} \leq \mathrm{Der}(\mathfrak{g})$. *If* $H^0(\mathcal{D}, \mathfrak{g}) = 0$, *we have* $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + \dim(\mathcal{D})$.

PROOF*:* Consider the extension $\widehat{\mathfrak{g}} = \mathcal{D} \ltimes \mathfrak{g}$ of $\mathfrak{g}$. Let us first show that $\mathfrak{z}(\mathcal{D} \ltimes \mathfrak{g}) = \mathfrak{z}(\mathfrak{g})^{\mathcal{D}}$. Let $z = (\delta, x)$ be a central element of the extension. Then for every (θ, y) in $\widehat{\mathfrak{g}}$, we have

$$\begin{aligned} (0,0) &= [(\delta, x), (\theta, y)] \\ &= ([\delta, \theta], \delta(y) - \theta(x) + [x, y]). \end{aligned}$$

For $y = 0$, we obtain $\theta(x) = 0$ so that x is an invariant. Since $\mathcal{D}$ contains all inner derivations, also $[x, y] = 0$. We then have $\delta(y) = 0$ for all y in $\mathfrak{g}$. This means that δ is the zero map and $(\delta, x) = (0, x) \in \mathfrak{z}(\mathfrak{g})^{\mathcal{D}}$. Conversely, $\mathfrak{z}(\mathfrak{g})^{\mathcal{D}} \leq \mathfrak{z}(\mathcal{D} \ltimes \mathfrak{g})$. Since $\mathrm{Inn}(\mathfrak{g}) \leq \mathcal{D}$, all invariants are central and $H^0(\mathcal{D}, \mathfrak{g}) = H^0(\mathcal{D}, \mathfrak{z}(\mathfrak{g}))$. The adjoint representation of $\widehat{\mathfrak{g}}$ is faithful of degree $\dim(\mathfrak{g}) + \dim(\mathcal{D}) \leq \dim(\mathfrak{g}) + \dim(\mathfrak{g}^2)$. By noting that $\mathfrak{g} \leq \widehat{\mathfrak{g}}$ we obtain the desired inequality.

□

Corollary 4.1.1.3. *A Lie algebra* $\mathfrak{g}$ *satisfying* $H^0(\mathrm{Der}(\mathfrak{g}), \mathfrak{g}) = 0$ *also satisfies* $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + \dim(\mathfrak{g})^2$.

4.2 Classes of interest

Characteristically nilpotent Lie algebras So which Lie algebras have a nice derivation? According to Jacobson, every Lie algebra with an invertible derivation is necessarily nilpotent. The two previous examples of abelian and two-step nilpotent Lie algebras would suggest that the converse also holds. Namely, that every nilpotent Lie algebra has a non-singular derivation. That this is far from true was shown by Dixmier and Lister.

Definition 4.2.0.1 (Dixmier, Lister)**.** A Lie algebra is *characteristically nilpotent*, a CNLA, if all its derivations are nilpotent.

Such a Lie algebra cannot have any invertible derivations. It is nilpotent since all inner derivations are nilpotent. Since every nilpotent Lie algebra of class at most two admits a non-singular derivation, a CNLA must be of class at least three. In their paper [DiLi], Dixmier and Lister produced a first example of a characteristically nilpotent Lie algebra and it has minimal class: it is nilpotent of dimension 7 and class 3. Later, it was shown that there are in fact many CNLA's. Even more surprisingly, there are Lie algebras $\mathfrak{n}$ such that both $\mathfrak{n}$ and $\mathrm{Der}(\mathfrak{n})$ are CNLA.

For which Lie algebras $\mathfrak{g}$ does the zeroth cohomology $H^0(\mathrm{Der}(\mathfrak{g}), \mathfrak{g})$ vanish? Engel's theorem tells us that every characteristically nilpotent Lie algebra has a non-trivial invariant with respect to the derivation algebra. So corollary 4.1.1.3 cannot be used to construct faithful representations of a low degree. It does not imply however that the μ-invariant is large [Schn]:

> **Theorem** [Scheuneman] Every three-step nilpotent Lie algebra admits an affine structure.

Free nilpotent Lie algebras The free nilpotent Lie algebras $f_{g,c}$ of class c with g generators are not characteristically nilpotent. This is easy to see: every free nilpotent Lie algebra is graded by the positive integers and hence admits an invertible derivation. We have in particular that $\mu(f_{g,c}) \leq \dim(f_{g,c}) + 1$. If the nilpotency class is low, the μ-invariant is smaller. Can we explicitly compute $\mu(f_{g,c})$?

