Scattering Theory for Jacobi Operators and Applications to Completely Integrable Systems

Johanna Michor

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Betreut von

a.Prof. Dr. Gerald Teschl

Universität Wien

Begutachtet von

Prof. Dr. Iryna Egorova

Kharkiv National University

und

a.Prof. Dr. Maria Hoffmann-Ostenhof

Universität Wien

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Abstract

In this thesis we develop direct and inverse scattering theory for Jacobi operators which are short range perturbations of quasi-periodic finite-gap operators. We show existence of transformation operators, investigate their properties, derive the corresponding Gel'fand-Levitan-Marchenko equation, and find minimal scattering data which determine the perturbed operator uniquely. Then we apply this knowledge to solve the associated initial value problem of the Toda hierarchy via the inverse scattering transform.

Contents

Preface

For a square summable complex valued sequence $f(n)_{n\in\mathbb{Z}}$ the Jacobi operator H is defined by

$$Hf(n) = a(n)f(n+1) + b(n)f(n) + a(n-1)f(n-1),$$

where $a(n)$, $b(n)$ are real valued and bounded. Classical scattering theory is concerned with the reconstruction of a given Jacobi operator H, which is a short range perturbation of the *free* Jacobi operator H_0 associated with the coefficients $a(n) = \frac{1}{2}$, $b(n) = 0$.

We are interested in generalizing this concept by replacing the free Jacobi operator with a *quasi-periodic* one. Quasi-periodic Jacobi operators arise in the setting of reflectionless Jacobi operators with finitely many spectral gaps. Its coefficients $a_q(n)$, $b_q(n)$ are expressible in terms of the Riemann theta function, hence they are quasi-periodic. A special case of quasi-periodic Jacobi operators are *periodic* ones, i.e.

$$a(n+N) = a(n), \quad b(n+N) = b(n), \quad \forall\, n \in \mathbb{Z}, \quad N \in \mathbb{N}.$$

Let H_q be a given quasi-periodic Jacobi operator and H a perturbation of H_q satisfying the short range assumption

$$\sum_{n\in\mathbb{Z}} |n| \Big(|a(n) - a_q(n)| + |b(n) - b_q(n)| \Big) < \infty.$$

With this assumption we prove the existence of *Jost solutions*, i.e. solutions of

$$a(n)\psi_\pm(w,n+1) + b(n)\psi_\pm(w,n) + a(n-1)\psi_\pm(w,n-1) = z\psi_\pm(w,n),$$

where z and $w(z)$ denote the spectral parameter and the *quasi-momentum map*, respectively. The Jost solutions $\psi_\pm(w,n)$ asymptotically look like the quasi-periodic solutions, that is,

$$\psi_\pm(w,n) = \psi_{q,\pm}(w,n) \quad \text{as } n \to \pm\infty,$$

where $\psi_{q,\pm}(w,.)$ denote the *Baker-Akhiezer* functions. The Jost solutions are used to define the *scattering data*

$$S_\pm(H) = \big\{ R_\pm(w), |w| = 1; (\rho_j, \gamma_{\pm,j}), 1 \le j \le q \big\}$$

for the pair (H, H_q) via the scattering relations

$$T(w)\psi_\pm(w,n) = \overline{\psi_\mp(w,n)} + R_\mp(w)\psi_\mp(w,n), \qquad |w| = 1,$$

and

$$\frac{1}{\gamma_{\pm,j}} = \sum_{n\in\mathbb{Z}} |\hat{\psi}_\pm(\rho_j,n)|^2.$$

The functions $T(w)$ and $R_\pm(w)$ are known as *transmission* and *reflection coefficients*. We find minimal scattering data which determine the perturbed operator uniquely. In the inverse scattering step we reconstruct H from its scattering data $S_\pm$ and H_q.

In addition to being of interest of its own, scattering theory can also be used to solve the initial value problem of the Toda equation. The Toda lattice is a simple model for a nonlinear one-dimensional crystal,

$$m\frac{d^2}{dt^2}x(n,t) = V'(x(n+1,t)-x(n,t)) - V'(x(n,t)-x(n-1,t)), \tag{1}$$

where the interaction potential is exponential, $V(r) = e^{-r}+r-1$. Here $x(n,t)$ denotes the deviation of the n-th particle (with mass m) from its equilibrium position. The important property of the Toda equation (1) is the existence of *soliton* solutions, that is, pulslike waves which spread in time without changing their size and shape. The existence of such solutions is usually related to complete integrability of the system.

The key to methods of solving the Toda equations based on spectral and inverse spectral theory for the Jacobi operator is its reformulation as a Lax pair. We solve the initial value problem of the Toda equations with asymptotically quasi-periodic initial conditions using a procedure known as inverse scattering transform. This method consists of three steps, namely to find the scattering data of the initial conditions, to find the time evolution of the scattering data, and finally, to reconstruct the potential from the (time dependent) scattering data.

Scattering theory with constant background was first developed on an informal level by Case [8] – [13] in 1973. Guseinov [27] gave necessary and sufficient conditions for the scattering data to determine H uniquely under the assumption

$$\sum_{n\in\mathbb{Z}} |n|\Big(|a(n)-\frac{1}{2}| + |b(n)|\Big) < \infty.$$

Further extensions are due to Guseinov [28], [29], and Teschl [47]. The inverse scattering transform was first derived by Gardner et al. [23] in 1967 to solve the Korteweg-de Vries equation. This method has been generalized to the Toda equation by Flaschka [21], who also worked out the inverse procedure in the reflectionless case, with further contributions by Boutet de Monvel et al. [4]. Additional results and an extension of the method to the entire Toda hierarchy are due to Teschl [46], [47].

The investigation of scattering theory with periodic background has only recently been started by Boutet de Monvel and Egorova [6], by Bazargan and Egorova [2], and by Volberg and Yuditskii [54], who treat the case where H has a homogeneous spectrum and is of Szegö class. Applications to the Toda lattice have been given by Boutet de Monvel and Egorova [5].

Jacobi operators can be viewed as the discrete analogue of Sturm-Liouville operators and their investigation has many similarities with Sturm-Liouville theory. First results in the case of periodic Sturm-Liouville operators have been obtained by Firsova [19]. For further results including potentials with different spatial asymptotics see Gesztesy et al. [25].

Spectral theory for quasi-periodic Jacobi operators and a complete algebro-geometric treatment of the Toda and Kac-van Moerbeke hierarchies can be found in Bulla et al. [7] and Teschl [48]. The reader is recommended to have a look at the monograph [48] by Teschl as this is the main reference for the basic methods and concepts of this thesis.

Our approach uses heavily the fact that the Baker-Akhiezer function is a meromorphic function on the Riemann surface associated with the problem. This strategy gives a more streamlined treatment and more elegant proofs even in the special cases which were previously known. In this respect it is important to emphasize that, in contradistinction to the constant background case, the upper sheet of our Riemann surface is not simply connected and in particular not isomorphic to the unit ball.

Contents

After a brief introduction to Jacobi operators, Chapter 1 collects some well-known facts from Riemann surfaces and introduces the Baker-Akhiezer function, following Bulla et al. [7] and Teschl [48]. A complete characterization of the solutions of the Jacobi equation with quasi-periodic coefficients is given by Egorova, Michor, and Teschl [16]. In particular, the second solution at the band edges is derived using the expression of the Baker-Akhiezer function in terms of the Riemann theta function. We investigate the quasi-momentum map, which maps the spectrum of the unperturbed operator on the unit circle. In the periodic case, where the integrals can be explicitly computed, this was first done by Percolab [39]. The properties of the Baker-Akhiezer function as a meromorphic function on the Riemann surface are presented.

In Chapter 2 we prove existence of the Jost solutions and use them to completely determine the spectrum of the perturbed operator. In the periodic case, existence of Jost solutions was first shown by Geronimo and Van Assche [26] and Teschl [44]. Existence of the transformation operators as well as the crucial decay estimate on their coefficients is established in [16]. For periodic operators, this was first done by Boutet de Monvel and Egorova [6] in the special case where all spectral gaps are open. We derive relations between the coefficients and the kernels of the transformation operators which are vital for the inverse scattering step. We describe the properties of the scattering matrix. In [16] it is shown that the transmission coefficient can be reconstructed from the reflection coefficient, which was not previously known even in the periodic case. We develop the analog of the Gel'fand-Levitan-Marchenko theory for perturbations of periodic and quasi-periodic operators, a procedure which allows the reconstruction of the perturbed operator from its scattering data. By taking the Fourier transform of the scattering relations we derive a discrete integral equation for the kernels of the transformation operator, the Gel'fand-Levitan-Marchenko equation. We show that the Gel'fand-Levitan-Marchenko equation has a unique solution and prove positivity of the associated operator. A decay estimate on the kernel of the Gel'fand-Levitan-Marchenko operator is derived. In addition, we formulate necessary conditions for the scattering data to uniquely determine the Jacobi operator.

In Chapter 3 we show that our necessary conditions for the scattering data are also sufficient and we present the inverse scattering procedure.

To illustrate our results we study in Chapter 4 the periodic Jacobi operator associated with $a(n) = 1$, $b(2n) = -1$, $b(2n+1) = 1$, $n \in \mathbb{Z}$, as an example. We perturb this operator at $n = -1$ and explicitly compute the main functions for this case.

Chapter 5 introduces to the Toda hierarchy. We verify that two arbitrary bounded solutions of the Toda system whose initial conditions satisfy the short range assumption satisfy it for all $t \in \mathbb{R}$. Then we use the scattering theory to solve the initial value problem of the Toda equation with asymptotically quasi-periodic initial conditions via the inverse scattering transform. In particular, we derive the time evolution of the scattering data.

In Chapter 6 trace formulas are applied to scattering theory with quasi-periodic background. In the periodic case, this was first done by Teschl [45]. Finally, we investigate the connection between the transmission coefficient and Krein's spectral shift theory [30].

The Appendix compiles some facts for periodic Jacobi operators and is included for easy reference. We present a different approach to the transformation operator.

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Chapter 1

Jacobi operators with quasi-periodic coefficients

1.1 Jacobi operators

In this section we establish notation and recall some of the basic facts on Jacobi operators needed in the sequel. Detailed accounts of this material can be found, for instance, in [48].

We denote by $\ell^p(M)$, where $1 \le p \le \infty$ and $M = \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$, etc. the space of p-summable respectively bounded complex valued sequences $f = \{f(n)\}_{n\in M}$ and by $\ell^p(M, \mathbb{R})$ the corresponding restrictions to real valued sequences. We abbreviate by $\ell^p_\pm(\mathbb{Z})$ the set of sequences in $\ell(\mathbb{Z})$ which are ℓ^p near $\pm\infty$, i.e. sequences whose restriction to $\ell(\pm\mathbb{N})$ belongs to $\ell^p(\pm\mathbb{N})$. The scalar product and norm in the Hilbert space $\ell^2(M)$ will be denoted by

$$\langle f, g\rangle = \sum_{n\in M} \overline{f(n)} g(n), \qquad \|f\| = \sqrt{\langle f, f\rangle}, \qquad f, g \in \ell^2(M).$$

Let a, $b \in \ell(\mathbb{Z}, \mathbb{R})$ be two sequences satisfying

$$a(n) \in \mathbb{R}\backslash\{0\}, \quad b(n) \in \mathbb{R}.$$

We introduce the second order, symmetric difference equation

$$\begin{array}{rrcl} \tau: & \ell(\mathbb{Z}) & \to & \ell(\mathbb{Z}) \\ & f(n) & \mapsto & a(n)f(n+1) + b(n)f(n) + a(n-1)f(n-1), \end{array} \tag{1.1}$$

and the corresponding eigenvalue problem which is referred to as *Jacobi difference equation*

$$\tau u = zu, \quad u \in \ell(\mathbb{Z}), \quad z \in \mathbb{C}. \tag{1.2}$$

Definition 1.1. Associated with a, $b \in \ell^\infty(\mathbb{Z}, \mathbb{R})$, $a(n) \neq 0$, is the *Jacobi operator* H

$$\begin{array}{rrcl} H: & \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ & f & \mapsto & \tau f. \end{array}$$

The *Wronskian* of two sequences f, $g \in \ell(\mathbb{Z})$ is defined by

$$W_n(f,g) = a(n)\big(f(n)g(n+1) - f(n+1)g(n)\big).$$

Green's formula ([48], eq. (1.20))

$$\sum_{j=m}^{n} \Big(f(\tau g) - (\tau f)g\Big)(j) = W_n(f,g) - W_{m-1}(f,g), \qquad f,g \in \ell(\mathbb{Z}), \tag{1.3}$$

shows that the Jacobi operator H associated with τ is self-adjoint, that is,

$$\langle f, Hg\rangle = \langle Hf, g\rangle, \qquad f,g \in \ell^2(\mathbb{Z}),$$

since $\lim_{n\to\pm\infty} W_n(f,g) = 0$ for $f,g \in \ell^2(\mathbb{Z})$. Evaluating (1.3) in the special case where f and g both solve the Jacobi difference equation $\tau u = zu$ yields that the Wronskian is constant (i.e. does not depend on n) in this case.

Let $\sigma(H)$ denote the spectrum and $\rho(H) = \mathbb{C}\backslash\sigma(H)$ the resolvent set of H. The matrix elements of the *resolvent* $(H-z)^{-1}$, $z \in \rho(H)$, are called *Green function*

$$G(z,n,m) = \langle \delta_n, (H-z)^{-1}\delta_m\rangle, \qquad z \in \rho(H),$$

where

$$\delta_n(m) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

is the standart basis of $\ell(\mathbb{Z})$. The solution $u_\pm(z,n)$ of (1.2) which is square summable near $\pm\infty$ exists for $z \in \mathbb{C}\backslash\sigma_{ess}(H)$. Here $\sigma_{ess}(.)$ denotes the essential spectrum. This solutions allow a more explicit representation of the Green function

$$G(z,n,m) = \frac{1}{W(u_-(z),u_+(z))}\begin{cases} u_+(z,n)u_-(z,m) & \text{for } m \le n \\ u_+(z,m)u_-(z,n) & \text{for } n \le m, \end{cases} \qquad z \in \rho(H).$$

1.2 Quasi-periodic Jacobi operators and Riemann surfaces

We give a short survey of the theory and results following the presentation in [48]. Let $g \in \mathbb{N}_0$ and

$$E_0 < E_1 < E_2 < \cdots < E_{2g+1} \tag{1.4}$$

be given real numbers. Define the function

$$R_{2g+2}(z) = \prod_{j=0}^{2g+1}(z - E_j).$$

We choose $R_{2g+2}^{1/2}(z)$ as the fixed branch

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1}\sqrt{z-E_j},$$

where the square root is defined as

$$\sqrt{z} = |\sqrt{z}|e^{\mathrm{i}\arg(z)/2}, \qquad \arg(z) \in (-\pi,\pi], \qquad z \in \mathbb{C}.$$

The surface $\mathbb{M}$ associated with $R_{2g+2}^{1/2}(z)$ is a *compact Riemann surface* (i.e. a one-complex-dimensional, compact, second countable, connected Hausdorff space together with a holomorphic structure), *hyperelliptic*, and of genus g, that is, it can be described as a two sheeted covering of the Riemann sphere ($\cong \mathbb{C}\cup\{\infty\}$) branched at $2g+2$ points. A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The points $\{(E_j, 0), 0 \le j \le 2g+1\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \Big\{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C}\backslash \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]\Big\} \subset \mathbb{M}$$

are called upper, lower sheet, respectively.

Let $\{a_j, b_j\}_{j=1}^{g}$ be loops on the surface $\mathbb{M}$ representing the canonical generators of $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counter-clock wise on the upper sheet, with pairwise intersection indices given by

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{ij}, \qquad 1 \le i, j \le g.$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^{g}$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}}, \tag{1.5}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}.$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \qquad \int_{b_j} \zeta_k = \tau_{j,k}, \qquad \tau_{j,k} = \tau_{k,j}, \qquad 1 \le j, k \le g. \tag{1.6}$$

Having these preparations out of the way we define the following set of real numbers, the *Dirichlet eigenvalues*.

Hypothesis H. 1.2. *Let*

$$(\hat{\mu}_j(n_0))_{j=1}^{g} = (\mu_j(n_0), \sigma_j(n_0))_{j=1}^{g}$$

be a list of pairs satisfying

$$\begin{aligned} \mu_j(n_0) &\in (E_{2j-1}, E_{2j}) \text{ and } \sigma_j(n_0) \in \{\pm 1\} \text{ or} \\ \mu_j(n_0) &\in \{E_{2j-1}, E_{2j}\} \text{ and } \sigma_j(n_0) = 0. \end{aligned}$$

Note that in particular, $\mu_1(n_0) \neq E_0$ *and* $\mu_g(n_0) \neq E_{2g+1}$.

For each given list $(\hat{\mu}_j(n_0))_{j=1}^{g}$ we can recursively define a list $(\hat{\mu}_j(n))_{j=1}^{g}$ for all $n \in \mathbb{Z}$ again fulfilling Hypothesis 1.2 ([48], Theorem 8.13), if we follow the procedure described in [48], Section 8.3. In addition, we get two real and bounded sequences $a_q(n)$, $b_q(n)$ defined by

$$\begin{aligned} a_q(n)^2 &= \frac{1}{2}\sum_{j=1}^{g}\hat{R}_j(n) + \frac{1}{8}\sum_{j=0}^{2g+1}E_j^2 - \frac{1}{4}\sum_{j=1}^{g}\mu_j(n)^2 - \frac{b_q(n)^2}{4} > 0, \\ b_q(n) &= \frac{1}{2}\sum_{j=0}^{2g+1}E_j - \sum_{j=1}^{g}\mu_j(n), \end{aligned} \tag{1.7}$$

where

$$R_j(n) = \lim_{z\to\mu_j(n)}(z-\mu_j(n))\frac{-\prod_{l=0}^{2g+1}\sqrt{z-E_l}}{\prod_{l=1}^{g}(z-\mu_l(n))}, \qquad \hat{R}_j(n) = \sigma_j(n)R_j(n). \tag{1.8}$$

Without loss of generality we choose $a_q(n) > 0$ for all $n \in \mathbb{Z}$. The Jacobi operator H_q associated with a_q, b_q is called *quasi-periodic*

$$\begin{aligned} H_q: \quad \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z}) \\ f(n) &\mapsto a_q(n)f(n+1) + b_q(n)f(n) + a_q(n-1)f(n-1). \end{aligned}$$

Next, we introduce the polynomials

$$\begin{aligned} G_g(z,n_0) &= \prod_{j=1}^{g}(z-\mu_j(n_0)) \\ H_{g+1}(z,n_0) &= \sum_{j=1}^{g}\hat{R}_j(n_0)\prod_{k\neq j}(z-\mu_k(n_0)) + (z-b_q(n_0))G_g(z,n_0) \\ &= G_g(z,n_0)\Big(z-b_q(n_0)+\sum_{j=1}^{g}\frac{\hat{R}_j(n_0)}{z-\mu_j(n_0)}\Big). \end{aligned} \tag{1.9}$$

Using this notation we define two meromorphic functions on the Riemann surface $\mathbb{M}$,

$$\phi(p,n) = \frac{H_{g+1}(p,n) + R_{2g+2}^{1/2}(p)}{2a_q(n)G_g(p,n)} = \frac{2a_q(n)G_g(p,n+1)}{H_{g+1}(p,n) - R_{2g+2}^{1/2}(p)} \tag{1.10}$$

and the *Baker-Akhiezer function*

$$\psi_q(p,n,n_0) = \prod_{j=n_0}^{n-1}{}^{*}\ \phi(p,j) = \begin{cases} \prod_{j=n_0}^{n-1}\phi(p,j) & \text{for } n > n_0 \\ 1 & \text{for } n = n_0 \\ \prod_{j=n}^{n_0-1}(\phi(p,j))^{-1} & \text{for } n < n_0. \end{cases} \tag{1.11}$$

We denote by $\phi_\pm(z,n)$, $\psi_{q,\pm}(z,n,n_0)$ the chart expressions (branches) of $\phi(p,n)$, $\psi_q(p,n,n_0)$ in the charts $(\Pi_\pm,\pi)$, that is,

$$\begin{aligned} \phi_\pm(z,n) &= \frac{H_{g+1}(z,n) \pm R_{2g+2}^{1/2}(z)}{2a_q(n)G_g(z,n)} \\ &= \frac{1}{2a_q(n)}\Big(z-b_q(n)+\sum_{j=1}^{g}\frac{\hat{R}_j(n)}{z-\mu_j(n)} \mp \frac{\prod_{j=0}^{2g+1}\sqrt{z-E_j}}{\prod_{j=1}^{g}(z-\mu_j(n))}\Big). \end{aligned} \tag{1.12}$$

The two branches $\psi_{q,\pm}(z,n,n_0)$ of the Baker-Akhiezer function are solutions of the Jacobi difference equation $\tau_q u = zu$, $z \in \mathbb{C}$, where τ_q is the difference expression associated with H_q,

$$a_q(n)\psi_{q,\pm}(z,n+1,n_0) + b_q(n)\psi_{q,\pm}(z,n,n_0) + a_q(n-1)\psi_{q,\pm}(z,n-1,n_0) = z\psi_{q,\pm}(z,n,n_0).$$

Moreover, $\psi_{q,\pm}(z,n,n_0)$ are square summable near $\pm\infty$ by [48], Theorem 8.17. Equation (1.10) immediately implies

$$\psi_{q,+}(z,n,n_0)\psi_{q,-}(z,n,n_0) = \frac{G_g(z,n)}{G_g(z,n_0)} = \prod_{j=1}^{g} \frac{z-\mu_j(n)}{z-\mu_j(n_0)}. \tag{1.13}$$

The branch $\psi_{q,\sigma_j}(z,n,n_0)$ has a first order pole at $\mu_j(n_0)$ if $\mu_j(n_0)$ is away from the band edges

$$\lim_{z\to\mu_j(n_0)} (z-\mu_j(n_0))\psi_{q,\sigma_j}(z,n,n_0) = \psi_{q,\sigma_j}(\mu_j(n_0),n,1)\frac{\hat{R}_j(n_0)}{a_q(n_0)}$$

(use (1.12) and $\psi_{q,\pm}(z,n,n_0) = \psi_{q,\pm}(z,n,1)\phi_\pm(z,n_0)$) and both branches have a square root singularity if $\mu_j(n_0)$ coincides with a band edge E_l

$$\lim_{z\to\mu_j(n_0)} \sqrt{z-\mu_j(n_0)}\psi_{q,\pm}(z,n,n_0) = \pm\frac{\mathrm{i}^l \prod_{k\neq l}\sqrt{|E_l-E_k|}}{2a_q(n_0)\prod_{k\neq j}\sqrt{E_l-\mu_k(n_0)}}\psi_{q,+}(E_l,n,1).$$

The Wronskian is independent of n for any solutions of the Jacobi equation with the same parameter z, therefore

$$\begin{aligned} W_{n=n_0}(\psi_{q,+}(z,n,n_0),\psi_{q,-}(z,n,n_0)) &= a_q(n_0)(\phi_-(z,n_0)-\phi_+(z,n_0)) \\ &= \frac{\prod_{j=0}^{2g+1}\sqrt{z-E_j}}{\prod_{j=1}^{g}(z-\mu_j(n_0))}. \end{aligned} \tag{1.14}$$

The Green function of H_q reads

$$G(z,n,n) = \frac{\psi_{q,+}(z,n,n_0)\psi_{q,-}(z,n,n_0)}{W(\psi_{q,-},\psi_{q,+})} = \frac{\prod_{j=1}^{g}(z-\mu_j(n))}{-\prod_{j=0}^{2g+1}\sqrt{z-E_j}}, \qquad z\in\mathbb{C}\backslash\sigma(H_q).$$

The spectrum of H_q is purely absolutely continuous and consists of $g+1$ bands,

$$\sigma(H_q) = \bigcup_{j=0}^{g}[E_{2j},E_{2j+1}] = \sigma_{ac}(H_q), \qquad \sigma_{sc}(H_q) = \emptyset = \sigma_p(H_q), \tag{1.15}$$

where $\sigma_{ac}(.)$, $\sigma_{sc}(.)$, $\sigma_p(.)$ denote the absolutely continuous, singularly continuous, and point spectrum (set of eigenvalues). If $\mu_j(n)\neq\mu_j(n_0)$, then $\psi_{q,\sigma_j(n)}(\mu_j(n),n,n_0)=0$ if $\sigma_j(n)^2=1$ and $\psi_{q,-}(\mu_j(n),n,n_0)=\psi_{q,+}(\mu_j(n),n,n_0)=0$ if $\sigma_j(n)=0$. The case where $E_{2j}=E_{2j+1}$, that is, H_q has eigenvalues, is studied in [48], Section 8.3.

