universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

Isomorphisms of Algebras of Smooth and Generalized Functions

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag.rer.nat.)

Verfasserin: Annegret Burtscher
Matrikelnummer: 0308854
Studienrichtung: A 405 Mathematik
Betreuer: Ao. Univ.-Prof. Michael Kunzinger

Wien, am 22. Mai 2009

Abstract

A well-known result in commutative Banach algebra theory states that two locally compact spaces X and Y are homeomorphic if and only if the C^*-algebras of continuous functions $\mathcal{C}_0(X)$ and $\mathcal{C}_0(Y)$ are algebraically isomorphic. Our aim is to construct a similar theory for algebras of smooth functions and Colombeau generalized functions. The underlying topological spaces are finite-dimensional smooth manifolds X and Y which are Hausdorff and second countable. We find that the non-zero multiplicative linear functions on $\mathcal{C}^\infty(X)$ and $\mathcal{G}(X)$ can be identified with the points in X and the compactly supported generalized points $\widetilde{X}_c$, respectively. Moreover, we prove that algebra isomorphisms $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ are characterized by diffeomorphisms from Y to X, a fact that holds even for manifolds that are not second countable. The same question for Colombeau algebras leads to c-bounded generalized functions $\mathcal{G}[Y, X]$ which again completely determine the algebra isomorphisms $\mathcal{G}(X) \to \mathcal{G}(Y)$.

Zusammenfassung

Ein bekanntes Resultat in der Theorie kommutativer Banachalgebren besagt, dass zwei lokal kompakte Räume X und Y genau dann homöomorph sind, wenn die C^*-Algebren der stetigen Abbildungen $\mathcal{C}_0(X)$ und $\mathcal{C}_0(Y)$ algebraisch isomorph sind. Es ist unser Ziel, analoge Aussagen auch für Algebren glatter Abbildungen bzw. Colombeaualgebren zu zeigen. Die zugrundeliegenden topologischen Räume werden in diesem Fall endlich-dimensionale glatte Mannigfaltigkeiten X und Y sein, die Hausdorff sind und das zweite Abzählbarkeitsaxiom erfüllen. Wir werden sehen, dass nichttriviale multiplikative lineare Funktionale auf $\mathcal{C}^\infty(X)$ bzw. $\mathcal{G}(X)$ mit Punkten in X bzw. kompakt getragenen verallgemeinerten Punkten $\widetilde{X}_c$ identifiziert werden können. Zudem werden wir beweisen, dass Algebraisomorphismen $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ bereits durch Diffeomorphismen von Y nach X charakterisiert sind. Letzteres gilt sogar für Mannigfaltigkeiten, die das zweite Abzählbarkeitsaxiom nicht erfüllen. Im Zusammenhang mit Colombeau verallgemeinerten Funktionen führt uns diese Fragestellung zu kompakt beschränkten verallgemeinerten Funktionen $\mathcal{G}[Y, X]$, welche die Algebraisomorphismen $\mathcal{G}(X) \to \mathcal{G}(Y)$ wiederum komplett beschreiben.

Acknowledgement

I wish to thank my supervisor Michael Kunzinger for his guidance, enthusiasm and care during the development of my thesis. In particular, I am grateful that he introduced me to the beauty of semi-Riemannian geometry and Colombeau theory.

Also, I would like to thank all members of the DIANA research group for excellent courses and many useful discussions over the years. It was a pleasure to share an office with James Grant and Shantanu Dave. I highly appreciate their company and support.

Above all, I owe my parents, Reingard and Herbert Burtscher, a debt of gratitude for supporting me in everything I wanted to explore. Thanks a lot also to Martin for his backing and patience during the years.

Contents

Chapter 0

Introduction

0.1 Historical background

The historical background for this thesis can be seen in classical Banach algebra theory, a field that was initiated by I. M. Gelfand in 1939. A *Banach algebra* $\mathcal{A}$ is a Banach space (over $\mathbb{C}$) and an algebra such that $\|ab\| \leq \|a\|\|b\|$ holds for all $a, b \in \mathcal{A}$. The *carrier space* $\hat{\mathcal{A}}$ of $\mathcal{A}$ is the space of non-zero multiplicative linear functionals $\mathcal{A} \to \mathbb{C}$. For a commutative Banach algebra $\mathcal{A}$ we have that $\hat{\mathcal{A}}$ is Hausdorff and locally compact (resp. compact if $\mathcal{A}$ is unital) with respect to the weak-$*$ topology.

For unital and commutative Banach algebras, there is a 1-1 correspondence between elements in $\hat{\mathcal{A}}$ and maximal ideals in $\mathcal{A}$, given by

$$\begin{array}{rcl} \hat{\mathcal{A}} & \to & \text{maximal ideals of } \mathcal{A} \\ \varphi & \mapsto & \ker \varphi. \end{array}$$

This is basically due to the fact that for maximal ideals $\mathcal{M}$ in $\mathcal{A}$, the quotient space $\mathcal{A}/\mathcal{M}$ is a commutative and unital Banach algebra over $\mathbb{C}$ such that every element is invertible, and hence by the Gelfand-Mazur theorem isomorphic to $\mathbb{C}$.

Both sets, $\hat{\mathcal{A}}$ and the set of maximal ideals on $\mathcal{A}$, are called the *spectrum* of the algebra $\mathcal{A}$, denoted by $\Omega_{\mathcal{A}}$. For non-unital commutative Banach algebras the elements in $\hat{\mathcal{A}}$ correspond to so-called *maximal modular ideals*.

Moreover, we can consider the *Gelfand transformation* $\Gamma_{\mathcal{A}}$, which transforms elements in a commutative Banach algebra $\mathcal{A}$ into continuous functions. It is defined by

$$\begin{array}{rrcl} \Gamma_{\mathcal{A}} : & \mathcal{A} & \to & \text{bounded functions on } \Omega_{\mathcal{A}} \\ & a & \mapsto & \hat{a}, \end{array}$$

where the elements a in $\mathcal{A}$ are identified with bounded functions $\hat{a}$, where $\hat{a}(\varphi) := \varphi(a)$ for $\varphi \in \hat{\mathcal{A}}$. For commutative Banach algebras, $\Gamma_{\mathcal{A}}$ is mapped to $\mathcal{C}_0(\Omega_{\mathcal{A}})$. For algebras which in addition are unital, the range is $\mathcal{C}(\Omega_{\mathcal{A}})$.

If we consider different classes of commutative Banach algebras, then the Gelfand transformation will take a different form. For example, the Gelfand transformation of $L^1(\mathbb{R}^n)$ can be viewed as the Fourier transform.

For the algebra $\mathcal{C}(X)$, with X a compact and Hausdorff topological space, the Gelfand transformation is simply the identity since $\widehat{\mathcal{C}(X)}$ is homeomorphic to X. Similarly, this also holds for the algebra $\mathcal{C}_0(X)$ for X locally compact and Hausdorff. This is due to the identification

$$\text{point } p \text{ in } X \quad \longleftrightarrow \quad \text{maximal ideal in } \mathcal{C}_0(X),$$

where the ideal corresponds to functions vanishing at p. Moreover, the algebras $\mathcal{C}_0(X)$ and $\mathcal{C}_0(Y)$ are algebraically isomorphic (equivalently even isometrically isomorphic) if and only if X and Y are homeomorphic. The same result of course holds for compact and Hausdorff spaces X and Y and continuous functions $\mathcal{C}(X)$ and $\mathcal{C}(Y)$.

In general, we may consider a *commutative C^*-algebra* $\mathcal{A}$, i.e. a Banach algebra with an involution $*$ that satisfies $\|a^*a\| = \|a\|^2 \ \forall a \in \mathcal{A}$. The Gelfand transformation $\Gamma_{\mathcal{A}}$ on a C^*-algebra is not only an algebra homomorphism but an isometric $*$-isomorphism, i.e. $\mathcal{A} \cong \mathcal{C}_0(\Omega_{\mathcal{A}})$. This is the famous Gelfand-Naimark theorem. An immediate consequence is the equivalence of the following statements:

(i) The commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are algebraically isomorphic.

(ii) $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic.

(iii) $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are homeomorphic.

While the equivalence of (i) and (iii) can be shown directly also for commutative Banach algebras, the equivalence with (ii) is due to the C^*-identity.

Hence there is a strong connection between algebraic properties of $\mathcal{A}$ and topological and geometrical properties of $\hat{\mathcal{A}}$. From a categorial point of view this means that the category of commutative C^*-algebras (with homomorphisms) and the category of Hausdorff and locally compact spaces (with proper maps, i.e. continuous maps such that the preimages of compact sets are compact) are equivalent.

For further reading on classical Banach algebra theory see for example [Lar73], [Hel93] and [Dav96]. The idea of representing spaces with algebras has had a remarkable impact on topology as well as analysis. Banach $\widetilde{\mathbb{C}}$-algebras and Colombeau C^*-algebras have recently been investigated in [Ver08].

0.2 Recent surveys

The same paradigm as mentioned above extends to algebras of smooth functions $\mathcal{C}^\infty(X)$ and Colombeau algebras $\mathcal{G}(X)$ on a manifold X. Colombeau algebras are algebras of generalized functions which have been developed to bypass the non-multiplicativity of distributions in order to be able to study non-linear partial differential equations. The basic theory on special Colombeau algebras on manifolds necessary for our investigations is provided in chapter 5.

It is the aim of this thesis to present the following new results and discuss different approaches. Firstly, we will see that for finite-dimensional and smooth manifolds X and Y (Hausdorff and second countable)

(i) a non-zero multiplicative $\mathbb{C}$-linear functional $\mathcal{C}^\infty(X) \to \mathbb{C}$ is the evaluation ev_p for a unique point $p \in X$.

(ii) a non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functional $\mathcal{G}(X) \to \widetilde{\mathbb{C}}$ is the evaluation $\mathrm{ev}_{\tilde{p}}$ for a unique c-bounded generalized point $\tilde{p} \in X$.

In particular, these investigations lead to the 'spectral' space $\widetilde{X}_c$ for the Colombeau algebra $\mathcal{G}(X)$ similar to the space $\hat{A}$ in the context of Banach algebras.

The main results again correspond to the ones in Banach algebra theory and show that the algebras of smooth resp. generalized functions recover the geometry of the respective spaces of points:

(i) Algebra isomorphisms $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ are characterized as pullbacks by diffeomorphisms from Y to X.

(ii) Algebra ismorphisms $\mathcal{G}(X) \to \mathcal{G}(Y)$ are characterized as pullbacks by invertible c-bounded generalized functions from Y to X.

The generalization of (i) to manifolds that are not second-countable is a recent result due to Janez Mrčun [Mrč05] and Janusz Grabowski [Gra05]. A preprint by Hans Vernaeve [Ver06] deals with (ii) in the case of non-smooth dependence on ε. We will treat (ii) in the case of generalized functions with smooth dependence on ε. This leads to somewhat nicer and simpler algebraic results (i.e. that $\mathcal{G}_{\mathrm{ld}}[X,Y] = \mathcal{G}[X,Y]$), but requires more work. For a thorough discussion of the disparities see section 6.5.

In addition, there are also general advantages in considering smooth dependence on ε. In [KSV09] it is proved that $\mathcal{G}[\,.\,,Y]$ is a sheaf of sets (theorem 2.3) and that the continuous functions $\mathcal{C}(X,Y)$ can be embedded in $\mathcal{G}[X,Y]$ (theorem 3.1) in this setting. Moreover, proposition 12.2 in [Obe92] shows that non-zero polynomials P in one variable with complex coefficients satisfy $P(C) = 0$ in $\widetilde{\mathbb{C}}$ (with continuous dependence on ε) if and only if C is identical to a classical root of P. Besides, smooth dependence occurs naturally if we obtain regularization of distributions via convolution. In the diffeomorphism invariant full Colombeau algebra, smooth dependence is always given, to mention a few advantages.

0.3 Outline

For smooth manifolds X and Y and an algebra isomorphism $\Psi : \mathcal{A}(X) \to \mathcal{A}(Y)$ the main steps will be the same in both the smooth and generalized functions setting:

- Identify points in the manifolds X resp. Y via algebraic properties in $\mathcal{A}(X)$ resp. $\mathcal{A}(Y)$. This will be non-zero multiplicative linear functionals resp. characteristic sequences.

- Pick a point q in the second manifold Y and look at its algebraic equivalent on $\mathcal{A}(Y)$.
- Use the algebra isomorphism to create a unique analogue on $\mathcal{A}(X)$.
- Identify this algebraic construction on $\mathcal{A}(X)$ again with a point p in X.
- Prove that the map $\psi : Y \to X$ obtained by doing this for every point in Y resp. X really is bijective and unique, characterizes Ψ and respects the algebraic and differential structure (i.e. that the map ψ is a diffeomorphism resp. a c-bounded generalized function).

In chapter 1 we recall algebraic definitions and in chapter 2 our basic geometrical setting, i.e. finite-dimensional and smooth manifolds which are Hausdorff and second countable.

In chapter 3 we treat the case of algebras of smooth functions by following two different approaches – the classical one using maximal ideals in $\mathcal{C}^\infty(X)$ and a new approach by Janez Mrčun using so-called characteristic sequences of functions. It turns out that algebra isomorphisms between algebras of smooth functions are characterized as compositions with unique diffeomorphisms.

In chapter 4 we provide results in Riemannian geometry with a special focus on submanifolds in $\mathbb{R}^n$. Moreover, we show that the (squared) Riemannian distance is smooth in a neighborhood of the diagonal.

In chapter 5 we introduce the theory of Colombeau algebras on manifolds and discuss basic properties such as point value characterizations and invertibility of c-bounded generalized functions.

Finally, in chapter 6 we prove that algebra isomorphisms on Colombeau algebras are simply pullbacks by invertible c-bounded generalized functions. We follow Hans Vernaeve's approach in [Ver06] in a different setting. The main problem here is that we work in a function algebra over the ring $\widetilde{\mathbb{C}}$ of generalized numbers, which is not a field.

At the beginning of each chapter, a short summary introduces its content. The bibliography, a list of notation and an index are provided at the end.

Chapter 1

Algebras

Our main objects of interest are the function algebras $\mathcal{C}^\infty(X)$ and $\mathcal{G}(X)$ on manifolds X and the algebra isomorphisms between them. We begin by recalling some basic definitions and terminology. For more details see, e.g., [Bou98].

1.1 Algebras over rings

The structure of an algebra is induced by a module and by a bilinear operation:

Definition 1.1.1. Let R be a commutative and unital ring and $\mathcal{A}$ a module over R, where the addition is denoted by $+$. Then $\mathcal{A}$ is called an *R-algebra* (or algebra over R) if it is equipped with an additional bilinear operation $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (the so-called *multiplication*), such that the following compatibility conditions hold for all $a, b, c \in \mathcal{A}$, $\lambda, \mu \in R$:

$$\begin{aligned}(a + b) \cdot c &= a \cdot c + b \cdot c \\ c \cdot (a + b) &= c \cdot a + c \cdot b \\ (\lambda a) \cdot (\mu b) &= (\lambda\mu)(a \cdot b)\end{aligned}$$

The algebra $\mathcal{A}$ is called *unital* if

$$\exists 1 \in \mathcal{A} : 1 \cdot a = a \cdot 1 = a,$$

associative if

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

and *commutative* if

$$a \cdot b = b \cdot a.$$

Remark 1.1.2 (Algebras over fields)**.** If $R = K$ is a field, then $\mathcal{A}$ in the above definition is a vector space over K.

Definition 1.1.3. Let $\mathcal{A}$ be an R-algebra. It is called a *differential R-algebra* if a so-called *derivation* $\partial : \mathcal{A} \to \mathcal{A}$ satisfies for all $a, b \in \mathcal{A}$, $\lambda \in R$:

$$\begin{aligned}\partial(\lambda a) &= \lambda\partial(a) \\ \partial(a + b) &= \partial(a) + \partial(b) \\ \partial(a \cdot b) &= \partial(a) \cdot b + a \cdot \partial(b)\end{aligned}$$

Example 1.1.4. Let X be a manifold. The space of smooth functions $\mathcal{C}^\infty(X) = \mathcal{C}^\infty(X, \mathbb{C})$ (or $\mathcal{C}^\infty(X, \mathbb{R})$) equipped with the usual pointwise operations of functions is a unital, associative and commutative algebra over $\mathbb{C}$ (resp. $\mathbb{R}$). It is even a differential algebra.

A Colombeau algebra $\mathcal{G}(X)$ over a manifold X (see definition 5.1.1) is a unital, associative, commutative and differential $\widetilde{\mathbb{C}}$-algebra. See [GKOS01], section 1.2, for more details.

1.2 Algebra homomorphisms and isomorphisms

We wish to consider maps that preserve the algebra structure.

Definition 1.2.1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two R-algebras and $\Psi : \mathcal{A}_1 \to \mathcal{A}_2$ a map that satisfies for all $a, b \in \mathcal{A}_1$, $\lambda \in R$:

$$\begin{aligned} \Psi(\lambda a) &= \lambda \Psi(a) \\ \Psi(a + b) &= \Psi(a) + \Psi(b) \\ \Psi(a \cdot b) &= \Psi(a) \cdot \Psi(b). \end{aligned}$$

Then Ψ is called an *algebra homomorphism*. If Ψ is bijective, then it is called an *algebra isomorphism*.

Note that even for unital algebras we do not necessarily assume that an algebra homomorphism satisfies $\Psi(1) = 1$.

1.3 Ideals

Since each algebra $\mathcal{A}$ is also a ring $(\mathcal{A}, +, \cdot)$, it is sufficient to consider ideals in rings R.

Definition 1.3.1. An additive subgroup $I \subseteq R$ of a ring R is called (two-sided) *ideal* if $RI \subseteq I$ and $IR \subseteq I$. We denote this by $I \lhd R$.

An ideal J of a ring R is called *maximal ideal* if $J \neq R$ and for all ideals I of R with $J \subseteq I$ either $J = I$ or $I = R$ holds.

Hence a maximal ideal is not contained in any other proper ideal.

Example 1.3.2. The kernel $\ker \Psi$ of an algebra homomorphism $\Psi : \mathcal{A}_1 \to \mathcal{A}_2$ is a maximal ideal in $\mathcal{A}_1$.

Chapter 2

Smooth Manifolds

The underlying topological and geometrical structures for our analysis are smooth manifolds of finite dimension. Locally these are diffeomorphic to the Euclidean space $\mathbb{R}^n$. In order to make the transition from local structures to global structures, we use so-called partitions of unity. For a Hausdorff space their existence is equivalent to paracompactness.

In chapter 4 we will furthermore consider smooth manifolds that are equipped with a Riemannian metric.

2.1 General definitions

2.1.1 Smooth manifolds

Manifolds are topological spaces that are locally homeomorphic to the Euclidean space. Charts describe this property.

Definition 2.1.1. Let X be a set. A *chart* (u, U) is a bijective map u from a domain U in X to an open set $u(U)$ in $\mathbb{R}^n$. Two charts (u, U) and (v, V) are called *compatible* if the sets $u(U \cap V)$ and $v(U \cap V)$ are open in $\mathbb{R}^n$ and $u \circ v^{-1}$ is a $\mathcal{C}^\infty$-diffeomorphism.

An *atlas* of X is a family $\mathcal{A} = \{(u_\alpha, U_\alpha) | \alpha \in A\}$ of pairwise compatible charts that cover X, i.e. $X = \bigcup_{\alpha \in A} U_\alpha$. Two atlases are called *equivalent* if their union is again an atlas.

Definition 2.1.2. A *differentiable manifold* X is a set X furnished with an equivalence class of atlases, the so-called *differentiable structure* of X.

It can be shown that each atlas is contained in a unique *maximal atlas* (maximal w.r.t. the inclusion). Therefore we can assume that X is equipped with a maximal atlas $\mathcal{A}$ and equip X with the natural topology induced by $\mathcal{A}$.

We will simply use the term *manifolds* for differentiable manifolds that do not necessarily fulfill further topological properties. But since we mainly use the following type of manifolds, we also define:

Definition 2.1.3. A manifold X is called a *smooth manifold* if it is finite dimensional, smooth, second countable and Hausdorff.

Recall that a topological space X is called *second countable* if X has a countable basis. It is *Hausdorff* (or T_2, the second axiom of separation) if for two distinct points $x, y \in X$ there exist open sets U, V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Unless stated otherwise, e.g. as in section 3.2 where we do not require second countable, we will always work with smooth manifolds in the sense of the above definition. Generally they are denoted by X and Y.

All essential definitions are provided along the way. For further reading on differential geometry see [BC70], [Spi79] or [Mic08].

2.1.2 Tangent and cotangent spaces

For a manifold X and a point $p \in X$, the *tangent space* of X at p is denoted by T_pX.

Let Y be another manifold and $f : X \to Y$ a smooth function. The *tangent map* of f at p is denoted by $T_pf : T_pX \to T_{f(p)}Y$.

Let the set $\{ \frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p \}$ denote the basis of the tangent space T_pX w.r.t. a given chart $(u = (x^1, ..., x^n), U)$ at p in X. If e_i is the i-th standard unit vector of $\mathbb{R}^n$, then

$$\frac{\partial}{\partial x^i}\bigg|_p := (T_pu)^{-1}(e_i) \in T_pX \qquad \forall 1 \le i \le n.$$

The *cotangent space* of a manifold X at $p \in X$ is the dual of the tangent space, i.e. $(T_pX)^*$.

Similarly, the basis of $(T_pX)^*$ w.r.t. a given chart $(u = (x^1, ..., x^n), U)$ is denoted by $\{ dx^1\big|_p, ..., dx^n\big|_p \}$, which is the dual basis of $\{ \frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p \}$ above.

2.1.3 Vector bundles and smooth sections

Vector bundles are used in many constructions. They are defined by:

Definition 2.1.4. Let E and B be two manifolds. The triple (E, B, π) is called a *vector bundle* if $\pi : E \to B$ is smooth surjection such that for all $b \in B$ the following holds:

(i) The *fiber* $\pi^{-1}(b)$ is a vector space.

(ii) There exists an open neighborhood V of b and a diffeomorphism $\Phi : \pi^{-1}(V) \to V \times F'$, which is fiberwise linear (i.e. $\Phi|_{\pi^{-1}(b)}$ is linear $\forall b \in V$), such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\Phi} & V \times F' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} \\ V & \xrightarrow{\mathrm{id}} & V \end{array}$$

Definition 2.1.5. Let (E, B, π) be a vector bundle. A *section* of E is a map $X : B \to E$ that satisfies $\pi \circ X = \mathrm{id}$.

The set of smooth sections of E is denoted by $\Gamma(B, E)$, or simply by $\Gamma(E)$.

2.1.4 Tensors and tensor fields

Some vector bundles occur naturally on manifolds:

Definition 2.1.6. For a manifold X and T_pX the tangent space of X at $p \in X$, the *tangent bundle* of X is defined by

$$TX := \bigsqcup_{p\in X} T_pX = \bigcup_{p\in X} \{p\} \times T_pX.$$

In accordance with definition 2.1.4, (TX, X, π_X) is a vector bundle for the canonical projection $\pi_X : TX \to X$, $(p, v) \mapsto p$.

If $f : X \to Y$ is a smooth map between manifolds, then the *tangent map* $Tf : TX \to TX$ is defined by $Tf(p, v) := (f(p), T_pf(v))$ for all $p \in X$, $v \in T_pX$.

Definition 2.1.7. Let X be a manifold and let $(T_pX)^*$ be the cotangent space at $p \in X$. The *cotangent bundle* is defined by

$$T^*X := \bigsqcup_{p\in X} (T_pX)^* = \bigcup_{p\in X} \{p\} \times (T_pX)^*.$$

Generally we can consider the space of $\binom{r}{s}$-tensors on any vector space:

Definition 2.1.8. The space of $\binom{r}{s}$-*tensors* on a vector space E consists of $(r + s)$-linear mappings of the form

$$T^r_s(E) := L^{r+s}(\underbrace{E^*, ..., E^*}_{r}, \underbrace{E, ..., E}_{s}; \mathbb{R}).$$

On manifolds X we will write T^r_sX instead of $T^r_s(TX)$ throughout. In particular, $TX = T^1_0X$ and $T^*X = T^0_1X$.

Definition 2.1.9. For $t_1 \in T^{r_1}_{s_1}(E)$ and $t_2 \in T^{r_2}_{s_2}(E)$, the *tensor product* $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ is defined by

$$\begin{aligned} &t_1 \otimes t_2(\beta^1, ..., \beta^{r_1}, \gamma^1, ..., \gamma^{r_2}, f_1, ..., f_{s_1}, g_1, ..., g_{s_2}) \\ &\qquad := t_1(\beta^1, ..., \beta^{r_1}, f_1, ..., f_{s_1}) \cdot t_2(\gamma^1, ..., \gamma^{r_2}, g_1, ..., g_{s_2}) \end{aligned}$$

for all $\beta^i, \gamma^i \in E^*$ and all $f_j, g_j \in E$.

Definition 2.1.10. Let X be a manifold. Smooth sections of T^r_sX are the $\binom{r}{s}$-*tensor fields*. The space of such sections is denoted by $\mathcal{T}^r_s(X) = \Gamma(X, T^r_sX)$.

In particular, the smooth vector fields $X \to TX$ are smooth sections of TX, i.e. $\mathfrak{X}(X) = \Gamma(TX) = \mathcal{T}^1_0(X)$, and the space of one-forms is $\Omega^1(X) = \Gamma(T^*X) = \mathcal{T}^0_1(X)$.

2.2 Partitions of unity

Partitions of unity are a convenient tool to globalize local properties. Generally, they can be defined for topological manifolds. However, since we only work with smooth manifolds, we will require, in addition, that the functions are smooth.

A partition of unity is defined on an *open cover* $\mathcal{U}$ of a manifold X. That is a family of open sets $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ such that $X = \bigcup_{\alpha \in A} U_\alpha$.

Definition 2.2.1. Let X be a manifold and $\mathcal{U}$ an open cover of X. A *partition of unity* subordinate to $\mathcal{U}$ is a family $(\chi_\alpha)_{\alpha \in A}$ of smooth functions $\chi_\alpha : X \to \mathbb{R}^+$ such that

(i) $(\operatorname{supp} \chi_\alpha)_{\alpha \in A}$ is locally finite

(ii) $\forall \alpha \in A \exists U \in \mathcal{U}$ such that $\operatorname{supp} \chi_\alpha \subseteq U$

(iii) $\forall p \in X : \sum_{\alpha \in A} \chi_\alpha(p) = 1$.

It is possible to have partitions of unity that are subordinate to a given cover (i.e. the functions χ_α have the same index as the sets in the open cover) or to have compactly supported functions χ_α. Generally it is not possible to fulfill both properties, although on compact spaces it is.

Recall the following statements about the existence of smooth partitions of unity on manifolds. Both proofs may be found in [BC70], chapter 3.4.

Theorem 2.2.2 (Countable partition of unity)**.** *Let X be a smooth manifold and $\mathcal{U}$ an open cover of X. Then there exists a subordinate partition of unity $(\chi_n)_{n \in \mathbb{N}}$ such that each* $\operatorname{supp} \chi_n$ *is compact and contained in a chart neighborhood.*

If a countable partition of unity is not required, it is enough to have a paracompact manifold.

Definition 2.2.3. A topological space X is called *paracompact* if it is Hausdorff and every open cover of X admits a locally finite open refinement.

Theorem 2.2.4 (Subordinate partition of unity)**.** *Let X be a paracompact manifold and $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ an open cover of X. Then there exists a partition of unity $(\chi_\alpha)_{\alpha \in A}$ such that* $\operatorname{supp} \chi_\alpha \subseteq U_\alpha$ *for all $\alpha \in A$.*

Remark 2.2.5. It can be shown that a Hausdorff manifold is second countable if and only if it is paracompact and consists of countably many connected components (see [Hal06]). Thus theorem 2.2.4 is also true for smooth manifolds and may even be derived from theorem 2.2.2. On the other hand, theorem 2.2.2 cannot be extended to paracompact manifolds with uncountably many connected components.

Corollary 2.2.6. *Let X be a paracompact manifold, $X \supseteq U$ open and $U \supseteq V$ closed. There exists a smooth bump function $\chi : X \to \mathbb{R}$ such that $\chi|_V \equiv 1$ and $\chi|_{X \setminus U} \equiv 0$.*

Proof. The family $(U, X \setminus V)$ forms an open cover of X. By 2.2.4 there exists a subordinate partition of unity $(\chi_U, \chi_{X \setminus V})$. The function $\chi := \chi_U$ satisfies the required conditions. $\square$

Chapter 3

Isomorphisms of Algebras of Smooth Functions

Throughout, $\mathcal{C}^\infty(X)$ denotes the associative and commutative algebra of smooth and complex-valued functions on X. It is, however, clear that all proofs also work for real-valued functions.

The aim of this chapter is to prove that, for certain finite-dimensional differentiable manifolds X and Y, any algebra isomorphism $\Psi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ is given by composition with a unique diffeomorphism $\psi : Y \to X$, i.e. that $\Psi(f) = f \circ \psi$ for all $f \in \mathcal{C}^\infty(X)$.

The classical approach to this question was inspired by the work of Gelfand and Kolmogorov in [GK37] for continuous functions on compact sets. The idea is to identify non-zero multiplicative linear functionals on the algebra $\mathcal{C}^\infty(X)$ with the points in X. However, this requires X and Y to be Hausdorff and second-countable, since it strongly uses partitions of unity. This approach is the topic of section 3.1.

A different proof of this result has recently been given by Janez Mrčun [Mrč05]. He characterizes the points of X by so-called *characteristic sequences* of (complex-valued) smooth functions on X. This approach merely requires that X and Y be smooth Hausdorff manifolds. It will be discussed in section 3.2.

3.1 On Hausdorff and second countable manifolds

In [MS74], p. 11f, problem 1-C states that the real-valued smooth functions $\mathcal{C}^\infty(X)$ on a smooth manifold X can be made into a ring and that every point $p \in X$ determines a ring homomorphism $\mathcal{C}^\infty(X) \to \mathbb{R}$ (point evaluation) and hence a maximal ideal in $\mathcal{C}^\infty(X)$. If there is a countable basis for the topology of X, then every ring homomorphism $\mathcal{C}^\infty(X) \to \mathbb{R}$ is obtained in this way. Due to its originator this result is often referred to as *Milnor's exercise*.

Our aim in subsection 3.1.1 is to prove Milnor's exercise for second countable manifolds following some ideas in [AMR88], supplement 4.2C, and – with some restrictions – also for paracompact manifolds in subsection 3.1.2.

This result is then used in subsection 3.1.3 to define a map $\psi : Y \to X$ from an algebra isomorphism $\Psi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ such that Ψ is given as pullback under ψ.

3.1.1 Multiplicative linear functionals on smooth manifolds

We consider the functionals

$$\begin{aligned} \mathrm{ev}_p : \mathcal{C}^\infty(X) &\to \mathbb{C} \\ f &\mapsto f(p) \end{aligned}$$

for any differentiable manifold X, $p \in X$. It is obvious that these are non-zero algebra homomorphisms since the multiplication and addition of functions are defined pointwise.

We now show that the converse is also true for Hausdorff and second countable manifolds.

Theorem 3.1.1. *Let X be a smooth manifold and $\varphi : \mathcal{C}^\infty(X) \to \mathbb{C}$ be a non-zero algebra homomorphism. Then there exists a unique point $p \in X$ such that*

$$\varphi(f) = f(p) \qquad \forall f \in \mathcal{C}^\infty(X).$$

Proof of uniqueness. For any two points $p_1, p_2 \in X$, $p_1 \neq p_2$ there exits a bump function $f \in \mathcal{C}^\infty(X)$ which separates them:

The manifold X is locally compact, so there exists an open set U and a compact set $V \subset\subset U$ such that $p_1 \in V$ and $p_2 \notin U$. The bump function $f : X \to \mathbb{R}$ of corollary 2.2.6 fulfills the requirements since it is smooth and $f(p_1) = f|_V(p_1) = 1$ but $f(p_2) = f|_{X \setminus U}(p_2) = 0$.

Hence p must be unique if it exists. □

It remains to show that such a p exists for every non-zero algebra homomorphisms φ. To this end we will first derive some algebraic properties of algebra homomorphisms $\mathcal{C}^\infty(X) \to \mathbb{C}$.

Lemma 3.1.2. *Let X be a manifold, $p \in X$ and $\varphi : \mathcal{C}^\infty(X) \to \mathbb{C}$ be a non-zero algebra homomorphism. Then*

(i) $\varphi(1) = 1$ *and* $\varphi(c) = c \; \forall c \in \mathbb{C}$.

(ii) $\ker \varphi$ *is a maximal ideal in* $\mathcal{C}^\infty(X)$.

Proof. (i) Since φ is multiplicative, $\varphi(1) = \varphi(1^2) = \varphi(1)^2$ implies that $\varphi(1) = 0$ or $\varphi(1) = 1$. If $\varphi(1) = 0$ then $\varphi(f) = \varphi(1 \cdot f) = \varphi(1) \cdot \varphi(f) = 0 \; \forall f \in \mathcal{C}^\infty(X)$ which contradicts φ being non-zero. Hence $\varphi(1) = 1$ and by multiplicativity also $\varphi(c) = c$ for all constant functionals c.

(ii) If $f \in \ker \varphi$, $g \in \mathcal{C}^\infty(X)$ then $fg \in \ker \varphi$, too, hence $\ker \varphi$ is an ideal. Let I be an ideal of $\mathcal{C}^\infty(X)$ such that $\ker \varphi \subseteq I$. Since φ is a ring homomorphism $\varphi(I)$ is an ideal in the field $\mathbb{C}$, thus

$$\text{either} \quad \varphi(I) = \{0\} \quad \text{or} \quad \varphi(I) = \mathbb{C}.$$

If $\varphi(I) = \{0\} = \varphi(\ker \varphi)$ then $I \subseteq \ker \varphi$, hence $I = \ker \varphi$. If $\varphi(I) = \mathbb{C} = \varphi(\mathcal{C}^\infty(X))$ then for every $f \in \mathcal{C}^\infty(X)$ there exists a $g \in I$ such that $\varphi(f) = \varphi(g) \in \mathbb{C}$ and therefore $f - g \in \ker \varphi \subset I$, i.e. $f \in g + I \subseteq I$. This implies that $I = \mathcal{C}^\infty(X)$. Thus $\ker \varphi$ is a maximal ideal. □

Proof of existence in theorem 3.1.1. First of all, note that it is enough to show the existence of a $p \in X$ such that $\ker \varphi = \ker \mathrm{ev}_p$:

Clearly, if $\varphi(f) = f(p)$ for some p then $\ker \varphi = \ker \mathrm{ev}_p$. Conversely, if $\ker \varphi = \ker \mathrm{ev}_p$ for some $p \in X$ then $\varphi = \mathrm{ev}_p$: For $f \in \mathcal{C}^\infty(X)$ we have that $\varphi(f) = c = \varphi(c)$ by lemma 3.1.2 (i), and therefore $f - c \in \ker \varphi = \ker \mathrm{ev}_p$. Hence $\mathrm{ev}_p(f) = f(p) = c = \varphi(f)$.

It remains to be proved that

$$\ker \varphi = \ker \mathrm{ev}_p \quad \text{for some } p \in X. \tag{3.1}$$

Assume that $\ker \varphi \neq \ker \mathrm{ev}_p$ for all $p \in X$. By lemma 3.1.2 (ii) both sets are maximal ideals in $\mathcal{C}^\infty(X)$. Hence neither of them can be included in the other one. For every $p \in X$ we can therefore find an $f_p \in \ker \varphi$ such that $f_p(p) > 0$, even a relatively compact open neighborhood V_p of p such that $f_p|_{V_p} > 0$.

Let $(\chi_n)_{n \in \mathbb{N}}$ be a locally finite partition of unity subordinate to the cover $(V_p)_{p \in X}$ as in theorem 2.2.2. For all $n \in \mathbb{N}$ choose $p(n)$ such that $\operatorname{supp} \chi_n \subseteq V_{p(n)}$. We will write $V_n := V_{p(n)}$ and $f_n := f_{p(n)}$.

The next step is to show that $1 \in \ker \varphi$. Since $\ker \varphi$ is an ideal this will imply that $f = 1 \cdot f \in \ker \varphi \; \forall f \in \mathcal{C}^\infty(X)$, contradicting $\varphi \not\equiv 0$.

Consider

$$f := \sum_{n=1}^{\infty} a_n \chi_n f_n.$$

This function is in $\mathcal{C}^\infty(X)$ since the χ_n are locally finite. If, in addition,

$$0 < a_n < \frac{1}{n^2 \|\chi_n f_n\|_\infty},$$

then the series defining f converges uniformly (being majorized by $\sum \frac{1}{n^2}$). Since for any $p \in X$ there exists an $n_0 \in \mathbb{N}$ such that $\chi_{n_0}(p) > 0$, we have that $p \in \operatorname{supp} \chi_{n_0} \subseteq V_{n_0}$ and therefore also $f_{n_0}(p) > 0$. In particular $f > 0$ on X.

To show that $\varphi(f) = 0$ and hence conclude $1 = \frac{1}{f} \cdot f \in \ker \varphi$, we need to interchange φ with the summation.

This is done by a so-called *g-estimate*: Let $g \in \mathcal{C}^\infty(X)$. Either g is unbounded or we consider $\lambda > \|g\|_\infty$. Since $\varphi(\lambda) = \lambda$ by lemma 3.1.2 (i) and $\lambda \pm g \neq 0$ (vanishes nowhere) are both invertible functions on X we obtain $\lambda \pm \varphi(g) = \varphi(\lambda \pm g) \neq 0$. Thus $\pm\varphi(g) \neq \lambda$ for all such λ, hence

$$|\varphi(g)| \leq \|g\|_\infty.$$

Finally we apply this to f to conclude that $f \in \ker \varphi$. By uniform convergence and boundedness of all functions involved we obtain

$$\begin{aligned} |\varphi(f)| &= \left| \varphi(f) - \sum_{n=1}^{N} \varphi(a_n \chi_n f_n) \right| = \left| \varphi\left(f - \sum_{n=1}^{N} a_n \chi_n f_n \right) \right| \\ &\leq \left\| f - \sum_{n=1}^{N} a_n \chi_n f_n \right\|_\infty \to 0 \end{aligned}$$

as $N \to \infty$. □

3.1.2 Multiplicative linear functionals on paracompact manifolds

Although we cannot extend the above proof to paracompact manifolds with uncountably many connected components (cf. remark 2.2.5), we are able to decompose the multiplicative linear functionals on such a paracompact manifold X into their restrictions to the connected components X_ι of X and use theorem 3.1.1 there.