4.3 Open problems

4.3.1 The growth of the μ-invariant

The exact behaviour of the μ-invariant as a function of the dimension is not known. There are some obvious upper and lower bounds for μ. Consider for example the function $f(d) = \lceil \sqrt{d} \rceil$: then $f \circ \dim$ is clearly a lower bound (see also 3.2.3) for μ. For the general linear algebras $\mathfrak{gl}_m(\mathbb{C})$ and the affine Lie algebras $\mathfrak{aff}_m(\mathbb{C})$, it it is even an equality. So we cannot expect to find a lower bound that is much sharper than this f. The upper bound is more problematic. The proof for Ado's theorem in subsection 1.1.2 tells us that $\mu(\mathfrak{g})$ is $\mathrm{O}(2^{\dim(\mathfrak{g})})$.

Polynomial bounds Let us focus on the nilpotent Lie algebras, since these have been studied the most. We already mentioned that Milnor conjectured $\mu(\mathfrak{n})$ to be bounded by $\dim(\mathfrak{n}) + 1$ for all nilpotent Lie algebras $\mathfrak{n}$. We also mentioned that this guess was in fact wrong. The *worst* examples until now are nilpotent and satisfy $\dim(\mathfrak{n}) + 2 \leq \mu(\mathfrak{n})$. So there is no direct evidence suggesting that the μ-invariant increases much faster than the dimension. On the contrary, for many classes of Lie algebras, the μ-invariant will be much smaller. A suggestive selection of results:

Proposition 4.3.1.1. *Let $\mathfrak{g}$ be a complex Lie algebra of dimension d and with μ-invariant μ.*

- *If $\mathfrak{g}$ has an abelian radical, we have $\mu \leq d$.*
- *If $\mathfrak{g}$ has a radical that is two-step nilpotent, we have $\mu \leq d + 1$.*
- *If $\mathfrak{g}$ is at most three-step nilpotent, we have $\mu \leq d + 1$.*
- *If $\mathfrak{g}$ is graded by the positive integers, we have $\mu \leq d + 1$.*
- *If $H^0(\mathrm{Der}(\mathfrak{g}), \mathfrak{g}) = 0$, we have $\mu \leq d + d^2$.*
- *If $\mathfrak{g}$ is nilpotent of class c, we have $\mu \leq 1 + d^c$.*

And of course we have $\mu \leq d$ if $\mathfrak{g}$ has a trivial centre. These observations naturally suggest the following question. *Does the μ-invariant grow polynomially (linearly) as a function of the dimension?* And can the cohomology of a Lie algebra be used to compute the μ-invariant, or the other way around?

Lie algebras with a large μ-invariant Just as it is unclear how prove that the μ-invariant is not too large, it is also not at all clear how to show that a given Lie algebra has a large μ-invariant. The current world records are not very intimidating. So far, the best candidates come from a very specific class, that of the filiform nilpotent Lie algebras. The examples are all characteristically nilpotent. We present a family of Lie algebras here, for which we guess the μ is much larger than d. Let K be a field of characteristic zero. In this section we define a filiform Lie algebra $\mathfrak{f}_n$ in each dimension $n \geq 13$ having interesting properties concerning Lie algebra cohomology, affine structures and faithful representations. The ideas behind the construction of $\mathfrak{f}_n$ are explained in [Bu6] (see also [BEG]), where a family of Lie algebras is defined, of which $\mathfrak{f}_n$ is a specialisation. Define an index set $\mathcal{I}$ by

$$\mathcal{I}^0 = \{(k,s) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq k \leq [n/2],\, 2k+1 \leq s \leq n\},$$
$$\mathcal{I} = \begin{cases} \mathcal{I}^0 & \text{if } n \text{ is odd,} \\ \mathcal{I}^0 \cup \{(\frac{n}{2}, n)\} & \text{if } n \text{ is even.} \end{cases}$$