Associated with $\mathbb{M}$ is the *Riemann theta function*

$$\begin{aligned} \theta: \quad \mathbb{C}^g &\longrightarrow \mathbb{C} \\ \underline{z}=(z_1,\dots,z_g) &\longmapsto \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\left(\langle\underline{m},\underline{z}\rangle + \frac{\langle\underline{m},\underline{\tau}\,\underline{m}\rangle}{2}\right), \end{aligned}$$

where $\langle \underline{z}, \underline{z'}\rangle = \sum_{l=1}^{g} \overline{z_l} z'_l$ denotes the scalar product in $\mathbb{C}^g$ and $\underline{\tau} = (\tau_{j,k})_{1\le j,k\le g}$ the matrix of b-periods of the differentials (1.6). The Riemann theta function is holomorphic and has the fundamental properties ([17], VI, [48], A.5)

$$\begin{aligned} \theta(-\underline{z}) &= \theta(\underline{z}), \qquad z \in \mathbb{C}^g, \\ \theta(\underline{z} + \underline{m} + \underline{\tau}\,\underline{n}) &= \exp\left(-2\pi \mathrm{i}\,\langle \underline{n}, \underline{z}\rangle - \pi \mathrm{i}\,\langle \underline{n}, \underline{\tau}\,\underline{n}\rangle\right)\theta(\underline{z}), \qquad \underline{n}, \underline{m} \in \mathbb{Z}^g, \end{aligned}$$

that is, $\theta(\underline{z})$ is *quasi-periodic*. The Baker-Akhiezer function ψ_q and the function ϕ are expressible in terms of the Riemann theta function ([48], Theorem 9.2)

$$\begin{aligned} \phi(p,n) &= \sqrt{\frac{\theta(\underline{z}(n-1))}{\theta(\underline{z}(n+1))}}\frac{\theta(\underline{z}(p,n+1))}{\theta(\underline{z}(p,n))}\exp\left(\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right), \qquad (1.16) \\ \psi_q(p,n,n_0) &= \sqrt{\frac{\theta(\underline{z}(n_0-1))\theta(\underline{z}(n_0))}{\theta(\underline{z}(n-1))\theta(\underline{z}(n))}}\frac{\theta(\underline{z}(p,n))}{\theta(\underline{z}(p,n_0))}\exp\left((n-n_0)\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right), \end{aligned}$$

where $\omega_{\infty_+,\infty_-}$ is a normalized meromorphic differential with simple poles at $\infty_\pm$ and corresponding residua ± 1 (an abelian differential of the third kind, [48], (A.20)),

$$\omega_{\infty_+,\infty_-} = \frac{\prod_{j=1}^{g}(\pi - \lambda_j)}{R_{2g+2}^{1/2}} d\pi.$$

The hat indicates that we require the path of integration to lie in $\hat{M}$ (the fundamental polygon associated with $\mathbb{M}$). The base point p_0 has been chosen to be $(E_0, 0)$. The constants λ_j are determined by the normalization

$$\int_{a_j} \omega_{\infty_+,\infty_-} = 2\int_{E_{2j-1}}^{E_{2j}} \frac{\prod_{j=1}^{g}(z - \lambda_j)}{R_{2g+2}^{1/2}} dz = 0, \qquad (1.17)$$

which shows $\lambda_j \in (E_{2j-1}, E_{2j})$. We also introduced the abbreviation

$$\underline{z}(p,n) = \underline{\hat{A}}_{p_0}(p) - \underline{\hat{\alpha}}_{p_0}(D_{\underline{\hat{\mu}}(n)}) - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \qquad \underline{z}(n) = \underline{z}(\infty_+, n),$$

where $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors) (cf. [48], A.4)

$$\begin{aligned} \underline{A}_{p_0} : \quad M &\longrightarrow J(M) \\ p &\longmapsto \left[\int_{p_0}^{p} \underline{\zeta}\right] = \left[\left(\int_{p_0}^{p} \zeta_1, \dots, \int_{p_0}^{p} \zeta_g\right)\right], \\ \underline{\alpha}_{p_0} : \quad \mathrm{Div}(M) &\longrightarrow J(M) \\ \mathcal{D} &\longmapsto \sum_{p\in\mathbb{M}} \mathcal{D}(p)\underline{A}_{p_0}(p). \end{aligned}$$

A divisor $\mathcal{D}$ on $\mathbb{M}$ is a map $\mathcal{D} : \mathbb{M} \to \mathbb{Z}$ with $\mathcal{D}(p) \neq 0$ for only finitely many $p \in \mathbb{M}$. The set of all divisors on M is denoted by Div(M). The *Jacobian variety* of $\mathbb{M}$,

$$J(M) = \mathbb{C}^g / \{\underline{m} + \underline{\tau}\,\underline{n} \mid \underline{m}, \underline{n} \in \mathbb{Z}^g\},$$

is a compact, commutative, g-dimensional, complex Lie group. Abel's map $\underline{A}_{p_0}$ is injective and holomorphic and thus an embedding of $\mathbb{M}$ into $J(M)$ ([17], III.6.4). Finally, $\underline{\Xi}_{p_0}$ denotes the vector of Riemann constants,

$$\underline{\hat{\Xi}}_{p_0} = (\underline{\hat{\Xi}}_{p_0,1}, \dots, \underline{\hat{\Xi}}_{p_0,g}), \qquad \underline{\hat{\Xi}}_{p_0,j} = \frac{1 - \sum_{k=1}^{g} \tau_{j,k}}{2}, \qquad p_0 = (E_0, 0).$$

By (a special case of) Riemann's vanishing theorem $\theta(\underline{z}(p,n))$ has zeros precisely at $\hat{\mu}_j(n)$, $1 \le j \le g$,

$$\theta(\underline{z}(p,n)) = 0 \qquad \Leftrightarrow \qquad p \in \{\hat{\mu}_j(n)\}_{j=1}^{g}. \tag{1.18}$$

In addition, [48], (9.27), (9.28), show

$$\theta(\underline{z}(p,m))\theta(\underline{z}(p,n)) > 0, \qquad C_1 \le |\theta(\underline{z}(n))| \le C_2$$

for $\pi(p) \in (-\infty, E_0) \cup (E_{2g+1}, \infty) \cup \{\infty\}$, m, $n \in \mathbb{Z}$, and positive constants $C_1 \le C_2$.

The sequences $a_q(n)$, $b_q(n)$ can also be expressed in terms of the theta function (e.g. [48], Theorem 9.4)

$$\begin{aligned} a_q(n)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1))\theta(\underline{z}(n-1))}{\theta(\underline{z}(n))^2}, \\ b_q(n) &= \tilde{b} + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln\left(\frac{\theta(\underline{w}+\underline{z}(n))}{\theta(\underline{w}+\underline{z}(n-1))}\right)\Big|_{\underline{w}=0}, \end{aligned} \tag{1.19}$$

and hence are quasi-periodic with g periods. The constants $\tilde{a}$, $\tilde{b}$ depend only on the Riemann surface

$$\begin{aligned} \tilde{a} &= \lim_{\lambda\to\infty} \frac{1}{\lambda} \exp\left(\int_{E_0}^{\lambda} \frac{\prod_{j=1}^{g}(z-\lambda_j)}{R_{2g+2}^{1/2}(z)} dz\right) > 0, \\ \tilde{b} &= \frac{1}{2}\sum_{j=0}^{2g+2} E_j - \sum_{j=1}^{g} \lambda_j. \end{aligned} \tag{1.20}$$

The following criterion for a_q, b_q to be periodic, that is,

$$a_q(n+N) = a_q(n), \qquad b_q(n+N) = b_q(n), \qquad \forall n \in \mathbb{Z},\ N \in \mathbb{N},$$

is given in [31], Chapter 2 ([48], Theorem 9.6).

Theorem 1.3. *A necessary and sufficient condition for $a_q(n)$, $b_q(n)$ to be periodic is that $R_{2g+2}(z)$ is of the form*

$$\frac{1}{4A_p^2} R_{2g+2}(z) Q(z)^2 = \Delta(z)^2 - 1,$$

where $Q(z)$ (cf. (A.8)) is a polynomial with leading coefficient one. The period N is given by

$$N = \deg(Q) + g + 1.$$

In the periodic case,

$$\tilde{a} = |A_p|^{1/N} := \left(\prod_{j=1}^{N} a_q(j)\right)^{1/N}, \qquad \tilde{b} = \sum_{j=1}^{N} b_q(j),$$

and

$$\omega_{\infty_+,\infty_-} = \frac{1}{N} \frac{2A_p\Delta' d\pi}{R_{2g+2}^{1/2} Q} = \frac{1}{N} \frac{\Delta' d\pi}{(\Delta^2-1)^{1/2}}, \tag{1.21}$$

where Δ denotes the Hill discriminant (A.4). By Floquet's theory (cf. Theorem A.2) the solutions of the periodic Jacobi equation satisfy

$$\psi_{q,\pm}(z, n+N, n_0) = m^{\pm}(z)\psi_{q,\pm}(z,n,n_0), \qquad \Delta(z)^2 \neq 1,$$

where $m^{\pm}(z)$ are the two branches of the Floquet multiplier $m(p)$ (cf. (A.5))

$$m(p) = \psi(p,N) = (-1)^N \operatorname{sgn}(A_p) \exp\left(N \int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right). \tag{1.22}$$

For further information on quasi-periodic Jacobi operators and proofs of the results discussed here we refer the reader to [48], Section 9.

1.3 The Baker-Akhiezer solutions and the quasi-momentum map

Before we collect some basic properties of $\psi_q(p,n,n_0)$, let us introduce the following *fundamental solutions* c, $s \in \ell(\mathbb{Z})$ of the Jacobi equation

$$\tau_q c(z,.,n_0) = z c(z,.,n_0), \qquad \tau_q s(z,.,n_0) = z s(z,.,n_0)$$

satisfying the initial conditions

$$c(z,n_0,n_0) = s(z,n_0+1,n_0) = 1, \qquad c(z,n_0+1,n_0) = s(z,n_0,n_0) = 0.$$

In what follows, we set $n_0 = 0$ for simplicity and omit it, i.e. $\psi_q(p,n) := \psi_q(p,n,0)$ and $\mu_j := \mu_j(0)$. According to (1.11), $\psi_{q,\pm}(z,0,0) = 1$, and [48], (2.18), implies

$$\begin{aligned} \psi_{q,\pm}(z,n) &= c(z,n) \mp a_q(0)\tilde{m}_{\pm}(z)s(z,n) \\ &= c(z,n) + \phi_{\pm}(z)s(z,n), \end{aligned} \tag{1.23}$$

where $\tilde{m}_\pm$ are the following pair of *Weyl m-functions* ([48], (2.12))

$$\tilde{m}_{\pm}(z,n) = \mp \frac{\psi_{q,\pm}(z,n+1)}{a_q(n)\psi_{q,\pm}(z,n)}, \qquad \tilde{m}_{\pm}(z) = \tilde{m}_{\pm}(z,0). \tag{1.24}$$

Recall that $\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]$.

Lemma 1.4. *The Baker-Akhiezer solutions $\psi_{q,\pm}(z,n)$ have the following properties.*

(i). *$\psi_q(p,n)$ is real for $\pi(p) = z \in \mathbb{R}\backslash\sigma(H_q)$.*

(ii). *For $z \in \mathbb{R}$,*

$$|\psi_{q,\pm}(z,n)| \leq M|w(z)|^{\pm n}, \qquad M > 0,$$

where

$$w(z) = \exp\left(\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right), \qquad p = (z, \pm R_{2g+2}^{1/2}(z)), \tag{1.25}$$

and $|w(z)| = 1$ for $z \in \sigma(H_q)$, $|w(z)| < 1$ for $z \in \mathbb{R}\backslash\sigma(H_q)$. The function $|w(z)|$ has minima precisely at the points λ_j, $1 \leq j \leq g$, and $|w(z)| \to 0$ for $|z| \to \infty$.

(iii). *For* $p \in \Pi_\pm$,

$$|\psi_{q,\pm}(z,n)| \le M \left|\frac{\tilde a}{z}\right|^{\pm n}, \qquad M > 0,$$

where $\tilde a$ *depends only on the Riemann surface, that is, on* $\{E_j\}_{j=0}^{2g+1}$.

Proof. For (i) and (ii) see [48], Lemma 9.3.

(iii). Since $\theta(\underline{z}(p,n))$ is quasi-periodic and hence bounded with respect to $n \in \mathbb{Z}$ we can estimate this terms by a constant. The path of integration may intersect b-cycles since all a-periods are zero (arguing via homology theory). Suppose $p \in \Pi_\pm$, then ([48], (9.42))

$$\begin{aligned} \exp\Big(\int_{p_0}^{p} \omega_{\infty_+,\infty_-}\Big) &= \exp\Big(\int_{E_0}^{\lambda} \frac{\prod_{j=1}^{g}(z-\lambda_j)}{\mp\prod_{j=0}^{2g+1}\sqrt{z-E_j}} dz\Big) \\ &= -\Big(\frac{\tilde a}{z}\Big)^{\pm 1}\Big(1 + \frac{\tilde b}{z} + \frac{\tilde c}{z^2} + O(\frac{1}{z^3})\Big)^{\pm 1}, \end{aligned}$$

where $\tilde a$ and $\tilde b$ are defined in (1.20). □

Our next aim is to find linearly independent solutions of $\tau_q u = zu$ for all $z \in \mathbb{C}$. The Wronskian (1.14) shows that $\psi_{q,\pm}(z,n)$ are linearly dependent at the band edges E_j, $0 \le j \le 2g+1$.

Lemma 1.5. ([16]). *The solutions of* $\tau_q u = zu$ *can be characterized as follows.*

(i). *If* $R_{2g+2}(z) \neq 0$, *there exist two solutions satisfying*

$$\psi_{q,\pm}(z,n) = \theta_\pm(z,n) w(z)^{\pm n}, \qquad w(z) = \exp\Big(\int_{p_0}^{(z,+)} \hat\omega_{\infty_+,\infty_-}\Big),$$

with $\theta_\pm(z,n)$ *quasi-periodic.*

(ii). *If* $R_{2g+2}(z) = 0$, $z = E_j$, *there are two solutions satisfying*

$$\psi_q(E_j,n) = \psi_{q,+}(E_j,n) = \psi_{q,-}(E_j,n), \qquad \hat\psi_q(E_j,n) = \psi_q(E_j,n)(\hat\theta_j(n) + n), \tag{1.26}$$

where $\hat\theta_j(n)$ *is quasi-periodic.*

Proof. (ii). By (1.12), $\psi_{q,+}(E_j,n) = \psi_{q,-}(E_j,n)$. We construct a second linearly independent solution at $z = E = E_l$ using

$$s(E,n) = \lim_{z\to E} a_q(0) \frac{\psi_{q,+}(z,n) - \psi_{q,-}(z,n)}{W(\psi_{q,-}(z), \psi_{q,+}(z))},$$

which we obtain by (1.23),

$$\psi_{q,+}(z,n) - \psi_{q,-}(z,n) = -a_q(0)\big(\tilde m_+(z) + \tilde m_-(z)\big)s(z,n),$$

and $W(\psi_{q,+}(z), \psi_{q,-}(z)) = a_q(0)^2(\tilde m_+(z) + \tilde m_-(z))$. Without loss of generality we assume that E_j does not coincide with one of the Dirichlet eigenvalues μ_j (otherwise shift the base point n_0). To derive an expression for $\psi_{q,\pm}(z)$ at $z = E + \epsilon^2$ we start with

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{E_l + \epsilon^2 - E_j} = -\epsilon \prod_{j \neq l} \sqrt{E_l + \epsilon^2 - E_j} = \epsilon(\tilde{R}(E_l) + O(\epsilon^2)).$$

Moreover,

$$W(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{R_{2g+2}^{1/2}(z)}{G(z,0)} = \frac{\tilde{R}(E_l)}{G(E_l + \epsilon^2, 0)} \epsilon(1 + O(\epsilon^2)).$$

For $p = (E + \epsilon^2, \pm R_{2g+2}^{1/2}(E + \epsilon^2))$,

$$\begin{aligned} \int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-} &= \int_{p_0}^{E} \hat{\omega}_{\infty_+,\infty_-} \pm \int_0^{\epsilon^2} \frac{\prod_{j=1}^{g}(E + x - \lambda_j)}{\sqrt{x}\tilde{R}(E+x)} dx \\ &= \int_{p_0}^{E} \hat{\omega}_{\infty_+,\infty_-} \pm \frac{2\prod_{j=1}^{g}(E - \lambda_j)}{\tilde{R}(E)} \epsilon + O(\epsilon^3) =: \int_{p_0}^{E} \hat{\omega} \pm \beta\epsilon + O(\epsilon^3). \end{aligned}$$

By (1.5),

$$\begin{aligned} \underline{z}(p,n) &= \int_{p_0}^{p} \underline{\zeta} - \underline{\hat{\alpha}}_{p_0}(D_{\underline{\hat{\mu}}(n)}) - \underline{\hat{\Xi}}_{p_0} = \underline{z}(E,n) \pm \sum_{j=1}^{g} \underline{c}(j) \int_0^{\epsilon^2} \frac{(E+x)^{j-1} d\pi}{\sqrt{x}\tilde{R}(E+x)} dx \\ &= \underline{z}(E,n) \pm \sum_{j=1}^{g} \underline{c}(j) \frac{2E^{j-1}}{\tilde{R}(E)} \epsilon + O(\epsilon^3) =: \underline{z}(E,n) \pm \underline{\gamma}\, \epsilon + O(\epsilon^3) \end{aligned}$$

and

$$\theta(\underline{z}(p,n)) = \theta(\underline{z}(E,n)) \pm \frac{\partial\theta}{\partial \underline{z}}(\underline{z}(E,n))\, \underline{\gamma}\, \epsilon + O(\epsilon^3) =: \theta(n) \pm \theta'(n)\, \epsilon + O(\epsilon^2).$$

We obtain for the Baker-Akhiezer functions

$$\begin{aligned} \psi_{q,\pm}(z,n) &= C(n,0) \frac{\theta(\underline{z}(p,n))}{\theta(\underline{z}(p,0))} \exp\Big(n \int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\Big) \\ &= \psi_{q,\pm}(E,n)\Big(1 \pm \Big(\frac{\theta'(n)}{\theta(n)} - \frac{\theta'(0)}{\theta(0)}\Big)\epsilon + O(\epsilon^2)\Big) \exp(\pm n\beta + O(\epsilon^3)) \\ &= \psi_{q,\pm}(E,n)\Big(1 \pm (\tilde{\theta}(n) + n\beta)\epsilon + O(\epsilon^2)\Big), \end{aligned}$$

where we abbreviated $\tilde{\theta}(n) = \frac{\theta'(n)}{\theta(n)} - \frac{\theta'(0)}{\theta(0)}$. Finally,

$$\begin{aligned} s(E,n) &= a_q(0) G(E,0) \psi_{q,\pm}(E,n) \lim_{\epsilon \downarrow 0} \frac{2(\tilde{\theta}(n) + n\beta)\epsilon + O(\epsilon^2)}{\tilde{R}(E)\epsilon} \\ &= \frac{2a_q(0)G(E,0)}{\tilde{R}(E)} \psi_q(E,n)(\tilde{\theta}(n) + n\beta), \end{aligned} \tag{1.27}$$

thus a second linearly independent solution is given by

$$\hat{\psi}_q(E,n) = \psi_q(E,n)(\hat{\theta}(n) + n),$$

where

$$\hat{\theta}(n) = \frac{1}{\prod_{j=1}^{g}(E-\lambda_j)} \sum_{j,k=1}^{g} E^j c_k(j) \frac{\partial}{\partial w_k} \ln \theta(z(E,n)+w_j). \tag{1.28}$$

Note that $\tilde{\theta}(1) = \frac{\tilde{R}(E)}{H(E,0)} - \beta$ since

$$1 = s(E,1) = \frac{2a(0)G(E,0)}{\tilde{R}(E)} \phi(E,1)(\tilde{\theta}(1)+\beta).$$

□

Remark 1.6. *(i). Since $\psi_q(z,n)$ has a singularity if $z=\mu_j$, the solutions in Lemma 1.5 are not well-defined for those z. However, one can either remove the singularities of $\psi_q(z,n)$ or choose a different normalization point $n_0 \neq 0$ to see that solutions of the above type exist for every z.*

(ii). In the periodic case Floquet theory tells you that there are two possible cases at a band edge: Either two (linearly independent) periodic solutions or one periodic and one linearly growing solution. The above lemma shows that the first case happens if the corresponding gap is closed and the second if the gap is open.

To understand the properties of the Baker-Akhiezer solutions we have to investigate the *quasi-momentum map*

$$w(z) = \exp\left(\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right), \qquad p=(z,+). \tag{1.29}$$

Since $\lambda_j \in (E_{2j-1}, E_{2j})$, the integrand is a Herglotz function, that is, a holomorphic function $F: \mathbb{C}^+ \to \mathbb{C}^+$, and admits the following representation (cf. [48], App. B)

$$\frac{\prod_{j=1}^{g}(z-\lambda_j)}{R_{2g+2}^{1/2}(z)} = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} d\tilde{\mu}(\lambda)$$

with the probability measure

$$d\tilde{\mu}(\lambda) = \frac{\prod_{j=1}^{g}(\lambda-\lambda_j)}{\pi \mathrm{i} R_{2g+2}^{1/2}(\lambda)} \chi_{\sigma(H_q)}(\lambda) d\lambda.$$

Hence

$$\begin{aligned} g(z,\infty) &= \int_{p_0}^{p} \omega_{\infty_+,\infty_-} = \int_{E_0}^{z} \int_{-\infty}^{\infty} \frac{1}{\lambda-\zeta} d\tilde{\mu}(\lambda) d\zeta \\ &= \int_{-\infty}^{\infty} \ln\left(\frac{\lambda - E_0}{\lambda - z}\right) d\tilde{\mu}(\lambda). \end{aligned}$$

In particular, note that $-\mathrm{Re}(g(z,\infty))$ is the Green's function of the upper sheet Π_+ with a pole at ∞_+ and $\tilde{\mu}$ is the equilibrium measure of the spectrum (see [52], Theorem III.37). We will abbreviate $g(z) = g(z,\infty)$.

The asymptotic expansion of $\exp(g(z))$ is given by ([48], (9.42))

$$\exp\left(\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right) = -\frac{\tilde{a}}{z}\left(1 + \frac{\tilde{b}}{z} + O(\frac{1}{z^2})\right), \qquad z \to \pm\infty, \tag{1.30}$$

with $\tilde{a}$, $\tilde{b}$ defined in (1.20).

Theorem 1.7. *The map g is a bijection from the upper (resp. lower) half plane $\mathbb{C}^{\pm} = \{z \in \mathbb{C} \mid \pm Im(z) > 0\}$ to*

$$S^{\pm} = \{z \in \mathbb{C} \mid \pm Re(z) < 0, 0 < Im(z) < \pi\} \setminus \bigcup_{j=1}^{g} [g(\lambda_j), g(E_{2j+1})]$$

such that $\sigma(H_q) = \{z \mid Re(z) = 0\}$.

Proof. Consider the Jacobian matrix of $g(z)$

$$d(g) = \begin{pmatrix} \mathrm{Re}(\tilde{g}) & \mathrm{Im}(\tilde{g}) \\ -\mathrm{Im}(\tilde{g}) & \mathrm{Re}(\tilde{g}) \end{pmatrix}.$$

Due to the Herglotz property, $\mathrm{Im}(\tilde{g}) > 0$, and $\mathrm{i}g(z)$ has positive diagonal entries. Thus it satisfies the conditions of [38], Theorem 1(b) in Chapter VI, which shows that $\mathrm{i}g(z)$ is one-to-one.

To prove that $g(z)$ is surjective, we first show that the boundary of $\mathbb{C}^+$ is mapped to the boundary of S^+. Note that $g(\lambda)$ is negative for $\lambda < E_0$ and purely imaginary for $\lambda \in [E_0, E_1]$. At E_1, the real part starts to decrease from zero until it hits its minimum at λ_1 and increases again until it becomes 0 at E_2 (since all a-periods are zero, cf. (1.17)), while the imaginary part remains constant. Proceeding like this we move along the boundary of S^+ as λ moves along the real line. For $\lambda > E_{2g+1}$, $g(\lambda)$ is again negative.

Since $dg \neq 0$, $g(z)$ is a local diffeomorphism and hence an open map. Suppose $w \in S^+$ lies not in the image of $g(.)$. Then w is either a boundary point of $g(\mathbb{C}^+)$ or there exists an open set $w \in U_w \subset S^+$ with $\partial(\overline{U_w}) \subset \partial(g(\mathbb{C}^+))$. W.l.o.g. let w be the boundary point, that is, there exists $(x_n)_n \in g(\mathbb{C}^+)$ with $x_n \to w$. Since g^{-1} is continuous, $g^{-1}(x_n) \to g^{-1}(w)$. The point $g^{-1}(w) \in \mathbb{C}^+$, because $\partial(\mathbb{C}^+)$ is mapped to $\partial(S^+)$. Since $g(.)$ is a local diffeomorphism, the open sets $U_{g^{-1}(x_n)}$, $U_{g^{-1}(w)}$ must be mapped to open sets in $g(\mathbb{C}^+)$, which is a contradiction to $w \notin g(\mathbb{C}^+)$. □

Remark 1.8. *In the special case where H_q is periodic the quasi-momentum is given by $w(z) = \exp(iN^{-1} \arccos \Delta(z))$, where $\Delta(z)$ is the Floquet discriminant, and our result is due to [39].*

Therefore the map

$$\begin{aligned} w: \mathbb{C}^{\pm} &\longrightarrow W^{\pm} = \{w \in \mathbb{C} \mid |w| < 1, \pm\mathrm{Im}(w) > 0\} \setminus \bigcup_{j=1}^{g} [w(\lambda_j), w(E_{2j+1})] \\ z &\longmapsto \exp(g(z)) \end{aligned}$$

is bijective. Denote $W = W^+ \cup W^- \cup (-1,1)\setminus\{0\}$. If we identify the corresponding points on the slits $[w(\lambda_j), w(E_{2j+1})]$ we obtain a Riemann surface $\mathbb{W}$ which is isomorphic to the upper sheet Π_+.

Remark 1.9. *In [39], the largest band edge E_{2g+1} is chosen for the base point p_0 instead of E_0 and w will map $\mathbb{C}^{\pm} \to W^{\mp}$ in this case. Moreover, in the periodic case the slits $[w(\lambda_j), w(E_{2j+1})]$ appear at equal angles $2\pi/N$, where N is the period.*

Since $z \mapsto \exp(g(z))$ is a bijection, we consider the Baker-Akhiezer functions $\psi_{q,\pm}$ as functions of the new parameter w whenever convenient (for notational simplicity we set $\psi_{q,\pm}(\lambda(w), n) = \psi_{q,\pm}(w, n)$ and similarly for other quantities). They inherit the following properties: $\psi_{q,\pm}(w, n) \in \ell^2_\pm(\mathbb{Z})$ for $|w| < 1$, $\psi_{q,\pm}(w, n)$ are real on the slits $[w(\lambda_j), w(E_{2j+1})]$, and have equal values in the (with respect to the real axis) symmetric points of the slits. The functions $\psi_{q,\pm}(w, n)$ are meromorphic in $\mathbb{W}$ and continuous up to the boundary with the only possible singularities at the images of the Dirichlet eigenvalues $w(\mu_j)$ and at 0. More precisely, denote by $M_\pm$ the sets of singularities of the Weyl m-functions $\tilde{m}_\pm(\lambda)$, i.e. $M_+ \cup M_- = \{\mu_j\}_{j=1}^g$ (see (1.11) and [48], Section 2.1). Note that $\mu_j \in M_+ \cap M_-$ if and only if $\mu_j = E_l$, in this case both $\tilde{m}_\pm(\lambda)$ have a square root singularity at μ_j. Then

(B1) $\psi_{q,\pm}(w, n) \in \ell^2_\pm(\mathbb{Z})$, $|w| < 1$, $w \notin \{w(\mu_j)\}$.

(B2) $\psi_{q,\pm}(w, n)$ are holomorphic in $\mathbb{W}\backslash(\{w(\mu_j)\}_{j=1}^g \cup \{0\})$ and continuous on the boundary $\partial W\backslash\{w(\mu_j)\}_{j=1}^g$.

(B3) $\psi_{q,\pm}(w, n)$ have a simple pole at $w(\mu_j)$ if $\mu_j \in M_\pm\backslash\{E_l\}$, no pole if $\mu_j \notin M_\pm$, and if $\mu_j = E_l$,

$$\psi_{q,\pm}(w, n) = \pm\frac{\mathrm{i}^l C(n)}{w - w_l} + O(1), \tag{1.31}$$

where $C(n)$ is bounded and real, $w_l := w(E_l)$.

(B4) $\psi_{q,\pm}(\overline{w}, n) = \psi_{q,\mp}(w, n) = \overline{\psi_{q,\pm}(w, n)}$ for $|w| = 1$.

Define for $w \in \mathbb{C}^+ \cap [w(\lambda_j), w(E_{2j+1})]$

$$\psi_q(w^\pm, .) := \lim_{\epsilon\to 0} \psi_q(we^{\mp\mathrm{i}\epsilon}, .), \qquad \psi_q(\overline{w}^\pm, .) := \lim_{\epsilon\to 0} \psi_q(\overline{w}e^{\pm\mathrm{i}\epsilon}, .),$$

then $\psi_{q,\pm}(w^\pm, n) = \psi_{q,\pm}(\overline{w}^\pm, n) \in \mathbb{R}$.

Abbreviate $\tilde{m}_\pm(\lambda) = \lim_{\epsilon\downarrow 0} \tilde{m}_\pm(\lambda + \mathrm{i}\epsilon, 0)$, $\lambda \in \mathbb{R}$. Then (B4) is equivalent to $\mathrm{Im}(\tilde{m}_+(\lambda)) = \mathrm{Im}(\tilde{m}_-(\lambda))$, that is, the quasi-periodic Jacobi operator is *reflectionless* (e.g. [48], Lemma 8.1).