Definition 3.1.3 (Restrictions φ_ι)**.** Let X be a manifold and X_ι, $\iota \in I$, be the (possibly uncountably many) connected components of X. First of all we decompose $\varphi : \mathcal{C}^\infty(X) \to \mathbb{C}$ into (possibly zero) multiplicative linear functionals

$$\begin{aligned} \varphi_\iota : \mathcal{C}^\infty(X_\iota) &\to \mathbb{C} \\ f_\iota &\mapsto \varphi(f^\iota), \end{aligned}$$

where

$$f^\iota(x) := \begin{cases} f_\iota(x) & x \in X_\iota \\ 0 & \text{else} \end{cases} .$$

Obviously, each $f \in \mathcal{C}^\infty(X)$ can be written uniquely as a sum $f = \sum_{\iota \in I} f^\iota$, where $f_\iota := f|_{X_\iota}$ in the above definition of f^ι. We may write

$$\mathcal{C}^\infty(X) = \bigoplus_{\iota \in I} \mathcal{C}^\infty(X_\iota)$$

and will not distinguish between $f_\iota \in \mathcal{C}^\infty(X_\iota)$ and $f^\iota \in \mathcal{C}^\infty(X)$ in the following.

To simplify the proof we first verify the following lemmas.

Lemma 3.1.4. *Let X be a paracompact manifold and φ be a multiplicative linear functional on $\mathcal{C}^\infty(X)$ with $\varphi|_{\mathcal{C}_c^\infty(X)} \equiv \mathrm{ev}_p$ for a $p \in X$. Then $\varphi \equiv \mathrm{ev}_p$ everywhere.*

Proof. Let χ be a bump function at p (which exists by corollary 2.2.6 since X is locally compact), i.e. $\chi \in \mathcal{C}_c^\infty(X)$, $\chi(p) = 1$. For $f \in \mathcal{C}^\infty(X)$ we have $\chi \cdot f \in \mathcal{C}_c^\infty(X)$. Hence

$$\begin{aligned} \varphi(f) &= 1 \cdot \varphi(f) = \chi(p) \cdot \varphi(f) = \varphi(\chi) \cdot \varphi(f) = \varphi(\chi \cdot f) \\ &= \mathrm{ev}_p(\chi \cdot f) = \chi(p) \cdot f(p) = 1 \cdot f(p) = f(p) \end{aligned}$$

and therefore $\varphi \equiv \mathrm{ev}_p$ on $\mathcal{C}^\infty(X)$. □

Lemma 3.1.5. *Let X be a manifold and $\varphi : \mathcal{C}^\infty(X) \to \mathbb{C}$ a multiplicative linear functional. The following statements are equivalent:*

(i) $\varphi_\iota \equiv 0 \;\; \forall \iota \in I$

(ii) $\varphi|_{\mathcal{C}_c^\infty(X)} \equiv 0$

Proof. (i ⇒ ii) If $\varphi_\iota \equiv 0$ for all $\iota \in I$, then obviously $\varphi|_{\mathcal{C}_c^\infty(X)} \equiv 0$:

$$\varphi(f) = \varphi\left(\sum_{\iota \in H} f^\iota\right) \stackrel{!}{=} \sum_{\iota \in H} \varphi_\iota(f_\iota) = 0$$

since $f = \sum_{\iota \in H} f^\iota \in \mathcal{C}_c^\infty(X)$ is a finite sum because the compact support of f may be covered by finitely many X_ι, $\iota \in H \subseteq I$ finite.

(ii $\Rightarrow$ i) Suppose that $\varphi|_{\mathcal{C}_c^\infty(X)} \equiv 0$ but that there exists an $\iota \in I$ such that φ_ι is non-zero. Hence $\varphi_\iota \equiv \mathrm{ev}_{p_\iota}$ for some $p_\iota \in X_\iota$ by theorem 3.1.1. Let $\chi_\iota \in \mathcal{C}_c^\infty(X_\iota)$ be a bump function around p_ι. Then $1 = \varphi_\iota(\chi_\iota) = \varphi(\chi_\iota) \neq 0$, a contradiction. □

Corollary 3.1.6. *Let X be a paracompact manifold and $\varphi : \mathcal{C}^\infty(X) \to \mathbb{C}$ be a multiplicative linear functional which is non-zero on $\mathcal{C}_c^\infty(X)$. Then there exists a unique point $p \in X$ such that*

$$\varphi(f) = f(p) \qquad \forall f \in \mathcal{C}^\infty(X).$$

Proof. Let X_ι, $\iota \in I$, be the (possibly uncountably many) connected components of X and φ_ι, f_ι etc. as in definition 3.1.3.

By theorem 3.1.1 either $\varphi_\iota \equiv \mathrm{ev}_{p_\iota}$ for some $p_\iota \in X_\iota$ or $\varphi_\iota \equiv 0$. We will prove that there is exactly one $\varphi_\iota \equiv \mathrm{ev}_{p_\iota}$ and all the others are zero:

Uniqueness: Suppose there exist $\iota \neq \kappa \in I$ such that $\varphi_\iota \equiv \mathrm{ev}_{p_\iota}$ for $p_\iota \in X_\iota$ and $\varphi_\kappa \equiv \mathrm{ev}_{p_\kappa}$ for $p_\kappa \in X_\kappa$. Let $\chi_\iota \in \mathcal{C}_c^\infty(X_\iota)$ be a bump function around p_ι as in corollary 2.2.6, then $\varphi(\chi_\iota) = \varphi_\iota(\chi_\iota) = \chi_\iota(p_\iota) = 1$. Define $\chi_\kappa \in \mathcal{C}_c^\infty(X_\kappa)$ analogously. Then $0 = \varphi(0) = \varphi(\chi_\iota \cdot \chi_\kappa) = \varphi(\chi_\iota) \cdot \varphi(\chi_\kappa) = 1 \cdot 1 = 1$, a contradiction.

Existence: Since φ is non-zero on $\mathcal{C}_c^\infty(X)$ there exists a $\varphi_\kappa \not\equiv 0$ by lemma 3.1.5, and by theorem 3.1.1 $\varphi_\kappa \equiv \mathrm{ev}_{p_\kappa}$ for some $p_\kappa \in X_\kappa$. Due to lemma 3.1.4 it remains to show that $\varphi|_{\mathcal{C}_c^\infty(X)} \equiv \mathrm{ev}_p$ for $p = p_\kappa \in X$. To this end consider $f \in \mathcal{C}_c^\infty(X)$, i.e. $f = \sum_{\iota \in H} f^\iota$ for a finite set $H \subseteq I$ (assume w.l.o.g. that $\kappa \in H$):

$$\varphi(f) = \varphi\left(\sum_{\iota \in H} f^\iota\right) \overset{!}{=} \sum_{\iota \in H} \varphi(f^\iota) = \sum_{\iota \in H} \varphi_\iota(f_\iota) = \varphi_\kappa(f_\kappa) = f(p),$$

because for all $\iota \neq \kappa$ we have that $\varphi_\iota \equiv 0$ by uniqueness. □

Remark 3.1.7. Note that the assumption 'non-zero on $\mathcal{C}_c^\infty(X)$' is much stronger than simply 'non-zero on $\mathcal{C}^\infty(X)$' as in theorem 3.1.1.

3.1.3 Algebra isomorphisms

Algebra isomorphisms are pullbacks by diffeomorphisms:

Theorem 3.1.8. *Let X and Y be smooth manifolds and $\Psi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ an algebra isomorphism. Then Ψ is the pullback by a unique diffeomorphism $\psi : Y \to X$, i.e.*

$$\Psi(f) = f \circ \psi \quad \forall f \in \mathcal{C}^\infty(X).$$

Moreover, $\dim X = \dim Y$.

Proof. Existence: For each $q \in Y$ define the algebra homomorphisms $\varphi_q : \mathcal{C}^\infty(X) \to \mathbb{C}$ by $\varphi_q(f) := \Psi(f)(q)$. Since Ψ is non-zero we conclude that $\Psi(1) = 1$ as in lemma 3.1.2 (i), thus φ_q is non-zero. By theorem 3.1.1 there exists a unique point $p \in X$ such that $\varphi_q = \mathrm{ev}_p$. We shall see that

$$\begin{aligned} \psi : Y &\to X \\ q &\mapsto p \end{aligned}$$

is the required diffeomorphism. Since $\Psi(f)(q) = \varphi_q(f) = \mathrm{ev}_{\psi(q)}(f) = (f \circ \psi)(q) = \psi^* f(q)$ for all $f \in \mathcal{C}^\infty(X)$ and all $q \in Y$, Ψ is the pullback via ψ.

Define $\sigma : X \to Y$ analogously via Ψ^{-1}, i.e. $\Psi^{-1}(g)(p) = \mathrm{ev}_{\sigma(p)}(g) = (g \circ \sigma)(p) = \sigma^* g(p)$ for all $g \in \mathcal{C}^\infty(Y)$ and all $p \in X$. Since $g = (\Psi \circ \Psi^{-1})(g) = \Psi(g \circ \sigma) = g \circ \sigma \circ \psi$, we conclude that $\sigma \circ \psi = \mathrm{id}_Y$. Analogously $\psi \circ \sigma = \mathrm{id}_X$. Thus ψ is bijective.

Both ψ and ψ^{-1} are smooth by proposition 3.1.9 (see below), since the smoothness of functions is preserved under the composition with ψ as well as with ψ^{-1}. Thus ψ is also a diffeomorphism.

Uniqueness: Suppose ψ and ρ are both such diffeomorphisms with $p_1 = \psi(q) \neq \rho(q) = p_2$ for some $q \in Y$. As in the proof of uniqueness of theorem 3.1.1 there exists an $f \in \mathcal{C}^\infty(X)$ such that $f(p_1) \neq f(p_2)$. Therefore $(f \circ \psi)(q) = f(p_1) \neq f(p_2) = (f \circ \rho)(q)$ which means that ψ and ρ cannot belong to the same Ψ.

It remains to be shown that $\dim X = \dim Y$. By differentiating $\psi \circ \psi^{-1} = \mathrm{id}_X$ at any $p \in X$ we obtain $T_{\psi^{-1}(p)}\psi \circ T_p\psi^{-1} = T_p \,\mathrm{id}_X = \mathrm{id}_{T_pX}$. Thus

$$\begin{aligned} \dim X &= \dim T_pX = \mathrm{rk}(\mathrm{id}_{T_pX}) = \mathrm{rk}(T_{\psi^{-1}(p)}\psi \circ T_p\psi^{-1}) \\ &\leq \mathrm{rk}(T_p\psi^{-1}) = \dim(\mathrm{im}\, T_p\psi^{-1}) \leq \dim T_{\psi^{-1}(p)}Y = \dim Y. \end{aligned}$$

Similarly $\dim Y \leq \dim X$. Hence we have equality. □

Proposition 3.1.9. *Let X, Y be manifolds and $\psi : Y \to X$. Then $\psi \in \mathcal{C}^\infty(Y, X)$ if and only if for all $f \in \mathcal{C}^\infty(X) : f \circ \psi \in \mathcal{C}^\infty(Y)$.*

Proof. ($\Rightarrow$) holds since the composition of smooth mappings is smooth.

($\Leftarrow$) It remains to be shown that $u \circ \psi \in \mathcal{C}^\infty(\psi^{-1}(U), \mathbb{R}^m)$ for a chart (u, U) at any $p \in X$, i.e. $u^i \circ \psi \in \mathcal{C}^\infty(\psi^{-1}(U), \mathbb{R})$ for all $i = 1, ..., m$. By corollary 2.2.6 there exits a bump function $\chi \in \mathcal{C}^\infty(X)$ such that $\chi(p) = 1$ and $\chi|_{X \setminus U} \equiv 0$. The assumption implies that $u^i \circ \psi = (u^i \cdot \chi) \circ \psi \in \mathcal{C}^\infty(Y)$ for all $i = 1, ..., m$. □

Remark 3.1.10. Unfortunately we cannot prove theorem 3.1.8 for paracompact manifolds in the same way because we had to strengthen our assumptions in corollary 3.1.6 (non-zero on $\mathcal{C}_c^\infty(X)$ instead of non-zero) as already pointed out in remark 3.1.7.

However, in the next section we will provide a theory which leads to the same result about algebra isomorphisms on smooth functions without assuming that X is paracompact.

3.2 On Hausdorff manifolds

In the previous section we strongly used the fact that the manifolds are second countable in order to identify the points of such manifolds with non-zero multiplicative linear functionals on the algebra of smooth functionals.

At the 'International Euroschool on Poisson Geometry, Deformation Quantisation and Group Representations' in Brussels in June 2003, Alan Weinstein posed the question of whether an analogous result on algebra ismorphisms of smooth functions holds true in a more general setting.

Janez Mrčun showed in [Mrč05] that also the isomorphisms between algebras of smooth functions on Hausdorff manifolds, which are not necessarily second countable, paracompact or connected, are induced by diffeomorphisms of the underlying manifolds. He works with so-called *characteristic sequences* and we will discuss his approach in the following.

Shortly after Mrčun's paper Janusz Grabowski independently proved the same result by using so-called *distinguished ideals*. He follows some ideas of [GK37]. For more details on Grabowski's approach see [Gra05].

3.2.1 Characteristic sequences of smooth functionals

First of all we define the main object of this section:

Definition 3.2.1. Let X be a manifold, $p \in X$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of smooth functions $f_n : X \to \mathbb{C}$. $(f_n)_n$ is called a *characteristic sequence of functions* on X at p if

(i) $f_n f_{n+1} = f_{n+1}$ for all $n \in \mathbb{N}$ and

(ii) the sequence of supports $(\operatorname{supp} f_n)_n$ is a fundamental system of neighborhoods of $p \in X$.

Recall the definition of a fundamental system of neighborhoods:

Definition 3.2.2. Let X be a topological space, $x \in X$ and $\mathcal{U}_x$ a neighborhood system of x, i.e. $\mathcal{U}_x := \{U \,|\, U \text{ is neighborhood of } x\}$. A subsystem $\mathcal{W}_x \subseteq \mathcal{U}_x$ is called *fundamental system of neighborhoods of* x if

$$\forall U \in \mathcal{U}_x \, \exists W \in \mathcal{W}_x : \, (x \in) W \subseteq U.$$

For T_1 topological spaces (e.g., manifolds) only x is contained in the intersection of all such neighborhoods:

Proposition 3.2.3. *Let X be a T_1 topological space, $x \in X$ and $\mathcal{W}_x$ a fundamental system of neighborhoods of x. Then*

$$\bigcap_{W\in\mathcal{W}_x} W = \{x\} .$$

Proof. ($\supseteq$) Since all $W \in \mathcal{W}_x$ are neighborhoods of x it is obvious that $x \in \bigcap W$.

($\subseteq$) Conversely, suppose there exists $y \neq x$ in the intersection. Since X is T_1 there exists a neighborhood U of x such that $y \notin U$. Hence by 3.2.2 there must be a $W \in \mathcal{W}_x$ such that $W \subseteq U$. In particular, $y \notin W$ and therefore $y \notin \bigcap W$, a contradiction. □

We are now ready to prove some basic properties of characteristic sequences of functions.

Proposition 3.2.4. *Let X be a manifold and $(f_n)_n$ a characteristic sequence of functions on X at $p \in X$. Then*

(i) $f_n|_{\operatorname{supp} f_{n+1}} \equiv 1$, *in particular* $\operatorname{supp} f_{n+1} \subseteq \operatorname{int}(\operatorname{supp} f_n) \; \forall n \in \mathbb{N}$

(ii) $f_n(p) = 1 \; \forall n \in \mathbb{N}$

(iii) For any $p \in X$ there exists a characteristic sequence of functions.

Proof. (i) holds since property (i) of definition 3.2.1 implies that $f_n(q) = 1$ whenever f_{n+1} is non-zero. Moreover, $f_n(q) = 1$ for all $q \in \operatorname{supp} f_{n+1}$, since $f_n^{-1}(\{1\})$ has to be closed in X.

(ii) follows by the argument in (i) and property (ii) of definition 3.2.1, since $p \in \operatorname{supp} f_n \ \forall n \in \mathbb{N}$.

(iii) Let (u, U) be a chart of X at $p \in X$, and w.l.o.g. assume that $u(p) = 0$ and $\overline{B_2(0)} \subseteq u(U) \subseteq \mathbb{R}^k$. We are going to construct a characteristic sequence of functions on $\mathbb{R}^k$ at 0, and will then pull it up to X via u^{-1}:

By corollary 2.2.6 there exists a smooth bump function $\chi : \mathbb{R}^k \to \mathbb{R}$ such that $\chi|_{\overline{B_1(0)}} \equiv 1$ and $\operatorname{supp} \chi \subseteq \overline{B_2(0)} \subseteq u(U)$. Define the sequence $(\chi_n)_n$ by

$$\chi_n(x) := \chi(2^n x).$$

Then clearly $\chi_n|_{\overline{B_{\frac{1}{2^n}}(0)}} \equiv 1$ and $\operatorname{supp}(\chi_n) \subseteq \overline{B_{\frac{1}{2^{n-1}}}(0)}$. Now let

$$\begin{aligned} f_n : X &\to \mathbb{R} \subseteq \mathbb{C} \\ q &\mapsto \begin{cases} (\chi_n \circ u)(q) & q \in U \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Since χ_n and u are smooth and $\operatorname{supp} \chi_n \subseteq u(U)$, f_n is a smooth function. It remains to prove (i) and (ii) of definition 3.2.1:

(i) By the above, $f_n(q) = \chi_n(u(q)) = 1$ if $q \in u^{-1}(\overline{B_{\frac{1}{2^n}}(0)})$. On the other hand $\operatorname{supp} f_{n+1} = u^{-1}(\operatorname{supp} \chi_{n+1}) \subseteq u^{-1}(\overline{B_{\frac{1}{2^n}}(0)})$ since u is a homeomorphism. Thus $f_n \equiv 1$ on $\operatorname{supp} f_{n+1}$, and $f_n f_{n+1} = f_{n+1}$ holds for all $n \in \mathbb{N}$.

(ii) Since $u^{-1}(B_{\frac{1}{2^n}}(0))$ is open and contained in $\operatorname{supp} f_n$, $\operatorname{supp} f_n$ is a neighborhood of $p = u^{-1}(0)$. Let V be any neighborhood of p. Then the set $u(U \cap V)$ is a neighborhood of $0 \in \mathbb{R}^k$, hence there exists an $m \in \mathbb{N}$ such that $\overline{B_{\frac{1}{2^{m-1}}}(0)} \subseteq u(U \cap V)$. Finally, $\operatorname{supp} f_m \subseteq u^{-1}(\overline{B_{\frac{1}{2^{m-1}}}(0)}) \subseteq U \cap V \subseteq V$ and we are done. ☐

For *Hausdorff manifolds* (i.e. differentiable manifolds that are Hausdorff) we may rewrite (ii) of definition 3.2.1 in the following way.

Lemma 3.2.5 (Characterization of characteristic sequences)**.** *Let X be a Hausdorff manifold and $(f_n)_n$ a sequence of complex-valued smooth functions on X satisfying $f_n f_{n+1} = f_{n+1}$ for all $n \in \mathbb{N}$. Then $(f_n)_n$ is a characterstic sequence of functions at $p \in X$ if and only if*

(i) $\bigcap_{n=1}^{\infty} \operatorname{supp} f_n = \{p\}$ *and*

(ii) $\exists m \in \mathbb{N} :\ \operatorname{supp} f_m$ *is compact*

are satisfied.

Proof. (⇒) Since X is locally compact there exists a compact neighborhood K of p. By (ii) of 3.2.1 $(\operatorname{supp} f_n)_n$ is a fundamental system of neighborhoods of p, hence there exists an $m \in \mathbb{N}$ such that $\operatorname{supp} f_m \subseteq K$. Thus $\operatorname{supp} f_m$ is compact itself and (ii) is fulfilled. Property (i) is proposition 3.2.3.

($\Leftarrow$) Let U be any open neighborhood of p and assume that no $\operatorname{supp} f_n$, $n \in \mathbb{N}$, is contained in U. Suppose w.l.o.g. that $\operatorname{supp} f_1$ is compact. By our assumptions and proposition 3.2.4 (i) (which only requires $f_n f_{n+1} = f_{n+1}$) the family $\mathcal{F} = (\operatorname{supp} f_n \setminus U)_{n\in\mathbb{N}}$ has the finite intersection property (see definition 3.2.6 below). On the other hand

$$\emptyset = \{p\} \setminus U \overset{\text{(i)}}{=} (\bigcap_{n\in\mathbb{N}} \operatorname{supp} f_n) \setminus U = \bigcap_{n\in\mathbb{N}} \operatorname{supp} f_n \setminus U,$$

which contradicts theorem 3.2.7 below. ☐

In the last paragraph we used theorem 3.1.1 of [Eng77] which characterizes compact Hausdorff spaces via the finite intersection property:

Definition 3.2.6. A family $\mathcal{F} = (F_s)_{s\in S}$ of subsets of a set X has the *finite intersection property* if $\mathcal{F} \neq \emptyset$ and $\bigcap_{s\in T} F_s \neq \emptyset$ for every finite set $T \subseteq S$.

Theorem 3.2.7. *A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

3.2.2 Algebra isomorphisms

We have seen that we can construct a characteristic sequence of functions at each point of a manifold. The aim of the next (quite fundamental) lemma is to prove that characteristic sequences are compatible with algebra isomorphisms. In particular, the image of a characteristic sequence is again a characteristic sequence and points are preserved in a sense specified below. This is then used to define a map between the two manifolds which will be the desired diffeomorphism.

Lemma 3.2.8 (Compatibility of characteristic sequences with algebra isomorphisms)**.** *Let X and Y be Hausdorff manifolds and $\Psi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ an algebra isomorphism. Then:*

(i) If $(f_n)_n$ is a characteristic sequence of functions on X at $p \in X$, then $(\Psi(f_n))_n$ is a characteristic sequence of functions on Y at a unique point $q \in Y$.

(ii) If $(f_n)_n$ and $(f'_n)_n$ are two characteristic sequences of functions on X at the same point $p \in X$, then $(\Psi(f_n))_n$ and $(\Psi(f'_n))_n$ are characteristic sequences of functions on Y at the same point $q \in Y$.

Proof of (i). Let g_n, $n \in \mathbb{N}$, denote the smooth functions $\Psi(f_n) : Y \to \mathbb{C}$. Obviously $g_n g_{n+1} = \Psi(f_n)\Psi(f_{n+1}) = \Psi(f_n f_{n+1}) = \Psi(f_{n+1}) = g_{n+1}$ since Ψ is algebra isomorphism and $(f_n)_n$ a characteristic sequence of functions. Thus by the proof of 3.2.4 (i)

$$g_n|_{\operatorname{supp} g_{n+1}} \equiv 1 \text{ and } \operatorname{supp} g_{n+1} \subseteq \operatorname{int}(\operatorname{supp} g_n) \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Note that each g_n is non-zero: Otherwise $f_k \equiv 0\, \forall k \geq n$ since Ψ is injective, and the sets $\operatorname{supp} f_n = \emptyset$, $k \geq n$, would not be neighborhoods of p, a contradiction to 3.2.1 (ii).

We are going to prove that $(g_n)_n$ is a characteristic sequence of functions on Y at a point $q \in Y$ using the characterization of lemma 3.2.5.

Let

$$K := \bigcap_{n=1}^{\infty} \operatorname{supp} g_n.$$

We have to prove that $K = \{q\}$ for some $q \in Y$. The first step is to show that K is not empty: Assume that K is empty. Since each g_n is non-zero (i.e. $\operatorname{supp} g_n \neq \emptyset$) and $\operatorname{supp} g_k \subseteq \operatorname{int}(\operatorname{supp} g_n)$ $\forall k > n$ by (3.2) we may find a strictly increasing subsequence $(i_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\operatorname{supp} g_{i_k} \setminus \operatorname{supp} g_{i_{k+1}} \neq \emptyset$ for all $k \in \mathbb{N}$. Then the sets

$$V_k := \operatorname{int}(\operatorname{supp} g_{i_k}) \setminus \operatorname{supp} g_{i_{k+1}} \quad (\neq \emptyset)$$

are open and disjoint subsets of Y. They are nonempty because $\emptyset = \overline{\emptyset} = \overline{V_k} = \overline{\operatorname{int}(\operatorname{supp} g_{i_k}) \setminus \operatorname{supp} g_{i_k+1}} = \operatorname{supp} g_{i_k} \setminus \operatorname{int}(\operatorname{supp} g_{i_k+1}) \supseteq \operatorname{supp} g_{i_k} \setminus \operatorname{supp} g_{i_k+1} \neq \emptyset$, which is a contradiction. Furthermore, the family $(V_k)_k$ is locally finite in Y: Assume that there exists a $q' \in Y$ such that each neighborhood of q' intersects infinitely many V_k. Then each neighborhood of q' intersects infinitely many $\operatorname{supp} g_{i_k}$, and hence all of the sets $\operatorname{supp} g_i$ (due to $i_k \to \infty$ and (3.2)). Thus $q' \in K$, a contradiction to K being empty.

The next idea is to construct an $f \in \mathcal{C}^\infty(X)$ and a converging sequence $(p_k)_k$ of points in X such that $\lim_{k\to\infty} f(p_k)$ does not exist (hence contradicting the fact that f is continuous):

Since $V_k \neq \emptyset$ there exists a $q_k \in V_k$ for each $k \in \mathbb{N}$. By proposition 3.2.9 we can find $h_k, v_k \in \mathcal{C}^\infty(Y)$ such that $\operatorname{supp} v_k$ is a non-empty and compact subset of V_k, $\operatorname{supp} h_k \subseteq V_k$ and for each $l \in \mathbb{N}_0$ we have $h_{2l}|_{\operatorname{supp} v_{2l}} \equiv 1$ and $h_{2l+1}|_{\operatorname{supp} v_{2l+1}} \equiv 0$ (let $c_{2l} := 1$ and $c_{2l+1} := 0$). As the V_k are disjoint and locally finite and $\operatorname{supp} h_k \subseteq V_k$ for all k, the function

$$\begin{aligned} g : Y &\to \mathbb{R} \subseteq \mathbb{C} \\ q &\mapsto \begin{cases} h_k(q) & q \in V_k \\ 0 & \text{else} \end{cases} \end{aligned}$$

is well-defined and smooth on Y. Furthermore we have

$$g v_{2k} \equiv v_{2k} \quad \text{and} \quad g v_{2k+1} \equiv 0. \tag{3.3}$$

Moreover,

$$g_{i_k} v_k \not\equiv 0 \tag{3.4}$$

for all $k \in \mathbb{N}$, since $\emptyset \neq \operatorname{supp} v_k \subseteq V_k \subseteq \operatorname{supp} g_{i_k}$. Let

$$f := \Psi^{-1}(g) \quad \text{and} \quad u_k := \Psi^{-1}(v_k).$$

Since Ψ^{-1} is an algebra isomorphism, we have $f_{i_k} u_k = \Psi^{-1}(g_{i_k} v_k) \not\equiv 0$ by (3.4). Therefore we can find a $p_k \in X$ for each $k \in \mathbb{N}$ such that

$$(f_{i_k} u_k)(p_k) \neq 0. \tag{3.5}$$

Thus $p_k \in \operatorname{supp} f_{i_k}$, which implies that $(p_k)_k$ converges to $p \in X$ since $(f_n)_n$ is a characteristic sequence of functions at p. On the other hand, $f u_{2l} \equiv u_{2l}$

and $fu_{2l+1} \equiv 0$ by (3.3). By (3.5), $u_k(p_k) \neq 0\ \forall k \in \mathbb{N}$, hence $f(p_{2l}) = 1$ and $f(p_{2l+1}) = 0$, a contradiction to the continuity of f. Hence $K \neq \emptyset$.

Take a point $q \in K$. We shall see that $K = \{q\}$ and that $(g_n)_n$ is a characteristic sequence of functions on Y at q. Let V be an open neighborhood of q in Y. Choose a characteristic sequence of functions $(\beta_n)_n$ on Y at q (this exists by proposition 3.2.4 (iii)) such that $\operatorname{supp}\beta_1 \subseteq V$. Set $\alpha_n := \Psi^{-1}(\beta_n)$ and

$$\gamma_n := \alpha_n f_n .$$

We have

$$\gamma_n \gamma_{n+1} = \gamma_{n+1} \tag{3.6}$$

since $\gamma_n\gamma_{n+1} = \alpha_n f_n \alpha_{n+1} f_{n+1} = \alpha_n\alpha_{n+1} f_n f_{n+1} = \Psi^{-1}(\beta_n\beta_{n+1}) f_n f_{n+1} = \Psi^{-1}(\beta_{n+1}) f_{n+1} = \alpha_{n+1} f_{n+1} = \gamma_{n+1}$. Furthermore,

$$\beta_n(q) \overset{3.2.4\text{ (ii)}}{=} 1 \overset{(3.2)}{=} g_n(q) \;\Rightarrow\; \gamma_n = \alpha_n f_n = \Psi^{-1}(\beta_n g_n) \not\equiv 0$$

for all $n \in \mathbb{N}$. These properties imply that $(\operatorname{supp}\gamma_n)_n$ is a descending sequence of non-empty sets. Using lemma 3.2.5 we will see that $(\gamma_n)_n$ actually is a characteristic sequence of functions: As $(f_n)_n$ is a characteristic sequence, there exists an $m \in \mathbb{N}$ such that $\operatorname{supp} f_n$ is compact for all $n \geq m$ by 3.2.4 (i) and 3.2.5 (ii). Thus

$$\operatorname{supp}\gamma_n \text{ is compact for all } n \geq m, \tag{3.7}$$

since $\operatorname{supp}\gamma_n \subseteq \operatorname{supp} f_n$ by the definition of γ_n. Furthermore we may apply theorem 3.2.7 to $(\operatorname{supp}\gamma_n)_n$ which has the finite intersection property, and therefore conclude that $\emptyset \neq \bigcap_{n\in\mathbb{N}} \operatorname{supp}\gamma_n \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{supp} f_n \overset{3.2.5\text{ (i)}}{=} \{p\}$, i.e.

$$\bigcap_{n\in\mathbb{N}} \operatorname{supp}\gamma_n = \{p\} . \tag{3.8}$$

Now (3.6)–(3.8) and lemma 3.2.5 imply that $(\gamma_n)_n$ is a characteristic sequence of functions at p in X. In particular, $\operatorname{supp}\gamma_2$ is a neighborhood of p. Since $(f_n)_n$ is a characteristic sequence of functions we can choose a $j \geq 2$ such that $\operatorname{supp} f_j \subseteq \operatorname{supp}\gamma_2$, which implies $\gamma_1 f_j = f_j$ because $\gamma_1|_{\operatorname{supp}\gamma_2} \equiv 1$ by 3.2.4 (i). Hence $\beta_1 g_j \overset{(3.2)}{=} \beta_1 g_1 g_j = \Psi(\alpha_1 f_1)\Psi(f_j) = \Psi(\gamma_1 f_j) = \Psi(f_j) = g_j$ and therefore

$$\operatorname{supp} g_j \subseteq \operatorname{supp}\beta_1 \subseteq V \text{ for all } j \geq 2.$$

This shows that $(\operatorname{supp} g_n)_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods at q in Y. Thus $(g_n)_n = (\Psi(f_n))_n$ is a characteristic sequence of functions on Y at q. In particular $K = \{q\}$ by lemma 3.2.5 (i), which implies uniqueness of q. □

In the previous proof of part (i) we have used the following bump functions:

Proposition 3.2.9. *Let Y be a Hausdorff manifold, $Y \supseteq V$ open, $q \in V$ and $c \in \mathbb{R}$. Then there exist $h, v \in \mathcal{C}^\infty(Y)$ such that*

(i) $v(q) > 0$ (hence v non-zero) and $\operatorname{supp} v$ is a compact subset of V

(ii) $h|_{\operatorname{supp} v} \equiv c$ and $\operatorname{supp} h \subseteq V$.

Proof. Let (w, W) be a chart neighborhood of q such that $w(W) \subseteq V$. By corollary 2.2.6 ($\mathbb{R}^n$ is a paracompact manifold) there exist two auxiliary functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ such that $f_1(w(q)) = 1$, $\operatorname{supp} f_1$ is a compact subset of $w(W)$, $f_2|_{\operatorname{supp} f_1} \equiv 1$ and $\operatorname{supp} f_2 \subseteq w(W)$. Finally,

$$\begin{array}{rcl} v : Y & \to & \mathbb{R}_0^+ \\ y & \mapsto & \begin{cases} (f_1 \circ w)(y) & y \in W \\ 0 & y \in Y \setminus w^{-1}(\operatorname{supp} f_1) \end{cases} \end{array}$$

and

$$\begin{array}{rcl} h : Y & \to & \mathbb{R} \\ y & \mapsto & \begin{cases} c \cdot (f_2 \circ w)(y) & y \in W \\ 0 & y \in Y \setminus w^{-1}(\operatorname{supp} f_2) \end{cases} \end{array}$$

are the required smooth functions on Y. The support of v is compact because w is a homeomorphism. □

Finally, it remains to prove 3.2.8 (ii).

Proof of (ii). As in (i) we write $g_n := \Psi(f_n)$ and $g'_n := \Psi(f'_n)$ for all $n \in \mathbb{N}$. Part (i) implies that $(g_n)_n$ and $(g'_n)_n$ are characteristic sequences on Y at points q and q' of Y. Assume that $q \neq q'$. Since Y is Hausdorff and by definition 3.2.1 (ii) we can choose an $m \in \mathbb{N}$ such that $\operatorname{supp} g_m \cap \operatorname{supp} g'_m = \emptyset$, hence $g_m g'_m \equiv 0$. This implies that $f_m f'_m = \Psi^{-1}(g_m g'_m) \equiv 0$, a contradiction to 3.2.4 (ii), i.e. $f_m(p) f'_m(p) = 1$. □

Remark 3.2.10. Note that lemma 3.2.8 implies that an equivalence class of characteristic sequences at a point in X maps under an algebra isomorphism to an equivalence class of characterstic sequences at a point in Y. As such, we can identify points in X and Y with equivalence classes of characteristic sequences, and such an identification is compatible with the action of algebra isomorphisms.

Again we can prove that algebra isomorphisms are pullbacks by diffeomorphisms in the case of smooth functions on Hausdorff manifolds:

Theorem 3.2.11. *Let X and Y be Hausdorff manifolds and $\Psi : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$ an algebra isomorphism. Then Ψ is given by composition with a unique diffeomorphism $\psi : Y \to X$, i.e.*

$$\Psi(f) = f \circ \psi \quad \forall f \in \mathcal{C}^\infty(X).$$

Moreover, $\dim X = \dim Y$ *in this case.*

Proof. For any $q \in Y$, we may choose a characteristic sequence $(g_n)_n$ of functions on Y at q using proposition 3.2.4 (iii). By lemma 3.2.8 (i) we know that $(\Psi^{-1}(g_n))_n$ then is a characteristic sequence of functions on X at a point $p \in X$. By 3.2.8 (ii), p is independent of the choice of $(g_n)_n$, thus by defining $\psi(q) := p$ we obtain a map

$$\psi : Y \to X.$$

We shall see that ψ meets the demands. First of all, Ψ is is given by the composition with ψ, i.e. $\Psi(f)(q) = f(\psi(q))$ for all $f \in \mathcal{C}^\infty(X)$ and all $q \in Y$:

We observe a simpler case first. Let $h \in \mathcal{C}^\infty(X)$ such that $\Psi(h)(q) = 0$, then also $h(\psi(q)) = 0$: Suppose $h(\psi(q)) \neq 0$, then $h|_V$ is vanishing nowhere on an open neighborhood V of $\psi(q)$ in X. Since $(\Psi^{-1}(g_n))_n$ is a characteristic sequence of functions at $\psi(q) = p$, we can choose $m \in \mathbb{N}$ such that $\operatorname{supp}\Psi^{-1}(g_m)$ is a compact subset of V and define the smooth function

$$\begin{array}{rcl} f: X & \to & \mathbb{C} \\ x & \mapsto & \begin{cases} \frac{\Psi^{-1}(g_m)(x)}{h(x)} & x \in V \\ 0 & x \in X \setminus \operatorname{supp}\Psi^{-1}(g_m) \end{cases} . \end{array}$$

This implies $1 \overset{3.2.4\,\text{(ii)}}{=} g_m(q) = \Psi(\Psi^{-1}(g_m))(q) = \Psi(hf)(q) = \Psi(h)(q)\Psi(f)(q)$, a contradiction to $\Psi(h)(q) = 0$.

For the general case take any $f \in \mathcal{C}^\infty(X)$. We have

$$\Psi(f - \Psi(f)(q)1)(q) = \Psi(f)(q) - \Psi(f)(q) = 0$$

as $\Psi(1) = 1$ for any algebra isomorphism. By the previous argument this yields $(f - \Psi(f)(q)1)(\psi(q)) = 0$. Thus

$$\Psi(f)(q) = f(\psi(q)).$$

Uniqueness of ψ follows as in the proof of theorem 3.1.8 (using that X is Hausdorff and a bump function of proposition 3.2.9).

Analogously, Ψ^{-1} is given by a composition with a map $\sigma : X \to Y$. In particular, $g = \Psi(\Psi^{-1}(g)) = \Psi(g \circ \sigma) = g \circ \sigma \circ \psi$ for all $g \in \mathcal{C}^\infty(Y)$, hence $\sigma \circ \psi = \mathrm{id}_Y$. Analogously, $\psi \circ \sigma = \mathrm{id}_X$. Thus $\sigma = \psi^{-1}$.

Both, ψ and ψ^{-1} are smooth by proposition 3.1.9, again as in the proof of 3.1.8. Therefore ψ is the required diffeomorphism.

That $\dim X = \dim Y$ is also shown as in the proof of theorem 3.1.8. □

Obviously, this last proof is quite similar to the proof of theorem 3.1.8, although it uses characteristic sequences instead of non-zero multiplicative linear functionals to interpret the points in the manifolds.

Remark 3.2.12. Note that we may use $\mathbb{R}$ instead of $\mathbb{C}$ following the same proofs. Furthermore, an analogous result holds true for algebra isomorphisms of smooth functions with compact support. See [Mrč05] for more details.

Mrčun and Šemrl further showed in [MŠ07] that a similar result holds for differentiable instead of smooth functions. More precisely, they proved that any multiplicative bijection between algebras of differentiable functions (defined on differentiable manifolds of positive dimension) is automatically an algebra isomorphism, which again is given by composition with a unique diffeomorphism:

Theorem 3.2.13. *Let X and Y be Hausdorff $\mathcal{C}^r$-manifolds of positive dimension, $1 \leq r < \infty$. Then for any multiplicative bijection $\mathcal{B} : \mathcal{C}^r(X) \to \mathcal{C}^r(Y)$ there exists a unique $\mathcal{C}^r$-diffeomorphism $\beta : Y \to X$ such that*

$$\mathcal{B}(f) = f \circ \beta \quad \forall f \in \mathcal{C}^r(X).$$

In particular, the map $\mathcal{B}$ is an algebra isomorphism.

They also mention, without further comment, that their method used in [MŠ07] does not work for $r = \infty$.

Chapter 4

Some (Semi-)Riemannian Geometry

In what follows, we will discuss the geometrical and topological setting of the chapters to come, namely finite dimensional, smooth, second countable and Hausdorff manifolds which are equipped with a Riemannian metric. Readers who are familiar with the exponential map, normal and convex neighborhoods, normal coordinates etc. may omit section 4.1 in this summary and only come back to it later if necessary. For a general introduction to semi-Riemannian geometry see e.g. [O'N83].