Now fix $n \geq 13$. We define a filiform Lie algebra $\mathfrak{f}_n$ of dimension n over K as follows. For $(k,s) \in \mathcal{I}$ let $\alpha_{k,s}$ be a set of parameters, subject to the following conditions: all $\alpha_{k,s}$ are zero, except for the following ones:

$$\alpha_{\ell,2\ell+1} = \frac{3}{\binom{\ell}{2}\binom{2\ell-1}{\ell-1}}, \quad \ell = 2,3,\dots,\lfloor \tfrac{n-1}{2} \rfloor,$$

$$\alpha_{3,n-4} = 1,$$

$$\alpha_{4,n-2} = \frac{1}{7} + \frac{10}{21}\frac{(n-7)(n-8)}{(n-4)(n-5)},$$

$$\alpha_{4,n} = \begin{cases} \frac{22105}{15246}, & \text{if } n = 13, \\ 0 & \text{if } n \geq 14, \end{cases}$$

and

$$\alpha_{5,n} = \frac{1}{42} - \frac{70(n-8)}{11(n-2)(n-3)(n-4)(n-5)} + \frac{25}{99}\frac{(n-6)(n-7)(n-8)}{(n-2)(n-3)(n-4)}$$
$$+ \frac{5}{66}\frac{(n-5)(n-6)}{(n-2)(n-3)} - \frac{65}{1386}\frac{(n-7)(n-8)}{(n-4)(n-5)}.$$

Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{f}_n$ and define the Lie brackets as follows:

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, n-1$$

$$[e_i, e_j] = \sum_{r=1}^{n} \Bigg(\sum_{\ell=0}^{\lfloor \frac{j-i-1}{2} \rfloor} (-1)^{\ell} \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell,\, r-j+i+2\ell+1} \Bigg) e_r,$$

$2 \leq i < j \leq n.$

One could also consider Lie algebras $\mathfrak{s} \ltimes \mathfrak{r}$ with a non-trivial Levi-complement. Is it possible to choose the action of $\mathfrak{s}$ on $\mathfrak{r}$ in such a way that $\mu(\mathfrak{s} \ltimes \mathfrak{r})$ is large? In this context it is necessary to analyse the structure of the derivation algebras $\mathrm{Der}(\mathfrak{g})$, in particular the maximal semisimple subalgebra $\frac{\mathrm{Der}(\mathfrak{g})}{\mathrm{rad}(\mathrm{Der}(\mathfrak{g}))}$. Characteristically nilpotent and abelian Lie algebras are then immediately excluded as candidates for the radical $\mathfrak{r}$.

On the other hand one could ask for the general properties of Lie algebras $\mathfrak{g}$ for which $\mu(\mathfrak{g})$ is large compared to $\dim(\mathfrak{g})$. It is also not clear whether the class of all Lie algebras with a large μ-invariant is itself large, say $\dim + m \leq \mu$ for some m in $\mathbb{N}$ (since almost all Lie algebras $\mathfrak{g}$ for which the invariant has been computed so far, satisfy $\mu(\mathfrak{g}) \leq \dim(\mathfrak{g}) + 1$). Proposition 4.3.1.1 suggests that almost all Lie algebras with an abelian radical have a "small" μ-invariant, which could be made more precise after a combinatorical analysis.

4.3.2 Role played by the field

In this thesis we have mainly focussed our attention on Lie algebras over the field of the complex numbers. The fact that the characteristic of the field was zero and that the field was algebraically closed, was crucial. We can naturally also ask what happens if one of these conditions fails. Counter-intuitive phenomena may occur.

Fields of prime characteristic A Lie algebra over a field of prime characteristic need not have a Levi-decomposition. Simplicity and the existence of derivations can be significantly different from the characteristic-zero case.

Real Lie algebras Even for the semisimple Lie algebras the behaviour of the μ-invariant will be different. Weyl's theorem does not necessarily hold.

Recall that the invariant was additive in the complex case and that it need not be so if the Lie algebras are real (see theorem 2.2.2.1 and remark 3). Lie's theorem fails for solvable Lie algebras in general. The counter-example in 1.3.2 used the fact that the field of the real numbers is not algebraically closed. Does this imply that the μ-invariant will be significantly bigger in the real case?

4.3.3 Groups

There is a close connection between groups and Lie algebras. It seems natural to want to exploit this connection. Can we use the results from group theory, and the corresponding μ-invariants to obtain better bounds for Lie algebras? We also include a proof of proposition 1.4.2

> **Proposition** Consider a finitely generated abelian group A with torsion subgroup T. Let t be the number of invariant factors of T. Then $\mu(A, \mathbb{C}) = \mu(T, \mathbb{C}) = t$.