In order to find an orthonormal basis for the Hilbert space $L^2(\sigma(H_q), \mathbb{C}^2, d\lambda)$ we consider the *eigenfunction expansion* for H_q. We choose

$$\underline{U}(\lambda, n) = \begin{pmatrix} \psi_{q,+}(\lambda, n) \\ \psi_{q,-}(\lambda, n) \end{pmatrix} = V(\lambda) \begin{pmatrix} c(\lambda, n) \\ s(\lambda, n) \end{pmatrix}$$

as a new basis for the space of solutions, where

$$V(\lambda) = \begin{pmatrix} 1 & -a_q(0)\tilde{m}_+(\lambda) \\ 1 & a_q(0)\tilde{m}_-(\lambda) \end{pmatrix}.$$

Since the spectrum of H_q is purely absolutely continuous, this choice of basis diagonalizes the matrix measure $d\rho_{ac}$ given in [48], (2.150),

$$d\tilde{\rho}_{ac}(\lambda) = (V^{-1}(\lambda))^* d\rho_{ac}(\lambda) V^{-1}(\lambda) = \frac{1}{4a_q(0)^2\pi \mathrm{Im}(\tilde{m}_+(\lambda))} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\lambda.$$

We have a corresponding Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2, d\tilde{\rho}_{ac})$ with scalar product given by

$$\langle \underline{F}, \underline{G}\rangle_{L^2} = \sum_{i,j=0}^{1} \int_{\mathbb{R}} \overline{F_i(\lambda)} G_j(\lambda) (d\tilde{\rho}_{ac}(\lambda))_{i,j} = \int_{\mathbb{R}} \overline{\underline{F}(\lambda)} \underline{G}(\lambda) d\tilde{\rho}_{ac}(\lambda).$$

The vector valued functions $\underline{U}(\lambda, n)$ are orthogonal with respect to $d\tilde{\rho}_{ac}$

$$\langle \underline{U}(\lambda, m), \underline{U}(\lambda, n)\rangle_{L^2} = \frac{1}{4a_q(0)^2\pi}\int_{\mathbb{R}} \overline{\underline{U}(\lambda, m)}\underline{U}(\lambda, n)\frac{1}{\mathrm{Im}(\tilde{m}_+(\lambda))}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\lambda = \delta_{m,n}. \tag{1.32}$$

Now we can read off the normalization. Set

$$\tilde{V}(\lambda) = \sqrt{\frac{1}{4a_q(0)^2\pi\mathrm{Im}(\tilde{m}_+(\lambda))}}V(\lambda), \qquad \tilde{\underline{U}}(\lambda, n) = \tilde{V}(\lambda)\begin{pmatrix} c(\lambda, n) \\ s(\lambda, n) \end{pmatrix}.$$

Transforming the matrix measure to this normalized basis

$$(\tilde{V}^{-1}(\lambda))^* d\rho_{ac}(\lambda)\tilde{V}^{-1}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\lambda$$

yields that $\tilde{\underline{U}}(\lambda, .)$ are orthogonal with respect to the Lebesgue measure $d\lambda$

$$\langle \tilde{\underline{U}}(\lambda, m), \tilde{\underline{U}}(\lambda, n)\rangle_{L^2} = \int_{\mathbb{R}} (\overline{\tilde{\psi}_{q,+}(\lambda, m)}\tilde{\psi}_{q,+}(\lambda, n) + \overline{\tilde{\psi}_{q,-}(\lambda, m)}\tilde{\psi}_{q,-}(\lambda, n))d\lambda = \delta_{m,n}. \tag{1.33}$$

Lemma 1.10. *The vector valued functions*

$$\tilde{\underline{U}}(\lambda, n) = \begin{pmatrix} \tilde{\psi}_{q,+}(\lambda, n) \\ \tilde{\psi}_{q,-}(\lambda, n) \end{pmatrix} = \sqrt{\frac{1}{4a_q(0)^2\pi\mathrm{Im}(\tilde{m}_+(\lambda))}}\begin{pmatrix} \psi_{q,+}(\lambda, n) \\ \psi_{q,-}(\lambda, n) \end{pmatrix}$$

form an orthonormal basis for the Hilbert space $L^2(\sigma(H_q), \mathbb{C}^2, d\lambda)$.

We even obtain a unitary transformation $\overline{U} : \ell^2(\mathbb{R}) \to L^2(\mathbb{R}, \mathbb{C}^2, d\lambda)$ defined by

$$\begin{aligned} (\overline{U}f)(\lambda) &= \sum_{n\in\mathbb{Z}} f(n)\tilde{\underline{U}}(\lambda, n), \\ (\overline{U}^{-1}\underline{F})(n) &= \int_{\mathbb{R}} \tilde{\underline{U}}(\lambda, n)\underline{F}(\lambda)d\lambda, \end{aligned}$$

which maps the operator H_q to the multiplication operator by λ (compare with [48], Section 7.5)

$$\overline{U}H_q\overline{U}^{-1} = \tilde{H}, \quad \text{where} \quad \tilde{H}\underline{F}(\lambda) = \lambda\underline{F}(\lambda), \quad \underline{F}(\lambda) \in L^2(\mathbb{R}, \mathbb{C}^2, d\lambda).$$

Using our map $w(z) = \exp(\int_{p_0}^{(z,+)} \hat{\omega}_{\infty_+,\infty_-})$ we can transform the orthonormal basis of Lemma 1.10 into an orthonormal basis on the unit circle. By (1.12), (1.11), we have $\overline{\psi_{q,\pm}(\lambda, n)} = \psi_{q,\mp}(\lambda, n)$ for $\lambda \in \sigma(H_q)$ and thus $\overline{\psi_{q,\pm}(w, n)} = \psi_{q,\mp}(w, n)$ for $|w| = 1$. Denote

$$\exp\left(\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-}\right) = \exp(\mathrm{i}\theta) = w,$$

then

$$\frac{d\theta}{dz} = \frac{1}{\mathrm{i}}\frac{\prod_{j=1}^{g}(z - \lambda_j)}{-\prod_{j=0}^{2g+1}\sqrt{z - E_j}}, \qquad \frac{dw}{dz} = w\frac{\prod_{j=1}^{g}(z - \lambda_j)}{-\prod_{j=0}^{2g+1}\sqrt{z - E_j}}, \tag{1.34}$$

and (1.32) becomes

$$\begin{aligned}
\delta_{m,n} &= \frac{1}{4a_q(0)^2\pi}\int_{\mathbb{R}} \overline{\underline{U}(\lambda,m)}\underline{U}(\lambda,n)\frac{1}{\mathrm{Im}(\tilde{m}_+(\lambda))}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} d\lambda\\
&= \frac{1}{4a_q(0)^2\pi}\int_{\sigma(H_q)} (\overline{\psi_{q,+}(\lambda,m)}\psi_{q,+}(\lambda,n) + \overline{\psi_{q,-}(\lambda,m)}\psi_{q,-}(\lambda,n))\frac{d\lambda}{\mathrm{Im}(\tilde{m}_+(\lambda))}\\
&= \frac{1}{4a_q(0)^2\pi}\int_0^{\pi} \Big(\overline{\psi_{q,+}(\lambda(e^{\mathrm{i}\theta}),m)}\psi_{q,+}(\lambda(e^{\mathrm{i}\theta}),n)\\
&\quad +\overline{\psi_{q,-}(\lambda(e^{\mathrm{i}\theta}),m)}\psi_{q,-}(\lambda(e^{\mathrm{i}\theta}),n)\Big)\frac{1}{\mathrm{Im}(\tilde{m}_+(e^{\mathrm{i}\theta}))}\frac{-\mathrm{i}\prod_{j=0}^{2g+1}\sqrt{e^{\mathrm{i}\theta}-E_j}}{\prod_{j=1}^{g}(e^{\mathrm{i}\theta}-\lambda_j)}d\theta.
\end{aligned} \tag{1.35}$$

Lemma 1.11. *Both functions $\psi_{q,+}(w,n)$ and $\psi_{q,-}(w,n)$ form orthonormal bases in the Hilbert space $L^2(S^1, \frac{1}{2\pi\mathrm{i}}d\omega)$, where*

$$d\omega(w) = \prod_{j=1}^{g}\frac{\lambda(w)-\mu_j}{\lambda(w)-\lambda_j}\frac{dw}{w}.$$

Remark 1.12. *Equivalently, both functions*

$$\sqrt{\prod_{j=1}^{g}\frac{\lambda(w)-\mu_j}{\lambda(w)-\lambda_j}}\psi_{q,\pm}(w,n),$$

form orthonormal bases in the Hilbert space $L^2(S^1, \frac{1}{2\pi\mathrm{i}}d\lambda)$.

Proof. Abbreviate

$$\tilde{\psi}_{q,\pm}(w,n) = \left(-\frac{\mathrm{i}\prod_{j=0}^{2g+1}(\lambda(w)-E_j)^{1/2}}{2a_q(0)^2\mathrm{Im}(\tilde{m}_+(\lambda(w)))\prod_{j=1}^{g}(\lambda(w)-\lambda_j)}\right)^{1/2}\psi_{q,\pm}(w,n), \tag{1.36}$$

then (1.35) is equal to

$$\begin{aligned}
\delta_{m,n} &= \frac{1}{2\pi}\int_0^{\pi}(\overline{\tilde{\psi}_{q,+}(\lambda(e^{\mathrm{i}\theta}),m)}\tilde{\psi}_{q,+}(\lambda(e^{\mathrm{i}\theta}),n) + \overline{\tilde{\psi}_{q,-}(\lambda(e^{\mathrm{i}\theta}),m)}\tilde{\psi}_{q,-}(\lambda(e^{\mathrm{i}\theta}),n))d\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{\tilde{\psi}_{q,+}(\lambda(e^{\mathrm{i}\theta}),m)}\tilde{\psi}_{q,+}(\lambda(e^{\mathrm{i}\theta}),n)d\theta\\
&= \frac{1}{2\pi\mathrm{i}}\int_{|w|=1}\overline{\tilde{\psi}_{q,+}(w,m)}\tilde{\psi}_{q,+}(w,n)\frac{dw}{w}.
\end{aligned}$$

Since the Weyl m-functions differ from $\phi_\pm(z,n)$ only by a constant (cf. (1.24)), (1.12) implies

$$\tilde{m}_+(z) = -\frac{\phi_+(0)}{a_q(0)} = -\frac{1}{2a_q(0)^2}\left(z - b(0) + \sum_{j=1}^{g}\frac{\hat{R}_j(0)}{z-\mu_j} - \frac{\prod_{j=0}^{2g+1}\sqrt{z-E_j}}{\prod_{j=1}^{g}(z-\mu_j)}\right).$$

Therefore

$$\mathrm{Im}(\tilde{m}_+(z)) = \frac{\prod_{j=0}^{2g+1}\sqrt{z-E_j}}{\mathrm{i}2a_q(0)^2\prod_{j=1}^{g}(z-\mu_j)}, \qquad z\in\sigma(H_q), \tag{1.37}$$

$\mathrm{Im}(\tilde{m}_+(z)) = 0$ for $z \in \mathbb{R}\backslash\sigma(H_q)$, and

$$\tilde{\psi}_{q,\pm}(w,n) = \sqrt{\prod_{j=1}^{g} \frac{\lambda(w) - \mu_j}{\lambda(w) - \lambda_j}}\psi_{q,\pm}(w,n).$$

□

Observe that $d\omega$ is meromorphic on $\mathbb{W}$ with a simple pole at $w = 0$. In particular, there are no poles at $w(\lambda_j)$.

In the periodic case,

$$w = \exp\left(\int_{p_0}^{p} \frac{\Delta'(\zeta)}{N(\Delta^2(\zeta)-1)^{1/2}} d\zeta\right) =: \exp\left(\int_{p_0}^{p} \tilde{g}(\zeta) d\zeta\right). \tag{1.38}$$

Lemma 1.13. *Let $\lambda \in \sigma(H_p)$ and let $\psi_{p,\pm}(\lambda)$ be the Floquet solutions, then*

$$\|\psi_{p,\pm}(\lambda)\|_N^2 := \sum_{n=1}^{N} |\psi_{p,\pm}(\lambda,n)|^2 = 2a_p(0)^2 N \mathrm{Im}(\tilde{m}_+(\lambda))\mathrm{Im}(\tilde{g}(\lambda)), \tag{1.39}$$

where $\tilde{g}$ is defined by (1.38).

Proof. The first resolvent identity ([48], (2.4))

$$\begin{aligned} \mathrm{Im}(\tilde{m}_+(z)) &= \mathrm{Im}(z)\langle\delta_1, (H_{q,+} - \overline{z})^{-1}(H_{q,+} - z)^{-1}\delta_1\rangle \\ &= \mathrm{Im}(z)\|(H_{q,+} - z)^{-1}\delta_1\|^2 \\ &= \frac{\mathrm{Im}(z)}{a_p(0)^2} \sum_{j=1}^{\infty} |\psi_{q,+}(z,j)|^2 \end{aligned} \tag{1.40}$$

holds for arbitrary Jacobi operators. Since we are interested in Floquet solutions, $\psi_{p,+}(z,n) = p_+(z,n)e^{\mathrm{i}nq(z)}$ by Theorem A.2 and

$$\begin{aligned} \sum_{j=1}^{\infty} |\psi_{p,+}(z,j)|^2 &= \sum_{j=1}^{N} \left|p_+(z,j)e^{\mathrm{i}jq(z)}\right|^2 \sum_{k=0}^{\infty} \left|e^{\mathrm{i}Nkq(z)}\right|^2 \\ &= \|\psi_{p,+}(z)\|_N^2 \frac{1}{1 - |e^{\mathrm{i}Nq(z)}|^2}. \end{aligned}$$

For $\lambda \in \sigma(H_q)$,

$$\begin{aligned} \left|e^{\mathrm{i}Nq(\lambda+\mathrm{i}\epsilon)}\right|^2 &= \exp\left(2N\mathrm{Re}\int_{E_0}^{\lambda+\mathrm{i}\epsilon} \tilde{g}(\zeta)d\zeta\right) \\ &= \exp\left(2N\mathrm{Re}\int_{E_0}^{\lambda} \tilde{g}(\zeta)d\zeta + 2N\mathrm{Re}\int_{\lambda}^{\lambda+\mathrm{i}\epsilon} \tilde{g}(\zeta)d\zeta\right) \\ &= \exp\left(0 + 2N\mathrm{Re}\int_{0}^{\epsilon} \tilde{g}(\lambda+\mathrm{i}\zeta)\mathrm{i}d\zeta\right) \\ &= \exp\left(-2N\mathrm{Im}\int_{0}^{\epsilon} \tilde{g}(\lambda+\mathrm{i}\zeta)d\zeta\right) \\ &\approx 1 - 2N\mathrm{Im}(\tilde{g}(\lambda))\epsilon + O(\epsilon^2), \end{aligned}$$

and (1.40) yields

$$\mathrm{Im}(\tilde{m}_+(\lambda)) = \frac{\|\psi_{p,+}(\lambda)\|_N^2}{2a_p(0)^2 N \mathrm{Im}(\tilde{g}(\lambda))}, \qquad \lambda \in \sigma(H_p). \tag{1.41}$$

□

Corollary 1.14. ([6]). *In the periodic case, we infer for the norm*

$$\|\psi_{p,\pm}(\lambda)\|_N^2 = N \prod_{j=1}^{N-1} \frac{\lambda - \lambda_j}{\lambda - \mu_j}. \tag{1.42}$$

The normalized Floquet solutions

$$\tilde{\psi}_{p,\pm}(w,n) := \psi_{p,\pm}(w,n) \frac{\sqrt{N}}{\|\psi_{p,\pm}(w)\|_N}$$

form orthonormal bases in the Hilbert space $L^2(S^1, \frac{1}{2\pi \mathrm{i}} d\lambda)$.

Proof. Insert (1.37) and

$$\mathrm{Im}(\tilde{g}(\lambda)) = \frac{1}{N} \mathrm{Im}\left(\frac{\Delta'(\lambda)}{(\Delta^2(\lambda)-1)^{1/2}} \right) = \mathrm{i} \frac{\prod_{j=1}^{N-1}(\lambda - \lambda_j)}{\prod_{j=0}^{2N-1} \sqrt{\lambda - E_j}}$$

into (1.39) to obtain the norm. For the second assertion proceed as in (1.35) and Lemma 1.11. □

Lemma 1.15. *The functions* $\psi_{q,\pm}(w,n)$ *have the following asymptotic behavior as* $w \to 0$

$$\psi_{q,\pm}(w,n) = (-1)^n \left(\frac{\prod_{j=0}^{n-1} a_q(j)}{\tilde{a}^n} \right)^{\pm 1} w^{\pm n} + O(w^{\pm n + 1}),$$

with $\tilde{a} \in \mathbb{R}$ *as in (1.20). In the periodic case,* $\tilde{a} = \left| \prod_{j=1}^{N} a_p(j) \right|^{1/N}$.

Proof. The functions $\phi_\pm$ have the following expansions near $\pm\infty$ ([48], Theorem 9.4)

$$\phi_\pm(z,n) = \Big(\frac{a_q(n)}{z} \Big)^{\pm 1} \Big(1 \pm \frac{b_q(n + {}^1_0)}{z} + O(\frac{1}{z^2}) \Big), \qquad z \to \pm\infty,$$

and therefore

$$\begin{aligned} \psi_{q,\pm}(z,n) &= \prod_{j=0}^{n-1} \phi_\pm(z,j) \\ &= \Big(\frac{\prod_{j=0}^{n-1} a_q(j)}{z^n} \Big)^{\pm 1} \Big(1 \pm \frac{\sum_{j=0}^{n-1} b_q(j + {}^1_0)}{z} + O(\frac{1}{z^2}) \Big), \qquad z \to \pm\infty. \end{aligned} \tag{1.43}$$

Equation (1.30) implies

$$\frac{1}{z} = -\frac{w}{\tilde{a}}(1 + O(w)), \qquad w \to 0, \tag{1.44}$$

hence

$$\psi_{q,\pm}(w,n) = \prod_{j=0}^{n-1} a_q(j)^{\pm 1} \Big(-\frac{w}{\tilde{a}}(1+O(w)) \Big)^{\pm n}$$
$$= (-1)^n \Big(\frac{\prod_{j=0}^{n-1} a_q(j)}{\tilde{a}^n} \Big)^{\pm 1} w^{\pm n}(1+O(w)), \qquad w \to 0.$$

In the periodic case, (A.4) and (A.6) imply that

$$\Delta(z) = \frac{1}{2A_p} z^N + O(z^{N-1}) = \frac{w^N + w^{-N}}{2}, \qquad w \to 0, \tag{1.45}$$

and thus $z = A_p^{1/N} w^{-1} + O(1)$. Inserting this in

$$\psi_{p,-}(z,n) = m_-(z) s_p(z,n) + c_p(z,n) = \frac{z^n}{\prod_{j=0}^{n-1} a_p(j)} + O(z^{n-1}), \qquad z \to \infty,$$

yields the desired result. The formula for $\psi_{p,+}(w,n)$ follows then from (1.13) which shows $\psi_{p,+}(z,n)\psi_{p,-}(z,n) = 1 + O(z^{-1})$. The result in the periodic case was first given in [6]. □

Chapter 2

Direct scattering theory for quasi-periodic Jacobi operators

2.1 Existence of Jost solutions

Suppose that $a_q(n)$, $b_q(n)$ are given quasi-periodic sequences with corresponding Jacobi operator H_q as discussed in Chapter 1. In this section we want to study short-range perturbations H of H_q associated with sequences a, b satisfying $a(n) \to a_q(n)$ and $b(n) \to b_q(n)$ as $|n| \to \infty$. More precisely, we will make the following assumption throughout this work.

Hypothesis H. 2.1. *Let H be a perturbation such that*

$$\sum_{n\in\mathbb{Z}} |n| \Big(|a(n) - a_q(n)| + |b(n) - b_q(n)| \Big) < \infty. \tag{2.1}$$

We first establish existence of *Jost solutions*, that is, solutions of the perturbed system which asymptotically look like the Baker-Akhiezer solutions. For the proof we will need the *Volterra sum equation*.

Lemma 2.2. ([48], Lemma 7.8). *Consider the Volterra sum equation*

$$f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m) f(m).$$

If there is a sequence $\hat{K}(n,m)$ such that

$$|K(n,m)| \le \hat{K}(n,m), \qquad \hat{K}(n+1,m) \le \hat{K}(n,m), \qquad \hat{K}(n,.) \in \ell^1(0,\infty),$$

then, for given $g \in \ell^\infty(0,\infty)$, there is an unique solution $f \in \ell^\infty(0,\infty)$ fulfilling

$$|f(n)| \le \left(\sup_{m>n} |g(m)| \right) \exp\left(\sum_{m=n+1}^{\infty} \hat{K}(n,m) \right).$$

If $g(n)$ and $K(n,m)$ depend continuously (resp. holomorphically) on a parameter and if $\hat{K}$ does not, then $f(n)$ also depends continuously (resp. holomorphically) on that parameter.

Lemma 2.3. *Assume (H.2.1). Then there exist solutions $\psi_\pm(z,.)$, $z \in \mathbb{C}$, of $\tau\psi = z\psi$ satisfying*

$$\lim_{n\to\pm\infty} \left| w(z)^{\mp n}(\psi_\pm(z,n) - \psi_{q,\pm}(z,n)) \right| = 0, \tag{2.2}$$

where $\psi_{q,\pm}(z,.)$ are the Baker-Akhiezer functions and $w(z) = \exp(\int_{p_0}^{p} \hat{\omega}_{\infty_+,\infty_-})$ (cf. (1.25)). Moreover, $\psi_\pm(z,.)$ are continuous respectively holomorphic with respect to z whenever $\psi_{q,\pm}(z,.)$ are and they inherit the properties (B1)-(B3) where

$$\psi_\pm(z,n) = \frac{\mathrm{i}^l C_\pm(n)}{\sqrt{z-\mu_j}} + O(1) \tag{2.3}$$

if μ_j coincides with a band edge E_l.

Proof. We only prove the claim for $\psi_+(z,.)$. If ψ satisfies the inhomogeneous Jacobi equation

$$(\tau_q - z)\psi = g \tag{2.4}$$

with $g = (\tau_q - \tau)\psi$, then $\tau\psi = z\psi$. By the general theory (e.g. [48], Chapter 1), the solution of (2.4) can be completely reduced to the solution of the corresponding homogeneous Jacobi equation $(\tau_q - z)\psi_q = 0$. Define

$$K_q(z,n,m) = \frac{s_q(z,n,m)}{a_q(m)},$$

where $s_q(z,.,m)$ is the fundamental solution of $\tau_q s = zs$ with initial conditions $s_q(z,m,m) = 0$, $s_q(z,m+1,m) = 1$. Suppose that $\psi_+(z,.)$ satisfies

$$\psi_+(z,n) = \psi_{q,+}(z,n) - \sum_{m=n+1}^{\infty} K_q(z,n,m)((\tau_q - \tau)\psi_+(z))(m), \tag{2.5}$$

then $\psi_+(z,.)$ fulfills (2.4) and (2.2) as can be computed directly. Green's formula (1.3) implies for $\tau - \tau_q =: \tilde{\tau}$

$$\begin{aligned}\sum_{m=n+1}^{\infty} K_q(z,n,m)(\tilde{\tau}\psi_+(z))(m) &= \tilde{W}_\infty(K_q(z,n),\psi_+(z)) - \tilde{W}_n(K_q(z,n),\psi_+(z)) \\ &\quad + \sum_{m=n+1}^{\infty} (\tilde{\tau}K_q(z,n,.))(m)\psi_+(z,m),\end{aligned}$$

where $\tilde{W}_\infty(K_q,\psi_+) = 0$ and $\tilde{W}_n(K_q,\psi_+) = (a(n) - a_q(n))a_q(n)^{-1}\psi_+(z,n)$. We set

$$\hat{\psi}_+(z,n) := w(z)^{-n}\psi_+(z,n)$$

to get a sequence which is bounded near $+\infty$ and obtain

$$\frac{a(n)}{a_q(n)}\hat{\psi}_+(z,n) = \hat{\psi}_{q,+}(z,n) + \sum_{m=n+1}^{\infty} w(z)^{m-n}\tilde{K}_q(z,n,m)\hat{\psi}_+(z,m), \tag{2.6}$$

where

$$\begin{aligned}\tilde{K}_q(z,n,m) &:= ((\tau-\tau_q)K_q(z,n,.))(m)\\ &= \frac{s_q(z,n,m+1)}{a_q(m+1)}\big(a(m)-a_q(m)\big)+\frac{s_q(z,n,m)}{a_q(m)}\big(b(m)-b_q(m)\big)\\ &\quad +\frac{s_q(z,n,m-1)}{a_q(m-1)}\big(a(m-1)-a_q(m-1)\big). \qquad (2.7)\end{aligned}$$

If we can apply the Volterra sum equation (Lemma 2.2) to (2.6), the proof will be finished. To do so, we need an estimate for $w(z)^{m-n}\tilde{K}_q(z,n,m)$ or, equivalently, for $s_q(z,n,.)$. The Wronskian (1.14) implies that $\psi_{q,+}(z,n,m)$, $\psi_{q,-}(z,n,m)$ are linearly independent for $z\neq E_j$. In this case,

$$s_q(z,n,m)=\alpha_1\psi_{q,+}(z,n,m)+\alpha_2\psi_{q,-}(z,n,m), \qquad z\neq E_j,$$

for some constants $\alpha_{1,2}$. By Lemma 1.4, $|\psi_{q,\pm}(z,n,m)|\le M|w(z)|^{\pm(n-m)}$, thus

$$|w(z)^{m-n}\hat{K}_q(z,n,m)|\le c(m)M'\big(1+|w(z)|^{2(m-n)}\big)=:K(z,n,m),$$

where $c(m)=|a(m)-a_q(m)|+|b(m)-b_q(m)|+|a(m-1)-a_q(m-1)|$ and M' depends on $\inf\{a_p(m)\}$ and $|w(z)|$. By H.2.1, $c(.)\in\ell^1(\mathbb{Z},\mathbb{R})$ and Lemma 1.4 shows that $|w(z)|\le 1$ for $z\in\mathbb{R}$ and $|w(z)|^m=O(|z|^{-m})$ for $p\in\Pi_+$ with $\pi(p)=z$. Hence $K(z,n,.)\in\ell^1(\mathbb{N})$ as desired. Since $w(z)$ is continuous with respect to z, $K(z,n,m)$ can be chosen independent of z as long as z varies in compacts. The claim about continuity (resp. holomorphy) of $\psi_\pm(z,.)$ follows then from Lemma 2.2.