Besides the short introduction to Riemannian geometry in 4.1, we will prove in section 4.2 that the square of any Riemannian distance function on a manifold X is smooth in both variables on a neighborhood of the diagonal $\triangle_X \subseteq X \times X$. This is a remarkable result in its own right, although not fundamental in what follows.

Rather fundamental for the proofs to come is section 4.3 where we investigate the link between the Riemannian and the Euclidean metric for smooth submanifolds X of $\mathbb{R}^n$. This is interesting because the Whitney embedding theorem in section 4.4 states that any finite dimensional manifold (in particular any Riemannian manifold) can be viewed as a submanifold of $\mathbb{R}^m$.

4.1 (Semi-)Riemannian manifolds

4.1.1 Metric tensors

In order to be able to investigate intrinsic properties of a smooth manifold as defined in 2.1.3, e.g. curvature and length, we have to equip a smooth manifold X with a scalar product on the tangent space:

Recall that a bilinear form $b : V \times V \to \mathbb{R}$ on a finite dimensional vector space V is called *non-degenerate* if

$$b(v, w) = 0 \ \forall w \in V \Rightarrow v = 0.$$

and *positive* resp. *negative definite* if

$$b(v,v) > 0 \text{ resp. } b(v,v) < 0 \ \forall v \in V, v \neq 0.$$

It is furthermore called *symmetric* if

$$b(v,w) = b(w,v) \ \forall v,w \in V.$$

The *index* of a symmetric bilinear form b on V is defined by

$$\nu := \max\{\dim W \mid W \text{ is a subspace of } V \text{ such that } \ b|_W \text{ is negative definite}\}.$$

Definition 4.1.1. Let X be a smooth manifold. A *metric tensor* g on X is a symmetric and non-degenerate smooth $\binom{0}{2}$-tensor field with constant index.

This means that $g \in \mathcal{T}_2^0(X)$, and $g(p) : T_pX \times T_pX \to \mathbb{R}$ is symmetric and non-degenerate for all $p \in X$.

Note that we will use Einstein's *summation convention* throughout. It says that if an index appears twice, as a lower and an upper index, then we sum over all its possible values.

Recall that, w.r.t. a given chart $(u = (x^1, ..., x^n), U)$ at p in X, the set $\{\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p\}$ denotes the basis of the tangent space T_pX and $\{dx^1\big|_p, ..., dx^n\big|_p\}$ the basis of the dual $(T_pX)^*$.

Moreover, $\otimes$ is the tensor product as defined in 2.1.9.

Remark 4.1.2 (Notation)**.** Often we will write $\langle\, .\,,.\rangle$ for $g(p)$ to emphasize that it is a scalar product on T_pX. Also vector fields can be inserted in g, i.e. $g(V,W) = \langle V,W\rangle \in \mathcal{C}^\infty(X,\mathbb{R})$ for $V,W \in \mathfrak{X}(X)$.

For a chart (u,U) of X, $u = (x^1, ..., x^n)$, the components g_{ij} of g w.r.t. this chart are given by $g_{ij} := \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$. Thus for $V = V^i \frac{\partial}{\partial x^i}, W = W^j \frac{\partial}{\partial x^j} \in \mathfrak{X}(X)$ we have that

$$g(V,W) = \langle V,W\rangle = V^i W^j g_{ij}$$
$$g|_U = g_{ij}\, dx^i \otimes dx^j.$$

The matrix of $g(p)$ w.r.t. an orthonormal basis $\{e_1, ..., e_n\}$ in T_pX is diagonal and invertible, since g is non-degenerate. We have that

$$g(p)(e_i, e_j) = \delta_{ij}\varepsilon_i,$$

where δ_{ij} is the Kronecker delta and $\varepsilon_i = g(p)(e_i,e_i) = \pm 1$. W.l.o.g. we assume that $\{e_1, ..., e_n\}$ is ordered in a way such that the *signature* $(\varepsilon_1, ..., \varepsilon_n)$ starts with the minus signs.

Definition 4.1.3. The *norm* of a vector $v \in T_pX$ is defined as

$$|v| := |g(p)(v,v)|^{\frac{1}{2}}.$$

Semi-Riemannian manifolds are now defined as follows:

Definition 4.1.4. A *semi-Riemannian manifold* is a smooth manifold X endowed with a metric tensor g.

It is called *Riemannian manifold* if $\nu = 0$ and *Lorentzian manifold* if $\nu = 1$ and $n \geq 2$.

We will denote a semi-Riemannian manifold by either X or (X, g) in the following.

Remark 4.1.5. Using a partition of unity, it is easy to prove that every smooth manifold (second countable and Hausdorff) admits a Riemannian metric tensor, cf. [O'N83], lemma 5.25. In particular, every such manifold is metrizable by [O'N83], proposition 5.18.

4.1.2 Riemannian distance

On a Riemannian manifold we can measure the distance of two points via curve length on the manifold:

Definition 4.1.6. Let (X, g) be a semi-Riemannian manifold and $c : [a, b] \to X$ a piecewise smooth curve on X. The *arc length* of c is defined by

$$L(c) := \int_a^b |\langle c'(t), c'(t)\rangle|^{\frac{1}{2}}$$

with $|\langle c'(t), c'(t)\rangle|^{\frac{1}{2}} = \left|\sum_{i,j=1}^n g_{ij}(c(t))\frac{d(x^i \circ c)}{dt}(t)\frac{d(x^j \circ c)}{dt}(t)\right|^{\frac{1}{2}}$ where $(x^1, ..., x^n)$ denotes a chart.

Definition 4.1.7. Let (X, g) be a connected Riemannian manifold, $p, q \in X$ and $\Omega(p, q)$ the set of piecewise smooth curves from p to q. Then the *Riemannian distance* $d_g(p, q)$ from p to q is defined as

$$d_g(p, q) \ := \inf_{c\in\Omega(p,q)} L(c).$$

It can be shown that d_g is a metric on X that is compatible with the topology, cf. [O'N83], proposition 5.18. Moreover, two Riemannian distances are locally equivalent:

Lemma 4.1.8. *Let X be a smooth manifold and g, h two Riemannian metrics that induce the respective Riemannian distances d_g, d_h. Then*

$$\forall K, L \subset\subset X\, \exists C > 0 \text{ such that } d_h(p, q) \leq C d_g(p, q)\, \forall p \in K, q \in L.$$

Proof. See [GKOS01], lemma 3.2.5, or [Nig06], lemma 3.13. □

4.1.3 The Levi-Civita connection

The Levi-Civita connection on a semi-Riemannian manifold X is denoted by $\nabla : \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathfrak{X}(X)$, where $\mathfrak{X}(X) := \Gamma(X, TX)$ is the set of smooth vector fields on X. It exists and is uniquely determined by the properties (∇1)–(∇5) for $U, V, W \in \mathfrak{X}(X)$, see e.g. [O'N83], theorem 3.11:

(∇1) $\nabla_U V$ is $\mathcal{C}^\infty(X)$-linear in U
(∇2) $\nabla_U V$ is $\mathbb{R}$-linear in V
(∇3) $\nabla_U(fV) = U(f)V + f\nabla_U V$ for all $f \in \mathcal{C}^\infty(X)$
(∇4) $[U, V] = \nabla_U V - \nabla_V U$
(∇5) $W\langle U, V\rangle = \langle \nabla_W U, V\rangle + \langle U, \nabla_W V\rangle$.

If a map ∇ just satisfies (∇1)–(∇3) it is called a (linear) *connection* on the manifold X. The vector field $\nabla_U V$ is called *covariant derivative* of V w.r.t. U for the connection ∇.

4.1.4 Christoffel symbols

Definition 4.1.9. Let X be a semi-Riemannian manifold and (u, U) a chart of X, $u = (x^1, ..., x^n)$. The *Christoffel symbols* w.r.t. u are the smooth functions ${}^u\Gamma^k_{ij} : U \to \mathbb{R}$ that satisfy

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = {}^u\Gamma^k_{ij} \frac{\partial}{\partial x^k} \qquad \forall 1 \leq i, j \leq n. \tag{4.1}$$

Remark 4.1.10 (Basic properties of ${}^u\Gamma^k_{ij}$)**.**

(i) Property (∇4) implies that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$ for all $1 \leq i, j \leq n$, hence ${}^u\Gamma^k_{ij} = {}^u\Gamma^k_{ji}$ for all $1 \leq i, j, k \leq n$.

(ii) The Christoffel symbols are not the components of a tensor field w.r.t. the local coordinate system and so do not transform like a tensor under coordinate transformations.

(iii) If ${}^pw = ({}^pw^1, ..., {}^pw^n)$ denotes a normal coordinate system at $p \in X$ (see remark 4.1.18 below), then

$${}^{{}^pw}\Gamma^k_{ij}(p) = 0 \qquad \forall 1 \leq i, j, k \leq n,$$

cf. [O'N83], proposition 3.33.

4.1.5 Geodesics

There is a special type of curve on a semi-Riemannian manifold X that is fundamental for the geometry of X:

Definition 4.1.11. Let X be a semi-Riemannian manifold and $I \subseteq \mathbb{R}$ an interval. A *geodesic* is a curve $\gamma : I \to X$, whose tangent vector field γ' is parallel along γ, i.e. $\gamma'' := \nabla_{\gamma'}\gamma' = 0$.

A geodesic is uniquely determined by an ordinary differential equation of second order, and hence by its initial conditions $c(0) = p$ and $c'(0) = v$:

Proposition 4.1.12 (Geodesic equation)**.** *Let X be a semi-Riemannian manifold and $(u = (x^1, ..., x^n), U)$ a chart of X. A curve $c : I \to U$ is a geodesic of X if and only if its local coordinate functions $x^i \circ c$ satisfy the following* geodesic equation

$$\frac{d^2(x^k \circ c)}{dt^2} + \sum_{i,j=1}^{n} {}^u\Gamma^k_{ij}(c) \frac{d(x^i \circ c)}{dt} \frac{d(x^j \circ c)}{dt} = 0$$

for $1 \leq k \leq n$.

Proof. See [O'N83], corollary 3.21. □

As we will see later, geodesics describe locally the shortest curve between two points. They are very important for many applications, e.g. general relativity.

4.1.6 The exponential map

The exponential map is defined as follows.

Definition 4.1.13. Let X be a semi-Riemannian manifold, $p \in X$ and

$$\begin{aligned}\mathcal{D}_p := \{v \in T_pX \,|\, &\text{the geodesic } c_v \text{ with initial conditions} \\ &c_v(0) = p \text{ and } c_v'(0) = v \text{ is defined on } [0,1]\}.\end{aligned}$$

The *exponential map of X at p* is defined as

$$\begin{aligned}\exp_p : T_pX \supseteq \mathcal{D}_p &\rightarrow X \\ v &\mapsto c_v(1).\end{aligned}$$

A well-known result in semi-Riemannian geometry states that $\exp_p$ is a local diffeomorphism:

Theorem 4.1.14. *Let X be a semi-Riemannian manifold and $p \in X$. Then there exists a neighborhood V of 0 in T_pX and a neighborhood U of p in X such that $\exp_p : V \rightarrow U$ is a diffeomorphism.*

Proof. See [O'N83], proposition 3.30. □

The exponential maps $\exp_p$ at different points $p \in X$ can also be put together nicely:

Definition 4.1.15. Let X be a semi-Riemannian manifold. Then

$$\mathcal{D} := \{v \in TX \,|\, c_v \text{ exists at least on } [0,1]\},$$

i.e. $\mathcal{D}_p = \mathcal{D} \cap T_pX$ for each $p \in X$. The map E is defined via the footpoint map $\pi : TX \rightarrow X$ ($v \in T_pX$ is mapped to $\pi(v) = p$) by

$$\begin{aligned}\mathrm{E} : TX \supseteq \mathcal{D} &\rightarrow X \times X \\ v &\mapsto (\pi(v), \exp_{\pi(v)}(v)).\end{aligned}$$

Obviously, $\mathcal{D}$ is the maximal domain of E. It can be shown that $\mathcal{D}$ is open in TX and that $\mathcal{D}_p$ is open and star-shaped at $0 \in T_pX$ for all $p \in X$. If X is geodesically complete then $\mathcal{D} = TX$. Moreover, E is a local diffeomorphism:

Theorem 4.1.16. *Let X be a semi-Riemannian manifold. Then the map $\mathrm{E} : \mathcal{V} \rightarrow \mathcal{U}$ is a diffeomorphism of a neighborhood $\mathcal{V}$ of $(TX)_0$ in TX onto some neighborhood $\mathcal{U}$ of $\triangle_X$ in $X \times X$.*

Here, $\triangle_X := \{(p,p) \,|\, p \in X\}$ is called the *diagonal* of X, and $(TX)_0 := \{0_p \,|\, p \in X\}$ the *zero section* of TX.

Proof. See [O'N83], lemma 5.6 and the remark thereafter, or [Kun06], theorem 2.4.6. □

4.1.7 Normal neighborhoods

Definition 4.1.17. Let X, U and V be as in theorem 4.1.14 and suppose furthermore that V is star-shaped around 0. Then U is called a *normal neighborhood* of p.

Remark 4.1.18. On a normal neighborhood U of $p \in X$ there exists a distinguished type of coordinates, so-called *normal coordinates*, which, as we shall see, are useful for many calculations. A normal coordinate system is induced by the exponential map $\exp_p : V \to U$ which is a diffeomorphism by theorem 4.1.14. To each $q \in U$ we assign the coordinates of $\exp_p^{-1}(q) \in V \subseteq T_pX$ w.r.t. an orthonormal basis $\{e_1, ..., e_n\}$ of T_pX. More precisely,

$$\exp_p^{-1}(q) = \sum_{i=1}^{n} {}^p w^i(q) e_i \qquad \forall q \in V,$$

where ${}^p w = ({}^p w^1, ..., {}^p w^n)$ denote the normal coordinate system at p.

Definition 4.1.19. Let (X, g) be a semi-Riemannian manifold, $p \in X$ and U a normal neighborhood of p. Then the function

$$\begin{array}{rcl} r : U & \to & \mathbb{R}^+ \\ q & \mapsto & r(q) := |\exp_p^{-1}(q)| \end{array}$$

is called *radius function* on U of p. Recall that $|v| := |g(p)(v, v)|^{\frac{1}{2}}$.

In terms of normal coordinates $({}^p w^1, ..., {}^p w^n)$, the radius function r at p is given by

$$r = \left| -\sum_{i=1}^{\nu} ({}^p w^i)^2 + \sum_{i=\nu+1}^{n} ({}^p w^i)^2 \right|^{\frac{1}{2}},$$

where ν denotes the index of (X, g). In particular, r is smooth whereever it is non-zero, i.e. everywhere except at p and the local null-cone.

Proposition 4.1.20. *Let X be a semi-Riemannian manifold. If U is a normal neighborhood of $p \in X$, then for each point $q \in U$ there exists a unique geodesic $\gamma_{pq} : [0, 1] \to U$ from p to q in U. Furthermore, $\gamma'_{pq}(0) = \exp_p^{-1}(q)$ and*

$$L(\gamma_{pq}) = r(q) = |\exp_p^{-1}(q)|,$$

where r denotes the radius function on U of p. This curve γ_{pq} is called a radial geodesic.

Proof. See [O'N83], proposition 3.31 and lemma 5.13. □

An even stronger result holds for Riemannian manifolds:

Proposition 4.1.21. *Let (X, g) be a Riemannian manifold and $p \in X$.*

(i) For sufficiently small $\varepsilon > 0$, the ε-neighborhood

$$U_\varepsilon(p) := \{q \in X \mid d_g(p, q) < \varepsilon\}$$

is normal.

(ii) For a normal ε-neighborhood $U_\varepsilon(p)$ the radial geodesic γ_{pq} from p to $q \in U_\varepsilon(p)$ is the unique shortest curve in X from p to q. In particular

$$L(\gamma_{pq}) = r(q) = |\exp_p^{-1}(q)| = d(p,q).$$

Sketch of proof. We may assume for a moment that there exists a normal (even convex) neighborhood U of p in X, a fact which will be proved explicitly later in theorem 4.2.4. Then for $\varepsilon > 0$ sufficiently small the neighborhood

$$V_\varepsilon(p) := \{v \in T_pX \mid |v| < \varepsilon\}, \tag{4.2}$$

with $|v| = |g(p)(v,v)|^{\frac{1}{2}}$, of 0 in T_pX is contained in $V = \exp_p^{-1}(U)$. It can be shown that $U_\varepsilon(p)$ as in (i) is the image of such a $V_\varepsilon(p)$ under the exponential map $\exp_p$, thus obviously a normal neighborhood of p. Furthermore it can be shown that (ii) holds for these ε. See [O'N83], proposition 5.16, for more details. □

4.1.8 Convex neighborhoods

Definition 4.1.22. An open subset U of a semi-Riemannian manifold X is called *geodesically convex* if U is a normal neighborhood of each of its points.

We will see in theorem 4.2.4 below that every point p of a semi-Riemannian manifold X possesses a base of convex neighborhoods. See also [O'N83], proposition 5.7.

Remark 4.1.23 (Definition of convexity)**.** Be aware, that different authors use different notions for convexity. For example in [GKM68], section 5.2, a convex set U is an open subset of a connected Riemannian manifold, such that for any two points $p, q \in G$ there exists a geodesic γ_{pq} with $L(\gamma_{pq}) = d_g(p,q)$ that lies entirely in G (not necessarily unique though).

On the other hand, e.g. in [dC92], theorem 3.7 resp. remark 3.8, geodesically convex sets as in definition 4.1.22 are called *totally normal*.

We mainly follow the terminology of [O'N83]. Additional definitions of convexity are introduced in the following section 4.2 where needed.

4.2 Smoothness of the Riemannian distance

In this section we will prove that the Riemannian distance d_g on a Riemannian manifold (X,g) is smooth on a neighborhood U of the diagonal $\triangle_X$ except on the diagonal itself (because $\sqrt{\ }$ is not smooth in 0), and that ${d_g}^2$ is smooth on all of U.

The proofs strongly rely on normal coordinates on convex neighborhoods as introduced in remark 4.1.18 and definition 4.1.22. In order to show that every point in a semi-Riemannian manifold possesses a base of convex neighborhoods and that all subsets of a certain type are also convex, we will require certain smoothness properties of Christoffel symbols and normal coordinate charts. Therefore we begin by investigating a few of their properties.

Coming back to Riemannian manifolds, it follows that sufficiently small balls $B_r(p) \subseteq X$ (and also those contained in them) are even *strongly convex* – a concept that is defined in 4.2.6. As each point has a strongly convex neighborhood we can define a strictly positive and also continuous so-called *convexity radius* κ on X. This finally allows us to define an open neighborhood on which the Riemannian distance is smooth in the sense mentioned before.

Be aware that both terms, strongly convex and convexity radius, are used differently by some authors.

4.2.1 Christoffel symbols with respect to normal coordinates

The following lemma shows how the Christoffel symbols transform for different charts.

Lemma 4.2.1. *Let X be a semi-Riemannian manifold and $(u = (x^1, ..., x^n), U)$ and $(v = (y^1, ..., y^n), V)$ different charts of X at $p \in U \cap V$. Then for all $1 \leq i, j, k \leq n$,*

$$ {}^u\Gamma^k_{ij}(p) = \frac{\partial x^k}{\partial y^l}\bigg|_p \left(\frac{\partial^2 y^l}{\partial x^i \partial x^j}\bigg|_p + {}^v\Gamma^l_{rs}(p)\, \frac{\partial y^r}{\partial x^i}\bigg|_p \frac{\partial y^s}{\partial x^j}\bigg|_p \right). \tag{4.3}$$

Proof. By [O'N83], lemma 1.14, $\frac{\partial}{\partial x^m} = \frac{\partial y^r}{\partial x^m}\frac{\partial}{\partial y^r}$ for all $1 \leq m \leq n$. Thus by definition 4.1.9,

$$ {}^u\Gamma^m_{ij}\frac{\partial y^l}{\partial x^m}\frac{\partial}{\partial y^l} \overset{(4.1)}{=} \nabla_{\frac{\partial y^r}{\partial x^i}\frac{\partial}{\partial y^r}} \frac{\partial y^s}{\partial x^j}\frac{\partial}{\partial y^s} \overset{(\nabla 1)}{=} \frac{\partial y^r}{\partial x^i}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial y^s}{\partial x^j}\frac{\partial}{\partial y^s}. \tag{4.4}$$

On the other hand, also $\frac{\partial}{\partial y^r} = \frac{\partial x^h}{\partial y^r}\frac{\partial}{\partial x^h}$ for all $1 \leq r \leq n$, hence

$$\begin{aligned} {}^u\Gamma^m_{ij}\frac{\partial y^l}{\partial x^m}\frac{\partial}{\partial y^l} &\overset{(4.4),(\nabla 3)}{=} \frac{\partial y^r}{\partial x^i}\left(\frac{\partial x^h}{\partial y^r}\frac{\partial^2 y^s}{\partial x^j \partial x^h}\frac{\partial}{\partial y^s} + \frac{\partial y^s}{\partial x^j}\nabla_{\frac{\partial}{\partial y^r}}\frac{\partial}{\partial y^s}\right) \\ &\overset{(4.1)}{=} \frac{\partial y^r}{\partial x^i}\left(\frac{\partial x^h}{\partial y^r}\frac{\partial^2 y^s}{\partial x^j \partial x^h}\frac{\partial}{\partial y^s} + \frac{\partial y^s}{\partial x^j}{}^v\Gamma^l_{rs}\frac{\partial}{\partial y^l}\right) \\ &= \left(\frac{\partial y^r}{\partial x^i}\frac{\partial x^h}{\partial y^r}\frac{\partial^2 y^l}{\partial x^j \partial x^h} + \frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}{}^v\Gamma^l_{rs}\right)\frac{\partial}{\partial y^l}. \end{aligned} \tag{4.5}$$

In particular, since the $\frac{\partial}{\partial y^l}$ form a basis in the tangent space,

$$\begin{aligned} \frac{\partial y^l}{\partial x^m}{}^u\Gamma^m_{ij} &\overset{(4.5)}{=} \underbrace{\frac{\partial y^r}{\partial x^i}\frac{\partial x^h}{\partial y^r}}_{=\delta_{ih}}\frac{\partial^2 y^l}{\partial x^j \partial x^h} + \frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}{}^v\Gamma^l_{rs} \\ &= \frac{\partial^2 y^l}{\partial x^j \partial x^i} + \frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}{}^v\Gamma^l_{rs}, \end{aligned} \tag{4.6}$$

which by multiplication with $\frac{\partial x^k}{\partial y^l}$ and summation over l finally leads to

$$ {}^u\Gamma^k_{ij} = \underbrace{\frac{\partial x^k}{\partial y^l}\frac{\partial y^l}{\partial x^m}}_{=\delta_{km}}{}^u\Gamma^m_{ij} \overset{(4.6)}{=} \frac{\partial x^k}{\partial y^l}\left(\frac{\partial^2 y^l}{\partial x^j \partial x^i} + {}^v\Gamma^l_{rs}\frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}\right). \qquad \square $$

Remark 4.2.2 (Orthonormal frame fields)**.** Let X be a semi-Riemannian manifold of dimension n and $o \in X$. An orthonormal basis $\{e_1, ..., e_n\}$ of T_oX is called a *frame* on X at o. A *frame field* $\{E_1, ..., E_n\}$ consists of smooth orthonormal vector fields $E_i \in \mathfrak{X}(X)$, $1 \leq i \leq n$, and hence assigns a frame $\{ E_1|_o, ..., E_n|_o\}$ to each point $o \in X$.

Frame fields may not exist globally on X. However, locally on a (normal) neighborhood U of $o \in X$, they may be constructed by parallel transport along geodesics by [O'N83], corollary 3.46. Any vector field $W \in \mathfrak{X}(X)$ can be written as a sum

$$W = \sum_{i=1}^{n} \varepsilon_i \langle W, E_i \rangle E_i,$$

where $\varepsilon_i := \langle E_i, E_i \rangle$.

For any $p \in U$ we denote the *normal coordinates* on a neighborhood W_p of p w.r.t. the orthonormal frame field $E_1, ..., E_n$ by ${}^p w = ({}^p w^1, ..., {}^p w^n)$. Since $\exp_p^{-1}(q) = {}^p w^i(q)\ E_i|_p \in T_pX$ for any $q \in W_p$, we have that

$${}^p w(q) := \Big(\varepsilon_i \langle \exp_p^{-1}(q),\, E_i|_p \rangle \Big)_{i=1}^{n} = ({}^p w^1(q), ..., {}^p w^n(q))). \tag{4.7}$$

By theorem 4.1.16, the exponential map $\mathrm{E} : \mathcal{V} \to \mathcal{U}$, $\mathrm{E}(v_p) := (p, \exp_p(v))$, is a diffeomorphism from a neighborhood $\mathcal{V}$ of TX_0 in TX onto a neighborhood $\mathcal{U}$ of $\triangle_X$ in $X \times X$. For $p \in X$ let

$$\mathcal{U}(p) := \{ q \in X \,|\, (p, q) \in \mathcal{U} \} \subseteq X.$$

For Christoffel symbols w.r.t. normal coordinates additional properties hold:

Lemma 4.2.3. *Let X be a semi-Riemannian manifold, $o \in X$ and $\{E_1, ..., E_n\}$ a local frame field on a neighborhood $\widetilde{U}$ of o such that $\widetilde{U} \times \widetilde{U} \subseteq \mathcal{U}$ ($\mathcal{U}$ as in 4.1.16). Let U be a neighborhood of o with $\overline{U} \subset\subset \widetilde{U}$. Then the following properties hold:*

(i) The normal coordinate charts ${}^p w$ are defined on $\widetilde{U}$ for all $p \in \widetilde{U}$.

(ii) The map $(p, q) \mapsto {}^p w(q)$, $\widetilde{U} \times \widetilde{U} \to \mathbb{R}^n$ is smooth.

(iii) There exists a neighborhood V of 0 in $\mathbb{R}^n$, such that the map

$$\begin{aligned} f : U \times U \times V &\to \mathbb{R}^n \\ (p, q, r) &\mapsto f_{pq}(r) := ({}^p w \circ ({}^q w)^{-1})(r) \end{aligned}$$

is smooth.

(iv) There exists a neighborhood $W \subseteq U$ of o, such that

$${}^{{}^p w}\Gamma^k_{ij}(q) = D_l f^k_{po}\big|_{{}^o w(q)} \Big(D_{ij} f^l_{op}\big|_{{}^p w(q)} + {}^{{}^o w}\Gamma^l_{rs}(q)\ D_i f^r_{op}\big|_{{}^p w(q)}\ D_j f^s_{op}\big|_{{}^p w(q)} \Big)$$

for all $p, q \in W$.

(v) For each $\varepsilon > 0$ there exists a neighborhood $U_\varepsilon \subseteq U$ of o, such that

$$\Big| {}^{{}^p w}\Gamma^k_{ij}(q) - {}^{{}^o w}\Gamma^k_{ij}(o) \Big| < \varepsilon$$

for all $p, q \in U_\varepsilon$ and all $1 \leq i, j, k \leq n$.

Proof. (i) First note that such a local frame $\{E_1, ..., E_n\}$ on some neighborhood $\widetilde{U}$ of o exists by remark 4.2.2. As $\widetilde{U} \times \widetilde{U} \subseteq \mathcal{U}$ by assumption, we have that $\widetilde{U} \subseteq \mathcal{U}(p)$ for all $p \in \widetilde{U}$. In particular, the inverse of the exponential map E exists on $\widetilde{U} \times \widetilde{U}$ and for all $p \in \widetilde{U}$ we have

$$\mathrm{E}^{-1}\big|_{\{p\}\times\mathcal{U}(p)} = \exp_p^{-1} \tag{4.8}$$

on $\mathcal{U}(p) \supseteq \widetilde{U}$ by theorem 4.1.16. Thus the charts ${}^p w$ are defined on $\widetilde{U}$.

(ii) By the definition (4.7) of ${}^p w$ in remark 4.2.2, (i) implies that

$${}^p w(q) = \Big(\varepsilon_i \langle \exp_p^{-1}(q),\, E_i|_p\rangle\Big)_{i=1}^n \overset{(4.8)}{=} \Big(\varepsilon_i \langle \mathrm{E}^{-1}(p,q),\, E_i|_p\rangle\Big)_{i=1}^n$$

is smooth on $\widetilde{U} \times \widetilde{U}$.

(iii) Since by assumption all E_i, $1 \le i \le n$, are smooth vector fields on $\widetilde{U}$, the map $\tau : \widetilde{U} \times \mathbb{R}^n \to TX$ defined by $\tau(q,r) := \sum_{i=1}^n r^i\, E_i|_q$ is smooth, too. Moreover, $\tau(\overline{U} \times \{0\}) \subseteq \mathcal{V}$. Thus $\tau^{-1}(\mathcal{V})$ is an open neighborhood of the compact set $\overline{U} \times \{0\}$ by continuity of τ. In particular, there is a neighborhood V of 0 in $\mathbb{R}^n$ that satisfies $\tau(U \times V) \subseteq \mathcal{V}$. Finally, f is well-defined and smooth on $U \times U \times V$, since

$$\begin{aligned} f_{pq}(r) &= ({}^p w \circ ({}^q w)^{-1})(r) \\ &= {}^p w\left(\exp_q\left(\sum_{i=1}^n r^i\, E_i|_q\right)\right) \\ &= {}^p w\left((\mathrm{pr}_2 \circ \mathrm{E} \circ \tau)(q,r)\right) \end{aligned}$$

is smooth by (ii) and the fact that pr_2, E and τ are smooth.

(iv) Let V be a neighborhood of $0 \in \mathbb{R}^n$ as in (iii). Obviously, ${}^o w(o) = 0 \in V$. Thus by continuity in both variables by (ii) there exist neighborhoods W_1 and W_2 of o such that ${}^p w(q) \in V$ $\forall p \in W_1$ $\forall q \in W_2$. Let W be the neighborhood

$$W := W_1 \cap W_2 \subseteq U$$

of o. Since W is contained in the domains of all charts ${}^x w$ $(x \in U)$, the Christoffel symbols ${}^{^p w}\Gamma^k_{ij}(q)$ and ${}^{^o w}\Gamma^l_{rs}(q)$ exist for all $p, q \in W$. Moreover, ${}^p w(q), {}^o w(q) \in V$ by the above construction. Hence we may apply lemma 4.2.1 and (iii) to $(p, o, {}^o w(q))$ and $(o, p, {}^p w(q))$ to finally obtain

$$\begin{aligned} {}^{^p w}\Gamma^k_{ij}(q) &\overset{(4.3)}{=} \frac{\partial\, {}^p w^k}{\partial\, {}^o w^l}\bigg|_q \left(\frac{\partial^2\, {}^o w^l}{\partial\, {}^p w^i\, \partial\, {}^p w^j}\bigg|_q + {}^{^o w}\Gamma^l_{rs}(q)\, \frac{\partial\, {}^o w^r}{\partial\, {}^p w^i}\bigg|_q \frac{\partial\, {}^o w^s}{\partial\, {}^p w^j}\bigg|_q \right) \\ &\overset{\text{(iii)}}{=} D_l f^k_{po}\big|_{{}^o w(q)} \left(D_{ij} f^l_{op}\big|_{{}^p w(q)} + {}^{^o w}\Gamma^l_{rs}(q)\, D_i f^r_{op}\big|_{{}^p w(q)}\, D_j f^s_{op}\big|_{{}^p w(q)} \right) \end{aligned}$$

for all $p, q \in W$.

(v) Let V as in (iii) and W as in (iv). Again, f_{op} and f_{po} are smooth on $W \times W \times V$. Moreover, $f_{oo}(r) = ({}^o w \circ ({}^o w)^{-1})(r) = r$ for all $r \in V$. Thus

$$D_l f^k_{oo} = \delta_{kl} \quad \text{and} \quad D_{ij} f^l_{oo} = 0 \qquad \forall 1 \le i, j, k, l \le n.$$

Obviously, ${}^{p}{}^{w}\Gamma^k_{ij}(q)$, ${}^{o}{}^{w}\Gamma^k_{ij}(q)$ and all terms in (iv) are well-defined for $p, q \in W$. By smoothness of f, we conclude that

$$\begin{aligned} {}^{p}{}^{w}\Gamma^k_{ij}(q) &\overset{\text{(iv)}}{=} \underbrace{D_l f^k_{po}\big|_{{}^{o}w(q)}}_{\to\delta_{lm}} \Big(\underbrace{D_{ij} f^l_{op}\big|_{{}^{p}w(q)}}_{\to 0} + {}^{o}{}^{w}\Gamma^l_{rs}(q)\, \underbrace{D_i f^r_{op}\big|_{{}^{p}w(q)}}_{\to\delta_{ir}} \underbrace{D_j f^s_{op}\big|_{{}^{p}w(q)}}_{\to\delta_{js}} \Big). \\ &\to {}^{o}{}^{w}\Gamma^k_{ij}(q) \end{aligned}$$

for $p \to o$ and $q \in W$. Since the Christoffel symbols are smooth maps, we also have that ${}^{o}{}^{w}\Gamma^k_{ij}(q) \to {}^{o}{}^{w}\Gamma^k_{ij}(o)$. Thus for each $\varepsilon > 0$ there exists a neighborhood U_ε of o such that

$$\left| {}^{p}{}^{w}\Gamma^k_{ij}(q) - {}^{o}{}^{w}\Gamma^k_{ij}(o) \right| < \varepsilon$$

for all $p, q \in U_\varepsilon$ and all $1 \le i, j, k \le n$. □

4.2.2 Convex neighborhoods with respect to normal coordinates

Again, let $({}^{p}w = ({}^{p}w^1, ..., {}^{p}w^n), W_p)$ denote a chart in normal coordinates at p, cf. remark 4.2.2. We are now going to investigate certain ε-balls at $p \in X$, defined by

$$\mathcal{N}_\varepsilon(p) := \{q \in W_p \mid N^p(q) < \varepsilon\}\,, \tag{4.9}$$

where $N^p(q) := \sum_{i=1}^n {}^{p}w^i(q)^2$ is the squared Euclidean radius of ${}^{p}w(q)$ (even for semi-Riemannian manifolds).

Theorem 4.2.4. *Let X be a semi-Riemannian manifold and let $o \in X$. Then there exists some $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ the following holds:*

(i) The subset $\mathcal{N}_\varepsilon(o)$ of X is geodesically convex. In particular, o has a basis of convex neighborhoods.

(ii) For all $p \in \mathcal{N}_\varepsilon(o)$ and all $\delta > 0$ with $\mathcal{N}_\delta(p) \subseteq \mathcal{N}_\varepsilon(o)$, $\mathcal{N}_\delta(p)$ is also geodesically convex.

Proof. (Modified version of [O'N83], proposition 5.7). By theorem 4.1.16, the map $\mathrm{E} : \mathcal{V} \to \mathcal{U}$, $\mathrm{E}(v_p) := (p, \exp_p(v))$, is a diffeomorphism from a neighborhood $\mathcal{V}$ of the zero section TX_0 in TX onto a neighborhood $\mathcal{U}$ of the diagonal $\triangle_X$ in $X \times X$. Let U be a neighborhood of $o \in X$ as in lemma 4.2.3, in particular $U \times U \subseteq \mathcal{U}$.

For each $p \in U$ we define a (symmetric) tensor field ${}^{p}B \in \mathcal{T}^0_2(U)$ by its components

$${}^{p}B_{ij}(q) := \delta_{ij} - \sum_{k=1}^n {}^{p}{}^{w}\Gamma^k_{ij}(q) \cdot {}^{p}w^k(q) \tag{4.10}$$

for any $q \in U$. This is well-defined by 4.2.3 (i). Recall that ${}^{o}{}^{w}\Gamma^k_{ij}(o) = 0$ for a normal coordinate system by remark 4.1.10 (iii), so ${}^{o}B_{ij}(o) = \delta_{ij}$. Furthermore, $(p, q) \mapsto {}^{p}w(q)$ is smooth and ${}^{p}{}^{w}\Gamma^k_{ij}(q) \to {}^{o}{}^{w}\Gamma^k_{ij}(o)$ by lemma 4.2.3 (ii) and (v). Hence we may suppose w.l.o.g. that U is so small that ${}^{p}B(q)$ is positive definite for all $p, q \in U$.

For $\varepsilon' > 0$ sufficiently small we have that $\overline{\mathcal{N}_{\varepsilon'}(o)} \subseteq U$ since these sets are diffeomorphic to open balls $B_{\sqrt{\varepsilon'}}(0)$ in $\mathbb{R}^n$ under normal coordinates (hence form a basis of neighborhoods of o in X). Since by the above E is a diffeomorphism on $U \times U \subseteq \mathcal{U}$, there exists an open neighborhood $V_{\varepsilon'}$ of $0 \in T_oX$ in TX such that $\mathrm{E} : V_{\varepsilon'} \to \mathcal{N}_{\varepsilon'}(o) \times \mathcal{N}_{\varepsilon'}(o)$ is a diffeomorphism. We may choose an open neighborhood $\widetilde{V}_{\varepsilon'}$ of 0_o in TX such that $[0,1] \cdot \widetilde{V}_{\varepsilon'} \subseteq \widetilde{V}_{\varepsilon'} \subseteq V_{\varepsilon'}$ where the multiplication is only applied to the vector component (e.g. let $\widetilde{V}_{\varepsilon'} := (T\,{}^o w)^{-1}(B_{r_1}(0) \times B_{r_2}(0))$ for $r_1, r_2 > 0$ sufficiently small), i.e. $\widetilde{V}_{\varepsilon'}$ is star-shaped in the vector component.

Let $\varepsilon_0 \in (0, \varepsilon')$ such that $\mathcal{N}_{\varepsilon_0}(o) \times \mathcal{N}_{\varepsilon_0}(o) \subseteq \mathrm{E}(\widetilde{V}_{\varepsilon'})$. For $\varepsilon \in (0, \varepsilon_0]$ we also have that $\mathcal{N}_{\varepsilon}(o) \times \mathcal{N}_{\varepsilon}(o) \subseteq \mathcal{N}_{\varepsilon_0}(o) \times \mathcal{N}_{\varepsilon_0}(o) \subseteq \mathrm{E}(\widetilde{V}_{\varepsilon'})$. We set

$$W_\varepsilon := \mathrm{E}^{-1}(\mathcal{N}_\varepsilon(o) \times \mathcal{N}_\varepsilon(o)) \subseteq \widetilde{V}_{\varepsilon'} \subseteq V_{\varepsilon'} \subseteq \mathcal{V}. \tag{4.11}$$

Then obviously $\mathrm{E} : W_\varepsilon \to \mathcal{N}_\varepsilon(o) \times \mathcal{N}_\varepsilon(o)$ is a diffeomorphism and

$$\mathrm{E}([0,1] \cdot W_\varepsilon) \subseteq U \times U \subseteq \mathcal{U} \tag{4.12}$$

since

$$\begin{aligned} \mathrm{E}([0,1] \cdot W_\varepsilon) &\subseteq \mathrm{E}([0,1] \cdot \widetilde{V}_{\varepsilon'}) \subseteq \mathrm{E}(\widetilde{V}_{\varepsilon'}) \subseteq \mathrm{E}(V_{\varepsilon'}) \\ &= \mathcal{N}_{\varepsilon'}(o) \times \mathcal{N}_{\varepsilon'}(o) \subseteq U \times U \subseteq \mathcal{U} \end{aligned}$$

(again, the multiplication is only done in the vector component). Moreover, $\overline{\mathcal{N}_\varepsilon(o)} \subseteq \overline{\mathcal{N}_{\varepsilon'}(o)} \subseteq U$ since $\varepsilon < \varepsilon'$ and ε' was chosen sufficiently small before.