In the proof we will identify $GL_1(\mathbb{C})$ with $\mathbb{C}^*$ and we denote by $\Gamma_q \leq \mathbb{C}^*$ the group of q'th-roots of unity. It is isomorphic to $\mathbb{Z}_q$. The proof does not use the *essential dimension*.

> PROOF: First suppose the group is free of rank r. Take r multiplicatively independent unit elements $x_1, \ldots, x_r$. Then the one-dimensional representation $\chi : \mathbb{Z}^r \longrightarrow \mathbb{C}^* : (z_1, \ldots, z_r) \longmapsto x_1^{m_1} \cdots x_r^{m_r}$ is faithful and we conclude that $\mathrm{rdim}(\mathbb{Z}^r) = 1$.
>
> Now suppose that the group A is finite abelian and non-trivial. Every representation $\rho : A \longrightarrow GL_n(\mathbb{C})$ satisfies $\rho(A)^{\exp(A)} = \mathbb{1}_n$. The Jordan canonical form implies that for every element $a \in A$, $\rho(a)$ can be diagonalised. Since all of these operators commute, they can be diagonalised simultaneously. We conclude that every representation of degree n actually maps into $(\Gamma_{\exp(A)})^n \leq GL_n(\mathbb{C})$.
>
> Since A is finite abelian and non-trivial, it can be written as the direct product $\mathbb{Z}_{a_1} \times \ldots \times \mathbb{Z}_{a_t}$ for some unique $a_1, \ldots, a_t \in \mathbb{N} \setminus \{0, 1\}$ such that each a_i divides its successor. We will prove

that $\mathrm{rdim}(A) = t$. For every a_i, take an a_i-th primitive root of unity ζ_{a_i}. The representation

$$\chi : \mathbb{Z}_{a_1} \times \ldots \times \mathbb{Z}_{a_t} \longrightarrow GL_1 \times \ldots \times GL_1 : (z_1, \ldots, z_t) \longmapsto (\zeta_{a_1}^{z_1}, \ldots, \zeta_{a_t}^{z_t})$$

is faithful of degree t so that $\mathrm{rdim}(A) \leq t$. For the converse inequality we proceed as follows. Take a prime p dividing a_1. This prime divides all of the a_i. Cauchy's theorem implies that $A_p = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p = \mathbb{Z}_p^t$ is a subgroup of A. In particular, $\mathrm{rdim}(A_p) \leq \mathrm{rdim}(A)$. Any embedding of A into $GL_n(\mathbb{C})$ induces an embedding of A_p into $GL_n(\mathbb{C})$ by restriction. As was remarked above, it is even an embedding of $A_p \cong \mathbb{Z}_p^t$ into $(\Gamma_p)^n \cong \mathbb{Z}_p^n$. This implies that $t \leq n$ and we conclude that $\mathrm{rdim}(A_p) = t = \mathrm{rdim}(A)$.

Finally, suppose that A is a finitely generated abelian group. Then A is the direct product of its torsion free part A_F of rank r and its torsion subgroup A_T with t invariant factors. We may suppose that A_T and A_F are non-trivial. Take a faithful representation $\chi : A_T \longrightarrow \Gamma^t_{\exp(A_T)} \leq GL_t(\mathbb{C})$ of A_T and a faithful representation $\psi : A_F \longrightarrow GL_1(\mathbb{C})$ as above. Then the product,

$$\chi \otimes \psi : A_T \times A_F \longrightarrow GL_t(C) \otimes GL_1(\mathbb{C}) : (u, v) \longmapsto \chi(u) \cdot \psi(v)$$

is representation of degree $t \times 1 = t$. Furthermore, it is faithful. Suppose namely that $\chi(u) \cdot \psi(v) = (\chi \otimes \psi)(u; v) = \mathbb{1}$. Then every power of this transformation is the identity and in particular, $\mathbb{1} = (\chi(u) \cdot \psi(v))^{\exp(A_T)} = (\chi(u))^{\exp(A_T)} \cdot (\psi(v))^{\exp(A_T)} = (\psi(v))^{\exp(A_T)}$. This implies that v is contained in the torsion subgroup of the torsion-free group A_F, so that $v = \mathbb{1}_{A_F}$. Then $\chi(u) = \chi(u) \cdot \psi(v) = (\chi \otimes \psi)(u; v) = \mathbb{1}$. Since χ is faithful, $u = \mathbb{1}_{A_T}$ and in particular $(u; v) = \mathbb{1}_A$. We conclude that the the inequalities in $t = \mathrm{rdim}(A_T) \leq \mathrm{rdim}(A) \leq \mathrm{rdim}(A_T) = t$ are actually equalities.