For $z=E$, (1.27) implies

$$s_q(E,n,m)=\frac{2a_q(m)G(E,m)}{\tilde{R}(E)}\psi_q(E,n,m)\big(\tilde{\theta}(n,m)+(n-m)\beta\big)$$

and

$$|w(E)^{m-n}\hat{K}_q(E,n,m)|\le Cc(m)\big|\tilde{\theta}(n,m)+(n-m)\beta\big|\in\ell^1(\mathbb{N},\mathbb{R})$$

finishes the proof. □

Remark 2.4. *Note that $a(n)\neq 0$ for $n\in\mathbb{Z}$, since*

$$a(n)=\frac{W(\psi_+(\lambda),\overline{\psi_+(\lambda)})}{\psi_+(\lambda,n)\overline{\psi_+(\lambda,n+1)}-\psi_+(\lambda,n+1)\overline{\psi_+(\lambda,n)}}.$$

Moreover, $\hat{\mu}_1(n)\neq(E_0,0)$ for $n\in\mathbb{Z}$ by H.1.2, thus (1.18) implies

$$\psi_q(p_0,n)=C(n,0)\frac{\theta(\underline{z}(p_0,n))}{\theta(\underline{z}(p_0,0))}\neq 0 \quad \forall n\in\mathbb{Z}. \qquad (2.8)$$

Next, we want to establish the connection between the spectra of H and H_q. The difference expression τ is called *oscillatory* if one (hence any) solution of $\tau u=0$ has an infinite number of nodes. A point $n\in\mathbb{Z}$ is called a *node* of u if either ([44])

$$u(n)=0 \quad \text{or} \quad a(n)u(n)u(n+1)>0.$$

In the special case $a(n)<0$, $n\in\mathbb{Z}$, a node is precisely a sign flip of u.

Theorem 2.5. *Assume (H.2.1).*

(i). $\sigma_{ess}(H) = \sigma(H_q)$.

(ii). *The point spectrum of H is finite and confined to the spectral gaps of H_q, i.e.* $\sigma_p(H) \subset \mathbb{R}\backslash\sigma(H_q)$.

(iii). *The essential spectrum of H is purely absolutely continuous,* $\sigma_{ess}(H) = \sigma_{ac}(H)$.

Proof. We essentially follow the proof of [48], Theorem 7.11.

(i). The essential spectrum only depends on the asymptotic behavior of the sequences $a(n)$, $b(n)$ ([48], Lemma 3.9), therefore $\sigma_{ess}(H) = \sigma_{ess}(H_q)$.

(ii). We consider the solutions $\psi_\pm(\lambda, .)$ of $\tau\psi = \lambda\psi$ for $\lambda \in \sigma(H_q)$ found in Lemma 2.3. Since $\psi_\pm(\lambda, .)$, $\lambda \in \sigma(H_q)$, are bounded and do not vanish near $\pm\infty$, there are no eigenvalues in the essential spectrum of H. Furthermore,

$$\lim_{n\to\pm\infty} \left| w(z)^{\mp n}(\psi_\pm(E_0, n) - \psi_{q,\pm}(E_0, n))\right| = 0$$

implies that $H - E_0$ is non-oscillatory since by (2.8) and (1.26), $|\psi_{q,\pm}(E_0, n)| \geq \epsilon > 0$, $n \in \mathbb{Z}$. By [48], Corollary 4.11 (Remark 4.12), there are only finitely many eigenvalues below E_0 (above E_{2g+1}) if $H - E_0$ is non-oscillatory. Applying [48], Corollary 4.20, in each spectral gap (E_{2j-1}, E_{2j}), $1 \leq j \leq g$, shows that the number of eigenvalues in the gaps is finite as well.

(iii) follows from (i) since $\sigma(H_q)$ is purely absolutely continuous (cf. (1.15)). □

2.2 The transformation operator

We define the kernel of the transformation operator as the Fourier coefficients of the Jost solutions $\psi_\pm(w, n)$ with respect to the orthonormal system given in Lemma 1.11, $\{\psi_{q,\pm}(w, n)\}_{n\in\mathbb{Z}}$,

$$K_\pm(n, m) := \frac{1}{2\pi \mathrm{i}} \int_{|w|=1} \psi_\pm(w, n)\psi_{q,\mp}(w, m)d\omega(w). \tag{2.9}$$

Since $\psi_\pm(w, .)$ have the same value on the (with respect to the real axis) symmetric points of the slits $[w(\lambda_j), w(E_{2j+1})]$, the functions $\psi_\pm(w, n)\psi_{q,\mp}(w, m)$ have equal values on the slit sides $\mathcal{S}_{k,\pm}$ and

$$\frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{g} \left(\int_{\mathcal{S}_{k,+}} \psi_\pm(w, n)\psi_{q,\mp}(w, m)d\omega(w) + \int_{\mathcal{S}_{k,-}} \psi_\pm(w, n)\psi_{q,\mp}(w, m)d\omega(w) \right) = 0. \tag{2.10}$$

The functions $\psi_\pm(w, n)$, $\psi_{q,\mp}(w, m)$ are holomorphic in W and continuous up to $\partial W\backslash\{w(\mu_j\}$. By the Cauchy theorem, $K_\pm(n, m)$ equals the residue at $w = 0$,

$$K_\pm(n, m) = \mathrm{Res}_0 \frac{1}{w}\psi_\pm(w, n)\psi_{q,\mp}(w, m).$$

In particular, since $\psi_\pm(w, n)\psi_{q,\mp}(w, m) = O(w^{\pm(n-m)})$, we conclude

$$K_\pm(n, m) = 0, \qquad \pm(m - n) < 0. \tag{2.11}$$

Lemma 2.6. ([16]). *Assume H.2.1. The Jost solutions $\psi_\pm(w,n)$ can be represented as*

$$\psi_\pm(w,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_{q,\pm}(w,m), \qquad |w|=1, \tag{2.12}$$

where the kernels $K_\pm(n,.)$ satisfy $K_\pm(n,m)=0$ for $\pm m < \pm n$ and

$$|K_\pm(n,m)| \le C \sum_{j=[\frac{m+n}{2}]\pm 1}^{\pm\infty} \Big(|a(j)-a_q(j)| + |b(j)-b_q(j)|\Big), \quad \pm m > \pm n. \tag{2.13}$$

The constant C depends only on H_q and the value of the sum in (2.1).

Proof. We prove the estimate for $K_+(n,m)$ and omit "+" and "z" whenever possible. Define $\varphi(n) = \psi(n)K(n,n)^{-1}$, then φ fulfills

$$z\varphi(n) = \frac{a(n)^2}{a_q(n)}\varphi(n+1) + b(n)\varphi(n) + a_q(n-1)\varphi(n-1) =: (\hat{H}\varphi)(n).$$

We abbreviate

$$\tilde{a}(m) = \frac{a(m)^2}{a_q(m)} - a_q(m), \qquad \tilde{b}(m) = b(m) - b_q(m)$$

and proceed for $(H_q - z)\varphi = (H_q - \hat{H})\varphi$ as in the proof of Lemma 2.3,

$$\begin{aligned}
\varphi(n) &= \psi_q(n) - \sum_{m=n+1}^{\infty} K_q(n,m)((H_q-\hat{H})\varphi)(m) \\
&= \psi_q(n) + \sum_{m=n+1}^{\infty} \frac{s_q(n,m)}{a_q(m)}\big(\tilde{a}(m)\varphi(m+1) + \tilde{b}(m)\varphi(m)\big) \\
&= \psi_q(n) + \sum_{m=n+1}^{\infty} J(n,m)\varphi(m),
\end{aligned}$$

where

$$J(z,n,m) = \tilde{a}(m-1)\frac{s_q(z,n,m-1)}{a_q(m-1)} + \tilde{b}(m)\frac{s_q(z,n,m)}{a_q(m)}.$$

On the other hand, $\varphi(n)$ is formally given by

$$\varphi(n) = \sum_{m=n}^{\infty} \kappa(n,m)\psi_q(m), \qquad \kappa(n,m) = \frac{K(n,m)}{K(n,n)},$$

therefore

$$\sum_{m=n}^{\infty} \kappa(n,m)\psi_q(m) = \sum_{m=n+1}^{\infty} J(n,m)\psi_q(m) + \sum_{m=n+1}^{\infty}\sum_{l=m+1}^{\infty} J(n,m)\kappa(m,l)\psi_q(l). \tag{2.14}$$

Multiplying both sides of (2.14) by $\psi_{q,-}(k)$ and integrating over the unit circle yields

$$\kappa(n,k) = \sum_{m=n+1}^{\infty} \Gamma(n,m,m,k) + \sum_{m=n+1}^{\infty}\sum_{l=n+1}^{\infty} \Gamma(n,m,l,k)\kappa(m,l), \tag{2.15}$$

where

$$\Gamma(n,m,l,k) = \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} J(w,n,m)\psi_{q,+}(w,l)\psi_{q,-}(w,k)d\omega(w).$$

Using [48], (1.50),

$$\frac{s_q(n,m)}{a(m)} = \frac{\psi_{q,+}(m)\psi_{q,-}(n) - \psi_{q,+}(n)\psi_{q,-}(m)}{W(\psi_{q,+},\psi_{q,-})},$$

we obtain

$$\Gamma(n,m,l,k) = \tilde{b}(m)\Gamma_q(n,m,l,k) + \tilde{a}(m)\Gamma_q(n,m-1,l,k)$$

with

$$\begin{aligned}
\Gamma_q(n,m,l,k) &= \Gamma_0(m,n,l,k) - \Gamma_0(n,m,l,k), \\
\Gamma_0(n,m,l,k) &= \frac{1}{2\pi\mathrm{i}} \int_{w(\gamma)} \frac{\psi_{q,+}(w,n)\psi_{q,-}(w,m)\psi_{q,+}(w,l)\psi_{q,-}(w,k)}{W(\psi_{q,+},\psi_{q,-})} d\omega(w) \\
&= \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{\psi_{q,+}(z,n)\psi_{q,-}(z,m)\psi_{q,+}(z,l)\psi_{q,-}(z,k)}{W(\psi_{q,+},\psi_{q,-})} \frac{\prod(z-\mu_j)}{R^{1/2}_{2g+2}(z)} dz \\
&= \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{\psi_{q,+}(z,n)\psi_{q,-}(z,m)\psi_{q,+}(z,l)\psi_{q,-}(z,k)}{W(\psi_{q,+},\psi_{q,-})^2} dz. \qquad (2.16)
\end{aligned}$$

Here γ is a path on the upper sheet encircling the spectrum. The integrand of Γ_0 is meromorphic on the Riemann surface $\mathbb{M}$ with poles of order one at E_j and poles of order $O(z^{\pm(n-m+l-k)-2})$ near $\infty_\pm$ (there are no poles at the Dirichlet eigenvalues μ_j). We apply the residue theorem twice, first on the side of γ including ∞_+, then on the other side including the spectrum (and thus ∞_-)

$$\begin{aligned}
\Gamma_0(n,m,l,k) &= -\mathrm{Res}_{\infty_+} \frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2} \\
&= \left(\mathrm{Res}_{\infty_-} + \sum_{j=0}^{2g+1} \mathrm{Res}_{E_j}\right) \left(\frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2}\right).
\end{aligned}$$

The order of the poles at $\infty_\pm$ implies

$$\Gamma_0(n,m,l,k) = \begin{cases} \sum_{j=0}^{2g+1} \mathrm{Res}_{E_j} \frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2} & n-m+l-k<0 \\ 0 & n-m+l-k \geq 0, \end{cases}$$

which shows that $\Gamma_0(n,m,l,k)$ is real and bounded since $\psi_{q,+}(E,.) = \psi_{q,-}(E,.)$ are (if $\mu_j = E_l$, use (B3)). Together with (2.16) this yields

$$\Gamma_0(n,m,l,k) = -\overline{\Gamma_0(m,n,k,l)} = -\Gamma_0(m,n,k,l) = -\Gamma_0(n,m,k,l).$$

Moreover,

$$\begin{aligned}
&\Gamma_q(n,m,l,k) = 0, \qquad l-k \geq |m-n|, \\
&\Gamma_q(n,m,l,k) = -\Gamma_q(m,n,k,l) = \Gamma_q(n,m,k,l),
\end{aligned}$$

which then implies

$$\Gamma_q(n,m,l,k) = \begin{cases} \mathrm{sign}(n-m)\sum_{j=0}^{2g+1} \mathrm{Res}_{E_j} \frac{\psi_{q,+}(n)\psi_{q,-}(m)\psi_{q,+}(l)\psi_{q,-}(k)}{W(\psi_{q,+},\psi_{q,-})^2} & |l-k|<|m-n| \\ 0 & |l-k|\geq |m-n| \end{cases} \tag{2.17}$$

and $\Gamma(n,m,l,k)=0$ for $|l-k|\geq m-n$ if $m>n$. Note that the residue at E_j is given by

$$\frac{2\prod_{l=1}^{g}(E_j-\mu_l)^2}{\prod_{l\neq j}(E_j-E_l)}\psi_q(E_j,n)\psi_q(E_j,m)\psi_q(E_j,l)\psi_q(E_j,k). \tag{2.18}$$

Now we obtain for $\kappa(n,k)$

$$\begin{aligned} \kappa(n,k) &= \sum_{m=n+1}^{\infty}\Gamma(n,m,m,k) + \sum_{m=n+1}^{\infty}\sum_{l=m+1}^{\infty}\Gamma(n,m,l,k)\kappa(m,l) \qquad (2.19) \\ &= \sum_{m=[\frac{n+k}{2}]+1}^{\infty}\Gamma(n,m,m,k) + \sum_{m=n+1}^{\infty}\sum_{l=n+k-m+1}^{m+k-n-1}\Gamma(n,m,l,k)\kappa(m,l), \end{aligned}$$

since $\Gamma(n,m,m,k)\neq 0$ only if $|m-k|<m-n$ implying $m>\frac{n+k}{2}$. In the third sum of (2.19) we need that $|m+\delta-k|<m-n$ for $\delta\geq 1$ which yields $\delta<k-n$ and $\delta>n+k-2m$. Two remarks might be in order: $m+k-n-1\geq n+k-m+1$ since $m-n\geq n-m+2$, and the starting point $l=n+k-m+1$ of the third sum actually has a lower limit, namely $m\leq\frac{n+k}{2}$, since we require $l\geq m+1$ for $\kappa(m,l)\neq 0,1$. Note that

$$\begin{aligned} \left|\sum_{m=[\frac{n+k}{2}]+1}^{\infty}\Gamma(n,m,m,k)\right| &\leq D\sum_{m=[\frac{n+k}{2}]+1}^{\infty}|\tilde{b}(m)+\tilde{a}(m)| =: \hat{q}(\tfrac{n+k}{2}), \\ \left|\sum_{l=n+k-m+1}^{m+k-n-1}|\Gamma(n,m,l,k)|\right| &\leq D(m-n-1)|\tilde{b}(m)+\tilde{a}(m)| =: \hat{c}(m)\in\ell^1(\mathbb{Z},\mathbb{R}), \end{aligned}$$

where D is the estimate provided by (2.17), (2.18). We set up the following iteration procedure

$$\begin{aligned} \kappa_0(n,k) &= \sum_{m=[\frac{n+k}{2}]+1}^{\infty}\Gamma(n,m,m,k), \\ \kappa_j(n,k) &= \sum_{m=n+1}^{\infty}\sum_{l=n+k-m+1}^{m+k-n-1}\Gamma(n,m,l,k)\kappa_{j-1}(m,l) \end{aligned}$$

and claim

$$|\kappa_j(n,k)|\leq \hat{q}(\tfrac{n+k}{2})\frac{\left(\sum_{m=n+1}^{\infty}\hat{c}(m)\right)^j}{j!}.$$

This follows from

$$\begin{aligned}
|\kappa_{j+1}(n,k)| &\leq \sum_{m=n+1}^{\infty} \sum_{l=n+k-m+1}^{m+k-n-1} |\Gamma(n,m,l,k)| \hat{q}(\tfrac{m+l}{2}) \frac{\left(\sum_{i=m+1}^{\infty} \hat{c}(i)\right)^j}{j!} \\
&\leq \frac{1}{j!} \sum_{m=n+1}^{\infty} \hat{q}(\tfrac{n+k+1}{2}) \left(\sum_{i=m+1}^{\infty} \hat{c}(i) \right)^j \hat{c}(m) \\
&\leq \frac{\hat{q}(\frac{n+k}{2})}{(j+1)!} \sum_{m=n+1}^{\infty} \left(\left(\sum_{i=m}^{\infty} \hat{c}(i) \right)^{j+1} - \left(\sum_{i=m+1}^{\infty} \hat{c}(i) \right)^{j+1} \right) \\
&= \frac{\hat{q}(\frac{n+k}{2})}{(j+1)!} \left(\sum_{m=n+1}^{\infty} \hat{c}(m) \right)^{j+1},
\end{aligned}$$

where we have used $\hat{q}(n+1) \leq \hat{q}(n)$ and

$$(S+s)^{j+1} - S^{j+1} = S^{j+1}\Big(\Big(1+\frac{s}{S}\Big)^{j+1} - 1\Big) = sS^j \sum_{l=0}^{j} \Big(1+\frac{s}{S}\Big)^l \geq (j+1)sS^j$$

with $s = \hat{c}(m) \geq 0$, $S = \sum_{i=m+1}^{\infty} \hat{c}(i) \geq 0$. Hence the iteration converges and implies the estimate

$$|\kappa(n,k)| = \left| \sum_{j=0}^{\infty} \kappa_j(n,k) \right| \leq \hat{q}(\tfrac{n+k}{2}) \exp\left(\sum_{m=n+1}^{\infty} \hat{c}(m) \right). \tag{2.20}$$

□

Associated with $K_\pm(n,m)$ is the operator

$$(\mathcal{K}_\pm f)(n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m) f(m), \qquad f \in \ell^\infty_\pm(\mathbb{Z}),$$

which acts as a *transformation operator* for the pair τ, τ_q.

Theorem 2.7. *Let τ_q and τ be the quasi-periodic and perturbated Jacobi difference expression, respectively. Then*

$$\tau \mathcal{K}_\pm f = \mathcal{K}_\pm \tau_q f, \qquad f \in \ell^\infty_\pm(\mathbb{Z}).$$

Proof. It suffices to show that $HK_\pm = K_\pm H_q$. Indeed,

$$\begin{aligned}
HK_\pm(n,m) &= \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} H\psi_\pm(w,n)\psi_{q,\mp}(w,m) d\omega(w) \\
&= \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \lambda(w)\psi_\pm(w,n)\psi_{q,\mp}(w,m) d\omega(w) \\
&= \frac{1}{2\pi\mathrm{i}} \int_{|w|=1} \psi_\pm(w,n) H_q \psi_{q,\mp}(w,m) d\omega(w).
\end{aligned} \tag{2.21}$$

□

Corollary 2.8. *For $n \in \mathbb{Z}$ we have*

$$\begin{aligned} \frac{a(n)}{a_q(n)} &= \frac{K_+(n+1,n+1)}{K_+(n,n)} = \frac{K_-(n,n)}{K_-(n+1,n+1)}, \\ b(n) - b_q(n) &= a_q(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)} \\ &= a_q(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}, \end{aligned} \tag{2.22}$$

and

$$\prod_{m=-\infty}^{\infty} \frac{a_q(m)}{a(m)} = K_+(n,n)K_-(n,n). \tag{2.23}$$

Proof. Consider the equation of the transformation operator $H\mathcal{K}_\pm = \mathcal{K}_\pm H_q$, which is equivalent to (cf. (2.21))

$$\begin{aligned} &a(n-1)K_\pm(n-1,m) + b(n)K_\pm(n,m) + a(n)K_\pm(n+1,m) = \\ &a_q(m-1)K_\pm(n,m-1) + b_q(m)K_\pm(n,m) + a_q(m)K_\pm(n,m+1). \end{aligned}$$

Evaluating at $m = n$ we obtain the first equation and at $m = n \mp 1$ the second. Equations (2.22) then imply

$$K_+(n,n) = \prod_{j=n}^{\infty} \frac{a_q(j)}{a(j)}, \qquad K_-(n,n) = \prod_{j=-\infty}^{n-1} \frac{a_q(j)}{a(j)}, \tag{2.24}$$

and

$$\begin{aligned} K_+(n,n+1) &= \frac{K_+(n,n)}{a_q(n)} \sum_{j=n+1}^{\infty} (b_q(j) - b(j)), \\ K_-(n,n-1) &= \frac{K_-(n,n)}{a_q(n-1)} \sum_{j=-\infty}^{n-1} (b_q(j) - b(j)). \end{aligned}$$

□

Remark 2.9. *As absolute convergent sums of $\psi_{q,\pm}(z,.)$ the Jost solutions $\psi_\pm(z,.)$ have the following behavior as $z \to \infty$ (use (1.43))*

$$\psi_\pm(z,n) = \frac{1}{A_\pm(0)} \left(\frac{\prod_{j=0}^{n-1} a(j)}{z^n} \right)^{\pm 1} \left(1 + \Big(B_\pm(0) \pm \sum_{j=1}^{n} b(j - {\scriptstyle 0 \atop 1}) \Big) \frac{1}{z} + O(\frac{1}{z^2}) \right), \tag{2.25}$$

where

$$\begin{aligned} A_+(n) &= \prod_{j=n}^{\infty} \frac{a(j)}{a_q(j)}, \qquad B_+(n) = \sum_{j=n+1}^{\infty} (b_q(j) - b(j)), \\ A_-(n) &= \prod_{j=-\infty}^{n-1} \frac{a(j)}{a_q(j)}, \qquad B_-(n) = \sum_{j=-\infty}^{n-1} (b_q(j) - b(j)). \end{aligned}$$

In terms of w,

$$\psi_\pm(w,n) = (-1)^n \left(\frac{\prod_{j=0}^{n-1} a(j)}{\tilde{a}^n} \right)^{\pm 1} w^{\pm n} \Big(\frac{1}{A_\pm(0)} + O(w) \Big), \quad w \to 0. \tag{2.26}$$

In the notation introduced above,

$$K_\pm(n,n) = \frac{1}{A_\pm(n)}, \qquad K_\pm(n,n\pm 1) = \frac{B_\pm(n)}{A_\pm(n) a_q(n-{0 \atop 1})}.$$

2.3 The scattering matrix

Let H_q be a given quasi-periodic Jacobi operator and H a perturbation of H_q satisfying Hypothesis H.2.1. To set up *scattering theory* for the pair (H, H_q) we proceed in the same manner as for the free Jacobi operator associated with constant sequences $a(n) = 1/2$, $b(n) = 0$ (cf. [48], Chapter 10). In the case of periodic background, the scattering matrix has been set up in [45], see also [15].

The first step is to show linear independence of the Jost solutions $\psi_\pm(\lambda)$, $\overline{\psi_\pm(\lambda)}$ for λ in the interior of $\sigma(H_q)$. The Wronskian of $\psi_\pm(\lambda)$ can be evaluated as $n \to \pm\infty$

$$\begin{aligned} W(\psi_\pm(\lambda), \overline{\psi_\pm(\lambda)}) &= W_q(\psi_{q,\pm}(\lambda), \psi_{q,\mp}(\lambda)) \\ &= a_q(0)(\psi_{q,\pm}(\lambda,0)\psi_{q,\mp}(\lambda,1) - \psi_{q,\pm}(\lambda,1)\psi_{q,\mp}(\lambda,0)) \\ &= \mp \frac{R_{2g+2}^{1/2}(\lambda)}{\prod_{j=1}^g (\lambda-\mu_j)}, \qquad \lambda \in \sigma(H_q). \end{aligned} \tag{2.27}$$

It coincides with the Wronskian of the Floquet solutions in the periodic case

$$\begin{aligned} W(\psi_{p,\pm}(\lambda), \psi_{p,\mp}(\lambda)) &= a_p(0)\big(m_\pm(\lambda) - m_\mp(\lambda)\big)(c_p(\lambda,1)s_p(\lambda,0) - c_p(\lambda,0)s_p(\lambda,1)) \\ &= \mp \frac{2a_p(0)(\Delta(\lambda)^2-1)^{1/2}}{s_p(\lambda,N)} \\ &= \pm \frac{\prod_{j=0}^{2N-1}\sqrt{\lambda-E_j}}{\prod_{j=1}^{N-1}(\lambda-\mu_j)}, \qquad \lambda \in \sigma(H_p). \end{aligned}$$

Hence $\psi_\pm(\lambda)$, $\overline{\psi_\pm(\lambda)}$ are linearly independent for λ in the interior of $\sigma(H_q)$ and we consider the *scattering relations*

$$\psi_\pm(\lambda,n) = \alpha(\lambda)\overline{\psi_\mp(\lambda,n)} + \beta_\mp(\lambda)\psi_\mp(\lambda,n), \qquad \lambda \in \sigma(H_q), \tag{2.28}$$

where

$$\begin{aligned} \alpha(\lambda) &= \frac{W(\psi_\mp(\lambda),\psi_\pm(\lambda))}{W(\psi_\mp(\lambda),\overline{\psi_\mp(\lambda)})} = \frac{\prod_{j=1}^g(\lambda-\mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_-(\lambda),\psi_+(\lambda)), \qquad (2.29) \\ \beta_\pm(\lambda) &= \frac{W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})}{W(\psi_\pm(\lambda),\overline{\psi_\pm(\lambda)})} = \mp\frac{\prod_{j=1}^g(\lambda-\mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)}). \end{aligned}$$

While $\alpha(\lambda)$ is only defined for $\lambda \in \sigma(H_q)$, the right hand side of (2.29) may be used as a definition for $\lambda \in \mathbb{C}\backslash\{E_j\}$. Therefore $\alpha(w)$ can be continued as a holomorphic function on $\mathbb{W}$ and it is continuous up to the boundary except possibly at the band edges.

Remark 2.10. *Note that $\alpha(\lambda)$ does not depend on the normalization of $\psi_\pm(\lambda)$ at the base point $n_0 = 0$ whereas $\beta_\pm = \beta_{\pm,0}$ does. Using*

$$\psi_\pm(z,n,n_0) = \psi_{q,\pm}(z,n_0)^{-1}\psi_\pm(z,n)$$

and

$$W((\psi_+(\lambda),\psi_-(\lambda)) = \prod_{j=1}^{g} \frac{\lambda-\mu_j(n_0)}{\lambda-\mu_j} W((\psi_+(\lambda,.,n_0),\psi_-(\lambda,.,n_0))$$

we see

$$\beta_{\pm,0}(\lambda) = \frac{\psi_{q,\mp}(\lambda,n_0)}{\psi_{q,\pm}(\lambda,n_0)} \beta_{\pm,n_0}(\lambda).$$

A direct calculation shows

$$\alpha(\overline{w}) = \overline{\alpha(w)}, \qquad \beta_\pm(\overline{w}) = \overline{\beta_\pm(w)} = -\beta_\mp(w), \qquad |w|=1, \tag{2.30}$$

and the Plücker identity (cf. [48], (2.169)) implies

$$|\alpha(w)|^2 = 1 + |\beta_\pm(w)|^2, \qquad |w|=1. \tag{2.31}$$

The point spectrum $\sigma_p(H)$ of H is finite and confined to the spectral gaps of H_q by Theorem 2.5. We will denote the eigenvalues of H by

$$\sigma_p(H) = \{\rho_j\}_{j=1}^q.$$

They coincide with the zeros of the Wronskian,

$$\sigma_p(H) = \{z \mid W(\psi_-(z),\psi_+(z)) = 0\} \backslash \{E_j\}_{j=0}^{2g+1}.$$

By (2.29), $\alpha(w)$ has simple zeros at ρ_j and possible singularities at E_j.