It remains to be proved that (i) and (ii) hold for all $\varepsilon \in (0, \varepsilon_0]$.

(i) Let $\varepsilon \in (0, \varepsilon_0]$ and $p \in \mathcal{N}_\varepsilon(o)$. For

$$W_\varepsilon(p) := W_\varepsilon \cap T_pX. \tag{4.13}$$

we obtain

$$\mathrm{E}(W_\varepsilon(p)) = \{p\} \times \mathcal{N}_\varepsilon(o). \tag{4.14}$$

Here, the inclusion $(\subseteq)$ is obvious because of the definition (4.11) of W_ε. To show that also $(\supseteq)$ holds, let $(p, r) \in \{p\} \times \mathcal{N}_\varepsilon(o)$. Again by (4.11) there exists $w \in W_\varepsilon$ with $(p, r) = \mathrm{E}(w) = (\pi(w), \exp_{\pi(w)}(w))$. In particular, $\pi(w) = p$, thus $w \in T_pX$ and $(p, r) = \mathrm{E}(w) \in \mathrm{E}(W_\varepsilon \cap T_pX) = \mathrm{E}(W_\varepsilon(p))$.

Hence $\mathrm{E}\,|_{W_\varepsilon(p)}$ is a diffeomorphism onto $\{p\} \times \mathcal{N}_\varepsilon(o)$, i.e. $\exp_p : W_\varepsilon(p) \to \mathcal{N}_\varepsilon(o)$ is also a diffeomorphism. To see that $\mathcal{N}_\varepsilon(o)$ is a normal neighborhood of p it remains to be shown that $W_\varepsilon(p)$ is star-shaped around $0 \in T_pX$. Since $p \in \mathcal{N}_\varepsilon(o)$ was arbitrary it then follows by definition 4.1.22 that $\mathcal{N}_\varepsilon(o)$ is actually convex which proves (i).

Let $0 \neq v \in W_\varepsilon(p)$. Then v is of the form $\mathrm{E}^{-1}(p, q) = \exp_p^{-1}(q)$ for some $p \neq q \in \mathcal{N}_\varepsilon(o)$ by (4.14). By the definition of $\exp_p$ in 4.1.13, $c_v : [0,1] \to X$ is a geodesic from p to q. Moreover, $c_v(t) = \exp_p(tv)$ $\forall t \in [0,1]$ since a geodesic is uniquely determined by its initial data. By (4.12) and the fact that $W_\varepsilon(p) \subseteq W_\varepsilon$ we have that $c_v(t) = \exp_p(tv) \in U$ for all $t \in [0,1]$.

Suppose for a moment that c_v lies entirely in $\mathcal{N}_\varepsilon(o)$. Then $tv \in W_\varepsilon(p)$ for all $t \in [0,1]$: Suppose not, then choose $s \in [0,1]$ such that $w := sv \notin W_\varepsilon(p)$. For $s_0 := \sup\{t \in [0,1] \,|\, tw \in W_\varepsilon(p)\}$ we get that $s_0 w \in \partial W_\varepsilon(p)$. By assumption, $(\exp_p^{-1} \circ c_v)([0,1]) \subset\subset \exp_p^{-1}(\mathcal{N}_\varepsilon(o)) \overset{(4.14)}{=} W_\varepsilon(p)$. Therefore there exists some

$t_0 < s_0$ with $t_0 w \in W_\varepsilon(p) \setminus (\exp_p^{-1} \circ c_v)([0,1])$. On the other hand we have that $\exp_p^{-1} \circ c_v = t \mapsto tv$ by the above. Thus $t_0 w = t_0 sv \in [0,1]v$ and therefore $(\exp_p^{-1} \circ c_v)(t_0 s) \in (\exp_p^{-1} \circ c_v)([0,1])$, a contradiction.

Now it only remains to be proved that $c_v([0,1]) \subseteq \mathcal{N}_\varepsilon(o)$: Suppose to the contrary that c_v actually leaves $\mathcal{N}_\varepsilon(o)$. By definition (4.9) we then have a $t \in (0,1)$ with $N^o(c_v(t)) \geq \varepsilon$. Since $p, q \in \mathcal{N}_\varepsilon(o)$ we have that $N^o(p), N^o(q) < \varepsilon$. As $N^o \circ c_v : [0,1] \to \mathbb{R}$ is continuous on a compact set it must attain a maximum at some $t_{\max} \in (0,1)$. Furthermore, $N^o \circ c_v$ is smooth, so we should obtain that $\frac{d}{dt}(N^o \circ c_v)(t_{\max}) = 0$ and $\frac{d^2}{dt^2}(N^o \circ c_v)(t_{\max}) < 0$:

$$\begin{aligned}
\frac{d^2}{dt^2}(N^o \circ c_v) &= 2 \sum_{i=1}^{n} \left[\left(\frac{d\, {}^o c_v^i}{dt} \right)^2 + {}^o c_v^i \underbrace{\frac{d^2\, {}^o c_v^i}{dt^2}}_{\overset{(*)}{=} -{}^{o}w\Gamma^i_{rs}(c_v) \frac{d\, {}^o c_v^r}{dt} \frac{d\, {}^o c_v^s}{dt}} \right] \\
&= 2 \sum_{i,j=1}^{n} \left(\delta_{ij} - \sum_{k=1}^{n} {}^{o}w\Gamma^k_{ij}(c_v)\, {}^o c_v^k \right) \frac{d\, {}^o c_v^i}{dt} \frac{d\, {}^o c_v^j}{dt} \\
&\overset{(4.10)}{=} 2 \sum_{i,j=1}^{n} {}^o B_{ij}(c_v) \frac{d\, {}^o c_v^i}{dt} \frac{d\, {}^o c_v^j}{dt}
\end{aligned}$$

where ${}^o c_v := {}^o w \circ c_v$ and $(*)$ is the geodesic equation 4.1.12. Hence $0 < \frac{d^2}{dt^2}(N^o \circ c_v)(t_{\max}) = 2\, {}^o B(c_v(t_{\max}))({}^o c_v{}'(t_{\max}), {}^o c_v{}'(t_{\max}))$, a contradiction to the positive definiteness of ${}^o B(c_v(t_{\max}))$ (recall that $c_v(t_{\max}) \in U$ by the above).

Since $v \in W_\varepsilon(p)$ was arbitrary, $W_\varepsilon(p)$ is indeed star-shaped.

(ii) Let $p \in \mathcal{N}_\varepsilon(o)$ and $\delta > 0$ such that $\mathcal{N}_\delta(p) \subseteq \mathcal{N}_\varepsilon(o)$. Let $p' \in \mathcal{N}_\delta(p)$ and $W_\varepsilon(p')$ for $\varepsilon \in (0, \varepsilon_0]$ as in (4.13). As in (i) we have that $\exp_{p'} : W_\varepsilon(p') \to \mathcal{N}_\varepsilon(o)$ is a diffeomorphism. Since $p' \in \mathcal{N}_\delta(p) \subseteq \mathcal{N}_\varepsilon(o)$, therefore there exists a subset $W_{p'} \subseteq W_\varepsilon(p')$ such that $\exp_{p'} : W_{p'} \to \mathcal{N}_\delta(p)$ is also a diffeomorphism.

It remains to be shown that $W_{p'}$ is star-shaped around $0 \in T_{p'}X$. Thus take any $v \in W_{p'}$. Then $v = \exp_{p'}^{-1}(q')$ for some $q' \in \mathcal{N}_\delta(p) \subseteq \mathcal{N}_\varepsilon(o)$. In (i) we have seen that c_v is a geodesic from p' to q' with $c_v(t) = \exp_{p'}(tv)$ that lies entirely in $\mathcal{N}_\varepsilon(o) \subseteq U$ (simply replace p, q there with p', q'). Again as in (i), it follows that c_v lies entirely in $\mathcal{N}_\delta(p)$ and that therefore W_q is star-shaped (replace $o, p, \varepsilon, W_\varepsilon(p)$ by p, q, δ, W_q). □

4.2.3 The distance on Riemannian manifolds

We now consider a Riemannian manifold (X, g). In this case, the proof of 4.1.21 implies that for sufficiently small ε

$$\mathcal{N}_\varepsilon(p) \overset{(4.9)}{=} \exp_p(V_{\sqrt{\varepsilon}}(p)) \overset{\text{proof of } 4.1.21}{=} U_{\sqrt{\varepsilon}}(p) \overset{4.1.21}{=} \{q \in X \mid d_g(p,q) < \sqrt{\varepsilon}\} =: B_{\sqrt{\varepsilon}}(p),$$

where d_g denotes the Riemannian distance induced by the Riemannian metric g on X. Thus we may replace all $\mathcal{N}_\varepsilon(p)$ in theorem 4.2.4 by some $B_r(p)$ in this case. We therefore have:

Corollary 4.2.5. *Let X be a Riemannian manifold and let $o \in X$. Then there exists some $r_0 > 0$ such that for all $r \in (0, r_0]$ we have:*

(i) $B_r(o)$ *is geodesically convex. In particular, every point* o *has a base of convex balls.*

(ii) $B_s(p)$ *is geodesically convex whenever* $B_s(p) \subseteq B_r(o)$*.*

In order to show that the Riemannian distance d_g^2 is smooth on a certain neighborhood of the diagonal we need to introduce a new type of neighborhoods and find a continuous function that will describe the boundary of open balls later.

Definition 4.2.6. Let X be a Riemannian manifold. An open subset U of X is called *strongly convex* if U itself and any open ball contained in U is convex.

Note that each ball $B_r(o)$ with $r \in (0, r_0]$ is strongly convex by corollary 4.2.5. Thus we actually proved the existence of strongly convex neighborhoods for each point of the Riemannian manifold.

Definition 4.2.7. Let X be a Riemannian manifold and let $p \in X$. The *convexity radius* $\kappa : X \to \mathbb{R}^+$ is defined by

$$\kappa(p) := \sup\{r \geq 0 \mid B_r(p) \text{ is strongly convex}\}$$

Remark 4.2.8 (Properties of $B_r(p)$ and κ)**.**

(i) Note that, by corollary 4.2.5,

$$\kappa(p) > 0 \qquad \forall p \in X.$$

(ii) Moreover, if $B_r(p)$ is strongly convex, then also $B_s(q) \subseteq B_r(p)$ is, because every ball contained in $B_s(q)$ is also contained in $B_r(p)$ and hence convex.

Remark 4.2.9 (Similar notions)**.** As already mentioned in remark 4.1.23, convexity e.g. in the sense of [GKM68] is defined differently from in [O'N83] which we used as a basis. Those authors also use a different notion of strong convexity. Namely, they define a subset G to be *strongly convex* if it is convex (in their sense), all open balls contained in G are also convex (in their sense) and if all points $p, q \in G$ can be joined by a unique geodesic. It is shown in [GKM68], section 5.2, that each point in the Riemannian manifold has a strongly convex neighborhood.

The definition of the *convexity radius* r in [GKM68] is the same as ours using, however, their notion of strong convexity.

Since their definition of strong convexity is weaker, it is obvious that $\kappa(p) \leq r(p)$ for all $p \in X$. In [GKM68] it is also shown that r is smaller than the *injectivity radius* inj. Thus

$$\kappa(p) \leq r(p) \leq \operatorname{inj}(p) \qquad \forall p \in X.$$

Lemma 4.2.10. *Let* (X, g) *be a Riemannian manifold. Then the convexity radius* κ *is (uniformly) continuous on* X*.*

Proof. Let $p, q \in X$. We distinguish two cases:

If $q \in B_{\kappa(p)}(p)$ then by definition of $\kappa(p)$ and strong convexity we have that

$$B_r(q) \subseteq B_{\kappa(p)}(p) \text{ is convex for all } r < \kappa(p) - d_g(p, q), \tag{4.15}$$

since $d_g(x,p) \leq d_g(x,q) + d_g(p,q) \leq \kappa(p)$ must hold for all $x \in B_r(q)$. Such $B_r(q)$ are even strongly convex by 4.2.8 (ii). In particular, $r \leq \kappa(q)$ and therefore by (4.15) (since this holds for all such r) $\kappa(q) \geq \kappa(p) - d_g(p,q)$, hence

$$\kappa(p) - \kappa(q) \leq d_g(p,q).$$

If $q \notin B_{\kappa(p)}(p)$ then $d_g(p,q) \geq \kappa(p) \geq \kappa(p) - \kappa(q)$.

Thus by symmetry in p and q we have that

$$|\kappa(p) - \kappa(q)| \leq d_g(p,q) \qquad \forall p,q \in X,$$

hence κ is uniformly continuous on X. □

Theorem 4.2.11 (Smoothness of the Riemannian distance)**.** *Let (X,g) be a Riemannian manifold and d_g the Riemannian distance. Then there exists a neighborhood U of the diagonal $\triangle_X \subseteq X \times X$ such that*

(i) $(p,q) \mapsto d_g(p,q)$ is smooth on $U \setminus \triangle_X$

(ii) $(p,q) \mapsto d_g(p,q)^2$ is smooth on U.

Proof. By theorem 4.1.16 the map $\mathrm{E} : \mathcal{V} \to \mathcal{U}$, $\mathrm{E}(v_p) := (p, \exp_p(v))$, is a diffeomorphism. Set

$$U := \mathcal{U} \cap \{(p,q) \in X \mid d_g(p,q) < \kappa(p)\}$$

Since κ is continuous on X by lemma 4.2.10 and $\kappa > 0$ by remark 4.2.8 (i), U is an open neighborhood of the diagonal $\triangle_X$ in $X \times X$.

Let $(p_0, q_0) \in U$, so that $q_0 \in B_{\kappa(p_0)}(p_0)$. By remark 4.2.2 there exists an orthonormal frame field $\{E_1, ..., E_n\}$ on a neighborhood $W \subseteq B_{\kappa(p_0)}(p_0)$ of p_0. Thus for all $(p,q) \in W \times B_{\kappa(p_0)}(p_0)$ we have that

$$d_g(p,q)^2 \overset{4.1.21}{=} |\exp_p^{-1}(q)|^2_{T_pX} = |\,\mathrm{E}^{-1}(p,q)|^2_{T_pX} = \sum_{i=1}^{n} \langle\, \mathrm{E}^{-1}(p,q),\, E_i|_p \,\rangle^2,$$

where $\langle .\,,.\rangle$ denotes the Riemannian metric $g(p)$. Since this expression is smooth, we have that (ii) holds. As the square root is smooth away from zero, (i) is also true. □

Remark 4.2.12. It is easier to prove that $q \mapsto d_g(p,q)^2$ is smooth on each normal neighborhood U of p. This is due to the fact that we can write the Riemannian distance in normal coordinates $({}^pw^1, ..., {}^pw^n)$ as

$$d_g^2(p,q) = \sum_{i=1}^{n} {}^pw^i(q)^2.$$

We even see that $q \mapsto d_g(p,q)$ is smooth on $U \setminus \{p\}$. This conclusion may also be found in [Pet06], theorem 29, p. 177.

4.3 The Riemannian distance on submanifolds

In this section we assume that X is a smooth submanifold of $\mathbb{R}^n$. If we equip X with the Riemannian metric induced by the Euclidean metric in $\mathbb{R}^n$ we obtain some additional structure. First, let us recall a well-known theorem from topology:

Theorem 4.3.1 (Lebesgue covering theorem)**.** *Let X be a compact metric space and $\mathcal{U}$ an open cover of X. Then there exists a $\delta > 0$, the so-called* Lebesgue number *of $\mathcal{U}$, such that for all $x \in X$ there exists $U \in \mathcal{U}$ such that $B_\delta(x) \subseteq U$.*

Proof. See for example [Eng77], theorem 4.3.31, or [Kri04], theorem 5.1.5. □

This result will only be used later in connection with local properties given on an open cover of a compact subset of a submanifold, like the following.

Lemma 4.3.2 (Riemannian vs. Euclidean metric)**.** *Let X be a smooth and connected submanifold of $\mathbb{R}^n$ and $K \subset\subset X$. Let g be the Riemannian metric on X induced by the Euclidean metric in $\mathbb{R}^n$. Then*

$$\exists C > 0 \; \forall p, q \in K : \; |p - q| \leq d_g(p, q) \leq C|p - q|. \tag{4.16}$$

First proof. By definition,

$$d_g(p, q) := \inf\{L(c) \,|\, c \text{ a piecewise smooth curve in } X \text{ from } p \text{ to } q\}$$

is the Riemannian distance from p to q, where $L(c)$ denotes the arc length of c. Thus $|p - q| \leq d_g(p, q)$.

It remains to be shown that $d_g(p, q) \leq C|p - q|$ for some $C > 0$ and $p, q \in K \subset\subset X$. We start this by proving that the inclusion of X in $\mathbb{R}^n$ is locally Lipschitz continuous. Let $o \in X$. By the proof of 4.2.11, there exists a neighborhood W of o in X such that for all $p, q \in W$ we have that

$$d_g(p, q) = |\,\mathrm{E}^{-1}(p, q)|_{T_pX} = |\,\mathrm{E}^{-1}(p, q)|, \tag{4.17}$$

where $|\,.\,|$ denotes the Euclidean norm (recall that g is induced by the Euclidean metric in $\mathbb{R}^n$). As E^{-1} is smooth (E is a diffeomorphism), there exists an open neighborhood V of o in $\mathbb{R}^n$ and a smooth map $e : V \times V \to TX \subseteq \mathbb{R}^{n^2}$ such that $e|_{(V\times V)\cap(X\times X)} = \mathrm{E}^{-1}|_{(V\times V)\cap(X\times X)}$. W.l.o.g. we may assume that V is relatively compact and convex, and that W is contained in $V \cap X$. Since $e(p, p) = \mathrm{E}^{-1}(p, p) = \exp_p^{-1}(p) = 0$ for any $p \in W$ we have that

$$\begin{aligned} d_g(p, q) &\overset{(4.17)}{=} |\,\mathrm{E}^{-1}(p, q)| = |e(p, q) - e(p, p)| \\ &\leq |p - q| \int_0^1 |(D_2e)(p + t(q - p))|dt \leq |p - q| \cdot \|D_2e\|_{L^\infty(\overline{V})} \end{aligned}$$

for all $p, q \in V$. In the last line the mean value theorem was applied using the fact that V is convex. Thus $d_g(p, q) \leq C|p - q|$ holds locally for $p, q \in W$ and $C := \|D_2e\|_{L^\infty(\overline{V})}$.

Suppose now that (4.16) is false. Thus for each $m \in \mathbb{N}$ there exist points $p_m, q_m \in X$ such that $d_g(p_m, q_m) > m|p_m - q_m| \geq 0$. Since K is compact in X, there exists a subsequence $(m_k)_k$ such that $p_{m_k} \to p \in K$ and $q_{m_k} \to q \in K$. Hence $p = q$, because for any $\varepsilon > 0$ and k sufficiently large we have

$$\begin{aligned} |p-q| &\leq |p - p_{m_k}| + |p_{m_k} - q_{m_k}| + |q - q_{m_k}| \\ &\leq d_g(p, p_{m_k}) + \tfrac{1}{m_k} d_g(p_{m_k}, q_{m_k}) + d_g(q, q_{m_k}) \\ &\leq \underbrace{(1 + \tfrac{1}{m_k})}_{\leq 2} \underbrace{d_g(p, p_{m_k})}_{\leq \varepsilon} + \underbrace{\tfrac{1}{m_k} d_g(p,q)}_{\leq \varepsilon} + \underbrace{(1 + \tfrac{1}{m_k})}_{\leq 2} \underbrace{d_g(q, q_{m_k})}_{\leq \varepsilon} \\ &\leq 5\varepsilon. \end{aligned}$$

On the other hand, there exists a neighborhood U of p as above, such that $d_g(p_{m_k}, q_{m_k}) \leq C|p_{m_k} - q_{m_k}|$ for some $C > 0$ and k sufficiently large. Thus

$$0 < d_g(p_{m_k}, q_{m_k}) \leq C|p_{m_k} - q_{m_k}| \leq \tfrac{C}{m_k} d_g(p_{m_k}, q_{m_k}),$$

a contradiction for $m_k > C$. Thus we also have that there exists some $C > 0$ such that for $p, q \in K$

$$d_g(p,q) \leq C|p-q|. \qquad \square$$

Remark 4.3.3. The limiting value of C in 4.3.2 can be computed as follows. By using a so-called tubular neighborhood U of $X \times X$ (which exists for every submanifold M of $\mathbb{R}^n$ without boundary by [Hir76], theorem 4.5.1) with smooth retraction $r = (\exp^{\perp})^{-1} : U \to X \times X$ (i.e. $r|_{X \times X} = \mathrm{id}_{X \times X}$), it can be shown that $C \to 1$ for $p, q \to o$: In this case we can additionally assume that $V \times V \subseteq U$ and explicitly write $e : V \times V \to TX$ as $e = \mathrm{E}^{-1} \circ r$. For $p, q \to o$ one can obtain that

$$D_2 e(p,q) \to D_2 e(o,o) = D_2(\mathrm{E}^{-1} \circ r)(o,o) = \mathrm{id}_{T_o X},$$

using the properties of E and r. Thus

$$C = \|D_2 e\|_{L^\infty(\overline{V})} \to \|D_2 e(o,o)\| = \|\,\mathrm{id}_{T_o X}\,\| = 1.$$

There is an alternative approach for the proof of lemma 4.3.2 by Hans Vernaeve [Ver09] that shows the second inequality in (4.16) locally without using the map E:

Second proof. Again, we will show the inequality locally around some $o \in X$ first. By corollary 4.2.5, there exists a geodesically convex and relatively compact neighborhood $B_r(o)$ of o in X w.r.t. the induced Riemannian metric g. W.l.o.g. let $(u, U = B_{\frac{r}{4}}(o))$ be a chart in normal coordinates at o in X. Thus for any $p, q \in U$ we have that

$$d_g(p,q) \leq d_g(p,o) + d_g(q,o) \leq \frac{r}{4} + \frac{r}{4} = \frac{r}{2},$$

hence in particular that $q \in B_{\frac{r}{2}}(p) \subseteq B_r(o)$. By proposition 4.1.21 we furthermore have that

$$d_g(p,q) = L(\gamma_{pq}) = \int_0^1 |\gamma_{pq}'(t)| dt,$$

where $\gamma_{pq} : [0,1] \to B_r(o)$ denotes the radial geodesic from p to q. Thus

$$\begin{aligned} |u(p)-u(q)| &\leq \int_0^1 |(u\circ\gamma_{pq})'(t)|dt \\ &\leq \int_0^1 \|T_{\gamma_{pq}(t)}u\||\gamma_{pq}'(t)|dt \\ &\leq L\, d_g(p,q) \end{aligned} \tag{4.18}$$

for $L := \sup_{t\in[0,1]} \|T_{\gamma_{pq}(t)}u\| > 0$ and all $p, q \in U$, since u is a diffeomorphism on $B_r(o)$ and $\overline{U} \subset B_r(o)$.

Now let $0 < \varepsilon < \frac{1}{2L}$ and choose $\delta > 0$ such that $B_\delta(o) \subseteq U$ and

$$\|T_x u^{-1} - T_o u^{-1}\| < \frac{\varepsilon}{2} \qquad \forall x \in B_\delta(o), \tag{4.19}$$

which is again possible because $u^{-1} : T_oX \supseteq u(U) \to X$ is smooth. Since we use normal coordinates, we have that $u(B_\delta(o)) = V_\delta(o) = \{v \in T_oX \,|\, |v| < \delta\}$ as in (4.2). This implies that the curve

$$\begin{aligned} c:\ [0,1] &\to u(B_\delta(o)) \subseteq \mathbb{R}^k \\ t &\mapsto u(p) + t(u(q)-u(p)) \end{aligned}$$

and hence also the curve $u^{-1}\circ c$ in $B_\delta(o) \subseteq U$ are well-defined and smooth. Moreover, since the tangential maps can be viewed as linear maps $T_{c(t)}u^{-1} : \mathbb{R}^k \to \mathbb{R}^k$, we have that

$$\begin{aligned} |(u^{-1}\circ c)'(t) - (u^{-1}\circ c)'(0)| &= |T_{c(t)}u^{-1}\cdot c'(t) - T_{c(0)}u^{-1}\cdot c'(0)| \\ &= |(T_{c(t)}u^{-1} - T_{c(0)}u^{-1})(u(q)-u(p))| \\ &\leq \|T_{c(t)}u^{-1} - T_{c(0)}u^{-1}\||u(q)-u(p)| \\ &\overset{(4.19)}{<} \varepsilon|u(q)-u(p)|. \end{aligned} \tag{4.20}$$

as $\|T_{c(t)}u^{-1} - T_{c(0)}u^{-1}\| \leq \|T_{c(t)}u^{-1} - T_ou^{-1}\| + \|T_ou^{-1} - T_{c(0)}u^{-1}\|$ by the triangle inequality. Thus

$$\begin{aligned} |(u^{-1}\circ c)'(t)| &\leq |(u^{-1}\circ c)'(t) - (u^{-1}\circ c)'(0)| + |(u^{-1}\circ c)'(0)| \\ &\overset{(4.20)}{<} \varepsilon|u(q)-u(p)| + |(u^{-1}\circ c)'(0)|, \end{aligned} \tag{4.21}$$

and therefore

$$d_g(p,q) \leq \int_0^1 |(u^{-1}\circ c)'(t)| \overset{(4.21)}{<} \varepsilon|u(q)-u(p)| + |(u^{-1}\circ c)'(0)| \tag{4.22}$$

for all $p, q \in B_\delta(o)$. On the other hand, by the fundamental theorem of calculus for curves (see e.g. [Kri04], theorem 5.5.18), and again the triangle inequality,

$$\begin{aligned} |p-q| &= |(u^{-1}\circ c)(1) - (u^{-1}\circ c)(0)| \\ &= \left|\int_0^1 (u^{-1}\circ c)'(t)dt\right| \\ &= \left|\int_0^1 (u^{-1}\circ c)'(t) - (u^{-1}\circ c)'(0)dt + (u^{-1}\circ c)'(0)\right| \\ &\geq |(u^{-1}\circ c)'(0)| - \int_0^1 |(u^{-1}\circ c)'(t) - (u^{-1}\circ c)'(0)|dt \\ &\overset{(4.20)}{>} |(u^{-1}\circ c)'(0)| - \varepsilon|u(q)-u(p)|. \end{aligned} \tag{4.23}$$

Finally we can combine (4.22) and (4.23) to obtain

$$\begin{aligned} d_g(p,q) &\overset{(4.22)}{<} \varepsilon|u(q)-u(p)| + |(u^{-1}\circ c)'(0)| \\ &\overset{(4.23)}{<} |p-q| + 2\varepsilon|u(q)-u(p)| \\ &\overset{(4.18)}{\leq} |p-q| + 2L\varepsilon\, d_g(p,q) \end{aligned}$$

for $p, q \in B_\delta(o)$. As we had set $\varepsilon < \frac{1}{2L}$, we have that for $p, q \in B_\delta(o)$

$$d_g(p,q) < \frac{1}{1-2L\varepsilon}|p-q|$$

with $C := \frac{1}{1-2L\varepsilon} > 1$. We have that $C \to 1$ for $U \to \{o\}$ (via appropriate choices of ε and δ).

The global result on a compact set K is proved by contradiction as in the first proof of lemma 4.3.2. □

Remark 4.3.4. Note that different Riemannian distances are always equivalent by lemma 4.1.8. Thus (4.16) holds for any other Riemannian distance on X as well, and not just for the one induced by the Euclidean metric on $\mathbb{R}^n$.

Remark 4.3.5 (Metric structure)**.** It is of general interest to study the length structure on a (path) metric space by comparing the distances in terms of the metric and an induced length metric. For a submanifold X of $\mathbb{R}^n$ the length metric may be the Riemannian one induced by the Euclidean metric in $\mathbb{R}^n$, i.e. d_g as in 4.3.2. In particular, if we look at

$$\mathrm{distort}(X) = \sup \frac{(\text{length dist})|_X}{\mathrm{dist}|_X},$$

some interesting results follow. For example, if $\mathrm{distort}(X) < \frac{\pi}{2}$ for a compact subset X of $\mathbb{R}^n$, then X is simply connected. See e.g. [Gro99] for a further discussion of metric structures for (non-)Riemannian spaces.

4.4 The Whitney embedding theorem

Many theorems are easier to prove in the setting of smooth submanifolds of $\mathbb{R}^n$ with the additional structure of the Euclidean metric and global coordinates. In order to generalize such results to smooth manifolds, we require the following theorem.

Theorem 4.4.1 (Whitney embedding theorem)**.** *Every n-dimensional smooth manifold embeds smoothly in $\mathbb{R}^{2n+1}$.*

Proof. See for example [GP74], p. 53, for a proof of this earlier version from 1936. In 1944 Whitney improved this result by one dimension using the so-called *Whitney trick*, showing that every n-dimensional manifold embeds in $\mathbb{R}^{2n}$. □

There is another embedding theorem by John Nash which states that every n-dimensional Riemannian manifold can be isometrically embedded in an $\mathbb{R}^m$. For further reading see e.g. [HH06].

Chapter 5

Colombeau Generalized Functions on Manifolds

In this chapter the main definitions and results of special Colombeau algebras on smooth manifolds X (in the sense of chapter 4) are recalled. In section 5.1 the basic definitions for the spaces of generalized numbers $\widetilde{\mathbb{C}}$, generalized functions $\mathcal{G}(X)$ and compactly supported points $\widetilde{X}_c$ are summarized. For further reading see [GKOS01] and [Nig06], but note that we always assume smooth dependence on the index ε.

The relevant point value characterizations are given in section 5.2, and in section 5.3 the sharp topology is introduced in a way suitable for its later use in chapter 6.

Compactly bounded generalized functions $\mathcal{G}[X, Y]$ between smooth manifolds X and Y are defined in section 5.4. Here, the case of smooth submanifolds of $\mathbb{R}^m$ resp. $\mathbb{R}^n$ is treated separately. We can reduce our final investigations of algebra isomorphisms to this case by Whitney's embedding theorem 4.4.1, as mentioned earlier. Most of the ideas for these proofs (although for non-smooth dependence on ε) are adapted from [Ver06]. Moreover, the intrinsic characterizations of $\mathcal{G}[X, Y]$ as provided in [KSV03] are recalled.

Finally, section 5.5 deals with the composition and invertibility of compactly bounded generalized functions.

5.1 General definitions

The theory of distributions, one of the first generalizations of classical functions, is widely used to treat linear partial differential equations. However, due to the famous impossibility result of Laurent Schwartz in [Sch54], distributions cannot be multiplied in a way that preserves the classical pointwise multiplication of continuous functions. This is clearly a significant restriction and therefore there arose a need to develop algebras of generalized functions that should contain the space of distributions.

The theory of generalized functions initiated by Jean-François Colombeau in [Col84] and [Col85] resolves the problem of non-multiplicativity of distributions by embedding the space of distributions in an associative and commutative differential algebra, only demanding that the pointwise multiplication of smooth functions to be preserved.

More precisely, this is achieved by looking at nets of smooth functions with certain asymptotic estimates. We will only consider the case of the special Colombeau algebra here, using the terminology introduced in [GKOS01]. In contrast to [GKOS01], however, we will adopt smooth dependence on the index ε.

The theory of Colombeau generalized functions has numerous applications in mathematics and physics. Among others, they are e.g. useful in the study of non-linear partial differential equations, non-smooth differential geometry and the theory of relativity.

5.1.1 Colombeau generalized functions

Let $\Omega \subseteq \mathbb{R}^n$ be open. The definition of generalized functions arises somewhat naturally from the wish to embed the space of distributions $\mathcal{D}'(\Omega)$ via regularization into a differential algebra that consists of nets of smooth functions. For a thorough discussion see [GKOS01], section 1.2.1.

Assuming that the reader is familiar with these concepts, we will immediately jump at our subject of interest – the algebra of generalized functions on smooth manifolds.

Let X be a smooth manifold. Recall that $\Gamma(X, E)$ is the space of smooth sections of a vector bundle (X, E, π), cf. definition 2.1.5. In what follows, the set of linear differential operators $P : \Gamma(X, E) \to \Gamma(X, E)$ for $E = X \times \mathbb{R}$ is denoted by $\mathcal{P}(X) = \mathcal{P}(X, E)$.

As before, the set $\mathcal{C}^\infty([0, 1) \times X)$ denotes the set of smooth functions $[0, 1) \times X \to \mathbb{C}$ (smooth in both variables). Note that we could, however, replace $\mathbb{C}$ by $\mathbb{R}$ everywhere.

The Landau notation (or Big-Oh) is used to describe the asymptotic behavior of functions:

$$f = O(g) \text{ as } x \to x_0 :\Leftrightarrow \exists C : |f(x)| \leq |Cg(x)| \text{ for all } x \text{ sufficiently close to } x_0.$$

Definition 5.1.1. Let X be a smooth manifold. The spaces of *moderate functions*, $\mathcal{E}_M(X)$, and *negligible functions*, $\mathcal{N}(X)$, are defined by

$$\begin{aligned} \mathcal{E}_M(X) :=& \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0, 1] \times X) \mid \forall K \subset\subset X \, \forall P \in \mathcal{P}(X) \, \exists N \in \mathbb{N} : \\ & \sup_{x \in K} |P u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\} \\ \mathcal{N}(X) :=& \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X) \mid \forall K \subset\subset X \, \forall m \in \mathbb{N} : \\ & \sup_{x \in K} |u_\varepsilon(x)| = O(\varepsilon^m) \text{ as } \varepsilon \to 0\} \end{aligned}$$

The *special Colombeau algebra* on X is defined as the quotient of $\mathcal{E}_M(X)$ and $\mathcal{N}(X)$, i.e.

$$\mathcal{G}(X) := \mathcal{E}_M(X)/\mathcal{N}(X).$$

The equivalence classes in $\mathcal{G}(X)$ are the *generalized functions* on X, denoted by $u = [(u_\varepsilon)_\varepsilon]$.

Remark 5.1.2 (Basic properties)**.** By applying Peetre's theorem it can be shown that instead of using $\mathcal{P}(X)$ above, $\mathcal{E}_M(X)$ can be described via Lie derivatives or charts and $\mathcal{E}_M(\Omega)$. Moreover, because $\mathcal{E}_M(X)$ and $\mathcal{N}(X)$ are invariant under the action of $P \in \mathcal{P}(X)$, all Pu, $u \in \mathcal{G}(X)$, are well-defined elements of $\mathcal{G}(X)$. See [GKOS01], section 3.2.1, for more details.

Remark 5.1.3 (Embeddings)**.** For a given atlas $\mathcal{A}$ of a smooth manifold X, $\iota_\mathcal{A} : \mathcal{D}'(X) \to \mathcal{G}(X)$, defined via a smooth partition of unity subordinate to the open cover consisting of the chart domains in $\mathcal{A}$ and a fixed mollifier $\rho \in \mathcal{S}(\mathbb{R}^n)$, is a linear embedding that coincides with the constant embedding σ on $\mathcal{C}^\infty(X)$. See [GKOS01], theorem 3.2.10.

5.1.2 Generalized numbers

The scalars in this theory cannot simply be complex numbers. By inserting points in generalized functions, we obtain so-called generalized numbers.

Definition 5.1.4.

$$\begin{aligned}\mathcal{E}_M &:= \{(r_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1]) \mid \exists N \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}\\ \mathcal{N} &:= \{(r_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1]) \mid \forall m \in \mathbb{N} : |r_\varepsilon| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}\end{aligned}$$

The *ring of generalized numbers* is defined by

$$\widetilde{\mathbb{C}} := \mathcal{E}_M/\mathcal{N}$$

and its elements are denoted by $r = [(r_\varepsilon)_\varepsilon]$.

Remark 5.1.5 ($\widetilde{\mathbb{C}}$ is only a ring)**.** $\widetilde{\mathbb{C}}$ is not a field although $\mathbb{C}$ is. Consider $r_\varepsilon := \sin\left(\frac{1}{\varepsilon}\right)$ for $\varepsilon \in (0,1]$. Clearly, $(r_\varepsilon)_\varepsilon \in \mathcal{E}_M$ since $|\sin|$ is bounded by 1. Since $r_{\varepsilon_n} = 1$ for $\varepsilon_n = \frac{2}{(2n+1)\pi}$ ($n \in \mathbb{N} \cup \{0\}$) and $\varepsilon_n \searrow 0$, we have that $(r_\varepsilon)_\varepsilon \notin \mathcal{N}$.

On the other hand, $r_{\varepsilon_k} = 0$ for $\varepsilon_k = \frac{1}{k\pi}$ $\forall k \in \mathbb{N}$, thus r_{ε_k} vanishes for infinitely many ε_k that converge to 0. Suppose that there exists $s = [(s_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{C}}$ such that $r \cdot s = 1$, i.e. $r_\varepsilon s_\varepsilon + n_\varepsilon = 0$ for $(n_\varepsilon)_\varepsilon \in \mathcal{N}$. Then obviously $n_{\varepsilon_k} = 1$ for all $k \in \mathbb{N}$. Thus $(n_\varepsilon)_\varepsilon \notin \mathcal{N}$, a contradiction.

5.1.3 Compactly supported generalized points

Although we can simply insert points $p \in X$ into generalized functions $[(u_\varepsilon)_\varepsilon]$ to obtain generalized numbers $[(u_\varepsilon(p))_\varepsilon]$, this is not sufficient to characterize generalized functions. The analogue of generalized points resolves this problem.

Definition 5.1.6. Let (X, g) be a Riemannian manifold with induced Riemannian distance d_g. The set of *compactly supported generalized points on X* is defined by

$$\widetilde{X}_c := \{(p_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1], X) \mid \exists K \subset\subset X\, \exists \varepsilon_0 > 0\, \forall \varepsilon < \varepsilon_0 : p_\varepsilon \in K\}/_\sim$$

where two nets $(p_\varepsilon)_\varepsilon$ and $(q_\varepsilon)_\varepsilon$ are called equivalent, i.e. $(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon$, if

$$(p_\varepsilon)_\varepsilon \sim (q_\varepsilon)_\varepsilon :\Longleftrightarrow d_g(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)\, \forall m \in \mathbb{N} \text{ as } \varepsilon \to 0.$$

Note that this latter definition is independent of the choice of g since two Riemannian distances are equivalent on any compact set of X, cf. lemma 4.1.8. Furthermore it makes sense to write $(d_g^2(p_\varepsilon, q_\varepsilon))_\varepsilon \in \mathcal{N}$ rather than $d_g(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m)$ $\forall m \in \mathbb{N}$, because for small distances the (squared) Riemannian distance is smooth by theorem 4.2.11.