□

4.4 The algorithm for $\mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{a}$

Algorithm In subsection 3.2.1 we described how one can compute the μ_0-invariant for Lie algebras with an abelian radical and a Levi-complement isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. We now implement this method in the following *Mathematica*-algorithm. This Lie algebra is completely determined by its defining representation δ, which in turn is described by the multiplicities of any decomposition into irreducible subrepresentations. Let $(m_1, \ldots, m_l)$ be these multiplicities. The algorithm will test if $\mathfrak{g}$ has a faithful representation of dimension *TestValue*. If not, it will test this for *TestValue* $+1$, *TestValue* $+2$ and so on until it finds a faithful representation.

INPUT

- $m = (m_1, \ldots, m_l)$.
- *TestValue*.
- *TestValueStop*.

It then gives as output the faithful representation as $\{x\}\{y\}\{M\}$, where

OUTPUT

- $x = (x_1, \ldots, x_k)$,
- $y = (y_1, \ldots, y_l)$,
- $M = (M_1, \ldots, M_t)$.

Here, x represents the multiplicities of the first representation of $\mathfrak{s}$, y represents the multiplicities of the second representation of $\mathfrak{s}$ - see subsection 3.2.1. The list M gives the multiplicities of $M = \langle x, y \rangle \geq m$.

ALGORITHM:

```
MultiplicityList = {0,0,1,8};
TestValue = 2; TestValueStop = 20;

Print["Start."];
```

```
booleanFound = False;

While[TestValue <= TestValueStop && booleanFound == False,

ContentList = Range[Max[Length[MultiplicityList],
TestValue-1]];

t1 = 1; While[t1 <= Floor[TestValue /2],

SetPartitionsX = IntegerPartitions[TestValue - t1];
SetPartitionsY = IntegerPartitions[t1];
SetPartitionsXY = Tuples[{SetPartitionsX,SetPartitionsY}];

t2 = 1; While[t2 <= Length[SetPartitionsXY],

MultiPartX = Range[Max[SetPartitionsXY[[t2 , 1]]]]; t3 = 1;
While[t3 <= Length[MultiPartX], MultiPartX[[t3]] = {  t3  ,
Count[    SetPartitionsXY[[t2 , 1]]   ,t3]  };  t3 = t3+1;];

MultiPartY = Range[Max[SetPartitionsXY[[t2 , 2]]]]; t3 = 1;

While[t3 <= Length[MultiPartY], MultiPartY[[t3]] ={  t3  ,
Count[    SetPartitionsXY[[t2 , 2]]    ,t3]  };  t3 = t3+1;];

MultiPartXY = Tuples[{MultiPartX,MultiPartY}];

t = 1; While[t <= Length[ContentList],
ContentList[[t]] = 0;t = t + 1;];

t4 = 1; While[t4 <= Length[MultiPartXY],

If[MultiPartXY[[t4,1,2]] * MultiPartXY[[t4,2,2]] != 0,

t5 = Abs[MultiPartXY[[t4,1,1]] -
MultiPartXY[[t4,2,1]]] + 1;
```

```
While[t5 <= MultiPartXY[[t4,1,1]] +
MultiPartXY[[t4,2,1]] - 1,

ContentList[[t5]] = ContentList[[t5]]
+ MultiPartXY[[t4,1,2]] * MultiPartXY[[t4,2,2]];

t5 = t5 + 2;];

];

t4 = t4 + 1;];

booleanMultiplicity  = True; t = 1;
While[booleanMultiplicity == True
&& t <= Length[MultiplicityList],

If[MultiplicityList[[t]] > ContentList[[t]],
booleanMultiplicity = False;];
t = t + 1;];

If[booleanMultiplicity , Print[SetPartitionsXY[[t2 , 1]],

SetPartitionsXY[[t2 , 2]],":",ContentList];
Print["-----"]; booleanFound = True;];

t2 = t2 + 1;];

t1 = t1 + 1;];

TestValue = TestValue + 1;];

Print["Stop."];
```

The algorithm was used to compute the μ_0-invariant in Table 3.1 and it was also used to compute Table 3.2.