Our next aim is to study the behavior of $\alpha(\lambda)$ at the eigenvalues ρ_j, therefore we modify the Jost solutions $\psi_\pm(\lambda,n)$ according to their poles at μ_j and define the following eigenfunctions $\hat\psi_\pm(\lambda,.)$

$$\begin{aligned} \hat\psi_+(\lambda,.) &= \prod_{\mu_l\in M_+} (\lambda-\mu_l)\,\psi_+(\lambda,.), \\ \hat\psi_-(\lambda,.) &= \prod_{\mu_l\in M_-\backslash\{E_j\}} (\lambda-\mu_l)\,\psi_-(\lambda,.). \end{aligned} \tag{2.32}$$

Define $\hat\psi_{q,\pm}(\lambda,.)$ accordingly. Moreover, $\hat\psi_\pm(\rho_j,n) = c_j^\pm \hat\psi_\mp(\rho_j,n)$ with $c_j^+ c_j^- = 1$. The *norming constants* $\gamma_{\pm,j}$ are defined by

$$\frac{1}{\gamma_{\pm,j}} = \sum_{n\in\mathbb{Z}} |\hat\psi_\pm(\rho_j,n)|^2. \tag{2.33}$$

We discuss the derivative of $\alpha(\lambda)$ next.

Lemma 2.11. *For $\lambda\in\mathbb{C}\backslash\sigma_{ess}(H)$,*

$$\begin{aligned} \frac{d}{d\lambda}\alpha(\lambda) &= \frac{-1}{W_q(\psi_{q,-}(\lambda),\psi_{q,+}(\lambda))} \sum_{j\in\mathbb{Z}} \big(\psi_+(\lambda,j)\psi_-(\lambda,j) - \alpha(\lambda)\psi_{q,+}(\lambda,j)\psi_{q,-}(\lambda,j)\big) \\ &= -\alpha(\lambda) \sum_{n\in\mathbb{Z}} \big(G(\lambda,n,n) - G_q(\lambda,n,n)\big), \end{aligned}$$

where $G(\lambda,.,.)$, $G_q(\lambda,.,.)$ are the Green functions of H, H_q.

Proof. We claim that

$$w(\lambda)^{\mp n}\psi_\pm(\lambda,n) = \begin{cases} w(\lambda)^{\mp n}\psi_{q,\pm}(\lambda,n), & n\to\pm\infty, \\ \alpha(\lambda)w(\lambda)^{\mp n}\psi_{q,\pm}(\lambda,n), & n\to\mp\infty, \end{cases} \qquad \lambda\in\mathbb{C}\backslash\sigma_{ess}(H). \tag{2.34}$$

The first equation follows from (2.2). To obtain the asymptotic behavior of $\psi_+(n)$ as $n\to-\infty$ (the case for $n\to\infty$ follows similarly) we recall

$$\alpha(\lambda) = \frac{\prod_{j=1}^g(\lambda-\mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_-(\lambda),\psi_+(\lambda)).$$

By (2.27),

$$\alpha(\lambda)W_q(\psi_{q,-}(\lambda),\psi_{q,+}(\lambda)) = W(\psi_-(\lambda),\psi_+(\lambda)). \tag{2.35}$$

For $n\to-\infty$,

$$\begin{aligned}\lim_{n\to-\infty} W(\psi_-,\psi_+) &= \lim_{n\to-\infty}\Big(W_q(\psi_-,\psi_+) \\ &\qquad - (a_q(n)-a(n))(\psi_-(n)\psi_+(n+1)-\psi_-(n+1)\psi_+(n))\Big) \\ &= \lim_{n\to-\infty} W_q(\psi_-,\psi_+),\end{aligned}$$

since the Wronskians are bounded and independent of n (if $\lambda=\mu_j$, use $\hat\psi_\pm$ instead of $\psi_\pm$). Therefore (2.35) is equal to

$$\lim_{n\to-\infty} W_q(\psi_{q,-}(\lambda),\alpha(\lambda)\psi_{q,+}(\lambda)) - \psi_+(\lambda,n)) = 0.$$

Asymptotically,

$$\psi_{q,-}(\lambda,n)\psi_+(\lambda,n) = C\psi_{q,-}(\lambda,n)\psi_{q,+}(\lambda,n)\big(1+O(1)\big), \qquad n\to-\infty,$$

and

$$\lim_{n\to-\infty} W_q(\psi_{q,-}(\lambda),\alpha(\lambda)\psi_{q,+}(\lambda)) - C\psi_{q,+}(\lambda,n)) = 0$$

shows $C=\alpha(\lambda)$, that is,

$$w(\lambda)^{-n}\psi_+(\lambda,n) = \alpha(\lambda)w(\lambda)^{-n}\psi_{q,+}(\lambda,n)\big(1+O(1)\big), \qquad n\to-\infty.$$

Green's formula implies ([48], eq. (2.29))

$$W_n(\psi_+(\lambda),\psi'_-(\lambda)) - W_{m-1}(\psi_+(\lambda),\psi'_-(\lambda)) = \sum_{j=m}^n \psi_+(\lambda,j)\psi_-(\lambda,j), \tag{2.36}$$

hence the derivative of the Wronskian can be written as

$$\begin{aligned}\frac{d}{d\lambda}W(\psi_-(\lambda),\psi_+(\lambda)) &= W_n(\psi'_-(\lambda),\psi_+(\lambda)) + W_n(\psi_-(\lambda),\psi'_+(\lambda)) \qquad (2.37)\\ = \quad & W_m(\psi'_-(\lambda),\psi_+(\lambda)) + W_n(\psi_-(\lambda),\psi'_+(\lambda)) - \sum_{j=m+1}^n \psi_+(\lambda,j)\psi_-(\lambda,j).\end{aligned}$$

We use (2.34) to evaluate the limits for $m\to-\infty$, $n\to\infty$ as before

$$\begin{aligned}W_m(\psi'_-(\lambda),\psi_+(\lambda)) &= \alpha(\lambda)W_m(\psi'_{q,-}(\lambda),\psi_{q,+}(\lambda)), \qquad m\to-\infty, \\ W_n(\psi_-(\lambda),\psi'_+(\lambda)) &= \alpha(\lambda)W_n(\psi_{q,-}(\lambda),\psi'_{q,+}(\lambda)), \qquad n\to\infty,\end{aligned}$$

and transform W_m again using (2.36)

$$W_m(\psi'_{q,-}(\lambda), \psi_{q,+}(\lambda)) = W_n(\psi'_{q,-}(\lambda), \psi_{q,+}(\lambda)) + \sum_{j=m+1}^{n} \psi_{q,+}(\lambda, j)\psi_{q,-}(\lambda, j).$$

Collecting the terms together we obtain

$$\begin{aligned} W'(\psi_-(\lambda), \psi_+(\lambda)) &= -\sum_{j\in\mathbb{Z}} \Big(\psi_+(\lambda, j)\psi_-(\lambda, j) - \alpha(\lambda)\psi_{q,+}(\lambda, j)\psi_{q,-}(\lambda, j)\Big) \\ &\quad +\alpha(\lambda) W'_q(\psi_{q,-}(\lambda)\psi_{q,+}(\lambda)). \end{aligned}$$

With obvious notation,

$$\begin{aligned} \frac{d}{d\lambda}\alpha(\lambda) &= \frac{d}{d\lambda}\Big(\frac{W}{W_q}\Big) = \Big(\frac{1}{W_q}\Big)' W + \frac{1}{W_q} W' \\ &= -\frac{W'_q}{W_q^2} W + \frac{1}{W_q}\Big(-\sum_{j\in\mathbb{Z}}\Big(\psi_+\psi_- - \alpha\psi_{q,+}\psi_{q,-}\Big) + \alpha W'_q\Big) \\ &= -\frac{1}{W_q}\sum_{j\in\mathbb{Z}}\Big(\psi_+\psi_- - \alpha\psi_{q,+}\psi_{q,-}\Big). \end{aligned}$$

□

Corollary 2.12. *The derivative of $\alpha(\lambda)$ at the eigenvalues ρ_j is given by*

$$\frac{d}{d\lambda}\alpha(\lambda)\Big|_{\rho_j} = \frac{-1}{c_j^{\pm}\gamma_{\pm,j} R_{2g+2}^{1/2}(\rho_j)}. \tag{2.38}$$

Proof. Use (2.27) and $\alpha(\rho_j) = 0$, that is,

$$\frac{d}{d\lambda}\alpha(\lambda)\Big|_{\rho_j} = -\frac{\prod_{l=1}^{g}(\rho_j - \mu_l)}{R_{2g+2}^{1/2}(\rho_j)} \sum_{n\in\mathbb{Z}} \psi_-(\rho_j, n)\psi_+(\rho_j, n). \tag{2.39}$$

Now apply (2.33). □

Remark 2.13. *(i). Note that*

$$\frac{d}{d\lambda} W(\hat\psi_-(\lambda), \hat\psi_+(\lambda))\Big|_{\rho_j} = -\sum_{k\in\mathbb{Z}} \hat\psi_-(\rho_j, k)\hat\psi_+(\rho_j, k) = -\frac{1}{c_j^{\pm}\gamma_{\pm,j}}. \tag{2.40}$$

(ii). Equation (2.38) gives a connection between the left and right norming constants

$$\gamma_{+,j}\gamma_{-,j} = \frac{-1}{(\alpha'(\rho_j))^2 R_{2g+2}(\rho_j)}. \tag{2.41}$$

As a last preparation, we study the behavior of $\alpha(w)$ as $w \to 0$. By (2.26),

$$\begin{aligned} W(\psi_-(w), \psi_+(w)) &= a(0)\big(\psi_-(w,0)\psi_+(w,1) - \psi_-(w,1)\psi_+(w,0)\big) \\ &= \frac{1}{A}\tilde a w^{-1} + O(1) \end{aligned}$$

where we abbreviated $A = A_-(0)A_+(0)$. Moreover,

$$\frac{R_{2g+2}^{1/2}(\lambda(w))}{\prod_{j=1}^{g}(\lambda(w)-\lambda_j)} = \tilde{a}w^{-1} + O(1),$$

therefore $\alpha^{-1}(w)$ is bounded at 0 with

$$\alpha(0) = \frac{1}{A} = \prod_{j=-\infty}^{\infty} \frac{a_q(j)}{a(j)}. \tag{2.42}$$

We now define the *scattering matrix*

$$S(w) = \begin{pmatrix} T(w) & R_-(w) \\ R_+(w) & T(w) \end{pmatrix}, \qquad |w| = 1,$$

where $T(w) = \alpha^{-1}(w)$ and $R_\pm(w) = \alpha^{-1}(w)\beta_\pm(w)$ are called *transmission* and *reflection coefficients*. Equations (2.30) and (2.31) imply

Lemma 2.14. *The scattering matrix $S(w)$ is unitary. The coefficients $T(w)$, $R_\pm(w)$ are bounded for $|w| = 1$, continuous for $|w| = 1$ except at possibly $w_l = w(E_l)$, fulfill*

$$\begin{aligned} |T(w)|^2 + |R_\pm(w)|^2 &= 1, \qquad |w| = 1, && (2.43)\\ T(w)R_+(\overline{w}) + T(\overline{w})R_-(w) &= 0, \qquad |w| = 1, && (2.44) \end{aligned}$$

and $\overline{T(w)} = T(\overline{w})$, $\overline{R_\pm(w)} = R_\pm(\overline{w})$ for $|w| = 1$.

Moreover, $R_{2g+2}^{1/2}(w)T(w)^{-1}$ is continuous (in particular, $T(w)$ can only vanish at w_l) and

$$\begin{array}{ll} \lim\limits_{w\to w_l} R_{2g+2}^{1/2}(w)\frac{R_\pm(w)+1}{T(w)} = 0, & w_l \neq w(\mu_j) \\ \lim\limits_{w\to w_l} R_{2g+2}^{1/2}(w)\frac{R_\pm(w)-1}{T(w)} = 0, & w_l = w(\mu_j) \end{array}. \tag{2.45}$$

The transmission coefficient $T(w)$ has a meromorphic continuation to $\mathbb{W}$ with simple poles at $w(\rho_j)$,

$$\big(\mathrm{Res}_{\rho_j} T(\lambda)\big)^2 = \gamma_{+,j}\gamma_{-,j}R_{2g+2}(\rho_j). \tag{2.46}$$

Moreover, $T(z) \in \mathbb{R}$ as $z \in \mathbb{R}\backslash\sigma(H_q)$ and

$$T(0) = \frac{1}{K_+(n,n)K_-(n,n)} = \prod_{j\in\mathbb{Z}} \frac{a(j)}{a_q(j)},$$

where $K_\pm(n,n)$ are the kernels of the transformation operators.

Proof. To show (2.45) we use definition (2.29),

$$\begin{aligned} R_{2g+2}^{1/2}(\lambda)\frac{R_\pm(\lambda)+1}{T(\lambda)} &= \big(\alpha(\lambda)+\beta_\pm(\lambda)\big)R_{2g+2}^{1/2}(\lambda) \\ &= \prod_{j=1}^{g}(\lambda-\mu_j)\big(W(\psi_-(\lambda),\psi_+(\lambda)) \mp W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)})\big). \end{aligned}$$

There are two cases to distinguish: If $\mu_j \neq E_l$, then $\psi_\pm$ are continuous and real at $\lambda = E_l$ and the two Wronskians cancel. Otherwise, if $\mu_j = E_l$, $\psi_\pm$ are purely imaginary (by property (B3) of the Jost functions) and the two terms are equal in the limit

$$\lim_{\lambda\to\mu_j}\prod_j(\lambda-\mu_j)W(\psi_-(\lambda),\psi_+(\lambda)) = (-1)^l a(n)(C_-(n)C_+(n+1)-C_-(n+1)C_+(n))$$
$$= \mp\lim_{\lambda\to\mu_j}\prod_j(\lambda-\mu_j)W(\psi_\mp(\lambda),\overline{\psi_\pm(\lambda)}).$$

They add up to

$$2\lim_{\lambda\to\mu_j}\prod_{j=1}^{g}(\lambda-\mu_j)W(\psi_-(\lambda),\psi_+(\lambda)) = 2\lim_{\lambda\to\mu_j}\frac{R_{2g+2}^{1/2}(\lambda)}{T(\lambda)}.$$

□

The sets

$$S_\pm(H)=\big\{R_\pm(w),|w|=1;\,(\rho_j,\gamma_{\pm,j}),1\le j\le q\big\} \tag{2.47}$$

are called left/right *scattering data* for H.

We have already seen that we can compute $S_-(H)$ from $S_+(H)$ and vice versa with formulas (2.43), (2.44), and (2.46) if the transmission coefficient $T(w)$ is known. Thus our next aim is to show that the transmission coefficient can be reconstructed from either left or right scattering data.

Let $g(w,w_0)$ be the Green function associated with $\mathbb{W}$ and let

$$\mu(w,w_0)dw_0=\frac{\partial g}{\partial r}(w,r\mathrm{e}^{\mathrm{i}\theta})\Big|_{r=1^-}\mathrm{e}^{\mathrm{i}\theta}d\theta,\qquad w_0=\mathrm{e}^{\mathrm{i}\theta}, \tag{2.48}$$

be the corresponding harmonic measure on the boundary (see, e.g., [52]). Since W_0 is simply connected, we can choose a function $h(w,v)$ such that $\hat g(w,w_0)=g(w,w_0)+\mathrm{i}h(w,w_0)$ is analytic in W_0. Clearly $\hat g$ is only well defined up to an imaginary constant and it will not be analytic on $\mathbb{W}\backslash\{0\}$ in general. Similarly we can find a corresponding $\nu(w,w_0)$ and set $\hat\mu(w,w_0)=\mu(w,w_0)+\mathrm{i}\nu(w,w_0)$.

Lemma 2.15. ([16]). *Either one of the sets $S_\pm(H)$ determines the other and $T(w)$ via the Poisson-Jensen formula*

$$T(w)=\exp\left(\sum_{j=1}^{q}\hat g(w,w(\rho_j))\right)\exp\left(\frac12\int_{|w|=1}\ln(1-|R_\pm(\omega)|^2)\hat\mu(w,\omega)d\omega\right), \tag{2.49}$$

where the constant of $\hat g$ has to be chosen such that $T(0)>0$, and

$$\frac{R_-(w)}{R_+(\overline{w})}=-\frac{T(w)}{T(\overline{w})},\qquad \gamma_{+,j}\gamma_{-,j}=\frac{\big(\mathrm{Res}_{\rho_j}T(\lambda)\big)^2}{\prod_{l=0}^{2g+1}(\rho_j-E_l)}.$$

Proof. It suffices to prove the formula for $T(w)$ since evaluating the residua provides $\gamma_{\pm,j}$ together with $\{\rho_j\}$ and $\{E_l\}$. The formula for $T(w)$ holds by [53], Theorem 1, at least when taking absolute values. Since both sides are analytic and have equal absolute values, they can only differ by a constant of absolute value one. But both sides are positive at $w=0$ and hence this constant is one. □

Note that neither the Blaschke factors nor the outer function in (2.49) are single valued on $\mathbb{W}$ in general. In particular, the eigenvalues cannot be chosen arbitrarily, which was first observed in [32].

2.4 The Gel'fand-Levitan-Marchenko equations

In this section we want to derive a procedure which allows the reconstruction of the Jacobi operator H with asymptotically quasi-periodic coefficients from its scattering data $S_\pm(H)$. This will be achieved by deriving an equation for $K_\pm(n,m)$ which is generally known as Gel'fand-Levitan-Marchenko equation.

Since the kernels $K_\pm(n,m)$ are essentially the Fourier coefficients of the Jost solutions $\psi_\pm(w,n)$ we compute the Fourier coefficients of the scattering relations (2.28). Therefore we multiply

$$T(w)\psi_\mp(w,n) = \overline{\psi_\pm(w,n)} + R_\pm(w)\psi_\pm(w,n) \tag{2.50}$$

by $(2\pi\mathrm{i})^{-1}\psi_{q,\pm}(w,m)d\omega$, where $\pm m \geq \pm n$, and integrate around the unit circle. First we evaluate the right hand side of (2.50) using (2.9)

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{|w|=1}\overline{\psi_+(w,n)}\psi_{q,+}(w,m)d\omega(w) &= K_+(n,m), \qquad (2.51)\\ \frac{1}{2\pi\mathrm{i}}\int_{|w|=1}R_+(w)\psi_+(w,n)\psi_{q,+}(w,m)d\omega(w) &= \sum_{l=n}^{\infty}K_+(n,l)\tilde{F}^+(l,m),\end{aligned}$$

where

$$\tilde{F}^+(l,m) = \frac{1}{2\pi\mathrm{i}}\int_{|w|=1}R_+(w)\psi_{q,+}(w,l)\psi_{q,+}(w,m)d\omega(w). \tag{2.52}$$

Note that $\tilde{F}^+(l,m) = \tilde{F}^+(m,l)$ is real.

To evaluate the left hand side of (2.50) we use the residue theorem. Take a closed contour in W and let this contour approach ∂W. By Lemma 2.14, the function $T(w)\psi_-(w,n)\psi_{q,+}(w,m)$ is continuous on $\{|w|=1\}\backslash\{w(E_j)\}$ and meromorphic on W with simple poles at $w(\rho_j)$ and a pole at $w=0$ if $m=n$ (for $m>n$ it is bounded at 0). The function $T(w)\psi_-(w,n)\psi_{q,+}(w,m)$ has equal values on the slit sides, therefore the integral along the slits contributes nothing except at the poles $w(\rho_j)$ (compare (2.10)). The poles at $w(\mu_j)$ cancel with the zeros of $d\omega$ at this points. In summary, the only poles are at the eigenvalues ρ_j and at $w=0$ if $m=n$, hence

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{|w|=1}T(w)\psi_-(w,n)\psi_{q,+}(w,m)d\omega(w)\\ &\quad= \frac{\delta(n,m)}{K_+(n,n)} + \sum_{j=1}^{q}\mathrm{Res}_{\rho_j}\left(\frac{T(\lambda)\hat{\psi}_-(\lambda,n)\hat{\psi}_{q,+}(\lambda,m)}{R_{2g+2}^{1/2}(\lambda)}\right). \qquad (2.53)\end{aligned}$$

Here $\delta(n,m)$ is one for $m=n$ and zero else. Note that

$$\begin{aligned}\mathrm{Res}_{w=w_0}F(w) &= \lim_{w\to w_0}(w-w_0)F(w) = \lim_{z\to z_0}\frac{w(z)-w(z_0)}{z-z_0}(z-z_0)F(w(z))\\ &= w'(z_0)\mathrm{Res}_{z=z_0}F(w(z)),\end{aligned}$$

if the second limes exists. By (2.38) the residua at the eigenvalues are given by

$$\begin{aligned}
\mathrm{Res}_{\rho_j}\left(\frac{T(\lambda)\hat{\psi}_-(\lambda,n)\hat{\psi}_{q,+}(\lambda,m)}{\prod_{l=0}^{2g+1}\sqrt{\lambda-E_l}}\right) &= \lim_{\lambda\to\rho_j}(\lambda-\rho_j)\frac{\hat{\psi}_-(\lambda,n)\hat{\psi}_{q,+}(\lambda,m)}{\alpha(\lambda)\prod_{l=0}^{2g+1}\sqrt{\lambda-E_l}} \\
&= \frac{\hat{\psi}_-(\rho_j,n)\hat{\psi}_{q,+}(\rho_j,m)}{\alpha'(\rho_j)\prod_{l=0}^{2g+1}\sqrt{\rho_j-E_l}} \\
&= \gamma_{+,j}c_j^+\hat{\psi}_-(\rho_j,n)\hat{\psi}_{q,+}(\rho_j,m) \\
&= \gamma_{+,j}\hat{\psi}_+(\rho_j,n)\hat{\psi}_{q,+}(\rho_j,m). \qquad (2.54)
\end{aligned}$$

Collecting all terms yields

$$K_\pm(n,m)+\sum_{l=n}^{\pm\infty}K_\pm(n,l)\tilde{F}^\pm(l,m)=\frac{\delta(n,m)}{K_\pm(n,n)}-\sum_{j=1}^{q}\gamma_{\pm,j}\hat{\psi}_\pm(\rho_j,n)\hat{\psi}_{q,\pm}(\rho_j,m)$$

and we have thus proved the following result.

Theorem 2.16. *The kernels $K_\pm(n,m)$ of the transformation operator satisfy the* Gel'fand-Levitan-Marchenko equation *(GLM-equation)*

$$K_\pm(n,m)+\sum_{l=n}^{\pm\infty}K_\pm(n,l)F^\pm(l,m)=\frac{\delta(n,m)}{K_\pm(n,n)}, \qquad \pm m\geq\pm n, \tag{2.55}$$

where

$$F^\pm(l,m)=\tilde{F}^\pm(l,m)+\sum_{j=1}^{q}\gamma_{\pm,j}\hat{\psi}_{q,\pm}(\rho_j,l)\hat{\psi}_{q,\pm}(\rho_j,m) \tag{2.56}$$

and

$$\tilde{F}^\pm(l,m)=\frac{1}{2\pi\mathrm{i}}\int_{|w|=1}R_\pm(w)\psi_{q,\pm}(w,l)\psi_{q,\pm}(w,m)d\omega(w). \tag{2.57}$$

Defining the *Gel'fand-Levitan-Marchenko operator*

$$\mathcal{F}_n^\pm f(j)=\sum_{l=0}^{\pm\infty}F^\pm(n+l,n+j)f(l), \qquad f\in\ell^2(\mathbb{N}_0),$$

yields that the Gel'fand-Levitan-Marchenko equation is equal to

$$(1+\mathcal{F}_n^\pm)K_\pm(n,n+.)=(K_\pm(n,n))^{-1}\delta_0. \tag{2.58}$$

Our next aim is to study the Gel'fand-Levitan-Marchenko operator $\mathcal{F}_n^\pm$ in more detail. The structure of the Gel'fand-Levitan-Marchenko equation suggests that the estimate (2.13) for $K_\pm(n,m)$ should imply a similar estimate for $F^\pm(n,m)$.

Lemma 2.17.

$$|F^\pm(n,m)|\leq C\sum_{j=[\frac{n+m}{2}]\pm1}^{\pm\infty}\Big(|a(j)-a_q(j)|+|b(j)-b_q(j)|\Big), \tag{2.59}$$

where the constant C is of the same nature as in (2.13).