In some cases, we will single out a compact set K that satisfies $p_\varepsilon \in K$ for $\varepsilon < \varepsilon_0$ and $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$ by $K_{\tilde{p}}$ and call it a *compact support of* $\tilde{p}$.

If we focus our interest on smooth submanifolds X of $\mathbb{R}^n$ (which is always possible by the Whitney embedding theorem 4.4.1), we can determine how $\widetilde{X}_c$ is embedded in $\widetilde{\mathbb{R}^n} = \widetilde{\mathbb{R}}^n$.

Proposition 5.1.7 ($\widetilde{X}_c \hookrightarrow \widetilde{\mathbb{R}}^n$)**.** *Let X be a smooth submanifold of $\mathbb{R}^n$ and g the Riemannian metric induced by the Euclidean metric. Then the compactly supported points $\widetilde{X}_c$ are in 1-1 correspondence with the elements of $\widetilde{\mathbb{R}}^n$ that have a representative that consists of elements of K, for some $K \subset\subset X$. The injection is given by the identity map on the representatives.*

Proof. By definition, two compactly supported points $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ on X represent the same generalized point $\tilde{p} \in \widetilde{X}_c$ if and only if

$$d_g(p_\varepsilon, q_\varepsilon) = O(\varepsilon^m) \text{ for all } m \in \mathbb{N}. \tag{5.1}$$

Let $K \subset\subset X$ be a compact support of $(p_\varepsilon)_\varepsilon$. We may cover K by the connected components of X. By the Lebesgue covering theorem 4.3.1 there exists a Lebesgue number $\delta > 0$ of this cover. Thus for sufficiently small but fixed ε we have that p_ε and q_ε lie in the same connected component of X. According to lemma 4.3.2 this implies that (5.1) is equivalent to

$$|p_\varepsilon - q_\varepsilon| = O(\varepsilon^m) \text{ for all } m \in \mathbb{N}. \tag{5.2}$$

Thus $(p_\varepsilon)_\varepsilon$ and $(q_\varepsilon)_\varepsilon$ also represent the same element $\tilde{p}$ in $\widetilde{\mathbb{R}}^n$, which has a representative $(p_\varepsilon)_\varepsilon$ that consists of elements of a compact set $K \subset\subset X$. This map is injective due to the fact that two compactly supported points $(p_\varepsilon)_\varepsilon$, $(q_\varepsilon)_\varepsilon$ on X that satisfy (5.2) also represent the same element in $\widetilde{X}_c$, again by lemma 4.3.2. ☐

5.2 Point value characterizations

In [GKOS01], theorems 1.2.46 and 3.2.8, the generalized functions on open sets $\Omega \subseteq \mathbb{R}^n$ resp. smooth manifolds X are characterized via the compactly supported generalized points $\widetilde{\Omega}_c$ resp. $\widetilde{X}_c$, i.e. every element in $\mathcal{G}(\Omega)$ resp. $\mathcal{G}(X)$ is characterized by its point values.

Compared to the above definitions 5.1.1 and 5.1.4, the smooth dependence on ε was not required in [GKOS01]. In the following, however, we will deduce the same results for our definitions.

First we must show that $u(\tilde{x})$, $\tilde{x} \in \widetilde{\Omega}_c$, is a well-defined generalized number:

Proposition 5.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $u \in \mathcal{G}(\Omega)$ and $\tilde{x} \in \widetilde{\Omega}_c$. Then the generalized point value of u at $\tilde{x} = [(x_\varepsilon)_\varepsilon]$,*

$$u(\tilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon],$$

is a well-defined element of $\widetilde{\mathbb{C}}$.

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ be a representative of u and $(x_\varepsilon)_\varepsilon$ a representative of $\tilde{x}$. By definition of $\widetilde{\Omega}_c$ we know that there exist $K \subset\subset \Omega$ and $\varepsilon_0 > 0$ such that $x_\varepsilon \in K$ for $\varepsilon < \varepsilon_0$. Since $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ implies that

$$|u_\varepsilon(x_\varepsilon)| \leq \sup_{x \in K} |u_\varepsilon(x)| < \varepsilon^{-N}$$

for some $N \in \mathbb{N}$ and sufficiently small ε, we know that $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{E}_M$. If we let $(v_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$, then

$$|v_\varepsilon(x_\varepsilon)| \leq \sup_{x \in K} |v_\varepsilon(x)| < \varepsilon^{m}$$

for all $m \in \mathbb{N}$ and small ε. Hence $(v_\varepsilon(x_\varepsilon))_\varepsilon$ is also negligible. This proves that different representatives of $u \in \mathcal{G}(X)$ lead to the same element $u(\tilde{x})$ in $\widetilde{\mathbb{C}}$.

It remains to be shown that if we choose another representative $(y_\varepsilon)_\varepsilon$ of $\tilde{x}$, i.e. $(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon$, then $[u((x_\varepsilon)_\varepsilon)] = [u((y_\varepsilon)_\varepsilon)]$ in $\widetilde{\mathbb{C}}$. Since both $(x_\varepsilon)_\varepsilon$ and $(y_\varepsilon)_\varepsilon$ are compactly supported and $x_\varepsilon - y_\varepsilon$ tends to 0, there exists some $\varepsilon_0' > 0$ such that for all $\varepsilon < \varepsilon_0'$ we have that $\{x_\varepsilon + t(y_\varepsilon - x_\varepsilon) \,|\, t \in [0,1]\}$ remains in a compact subset L of Ω. For fixed $\varepsilon < \varepsilon_0'$ we obtain by the mean value theorem that

$$|u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)| \leq |x_\varepsilon - y_\varepsilon| \int_0^1 |(Du_\varepsilon)(x_\varepsilon + t(y_\varepsilon - x_\varepsilon))| dt.$$

Here, $(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon$ implies that $|x_\varepsilon - y_\varepsilon| < \varepsilon^m$ for all $m \in \mathbb{N}$ and sufficiently small ε. Since $u \in \mathcal{G}(\Omega)$ we also have that $|(Du_\varepsilon)(x_\varepsilon + t(y_\varepsilon - x_\varepsilon))| \leq \sup_{z \in L} |Du_\varepsilon(z)| < \varepsilon^{-N}$ for some $N > 0$. Hence $|u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon| = O(\varepsilon^{m'})$ for all $m' \in \mathbb{N}$ and sufficiently small ε, i.e. $(u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon))_\varepsilon \in \mathcal{N}$. ☐

Theorem 5.2.2 (Point value characterization in $\mathcal{G}(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in \mathcal{G}(\Omega)$. Then*

$$u = 0 \text{ in } \mathcal{G}(\Omega) \Longleftrightarrow u(\tilde{x}) = 0 \text{ in } \widetilde{\mathbb{C}} \text{ for all } \tilde{x} \in \widetilde{\Omega}_c.$$

Proof. ($\Rightarrow$) By proposition 5.2.1, $u(\tilde{x}) = [(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well-defined element of $\widetilde{\mathbb{C}}$. Since $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c$, there exist $K \subset\subset \Omega$ and $\varepsilon_0 > 0$ such that $x_\varepsilon \in K$ for $\varepsilon < \varepsilon_0$. The assumption $u = 0$ now implies that

$$|u_\varepsilon(x_\varepsilon)| \leqslant \sup_{x \in K} |u_\varepsilon(x)| < \varepsilon^{m}$$

is true for all $m \in \mathbb{N}$ and sufficiently small ε, i.e. $u(\tilde{x}) = 0$ in $\widetilde{\mathbb{C}}$.

($\Leftarrow$) Suppose that $u \neq 0$ in $\mathcal{G}(\Omega)$. By definition of $\mathcal{N}(\Omega)$ there exist $K \subset\subset \Omega$, $\varepsilon_k \searrow 0$, $M \in \mathbb{N}$ and $x_{\varepsilon_k} \in K$ such that

$$|u_{\varepsilon_k}(x_{\varepsilon_k})| > \varepsilon_k^M. \tag{5.3}$$

Since K is compact, we may assume that $(x_{\varepsilon_k})_k$ converges towards some x in K. By again extracting a subsequence, this convergence may assumed to be fast, i.e. such that for each $n \in \mathbb{N}$ the sequence $k^n(x_{\varepsilon_k} - x)$ is bounded (e.g. choose x_{ε_k} such that $|x_{\varepsilon_k} - x| < k^{-k}$ for all $k \in \mathbb{N}$). Let $(x_{\varepsilon_{k_m}})_m$ be the final subsequence.

The special curve lemma 5.2.6 ensures the existence of a continuous curve $\tilde{c} : [0,1] \to \mathbb{R}^n$ that is smooth on $(0,1]$ and satisfies $\tilde{c}(\varepsilon_{k_m}) = x_{\varepsilon_{k_m}}$ and $\tilde{c}(0) = x$. Since Ω is open, we have that $\overline{B_r(x)} \subset \Omega$ for some $r > 0$. Due to the continuity of $\tilde{c}$ at 0 we have that $x_\varepsilon := \tilde{c}(\varepsilon) \in \overline{B_r(x)}$ for ε sufficiently small. Hence $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c$, and (5.3) implies that $u(\tilde{x}) \neq 0$ in $\widetilde{\mathbb{C}}$, a contradiction to the assumption. □

The proof of theorem 5.2.2 ($\Leftarrow$) required the special curve lemma, which can be found in [KM97], section 2, p. 18:

Definition 5.2.3. Let E be a locally convex space and let $(x_n)_n$ a sequence in E. Then x_n *converges fast* to $x \in E$ if for each $k \in \mathbb{N}$ the sequence $n^k(x_n - x)$ is bounded.

Lemma 5.2.4 (Special curve lemma)**.** *Let E be a locally convex space and let $(x_n)_n$ be a sequence which converges fast to x in E. Then the infinite polygon through x_n can be parametrized as a smooth curve $c : \mathbb{R} \to E$ such that*

$$c(\tfrac{1}{n}) = x_n \quad \textit{and} \quad c(0) = x.$$

Proof. We are going to define c piecewise and need a smooth map χ to smooth out the polygon that connects the points $x_n \in E$. Let $\rho : \mathbb{R} \to \mathbb{R}$ be the map

$$\rho(t) := \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1,1) \\ 0 & \text{else} \end{cases}$$

and $\overline{\rho}(t) = \frac{\rho(t)}{\int_{-\infty}^{\infty} \rho(s)ds}$ the respective normed bump. Obviously, ρ and hence $\overline{\rho}$ are smooth with k-th derivatives $\overline{\rho}^{(k)}(0) = \overline{\rho}^{(k)}(1) = 0$. Then $\chi : \mathbb{R} \to [0,1]$,

$$\chi(t) := \int_{-\infty}^{t} 2\overline{\rho}(2s-1)ds,$$

is a well-defined and smooth map that satisfies $\chi|_{(-\infty,0]} = 0$, $\chi|_{[1,\infty)} = 1$ and $\chi^{(k)}(0) = \chi^{(k)}(1) = 0$ for all $k \in \mathbb{N}$. Finally, $c : \mathbb{R} \to E$ is defined by

$$c(t) := \begin{cases} x & t \leq 0 \\ x_{n+1} + \chi\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(x_n - x_{n+1}) & \frac{1}{n+1} \leq t \leq \frac{1}{n} \\ x_1 & t \geq 1 \end{cases} .$$

Obviously, $c(0) = x$ and $c(\frac{1}{n}) = x_{n+1} + \chi\left(\frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(x_n - x_{n+1}) = x_{n+1} + 1 \cdot (x_n - x_{n+1}) = x_n$.

It remains to be shown that c is smooth. Firstly, it is evident that c is continuous everywhere and smooth on each of the subintervals $(-\infty, 0)$, $(1, \infty)$

and all $(\frac{1}{n+1}, \frac{1}{n})$, $n \in \mathbb{N}$. The k-th derivative in $t \in (\frac{1}{n+1}, \frac{1}{n})$ can easily be computed using the chain rule:

$$c^{(k)}(t) = \chi^{(k)}\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(n(n+1))^k(x_n - x_{n+1}).$$

Since $\chi^{(k)}(0) = \chi^{(k)}(1) = 0$ by the above, we have that $c^{(k)}(t) \to 0$ for $t \to \frac{1}{n}$ $\forall k, n \in \mathbb{N}$. Thus by lemma 5.2.5 below, $c^{(k)}(\frac{1}{n}) = 0$ $\forall k, n \in \mathbb{N}$.

By assumption, x_n converges fast towards x in E. If $(p_\alpha)_{\alpha \in A}$ is a family of seminorms that generates the topology on E, then by the binomial theorem and the triangle inequality

$$\begin{aligned} p_\alpha((n(n+1))^k(x_n - x_{n+1})) &= n^k \sum_{l=0}^{k} \binom{k}{l} n^l p_\alpha(x_n - x_{n+1}) \\ &\leq \sum_{l=0}^{k} \binom{k}{l} n^{k+l} \left(p_\alpha(x_n - x) + p_\alpha(x_{n+1} - x)\right) \end{aligned}$$

converges to 0 for all $\alpha \in A$, $k \in \mathbb{N}$. Moreover, $\chi^{(k)}$ has compact support for $k > 0$ and hence is bounded globally. Thus $c^{(k)}(t) \to 0$ for $t \to 0$ and c is smooth on $\mathbb{R}$, again by lemma 5.2.5. □

Lemma 5.2.5. *Let $I \subseteq \mathbb{R}$ be an interval that contains 0 and let $c : I \to E$ be continuous and differentiable on $I \setminus \{0\}$. Assume that $c' : I \setminus \{0\} \to E$ has a continuous extension to $\mathbb{R}$. Then c is differentiable at 0 and $c'(0) = \lim_{t\to 0} c'(t)$.*

Proof. Let $a := \lim_{t\to 0} c'(t)$ and let A be the closed and convex hull of $\{c'(s) \mid 0 \neq s \in I\}$. By the generalized mean value theorem (cf. [KM97], 1.4, p. 10) we have that $c(t) - c(0) \in tA$, i.e. $\frac{c(t)-c(0)}{t} \in A$. Let U be any closed and convex 0-neighborhood. Since c' is continuously extendable, there exists $\delta > 0$ such that $c'(t) \in a + U$ $\forall 0 < |t| \leq \delta$. Thus $\frac{c(t)-c(0)}{t} - a \in U$, i.e. $c'(0) = a$. □

In our case we have to generalize the zero sequence $(\frac{1}{n})_n$ to any sequence $(\varepsilon_n)_n$ with $\varepsilon_n \searrow 0$:

Corollary 5.2.6 (Special curve lemma for any decreasing sequence)**.** *Let E be a locally convex space, $(\varepsilon_n)_n$ such that $\varepsilon_n \searrow 0$ and let $(x_{\varepsilon_n})_n$ be a sequence which converges fast to x in E. Then the infinite polygon through x_{ε_n} can be parametrized as a continuous curve $\tilde{c} : [0,1] \to E$ such that $\tilde{c}$ is smooth on $(0,1]$,*

$$\tilde{c}(\varepsilon_n) = x_{\varepsilon_n} \quad \textit{and} \quad \tilde{c}(0) = x.$$

Proof. We will construct a continuous function $f : [0,1] \to \mathbb{R}$ that satisfies $f(\varepsilon_n) = \frac{1}{n}$ and $f(0) = 0$ and is smooth on $(0,1]$, and then compose it with the curve c of lemma 5.2.4 to obtain the required $\tilde{c}$.

Define an open cover $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ of $(0,1]$ such that $U_n := (\varepsilon_{n+1}, \varepsilon_{n-1})$ for $n > 1$, $U_1 := (\varepsilon_2, 1]$. By theorem 2.2.4 there exists a smooth partition of unity $(\chi_n)_{n\in\mathbb{N}}$ subordinate to $\mathcal{U}$. Because $\chi_n^{(k)}(\varepsilon) = 0$ for $\varepsilon < \varepsilon_{n+1}$ and any $k \in \mathbb{N} \cup \{0\}$, each χ_n can be extended smoothly to $\varepsilon = 0$ by defining $\chi_n(0) := 0$. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(\varepsilon) := \sum_{n=1}^{\infty} \frac{1}{n}\chi_n(\varepsilon).$$

Obviously, f is smooth on $(0,1]$. Moreover,

$$|f(\varepsilon)| = \left|\sum_{n=1}^{\infty} \frac{1}{n}\chi_n(\varepsilon)\right| \le \sum_{n=m}^{\infty} \frac{1}{n}\chi_n(\varepsilon) \le \frac{1}{m} \underbrace{\sum_{n=m}^{\infty} \chi_n(\varepsilon)}_{\le 1} \le \frac{1}{m} \tag{5.4}$$

for $\varepsilon \le \varepsilon_m$. Thus $\lim_{\varepsilon\to 0} f(\varepsilon) = 0 = f(0)$ and f is continuous on $[0,1]$. This implies that $\tilde{c} = c \circ f$ is continuous on $[0,1]$, smooth on $(0,1]$ and satisfies $\tilde{c}(0) = c(0) \overset{5.2.4}{=} x$.

Furthermore, $f(\varepsilon_n) = \frac{1}{n}$ because $\chi_n(\varepsilon_n) = 1$ for each $n \in \mathbb{N}$. This implies

$$\tilde{c}(\varepsilon_n) = (c \circ f)(\varepsilon_n) = c\left(\tfrac{1}{n}\right) \overset{5.2.4}{=} x_{\varepsilon_n}.$$ □

Remark 5.2.7 ($\tilde{c}$ is not smooth at 0)**.** Generally it is not possible to have that f is smooth at 0. Let

$$\varepsilon_n := e^{-n}.$$

By definition of f and (5.4) we have that $0 \le f(\varepsilon) \le \frac{1}{m}$ for $\varepsilon \in [0, e^{-m}]$. In particular, $f(\varepsilon_m) = \frac{1}{m}$ since χ_m is the only non-zero summand at ε_m by the definition of $\mathcal{U}$. By the mean value theorem there exists an element $\xi_m \in [0, e^{-m}]$ such that

$$f'(\xi_m) = \frac{f(\varepsilon_m) - f(0)}{\varepsilon_m} = \frac{\frac{1}{m}}{e^{-m}} = \frac{e^m}{m}.$$

Thus $f'(\xi_m) \to \infty$ for $m \to \infty$ and f is not even differentiable at 0.

In some cases, however, $\tilde{c}$ of course may be smooth on all of $[0,1]$, e.g. if $\varepsilon_n = \frac{1}{n}$ as in the original special curve lemma 5.2.4.

Theorem 5.2.2 pertains to the local theory on $\mathbb{R}^n$. An analogous result holds true on a smooth manifold X by restricting to geodesically convex chart neighborhoods and applying 5.2.2 there:

Theorem 5.2.8 (Point value characterization in $\mathcal{G}(X)$)**.** *Let $u \in \mathcal{G}(X)$. Then*

$$u = 0 \text{ in } \mathcal{G}(X) \Longleftrightarrow u(\tilde{p}) = 0 \text{ in } \widetilde{\mathbb{C}} \text{ for all } \tilde{p} \in \widetilde{X}_c.$$

Proof. By [Nig06], proposition 3.16, we know that $u(\tilde{p}) := [(u_\varepsilon(p_\varepsilon))_\varepsilon]$ is a well-defined element of $\widetilde{\mathbb{C}}$. For the proof of 5.2.8 see also [Nig06], theorem 3.17. □

5.3 Continuity in $\mathcal{G}(X)$

Sharp topologies give a means of endowing spaces of Colombeau generalized functions with the structure of locally convex $\widetilde{\mathbb{C}}$-modules. For an in-depth treatment of this theory we refer to [Gar05b] and [Gar05a]. We shall only make use of the following particular result.

Proposition 5.3.1 (Continuity in the sharp topology)**.** *Let X be a smooth submanifold of $\mathbb{R}^m$, $u \in \mathcal{G}(X)$ and $K \subset\subset X$. Then*

$$\begin{aligned} &\forall l \in \mathbb{N}\, \exists n \in \mathbb{N}\, \exists \varepsilon_0 > 0\, \forall \varepsilon \le \varepsilon_0\, \forall p, q \in K : \\ &\quad |p - q| \le \varepsilon^n \Rightarrow |u_\varepsilon(p) - u_\varepsilon(q)| \le \varepsilon^l. \end{aligned} \tag{5.5}$$

Proof. First, let Ω be an open subset of $\mathbb{R}^k$, $w \in \mathcal{G}(\Omega)$ and $L \subset\subset \Omega$. As in corollary 2.2.6 there exists a smooth bump function $\chi : \mathbb{R}^k \to [0,1]$ such that $\chi|_L = 1$ and $\chi|_{\mathbb{R}^k \setminus \Omega} = 0$. Consider $\overline{w} := [(\overline{w}_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^k)$ defined by $\overline{w}_\varepsilon := w_\varepsilon \cdot \chi : \mathbb{R}^k \to \mathbb{C}$. If we denote the closed convex hull of L by $\mathrm{ch}(L)$, then we obtain for $x, y \in L$ by the mean value theorem that

$$\begin{aligned} |w_\varepsilon(x) - w_\varepsilon(y)| &= |\overline{w}_\varepsilon(x) - \overline{w}_\varepsilon(y)| \\ &\leq |x-y| \cdot \int_0^1 |D\overline{w}_\varepsilon(x + t(y-x))| dt \\ &\leq |x-y| \cdot \|D\overline{w}_\varepsilon\|_{L^\infty(\mathrm{ch}(L))}. \end{aligned} \tag{5.6}$$

Since $(\overline{w}_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^k)$ we know that $\|D\overline{w}_\varepsilon\|_{L^\infty(\mathrm{ch}(L))} \leq \varepsilon^{-N}$ for some $N \in \mathbb{N}$ and ε smaller than a certain $\varepsilon_0 > 0$. Given $l \in \mathbb{N}$, we set $n := l + N$ and conclude that

$$|w_\varepsilon(x) - w_\varepsilon(y)| \overset{(5.6)}{\leq} \|D\overline{w}_\varepsilon\|_{L^\infty(\mathrm{ch}(L))}|x-y| \leq \varepsilon^{-N}\varepsilon^n = \varepsilon^l$$

for $\varepsilon < \varepsilon_0$ and $|x - y| \leq \varepsilon^n$. Thus (5.5) holds in this case.

Now let X be a smooth submanifold of $\mathbb{R}^m$ and $K \subset\subset X$. Let g be the Riemannian metric on X induced by the Euclidean metric on $\mathbb{R}^m$. For each $p \in K$ there exists a geodesically convex neighborhood W_p such that $\overline{W_p} \subset\subset V_p$ for a chart (v_p, V_p) at p, by theorem 4.2.4. Since K is compact, it may be covered by finitely many such sets W_i, $i = 1, ..., k$. The respective charts are also denoted by (v_i, V_i), $i = 1, ..., k$. By the Lebesgue covering theorem 4.3.1, there exists a Lebesgue number $\delta > 0$ such that for each $p \in K$ there exists some $i \in \{1, ..., k\}$ with $B_\delta(p) \subseteq W_i$. W.l.o.g. we can assume that $\varepsilon_0 < \delta < 1$. Thus if $|p - q| \leq \varepsilon^n \leq \varepsilon_0^n < \delta$, then p and q belong to the same set $W_i \subseteq V_i$ and we can restrict to this case.

Consider $u_i := u \circ v_i^{-1} \in \mathcal{G}(v_i(V_i))$. Since $v_i(V_i)$ is an open subset of $\mathbb{R}^d$, we may apply (5.5) by the above and obtain that

$$\begin{aligned} &\forall l \in \mathbb{N}\, \exists h_i \in \mathbb{N}\, \exists \varepsilon_i > 0\, \forall \varepsilon \leq \varepsilon_i\, \forall p, q \in W_i : \\ &|v_i(p) - v_i(q)| \leq \varepsilon^{h_i} \Rightarrow |u_\varepsilon(p) - u_\varepsilon(q)| \leq \varepsilon^l. \end{aligned} \tag{5.7}$$

By [GKOS01], proof of lemma 3.2.6, there exists a constant $C_1 > 0$ such that $|v_i(p) - v_i(q)| \leq C_1 d_g(p,q)$ for all $p, q \in \overline{W_i}$. Moreover, by lemma 4.3.2, there exists a constant $C_2 > 0$ such that $d_g(p,q) \leq C_2|p - q|$ for all $p, q \in \overline{W_i}$. Thus

$$\exists C_1, C > 0\, \forall p, q \in W_i : \ |v_i(p) - v_i(q)| \leq C_1 d_g(p,q) \leq C|p-q|$$

if we set $C := C_1 C_2 > 0$. Therefore, by (5.7),

$$\begin{aligned} &\forall l \in \mathbb{N}\, \exists n_i \in \mathbb{N}\, \exists \varepsilon_i > 0\, \forall \varepsilon \leq \varepsilon_i\, \forall p, q \in W_i : \\ &|p - q| \leq \varepsilon^{n_i} \Rightarrow |u_\varepsilon(p) - u_\varepsilon(q)| \leq \varepsilon^l, \end{aligned}$$

where $n_i = h_i + s$ with $s \in \mathbb{N}$ such that $C < \varepsilon_i^{-s}$ (this ensures that $|v_i(p) - v_i(q)| \leq C|p - q| \leq \varepsilon_i^{-s}\varepsilon^{n_i} \leq \varepsilon^{h_i}$ for $\varepsilon < \varepsilon_i$). Finally, given $l \in \mathbb{N}$, set $\varepsilon_0 := \min_{1 \leq i \leq k} \varepsilon_i$ and $n := \max_{1 \leq i \leq k} n_i$. This gives (5.5). □

5.4 Manifold-valued generalized functions

In the previous sections we considered generalized functions $\mathcal{G}(X)$ with values in $\mathbb{C}$. Now we will introduce and analyze generalized functions $\mathcal{G}[X,Y]$ with values in a smooth manifold Y. The following is based on [KSV03] and [Nig06], chapter 4, which may also be consulted for further reading.

5.4.1 C-bounded generalized functions

While $\mathcal{E}_M(X)$, $\mathcal{N}(X)$ and $\mathcal{G}(X)$ in section 5.1 were defined intrinsically without reference to the notion of charts, this is not possible for c-bounded generalized functions where the range space is a smooth manifold Y. In particular, $\mathcal{G}[X,Y]$ cannot be defined as a quotient of moderate and negligible maps, since these are not well-defined once we leave the vector space setting – charts, Riemannian distances and an equivalence relation must be used instead.

However, simpler intrinsic characterizations for moderateness etc. are also available in this setting, see subsection 5.4.2 below.

In the case of smooth submanifolds $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ we can simply consider $\mathcal{G}(X)^n$. This will be examined in section 5.4.3.

Definition 5.4.1. Let X and Y be smooth manifolds. The space $\mathcal{E}_M[X,Y]$ of *compactly bounded (c-bounded) moderate maps* from X to Y is defined as the set of all $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1]\times X, Y)$ which satisfy the following properties:

(i) $\forall K \subset\subset X\ \exists K' \subset\subset Y\ \exists \varepsilon_0 > 0$ such that $\forall \varepsilon < \varepsilon_0 : u_\varepsilon(K) \subseteq K'$.

(ii) $\forall k \in \mathbb{N}_0\ \forall$ charts (a,A) in $X\ \forall$ charts (b,B) in $Y\ \forall L \subset\subset A\ \forall L' \subset\subset B$ $\exists N \in \mathbb{N}$ such that

$$\sup_{x\in L\cap u_\varepsilon^{-1}(L')} |D^{(k)}(b\circ u_\varepsilon \circ a^{-1})(a(x))| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0.$$

Property (i) in definition 5.4.1 is called *c-boundedness*.

In the absence of a linear structure, the concept of a set of negligible elements $\mathcal{N}[X,Y]$ similar to $\mathcal{N}(X)$ makes no sense. The equivalence relation on $\mathcal{E}_M[X,Y]$ must be defined directly:

Definition 5.4.2. Let X and Y be smooth manifolds. Two elements $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$ in $\mathcal{E}_M[X,Y]$ are called *equivalent*, $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$, if

(i) $\forall K \subset\subset X$ and one (hence any) Riemannian metric h on Y:

$$\sup_{p\in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) \to 0 \text{ as } \varepsilon \to 0.$$

(ii) $\forall k \in \mathbb{N}_0\ \forall m \in \mathbb{N}\ \forall$ charts (a,A) in $X\ \forall$ charts (b,B) in $Y\ \forall L \subset\subset A$ $\forall L' \subset\subset B$ we have that

$$\sup_{x\in L\cap u_\varepsilon^{-1}(L')\cap v_\varepsilon^{-1}(L')} |D^{(k)}(b\circ u_\varepsilon\circ a^{-1} - b\circ v_\varepsilon \circ a^{-1})(a(x))| = O(\varepsilon^m) \text{ as } \varepsilon \to 0.$$

The above definition is independent of the choice of the Riemannian metric h on Y as shown in [Nig06], proposition 3.35. Furthermore, $\sim$ really defines an equivalence relation on $\mathcal{E}_M[X,Y]$, cf. [Nig06], proposition 3.36.

Finally, the c-bounded generalized functions are defined as the quotient:

Definition 5.4.3. Let X and Y be smooth manifolds. The space of *compactly bounded* (c-bounded) *generalized functions* from X to Y is defined as

$$\mathcal{G}[X,Y] := \mathcal{E}_M[X,Y]/_\sim.$$

5.4.2 Intrinsic characterizations

The definitions of $\mathcal{E}_M[X,Y]$ and $\mathcal{G}[X,Y]$ were given in terms of charts. As in section 5.2 for elements in $\mathcal{G}(X)$, we will now provide point value characterizations for c-boundedness and intrinsic characterizations for moderate and negligible elements. See [KSV03], section 3, or [Nig06], chapter 4, for the proofs.

Proposition 5.4.4. *Let X and Y be smooth manifolds, $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1] \times X, Y)$. Then the following are equivalent:*

(i) $(u_\varepsilon)_\varepsilon$ *is c-bounded*

(ii) $(u_\varepsilon(p_\varepsilon))_\varepsilon \in \widetilde{Y}_c$ *for all* $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$.

Proof. See [KSV03], proposition 3.1 (i)⇔(iv). □

Proposition 5.4.5. *Let X and Y be smooth manifolds, $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1] \times X, Y)$. Then the following are equivalent:*

(i) $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M[X,Y]$ *(cf. definition 5.4.1)*

(ii) $(f \circ u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ *for all* $f \in \mathcal{C}^\infty(Y)$.

Proof. See [KSV03], proposition 3.2 (a)⇔(c). □

Proposition 5.4.6. *Let X and Y be smooth manifolds, $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{E}_M[X,Y]$. Then the following are equivalent:*

(i) $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$ *(cf. definition 5.4.2)*

(ii) For every Riemannian metric h on Y, every $m \in \mathbb{N}$ and every $K \subset\subset X$ we have

$$\sup_{p \in K} d_h(u_\varepsilon(p), v_\varepsilon(p)) = O(\varepsilon^m) \text{ as } \varepsilon \to 0.$$

(iii) $(f \circ u_\varepsilon - f \circ v_\varepsilon)_\varepsilon \in \mathcal{N}(X)$ *for all* $f \in \mathcal{C}^\infty(Y)$.

Proof. See [KSV03], proposition 3.3 (i)⇔(iii)⇔(v). □

A point value characterization is obtained from the above theorem and the following proposition:

Proposition 5.4.7. *Let X and Y be smooth manifolds, $u \in \mathcal{G}[X,Y]$ and $\tilde{p} \in \widetilde{X}_c$. Then the point value of u at $\tilde{p}$,*

$$u(\tilde{p}) := [(u_\varepsilon(p_\varepsilon))_\varepsilon],$$

is a well-defined element of $\widetilde{Y}_c$.

Proof. See [Nig06], proposition 4.8. ☐

Corollary 5.4.8 (Point value characterization in $\mathcal{G}[X,Y]$)**.** *Let X and Y be smooth manifolds, $u, v \in \mathcal{G}[X,Y]$. Then*

$$u = v \text{ in } \mathcal{G}[X,Y] \Longleftrightarrow u(\tilde{p}) = v(\tilde{p}) \text{ in } \widetilde{Y}_c \text{ for all } \tilde{p} \in \widetilde{X}_c.$$

Proof. See [Nig06], proposition 4.9, or [KSV03], theorem 3.5. ☐

5.4.3 C-bounded generalized functions on submanifolds

Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be smooth submanifolds. Of course we can consider elements in $\mathcal{G}[X,Y]$ using the definitions in 5.4.1. Alternatively, given a surrounding space, we can also study functions in $\mathcal{G}(X)^n$ and ask when they are elements in $\mathcal{G}[X,Y]$. This is done in the following.

As above, an element $u \in \mathcal{G}(X)^n$ is called *c-bounded into Y* if there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that

$$\forall K \subset\subset X \, \exists K' \subset\subset Y \, \exists \varepsilon_0 > 0 \text{ such that } \forall \varepsilon < \varepsilon_0 : \, u_\varepsilon(K) \subseteq K'.$$

We make use of an increasing sequence of compact sets which always exists on a manifold with a countable basis:

Definition 5.4.9. Let X be a manifold. Then $(K_n)_{n\in\mathbb{N}}$ is called an *exhaustion* of X *by compact sets* if $K_n \subseteq \mathrm{int}(K_{n+1})$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$.

Remark 5.4.10 (Construction of compact exhaustion)**.** Let X be a Hausdorff and second countable manifold. In particular, X is locally compact, hence there exists an open cover $\mathcal{U} = (U_p)_{p\in X}$ of X such that all $\overline{U_p}$ are compact. Since X is second countable we may choose a countable sub-cover $(U_n)_{n\in\mathbb{N}}$ of $\mathcal{U}$. The exhaustion by compact sets is now constructed inductively. First, let $K_1 := \overline{U_1}$. Assume that we have already defined $K_n \subset\subset X$. As K_n is compact we may find $n + 1 \leq k_{n+1} \in \mathbb{N}$ such that $K_n \subseteq \bigcup_{j=1}^{k_{n+1}} U_j$. Define

$$K_{n+1} := \bigcup_{j=1}^{k_{n+1}} \overline{U_j}.$$

We have that $K_{n+1} \subset\subset X$, $K_n \subseteq \mathrm{int}(K_{n+1})$ and $\bigcup_{n=1}^{\infty} K_n \supseteq \bigcup_{n=1}^{\infty} U_n = X$.

Proposition 5.4.11. *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be smooth submanifolds, $u \in \mathcal{G}(X)^n$. Then the following are equivalent:*

(i) u is c-bounded into Y.

(ii) $u(\widetilde{X}_c) \subseteq \widetilde{Y}_c$ *as a pointwise function on compactly supported generalized points.*

(iii) For one (hence any) representative $(u_\varepsilon)_\varepsilon$ *of* u *we have that*

$$\forall K \subset\subset X \,\exists K' \subset\subset Y \,\forall l \in \mathbb{N} : \sup_{p \in K} d(u_\varepsilon(p), K') = O(\varepsilon^l) \text{ as } \varepsilon \to 0,$$

where d *denotes the Euclidean metric in* $\mathbb{R}^n$.

Proof. (i ⇒ ii) Let $(p_\varepsilon)_\varepsilon$ be a representative of $\tilde{p} \in \widetilde{X}_c$. There exists a compact set $K \subset\subset X$ and $\varepsilon_1 > 0$ such that $p_\varepsilon \in K$ for all $\varepsilon < \varepsilon_1$. Since $u \in \mathcal{G}(X)^n$ is c-bounded into Y we furthermore have a representative $(u_\varepsilon)_\varepsilon$ of u, $K' \subset\subset Y$ and $\varepsilon_2 > 0$ such that $u_\varepsilon(K) \subseteq K'$ for $\varepsilon < \varepsilon_2$. Thus for all $\varepsilon < \varepsilon_0 := \min(\varepsilon_1, \varepsilon_2)$ we have that $u_\varepsilon(p_\varepsilon) \in K'$, i.e. $u(\tilde{p}) \in \widetilde{Y}_c$.

(ii ⇒ iii) Suppose to the contrary that

$$\begin{gathered}\exists K \subset\subset X \,\forall K' \subset\subset Y \,\exists l \in \mathbb{N} \,\exists (\varepsilon_k)_k, \varepsilon_k \searrow 0 \,\exists p \in K : \\ d(u_{\varepsilon_k}(p), K') \geq \varepsilon_k^l.\end{gathered} \tag{5.8}$$

Consider $(r_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]}$, defined by

$$r_\varepsilon := \sup_{p \in K} d(u_\varepsilon(p), Y).$$

In general, r_ε will not be smooth in ε. Nevertheless, we may speak of moderateness and negligibility for $(r_\varepsilon)_\varepsilon$. There are two distinct cases to consider in order to construct $\tilde{p} \in \widetilde{X}_c$ that satisfies $u(\tilde{p}) \notin \widetilde{Y}_c$:

- $(r_\varepsilon)_\varepsilon$ is not negligible: Hence there exists a sequence $(\varepsilon_k)_k$, $\varepsilon_k \searrow 0$, such that $r_{\varepsilon_k} \geq \varepsilon_k^l$ for some $l \in \mathbb{N}$. In particular, we have points $p_{\varepsilon_k} \in K$ such that $d(u_{\varepsilon_k}(p_{\varepsilon_k}), Y) \geq \varepsilon_k^l$, meaning that (5.8) also holds for Y instead of $K' \subset\subset Y$. In particular, $(u_{\varepsilon_k}(p_{\varepsilon_k}))_k$ does not approach Y sufficiently fast.

  We are going to extend the sequence $(p_{\varepsilon_k})_k$ to an element $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$. Since K is compact we may assume that $(p_{\varepsilon_k})_k$ converges towards some $p \in K$. Let (v, V) be a chart of X at p such that $\overline{V} \subset\subset X$ and $v(V)$ is convex. W.l.o.g. we have that $p_{\varepsilon_k} \in V$ for all $k \in \mathbb{N}$. As in the proof of theorem 5.2.2 we may assume that $(v(p_{\varepsilon_k}))_k$ converges fast towards $v(p) \in v(V) \subseteq \mathbb{R}^d$. Thus by the special curve lemma 5.2.6 we can extend $(v(p_{\varepsilon_k}))_k$ to a smooth net $(v(p_\varepsilon))_\varepsilon$ by setting $v(p_\varepsilon) := \tilde{c}(\varepsilon)$. Finally,

  $$p_\varepsilon := v^{-1}(\tilde{c}(\varepsilon))$$

  defines a compactly supported point $\tilde{p}$ in $\widetilde{X}_c$. The values p_{ε_k} stayed the same. Since $d(u_{\varepsilon_k}(p_{\varepsilon_k}), Y) \geq \varepsilon_k^l$ for some $l \in \mathbb{N}$ as above, we have that $u(\tilde{p}) \notin \widetilde{Y}_c$, a contradiction to (ii).