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Index

Curriculum Vitae

Name:	Wolfgang Alexander Siegfried Hugo R. Moens
E-mail:	Wolfgang.Moens@univie.ac.at
Nationality	Belgian
Date of birth:	19th of September 1984
Place of birth:	Leuven, Belgium

Primary school:	Haasrode, 1992-1994
	Laak, 1994-1996
Secondary school:	Houthalen, 1996-2002
Bachelor (Kandidaat):	Leuven, 2002-2004
in Mathematics:	magna cum laude
Master (Licentiaat):	Leuven, 2004-2006
in Mathematics:	cum laude
Master's thesis:	*"Construction of aspherical spaces and cohomology of T-groups."*
Supervisor:	Prof. Paul Igodt
Research Assistent and	Vienna, 2006-2009
PhD in Mathematics:	*"Faithful represetations of minimal degree for Lie algebras with an abelian radical."*
Supervisor:	Prof. Dietrich Burde

Teaching experience

- *"Exercises on group theory"*: University of Vienna, SS 2009. University of Vienna.
- *"Finitely generated abelian groups"*: University of Vienna, 2009.

List of publications

- *"Faithful representations of minimal degree for Lie algebras with an abelian radical."* Thesis at the University of Vienna. (2009)
- *"Minimal faithful representations of reductive Lie algebras."* Arch. Math. 89, 513-523 (2007), with D. Burde.
- *"Construction of aspherical spaces and cohomology of T-groups."* Thesis at the Katholieke Universiteit Leuven. (2006).

List of Talks

- *"Faithful representations of minimal degree for reductive Lie algebras."* - Discrete Groups and Geometric structures with applications. (Katholieke Universiteit Leuven Campus Kortrijk, 29.05.2008)
- *"Algebra featuring topology: A few facts on free groups"* - Colloquium for Master and PhD Students. (University of Vienna, 15.05.2008)
- *"Minimal faithful representations of reductive Lie algebras"* - Seminar for Differential geometry and Lie groups. (University of Vienna, 10.05.2007)
- *"T-groups: Cohomology and polynomial crystallographic actions."* - Seminar Sophus Lie XXXII. (Erwin Schroedinger Institute, 03.11.2006)
- *"T-groups: Cohomology and polynomial crystallographic actions."* - Seminar for Differential geometry and Lie groups. (University of Vienna, 19.10.2006)
- *"Construction of aspherical spaces and the cohomology of T-groups."* (Katholieke Universiteit Leuven, 2006)

In dieser Dissertation untersuchen wir die sogenannte μ-Invariante von Lie Algebren. Für eine endlich-dimensionale Lie Algebra $\mathfrak{g}$ ist sie die minimale Dimension eines treuen $\mathfrak{g}$-Moduls. Es ist bereits nicht-trivial zu zeigen, daß diese Invariante Werte in den natürlichen Zahlen annimmt, d.h., daß jede endlich-dimensionale Lie Algebra eine endlich-dimensionale treue Darstellung besitzt. Das wurde ursprünglich von Ado und Iwasawa bewiesen, und ist ein fundamentales Resultat. Es hat eine lange Geschichte. In dieser Arbeit geht es um eine Verfeinerung des Ado-Iwasawa-Theorems, und zwar in folgender Hinsicht:

Sei $\mathfrak{g}$ eine endlich-dimensionale Lie algebra. Berechne $\mu(\mathfrak{g})$ und finde einen treuen Modul dieser Dimension. Beschreibe die Eigenschaften treuer Moduln minimaler Dimension. Berechne obere und untere Schranken für $\mu(\mathfrak{g})$ als Funktion anderer Invarianten.

Im allgemeinen kann man keine explizite Formel für $\mu(\mathfrak{g})$ erwarten, insbesondere nicht für nilpotente Lie Algebren. Die Frage ist daher, ob man für reduktive bzw. halbeinfache Lie Algebren $\mu(\mathfrak{g})$ bestimmen kann. Tatsächlich gelingt dies für den Fall daß $\mathfrak{g}$ abelsch, einfach, halbeinfach oder reduktiv ist. Der Beweis dazu ist im wesentlichen kombinatorischer Natur und verwendet klassische Resultate der Darstellungstheorie für reduktive Lie-Algebren. Allgemeiner untersuchen wir die μ-Invariante auch für Lie Algebren deren auflösbares Radikal abelsch ist. Wir betrachten weitere Invarianten, die mit der μ-Invariante zusammenhängen. Abschliessend werden dazu einige spezielle Familien von solchen Lie Algebren im Detail betrachtet.