Proof. We abbreviate the estimate (2.13) for $K_+(n,m)$ by

$$|K_+(n,m)| \le C C_+(n+m) \tag{2.60}$$

where

$$C_+(n+m) = \sum_{j=[\frac{n+m}{2}]+1}^{\infty} c(j), \qquad c(j) = |a(j) - a_q(j)| + |b(j) - b_q(j)|. \tag{2.61}$$

Note that $C_+(n+1) \le C_+(n)$. Moreover, $C_+ \in \ell^1_+(\mathbb{Z}, \mathbb{R})$ since the summation by parts formula (e.g. [48], (1.18))

$$\sum_{m=n}^{N} g(m)(f(m+1)-f(m)) = g(N)f(N+1)-g(n-1)f(n)+\sum_{m=n}^{N}(g(m-1)-g(m))f(m) \tag{2.62}$$

implies for $g(m) = m$, $f(m) = C_+(m)$ that

$$\sum_{m=n}^{\infty} m\, c(m) = (n-1)C_+(n) + \sum_{m=n}^{\infty} C_+(m), \tag{2.63}$$

where we used

$$\lim_{n\to\infty} n\, C_+(n+1) \le \lim_{n\to\infty} \sum_{m=n}^{\infty} m\, c(m) = 0.$$

Solving the GLM-equation (2.55) for $F^+(n,m)$, $m > n$, we obtain

$$\begin{aligned} |F^+(n,m)| &\le \frac{1}{K_+(n,n)}\Big(|K_+(n,m)| + \sum_{l=n+1}^{\infty} \big|K_+(n,l)F^+(l,m)\big| \Big) \\ &\le C_1(n)\Big(C_+(n+m) + \sum_{l=n+1}^{\infty} C_+(n+l)\, \big|F^+(l,m)\big| \Big), \end{aligned}$$

where $C_1(n) = C\,|K_+(n,n)|^{-1} \to C$ for $n \to \infty$ by (2.24). For n large enough, i.e. $C_1(n)C_+(2n) < 1$, we apply the discrete Gronwall-type inequality ([48], Lemma 10.8),

$$\begin{aligned} |F^+(n,m)| &\le C_1(n)\Big(C_+(n+m) + \sum_{l=n+1}^{\infty} \frac{C_1(l)C_+(l+m)C_+(n+l)}{\prod_{k=n+1}^{l}(1 - C_1(k)C_+(n+k))} \Big) \\ &\le C_1(n)C_+(n+m)\Big(1 + \sum_{l=n+1}^{\infty} \frac{C_1(k)C_+(n+l)}{\prod_{k=n+1}^{l}(1 - C_1(n)C_+(n+k))} \Big). \end{aligned}$$

In summary we have

$$|F^+(n,m)| \le C_+(n+m)(C_1(n) + O(1)). \tag{2.64}$$

□

Corollary 2.18. *The Fourier coefficients $\tilde{F}^{\pm}(n,m)$ of the reflection coefficients $R^{\pm}(w)$ satisfy*

$$|\tilde{F}^{\pm}(l,m)| \le C \sum_{j=[\frac{l+m}{2}]\pm 1}^{\pm\infty} \Big(|a(j) - a_p(j)| + |b(j) - b_p(j)| \Big).$$

Proof. Recalling (2.56) we see that the finite sum consists of the functions $\hat{\psi}_{q,\pm}(\rho_j, n)$ which are bounded at ρ_j. □

Lemma 2.17 implies in turn

Lemma 2.19. *Let $F^{\pm}(n,m)$ be solutions of the Gel'fand-Levitan-Marchenko equation. Then*

$$\sum_{n=n_0}^{\pm\infty} |n| \left|F^{\pm}(n,n) - F^{\pm}(n\pm 1, n\pm 1)\right| \quad < \quad \infty, \tag{2.65}$$

$$\sum_{n=n_0}^{\pm\infty} |n| \left|a_q(n) F^{\pm}(n,n+1) - a_q(n-1) F^{\pm}(n-1,n)\right| \quad < \quad \infty. \tag{2.66}$$

Proof. We prove (2.66) for F^+ (F^- follows then analogously). Corollary 2.8 implies

$$b(n) - b_q(n) = a_q(n)\kappa_{+,1}(n) - a_q(n-1)\kappa_{+,1}(n-1), \tag{2.67}$$

where

$$\kappa_{+,j}(n) := \kappa_+(n,n+j) := \frac{K_+(n,n+j)}{K_+(n,n)}.$$

Abbreviate $F_j^+(n) = F^+(n+j,n)$. With this notation the GLM-equation (2.55) reads

$$\kappa_{+,l}(n) + F_l^+(n) + \sum_{j=1}^{\infty} \kappa_{+,j}(n) F_{j-l}^+(n+l) = \frac{\delta(l,0)}{K_+(n,n)^2}, \qquad l \geq 0.$$

Insert the GLM-equation for $F^+(n,n+1)$, $F^+(n-1,n)$ (recall $F^+(n,m) = F^+(m,n)$)

$$\begin{aligned} a_q(n) F_1^+(n) - a_q(n-1) F_1^+(n-1) \\ = \quad & -a_q(n)\kappa_{+,1}(n) + a_q(n-1)\kappa_{+,1}(n-1) \\ & - \sum_{j=1}^{\infty} \Big(a_q(n)\kappa_{+,j}(n) F_{j-1}^+(n+1) - a_q(n-1)\kappa_{+,j}(n-1) F_{j-1}^+(n) \Big). \end{aligned}$$

Since $-a_q(n)\kappa_{+,1}(n) + a_q(n-1)\kappa_{+,1}(n-1) = b_q(n) - b(n)$ and $\sum |n||b(n) - b_q(n)| < \infty$ by Hypothesis H.2.1, the only interesting part is the sum. For N, $J < \infty$,

$$\begin{aligned} & \sum_{n=n_0}^{N} n \sum_{j=1}^{J} \Big(a_q(n)\kappa_{+,j}(n) F_{j-1}^+(n+1) - a_q(n-1)\kappa_{+,j}(n-1) F_{j-1}^+(n) \Big) \\ & = \quad \sum_{j=1}^{J} \sum_{n=n_0}^{N} n \Big(a_q(n)\kappa_{+,j}(n) F_{j-1}^+(n+1) - a_q(n-1)\kappa_{+,j}(n-1) F_{j-1}^+(n) \Big) \\ & = \quad \sum_{j=1}^{J} \Big(N a_q(N)\kappa_{+,j}(N) F_{j-1}^+(N+1) - (n_0-1) a_q(n_0-1)\kappa_{+,j}(n_0-1) F_{j-1}^+(n_0) \\ & \qquad - \sum_{n=n_0}^{N} a_q(n-1)\kappa_{+,j}(n-1) F_{j-1}^+(n) \Big), \end{aligned} \tag{2.68}$$

where we used the summation by parts formula (2.62) with

$$g(n) = n, \qquad f(n) = a_q(n-1)\kappa_{+,j}(n-1) F_{j-1}^+(n).$$

Estimates (2.60), (2.64) imply for the first summand

$$\begin{aligned}\Big|\sum_{j=1}^{J} N a_q(N)\kappa_{+,j}(N)F^+_{j-1}(N+1)\Big| &\leq \sum_{j=1}^{J}|N|a_q(N)\tilde{C}C_+(2N+j)C_+(2N+j+1)\\ &\leq |N|a_q(N)\hat{C}C_+(2N+1),\end{aligned}$$

which holds uniformly in J. Comparing (2.63) we obtain that

$$\lim_{N\to\infty} N a_q(N)\hat{C}C_+(2N+1)=0.$$

Moreover,

$$\begin{aligned}&\lim_{N,J\to\infty}\Big|\sum_{j=1}^{J}\sum_{n=n_0}^{N} a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big|\\ \leq\ &\lim_{N,J\to\infty}\sum_{j=1}^{J}\sum_{n=n_0}^{N}\Big|a_q(n-1)\kappa_{+,j}(n-1)F^+_{j-1}(n)\Big|\\ \leq\ &\sum_{j=1}^{\infty}\sum_{n=n_0}^{\infty} a_q(n-1)\tilde{C}C_+(2n+j)C_+(2n+j+1)<\infty.\end{aligned}$$

Therefore $|n||a_q(n)F^+(n,n+1)-a_q(n-1)F^+(n-1,n)|\in\ell^1_+(\mathbb{Z},\mathbb{R})$ as desired. To apply Corollary 2.8 for F^- use the symmetry property $F^-(n,m)=F^-(m,n)$. For (2.65), inserting the GLM-equation yields

$$\begin{aligned}&F^+(n,n)-F^+(n+1,n+1)\\ &=K_+^{-2}(n,n)-K_+^{-2}(n+1,n+1)+\sum_{j=1}^{\infty}\Big(\kappa_{+,j}(n+1)F^+_j(n+1)-\kappa_{+,j}(n)F^+_j(n)\Big).\end{aligned}$$

By (2.24),

$$\begin{aligned}\big|K_+^{-2}(n,n)-K_+^{-2}(n+1,n+1)\big| &\leq \frac{|a(n)+a_q(n)|}{a(n)^2}\prod_{j=n+1}^{\infty}\frac{a(j)^2}{a_q(j)^2}|a(n)-a_q(n)|\\ &\leq C|a(n)-a_q(n)|,\end{aligned}$$

and the same considerations as above imply (2.65). □

Corollary 2.20. *The Fourier coefficients $\tilde{F}^\pm(n,m)$ of $R^\pm(w)$ satisfy*

$$\begin{aligned}\sum_{n=n_0}^{\pm\infty}|n|\Big|\tilde{F}^\pm(n,n)-\tilde{F}^\pm(n\pm1,n\pm1)\Big| &< \infty,\\ \sum_{n=n_0}^{\pm\infty}|n|\Big|a_q(n)\tilde{F}^\pm(n,n+1)-a_q(n-1)\tilde{F}^\pm(n-1,n)\Big| &< \infty.\end{aligned}$$

Remark 2.21. *The Gel'fand-Levitan-Marchenko equation is symmetric in $K_\pm(n,m)$ and $F^\pm(n,m)$, therefore we can invert the analysis done in Lemma 2.19 and obtain estimates for $K_\pm(n,m)$ starting with an analogue of estimate (2.59) for $F^\pm(n,m)$ and the estimates (2.65), (2.66) (cf. Lemma 3.1).*

Theorem 2.22. *The Gel'fand-Levitan-Marchenko operator $\mathcal{F}_n^{\pm} : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ is Hilbert-Schmidt for all $n \in \mathbb{Z}$. Moreover, $1 + \mathcal{F}_n^{\pm}$ is positive and hence invertible.*

In particular, the Gel'fand-Levitan-Marchenko equation (2.58) has a unique solution and $S_+(H)$ or $S_-(H)$ uniquely determine H.

Proof. That $\mathcal{F}_n^{\pm}$ is Hilbert-Schmidt is a straight-forward consequence of our estimate in Lemma 2.17. W.l.o.g. we consider $\mathcal{F}_0^+$ and the standart basis $\delta_n(j)$, then

$$\mathcal{F}_0^+ \delta_n(j) = F^+(n, j).$$

Moreover,

$$\begin{aligned} \sum_{n=0}^{\infty} \|\mathcal{F}_0^+ \delta_n(j)\|^2 &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} |F^+(n,j)|^2 \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} C_+(n+j)^2 \\ &\le \sum_{j,n=0}^{\infty} C_+(n) C_+(j) = \Big(\sum_{n=0}^{\infty} C_+(n) \Big)^2 < \infty, \end{aligned}$$

since $C_+(n+1) \le C_+(n)$ and $C_+ \in \ell^1_+(\mathbb{Z}, \mathbb{R})$ (cf. (2.61)).

Let $f \in \ell^2(\mathbb{N}_0, \mathbb{R})$ be real (which is no restriction since $F^+(n,l)$ is real and the real and imaginary part of (2.69) could be treated separately) and abbreviate

$$f_n(w) = \sum_{j=0}^{\infty} f(j) \psi_{q,+}(w, n+j), \qquad \hat{f}_n(w) = \sum_{j=0}^{\infty} f(j) \hat{\psi}_{q,+}(w, n+j).$$

Then

$$\begin{aligned} \sum_{j=0}^{\infty} f(j) \mathcal{F}_n^+ f(j) &= \sum_{j=0}^{\infty} f(j) \sum_{l=0}^{\infty} F^+(n+j, n+l) f(l) \qquad (2.69) \\ &= \frac{1}{2\pi i} \int_{|w|=1} R_+(w) \sum_{j,l=0}^{\infty} f(j) \psi_{q,+}(w, n+j) \psi_{q,+}(w, n+l) f(l) d\omega(w) \\ &\quad + \sum_{k=1}^{q} \sum_{j,l=0}^{\infty} f(j) \gamma_{+,k} \hat{\psi}_{q,+}(\rho_k, n+j) \hat{\psi}_{q,+}(\rho_k, n+l) f(l) \\ &= \frac{1}{2\pi i} \int_{|w|=1} R_+(w) f_n(w) f_n(w) d\omega(w) + \sum_{k=1}^{q} \gamma_{+,k} |\hat{f}_n(\rho_k)|^2 \\ &= \frac{1}{2\pi i} \int_{|w|=1} \tilde{R}_+(w) |f_n(w)|^2 d\omega(w) + \sum_{k=1}^{q} \gamma_{+,k} |\hat{f}_n(\rho_k)|^2, \end{aligned}$$

where

$$\tilde{R}_+(w) = R_+(w) f_n(w) \big(\overline{f_n(w)}\big)^{-1}, \qquad |\tilde{R}_+(w)| = |R_+(w)|.$$

The integral over the imaginary part vanishes since $\overline{\tilde{R}_+(w)} = \tilde{R}_+(\overline{w})$ and we replace the real part by (recall $|\tilde{R}_+(w)|^2 + |T(w)|^2 = 1$)

$$\begin{aligned} \mathrm{Re}(\tilde{R}_+(w)) &= \frac{1}{2}\big(|1 + \tilde{R}_+(w)|^2 - 1 - |\tilde{R}_+(w)|^2\big) \\ &= \frac{1}{2}\big(|1 + \tilde{R}_+(w)|^2 + |T(w)|^2\big) - 1. \qquad (2.70) \end{aligned}$$

Using

$$\sum_{j=0}^{\infty} |f(j)|^2 = \frac{1}{2\pi i} \int_{|w|=1} |f_n(w)|^2 d\omega(w)$$

equation (2.70) yields

$$\begin{aligned} \sum_{j=0}^{\infty} f(j)(1+\mathcal{F}_n^+)f(j) &= \sum_{k=1}^{q} \gamma_{+,k} |\hat{f}_n(\rho_k)|^2 \\ &+ \frac{1}{4\pi i} \int_{|w|=1} \big(|1+\tilde{R}_+(w)|^2 + |T(w)|^2\big) |f_n(w)|^2 d\omega(w), \end{aligned}$$

which establishes $1+\mathcal{F}_n^+ \geq 0$. According to Lemma 2.14, $|T(w)|^2 > 0$ a.e., therefore -1 is not an eigenvalue and $1+\mathcal{F}_n^+ \geq \epsilon_n$ for some $\epsilon_n > 0$.

□

To finish the direct scattering step for the Jacobi operator H with asymptotically quasi-periodic coefficients we summarize the properties of the scattering data $S_\pm(H)$.

Hypothesis H. 2.23. *The scattering data*

$$S_\pm(H) = \big\{ R_\pm(w), |w| = 1; (\rho_j, \gamma_{\pm,j}), 1 \leq j \leq q \big\}$$

satisfy the following conditions.

(i). *The reflection coefficients $R_\pm(w)$ are continuous except possibly at $w_l = w(E_l)$ and fulfill*

$$\overline{R_\pm(w)} = R_\pm(\overline{w}).$$

Moreover, $|R_\pm(w)| < 1$ for $w \neq w_l$ and

$$1 - |R_\pm(w)|^2 \geq C \prod_{l=0}^{2g+1} (w - w_l)^2. \tag{2.71}$$

The Fourier coefficients $\tilde{F}^\pm$ of $R^\pm(w)$ satisfy

$$\begin{aligned} &|\tilde{F}^\pm(n,m)| \leq \sum_{j=n+m}^{\pm\infty} q(j), \qquad q(j) \geq 0, \qquad |j| q(j) \in \ell^1(\mathbb{Z}, \mathbb{R}), \\ &\sum_{n=n_0}^{\pm\infty} |n| \big| \tilde{F}^\pm(n,n) - \tilde{F}^\pm(n\pm 1, n\pm 1) \big| < \infty, \\ &\sum_{n=n_0}^{\pm\infty} |n| \big| a_q(n) \tilde{F}^\pm(n,n+1) - a_q(n-1) \tilde{F}^\pm(n-1,n) \big| < \infty. \end{aligned}$$

(ii). *The values $\rho_j \in \mathbb{R} \backslash \sigma(H_q)$, $1 \leq j \leq q$, are distinct and the norming constants $\gamma_{\pm,j}$, $1 \leq j \leq q$, are positive.*

(iii). *The transmission coefficient $T(w)$ defined via equation (2.49) extends to a single valued function on $\mathbb{W}$ (i.e., it has equal values on the corresponding slits) and satisfies*

$$\begin{aligned} &\lim_{w \to w_l} (w - w_l) \tfrac{R_\pm(w)+1}{T(w)} = 0, \qquad w_l \neq w(\mu_j), \\ &\lim_{w \to w_l} (w - w_l) \tfrac{R_\pm(w)-1}{T(w)} = 0, \qquad w_l = w(\mu_j). \end{aligned} \tag{2.72}$$

(iv). *The consistency conditions*

$$\frac{R_-(w)}{R_+(\overline{w})} = -\frac{T(w)}{T(\overline{w})}, \qquad \gamma_{+,j}\,\gamma_{-,j} = \frac{\big(\mathrm{Res}_{\rho_j} T(\lambda)\big)^2}{\prod_{l=0}^{2g+1}(\rho_j - E_l)}.$$

Chapter 3

Inverse scattering theory for quasi-periodic Jacobi operators

In this chapter we want to invert the process of scattering theory, that is, we want to reconstruct the operator H from given sets $S_\pm$ and a given quasi-periodic Jacobi operator H_q. If $S_\pm(H)$ and H_q are known, we construct $F^\pm(l,m)$ via formula (2.56) and thus derive the Gel'fand-Levitan-Marchenko equation, which has a unique solution by Lemma 2.22. This solution

$$\begin{aligned} K_\pm(n,n) &= \langle \delta_0, (1+\mathcal{F}_n^\pm)^{-1}\delta_0\rangle^{1/2} \\ K_\pm(n,n\pm j) &= \frac{1}{K_\pm(n,n)}\langle \delta_j, (1+\mathcal{F}_n^\pm)^{-1}\delta_0\rangle \end{aligned}$$

is the kernel of the transformation operator. Since $1+\mathcal{F}_n^\pm$ is positive, $K_\pm(n,n)$ is positive and we set in accordance with Corollary 2.8

$$\begin{aligned} a_+(n) &= a_q(n)\frac{K_+(n+1,n+1)}{K_+(n,n)}, \qquad (3.1)\\ a_-(n) &= a_q(n)\frac{K_-(n,n)}{K_-(n+1,n+1)}, \\ b_+(n) &= b_q(n) + a_q(n)\frac{K_+(n,n+1)}{K_+(n,n)} - a_q(n-1)\frac{K_+(n-1,n)}{K_+(n-1,n-1)}, \\ b_-(n) &= b_q(n) + a_q(n-1)\frac{K_-(n,n-1)}{K_-(n,n)} - a_q(n)\frac{K_-(n+1,n)}{K_-(n+1,n+1)}. \end{aligned}$$

Let H_+, H_- be the associated Jacobi operators.

Lemma 3.1. *Suppose a given set S_+ (or S_-) satisfies (H.2.23). Then the sequences defined in (3.1) satisfy*

$$n|a_\pm(n) - a_q(n)|,\ n|b_\pm(n) - b_q(n)| \in \ell^1_\pm(\mathbb{Z},\mathbb{R}).$$

Moreover,

$$\psi_\pm(\lambda,n) = \sum_{m=n}^{\pm\infty} K_\pm(n,m)\psi_{q,\pm}(\lambda,m)$$

satisfies $\tau_\pm \psi_\pm = \lambda \psi_\pm$*, where* $K_\pm(n,m)$ *is the solution of the Gel'fand-Levitan-Marchenko equation.*

Proof. We only prove the statements for the "+" case. Define $F^+(n,m)$ by (cf. (2.56))

$$F^+(l,m) = \tilde{F}^+(l,m) + \sum_{j=1}^{q} \gamma_{+,j} \hat{\psi}_{q,+}(\rho_j, l) \hat{\psi}_{q,+}(\rho_j, m).$$

Hypothesis H.2.23 (i) implies

$$|F^+(n,m)| \le C \sum_{j=n+m}^{\infty} q(j) =: C_+(n+m), \tag{3.2}$$

$$\sum_{n=n_0}^{\infty} |n| \big| F^+(n,n) - F^+(n+1,n+1) \big| < \infty, \tag{3.3}$$

$$\sum_{n=n_0}^{\infty} |n| \big| a_q(n) F^+(n,n+1) - a_q(n-1) F^+(n-1,n) \big| < \infty, \tag{3.4}$$

since $\hat{\psi}_{q,+}(\rho_j, n)$ decay exponentially as $n \to \infty$ and $\sum_j \gamma_{+,j} \hat{\psi}_{q,+}(\rho_j, .) \hat{\psi}_{q,+}(\rho_j, .)$ form a telescopic sum. Note that $C_+(n+1) < C_+(n)$.

Set $\kappa_+(n,m) = K_+(n,m) K_+(n,n)^{-1}$. As in the proof of Lemma 2.17 we apply the Gronwall-type inequality on the GLM-equation for $m > n$ and obtain

$$\begin{aligned} |\kappa_+(n,m)| &\le |F^+(n,m)| + \sum_{l=n+1}^{\infty} |\kappa_+(n,l) F^+(l,m)| \\ &\le C_+(n+m) + \sum_{l=n+1}^{\infty} C_+(l+m) |\kappa_+(n,l)| \\ &\le C_+(n+m) + \sum_{l=n+1}^{\infty} \frac{C_+(l+m) C_+(l+m)}{\prod_{j=n+1}^{l} (1 - C_+(j+m))} \\ &\le C_+(n+m)(1 + O(1)). \end{aligned} \tag{3.5}$$

Now we have all estimates at our disposal to prove $n|b_+(n) - b_q(n)| \in \ell^1_+(\mathbb{Z}, \mathbb{R})$. By definition (cf. (3.1)),

$$b_+(n) - b_q(n) = a_q(n) \kappa_+(n,n+1) - a_q(n-1) \kappa_+(n-1,n).$$

We insert the GLM-equation for $\kappa_+(n,n+1)$, $\kappa_+(n-1,n)$ and use estimate (3.4), the summation by parts formula, and estimates (3.2), (3.5) in the same way as in Lemma 2.19. Similarly using (3.3) we see

$$\sum_{n=n_0}^{\infty} |n| \left| \frac{1}{K_+^2(n,n)} - \frac{1}{K_+^2(n+1,n+1)} \right| < \infty.$$

Equation (3.1) yields

$$\left| \frac{1}{K_+^2(n,n)} - \frac{1}{K_+^2(n+1,n+1)} \right| = \frac{1}{a_q(n)^2} \Big(\prod_{j=n+1}^{\infty} \frac{a_+(j)^2}{a_q(j)^2} \Big) \big| a_+(n)^2 - a_q(n)^2 \big|.$$

The product converges by (3.1) and therefore $|n||a_+(n)^2 - a_q(n)^2| \in \ell^1_+(\mathbb{Z},\mathbb{R})$.

Next we consider $\psi_+(\lambda, n)$. Abbreviate

$$\begin{aligned}(\Delta K_+)(n,m) &= a_q(n-1)\kappa_+(n-1,m) + a_+^2(n)a_q^{-1}(n)\kappa_+(n+1,m) \\ &-a_q(m-1)\kappa_+(n,m-1) - a_q(m)\kappa_+(n,m+1) + (b_+(n) - b_q(m))\kappa_+(n,m).\end{aligned} \tag{3.6}$$

$\Delta K_+ = 0$ is equivalent to the operator equality $H_+K_+ = K_+H_q$, which in turn implies that $\psi_+(\lambda,n)$ satisfies $H_+\psi_+ = \lambda\psi_+$

$$H_+\psi_+ = H_+K_+\psi_{q,+} = K_+H_q\psi_{q,+} = K_+\lambda\psi_{q,+} = \lambda K_+\psi_{q,+} = \lambda\psi_+.$$

To show that $\Delta K_+ = 0$ we insert the GLM-equation into (3.6) and obtain

$$(\Delta K_+)(n,m) + \sum_{l=n+1}^{\infty} (\Delta K_+)(n,l)F^+(l,m) = 0, \qquad m > n+1. \tag{3.7}$$

In the calculations we used

$$\begin{aligned}&a_q(n-1)F^+(n-1,m) + b_q(n)F^+(n,m) + a_q(n)F^+(n+1,m) = \\ &a_q(m-1)F^+(n,m-1) + b_q(m)F^+(n,m) + a_q(m)F^+(n,m+1)\end{aligned}$$

which follows from (2.56). By Theorem 2.22 equation (3.7) has only the trivial solution $\Delta K_+ = 0$ and hence the proof is complete. □

Now we can prove the main result of this chapter.

Theorem 3.2. *Hypothesis H.2.23 is necessary and sufficient for a set S_+ (or S_-) to be the left/right scattering data of a unique Jacobi operator H associated with sequences a, b satisfying H.2.1.*

Proof. Necessity has been established in the previous chapter. By Lemma 3.1, we know existence of sequences $a_\pm$, $b_\pm$ and corresponding solutions $\psi_\pm(w,n)$ associated with S_+ (or S_-). Hence it remains to establish $a_+(n) = a_-(n)$ and $b_+(n) = b_-(n)$.

Consider the following part of the GLM-equation

$$\Phi_+(n,.) := \sum_{l=n}^{\infty} K_+(n,l)\tilde{F}^+(l,.) \in \ell^1_+(\mathbb{Z},\mathbb{R}).$$

Then by use of (2.51) and Lemma 1.11,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \Phi_+(n,m)\psi_{q,-}(w,m) &= \sum_{m\in\mathbb{Z}} \Big(\sum_{l=n}^{\infty} K_+(n,l)\tilde{F}^+(l,m)\Big)\psi_{q,-}(w,m) \\ &= \sum_{m\in\mathbb{Z}} \Big(\frac{1}{2\pi \mathrm{i}}\int_{|w|=1} R_+(w)\psi_+(w,n)\psi_{q,+}(w,m)d\omega(w)\Big)\psi_{q,-}(w,m) \\ &= \sum_{m\in\mathbb{Z}} \langle \psi_{q,-}(w,m), R_+(w)\psi_+(w,n)\rangle \psi_{q,-}(w,m) \\ &= R_+(w)\psi_+(w,n).\end{aligned}$$

On the other hand, inserting the GLM-equation yields for $|w|=1$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \Phi_+(n,m)\psi_{q,-}(w,m) = &\\
= \sum_{m=-\infty}^{n-1} \Phi_+(n,m)\psi_{q,-}(w,m) &+ \sum_{m=n}^{\infty} \Big[\delta(n,m)K_+^{-1}(n,n) - K_+(n,m)\\
&- \sum_{l=n}^{\infty} K_+(n,l) \sum_{j=1}^{q} \gamma_{+,j}\hat{\psi}_{q,+}(\rho_j,l)\hat{\psi}_{q,+}(\rho_j,m)\Big]\psi_{q,-}(w,m)\\
= \sum_{m=-\infty}^{n-1} \Phi_+(n,m)\psi_{q,-}(w,m) &+ \psi_{q,-}(w,n)K_+^{-1}(n,n) - \overline{\psi_+(w,n)}\\
&- \sum_{j=1}^{q} \gamma_{+,j}\hat{\psi}_+(\rho_j,n) \sum_{m=n}^{\infty} \hat{\psi}_{q,+}(\rho_j,m)\psi_{q,-}(w,m),
\end{aligned}$$

(recall the definition of $\hat{\psi}_{q,\pm}$ from (2.32)) and therefore

$$T(w)h_-(w,n) = \overline{\psi_+(w,n)} + R_+(w)\psi_+(w,n), \qquad |w|=1, \tag{3.8}$$

where

$$\begin{aligned}
h_-(w,n) = \frac{\psi_{q,-}(w,n)}{T(w)}\Bigg(&\frac{1}{K_+(n,n)} + \sum_{m=-\infty}^{n-1} \Phi_+(n,m)\frac{\psi_{q,-}(w,m)}{\psi_{q,-}(w,n)}\\
&+ \sum_{j=1}^{q} \gamma_{+,j}\hat{\psi}_+(\rho_j,n)\frac{W_{n-1}(\hat{\psi}_{q,+}(\rho_j),\psi_{q,-}(w))}{\psi_{q,-}(w,n)(\lambda(w)-\rho_j)}\Bigg),
\end{aligned} \tag{3.9}$$

since Green's formula (1.3) implies for $\lambda \in \sigma(H_q)$

$$(\lambda-\rho_j)\sum_{m=n}^{\infty} \hat{\psi}_{q,+}(\rho_j,m)\psi_{q,-}(\lambda,m) = -W_{n-1}(\hat{\psi}_{q,+}(\rho_j),\psi_{q,-}(\lambda)).$$

Similarly, we obtain

$$\begin{aligned}
h_+(w,n) = \frac{\psi_{q,+}(w,n)}{T(w)}\Bigg(&\frac{1}{K_-(n,n)} + \sum_{m=n+1}^{\infty} \Phi_-(n,m)\frac{\psi_{q,+}(w,m)}{\psi_{q,+}(w,n)}\\
&- \sum_{j=1}^{q} \gamma_{-,j}\hat{\psi}_-(\rho_j,n)\frac{W_{n}(\hat{\psi}_{q,-}(\rho_j),\psi_{q,+}(w))}{\psi_{q,+}(w,n)(\lambda(w)-\rho_j)}\Bigg)
\end{aligned} \tag{3.10}$$

with

$$\Phi_-(n,m) = \sum_{l=-\infty}^{n} K_-(n,l)\tilde{F}^-(l,m)$$

and

$$(\lambda-\rho_j)\sum_{m=-\infty}^{n} \hat{\psi}_{q,-}(\rho_j,m)\psi_{q,+}(\lambda,m) = W_n(\hat{\psi}_{q,-}(\rho_j),\psi_{q,+}(\lambda)), \qquad \lambda\in\sigma(H_q).$$

For $n \in \mathbb{Z}$, $|w| = 1$, we see that $h_\mp(w^{-1}, n) = \overline{h_\mp(w, n)}$, since $K_\pm(n, m)$ and $\Phi_\pm(n, m)$ are real. The functions $h_\mp(w, n)$ are continuous for $|w| = 1$, $w \neq w(E_j)$, since $T^{-1}(w)$ is continuous on this set by the Poisson-Jensen formula (2.49) ($|R_\pm(w)| < 1$ for $w \neq w(E_j)$ by H.2.23 (i)) and $\psi_{q,\mp}(w, m)$ are continuous on $\partial W \backslash \{w(\mu_j)\}$. The functions $h_\mp(w, n)$ have a meromorphic continuation to $\mathbb{W}\backslash\{0\}$ with the only possible poles at $w(\rho_j)$ and $w(\mu_j)$. At $w(\rho_j)$ there are no poles, due to the zeros of $T^{-1}(w)$ at $w(\rho_j)$. For $w = w(\mu_j)$, the functions $h_\pm(w, .)$ have the same type of singularity as $\psi_{q,\pm}(w, .)$. In summary, $h_\pm(w, n)$ have simple poles at $w(\mu_j)$ and are continuous at the boundary except possibly at $w(E_j)$.