- $(r_\varepsilon)_\varepsilon$ is negligible: Let $(K_j)_{j \in \mathbb{N}}$ be an exhaustion of Y by compact sets with $K_j \subseteq \mathrm{int}(K_{j+1})$ for all $j \in \mathbb{N}$. Such a $(K_j)_j$ exists by remark 5.4.10. By (5.8) we have $l_1 \in \mathbb{N}$, $\varepsilon_1 > 0$ and $p_{\varepsilon_1} \in K$ such that $d(u_{\varepsilon_1}(p_{\varepsilon_1}), K_1) \geq \varepsilon_1^{l_1}$. Since, by assumption, $(r_\varepsilon)_\varepsilon$ is negligible, we may assume w.l.o.g. that

$d(u_{\varepsilon_1}(p_{\varepsilon_1}), Y) \leq \sup_{p\in K} d(u_{\varepsilon_1}(p), Y) = r_{\varepsilon_1} < \varepsilon_1^{l_1}$ (this can be achieved possibly by decreasing ε_1). Inductively this can be done for all $j \in \mathbb{N}$, again by (5.8) – we obtain $l_j \in \mathbb{N}$, $\varepsilon_j > 0$ and $p_{\varepsilon_j} \in K$ such that $d(u_{\varepsilon_j}(p_{\varepsilon_j}), K_j) \geq \varepsilon_j^{l_j} > r_{\varepsilon_j}$. W.l.o.g. we may assume that $l_j > j$ and $\varepsilon_j < \min\{\varepsilon_{j-1}, \frac{1}{j}\}$ (for ε_j just pick an element of $(\varepsilon_k)_k$ w.r.t. K_j in (5.8) sufficiently small). Thus we obtain sequences $(\varepsilon_j)_j$, $\varepsilon_j \searrow 0$, $(l_j)_j$ in $\mathbb{N}$ and $(p_{\varepsilon_j})_j$ in K such that

$$d(u_{\varepsilon_j}(p_{\varepsilon_j}), Y) \leq r_{\varepsilon_j} < \varepsilon_j^{l_j} \leq d(u_{\varepsilon_j}(p_{\varepsilon_j}), K_j).$$

In particular, there exists $q_j \in Y \setminus K_j$ such that

$$d(u_{\varepsilon_j}(p_{\varepsilon_j}), q_j) \leq \varepsilon_j^{l_j} \leq \varepsilon_j^{j}, \tag{5.9}$$

because $d(u_{\varepsilon_j}(p_{\varepsilon_j}), Y) = \inf_{q\in Y} d(u_{\varepsilon_j}(p_{\varepsilon_j}), q)$. As in the first case, we can extend $(p_{\varepsilon_j})_j$ to an element $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$. By (ii), $u(\tilde{p}) \in \widetilde{Y}_c$. Then there exists a compact set $K' \subset\subset Y$ such that $u_\varepsilon(p_\varepsilon) \in K'$ for all ε sufficiently small. Since $(K_j)_{j\in\mathbb{N}}$ is a compact exhaustion, there exists $M \in \mathbb{N}$ such that $K' \subseteq K_M$. For $j > M$ we have that $K_M \subseteq \text{int}(K_j)$ and $d(q_j, K_M) \geq d(Y \setminus K_j, K_M) \geq d(Y \setminus K_{M+1}, K_M) = \delta > 0$. The sequence $(\varepsilon_j)_j$ converges strictly monotonically to zero, thus there exists $N \in \mathbb{N}$ such that for all $j \geq N$ we have that $\varepsilon_j^j \leq \varepsilon_N^N < \delta$. Putting this information together, we obtain that for j sufficiently large we have that $u_{\varepsilon_j}(p_{\varepsilon_j}) \notin K_M$, because

$$d(u_{\varepsilon_j}(p_{\varepsilon_j}), K_M) \geq d(q_j, K_M) - d(u_{\varepsilon_j}(p_{\varepsilon_j}), q_j) \overset{(5.9)}{\geq} \delta - \varepsilon_j^j > 0.$$

Thus $u(\tilde{p}) \notin \widetilde{Y}_c$, again a contradiction to (ii).

(iii $\Rightarrow$ i) Let $(u_\varepsilon)_\varepsilon$ be any representative of $u \in \mathcal{G}(X)^n$. Since Y is a smooth submanifold of $\mathbb{R}^n$ there exists a normal tubular neighborhood V of Y in $\mathbb{R}^n$ with an associated smooth retraction $r : V \to Y$ such that the unique closest point of $q \in V$ in Y is $r(q)$ by [Hir76], section 4.5. Moreover, u is c-bounded into V by (iii). Thus first of all, $r \circ u \in \mathcal{G}(X)^n$ is c-bounded into Y. Let $K \subset\subset X$ and $p \in K$. For ε sufficiently small we have that $u_\varepsilon(p) \in V$ and therefore $(r \circ u_\varepsilon)(p) \in Y$ is the closest point in Y to $u_\varepsilon(p)$. Hence (iii) implies that for some $K' \subset\subset Y$, all $l \in \mathbb{N}$ and ε sufficiently small,

$$\sup_{p\in K} |(r \circ u_\varepsilon)(p) - u_\varepsilon(p)| = \sup_{p\in K} d(u_\varepsilon(p), Y) \leq \sup_{p\in K} d(u_\varepsilon(p), K') = O(\varepsilon^l).$$

Thus $u = r \circ u \in \mathcal{G}(X)^n$ and is therefore c-bounded into Y itself. □

Now, given a c-bounded function in $\mathcal{G}(X)^n$ we have:

Proposition 5.4.12. *Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ be smooth submanifolds. Then every element $u \in \mathcal{G}(X)^n$ which is c-bounded into $Y \subseteq \mathbb{R}^n$ defines a unique element in $\mathcal{G}[X, Y]$.*

We show this by following the arguments in [KSV09], remark 2.4. A compact exhaustion is used to construct another representative of u with the required property.

Lemma 5.4.13. *Let X be a smooth manifold and $(K_n)_{n\in\mathbb{N}}$ an exhaustion of X by compact sets. Suppose we are given a decreasing, positive sequence $(\varepsilon_n)_n$. Then there exists a smooth function $\eta : X \to \mathbb{R}$ such that*

$$0 < \eta(p) \leq \varepsilon_n$$

for all $p \in K_n \setminus \operatorname{int}(K_{n-1})$ and all $n \in \mathbb{N}$ (set $K_0 := \emptyset$).

Proof. Set $U_n := \operatorname{int}(K_n) \setminus K_{n-2}$ for $n \geq 2$ (recall that $K_0 := \emptyset$) and $U_1 := \emptyset$. Then $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is an open cover of X and by theorem 2.2.4 there exists a subordinate partition of unity $(\chi_n)_{n\in\mathbb{N}}$. For $p \in X$ we define

$$\eta(p) := \sum_{n=1}^{\infty} \varepsilon_n \chi_n(p). \tag{5.10}$$

The functions χ_n are smooth and the sum is locally finite, hence η is smooth on X.

Let $p \in K_n \setminus \operatorname{int}(K_{n-1})$. By assumption, $\operatorname{int}(K_{n-1}) \supseteq \operatorname{int}(K_i) \supseteq U_i$ for $i \leq n-1$, thus $p \notin \bigcup_{i=1}^{n-1} U_i$. Furthermore, $K_n \cap U_i = K_n \cap (\operatorname{int}(K_i) \setminus K_{i-2}) \subseteq K_n \cap \complement K_{i-2} = \emptyset$ for $i \geq n+2$, hence $p \notin \bigcup_{i=n+2}^{\infty} U_i$. Thus altogether we obtain that

$$p \in K_n \setminus \operatorname{int}(K_{n-1}) \subseteq \bigcup_{i=1}^{\infty} U_i \setminus \bigcup_{\substack{i\leq n-1\\ i\geq n+2}} U_i \subseteq (U_n \cup U_{n+1}) \setminus \bigcup_{i\neq n,n+1} U_i$$

This implies that $\chi_n(p) + \chi_{n+1}(p) = 1$ and therefore

$$\eta(p) \overset{(5.10)}{=} \varepsilon_n \chi_n(p) + \varepsilon_{n+1}\chi_{n+1}(p).$$

By the assumptions on $(\varepsilon_n)_n$ this leads to $0 < \varepsilon_{n+1} = \varepsilon_{n+1}(\chi_n(p) + \chi_{n+1}(p)) \leq \varepsilon_n\chi_n(p) + \varepsilon_{n+1}\chi_{n+1}(p) = \eta(p) \leq \varepsilon_n(\chi_n(p) + \chi_{n+1}(p)) = \varepsilon_n$. □

Now we are ready to prove the proposition mentioned in the beginning:

Proof of 5.4.12. Let $(u_\varepsilon)_\varepsilon$ be any representative of the c-bounded function $u \in \mathcal{G}(X)^n$. Thus we have that

$$\forall K \subset\subset X \,\exists K' \subset\subset Y \,\exists \varepsilon_0 > 0 \text{ such that } \forall \varepsilon < \varepsilon_0 : \, u_\varepsilon(K) \subseteq K'. \tag{5.11}$$

We want to show that there exists a representative $(v_\varepsilon)_\varepsilon$ of u such that $v_\varepsilon(p) \in Y$ $\forall p \in X \,\forall \varepsilon \in (0,1]$. To this end, let $(K_j)_{j\in\mathbb{N}}$ be an exhaustion of $X \subseteq \mathbb{R}^m$ by compact sets (cf. remark 5.4.10). By (5.11) there exist $K'_j \subset\subset Y$ and $\varepsilon_j > 0$ such that

$$u_\varepsilon(K_j) \subseteq K'_j \tag{5.12}$$

for all $\varepsilon \leq \varepsilon_j$. W.l.o.g. assume that $\varepsilon_{j+1} \leq \varepsilon_j$ for all $j \in \mathbb{N}$. Then there exists a smooth function $\eta : X \to \mathbb{R}$ such that

$$0 < \eta(p) \leq \varepsilon_j \text{ for all } p \in K_j \setminus \operatorname{int}(K_{j-1}) \tag{5.13}$$

by lemma 5.4.13. This function η will later guarantee that $(v_\varepsilon)_\varepsilon$ always ends up in Y. Moreover, we need to ensure that $(v_\varepsilon)_\varepsilon$ is a representative of the same generalized function u. Therefore we need another smooth function $\nu : \mathbb{R}_0^+ \to [0,1]$ that satisfies $\nu(x) \leq x$ for all $x \in \mathbb{R}^+$ and

$$\nu(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1 & x \geq \frac{3}{2} \end{cases} . \tag{5.14}$$

To construct ν we consider the open cover $\mathcal{U}$ of $\mathbb{R}_0^+$ that consists of the sets $U_1 := [0,1)$, $U_2 := (\frac{1}{2}, \frac{3}{2})$ and $U_3 := (1, \infty)$. By theorem 2.2.4 there exists a subordinate smooth partition of unity (χ_1, χ_2, χ_3) such that $\operatorname{supp} \chi_i \subseteq U_i$ and $\sum_{i=1}^{3} \chi_i = 1$. Then $\nu(x) := x \cdot \chi_1(x) + \frac{1}{2} \cdot \chi_2(x) + \chi_3(x)$ meets all conditions mentioned before:

$$\begin{array}{ll} x \in [0, \frac{1}{2}] : & \nu(x) = x\chi_1(x) = x \\ x \in (\frac{1}{2}, 1] : & \nu(x) = x\chi_1(x) + \frac{1}{2}\chi_2(x) \leq x(\chi_1(x) + \chi_2(x)) = x \\ x \in (1, \frac{3}{2}] : & \nu(x) = \frac{1}{2}\chi_2(x) + \chi_3(x) \leq \chi_2(x) + \chi_3(x) = 1 \leq x \\ x \in (\frac{3}{2}, \infty) : & \nu(x) = \chi_3(x) = 1 \leq x \end{array}$$

Finally, let μ be the smooth function

$$\begin{aligned} \mu : \ (0,1] \times X &\to (0,1] \\ (\varepsilon, p) &\mapsto \eta(p)\, \nu\left(\frac{\varepsilon}{\eta(p)}\right) \end{aligned}$$

and set

$$v_\varepsilon(p) := u_{\mu(\varepsilon,p)}(p)$$

for $\varepsilon \in (0,1]$ and $p \in X$. Obviously, $(v_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1] \times X, \mathbb{R}^n)$. It remains to be shown that $(v_\varepsilon)_\varepsilon$ is another representative of $u \in \mathcal{G}(X)^n$ and $(v_\varepsilon)_\varepsilon \in \mathcal{E}_M[X,Y]$.

So let $K \subset\subset X$. Then $\mu(\varepsilon,p) = \eta(p)\frac{\varepsilon}{\eta(p)} = \varepsilon$ for $p \in K$ and $\varepsilon < \frac{\min_{q \in K} \eta(q)}{2}$ by (5.14). Thus finally $v_\varepsilon = u_\varepsilon$ on every $K \subset\subset X$, hence $(v_\varepsilon)_\varepsilon \in [(u_\varepsilon)_\varepsilon]$.

Now let p be an arbitrary point in X. Since $X = \bigcup_{i=1}^{\infty} K_i$ there exists a $j \in \mathbb{N}$ such that $p \in K_j \setminus \operatorname{int}(K_{j-1})$. Therefore, for any $\varepsilon > 0$, $\mu(\varepsilon,p) \leq \eta(p) \cdot 1 \leq \varepsilon_j$ by (5.13), which by (5.12) implies that $v_\varepsilon(p) \in v_\varepsilon(K_j) = u_{\mu(\varepsilon,p)}(K_j) \subseteq K_j' \subset\subset Y$. Altogether this ensures that $(v_\varepsilon)_\varepsilon$ is an element of $\mathcal{E}_M[X,Y]$.

Uniqueness is obvious, because each $[(w_\varepsilon)_\varepsilon] \in \mathcal{G}[X,Y]$ with this property has to satisfy $(w_\varepsilon)_\varepsilon \in [(u_\varepsilon)_\varepsilon]$ and is therefore equivalent to the constructed $(v_\varepsilon)_\varepsilon$. □

5.5 Composition of generalized functions

The composition in the sense $\mathcal{G}(Y) \circ \mathcal{G}[X,Y] \subseteq \mathcal{G}(X)$ is well-defined:

Proposition 5.5.1. *Let X and Y be smooth manifolds, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X,Y]$ and $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}(Y)$. Then the composition $v \circ u$, defined by*

$$v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon],$$

is a well-defined generalized function in $\mathcal{G}(X)$.

Proof. See [Nig06], theorem 5.3. □

The composition with smooth functions is also possible:

Proposition 5.5.2. *Let X, Y and Z be smooth manifolds, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X,Y]$, $i \in \mathcal{C}^\infty(Y,Z)$ and $j \in \mathcal{C}^\infty(Z,X)$. Then*

(i) $i \circ u := [(i \circ u_\varepsilon)_\varepsilon]$ *is a well-defined element in* $\mathcal{G}[X,Z]$

(ii) $u \circ j := [(u_\varepsilon \circ j)_\varepsilon]$ *is a well-defined element in* $\mathcal{G}[Z,Y]$.

Proof. See [Nig06], corollary 5.2. □

Remark 5.5.3. It is also possible to define the composition of two c-bounded generalized functions. Let X, Y and Z be smooth manifolds, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X,Y]$ and $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}[Y,Z]$. Then the composition $v \circ u$ is defined by

$$v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon].$$

By [Nig06], theorem 5.1, $v \circ u$ is a well-defined element in $\mathcal{G}[X,Z]$.

5.5.1 Invertibility in $\mathcal{G}[X,Y]$

Now we are also able to define invertibility for compactly bounded generalized functions in $\mathcal{G}[X,Y]$. Of course, we require an identity map first:

Definition 5.5.4. Let X be a smooth manifold. Then $\mathrm{id} \in \mathcal{G}[X,X]$, defined by

$$\mathrm{id} := [(\mathrm{id})_\varepsilon],$$

is called the *identity in* $\mathcal{G}[X,X]$.

In order to distinguish the identities of different underlying manifolds, we may also write $\mathrm{id}_{\mathcal{G}[X,X]} = [(\mathrm{id}_X)_\varepsilon]$ for the identity in $\mathcal{G}[X,X]$.

Definition 5.5.5. Let X and Y be smooth manifolds, $u \in \mathcal{G}[X,Y]$. Then u is called *invertible* if there exists $v \in \mathcal{G}[Y,X]$ such that

$$u \circ v = \mathrm{id}_{\mathcal{G}[Y,Y]} \quad \text{and} \quad v \circ u = \mathrm{id}_{\mathcal{G}[X,X]}\,.$$

Remark 5.5.6 (Existence of (local) inverse)**.**

(i) The inverse v of u may be denoted by u^{-1} although usually $v_\varepsilon \neq u_\varepsilon^{-1}$. In general, we do not know whether the u_ε's in $\mathcal{C}^\infty(X,Y)$ are diffeomorphisms or even bijective. However, for sufficiently small ε the latter has to be the case – at least locally around some point $p \in X$ – by the chain rule and the classical inverse function theorem.

(ii) The question arises of when a generalized function $\mathcal{G}[X,Y]$ is actually invertible. Suppose $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[X,Y]$, u_ε are diffeomorphisms for sufficiently small ε and $(u_\varepsilon^{-1})_\varepsilon \in \mathcal{E}_M[Y,X]$. This obviously implies that u is invertible with inverse $v := [(u_\varepsilon^{-1})_\varepsilon]$. The converse, however, may not hold.

(iii) For a thorough discussion of (local) invertibility of generalized functions $u \in \mathcal{G}[\Omega,\mathbb{R}^n]$, Ω an open subset of $\mathbb{R}^n$, see [Erl07]. Local invertibility there, for example, follows from additional assumptions such as so-called ca-injectivity, ca-surjectivity and that $\det Du$ is strictly non-zero, cf. theorem 3.37 there.

Chapter 6

Isomorphisms of Algebras of Generalized Functions

As in the case of algebras of smooth functions in chapter 3, our aim is to prove that any algebra isomorphism $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$ between the special Colombeau algebras defined on smooth manifolds is given by the pullback under a c-bounded generalized function $\psi \in \mathcal{G}[X, Y]$, i.e. that $\Psi(u) = u \circ \psi$ for all $u \in \mathcal{G}(X)$.

Our approach is based on the algebraic properties of non-zero multiplicative linear functionals $\varphi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$. As in section 3.1 we will identify the compactly supported generalized points with the ideals $\ker \varphi$. However, as will be shown in section 6.1, $\widetilde{\mathbb{C}}$ is not a field and $\ker \varphi$ is not a maximal ideal. Thus some adaptions are required.

The main ideas for these modifications are taken from [Ver06], where Hans Vernaeve proved a similar result for Colombeau algebras with non-smooth dependence on the index ε. In section 6.2 we discuss extensively the main changes required: invertibility and strictly non-zero w.r.t. so-called characterstic sets, and maximal ideals I w.r.t. the property $I \cap \widetilde{\mathbb{C}}1 = \{0\}$.

The proofs in section 6.3 and section 6.4 roughly follow the smooth case as presented in section 3.1, but of course are more involved in order to handle the algebraic peculiarities mentioned above.

Finally, a comparison between the definitions and proofs of [Ver06] and this chapter, i.e. between the smooth and non-smooth dependence of generalized functions on ε, is drawn.

6.1 General approach and problems

A fundamental ingredient of the proofs in section 3.1, in particular theorem 3.1.1, was that $\ker \varphi$ is a maximal ideal for non-zero multiplicative linear functionals $\varphi : \mathcal{C}^\infty(X) \to \mathbb{R}$. For obvious reasons, it is an ideal. It is maximal because the co-domain of φ, namely $\mathbb{R}$, is a field – as seen in lemma 3.1.2 (ii).

6.1.1 Surjectivity of multiplicative $\widetilde{\mathbb{C}}$-linear maps

For arbitrary commutative $\widetilde{\mathbb{C}}$-algebras with unit, the image of an idempotent under algebra homomorphisms reveal some of their structure:

Definition 6.1.1. An element e of a ring R resp. an algebra $\mathcal{A}$ is called an *idempotent* if $e^2 = e$. To exclude the trivial cases we assume that $e \neq 0$ and $e \neq 1$.

Lemma 6.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be commutative $\widetilde{\mathbb{C}}$-algebras with 1. Then the following holds:*

(i) If a multiplicative $\widetilde{\mathbb{C}}$-linear map $\Psi : \mathcal{A} \to \mathcal{B}$ is surjective, then $\Psi(1) = 1$.

(ii) A multiplicative $\widetilde{\mathbb{C}}$-linear functional $\varphi : \mathcal{A} \to \widetilde{\mathbb{C}}$ is surjective if and only if $\varphi(1) = 1$.

Proof. (i) By multiplicativity we have that $\Psi(1) = \Psi(1 \cdot 1) = \Psi(1)\Psi(1)$, hence either $\Psi(1) = 0$, $\Psi(1) = 1$ or $\Psi(1) = e$ for e an idempotent of $\mathcal{B}$. If $\Psi(1) = 0$, then $\Psi \equiv 0$ is not surjective because $0 \neq 1 \in \mathcal{B}$, a contradiction to the assumption. Suppose that $\Psi(1) = e$. Since Ψ is surjective we have for all $b \in \mathcal{B}$ that $b = \Psi(a) = \Psi(a)e = be$, thus $e = 1$ because the unit is unique. In particular, $\Psi(1) = 1$.

(ii) If we set $\mathcal{B} = \widetilde{\mathbb{C}}$, then $\varphi(1) = 1$ by (i). On the other hand, let $r = [(r_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{C}}$. Hence $r1 \in \mathcal{A}$ and $\varphi(r1) = r\varphi(1) = r \in \widetilde{\mathbb{C}}$ due to $\widetilde{\mathbb{C}}$-linearity. Thus φ is surjective. □

Our algebras $\widetilde{\mathbb{C}}$ and $\mathcal{G}(X)$ have different intrinsic properties though, which also have an impact on algebra homomorphisms:

Proposition 6.1.3. *There are no idempotents in $\widetilde{\mathbb{C}}$.*

Proof. Suppose that $(r_\varepsilon)_\varepsilon \in \mathcal{E}_M$ satisfies $r_\varepsilon r_\varepsilon = r_\varepsilon + n_\varepsilon$ for $(n_\varepsilon)_\varepsilon \in \mathcal{N}$. There are two possible solutions for the quadratic equation:

$$r_{\varepsilon,1} = \frac{1}{2} + \sqrt{\frac{1}{4} + n_\varepsilon} \quad \text{and} \quad r_{\varepsilon,2} = \frac{1}{2} - \sqrt{\frac{1}{4} + n_\varepsilon}. \tag{6.1}$$

Since $(n_\varepsilon)_\varepsilon$ is negligible, there exists $\varepsilon_0 > 0$ such that $|n_\varepsilon| < \frac{1}{8}\ \forall \varepsilon < \varepsilon_0$. Assume w.l.o.g. that there exists $\varepsilon' \in (0, \varepsilon_0]$ such that $r_{\varepsilon'} = r_{\varepsilon',1}$. Thus by continuity of $(r_\varepsilon)_\varepsilon$ in ε and the fact that the solutions in (6.1) for $\varepsilon < \varepsilon_0$ are separated by a neighborhood around $\frac{1}{2}$, we have that $r_\varepsilon = r_{\varepsilon,1}$ in a neighborhood of ε'.

Let $U_1 := \{\varepsilon \in (0, \varepsilon_0] \,|\, r_\varepsilon = r_{\varepsilon,1}\}$ and $U_2 := \{\varepsilon \in (0, \varepsilon_0] \,|\, r_\varepsilon = r_{\varepsilon,2}\}$. By the above, both sets are open (and therefore also closed), disjoint and $U_1 \cup U_2 = (0, \varepsilon_0]$. Thus by connectedness of $(0, \varepsilon_0]$ and the assumption that U_1 is non-empty, we have that $U_1 = (0, \varepsilon_0]$.

Hence,

$$|r_\varepsilon - 1| = \left| \frac{1}{2} + \sqrt{\frac{1}{4} + n_\varepsilon} - 1 \right| = \left| \sqrt{\frac{1}{4} + n_\varepsilon} - \frac{1}{2} \right| \leq \sqrt{\frac{1}{4}} + \sqrt{|n_\varepsilon|} - \frac{1}{2} < \varepsilon^m,$$

for any $m \in \mathbb{N}$ and ε sufficiently small. Thus $r = [(r_\varepsilon)_\varepsilon] = 1$ is not an idempotent. □

This also holds true in $\mathcal{G}(X)$:

Proposition 6.1.4. *Let X be a connected smooth manifold. Then there are no idempotents in $\mathcal{G}(X)$.*

Proof. Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(X)$ such that $u_\varepsilon \cdot u_\varepsilon = u_\varepsilon + n_\varepsilon$ for some $(n_\varepsilon)_\varepsilon \in \mathcal{N}(X)$.

Let $q \in U$. We first consider an open, relatively compact and connected neighborhood U of q in X. There are two possible solutions for the quadratic equation $u_\varepsilon(p) \cdot u_\varepsilon(p) = u_\varepsilon(p) + n_\varepsilon(p)$:

$$u_{\varepsilon,1}(p) = \frac{1}{2} + \sqrt{\frac{1}{4} + n_\varepsilon(p)} \quad \text{and} \quad u_{\varepsilon,2}(p) = \frac{1}{2} - \sqrt{\frac{1}{4} + n_\varepsilon(p)}. \tag{6.2}$$

As $(n_\varepsilon)_\varepsilon$ is negligible, there exists $\varepsilon_0 > 0$ such that $|n_\varepsilon(p)| < \frac{1}{8}$ for all $\varepsilon < \varepsilon_0$ and all $p \in U$. The set $(0, \varepsilon_0] \times U$ is again connected.

By continuity of u in ε and p, both of the sets

$$U_1 := \{(\varepsilon, p) \in (0, \varepsilon_0] \times U \mid u_\varepsilon(p) = u_{\varepsilon,1}(p)\}$$
$$U_2 := \{(\varepsilon, p) \in (0, \varepsilon_0] \times U \mid u_\varepsilon(p) = u_{\varepsilon,2}(p)\}$$

are open and, as they represent a disjoint union of $(0, \varepsilon_0] \times U$, also closed in $(0, \varepsilon_0] \times U$. Since the latter is connected we have that either $U_1 = (0, \varepsilon_0] \times U$ or $U_2 = (0, \varepsilon_0] \times U$. W.l.o.g. we may assume that it is U_1. Thus for any $p \in U$, any $m \in \mathbb{N}$ and sufficiently small ε we obtain that

$$\begin{aligned} |u_\varepsilon(p) - 1| &= \left| \frac{1}{2} + \sqrt{\frac{1}{4} + n_\varepsilon(p)} - 1 \right| = \left| \sqrt{\frac{1}{4} + n_\varepsilon(p)} - \frac{1}{2} \right| \\ &\leq \sqrt{\frac{1}{4}} + \sqrt{|n_\varepsilon(p)|} - \frac{1}{2} < \varepsilon^m. \end{aligned}$$

Therefore $u|_U = 1$ in $\mathcal{G}(U)$. In the case $U_2 = (0, \varepsilon_0] \times U$ we have that $u|_U = 0$.

Now consider

$$X_1 := \{p \in X \mid \exists \text{ neighborhood } V \text{ of } p \text{ such that } u|_V = 1\}$$
$$X_2 := \{p \in X \mid \exists \text{ neighborhood } V \text{ of } p \text{ such that } u|_V = 0\}.$$

Both sets are obviously open. Moreover, by the above, each $p \in X$ is either in X_1 or X_2. Thus $X = X_1 \cup X_2$ is a disjoint union. Connectedness of X implies that either $X = X_1$ or $X = X_2$, i.e. u is either 1 or 0 and therefore not an idempotent in $\mathcal{G}(X)$. ☐

Remark 6.1.5 (Non-zero vs. surjective)**.**

(i) As there are no idempotents in $\widetilde{\mathbb{C}}$ by proposition 6.1.3, lemma 6.1.2 (ii) implies that the non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functionals $\varphi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$ are exactly the surjective ones. In particular, $\varphi(1) = 1$ for φ non-zero.

(ii) Let Y be a connected manifold. By proposition 6.1.4 and lemma 6.1.2 (i) then this is also true for non-zero algebra homomorphisms $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$.

In the case of non-connected manifolds, however, there exist idempotents in $\mathcal{G}(Y)$ which are 0 and 1 on different components. Hence, in general, non-zero is not enough to obtain $\Psi(1) = 1$.

In order to have that 1 is mapped to 1 we will therefore only consider surjective algebra homomorphisms φ and Ψ in what follows. As an alternative we could also have restricted our investigations to connected manifolds.

6.1.2 $\ker\varphi$ is not a maximal ideal

Unfortunately, $\widetilde{\mathbb{C}}$ is only a ring but not a field by remark 5.1.5. This implies that the kernels of multiplicative, $\widetilde{\mathbb{C}}$-linear functionals φ on $\widetilde{\mathbb{C}}$-algebras $\mathcal{A}$ (e.g. $\mathcal{G}(X)$) are not maximal ideals:

Proposition 6.1.6. *Let $\mathcal{A}$ be a commutative $\widetilde{\mathbb{C}}$-algebra with unit and $\varphi : \mathcal{A} \to \widetilde{\mathbb{C}}$ a non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functional. Then $\ker\varphi$ is not a maximal ideal.*

Proof. Since $\widetilde{\mathbb{C}}$ is not a field by the above, there exists a non-invertible element $0 \neq r \neq 1$ in $\widetilde{\mathbb{C}}$ (e.g. r as in remark 5.1.5). As φ is non-zero we have that $\varphi(1) = 1$ by remark 6.1.5 (i). Therefore $\varphi(r1) = r\varphi(1) = r \neq 0$, i.e. $0 \neq r1 \notin \ker\varphi$. Obviously, $r\mathcal{A}$ is a non-trivial ideal in $\mathcal{A}$. Since $\ker\varphi$ is an ideal and the sum of ideals is again an ideal (cf. [Bou98], III, §1, 2.), we have that

$$I := \ker\varphi + r\mathcal{A} = \{a + rb \,|\, a \in \ker\varphi, b \in \mathcal{A}\}$$

is an ideal in $\mathcal{A}$. By $r1 \in I \setminus \ker\varphi$ it follows that I is strictly bigger than $\ker\varphi$.

Furthermore, suppose $1 = a + rb \in I$. Then $1 = \varphi(1) = \varphi(a + rb) = \varphi(rb) = r\varphi(b)$ would imply that r is invertible in $\widetilde{\mathbb{C}}$ with inverse $\varphi(b) \in \widetilde{\mathbb{C}}$, a contradiction. Thus there exists an ideal I in $\mathcal{A}$ such that $\ker\varphi \subsetneq I \subsetneq \mathcal{A}$. Hence, by definition 1.3.1, $\ker\varphi$ is not a maximal ideal. ☐

Thus we cannot simply proceed as we did in subsection 3.1.1. In order to proceed, we first require suitable modifications of invertibility and maximal ideals.

6.2 The new maximal ideals and invertibility

The following definitions are taken from or modified from those in [Ver06]. They shall ensure that we can proceed with our approach as in the setting of algebras of smooth functions.

As we will see in subsection 6.2.2, it is essentially clear from the proof of 6.1.6 above how the maximal ideals should be chosen.

The new definition of invertibility was originally inspired by concepts from Non-Standard Analysis[1]. Alternatively, however, we can view it as being motivated by corollary 6.2.8, where we show that by using the new invertibility $\widetilde{\mathbb{C}}$ retains certain properties of a field.

6.2.1 Invertibility with respect to a characteristic set

Characteristic sets and invertibility w.r.t. characteristic sets are defined as follows:

[1] as communicated by Hans Vernaeve at the 'Workshop on Generalized Functions and PDEs' in Innsbruck, February 2009

Definition 6.2.1. A set $S \subseteq (0,1]$ is called *characteristic* if there exists a sequence $\varepsilon_k \searrow 0$ such that $S = \{\varepsilon_k | k \in \mathbb{N}\}$.

Definition 6.2.2. Let $\mathcal{A}$ be a commutative $\widetilde{\mathbb{C}}$-algebra with unit 1 and S a characteristic set. Then $u \in \mathcal{A}$ is called *invertible with respect to* S (or S-invertible) if there exist $v \in \mathcal{A}$ and $r = [(r_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{C}}$ such that

$$uv = r1 \text{ in } \mathcal{A} \quad \text{and} \quad r|_S = 1 \text{ in } \widetilde{\mathbb{C}}.$$

Here $r|_S = 1$ means that for a representative $(r_\varepsilon)_\varepsilon \in \mathcal{E}_M$ of r and some $n = (n_\varepsilon)_\varepsilon \in \mathcal{N}$ we have that $r_{\varepsilon_k} = 1 + n_{\varepsilon_k}$ $\forall \varepsilon_k \in S$. Note also that v is not necessarily unique. Nevertheless we will refer to it as an *S-inverse of u*.

Note that the definition of invertibility w.r.t. S is slightly different in [Ver06]. For more details, see subsection 6.5.1.

In [GKOS01], theorem 1.2.38, it was shown that invertibility in $\widetilde{\mathbb{C}}$ is equivalent to being strictly non-zero. We are going to derive a similar result for S-invertibility. Hence we first have to define what S-strictly non-zero should mean.

Definition 6.2.3. Let S be a characteristic set. Then $r \in \widetilde{\mathbb{C}}$ is called *S-strictly non-zero in* $\widetilde{\mathbb{C}}$ if there exists a representative $(r_\varepsilon)_\varepsilon$ such that

$$\exists k_0 \in \mathbb{N}\, \exists m \in \mathbb{N} \text{ such that } \forall k > k_0 : |r_{\varepsilon_k}| > \varepsilon_k^m. \tag{6.3}$$

Remark 6.2.4. Definition 6.2.3 implies that if a generalized number is S-strictly non-zero, then this holds for any representative.

This can be seen by the following argument: Let $S = \{\varepsilon_k | k \in \mathbb{N}\}$ be a characteristic set, $r = [(r_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{C}}$ S-strictly non-zero and let $(s_\varepsilon)_\varepsilon$ be another representative of r. In particular we have for m as in (6.3), some $\varepsilon_0 > 0$ and all $\varepsilon < \varepsilon_0$ that

$$|r_\varepsilon - s_\varepsilon| < \varepsilon^{2m}.$$

Then

$$|s_{\varepsilon_k}| \geq |r_{\varepsilon_k}| - |r_{\varepsilon_k} - s_{\varepsilon_k}| > \varepsilon_k^m - \varepsilon_k^{2m} > \varepsilon_k^{2m}(\varepsilon_k^{-m} - 1) > \varepsilon_k^{2m}$$

for $\varepsilon_k < \varepsilon_0$ and $\varepsilon_k^{-m} > 2$, hence for k sufficiently large. Thus (6.3) holds for $(s_\varepsilon)_\varepsilon$ as well.

Proposition 6.2.5. *Let $r \in \widetilde{\mathbb{C}}$ and let S be a characteristic set. Then*

$$r \text{ is } S\text{-invertible} \Longleftrightarrow r \text{ is } S\text{-strictly non-zero.}$$

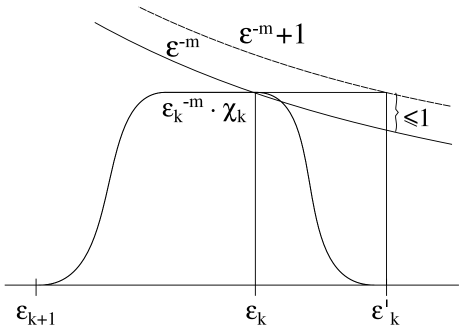
Proof. ($\Rightarrow$) Since $r = [(r_\varepsilon)_\varepsilon]$ is S-invertible there exist $(s_\varepsilon)_\varepsilon \in \mathcal{E}_M$ and $(n_\varepsilon)_\varepsilon \in \mathcal{N}$ such that $r_{\varepsilon_k} s_{\varepsilon_k} = 1 + n_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Now choose $N, k_0 \in \mathbb{N}$ such that $0 < |s_{\varepsilon_k}| < \varepsilon_k^{-N}$ and $|n_{\varepsilon_k}| < \frac{1}{2}$ for all $k > k_0$. Then

$$|r_{\varepsilon_k}| = \left| \frac{1 + n_{\varepsilon_k}}{s_{\varepsilon_k}} \right| \geq \frac{1}{|s_{\varepsilon_k}|}(1 - |n_{\varepsilon_k}|) > \frac{1}{2}\varepsilon_k^N$$

for all $k > k_0$.

($\Leftarrow$) Suppose $(r_\varepsilon)_\varepsilon$ is strictly non-zero on S, i.e. there exist $k_0, m \in \mathbb{N}$ such that for all $k > k_0$ we have that $|r_{\varepsilon_k}| > \varepsilon_k^m$. For each $k > k_0 + 1$ choose $\varepsilon'_k \in (\varepsilon_k, \varepsilon_{k-1})$ such that

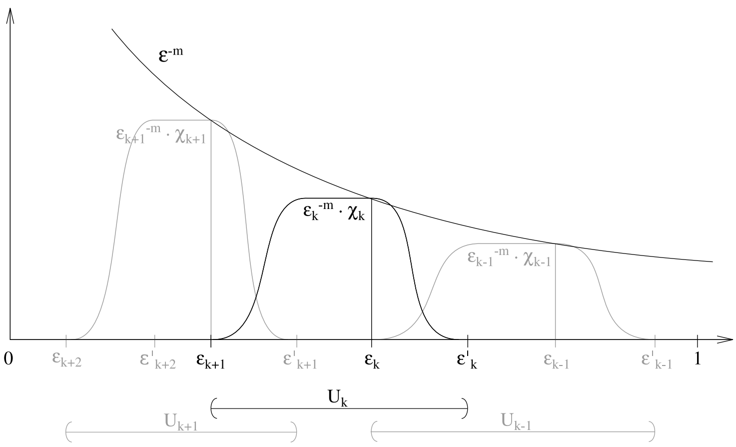
$$\varepsilon'_k{}^{-m} + 1 \geq \varepsilon_k^{-m} \tag{6.4}$$



For each $k > k_0 + 1$ define $U_k := (\varepsilon_{k+1}, \varepsilon'_k)$ and $U_{k_0+1} := (\varepsilon_{k_0+2}, 1]$. Hence $\mathcal{U} = (U_k)_{k>k_0}$ is an open cover of $(0, 1]$. By theorem 2.2.4 there exists a subordinate partition of unity $(\chi_k)_{k>k_0}$ to $\mathcal{U}$. We furthermore have that

$$\chi_k(\varepsilon_j) = \delta_{jk} \quad \forall j, k > k_0.$$

since $\operatorname{supp} \chi_k \subseteq U_k \subseteq (\varepsilon_{k+1}, \varepsilon_{k-1})$.