To study the behavior of $h_\pm(w, n)$ as $w \to 0$ we recall $z^{-1} = -w/\tilde{a}\,(1 + O(w))$. Lemma 1.15 implies for $z \to \infty$

$$\frac{W_{n-1}(\hat{\psi}_{q,+}(\rho_j), \psi_{q,-}(z))}{z - \rho_j} = \frac{z^{n-1}}{\prod_{j=0}^{n-2} a_q(j)} \Big(\hat{\psi}_{q,+}(\rho_j, n-1) + O(\frac{1}{z})\Big) = O(z^{n-1}),$$

$$-\frac{W_n(\hat{\psi}_{q,-}(\rho_j), \psi_{q,+}(z))}{z - \rho_j} = \frac{\prod_{j=0}^{n-1} a_q(j)}{z^{n+1}} \Big(\hat{\psi}_{q,-}(\rho_j, n+1) + O(\frac{1}{z})\Big) = O\Big(\frac{1}{z^{n+1}}\Big),$$

and

$$\sum_{m=n\mp 1}^{\mp\infty} \Phi_\pm(n, m)\psi_{q,\mp}(z, m)\psi_{q,\mp}^{-1}(z, n) = O\Big(\frac{1}{z}\Big), \qquad z \to \infty.$$

We conclude that

$$\lim_{w\to 0} h_\mp(w, n)\psi_{q,\pm}(w, n) = \frac{1}{T(0)K_\pm(n, n)}.$$

By (2.27),

$$\begin{aligned} \lim_{\lambda\to\rho_j} W(\hat{\psi}_{q,+}(\rho_j), \hat{\psi}_{q,-}(\lambda)) &= \lim_{\lambda\to\rho_j} W(\psi_{q,+}(\lambda), \psi_{q,-}(\lambda)) \prod_{l=1}^{g} (\lambda - \mu_l) \\ &= \prod_{l=0}^{2g+1} \sqrt{\rho_j - E_l}. \end{aligned} \tag{3.11}$$

Hypothesis H.2.23 (iv) implies the following behavior of $\hat{h}_\mp(\lambda, n)$ as $\lambda \to \rho_j$

$$\begin{aligned} \lim_{\lambda\to\rho_j} \hat{h}_\mp(\lambda, n) &= \pm\gamma_{\pm,j}\hat{\psi}_\pm(\rho_j, n) \lim_{\lambda\to\rho_j} \frac{W_{n-1}(\hat{\psi}_{q,\pm}(\rho_j), \hat{\psi}_{q,\mp}(\lambda))}{(\lambda - \rho_j)T(\lambda)} \\ &= \gamma_{\pm,j}\hat{\psi}_\pm(\rho_j, n)\big(\mathrm{Res}_{\rho_j} T(\lambda)\big)^{-1} \prod_{l=0}^{2g+1} \sqrt{\rho_j - E_l} \\ &= \sqrt{\frac{\gamma_{\pm,j}}{\gamma_{\mp,j}}}\hat{\psi}_\pm(\rho_j, n), \end{aligned} \tag{3.12}$$

where $\hat{h}_\pm(\lambda, n)$ are defined as in (2.32).

By virtue of the consistency condition $T(w)\overline{R_+(w)} = -\overline{T(w)}R_-(w)$ we obtain

$$\begin{aligned}
&\overline{h_\pm(w,n)} + R_\pm(w)h_\pm(w,n) = \\
&= \frac{1}{\overline{T(w)}}\Big(\psi_\mp(w,n) + \overline{R_\mp(w)\psi_\mp(w,n)}\Big) + \frac{R_\pm(w)}{T(w)}\Big(\overline{\psi_\mp(w,n)} + R_\mp(w)\psi_\mp(w,n)\Big) \\
&= \psi_\mp(w,n)\left(\frac{1}{\overline{T(w)}} + \frac{R_\pm(w)R_\mp(w)}{T(w)}\right) + \overline{\psi_\mp(w,n)}\left(\frac{\overline{R_\mp(w)}}{\overline{T(w)}} + \frac{R_\pm(w)}{T(w)}\right) \\
&= \psi_\mp(w,n)T(w), \qquad |w| = 1.
\end{aligned} \tag{3.13}$$

If we eliminate $R_\mp(w)$ from the last equation and (3.8) we see

$$\begin{aligned}
G(w,n) &:= T(w)R_{2g+2}^{-1/2}(w)\big(\hat\psi_+(w,n)\hat\psi_-(w,n) - \hat h_+(w,n)\hat h_-(w,n)\big) \\
&= \frac{\prod_j(\lambda(w)-\mu_j)}{R_{2g+2}^{1/2}(w)}\big(\overline{h_\pm(w,n)}\psi_\pm(w,n) - \overline{\psi_\pm(w,n)}h_\pm(w,n)\big),
\end{aligned} \tag{3.14}$$

for $|w| = 1$. Observe that $G(\overline{w},n) = \overline{G(w,n)} = G(w,n)$, $|w| = 1$, since $\overline{h_\pm}\psi_\pm - \overline{\psi_\pm}h_\pm$ and $R_{2g+2}^{-1/2}(w)$ are odd functions for $|w| = 1$. The function $G(w,n)$ can be continued analytically on $\mathbb{W}$ since the difference $\hat\psi_+\hat\psi_- - \hat h_+\hat h_-$ vanishes at the poles $w(\rho_j)$ of $T(w)$ by (3.12). Note that the product $\hat\psi_+\hat\psi_-$ and hence also $\hat h_+\hat h_-$ do not have poles at $w(\mu_j)$. At $w = 0$, $G(w,n) = O(w)$ as $\psi_+\psi_-$, h_+h_-, and $T(w)$ have no poles at 0. Moreover, since $\mathbb{W}$ is just the image of the upper sheet, we can extend it to a compact Riemann surface $\tilde{\mathbb{W}}$ by adding the image of the lower sheet. By $G(\overline{w},n) = G(w,n)$ we can extend G to $\tilde{\mathbb{W}}$ by setting $G(w,n) = G(w^{-1},n)$ for $|w| > 1$.

Now let us investigate the behavior at the band edges: If $w_l \neq w(\mu_j)$, we obtain by (2.72), (3.8), and real-valuedness of $\psi_\pm$ at the band edges that

$$\begin{aligned}
&\lim_{w\to w_l} R_{2g+2}^{1/2}(w)\prod_j(\lambda(w)-\mu_j)h_\mp(w,n)\overline{\psi_\mp(w,n)} \\
&\quad= \lim_{w\to w_l}\frac{R_{2g+2}^{1/2}\prod_j(\lambda-\mu_j)}{T}\left(\overline{\psi_\pm} + R_\pm\psi_\pm\right)\overline{\psi_\mp} \\
&\quad= \lim_{w\to w_l}\frac{R_{2g+2}^{1/2}\prod_j(\lambda-\mu_j)}{T}\left((R_\pm+1)\psi_\pm + \overline{\psi_\pm} - \psi_\pm\right)\overline{\psi_\mp} \;=\; 0.
\end{aligned}$$

If $w_l = w(\mu_j)$, we use $\psi_\pm(w,n) = \mathrm{i}^l C_\pm(n)(\lambda(w)-\mu_j)^{-1/2} + O(1)$ and the same calculation shows that

$$\begin{aligned}
&\lim_{w\to w_l} R_{2g+2}^{1/2}(w)\prod_j(\lambda(w)-\mu_j)h_\pm(w,n)\overline{\psi_\pm(w,n)} \\
&= \lim_{w\to w_l}\frac{R_{2g+2}^{1/2}}{T}\Big[(R_\pm+1)(-1)^{l+1}C_+(n)C_-(n) + 2(-1)^l C_+(n)C_-(n) + O(\sqrt{\lambda-\mu_j})\Big] \\
&= (-1)^{l+1}C_+(n)C_-(n)\lim_{w\to w_l}R_{2g+2}^{1/2}(w)\frac{R_\pm(w)-1}{T(w)} = 0
\end{aligned}$$

by (2.72). Consequently, $R_{2g+2}(w)G(w,n)$ is continuous at $w = w_l$ and vanishes at the band edges. Thus the singularities of $R_{2g+2}^{1/2}(w)G(w,n)$ at w_l are removable. Furthermore, $R_{2g+2}^{1/2}(w)G(w,n)$ is purely imaginary for $|w| = 1$ and real on the slits and hence must vanish at w_l by continuity. So the singularities of $G(w,n)$ at w_l are removable as well. Thus G is holomorphic on all of $\tilde{\mathbb{W}}$ and vanishes at $w=0$, that is, $G(w,n)\equiv 0$ which implies (compare Remark 2.9)

$$\begin{aligned}&\lim_{w\to 0}\big(\psi_+(w,n)\psi_-(w,n)-h_+(w,n)h_-(w,n)\big)\\&=K_+(n,n)K_-(n,n)-(T(0)^2K_+(n,n)K_-(n,n))^{-1}=0.\end{aligned}$$

Using (3.1) we finally obtain from $T(0)^2=\big(K_+(n,n)K_-(n,n)\big)^{-2}$ that

$$a_+(n)=a_-(n)\equiv a(n),\qquad \forall n\in\mathbb{Z}.$$

It remains to prove $b_+(n)=b_-(n)$. We consider now equations (3.8) and (3.13) at n and $n+1$. Proceeding as for $G(w,n)$ we show that

$$\begin{aligned}&T(w)R_{2g+2}^{-1/2}(w)\big(\hat{\psi}_+(w,n)\hat{\psi}_-(w,n+1)-\hat{h}_+(w,n+1)\hat{h}_-(w,n)\big)=\\&\frac{\prod_j(\lambda(w)-\mu_j)}{R_{2g+2}^{1/2}(w)}\big(\overline{h_+(w,n+1)}\psi_+(w,n)-\overline{\psi_+(w,n)}h_+(w,n+1)\big)\end{aligned}\tag{3.15}$$

is a constant equal to $-1/a(n)$, the only difference to the proof for $G(w,n)=0$ being that the term $R_{2g+2}^{-1/2}(w)\hat{\psi}_+(w,n)\hat{\psi}_-(w,n+1)$ does not vanish at $w=0$. By (2.26), we have in fact for $\lambda\to\infty$

$$a(n)\psi_+(\lambda,n)\psi_-(\lambda,n+1)=\frac{1}{A}\lambda+\frac{1}{A}\Big(B_+(0)+\sum_{j=1}^{n}b_+(j)+B_-(0)-\sum_{j=0}^{n}b_-(j)\Big)+O(\frac{1}{\lambda}),\tag{3.16}$$

where $A=A_+(0)A_-(0)$. Equation (3.15) yields

$$\begin{aligned}W(w,n)&:=a(n)\big(\psi_+(w,n)\psi_-(w,n+1)-h_+(w,n+1)h_-(w,n)\big)\\&=-\frac{R_{2g+2}^{1/2}(w)}{T(w)\prod_j(\lambda(w)-\mu_j)}\end{aligned}$$

and computing the asymptotics at $w=0$ (compare (3.16)) we see

$$0=W(w,n)-W(w,n-1)=\frac{1}{A}(b_+(n)-b_-(n)).$$

In particular, $b_+(n)=b_-(n)\equiv b(n)$.

Our operator H has the correct norming constants since (2.39) shows

$$\sum_{n\in\mathbb{Z}}\hat{\psi}_+(\rho_j,n)\hat{\psi}_-(\rho_j,n)=\big(\mathrm{Res}_{\rho_j}T(\lambda)\big)^{-1}\prod_{l=0}^{2g+1}\sqrt{\rho_j-E_l}\tag{3.17}$$

and by (3.12),

$$\sum_{n\in\mathbb{Z}}\hat{\psi}_\pm(\rho_j,n)\hat{\psi}_\pm(\rho_j,n)=\gamma_{\pm,j}^{-1}.$$

□

Chapter 4

A simple example

We want to illustrate our results and in particular the statements of Lemma 2.14 by explicitly computing the relevant functions of scattering theory for a simple example.

For the background operator we take the periodic Jacobi operator with period $N = 2$ associated with $a_q(n) = 1$ and $b_q(2n) = -1$, $b_q(2n+1) = 1$, $n \in \mathbb{Z}$. Let the fundamental solutions c, s of the Jacobi equation satisfy the initial conditions $c(z,1) = s(z,0) = 1$, $c(z,0) = s(z,1) = 1$, then $c(z,2) = -1$, $s(z,3) = z^2 - 2$, and the Hill discriminant (A.4) is equal to

$$\Delta(z) = \frac{1}{2}\Big(c(z,2) + s(z,3)\Big) = \frac{z^2-3}{2}.$$

Moreover,

$$\Delta(z)^2 - 1 = \frac{1}{4}(z^2-5)(z^2-1) = \frac{1}{4}R_4(z)$$

yields $R_4^{1/2}(z) = -\sqrt{z+\sqrt{5}}\sqrt{z+1}\sqrt{z-1}\sqrt{z-\sqrt{5}}$. The spectrum of H_q is therefore given by

$$\sigma(H_q) = [-\sqrt{5}, -1] \cup [1, \sqrt{5}].$$

Since $s(z,2) = z-1$, we obtain for the Dirichlet eigenvalue $\mu_1 = 1 = E_2$ (cf. (A.3)). The Floquet multipliers

$$m^{\pm}(z) = \Delta(z) \pm \frac{R_4^{1/2}(z)}{2}$$

are used to calculate the Weyl m-functions

$$\tilde{m}_{\pm}(z) = \frac{m^{\pm}(z) - c(z,2)}{s(z,2)} = \frac{z^2 - 1 \pm R_4^{1/2}(z)}{2(z-1)},$$

which are unbounded at $z = 1 = E_2$. The Floquet solutions are of the form

$$\psi_{q,\pm}(z,2n) = (m^{\pm}(z))^n, \quad \psi_{q,\pm}(z,2n+1) = \tilde{m}_{\pm}(z)(m^{\pm}(z))^n, \quad n \in \mathbb{Z}.$$

Since $\psi_{q,+}(z,2n)\psi_{q,-}(z,2n) = 1$ and $\psi_{q,+}(z,2n+1)\psi_{q,-}(z,2n+1) = (z+1)/(z-1)$, the Dirichlet eigenvalues are given by $\mu_1(2n) = 1$ and $\mu_1(2n+1) = -1$ for all $n \in \mathbb{Z}$.

Now we perturb our operator at $n = -1$, namely we set $a(n) = 1$, $b(n) = b_q(n)$ except for $b(-1) = 1 + b$. The Jost solutions $\psi_{\pm}$ of $Hu = zu$ satisfy $\psi_+(n) = \psi_{q,+}(n)$

for $n \geq 0$, $\psi_-(n) = \psi_{q,-}(n)$ for $n \leq -1$, and

$$\psi_+(z,-1) = \tilde{m}_-(z) = \frac{z^2 - 1 - R_4^{1/2}(z)}{2(z-1)},$$

$$\psi_-(z,0) = 1 - b\ m^+(z)\psi_{q,-}(z,1) = 1 - \frac{(1+z)b}{2} - \frac{R_4^{1/2}(z)b}{2(z-1)}.$$

For example, the limits at the band edge $E_2 = \mu_1 = 1$ are given by

$$\lim_{z\to 1+} \sqrt{z-1}\psi_+(z,-1) = \mathrm{i}\sqrt{2}, \qquad \lim_{z\to 1+} \sqrt{z-1}\psi_-(z,0) = \mathrm{i}\sqrt{2}b.$$

We compute the Wronskians

$$\begin{aligned}
W(\psi_\pm(z), \overline{\psi_\pm(z)}) &= \mp \frac{R_4^{1/2}(z)}{z-1}, \\
W(\psi_-(z), \psi_+(z)) &= \frac{(z+1)b + R_4^{1/2}(z)}{z-1}, \\
W(\psi_-(z), \overline{\psi_+(z)}) &= \frac{2(z+1)b}{(z-1)(z^2 - 3 - R_4^{1/2}(z))}, \\
W(\psi_+(z), \overline{\psi_-(z)}) &= -\frac{(z+1)(z^2-3-R_4^{1/2}(z))b}{2(z-1)},
\end{aligned}$$

and the coefficients $\beta_\pm$

$$\beta_\pm(z) = -\frac{(z+1)b}{R_4^{1/2}(z)} \left(\frac{2}{z^2 - 3 - R_4^{1/2}(z)} \right)^{\pm 1}.$$

The transmission coefficient

$$T(z) = \frac{R_4^{1/2}(z)}{(z+1)b + R_4^{1/2}(z)}$$

satisfies $\lim_{z\to\infty} T(z) = 1$. The reflection coefficients involve more terms

$$\begin{aligned}
R_+(z) &= -\frac{2(z+1)b}{\big(z^2 - 3 - R_4^{1/2}(z)\big)\big((z+1)b + R_4^{1/2}(z)\big)}, \\
R_-(z) &= -\frac{(z+1)\big(z^2 - 3 - R_4^{1/2}(z)\big)b}{2(z+1)b + 2R_4^{1/2}(z)},
\end{aligned}$$

but we obtain the desired result of Lemma 2.14,

$$\begin{aligned}
&\lim_{z\to c} R_4^{1/2}(z) \frac{R_\pm(z)+1}{T(z)} = 0, \qquad c = \pm\sqrt{5}, -1, \\
&\lim_{z\to 1} R_4^{1/2}(z) \frac{R_\pm(z)-1}{T(z)} = 0. \qquad (4.1)
\end{aligned}$$

To illustrate the proof of this result in Lemma 2.14, where we considered

$$R_{2g+2}^{1/2}(z) \frac{R_\pm(z)+1}{T(z)} = \prod_{j=1}^{g} (z-\mu_j) \big(W(\psi_-(z), \psi_+(z)) \mp W(\psi_\mp(z), \overline{\psi_\pm(z)}) \big),$$

we compute the limits for $z \to \mu_1 = E_2 = 1$

$$\lim_{z\to 1+} (z-1) W(\psi_-(z), \psi_+(z)) = 2b,$$
$$\lim_{z\to 1+} (z-1) W(\psi_\mp(z), \overline{\psi_\pm(z)}) = \mp 2b,$$

as well as for $z \to E_0 = -\sqrt{5}$

$$\lim_{z\to -\sqrt{5}} (z-1) W(\psi_-(z), \psi_+(z)) = \pm \lim_{z\to -\sqrt{5}} (z-1) W(\psi_\mp(\lambda), \overline{\psi_\pm(\lambda)}) = \frac{-4b}{1+\sqrt{5}}.$$

On the other hand, one can verify (4.1) directly by

$$\frac{R_\pm(z)+1}{T(z)} = 1 \mp \frac{(1+z)b}{2} - \frac{R_4^{1/2}(z) b}{2(z-1)}$$

and

$$R_4^{1/2}(z) \frac{R_\pm(z)+1}{T(z)} = R_4^{1/2}(z) \pm \frac{(1+z)b}{2} \big(\pm 5 \mp z^2 - R_4^{1/2}(z) \big),$$
$$R_4^{1/2}(z) \frac{R_\pm(z)-1}{T(z)} = -R_4^{1/2}(z) \pm \frac{(1+z)b}{2} \big(\pm 1 \mp z^2 - R_4^{1/2}(z) \big).$$

Chapter 5

Inverse scattering transform for the Toda hierarchy

5.1 Introduction to the Toda equation

We only give a brief overview here, for a detailed description we refer the reader for example to [7], [48], [49].

In 1955 Enrico Fermi, John Pasta, and Stanislaw Ulam [18] considered a simple model for a nonlinear one-dimensional crystal describing the motion of a chain of particles with nearest neighbor interaction. The Hamiltonian of such a system is given by

$$\mathcal{H}(p,q) = \sum_{n\in\mathbb{Z}} \Big(\frac{p(n,t)^2}{2} + V(q(n+1,t) - q(n,t)) \Big),$$

where $q(n,t)$ is the displacement of the n-th particle from its equilibrium position, $p(n,t)$ is its momentum (mass $m=1$), and $V(r)$ is the interaction potential.

Restricting the attention to finitely many particles (e.g., by imposing periodic boundary conditions) and to the harmonic interaction $V(r) = \frac{r^2}{2}$, the equations of motion form a linear system of differential equations with constant coefficients. The solution is then given by a superposition of the associated *normal modes*. It was general belief at that time that a generic nonlinear perturbation would yield to *thermalization*. That is, for any initial condition the energy should eventually be equally distributed over all normal modes. To investigate the rate of approach to the equipartition of energy Fermi, Pasta, and Ulam [18] carried out a computer experiment at Los Alamos. But the experiment indicated, instead of the expected thermalization, a quasi-periodic motion of the system! It was not until ten years later that Martin Kruskal and Norman Zabusky [58] discovered the connections with *solitons*.

This had a high influence on soliton mathematics and led to the search for a potential $V(r)$ for which the above system has soliton solutions. Considering additional formulas for elliptic functions Morikazu Toda came up with the choice

$$V(r) = \mathrm{e}^{-r} + r - 1.$$

The corresponding system is now known as the *Toda equation*, [50], [51].

This model is of course only valid as long as the relative displacement is not too large (i.e., at least smaller than the distance of the particles in the equilibrium position). For small displacements it is approximately equal to a harmonic interaction. The equation of motion in this case reads explicitly

$$\begin{aligned}
\frac{d}{dt}p(n,t) &= -\frac{\partial\mathcal{H}(p,q)}{\partial q(n,t)}\\
&= \mathrm{e}^{-(q(n,t)-q(n-1,t))} - \mathrm{e}^{-(q(n+1,t)-q(n,t))},\\
\frac{d}{dt}q(n,t) &= \frac{\partial\mathcal{H}(p,q)}{\partial p(n,t)} = p(n,t).
\end{aligned} \tag{5.1}$$

The important property of the Toda equation is the existence of so called *soliton solutions*, that is, pulslike waves which spread in time without changing their size and shape (cf. [49]). Existence of soliton solutions is usually related to complete integrability of the system. To see that the Toda system is integrable we use Flaschka's variables ([20])

$$a(n,t) = \frac{1}{2}\mathrm{e}^{-(q(n+1,t)-q(n,t))/2}, \qquad b(n,t) = -\frac{1}{2}p(n,t).$$

Then (5.1) yields

$$\begin{aligned}
\dot{a}(t) &= a(t)\Big(b^+(t) - b(t)\Big),\\
\dot{b}(t) &= 2\Big(a(t)^2 - a^-(t)^2\Big),
\end{aligned} \tag{5.2}$$

where we have used the abbreviation $f^{\pm}(n) = f(n{\pm}1)$. To show complete integrability it suffices to find a *Lax pair*, that is, two operators $H(t)$, $P(t)$ such that the *Lax equation*

$$\frac{d}{dt}H(t) = P(t)H(t) - H(t)P(t) \tag{5.3}$$

is equivalent to (5.2). It turns out that the right choice is

$$\begin{aligned}
H(t):\ \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z})\\
f(n) &\mapsto a(n,t)f(n+1) + b(n,t)f(n) + a(n-1,t)f(n-1)\\
P(t):\ \ell^2(\mathbb{Z}) &\to \ell^2(\mathbb{Z})\\
f(n) &\mapsto a(n,t)f(n+1) - a(n-1,t)f(n-1).
\end{aligned}$$

The Lax equation implies that the Jacobi operators $H(t)$ for different $t \in \mathbb{R}$ are all unitary equivalent. The Lax equation also holds for $H(t)^j - H_0^j$, where H_0 is the operator corresponding to the constant solution $a_0(n,t) = 1/2$, $b_0(n,t) = 0$. Hence taking traces shows that

$$\mathrm{tr}\big(H(t)^j - H_0^j\big), \qquad j \in \mathbb{N},$$

are conserved quantities too. In particular,

$$\begin{aligned}
\mathrm{tr}\big(H(t) - H_0\big) &= \sum_{n\in\mathbb{Z}} b(n,t) = -\frac{1}{2}\sum_{n\in\mathbb{Z}} p(n,t)\\
\mathrm{tr}\big(H(t)^2 - H_0^2\big) &= \sum_{n\in\mathbb{Z}} \left(b(n,t)^2 + \frac{a(n,t)^2}{2} - \frac{1}{2}\right) = \frac{1}{2}\mathcal{H}(p,q)
\end{aligned}$$

correspond to conservation of the total momentum and the total energy.

Further details can be found, e.g., in [48].

5.2 The Toda hierarchy

The Lax approach allows a straightforward generalization of the Toda equation by replacing P with more general operators P_{2r+2} of order $2r+2$.

Let our sequences a, b depend on an additional parameter $t \in \mathbb{R}$.

Hypothesis H. 5.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z} \times \mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

Associated with $a(t)$, $b(t)$ is the Jacobi operator $H(t)$. The idea of the Lax formalism is to find a finite, skew-symmetric operator $P_{2r+2}(t)$ such that the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{5.4}$$

holds, where $[P, H] = PH - HP$ is the commutator. By [48], Theorem 12.2, the *Lax operator* is given by

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} \big(2a(t) g_j(t) S^+ - h_j(t)\big) H(t)^{r-j} + g_{r+1}(t), \qquad r \in \mathbb{N}_0,$$

where $S^\pm f(n) = f(n \pm 1)$ and $(g_j(n,t))_{j=0}^{r+1}$, $(h_j(n,t))_{j=0}^{r+1}$ are defined as

$$g_j(n,t) = \sum_{l=0}^{j} c_{j-l} \langle \delta_n, H(t)^l \delta_n \rangle,$$

$$h_j(n,t) = 2a(n,t) \sum_{l=0}^{j} c_{j-l} \langle \delta_{n+1}, H(t)^l \delta_n \rangle + c_{j+1}$$

for some arbitrarily chosen constants $\{c_j\}_{j=0}^r$ with $c_0 = 1$. The Lax equation (5.4) is then equivalent to the *r-th Toda equation*

$$\begin{aligned} \mathrm{TL}_r(a(t), b(t))_1 &= \dot{a}(t) - a(t)\big(g_{r+1}^+(t) - g_{r+1}(t)\big) = 0, \\ \mathrm{TL}_r(a(t), b(t))_2 &= \dot{b}(t) - \big(h_{r+1}(t) - h_{r+1}^-(t)\big) = 0. \end{aligned} \tag{5.5}$$

Here the dot denotes the derivative with respect to t and $f^\pm(n) = f(n \pm 1)$. Varying $r \in \mathbb{N}_0$ yields the *Toda hierarchy*

$$\mathrm{TL}_r(a, b) = \big(\mathrm{TL}_r(a, b)_1, \mathrm{TL}_r(a, b)_2\big) = 0, \qquad r \in \mathbb{N}_0.$$

The well-known isospectrality theorem (cf. for instance [47], [48]) is one of the key results required for the inverse scattering transform.