By setting

$$s_\varepsilon := \sum_{k=k_0+1}^{\infty} \frac{1}{r_{\varepsilon_k}} \chi_k(\varepsilon) \tag{6.5}$$

we obtain a well-defined smooth function $s : (0, 1] \to \mathbb{C}$ which satisfies $rs = 1$ on S. It remains to be shows that $s \in \mathcal{E}_M$:

$$|s_\varepsilon| \leq \frac{1}{|r_{\varepsilon_{k-1}}|} + \frac{1}{|r_{\varepsilon_k}|} < \varepsilon_{k-1}^{-m} + \varepsilon_k^{-m} \overset{(6.4)}{\leq} \varepsilon_{k-1}^{-m} + \varepsilon'_k{}^{-m} + 1 < 2\varepsilon^{-m} + 1$$

for $\varepsilon \in [\varepsilon'_{k+1}, \varepsilon'_k]$, $k > k_0 + 1$, hence $|s_\varepsilon| = O(\varepsilon^{-m})$ as $\varepsilon \to 0$. □

Remark 6.2.6. Note that in general it is not enough to drop the ε'_k's and to simply define $U_k := (\varepsilon_{k+1}, \varepsilon_{k-1})$ in the previous proof. Although we would still be able to obtain $|s_{\varepsilon_k}| < \varepsilon_k^{-m}$ for all $k \in \mathbb{N}$, we do not have any control on the ε's in between. The following example shows that something could go badly wrong here if we choose a rather fast converging sequence $(\varepsilon_k)_k$ and bad bump functions.

Example 6.2.7 (Moderate sequences are not enough)**.** For all $k \in \mathbb{N}$ and any $m \in \mathbb{N}$ fixed (in particular $m > 0$) let

$$\varepsilon_k := e^{-k^k} \text{ and } s_{\varepsilon_k} := \varepsilon_k^{-m}.$$

Let $U_k := (\varepsilon_{k+1}, \varepsilon_{k-1})$ for $k > 1$ and $U_1 := (\varepsilon_2, 1]$. Furthermore, let $(\chi_k)_{k\in\mathbb{N}}$ be a partition of unity subordinate to the open cover $\mathcal{U} = (U_k)_{k\in\mathbb{N}}$, such that $\chi_k \equiv 1$ on $[\varepsilon_k, e^{-k^{k-1}}]$ for $k > 1$. This may be achieved by actually selecting a partition of unity subordinate to $\mathcal{V} = (V_k)_{k\in\mathbb{N}}$ for $V_k := (e^{-(k+1)^k}, \varepsilon_{k-1}) \subset U_k$, $V_1 := U_1$. As in (6.5) we define

$$s_\varepsilon := \sum_{k=1}^{\infty} s_{\varepsilon_k} \chi_k(\varepsilon),$$

hence $|s_{\varepsilon_k}| = \varepsilon_k^{-m}$ for all $k \in \mathbb{N}$. Suppose that $|s_\varepsilon| = O(\varepsilon^{-l})$ for some $l \in \mathbb{N}$. Then we should have

$$e^{mk^k} = \varepsilon_k^{-m} = |s_{\delta_k}| \leq \delta_k^{-l} = e^{lk^{k-1}}$$

for small $\delta_k = e^{-k^{k-1}}$, $k \in \mathbb{N}$. By taking the logarithm of this inequality we see that this implies $mk \leq l$. For all $k > \frac{l}{m}$ this is a contradiction, thus $(s_\varepsilon)_\varepsilon \notin \mathcal{E}_M$.

This shows that it actually depends on the partition of unity whether the join is moderate or not. By the ε'_k's in the proof of 6.2.5 we can control the partition of unity in such a way that we can connect any sequence of moderate values to a moderate number $(s_\varepsilon)_\varepsilon$ with the same rate of convergence.

Corollary 6.2.8. *Let $r \in \widetilde{\mathbb{C}}$. Then*

$$r \neq 0 \iff \exists \text{ characteristic set } S \text{ such that } r \text{ is invertible w.r.t. } S.$$

Proof. ($\Rightarrow$) By definition of $\mathcal{N}$ there exists some $M \in \mathbb{N}$ and a sequence $(\varepsilon_k)_k$ in $(0, 1]$ such that $\varepsilon_k \searrow 0$ and

$$|r_{\varepsilon_k}| > \varepsilon_k^M.$$

Hence r is strictly non-zero w.r.t. $S := \{\varepsilon_k | k \in \mathbb{N}\}$. Proposition 6.2.5 implies that r is S-invertible.

($\Leftarrow$) Suppose that $r = 0$. Then for any representative $(r_\varepsilon)_\varepsilon$ of r we have that $|r_\varepsilon| = O(\varepsilon^m)$ for all $m \in \mathbb{N}$. This is a contradiction to r being strictly non-zero on some characteristic set S (since $0 \in \overline{S}$). Hence again by proposition 6.2.5 we know that r cannot be S-invertible for any characteristic set S, a contradiction to the assumption. ☐

Remark 6.2.9 ($\widetilde{\mathbb{C}}$ is nearly a field)**.** The previous result shows that by using S-invertibility, all elements $\neq 0$ are invertible in a certain sense. Hence it is a construction that allows us to retain certain properties of a field.

The aim of the next proposition is to prove the same result as in 6.2.5 for $\widetilde{\mathbb{C}}$ in the case of generalized functions, following the same ideas:

Definition 6.2.10. Let X be a smooth manifold and S a characteristic set. An element $u \in \mathcal{G}(X)$ is called *S-strictly non-zero in $\mathcal{G}(X)$* if there exists a representative $(u_\varepsilon)_\varepsilon$ such that

$$\forall K \subset\subset X \, \exists k_0 \in \mathbb{N} \, \exists m \in \mathbb{N} \text{ such that } \forall k > k_0 : \inf_{p \in K} |u_{\varepsilon_k}(p)| > \varepsilon_k^m. \tag{6.6}$$

Again, it is enough that (6.6) holds for one representative as that it then holds for all, cf. remark 6.2.4.

Proposition 6.2.11. *Let X be a smooth manifold. Let $u \in \mathcal{G}(X)$ and let S be a characteristic set. Then*

$$u \text{ is } S\text{-invertible} \Longleftrightarrow u \text{ is } S\text{-strictly non-zero.}$$

Proof. ($\Rightarrow$) Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(X)$ be S-invertible, i.e. there exist $(v_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$, $(n_\varepsilon)_\varepsilon \in \mathcal{N}(X)$ such that

$$u_{\varepsilon_k} v_{\varepsilon_k} = 1 + n_{\varepsilon_k} \text{ for all } k \in \mathbb{N}. \tag{6.7}$$

Let $K \subset\subset X$ and choose $N \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ so that for all $k > k_0$ we have $\sup_{p\in K} |n_{\varepsilon_k}(p)| < \frac{1}{2}$ and $\sup_{p\in K} |v_{\varepsilon_k}(p)| < \varepsilon_k^{-N}$. Note that (6.7) then implies that for $p \in K$ and $k > k_0$ $|v_{\varepsilon_k}(p)| > 0$. Thus we have

$$|u_{\varepsilon_k}(p)| = \left| \frac{1 + n_{\varepsilon_k}(p)}{v_{\varepsilon_k}(p)} \right| \geq \frac{1}{|v_{\varepsilon_k}(p)|}(1 - |n_{\varepsilon_k}(p)|) > \frac{1}{2}\varepsilon_k^N,$$

in particular $\inf_{p\in K} |u_{\varepsilon_k}(p)| \geq \frac{1}{2}\varepsilon_k^N$ for $k > k_0$. By definition 6.2.10, u is S-strictly non-zero.

($\Leftarrow$) We will first show that for any open set Ω with $\overline{\Omega} \subset\subset X$ an S-inverse of $u|_\Omega \in \mathcal{G}(\Omega)$ exists. Afterwards, we construct a global S-inverse by glueing together all S-inverses of an open cover $(\Omega_m)_m$ of X.

So let Ω as above. Since u is S-strictly non-zero there exist $k_0, m \in \mathbb{N}$ such that for $k > k_0$ $\inf_{p\in\overline{\Omega}} |u_{\varepsilon_k}(p)| > \varepsilon_k^m$. Let $(\chi_k)_{k>k_0}$ and ε'_k as in the proof of proposition 6.2.5. For $p \in \Omega$ we define

$$v_\varepsilon(p) := \sum_{k=k_0+1}^{\infty} \frac{1}{u_{\varepsilon_k}(p)} \chi_k(\varepsilon). \tag{6.8}$$

Hence $uv = 1$ on S. Furthermore,

$$|v_\varepsilon(p)| \leq \frac{1}{|u_{\varepsilon_{k-1}}(p)|} + \frac{1}{|u_{\varepsilon_k}(p)|} < \varepsilon_{k-1}^{-m} + \varepsilon_k^{-m} \leq 2\varepsilon^{-m} + 1,$$

for $\varepsilon \in [\varepsilon'_{k+1}, \varepsilon'_k]$, $k > k_0 + 1$, again as in the proof of 6.2.5. A similar statement holds true for any derivative of $(v_\varepsilon)_\varepsilon$ since $\sup_{p\in\overline{\Omega}} |Pu_\varepsilon(p)| = O(\varepsilon^{-N_P})$ for all $P \in \mathcal{P}(X)$ as $\varepsilon \to 0$. Thus v is an S-inverse of u on Ω.

Since X is locally compact and second countable, there exists a countable open cover $(\Omega_m)_{m\in\mathbb{N}}$ by relatively compact open subsets of X. By theorem 2.2.4 we may choose a subordinate partition of unity $(\zeta_m)_{m\in\mathbb{N}}$ such that $\operatorname{supp} \zeta_m \subseteq$

Ω_m for all $m \in \mathbb{N}$. Obviously, if $u \in \mathcal{G}(X)$ is S-strictly non-zero, then all $u|_{\Omega_m} \in \mathcal{G}(\Omega_m)$ are S-strictly non-zero as well. Hence, as defined in (6.8), let $v_m = [(v_{m,\varepsilon})_\varepsilon] \in \mathcal{G}(\Omega_m)$ be an S-inverse of $u|_{\Omega_m}$ for each $m \in \mathbb{N}$. We shall see that

$$v := \sum_{m=1}^{\infty} \zeta_m v_m$$

is a global S-inverse of u. First of all, $v \in \mathcal{G}(X)$ since any compact set $K \subset\subset X$ may be covered by finitely many Ω_m's and all $[(\zeta_m v_{m,\varepsilon})_\varepsilon] \in \mathcal{G}(X)$. Choose $m_0 \in \mathbb{N}$ such that $\sum_{m=1}^{m_0} \zeta_m = 1$ on K. Then

$$(v_{\varepsilon_k} u_{\varepsilon_k})(p) = \sum_{m=1}^{m_0} \zeta_m(p) u_{\varepsilon_k}(p) v_{m,\varepsilon_k}(p) = 1$$

for all $p \in K$ and all $k \in \mathbb{N}$ as v_{m,ε_k} is the S-inverse of u_{ε_k} on $\Omega_m \supseteq \operatorname{supp} \zeta_m$. Hence v is an S-inverse of u. □

By making use of the point value characterization of generalized functions in section 5.2 we obtain a characterization of S-invertibility in terms of compactly supported points $\widetilde{X}_c$.

Theorem 6.2.12 (Characterization of S-invertibility in $\mathcal{G}(X)$)**.** *Let X be a smooth manifold and S a characteristic set. Let $u \in \mathcal{G}(X)$. Then the following are equivalent:*

(i) u is invertible w.r.t. S (in $\mathcal{G}(X)$)

(ii) $u(\tilde{p})$ is invertible w.r.t. S for all $\tilde{p} \in \widetilde{X}_c$ (in $\widetilde{\mathbb{C}}$)

Proof. (i ⇒ ii) Let $v = [(v_\varepsilon)_\varepsilon]$ be an S-inverse of $u = [(u_\varepsilon)_\varepsilon]$ in $\mathcal{G}(X)$, i.e. $uv = r1$ for some $r \in \widetilde{\mathbb{C}}$ with $r|_S = 1$. For $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$ this implies that $u_\varepsilon(p_\varepsilon) v_\varepsilon(p_\varepsilon) = r_\varepsilon + n_\varepsilon$ for some $n = (n_\varepsilon)_\varepsilon \in \mathcal{N}$. Hence $u(\tilde{p})v(\tilde{p}) = r$ in $\widetilde{\mathbb{C}}$ which by definition 6.2.2 means that $u(\tilde{p})$ is S-invertible in $\widetilde{\mathbb{C}}$.

(ii ⇒ i) Suppose $u \in \mathcal{G}(X)$ is not invertible w.r.t. S. Then, by proposition 6.2.11, u is not S-strictly non-zero. In particular (by setting $k_0 = m$ in the negation of (6.6))

$$\exists K \subset\subset X \, \forall m \in \mathbb{N} \, \exists k_m > m \, \exists p_{\varepsilon_{k_m}} \in K : |u_{\varepsilon_{k_m}}(p_{\varepsilon_{k_m}})| \leq \varepsilon_{k_m}^m. \tag{6.9}$$

Since K is compact, we may assume w.l.o.g. that $(p_{\varepsilon_{k_m}})_m$ converges towards some $p \in K$ w.r.t. a (hence any) Riemannian distance d_h. Let (v, V) be a chart of X at $p \in X$ such that $\overline{V} \subset\subset X$ and w.l.o.g. all $p_{\varepsilon_{k_m}} \in V$. As in the proof of theorem 5.2.2 we may assume that $(v(p_{\varepsilon_{k_m}}))_m$ converges fast to $v(p)$ in $v(V) \subseteq \mathbb{R}^n$. The special curve lemma 5.2.6 ensures the existence of a continuous curve $\tilde{c} : [0, 1] \to \mathbb{R}^n$ such that $\tilde{c}$ is smooth on $(0, 1]$, $\tilde{c}(\varepsilon_{k_m}) = v(p_{\varepsilon_{k_m}})$ and $\tilde{c}(0) = v(p)$. Hence

$$p_\varepsilon := v^{-1}(\tilde{c}(\varepsilon))$$

defines a compactly supported point $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$. In view of definition 6.2.3 take any $k_0', m' \in \mathbb{N}$ and let $M := \max\{k_0', m'\}$. By (6.9) there exists a $k_M > M \geq k_0'$ such that

$$|u_{\varepsilon_{k_M}}(p_{\varepsilon_{k_M}})| \leq \varepsilon_{k_M}^M \leq \varepsilon_{k_M}^{m'}.$$

Hence $u(\tilde{p})$ is not S-strictly non-zero in $\widetilde{\mathbb{C}}$, since by remark 6.2.4 it is enough to show this property for just one representative of $u(\tilde{p})$. Thus, by proposition 6.2.5, $u(\tilde{p}) \in \widetilde{\mathbb{C}}$ is not invertible w.r.t. S, a contradiction. □

6.2.2 Maximal ideals with respect to $\widetilde{\mathbb{C}}1$

Let $\mathcal{A}$ be a commutative $\widetilde{\mathbb{C}}$-algebra with unit 1 and $\varphi : \mathcal{A} \to \widetilde{\mathbb{C}}$ a non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functional. In the beginning of this chapter we have seen that – since $\widetilde{\mathbb{C}}$ is not a field – $\ker\varphi$ is not a maximal ideal in $\mathcal{A}$. We proved this by extending $\ker\varphi$ to a bigger but still non-trivial ideal I, using a non-invertible element $0 \neq r \in \widetilde{\mathbb{C}}$. In particular, $I \cap \widetilde{\mathbb{C}}1 \supseteq \langle r1\rangle \neq \{0\}$, where $\langle r1\rangle$ denotes the linear subspace of $\mathcal{A}$ generated by $r1$. In order to prevent the possibility of such constructions, it is therefore reasonable to restrict to ideals that satisfy $I \cap \widetilde{\mathbb{C}}1 = \{0\}$:

Definition 6.2.13. Let $\mathcal{A}$ be a $\widetilde{\mathbb{C}}$-algebra. An ideal $I \lhd \mathcal{A}$ is called *maximal w.r.t.* $\widetilde{\mathbb{C}}1$ (or $\widetilde{\mathbb{C}}1$-maximal ideal) if it is maximal w.r.t. the property $I\cap\widetilde{\mathbb{C}}1 = \{0\}$, i.e. if any other ideal $J \lhd \mathcal{A}$ with $I \subseteq J$ and $J \cap \widetilde{\mathbb{C}}1 = \{0\}$ equals I.

This definition makes sense for our problem because $\ker\varphi$ is a maximal ideal w.r.t. that property:

Proposition 6.2.14. *Let $\varphi : \mathcal{A} \to \widetilde{\mathbb{C}}$ be a non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functional on a commutative $\widetilde{\mathbb{C}}$-algebra $\mathcal{A}$ with unit* 1*. Then* $\ker\varphi$ *is an ideal maximal w.r.t.* $\widetilde{\mathbb{C}}1$.

Proof. Obviously, $\ker\varphi$ is an ideal.

Furthermore, $\ker\varphi \cap \widetilde{\mathbb{C}}1 = \{0\}$: Suppose to the contrary that there exists a $u \in \ker\varphi$ such that $u = r1$ with $0 \neq r \in \widetilde{\mathbb{C}}$. Since φ is non-zero, remark 6.1.5 (i) implies that $\varphi(1) = 1$. Hence by $\widetilde{\mathbb{C}}$-linearity $0 = \varphi(u) = r\varphi(1) = r \neq 0$, a contradiction.

Finally, $\ker\varphi$ is maximal w.r.t. this property: Suppose there exists an ideal $J \lhd \mathcal{A}$ with $J \cap \widetilde{\mathbb{C}}1 = \{0\}$ and $\ker\varphi \subsetneq J$, i.e. we have a $v \in J$ such that $\varphi(v) =: r \neq 0$ in $\widetilde{\mathbb{C}}$. Since $\varphi(r1) = r$ this implies that $\varphi(v - r1) = 0$, hence $v - r1 \in \ker\varphi \subsetneq J$. But this also means that $r1 = v - (v - r1) \in J$, a contradiction to $J \cap \widetilde{\mathbb{C}}1 = \{0\}$. □

Although $\ker\varphi$ is not a maximal ideal we have just proved that it is maximal w.r.t. $\widetilde{\mathbb{C}}1$. This was the main motivation for considering such ideals and also for introducing the notion of S-invertibility, since we have seen in subsection 3.1 that maximal ideals play an important role when working with multiplicative linear functionals and algebra isomorphisms.

The next result shows the first connection between S-invertibility and $\widetilde{\mathbb{C}}1$-maximal ideals. Further results will follow in section 6.3.

Lemma 6.2.15. *Let $\mathcal{A}$ be a commutative $\widetilde{\mathbb{C}}$-algebra with unit* 1 *and $I \lhd \mathcal{A}$. The following are equivalent:*

(i) $I \cap \widetilde{\mathbb{C}}1 = \{0\}$

(ii) For all characteristic sets S we have that if $u \in \mathcal{A}$ is invertible w.r.t. S then $u \notin I$.

Proof. (i $\Rightarrow$ ii) Suppose that $u \in \mathcal{A}$ is invertible w.r.t. S but $u \in I$. This implies that $uv = r1 \in I$ for $r \neq 0$, a contradiction to (i).

(ii $\Rightarrow$ i) Let $u \in I$ such that $u = r1$, $r \neq 0$. By corollary 6.2.8 there exists an S-inverse s of r. Then $v := s1$ is an S-inverse of u, hence $u \notin I$ by (ii), a contradiction. □

6.3 Multiplicative $\widetilde{\mathbb{C}}$-linear functionals

By the results in subsection 6.1.1 we know that surjective (or rather non-zero) multiplicative $\widetilde{\mathbb{C}}$-linear functionals $\varphi : \mathcal{A} \to \widetilde{\mathbb{C}}$ satisfy $\varphi(1) = 1$ and vice versa. In the previous subsection we derived that $\ker\varphi$ is a certain maximal ideal.

We will now establish more properties of these functionals φ for $\mathcal{A} = \mathcal{G}(X)$. We also prove a similar result to theorem 3.1.1 by following the same strategy.

Lemma 6.3.1. *Let X be a smooth submanifold of $\mathbb{R}^m$ and $I \lhd \mathcal{G}(X)$ an ideal. If*

$$\forall \tilde{p} \in \widetilde{X}_c \,\exists u_{\tilde{p}} \in I \text{ such that } u_{\tilde{p}}(\tilde{p}) \neq 0, \tag{6.10}$$

then $I \neq \ker\varphi$ for all non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functionals $\varphi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$.

Proof. Assume by contradiction that I is the kernel of such a functional φ. By proposition 6.2.14 we know that

$$I \cap \widetilde{\mathbb{C}}1 = \{0\}, \tag{6.11}$$

and by the homomorphism theorem for algebras (cf. [Bou98], III, §1, 2.) that $\mathcal{G}(X)/I = \mathcal{G}(X)/\ker\varphi \cong \operatorname{im}\varphi = \widetilde{\mathbb{C}}$, i.e.

$$I + \widetilde{\mathbb{C}}1 = \mathcal{G}(X). \tag{6.12}$$

In particular we consider the coordinate functions $x^i \in \mathcal{G}(X)$ for $i = 1, ..., m$. By (6.12) there exist $\lambda_i \in \widetilde{\mathbb{C}}$ such that $x^i - \lambda_i 1 \in I$ for $i = 1, ..., m$. We define the generalized function $v = [(v_\varepsilon)_\varepsilon]$ by

$$v := |x - \lambda|^2 := \sum_{i=1}^{m} (x^i - \lambda_i 1)(x^i - \overline{\lambda_i} 1) \in I, \tag{6.13}$$

where $\lambda = (\lambda_1, ..., \lambda_m) \in \widetilde{\mathbb{C}}^m$. We will show that for any such λ, this leads to a contradiction to lemma 6.2.15, more precisely that (ii) there is not satisfied although (i) is, by (6.11). To this end we distinguish three cases:

(i) $\lambda \in \widetilde{X}_c \subseteq \widetilde{\mathbb{R}}^m$

(ii) $\lambda \in \widetilde{X} \setminus \widetilde{X}_c \subseteq \widetilde{\mathbb{R}}^m$

(iii) $\lambda \in \widetilde{\mathbb{C}}^m \setminus \widetilde{X}$

(i) $\lambda \in \widetilde{X}_c \subseteq \widetilde{\mathbb{R}}^m$: Recall that the last inclusion holds by proposition 5.1.7. Assumption (6.10) ensures the existence of a $u_\lambda = [(u_{\lambda,\varepsilon})_\varepsilon] \in I$ such that $u_\lambda(\lambda) \neq 0$. We have that $|x - \lambda|^2 + |u_\lambda|^2 \in I$ since both summands are.

Furthermore, since $u_\lambda(\lambda) \neq 0$ in $\widetilde{\mathbb{C}}$, there exists a characteristic set $S := \{\varepsilon_k | k \in \mathbb{N}\}$ and some $M \in \mathbb{N}$ such that

$$|u_{\lambda,\varepsilon_k}(\lambda_{\varepsilon_k})| \geq \varepsilon_k^M. \tag{6.14}$$

We will show that $|x-\lambda|^2 + |u_\lambda|^2 \in I$ is S-strictly non-zero.

Continuity of $u_\lambda \in \mathcal{G}(X)$ in the sharp topology means that

$$\begin{gathered}\forall K \subset\subset X \,\forall l \in \mathbb{N}\,\exists n \in \mathbb{N}\,\exists \varepsilon_0 > 0\,\forall \varepsilon \leq \varepsilon_0 \,\forall p,q \in K: \\ |p-q| \leq \varepsilon^n \Rightarrow |u_{\lambda,\varepsilon}(p) - u_{\lambda,\varepsilon}(q)| \leq \varepsilon^l,\end{gathered} \tag{6.15}$$

cf. proposition 5.3.1. Consider $l > M$ fixed. Let $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$ and $K_{\tilde{p}}$ a compact support of $\tilde{p}$. If K_λ is a compact support of $\lambda \in \widetilde{X}_c$ then let $K := K_{\tilde{p}} \cup K_\lambda$ be the compact set in (6.15). Furthermore, assume w.l.o.g. that $p_\varepsilon, \lambda_\varepsilon \in K$ for $\varepsilon \leq \varepsilon_0 \leq \frac{1}{2}$. It then follows by (6.15) that

$$\begin{gathered}\exists n \in \mathbb{N}\,\exists \varepsilon_0 \in (0, \tfrac{1}{2}]\,\forall \varepsilon_k \leq \varepsilon_0 : \\ |p_{\varepsilon_k} - \lambda_{\varepsilon_k}| \leq \varepsilon_k^n \Rightarrow |u_{\lambda,\varepsilon_k}(p_{\varepsilon_k}) - u_{\lambda,\varepsilon_k}(\lambda_{\varepsilon_k})| \leq \varepsilon_k^l\end{gathered} \tag{6.16}$$

By eliminating the term $u_{\lambda,\varepsilon_k}(\lambda_{\varepsilon_k})$, we shall see that $|x-\lambda|^2 + |u_\lambda|^2$ is invertible w.r.t. S. First of all, (6.14) and (6.16) imply that

$$\begin{aligned}|u_{\lambda,\varepsilon_k}(p_{\varepsilon_k})| &\geq |u_{\lambda,\varepsilon_k}(\lambda_{\varepsilon_k})| - |u_{\lambda,\varepsilon_k}(p_{\varepsilon_k}) - u_{\lambda,\varepsilon_k}(\lambda_{\varepsilon_k})| \\ &\geq \varepsilon_k^M - \varepsilon_k^l = \varepsilon_k^M(1 - \varepsilon_k^{l-M}) \geq \varepsilon_k^M \varepsilon_k^{l-M} = \varepsilon_k^l\end{aligned}$$

for the above $l > M$, $\varepsilon_k \leq \varepsilon_0 \leq \frac{1}{2}$ and $|p_{\varepsilon_k} - \lambda_{\varepsilon_k}| \leq \varepsilon_k^n$. Altogether we obtain that

$$\begin{gathered}\exists \varepsilon_0 > 0\,\forall \varepsilon_k \leq \varepsilon_0 : \\ |p_{\varepsilon_k} - \lambda_{\varepsilon_k}|^2 + |u_{\lambda,\varepsilon_k}(p_{\varepsilon_k})|^2 \geq \begin{cases} \varepsilon_k^{2l} & \text{if } |p_{\varepsilon_k} - \lambda_{\varepsilon_k}| \leq \varepsilon_k^m \\ \varepsilon_k^{2m} & \text{if } |p_{\varepsilon_k} - \lambda_{\varepsilon_k}| \geq \varepsilon_k^m \end{cases}\end{gathered}$$

for some $l, n \in \mathbb{N}$. Thus $(|p_\varepsilon - \lambda_\varepsilon|^2 + |u_{\lambda,\varepsilon}(p_\varepsilon)|^2)_\varepsilon$ is strictly-nonzero w.r.t. S, hence by 6.2.5 S-invertible in $\widetilde{\mathbb{C}}$.

Since $\tilde{p} \in \widetilde{X}_c$ was arbitrary we conclude by theorem 6.2.12 that

$$|x-\lambda|^2 + |u_\lambda|^2 \text{ is invertible in } \mathcal{G}(X) \text{ w.r.t. } S, \tag{6.17}$$

a contradiction to lemma 6.2.15 (ii) because $|x-\lambda|^2 + |u_\lambda|^2 \in I$ as shown above.

(ii) $\lambda \in \widetilde{X} \setminus \widetilde{X}_c \subseteq \widetilde{\mathbb{R}}^m$ where

$$\widetilde{X} := \{\tilde{p} \in \widetilde{\mathbb{R}}^m \mid \exists \text{ representative } (p_\varepsilon)_\varepsilon \text{ of } \tilde{p} \text{ such that } \forall \varepsilon : p_\varepsilon \in X\}:$$

Let $(K_k)_{k\in\mathbb{N}}$ be an exhaustion of X by compact sets with $K_k \subseteq \mathrm{int}(K_{k+1})$ for all $k \in \mathbb{N}$ w.r.t. the trace topology of $\mathbb{R}^m$ on X (see remark 5.4.10 for a general construction of a compact exhaustion). We assumed that $\lambda \in \widetilde{X}$, hence we may choose a representative $(\lambda_\varepsilon)_\varepsilon$ of λ such that all $\lambda_\varepsilon \in X$. Consider the generalized function $v = [(v_\varepsilon)_\varepsilon] \in I$ defined by (6.13), i.e.

$$v_\varepsilon(p_\varepsilon) = |p_\varepsilon - \lambda_\varepsilon|^2$$

for $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$. Let $K \subset\subset X$ such that $p_\varepsilon \in K$ for all $\varepsilon < \varepsilon_0$. There exists an $N \in \mathbb{N}$ such that $K \subseteq K_N$.

Since $\lambda \notin \widetilde{X}_c$ there exists a characteristic set $S := \{\varepsilon_k | k \in \mathbb{N}\}$ such that

$$\lambda_{\varepsilon_k} \in X \setminus K_k \text{ for all } k \in \mathbb{N}.$$

Let ε_0' be the minimum of ε_0 and the Euclidean distance $d(X \setminus K_{N+1}, K_N) > 0$. Hence for $\varepsilon_k < \varepsilon_0'$ we have that $p_{\varepsilon_k} \in K_N$ and for $k > N$ that $\lambda_{\varepsilon_k} \in X \setminus K_k \subseteq X \setminus K_{N+1}$, i.e.

$$|p_{\varepsilon_k} - \lambda_{\varepsilon_k}|^2 \geq d(X \setminus K_{N+1}, K_N)^2 \geq {\varepsilon_0'}^2 > \varepsilon_k^2.$$

Thus $v(\tilde{p})$ is S-strictly non-zero by definition 6.2.3 (set $m = 2$ and $k_0 > N$ such that $\varepsilon_{k_0} \leq \varepsilon_0'$).

The compactly supported point $\tilde{p}$ was arbitrary and therefore, again by proposition 6.2.5 and theorem 6.2.12, we know that $v \in I$ is S-invertible in $\mathcal{G}(X)$. This contradicts lemma 6.2.15 (ii).

(iii) $\lambda \in \widetilde{\mathbb{C}}^m \setminus \widetilde{X}$: The assumption $\lambda \notin \widetilde{X}$ implies that for any representative $(\lambda_\varepsilon)_\varepsilon$ of λ we have that

$$(d(\lambda_\varepsilon, X))_\varepsilon \text{ is not negligible in } \widetilde{\mathbb{C}}.$$

Hence by corollary 6.2.8 there exists a characteristic set $S := \{\varepsilon_k | k \in \mathbb{N}\}$ such that $v(\tilde{p})$ is invertible w.r.t. to S for all $\tilde{p} \in \widetilde{X}_c \subseteq \widetilde{X}$ (we even have for any representative $(p_\varepsilon)_\varepsilon$ of $\tilde{p}$ that $p_\varepsilon \in X\ \forall \varepsilon$). This again contradicts lemma 6.2.15 (ii). □

Remark 6.3.2. Compare lemma 6.3.1 to the proof of existence in theorem 3.1.1, in particular where we proved (3.1), i.e. that

$$\ker \varphi = \ker \mathrm{ev}_p \quad \text{for some } p \in X$$

if $\varphi : \mathcal{C}^\infty(X) \to \mathbb{R}$ is a non-zero multiplicative linear functional. The previous result is a contrapositive analogue of this. The next proof makes use of the same idea as that of theorem 3.1.1. In order to apply lemma 6.3.1 we need to be able to reduce the general case to submanifolds of $\mathbb{R}^m$, which is achieved by Whitney's embedding theorem 4.4.1.

Obviously, for each $\tilde{p} \in \widetilde{X}_c$, $\mathrm{ev}_{\tilde{p}}$ is a multiplicative $\widetilde{\mathbb{C}}$-linear functional on $\mathcal{G}(X)$. For surjective functionals the converse is also true:

Theorem 6.3.3. *Let X be a smooth manifold and let $\varphi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$ be a non-zero multiplicative $\widetilde{\mathbb{C}}$-linear functional. Then there exists a unique $\tilde{p} \in \widetilde{X}_c$ such that*

$$\varphi(u) = u(\tilde{p}) \quad \forall u \in \mathcal{G}(X).$$

Proof. First, consider X to be a smooth submanifold of $\mathbb{R}^{2m+1}$. Then by lemma 6.3.1, there exists a $\tilde{q} \in \widetilde{X}_c$ such that

$$u(\tilde{q}) = 0 \quad \forall u \in \ker \varphi.$$

Obviously, $\ker\varphi \subseteq \ker \mathrm{ev}_{\tilde{q}}$. Furthermore, $\ker \mathrm{ev}_{\tilde{q}} \cap \widetilde{\mathbb{C}}1 = \{0\}$ and $\ker\varphi \cap \widetilde{\mathbb{C}}1 = \{0\}$, and both ideals are maximal w.r.t. that property by proposition 6.2.14. Thus

$$\ker\varphi = \ker \mathrm{ev}_{\tilde{q}}.$$

It remains to be shown that this already implies that $\varphi = \mathrm{ev}_{\tilde{q}}$. For any $u \in \mathcal{G}(X)$ we have that $u - u(\tilde{q}) \in \ker \mathrm{ev}_{\tilde{q}} = \ker\varphi$. Hence $\varphi(u) = \varphi(u - u(\tilde{q}) + u(\tilde{q})) = \varphi(u - u(\tilde{q})) + \varphi(u(\tilde{q})) = 0 + \varphi(u(\tilde{q})) = u(\tilde{q})$ since $\varphi(1) = 1$.

Now let X be any smooth manifold. By Whitney's embedding theorem 4.4.1 there exists a smooth embedding $j : X \to j(X) \subseteq \mathbb{R}^{2m+1}$. The map j may be interpreted as a generalized function $j = [(j)_\varepsilon] \in \mathcal{G}[X, j(X)]$. Then, by proposition 5.5.1, $\tilde{u} \circ j \in \mathcal{G}(X)$ for all $\tilde{u} \in \mathcal{G}(j(X))$. Moreover, $\widetilde{\varphi} : \mathcal{G}(j(X)) \to \widetilde{\mathbb{C}}$ defined by

$$\widetilde{\varphi}(\tilde{u}) := \varphi(\tilde{u} \circ j) = \varphi(j^*(\tilde{u})), \tag{6.18}$$

is a non-zero multiplicative linear functional, since φ is and $j^* : \mathcal{G}(j(X)) \to \mathcal{G}(X)$ is an algebra isomorphism. Hence by the above there exists $\tilde{q} \in \widetilde{j(X)}_c$ such that

$$\widetilde{\varphi}(\tilde{u}) = \tilde{u}(\tilde{q}) \quad \forall \tilde{u} \in \mathcal{G}(j(X)). \tag{6.19}$$

Let $\tilde{p} := j^{-1}(\tilde{q})$. Since $j^{-1} \in \mathcal{G}[j(X), X]$ we have that $\tilde{p} \in \widetilde{X}_c$ by proposition 5.4.7. Furthermore, by proposition 5.5.1, $u \circ j^{-1} \in \mathcal{G}(j(X))$ for any $u \in \mathcal{G}(X)$. Putting all this together we obtain that

$$\varphi(u) \overset{(6.18)}{=} \widetilde{\varphi}(u \circ j^{-1}) \overset{(6.19)}{=} u(j^{-1}(\tilde{q})) = u(\tilde{p}).$$

It remains to be proved that $\tilde{p} = [(p_\varepsilon)_\varepsilon] \in \widetilde{X}_c$ is uniquely determined by φ. Suppose there exists $\tilde{p} \neq \tilde{z} = [(z_\varepsilon)_\varepsilon] \in \widetilde{X}_c$ with $u(\tilde{p}) = u(\tilde{z})$ for all $u \in \mathcal{G}(X)$. Thus we have that $(d_g(p_\varepsilon, z_\varepsilon))_\varepsilon \notin \mathcal{N}$ for one (hence any) Riemannian metric g on X. This means that there exists a characteristic set $S := \{\varepsilon_k | k \in \mathbb{N}\}$ and $N \in \mathbb{N}$ so that

$$d_g(p_{\varepsilon_k}, z_{\varepsilon_k}) > \varepsilon_k^N \quad \forall k \in \mathbb{N}. \tag{6.20}$$

By theorem 4.2.11, d_g^2 is smooth on some neighborhood U of $\triangle_X \subseteq X \times X$. W.l.o.g. we may assume that p_{ε_k} converges to some p in X. Let K be a compact neighborhood of p such that $K \times K \subseteq U$. Assume that $p_{\varepsilon_k} \in K$ for all $k \in \mathbb{N}$. Regarding $(z_{\varepsilon_k})_k$ we can distinguish two cases:

- $z_{\varepsilon_k} \notin K$ for sufficiently large k: As p is contained in the interior of K we have an open neighborhood V of p such that $V \subset\subset \mathrm{int}(K)$. By corollary 2.2.6 there exists a smooth bump function $\chi_1 : X \to \mathbb{R}$ such that $\chi_1|_V \equiv 1$ and $\chi_1|_{X\setminus K} \equiv 0$. Let

  $$v := [(\chi_1)_\varepsilon] \in \mathcal{G}(X).$$

  Obviously, for k sufficiently large, $v_{\varepsilon_k}(p_{\varepsilon_k}) = 1$ and $v_{\varepsilon_k}(z_{\varepsilon_k}) = 0$, a contradiction to $v(\tilde{p}) = v(\tilde{z})$.

- $z_{\varepsilon_k} \in K$ for a subsequence of $(z_{\varepsilon_k})_k$: W.l.o.g. we may assume that $z_{\varepsilon_k} \in K$ for all $k \in \mathbb{N}$. Again, by corollary 2.2.6, there exists a smooth bump function $\chi_2 : X \times X \to \mathbb{R}$ such that $\chi_2|_{K\times K} \equiv 1$ and $\chi_2|_{(X\times X)\setminus U} \equiv 0$.

We may assume that $\operatorname{supp}\chi_2$ is compact and contained in U. Consider $w = [(w_\varepsilon)_\varepsilon]$ defined by

$$w_\varepsilon(x) := \chi_2(p_\varepsilon, x) \cdot d_g^2(p_\varepsilon, x)$$

Obviously, $(w_\varepsilon)_\varepsilon \in \mathcal{C}^\infty((0,1] \times X)$. Moreover,

$$\begin{aligned}\sup_{x\in X} |w_\varepsilon(x)| &\leq \|\chi_2\|_\infty \cdot \sup_{\substack{x\in X \text{ s.t.}\\ (p_\varepsilon,x)\in\operatorname{supp}\chi_2}} |d_g^2(p_\varepsilon, x)| \\ &\leq \|\chi_2\|_\infty \cdot \sup_{(y,x)\in\operatorname{supp}\chi_2} |d_g^2(y,x)| < \infty,\end{aligned}$$

since $\operatorname{supp}\chi_2 \subseteq U$ is compact. A similar computation can be done for all $(Pw_\varepsilon)_\varepsilon$, $P \in \mathcal{P}(X)$. Thus $(w_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$.

In addition, since $\chi_2(p_{\varepsilon_k}, p_{\varepsilon_k}) = \chi_2(p_{\varepsilon_k}, z_{\varepsilon_k}) = 1$ for all $k \in \mathbb{N}$,

$$w_{\varepsilon_k}(p_{\varepsilon_k}) = d_g^2(p_{\varepsilon_k}, p_{\varepsilon_k}) = 0$$

and

$$w_{\varepsilon_k}(z_{\varepsilon_k}) = d_g^2(p_{\varepsilon_k}, z_{\varepsilon_k}) \overset{(6.20)}{>} \varepsilon_k^{2N}.$$

Hence $w(\tilde{p}) \neq w(\tilde{z})$, a contradiction. □

6.4 Algebra homomorphisms and isomorphisms

Again, we will first restrict to the case of smooth submanifolds and show the general result later using the Whitney embedding theorem.

Theorem 6.4.1. *Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be smooth submanifolds and $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$.*

(i) If Ψ is a surjective homomorphism of Colombeau algebras, then there exists a unique $\psi \in \mathcal{G}[Y, X]$ such that

$$\Psi(u) = u \circ \psi \quad \forall u \in \mathcal{G}(X).$$

(ii) If Ψ is an algebra isomorphism then $\psi \in \mathcal{G}[Y, X]$ as in (i) is invertible (cf. 5.5.5) with inverse ψ^{-1} and

$$\Psi^{-1}(v) = v \circ \psi^{-1} \quad \forall v \in \mathcal{G}(Y).$$

In this case, $\dim X = \dim Y$.

Note that if we only consider unital algebra homomorphisms, i.e. assume that $\Psi(1) = 1$, then we can drop the term 'surjective' in (i). Alternatively we could also restrict to connected manifolds Y, cf. remark 6.1.5 (ii).

Proof of (i). Let $\tilde{q} \in \widetilde{Y}_c$. The map $\varphi_{\tilde{q}} := \operatorname{ev}_{\tilde{q}} \circ \Psi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$ is multiplicative and $\widetilde{\mathbb{C}}$-linear. Since Ψ is surjective we have by lemma 6.1.2 (i) that $\Psi(1) = 1$ and therefore also $\varphi_{\tilde{q}}(1) = (\operatorname{ev}_{\tilde{q}} \circ \Psi)(1) = \operatorname{ev}_{\tilde{q}}(1) = 1$. Thus $\varphi_{\tilde{q}}$ is surjective by lemma 6.1.2 (ii). By theorem 6.3.3 there exists a (unique) $\tilde{p} \in \widetilde{X}_c$ such that

$$\varphi_{\tilde{q}} = \operatorname{ev}_{\tilde{q}} \circ \Psi = \operatorname{ev}_{\tilde{p}}.$$

We join these compactly supported points by the map

$$\begin{array}{rcl} \widetilde{\psi}: \widetilde{Y}_c & \to & \widetilde{X}_c \\ \tilde{q} & \mapsto & \tilde{p} \end{array}$$

which we will extend to a function $\psi \in \mathcal{G}[Y, X]$ with the same properties.

So far we have shown that

$$\forall u \in \mathcal{G}(X)\, \forall \tilde{q} \in \widetilde{Y}_c : (\Psi(u))(\tilde{q}) = \varphi_{\tilde{q}}(u) = u(\widetilde{\psi}(\tilde{q})). \tag{6.21}$$

Applied to the coordinate functions $x^i \in \mathcal{G}(X)$, $x^i(p) = p_i$, $i = 1, ..., m$, this implies that

$$\psi := (\Psi(x^1), ..., \Psi(x^m)) \in \mathcal{G}(Y)^m \tag{6.22}$$

coincides with $(x^1, ..., x^m) \circ \widetilde{\psi} = \widetilde{\psi}$ at the generalized points in $\widetilde{Y}_c$. Since any generalized function is uniquely determined by its values on the compactly supported points by theorem 5.2.8, we know that ψ is the unique generalized function that coincides with $\widetilde{\psi}$. Thus, by $\psi(\widetilde{Y}_c) \subseteq \widetilde{X}_c$ and proposition 5.4.11 (i)$\Leftrightarrow$(ii), ψ is c-bounded into X. Hence by proposition 5.4.12, $\psi \in \mathcal{G}[Y, X]$ and therefore $u \circ \psi \in \mathcal{G}(Y)\, \forall u \in \mathcal{G}(X)$ by proposition 5.5.1. By (6.21) and again theorem 5.2.8, we have that

$$\forall u \in \mathcal{G}(X) : \Psi(u) = u \circ \psi \text{ in } \mathcal{G}(Y). \tag{6.23}$$

Furthermore, suppose there exists $\psi \neq \sigma \in \mathcal{G}[Y, X]$ which also satisfies $\Psi(u) = u \circ \sigma$ for all $u \in \mathcal{G}(X)$. Since $\mathcal{C}^\infty(X)$ is embedded in $\mathcal{G}(X)$, we have that $f \circ \psi = \Psi(f) = f \circ \sigma$ for all $f \in \mathcal{C}^\infty(X)$. This is equivalent to $\psi = \sigma$ by proposition 5.4.6, a contradiction. Hence $\psi \in \mathcal{G}[Y, X]$ is unique. □

Remark 6.4.2. Uniqueness in the previous result also follows by corollary 5.4.8, since ψ equals $\widetilde{\psi}$ on $\widetilde{Y}_c$ and $\widetilde{\psi}$ is uniquely determined by theorem 6.3.3 above.

It is also possible to just use the coordinate functions x^i as above (and not *all* smooth functions) – ψ is already completely determined by $\psi_i = x^i \circ \psi = \Psi(x^i)$, $i = 1, ..., m$.

Remark 6.4.3 (Smooth functions suffice)**.** From the above uniqueness argument we see that an algebra homomorphism $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$ is uniquely determined by its values on the space of smooth functions $\mathcal{C}^\infty(X)$.

Proof of 6.4.1 (ii). By (i) we may also find $\varphi \in \mathcal{G}[X, Y]$ such that

$$\forall v \in \mathcal{G}(Y) : \Psi^{-1}(v) = v \circ \varphi \text{ in } \mathcal{G}(X). \tag{6.24}$$

For $\psi_i := x^i \circ \psi$, x^i the coordinate functions as above, we have that

$$\begin{aligned} \psi \circ \varphi &= (\psi_1 \circ \varphi, ..., \psi_m \circ \varphi) \\ &\overset{(6.24)}{=} (\Psi^{-1}(\psi_1), ..., \Psi^{-1}(\psi_m)) \\ &\overset{(6.23)}{=} (\Psi^{-1}(\Psi(x^1)), ..., \Psi^{-1}(\Psi(x^m))) \\ &= (x^1, ..., x^m) \\ &= \mathrm{id}_{\mathcal{G}[X,X]}, \end{aligned} \tag{6.25}$$

and similarly

$$\varphi \circ \psi = \mathrm{id}_{\mathcal{G}[Y,Y]},$$

which yields $\varphi = \psi^{-1}$ (cf. definition 5.5.5).

It remains to be shown that $\dim X = \dim Y$. The equation $\psi \circ \varphi = \mathrm{id}_{\mathcal{G}[X,X]}$ in $\mathcal{G}[X,X]$ may also be viewed as $i \circ \psi \circ \varphi = i \circ \mathrm{id}_{\mathcal{G}[X,X]}$ in $\mathcal{G}(X)^m$, where $i : X \hookrightarrow \mathbb{R}^m$ denotes the embedding of the submanifold X into its surrounding space $\mathbb{R}^m$ (the composition is well-defined by proposition 5.5.2). Thus there exists $(n_\varepsilon)_\varepsilon \in \mathcal{N}(X)^m$ such that

$$i \circ \psi_\varepsilon \circ \varphi_\varepsilon = i \circ \mathrm{id}_X + n_\varepsilon = i + n_\varepsilon \tag{6.26}$$

for each $\varepsilon \in (0,1]$. Let $p \in X$ be fixed. By differentiating (6.26) and making use of the chain rule, cf. lemma 1.15, [O'N83], we obtain that

$$T_{(\psi_\varepsilon \circ \varphi_\varepsilon)(p)} i \circ T_{\varphi_\varepsilon(p)} \psi_\varepsilon \circ T_p \varphi_\varepsilon = T_p(i \circ \psi_\varepsilon \circ \varphi_\varepsilon) \overset{(6.26)}{=} T_p i + T_p n_\varepsilon \tag{6.27}$$

in $L(T_p X, \mathbb{R}^m)$. Since i is an embedding we have that $T_p i$ is regular, i.e. the rank $\mathrm{rk}(T_p i)$ is maximal and equals $\dim T_p X = \dim X \leq m$. For sufficiently small ε we also have that $\mathrm{rk}(T_p i + T_p n_\varepsilon) = \dim X$. Concerning the left-hand side, we use some basic results from linear algebra to obtain

$$\begin{aligned}
\dim X &= \mathrm{rk}(T_p i + T_p n_\varepsilon) \\
&\overset{(6.27)}{=} \mathrm{rk}(T_{(\psi_\varepsilon \circ \varphi_\varepsilon)(p)} i \circ T_{\varphi_\varepsilon(p)} \psi_\varepsilon \circ T_p \varphi_\varepsilon) \\
&\leq \min\{\mathrm{rk}(T_{(\psi_\varepsilon \circ \varphi_\varepsilon)(p)} i), \mathrm{rk}(T_{\varphi_\varepsilon(p)} \psi_\varepsilon), \mathrm{rk}(T_p \varphi_\varepsilon)\} \\
&\leq \mathrm{rk}(T_{\varphi_\varepsilon(p)} \psi_\varepsilon) \\
&= \dim(\mathrm{im}\, T_{\varphi_\varepsilon(p)} \psi_\varepsilon) \\
&\leq \dim T_{\varphi_\varepsilon(p)} Y \\
&= \dim Y
\end{aligned}$$

for small ε. By symmetry in ψ_ε and φ_ε, we also get that $\dim X \geq \dim Y$. Thus $\dim X = \dim Y$ and we are done. □

Finally, by making use of the Whitney embedding theorem, we obtain the same result for smooth manifolds X and Y:

Corollary 6.4.4. *Let X and Y be smooth manifolds and $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$.*

(i) If Ψ is a surjective homomorphism of Colombeau algebras, then there exists a unique $\psi \in \mathcal{G}[Y,X]$ such that

$$\Psi(u) = u \circ \psi \quad \forall u \in \mathcal{G}(X).$$

(ii) If Ψ is an algebra isomorphism then $\psi \in \mathcal{G}[Y,X]$ as in (i) is invertible with inverse ψ^{-1} and

$$\Psi^{-1}(v) = v \circ \psi^{-1} \quad \forall v \in \mathcal{G}(Y).$$

In this case, $\dim X = \dim Y$.

Proof. By the Whitney embedding theorem 4.4.1 there exist smooth embeddings $i : X \to i(X) \subseteq \mathbb{R}^m$ and $j : Y \to j(Y) \subseteq \mathbb{R}^n$ for some $m, n \in \mathbb{N}$.

(i) We consider the respective submanifold-version of Ψ,

$$\begin{array}{rcl} \widetilde{\Psi} : \mathcal{G}(i(X)) & \to & \mathcal{G}(j(Y)) \\ \widetilde{u} & \mapsto & \Psi(\widetilde{u} \circ i) \circ j^{-1}, \end{array} \tag{6.28}$$

which obviously is multiplicative, $\widetilde{\mathbb{C}}$-linear and satisfies $\widetilde{\Psi}(1) = \Psi(1 \circ i) \circ j^{-1} = \Psi(1) \circ j^{-1} = 1 \circ j^{-1} = 1$. By the previous theorem 6.4.1 (i) there exists some $\widetilde{\psi} \in \mathcal{G}[j(Y), i(X)]$ such that

$$\forall \widetilde{u} \in \mathcal{G}(i(X)) : \widetilde{\Psi}(\widetilde{u}) = \widetilde{u} \circ \widetilde{\psi} \text{ in } \mathcal{G}(j(Y)). \tag{6.29}$$

By proposition 5.5.2,

$$\psi := i^{-1} \circ \widetilde{\psi} \circ j \tag{6.30}$$

is a well-defined element in $\mathcal{G}[Y, X]$. Furthermore, for any $u \in \mathcal{G}(X)$,

$$\Psi(u) \stackrel{(6.28)}{=} \widetilde{\Psi}(u \circ i^{-1}) \circ j \stackrel{(6.29)}{=} (u \circ i^{-1} \circ \widetilde{\psi}) \circ j \stackrel{(6.30)}{=} u \circ \psi$$

holds in $\mathcal{G}(Y)$. Uniqueness follows as in the proof of theorem 6.4.1 (i) by using proposition 5.4.6.

(ii) Again, for $\widetilde{\Psi}^{-1} : \mathcal{G}(j(Y)) \to \mathcal{G}(i(X))$, defined by

$$\widetilde{\Psi}^{-1}(\widetilde{v}) := \Psi^{-1}(v \circ j) \circ i^{-1}$$

for $\widetilde{v} \in \mathcal{G}(j(Y))$, we find $\widetilde{\psi}^{-1} \in \mathcal{G}[i(X), j(Y)]$ resp.

$$\varphi := j^{-1} \circ \widetilde{\psi}^{-1} \circ i \in \mathcal{G}[X, Y] \tag{6.31}$$

by theorem 6.4.1 (ii) and proposition 5.5.2. Thus,

$$\begin{array}{rcl} \psi \circ \varphi & \stackrel{(6.30),(6.31)}{=} & (i^{-1} \circ \widetilde{\psi} \circ j) \circ (j^{-1} \circ \widetilde{\psi}^{-1} \circ i) \\ & = & i^{-1} \circ \mathrm{id}_{\mathcal{G}[i(X), i(X)]} \circ i \\ & \stackrel{5.5.2}{=} & \mathrm{id}_{\mathcal{G}[X,X]}, \end{array}$$

and similarly

$$\varphi \circ \psi \quad = \quad \mathrm{id}_{\mathcal{G}[Y,Y]} \,.$$

This yields $\varphi = \psi^{-1}$, hence $\psi \in \mathcal{G}[Y, X]$ is invertible.

By theorem 6.4.1 we have that $\dim i(X) = \dim j(Y)$. Since i and j are diffeomorphisms this implies that $\dim X = \dim Y$. $\square$

Remark 6.4.5. Note that proof 6.4.1 about Colombeau algebra isomorphisms on smooth submanifolds is a generalization of the smooth version, i.e. theorem 3.1.8 and 3.2.11. This is in accordance with the general outline mentioned in the introduction.

Remark 6.4.6 (Second countability)**.** In section 3.2 we have seen that Mrčun's approach via characteristic sequences eliminates the need for the Hausdorff manifolds X and Y to be second countable in the case of algebra isomorphisms $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(Y)$.

So far, second-countability of the manifolds X and Y cannot be omitted in the Colombeau setting as it is required for the existence of exhaustions by compact sets, see subsection 5.4.3 and the proof of lemma 6.3.1. Moreover, a countable partition of unity is used in the proof of proposition 6.2.11. On the other hand, it is easy to see that the majority of the results also holds for paracompact manifolds. It is conceivable that another approach could be more successful in showing that corollary 6.4.4 is also true on paracompact manifolds.

6.5 Differences between smooth and non-smooth dependence on ε

As already mentioned at the beginning of this chapter, the main ideas for the treatment of the algebra isomorphism problem in the Colombeau setting are taken from [Ver06]. In Vernaeve's paper, however, smooth dependence on the index ε as in the definitions of section 5.1 is not required. It is the aim of this section to point out the differences and overlaps, both in the results and proofs.

6.5.1 Idempotents in $\widetilde{\mathbb{C}}$

By [AJOS08], theorem 4.1, the idempotents in $\widetilde{\mathbb{C}}$ (or $\widetilde{\mathbb{R}}$) are characteristic functions in the non-smooth setting. More precisely, if χ_T denotes the characterstic function on $T \subseteq \mathbb{R}$ (i.e. $\chi|_T \equiv 1$ and $\chi|_{\complement T} \equiv 0$), then $e_S = [(\chi_S)_\varepsilon]$ for a set $S \subseteq (0,1]$, $0 \in \overline{S}$. On the other hand, each such e_S obviously is an idempotent element of $\widetilde{\mathbb{C}}$.

In the smooth setting there exist no idempotents in $\widetilde{\mathbb{C}}$. This already leads to the difference that, by remark 6.1.5 (i), all non-zero multiplicative linear functionals $\varphi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$ are exactly the surjective ones, while in the non-smooth setting surjectivity is equivalent to $\varphi(1) = 1$ (in general, $\varphi(1)$ might only be an idempotent).

The definition of $\widetilde{\mathbb{C}}1$-maximal ideals is the same in both cases. In contrast, understanding invertibility w.r.t. S in the smooth setting is slightly more involved. In the non-smooth case, an element u of a commutative $\widetilde{\mathbb{C}}$-algebra is called invertible w.r.t. S – here S can be any subset of $(0,1]$ with $0 \in \overline{S}$ – iff there exists $v \in \mathcal{A}$ such that $uv = e_S$. Recall that in our definition 6.2.1 of a characteristic set, S has to be *smaller*, i.e. only a strictly decreasing 0-sequence is allowed. This is due to the fact that in the smooth setting it is much more difficult to find an S-inverse for elements in $\widetilde{\mathbb{C}}$. While in the non-smooth case this can simply be achieved by setting it equal to 0 on $\complement S \cap (0,1]$, a smooth inverse is much harder to find – we even had to introduce the additional definition of S-strictly non-zero.

In the non-smooth case one can easily switch between S-invertibility and classical invertibility, cf. [Ver06], lemma 4.2:

Lemma 6.5.1. *Let $\mathcal{A}$ be a commutative $\widetilde{\mathbb{C}}$-algebra with unit* 1. *Let $u \in \mathcal{A}$ and $S \subseteq (0,1]$ with $0 \in \overline{S}$. Then*

$$u \text{ is invertible w.r.t. } S \iff ue_S + e_{S^c} \text{ is invertible},$$

where $S^c := \complement S \cap (0,1]$ resp. $e_{S^c} = 1 - e_S$.

Proof. ($\Rightarrow$) As u is S-invertible there exists some $v \in \mathcal{A}$ with $uv = e_S$. Then

$$(ue_S + e_{S^c})(ve_S + e_{S^c}) = \underbrace{\underbrace{uv}_{e_S}\ \underbrace{e_S^2}_{e_S}}_{e_S} + (u+v)\underbrace{e_S e_{S^c}}_{0} + \underbrace{e_{S^c}^2}_{e_{S^c}} = 1. \tag{6.32}$$

($\Leftarrow$) Suppose v is the inverse of $ue_S + e_{S^c}$. Then ve_S is an S-inverse of u because $(ue_S + e_{S^c})v \cdot e_S = 1 \cdot e_S = e_S$. ☐

Although we can define something similar to e_S also in the smooth case (as done in definition 6.2.2) to obtain a bump function $\bar{e}_S$ similar to a characteristic function, the calculation (6.32) still does not work because $\bar{e}_S \bar{e}_{S^c} \neq 0$. With this result at hand, however, the characterization of S-invertibility in $\mathcal{G}(X)$ w.r.t. point values is straightforward and immediately follows from the classical invertibility result in Colombeau theory. In the smooth case we proved theorem 6.2.12 directly on smooth manifolds.

Furthermore, the form of the multiplicative $\widetilde{\mathbb{C}}$-linear functionals $\varphi : \mathcal{G}(X) \to \widetilde{\mathbb{C}}$ is considerably more complicated in the non-smooth setting as we always have to drag along idempotents. In particular, if φ is not surjective but still non-zero, then $\varphi = e \cdot \mathrm{ev}_{\tilde{p}}$ for some idempotent $e \in \widetilde{\mathbb{R}}$ and some $\tilde{p} \in \widetilde{X}_c$, cf. [Ver06], theorem 4.5. The proofs in section 6.3 are exactly the same as in [Ver06].

6.5.2 Locally defined c-bounded generalized functions

The final results in section 6.4 are obtained in basically the same way as in [Ver06]. The results, however, differ. To understand this, the following definition is needed.

Definition 6.5.2. For smooth manifolds X and Y a *locally defined c-bounded generalized function* is an equivalence class $u = [(u_\varepsilon)_\varepsilon]$ whose representative $(u_\varepsilon)_\varepsilon$ is a net of smooth functions $X \supseteq X_\varepsilon \to Y$ such that

$$\forall K \subset\subset X\, \exists \varepsilon_0 > 0\, \forall \varepsilon < \varepsilon_0 :\ K \subseteq X_\varepsilon$$

and satisfies the usual c-boundedness condition and moderateness (cf. definition 5.4.1). The equivalence relation is defined as in 5.4.2. The set of locally defined c-bounded generalized functions from X to Y is denoted by $\mathcal{G}_{\mathrm{ld}}[X,Y]$.

Obviously, $\mathcal{G}[X,Y] \subseteq \mathcal{G}_{\mathrm{ld}}[X,Y]$. The converse inclusion may in general not hold and is still an open question. For some special cases, however, e.g. X compact, it can be proved that $\mathcal{G}[X,Y] = \mathcal{G}_{\mathrm{ld}}[X,Y]$.

The final result in [Ver06] is:

Theorem 6.5.3. *Let X, Y be smooth manifolds.*

(i) Let $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$ be a morphism of algebras. Then there exists $\psi \in \mathcal{G}_{ld}[Y, X]$ and $e \in \mathcal{G}(Y)$ an idempotent such that

$$\Psi(u) = e \cdot (u \circ \psi) \quad \forall u \in \mathcal{G}(X).$$

If $\Psi(1) = 1$, then $e = 1$ and ψ is uniquely determined.

(ii) If $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$ is an isomorphism of algebras, then the map ψ has an inverse $\psi^{-1} \in \mathcal{G}_{ld}[X, Y]$ such that Ψ^{-1} is given by composition with ψ^{-1}. As a map $\widetilde{X}_c \to \widetilde{Y}_c$, ψ is bijective. In this case, $\dim X = \dim Y$.

Here, ψ is an element of the larger space $\mathcal{G}_{\mathrm{ld}}[Y, X]$ rather than $\mathcal{G}[Y, X]$ as in corollary 6.4.4. This is due to our stronger proposition 5.4.12, which in the smooth case implies that on submanifolds X and Y, generalized functions $\mathcal{G}(X)^n$ that are c-bounded into $Y \subseteq \mathbb{R}^n$ already define a unique element in $\mathcal{G}[X, Y]$. This proof does not work in the non-smooth case as in general $(v_\varepsilon)_\varepsilon$ is not an element of $\mathcal{C}^\infty(X)^{(0,1]}$ there.

If $\mathcal{G}[Y, X] \subsetneq \mathcal{G}_{\mathrm{ld}}[Y, X]$ in the non-smooth setting, then a characterization of algebra homomorphisms $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$ as composition with elements in $\mathcal{G}[Y, X]$ is not possible as each element in $\mathcal{G}_{\mathrm{ld}}[Y, X]$ also defines such an algebra homomorphism.

Bibliography

[AJOS08] J. Aragona, S. O. Juriaans, O. R. B. Oliveira, and D. Scarpalezos, *Algebraic and geometric theory of the topological ring of Colombeau generalized functions*, Proc. Edinb. Math. Soc. (2) **51** (2008), no. 3, 545–564.

[AMR88] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, tensor analysis, and applications*, second ed., Applied Mathematical Sciences, vol. 75, Springer-Verlag, New York, 1988.

[BC70] F. Brickell and R. S. Clark, *Differentiable manifolds: an introduction*, The New University Mathematics Series, Van Nostrand Reinhold Company Ltd., London, 1970.

[Bou98] Nicolas Bourbaki, *Algebra I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.

[Col84] Jean-François Colombeau, *New generalized functions and multiplication of distributions*, North-Holland Mathematics Studies, vol. 84, North-Holland Publishing Co., Amsterdam, 1984.

[Col85] ______, *Elementary introduction to new generalized functions*, North-Holland Mathematics Studies, vol. 113, North-Holland Publishing Co., Amsterdam, 1985.

[Dav96] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[dC92] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992.

[Eng77] Ryszard Engelking, *General topology*, PWN—Polish Scientific Publishers, Warsaw, 1977.

[Erl07] Evelina Erlacher, *Local existence results in algebras of generalised functions*, Ph.D. thesis, Universität Wien, May 2007.

[Gar05a] Claudia Garetto, *Topological structures in Colombeau algebras: investigation of the duals of $\mathcal{G}_c(\Omega), \mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^n)$*, Monatsh. Math. **146** (2005), no. 3, 203–226.

[Gar05b] ______, *Topological structures in Colombeau algebras: topological $\widetilde{\mathbb{C}}$-modules and duality theory*, Acta Appl. Math. **88** (2005), no. 1, 81–123.

[GK37] I. M. Gelfand and A. N. Kolmogorov, *On rings of continuous functions on topological spaces*, C. R. Acad. Sci. URSS (N. S.) **22** (1937), 11–15.

[GKM68] D. Gromoll, W. Klingenberg, and W. Meyer, *Riemannsche Geometrie im Grossen*, Lecture Notes in Mathematics, No. 55, Springer-Verlag, Berlin, 1968.

[GKOS01] Michael Grosser, Michael Kunzinger, Michael Oberguggenberger, and Roland Steinbauer, *Geometric theory of generalized functions with applications to general relativity*, Mathematics and its Applications, vol. 537, Kluwer Academic Publishers, Dordrecht, 2001.

[GP74] Victor Guillemin and Alan Pollack, *Differential topology*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.

[Gra05] J. Grabowski, *Isomorphisms of algebras of smooth functions revisited*, Arch. Math. (Basel) **85** (2005), no. 2, 190–196.

[Gro99] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999.

[Hal06] Stefan Haller, *Differentialtopologie*, Lecture notes, January 2006, http://www.mat.univie.ac.at/∼stefan/teach.html.

[Hel93] A. Ya. Helemskii, *Banach and locally convex algebras*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1993.

[HH06] Qing Han and Jia-Xing Hong, *Isometric embedding of Riemannian manifolds in Euclidean spaces*, Mathematical Surveys and Monographs, vol. 130, American Mathematical Society, Providence, RI, 2006.

[Hir76] Morris W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1976, Graduate Texts in Mathematics, No. 33.

[KM97] Andreas Kriegl and Peter W. Michor, *The convenient setting of global analysis*, Mathematical Surveys and Monographs, vol. 53, American Mathematical Society, Providence, RI, 1997.

[Kri04] Andreas Kriegl, *Analysis 2*, Lecture notes, March 2004, http://www.mat.univie.ac.at/∼kriegl/Skripten/Analysis.html.

[KSV03] Michael Kunzinger, Roland Steinbauer, and James A. Vickers, *Intrinsic characterization of manifold-valued generalized functions*, Proc. London Math. Soc. (3) **87** (2003), no. 2, 451–470.

[KSV09] ______, *Sheaves of nonlinear generalized functions and manifold-valued distributions*, Trans. Amer. Math. Soc. (2009), to appear, Preprint arXiv: math.FA/0609358.

[Kun06] Michael Kunzinger, *Differentialgeometrie 2*, Lecture notes, 2006.

[Lar73] Ronald Larsen, *Banach algebras*, Marcel Dekker Inc., New York, 1973, Pure and Applied Mathematics, No. 24.

[Mic08] Peter W. Michor, *Topics in differential geometry*, Graduate Studies in Mathematics, vol. 93, American Mathematical Society, Providence, RI, 2008.

[Mrč05] Janez Mrčun, *On isomorphisms of algebras of smooth functions*, Proc. Amer. Math. Soc. **133** (2005), no. 10, 3109–3113.

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974, Annals of Mathematics Studies, No. 76.

[MŠ07] Janez Mrčun and Peter Šemrl, *Multiplicative bijections between algebras of differentiable functions*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 471–480.

[Nig06] Eduard Nigsch, *Colombeau generalized functions on manifolds*, Master's thesis, Technische Universität Wien, 2006.

[Obe92] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Pitman Research Notes in Mathematics Series, vol. 259, Longman Scientific & Technical, Harlow, 1992.

[O'N83] Barrett O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc., New York, 1983, With applications to relativity.

[Pet06] Peter Petersen, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006.

[Sch54] Laurent Schwartz, *Sur l'impossibilité de la multiplication des distributions*, C. R. Acad. Sci. Paris **239** (1954), 847–848.

[Spi79] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I*, second ed., Publish or Perish Inc., Wilmington, Del., 1979.

[Ver06] Hans Vernaeve, *Isomorphisms of algebras of Colombeau generalized functions*, Preprint arXiv: math.FA/0612646, 2006.

[Ver08] ______, *Banach* $\widetilde{\mathbb{C}}$*-algebras*, Preprint arXiv: math.FA/0811.1742, 2008.

[Ver09] ______, *Exercise in differential geometry*, Private communications, March 2009.

Notation

This list includes short descriptions for commonly used symbols in the text. The page numbers refer to either their definition or the first appearance in the text.

$\langle .,. \rangle$	the scalar product $g(p)$ on the tangent space T_pX of the semi-Riemannian manifold (X, g), p. 26
$\sim$	an equivalence relation
$\lhd$	ideal, p. 6
∇	the Levi-Civita connection on a semi-Riemannian manifold, p. 27
$\otimes$	tensor product, p. 9
$\subseteq$	subset
$\subset, \subsetneq$	proper subset
$V \subset\subset U$	$V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact
$\bigoplus$	direct sum of sets, p. 14
$\triangle_X$	the diagonal in $X \times X$, p. 29
$\overline{U}$	the closure of the set U
$[U, V]$	the Lie bracket of the vector fields U and V
$\\|f\\|_\infty$	the supremum norm of the function f, $\\|f\\|_\infty = \sup_{x \in X} \|f(x)\|$
$\|v\|$	the norm of the vector v, e.g. p. 26
$\mathcal{A}$	an algebra, p. 5
$\hat{\mathcal{A}}$	the carrier space of the algebra $\mathcal{A}$, p. 1
$\mathcal{A}(X)$	an algebra on the manifold X, p. 3
$B_r(p)$	the open ball at p with radius r
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^{(0,1]}$	the set of nets of complex numbers on the directed set $(0, 1]$, p. 57
$\widetilde{\mathbb{C}}$	the ring of generalized complex numbers, p. 47
$\complement U$	the complement of the set U

$\mathrm{ch}(U)$	the closed and convex hull of the set U, p. 53
$\mathcal{C}(X)$	the set of continuous functions on the topological space X, p. 1
$\mathcal{C}_0(X)$	the set of continuous functions on the topological space X that vanish at infinity, p. 1
$\mathcal{C}^\infty((0,1])$	the set of smooth functions $(0,1] \to \mathbb{C}$, p. 47
$\mathcal{C}^\infty((0,1] \times X)$	the set of smooth functions $(0,1] \times X \to \mathbb{C}$, p. 46
$\mathcal{C}^\infty(X)$	the set of smooth functions $X \to \mathbb{C}$, p. 11
$\mathcal{C}^\infty(X)^{(0,1]}$	the set of nets of smooth functions $X \to \mathbb{C}$ on the directed set $(0,1]$ (only smooth in X), p. 83
$\mathcal{C}_c^\infty(X)$	the set of compactly supported smooth functions $X \to \mathbb{C}$, p. 14
c_v	the geodesic with initial velocity v, p. 29
d	the Euclidean metric on $\mathbb{R}^n$
$\mathcal{D}$	the domain of the exponential map E, $\subseteq TX$, p. 29
δ_{ij}	the Kronecker delta
$\frac{df}{dt} = f'$	the differential of f w.r.t. t
$\frac{\partial f}{\partial x} = D_x f$	the partial derivative of f w.r.t. x
d_g	the Riemannian distance on a Riemannian manifold (X, g), p. 27
$\dim E$	the dimension of the vector space E
$\mathrm{distort}(X)$	the distortion of a Riemannian manifold X, p. 43
$D^{(k)} f$	the total derivative of f of order k, p. 54
$\mathcal{D}'(\Omega)$	the set of distributions on the open set $\Omega \subseteq \mathbb{R}^n$, p. 46
$\mathcal{D}_p$	the domain of the exponential map $\exp_p$, $\subseteq T_pX$, p. 29
E	the global exponential map on a semi-Riemannian manifold, p. 29
(E, B, π)	the vector bundle with $\pi : E \to B$, p. 8
$\mathcal{E}_M$	the set of moderate numbers, p. 47
$\mathcal{E}_M(\Omega)$	the set of moderate functions on an open set $\Omega \subseteq \mathbb{R}^n$
$\mathcal{E}_M(X)$	the set of moderate functions on a manifold X, p. 46
$\mathcal{E}_M[X, Y]$	the set of c-bounded moderate functions from X to Y, p. 54
e_S	an idempotent in $\widetilde{\mathbb{C}}$, p. 81
ev_p	the evaluation mapping at the (generalized) point p
$\exp_p$	the exponential map on a semi-Riemannian manifold at the point p, p. 29
$\Gamma_\mathcal{A}$	the Gelfand transformation on the Banach algebra $\mathcal{A}$, p. 1

${}^{u}\Gamma^{k}_{ij}$	the Christoffel symbols w.r.t. the chart (u, U), p. 28
γ_{pq}	the radial geodesic from p to q, p. 30
$\Gamma(E) = \Gamma(B, E)$	the set of smooth sections of a vector bundle (E, B, π), p. 9
g, h	metric tensors of semi-Riemannian manifolds, p. 26
$\mathcal{G}(X)$	the special Colombeau algebra on a manifold X, p. 47
$\mathcal{G}(X)^n$	the set of generalized functions from X to $\mathbb{C}^n$ resp. $\mathbb{R}^n$, p. 56
$\mathcal{G}[X, Y]$	the set of c-bounded generalized functions from X to Y, p. 55
$\mathcal{G}_{\mathrm{ld}}[X, Y]$	the set of locally defined c-bounded generalized functions from X to Y, p. 82
$\mathrm{id}_{\mathcal{G}[X,X]}$	the generalized identity on the space $\mathcal{G}[X, X]$, p. 61
id_X	the identity map on a manifold X, $\mathrm{id}(p) = p \ \forall p \in X$
$\mathrm{im}\,\varphi$	the image of the function φ
inj	the injectivity radius of a Riemannian manifold, p. 38
$\mathrm{int}(U)$	the interior of the set U
κ	the convexity radius of a Riemannian manifold, p. 38
$\ker\varphi$	the kernel of the function φ
$K_{\tilde{p}}$	a compact support of a generalized point $\tilde{p}$, p. 48
$L(c)$	the arc length of the curve c, p. 27
$L^\infty(\Omega)$	the set of essentially bounded functions on Ω
$\mathbb{N}$	the set of positive integers
$\mathcal{N}$	the set of negligible numbers, p. 47
$\mathcal{N}(\Omega)$	the set of negligible functions on an open set $\Omega \subseteq \mathbb{R}^n$
$\mathcal{N}(X)$	the set of negligible functions on a manifold X, p. 46
$\mathcal{N}_\varepsilon(p)$	the neighborhood at p in W_p with radius $N^p < \varepsilon$, p. 35
$N^p(q)$	the squared Euclidean radius of ${}^{p}w(q)$, p. 35
ν	the index of a symmetric bilinear form resp. metric tensor, p. 26
$O(.)$	Big-Oh, Landau symbol, p. 46
Ω	an open set in $\mathbb{R}^n$
$\Omega^1(X) = \Gamma(T^*X)$	the set of one-forms on the manifold X, p. 9
$\widetilde{\Omega}_c$	the set of compactly supported generalized points on an open set $\Omega \subseteq \mathbb{R}^n$, p. 48
φ	a multiplicative linear functional on an algebra
pr_i	the projection in the i-th direction

Ψ	an algebra homomorphism or isomorphism, p. 6
$\mathcal{P}(X)$	the set of linear differential operators on a manifold X, p. 46
r	the radius function at a point p on a semi-Riemannian manifold, p. 30
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, i.e. $(0, \infty)$
$\mathbb{R}_0^+$	the set of positive real numbers including 0, i.e. $[0, \infty)$
$\widetilde{\mathbb{R}}$	the ring of generalized real numbers
$\mathrm{rk}(A)$	the rank of the matrix A
S	a characteristic set, p. 67
$\operatorname{supp} f$	the support of the function f
Tf	the tangent map of a smooth function f, p. 9
T_pf	the tangent map of a smooth function f at p, p. 8
T_pX	the tangent space of a manifold X at $p \in X$, p. 8
$(T_pX)^*$	the cotangent space of a manifold X at $p \in X$, p. 8
$T^r_s(E)$	the set of $\binom{r}{s}$-tensors on a vector space E, p. 9
T^r_sX	the set of $\binom{r}{s}$-tensors on the tangent bundle TX of a manifold X, p. 9
$\mathcal{T}^r_s(X)$	the set of $\binom{r}{s}$-tensor fields on a manifold X, p. 9
TX	the tangent bundle of a manifold X, p. 9
T^*X	the cotangent bundle of a manifold X, p. 9
$(TX)_0$	the zero section of TX, p. 29
$\mathcal{U}$	an open cover of a manifold, p. 10
$U_\varepsilon(p)$	the normal ε-neighborhood at p in a Riemannian manifold, p. 30
(u, U)	a chart $u : X \supseteq U \to \mathbb{R}^n$ defined on the open set U, p. 7
$V_\varepsilon(p)$	the ε-neighborhood at 0 in the tangent space T_pX, p. 31
$({}^pw, W_p)$	a normal coordinate chart at the point p, p. 33
$\widetilde{X}_c$	the set of compactly supported generalized points on a manifold X, p. 47
(X, g)	a semi-Riemannian manifold X with metric tensor g, p. 27
$\mathfrak{X}(X) = \Gamma(TX)$	the set of smooth vector fields on the manifold X, p. 9
X, Y	manifolds, p. 7

Index

Curriculum Vitae

Personal Information

Name	Annegret Yvonne Burtscher
Date of Birth	25th of January, 1983
Place of Birth	Rum, Austria

Education

since 10/2003	Diploma/Bachelor studies at the University of Vienna Mathematics and Earth Sciences
06/2002	Matura (school leaving examination) passed with honors
1997–2002	HTL Rankweil, secondary technical college focus on Construction/Civil Engineering
1993–1997	Bundesrealgymnasium Bludenz, grammar school
1989-1993	Volksschule Nüziders, primary school

Employment

since 04/2009	Employed on the FWF project P20525 Global Analysis in Algebras of Generalized Functions
10/2006–02/2009	Tutor at the Faculty of Mathematics, University of Vienna
summer of 2008	Seminar for Applied Mathematics, ETH Zürich, Switzerland
summer of 2006	INA (oil and gas company), Zagreb, Croatia
summer of 2004	Versuchsanstalt Rankweil (experimental station), Austria
11/2002–08/2003	Arup, Morris Sisk Consortium, Co. Wicklow, Ireland N11 Euro-Route, drawing and surveying office, laboratory
summer of 2001	Seitz & Stark (surveyor's office), Offenburg, Germany
summer of 2000	Gemeinde Nüziders (building authorities), Austria
summer of 1999	Wucher (building contractor), Ludesch, Austria

Voluntary work

since 12/2006	Member of IAESTE Vienna
03–08/2008	Board member for Exchange Outgoing, IAESTE Vienna