Theorem 5.2. *Let $a(t), b(t)$ satisfy $\mathrm{TL}_r(a, b) = 0$ and (H.5.1). Then the Lax equation (5.4) implies existence of a* unitary propagator *$U_r(t,s)$ for $P_{2r+2}(t)$ such that*

$$H(t) = U_r(t,s) H(s) U_r(t,s)^{-1}, \qquad (t,s) \in \mathbb{R}^2.$$

Therefore all operators $H(t)$, $t \in \mathbb{R}$, are unitary equivalent and in particular,

$$\sigma(H) \equiv \sigma(H(t)) = \sigma(H(0)), \qquad \rho(H) \equiv \rho(H(t)) = \rho(H(0)).$$

In addition, if $\psi(s) \in \ell^2(\mathbb{Z})$ solves $H(s)\psi(s) = z\psi(s)$, then

$$\psi(t) = U_r(t, s)\psi(s)$$

satisfies

$$H(t)\psi(t) = z\psi(t), \qquad \frac{d}{dt}\psi(t) = P_{2r+2}(t)\psi(t). \tag{5.6}$$

Moreover, we will need the basic existence and uniqueness theorem for the Toda hierarchy ([46], Theorem 2.2, and [14], Prop. 8).

Theorem 5.3. *Suppose $(a_0, b_0) \in M = \ell^\infty \oplus \ell^\infty$. Then there exists a unique integral curve $t \mapsto (a(t), b(t))$ in $C^\infty(\mathbb{R}, M)$ of the Toda equations, that is,* $\mathrm{TL}_r(a(t), b(t)) = 0$, *such that $(a(0), b(0)) = (a_0, b_0)$.*

The *stationary* Toda hierarchy is characterized by $\dot{a} = \dot{b} = 0$ in (5.5) or, equivalently, by commuting difference expressions

$$[P_{2r+2}, H] = 0.$$

The stationary solutions of the Toda hierarchy are precisely the reflectionless finite-gap sequences discussed in Sections 1.2, 1.3 (cf. [48], Section 12.3).

If $P(t) = P$ is time-independent (in the case of stationary solutions), then the unitary propagator is given by Stone's theorem, that is, $U(t, s) = \exp\big((t - s)P\big)$.

5.3 Finite-gap solutions of the Toda hierarchy

In this section we present the reflectionless time-dependent finite-gap solutions for the Toda hierarchy constructed in [7], [48]. Their idea was to choose a stationary solution of $TL_r(a, b) = 0$ as the initial condition for $TL_s(a, b)$ for some $s \in \mathbb{N}_0$ and to consider the time evolution in the coordinates $\{E_j\}_{j=0}^{2r+1}$ and $\{(\mu_j(n), \sigma_j(n))\}_{j=1}^{r}$.

More precisely, we take the r-gap stationary solutions (1.7)

$$\begin{aligned} a_{q,0}(n)^2 &= \frac{1}{2}\sum_{j=1}^{g} \hat{R}_j(n) + \frac{1}{8}\sum_{j=0}^{2g+1} E_j^2 - \frac{1}{4}\sum_{j=1}^{g}\mu_j(n)^2 - \frac{b_{q,0}(n)^2}{4}, \\ b_{q,0}(n) &= \frac{1}{2}\sum_{j=0}^{2g+1} E_j - \sum_{j=1}^{g}\mu_j(n), \end{aligned} \tag{5.7}$$

as the initial condition for

$$\hat{TL}_s(a_q(t), b_q(t)) = 0, \qquad (a_q(0), b_q(0)) = (a_{q,0}, b_{q,0}), \tag{5.8}$$

where we denote the Toda equation (and all associated quantities) which gives rise to the time evolution with a hat, $\hat{TL}_s$. The sequences $(a_{q,0}, b_{q,0})$ are completely determined by the band edges $\{E_j\}_{j=0}^{2r+1}$ and the Dirichlet eigenvalues $\{(\mu_j(n_0), \sigma_j(n_0))\}_{j=1}^{r}$ at a fixed point $n_0 \in \mathbb{Z}$. The band edges E_j do not depend on t since the family of operators $H(t)$ is unitary equivalent. If we make the assumption that $(a_q(t), b_q(t))$ is reflectionless for all t (since we do not know wether the reflectionless property of the initial condition is preserved by the Toda flow), we obtain for the Green function

$$g(z, n_0, t) = \prod_{j=1}^{r}\big(z - \mu_j(n_0, t)\big) R_{2r+2}^{-1/2}(z).$$

The time evolution of the Dirichlet data follows now from the Lax equation

$$\frac{d}{dt}(H(t)-z)^{-1} = [P_{2s+2}(t), (H(t)-z)^{-1}]$$

and is given by ([48], (13.3))

$$\begin{aligned} \frac{d}{dt}\mu_j(n_0,t) &= -2\hat{G}_s(\mu_j(n_0,t),n_0,t)\sigma_j(n_0,t)R_j(n_0,t), \\ \hat{\mu}_j(n_0,0) &= \hat{\mu}_{0,j}(n_0), \quad 1 \le j \le r, \quad t \in \mathbb{R}. \end{aligned} \tag{5.9}$$

This system has a unique global solution $(\hat{\mu}_j(n_0,.))_{j=1}^r \in C^\infty(\mathbb{R}, M^D)$ for each initial condition satisfying [48], (H.8.12), (cf. [48], Theorem 13.1). This allows us to construct $a_q(n,t)$, $b_q(n,t)$ from $(\mu_j(n_0,t))_{j=1}^r$ as in [48], Sections 8.3, 13.1, which indeed solve the Toda equation $\hat{T}L_s(a_q,b_q)=0$.

Theorem 5.4. ([48], Theorem 13.2). *The solution $(a_q(t), b_q(t))$ of the $\hat{T}L_s$ equations (5.8) with r-gap initial conditions $(a_{q,0}, b_{q,0})$ as in (5.7) is given by*

$$\begin{aligned} a_q(n,t)^2 &= \frac{1}{2}\sum_{j=1}^r \hat{R}_j(n,t) + \frac{1}{8}\sum_{j=0}^{2r+1} E_j^2 - \frac{1}{4}\sum_{j=1}^r \mu_j(n,t)^2 - \frac{b_q(n,t)^2}{4}, \\ b_q(n,t) &= \frac{1}{2}\sum_{j=0}^{2r+1} E_j - \sum_{j=1}^r \mu_j(n,t), \end{aligned} \tag{5.10}$$

where $(\hat{\mu}_j(n,t))_{j=1}^r$ is the unique solution of (5.9).

Introduce

$$\phi(p,n,t) = \frac{H_{r+1}(p,n,t) + R_{2r+2}^{1/2}(p)}{2a_q(n,t)G_r(p,n,t)} = \frac{2a_q(n,t)G_r(p,n+1,t)}{H_{r+1}(p,n,t) - R_{2r+2}^{1/2}(p)}, \tag{5.11}$$

where $G_r(z,n,t)$, $H_r(z,n,t)$ are defined as in (1.9) with $\hat{\mu}_j(n,t)$ instead of $\hat{\mu}_j(n)$, $1 \le j \le r$. Their time derivative is computed in [48], eq. (13.18), (13.19),

$$\frac{d}{dt}G_r(z,n,t) = 2\Big(\hat{G}_s(z,n,t)H_{r+1}(z,n,t) - G_r(z,n,t)\hat{H}_{s+1}(z,n,t)\Big), \tag{5.12}$$

$$\frac{d}{dt}H_{r+1}(z,n,t) = 4a_q(n,t)^2\Big(\hat{G}_s(z,n,t)G_r(z,n+1,t) - G_r(z,n,t)\hat{G}_s(z,n+1,t)\Big).$$

The time-dependent Baker-Akhiezer function is given by ([48], eq. (13.24))

$$\tilde{\psi}_q(p,n,n_0,t) = \exp\left(\hat{\alpha}_s(p,n_0,t)\right)\prod_{m=n_0}^{n-1}\phi(p,m,t),$$

where

$$\hat{\alpha}_s(p,n_0,t) = \int_0^t \big(2a_q(n_0,x)\hat{G}_s(p,n_0,x)\phi(p,n_0,x) - \hat{H}_{s+1}(p,n_0,x)\big)dx. \tag{5.13}$$

It is straightforward to calculate that $\tilde{\psi}_q(p,n,n_0,t)$ satisfies

$$\tau_q(t)\tilde{\psi}_q(p,n,n_0,t) = \pi(p)\tilde{\psi}_q(p,n,n_0,t) \tag{5.14}$$

$$\begin{aligned} \frac{d}{dt}\tilde{\psi}_q(p,n,n_0,t) &= 2a_q(n,t)\hat{G}_s(p,n,t)\tilde{\psi}_q(p,n+1,n_0,t) \\ &\quad -\hat{H}_{s+1}(p,n,t)\tilde{\psi}_q(p,n,n_0,t) \\ &= \hat{P}_{q,2s+2}(t)\tilde{\psi}_q(p,.,n_0,t)(n). \end{aligned} \tag{5.15}$$

We set $n_0 = 0$ and omit it. By (5.11) and (5.12),

$$\begin{aligned}
\hat{\alpha}_{s,\pm}(z,t) &= \int_0^t \big(2a_q(0,x)\hat{G}_s(z,0,x)\phi_\pm(z,0,x) - \hat{H}_{s+1}(z,0,x)\big)dx \\
&= \int_0^t \left(\frac{\hat{G}_s(z,0,x)}{G_r(z,0,x)}(H_{r+1}(z,0,x) \pm R_{2r+2}^{1/2}(z)) - \hat{H}_{s+1}(z,0,x)\right)dx \\
&= \pm R_{2r+2}^{1/2}(z)\int_0^t \frac{\hat{G}_s(z,0,x)}{G_r(z,0,x)}dx + \int_0^t \frac{\frac{d}{dx}G_r(z,0,x)}{2G_r(z,0,x)}dx.
\end{aligned}$$

Therefore,

$$\exp\big(\hat{\alpha}_{s,\pm}(z,t)\big) = \tilde{\psi}_{q,\pm}(z,0,t) = \sqrt{\frac{G_r(z,0,t)}{G_r(z,0,0)}}\exp\left(\pm R_{2r+2}^{1/2}(z)\int_0^t \frac{\hat{G}_s(z,0,x)}{G_r(z,0,x)}dx\right). \tag{5.16}$$

Note that this implies for $\lambda \in \sigma(H_q)$,

$$\begin{aligned}
\overline{\tilde{\psi}_{q,\pm}(\lambda,n,t)} &= \tilde{\psi}_{q,\pm}(\overline{\lambda},n,t), \\
\hat{\alpha}_{s,\pm}(\lambda,t)^* &= \hat{\alpha}_{s,\mp}(\lambda,t) \neq \mp\hat{\alpha}_{s,\pm}(\lambda,t).
\end{aligned} \tag{5.17}$$

By [48], Lemma 12.15, the Wronskian of two solutions satisfying (5.14), (5.15) is independent of n and t, hence

$$\begin{aligned}
W(\tilde{\psi}_{q,+}(z,n,t),\tilde{\psi}_{q,-}(z,n,t)) &= a_q(0,t)e^{\hat{\alpha}_+(z,t)}e^{\hat{\alpha}_-(z,t)}\big(\phi_-(z,0,t) - \phi_+(z,0,t)\big) \\
&= \exp\big(\hat{\alpha}_+(z,t) + \hat{\alpha}_-(z,t)\big)\frac{-R_{2r+2}^{1/2}(z)}{\prod_{j=1}^r(z-\mu_j(0,t))} \\
&= \frac{-R_{2r+2}^{1/2}(z)}{\prod_{j=1}^r(z-\mu_j(0,0))}
\end{aligned}$$

and

$$\exp\big(\hat{\alpha}_+(z,t) + \hat{\alpha}_-(z,t)\big) = \prod_{j=1}^r \frac{z-\mu_j(0,t)}{z-\mu_j(0,0)}.$$

5.4 Inverse scattering transform

Our aim in this section is to solve the initial value problem of the Toda hierarchy using what is generally known as *inverse scattering transform.*

So far we know the time evolution of the unperturbed system. Our first aim is to consider whether the short-range assumption (5.18) holds for all $t \in \mathbb{R}$.

Lemma 5.5. *Suppose $a(n,t)$, $b(n,t)$ and $\bar{a}(n,t)$, $\bar{b}(n,t)$ are two arbitrary bounded solutions of the Toda system satisfying (5.18) for one $t_0 \in \mathbb{R}$, then (5.18) holds for all $t \in \mathbb{R}$, that is,*

$$\sum_{n\in\mathbb{Z}} |n|\Big(|a(n,t) - \bar{a}(n,t)| + |b(n,t) - \bar{b}(n,t)|\Big) < \infty. \tag{5.18}$$

Proof. Without loss of generality we assume that $t_0 = 0$. Consider the expression

$$\|(a(0),b(0))\|_* = \sum_{n\in\mathbb{Z}}(1+|n|)\big(|a(n,0) - \bar{a}(n,0)| + |b(n,0) - \bar{b}(n,0)|\big)$$

which remains finite at least for small t since there is a local solution of the Toda system with respect to $\|.\|_*$. We claim that the following estimate holds

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}(1+|n|)|g_r(n,t)-\bar{g}_r(n,t)| &\leq C_r\|(a(t),b(t))\|_*,\\
\sum_{n\in\mathbb{Z}}(1+|n|)|h_r(n,t)-\bar{h}_r(n,t)| &\leq C_r\|(a(t),b(t))\|_*, \qquad (5.19)
\end{aligned}$$

where $C_r = C\big(\|H(0)\|^r, \|\bar{H}(0)\|^r\big)$ is a positive constant due to H.5.1. Let us prove (5.19) by induction on r. It suffices to consider the case where $c_j = 0$, $1 \leq j \leq r$, since all involved sums are finite. In this case [48], Lemma 6.4, shows that $g_j(n,t)$, $h_j(n,t)$ can be recursively computed from $g_0(n,t) = 1$, $h_0(n,t) = 0$ via

$$\begin{aligned}
g_{j+1}(n,t) &= \frac{1}{2}\big(h_j(n,t)+h_j(n-1,t)\big)+b(n,t)g_j(n,t),\\
h_{j+1}(n,t) &= 2a(n,t)^2\sum_{l=0}^{j} g_{j-l}(n,t)g_l(n+1,t)-\frac{1}{2}\sum_{l=0}^{j} h_{j-l}(n,t)h_l(n,t).
\end{aligned}$$

Hence

$$\begin{aligned}
g_0(n,t)-\bar{g}_0(n,t)=0, &\qquad g_1(n,t)-\bar{g}_1(n,t)=b(n,t)-\bar{b}(n,t),\\
h_0(n,t)-\bar{h}_0(n,t)=0, &\qquad h_1(n,t)-\bar{h}_1(n,t)=2(a(n,t)^2-\bar{a}(n,t)^2),
\end{aligned}$$

and

$$\begin{aligned}
g_{r+1}(n,t)-\bar{g}_{r+1}(n,t) &= \big(b(n,t)-\bar{b}(n,t)\big)g_r(n,t)+\bar{b}(n,t)\big(g_r(n,t)-\bar{g}_r(n,t)\big)\\
&\quad +\frac{1}{2}\big(h_r(n,t)-\bar{h}_r(n,t)+h_r(n-1,t)-\bar{h}_r(n-1,t)\big),\\
h_{r+1}(n,t)-\bar{h}_{r+1}(n,t) &= \frac{1}{2}\sum_{l=0}^{r}\big(h_{r-l}(n,t)h_l(n,t)-\bar{h}_{r-l}(n,t)\bar{h}_l(n,t)\big)\\
&\quad +2a(n,t)^2\sum_{l=0}^{r} g_{r-l}(n,t)g_l(n+1,t)\\
&\quad -2\bar{a}(n,t)^2\sum_{l=0}^{r} \bar{g}_{r-l}(n,t)\bar{g}_l(n+1,t).
\end{aligned}$$

We use

$$\begin{aligned}
&g_k(n,t)g_l(m,t)-\bar{g}_k(n,t)\bar{g}_l(m,t)\\
&\quad = (g_k(n,t)-\bar{g}_k(n,t))g_l(m,t)+\bar{g}_k(n,t)(g_l(m,t)-\bar{g}_l(m,t))
\end{aligned}$$

to finish the claim.

Since both sequences a, b and $\bar{a}$, $\bar{b}$ are solutions of the Toda system, (5.5) yields

$$\begin{aligned}
&\dot{a}(t)-\dot{\bar{a}}(t) = a(t)\big(g^+_{r+1}(t)-g_{r+1}(t)\big)-\bar{a}(t)\big(\bar{g}^+_{r+1}(t)-g_{r+1}(t)\big)\\
&= (a(t)-\bar{a}(t))\big(\bar{g}^+_{r+1}(t)-\bar{g}_{r+1}(t)\big)-a(t)\big(\bar{g}^+_{r+1}(t)-g^+_{r+1}(t)+g_{r+1}(t)-\bar{g}_{r+1}(t)\big)
\end{aligned}$$

and thus

$$|a(n,t)-\bar a(n,t)|\le \big|a(n,0)-\bar a(n,0)\big|+C_1\int_0^t \big|a(n,s)-\bar a(n,s)\big|ds$$
$$+\|H(0)\|\int_0^t \big(|g_{r+1}(n+1,s)-\bar g_{r+1}(n+1,s)|+|g_{r+1}(n,s)-\bar g_{r+1}(n,s)|\big)ds,$$
$$|b(n,t)-\bar b(n,t)|\le |b(n,0)-\bar b(n,0)|$$
$$+\int_0^t (|h_{r+1}(n,s)-\bar h_{r+1}(n,s)|+|h_{r+1}(n-1,s)-\bar h_{r+1}(n-1,s)|)ds.$$

By (5.19),

$$\|(a(t),b(t))\|_*\le \|(a(0),b(0))\|_*+C\int_0^t \|(a(s),b(s))\|_* ds,$$

where $C=(C_1+2\|H(0)\|+2)C_{r+1}$. We apply Gronwall's inequality and obtain

$$\|(a(t),b(t))\|_*\le \|(a(0),b(0))\|_*\exp\big(Ct\big).$$

□

In our setting of quasi-periodic background, let $a(n,t)$, $b(n,t)$ be a solution of the Toda system satisfying

$$\sum_{n\in\mathbb{Z}}|n|\Big(|a(n,t)-a_q(n,t)|+|b(n,t)-b_q(n,t)|\Big)<\infty \tag{5.20}$$

for one (and hence for any) $t_0\in\mathbb{R}$. We develop scattering theory for the Jacobi operator $H(t)$ associated with $a(n,t)$, $b(n,t)$ as in Chapter 2 with the only difference that Jost solutions, transmission and reflection coefficients depend now on an additional parameter $t\in\mathbb{R}$. In particular,

$$\sigma(H(t))\equiv\sigma(H),\qquad \sigma_{ess}(H)=\bigcup_{j=0}^{g}[E_{2j},E_{2j+1}],\qquad \sigma_p(H)=\{\rho_j\}_{j=1}^q\subseteq\mathbb{R}\backslash\sigma_{ess}(H),$$

where $q\in\mathbb{N}$ is finite. The Jost solutions $\psi_\pm(z,n,t)$ satisfy

$$\big(\tau(t)-z\big)\psi_\pm(z,n,t)=0,\qquad \psi_\pm(z,n,t)=\sum_{j=n}^{\pm\infty}K_\pm(n,j,t)\psi_{q,\pm}(z,j,t),\qquad z\in\sigma(H_q).$$

Transmission $T(\lambda,t)$ and reflection $R_\pm(\lambda,t)$ coefficients are defined via the scattering relations

$$T(\lambda,t)\psi_\pm(\lambda,n,t)=\overline{\psi_\mp(\lambda,n,t)}+R_\mp(\lambda,t)\psi_\mp(\lambda,n,t),\qquad \lambda\in\sigma(H_q).$$

The scattering data is now given by

$$S_\pm(H(t))=\big\{R_\pm(w,t),|w|=1;(\rho_j,\gamma_{\pm,j}(t)),1\le j\le q\big\}.$$

How does the scattering data evolve with t?

Theorem 5.6. *Suppose $a(n,t)$, $b(n,t)$ is a solution of the Toda system satisfying (5.20) for one $t_0 \in \mathbb{R}$. The functions*

$$u_\pm(z,n,t) = \exp(\hat{\alpha}_{s,\pm}(z,t))\psi_\pm(z,n,t),$$

where $\psi_\pm(z,n,t)$ are the Jost solutions and $\hat{\alpha}_s(z,t) = \hat{\alpha}_s(z,0,t)$ is defined in (5.13) (cf. (5.16)), satisfy

$$H(t)u(z,n,t) = zu(z,n,t), \qquad \frac{d}{dt}u(z,n,t) = P_{2s+2}(t)u(z,n,t). \tag{5.21}$$

Moreover,

$$\begin{aligned} T(\lambda,t) &= T(\lambda,0), \\ R_\pm(\lambda,t) &= R_\pm(\lambda,0)\exp\Big(\pm 2R_{2r+2}^{1/2}(\lambda)\int_0^t \frac{\hat{G}_s(\lambda,0,x)}{G_r(\lambda,0,x)}dx\Big), \\ \gamma_{\pm,j}(t) &= \gamma_{\pm,j}(0)\big|\exp(\hat{\alpha}_{s,\pm}(\rho_j,t))\big|^2, \qquad 1 \le j \le q. \end{aligned}$$

Proof. As in Lemma 2.3 we show existence of the Jost solutions $\psi_\pm(z,n,t)$ satisfying $\tau(t)\psi_\pm(z,n,t) = z\psi_\pm(z,n,t)$ and $\lim_{n\to\pm\infty}\psi_\pm(z,n,t) = \psi_{q,\pm}(z,n,t)$. The solutions $\psi_\pm(z,n,t)$ are continuously differentiable with respect to t by the same arguments as for z and $\dot{\psi}_\pm(z,n,t) \approx \dot{\psi}_{q,\pm}(z,n,t)$ for $n \to \pm\infty$ (use (2.5), (5.5), and (5.19)). By (5.14),

$$\begin{aligned} \frac{d}{dt}\psi_{q,\pm}(z,n,t) &= \frac{d}{dt}\Big(\prod_{j=0}^{n-1}\phi_\pm(z,j,t)\Big) = \frac{d}{dt}\Big(\tilde{\psi}_{q,\pm}(z,n,t)\exp\big(-\hat{\alpha}_{s,\pm}(z,t)\big)\Big) \\ &= \hat{P}_{q,2s+2}(t)\psi_{q,\pm}(z,.,t)(n) - \frac{d}{dt}\big(\hat{\alpha}_{s,\pm}(z,t)\big)\psi_{q,\pm}(z,n,t). \end{aligned}$$

Lemma 12.16 in [48] implies for $z \in \rho(H)$ that the solution $u_\pm(z,n,t)$ of (5.21) with initial condition $\psi_\pm(z,n,0) \in \ell^2_\pm(\mathbb{Z})$ is square summable near $\pm\infty$ for all $t \in \mathbb{R}$, that is,

$$u_\pm(z,n,t) = C_\pm(t)\psi_\pm(z,n,t). \tag{5.22}$$

Evaluating

$$\dot{u}_+(z,n,t) = \dot{C}_+(t)\psi_+(z,n,t) + C_+(t)\dot{\psi}_+(z,n,t) = P_{2s+2}(t)C_+(t)\psi_+(z,n,t)$$

as $n \to \infty$ yields

$$\dot{C}_+(t) = \frac{d}{dt}\big(\hat{\alpha}_{s,\pm}(z,t)\big)C_+(t).$$

The general result for all $z \in \mathbb{C}$ now follows from continuity.

The time evolution for the scattering coefficients is given by

$$\begin{aligned} T(\lambda,t) &= \frac{W(\psi_-(\lambda,t),\overline{\psi_-(\lambda,t)})}{W(\psi_-(\lambda,t),\psi_+(\lambda,t))} \\ &= \frac{\exp(\hat{\alpha}_{s,+}(\lambda,t))\overline{\exp(\hat{\alpha}_{s,+}(\lambda,t))}}{\exp(\hat{\alpha}_{s,+}(\lambda,t))\exp(\hat{\alpha}_{s,-}(\lambda,t))}\frac{W(u_-(\lambda,t),\overline{u_-(\lambda,t)})}{W(u_-(\lambda,t),u_+(\lambda,t))} = T(\lambda,0), \end{aligned}$$

since $\overline{\hat{\alpha}_{s,\pm}(\lambda,t)} = \hat{\alpha}_{s,\mp}(\lambda,t)$ by (5.17) and the Wronskian of two solutions satisfying (5.21) does not depend on n or t by [48], Lemma 12.15. Similarly,

$$\begin{aligned} R_{\pm}(\lambda,t) &= \frac{\exp(\hat{\alpha}_{s,\pm}(\lambda,t))\exp(\overline{\hat{\alpha}_{s,\mp}(\lambda,t)})}{\exp(\hat{\alpha}_{s,\mp}(\lambda,t))\exp(\hat{\alpha}_{s,\pm}(\lambda,t))}\frac{W(u_{\mp}(\lambda,t),\overline{u_{\pm}(\lambda,t)})}{W(u_{\pm}(\lambda,t),u_{\mp}(\lambda,t))} \\ &= \exp\Big(\pm 2R_{2r+2}^{1/2}(\lambda)\int_0^t \frac{\hat{G}_s(\lambda,0,x)}{G_r(\lambda,0,x)}dx\Big)R_{\pm}(\lambda,0), \end{aligned}$$

where we used (5.16). It remains to consider $\gamma_{\pm,j}(t)$. By Theorem 5.2,

$$u_{\pm}(z,n,t) = U_s(t,0)u_{\pm}(z,n,0),$$

that is,

$$\exp(\hat{\alpha}_{s,\pm}(z,t))\psi_{\pm}(z,n,t) = U_s(t,0)\psi_{\pm}(z,n,0),$$

where $U_s(t,0)$ is the unitary propagator of $P_{2s+2}(t)$. At the eigenvalues ρ_j,

$$\big|\exp(\hat{\alpha}_{s,\pm}(\rho_j,t))\big|^2\sum_{n\in\mathbb{Z}}|\psi_{\pm}(\rho_j,n,t)|^2 = \frac{\big|\exp(\hat{\alpha}_{s,\pm}(\rho_j,t))\big|^2}{\gamma_{\pm,j}(t)} = \|U_s(t,0)\psi_{\pm}(\rho_j,n,0)\|^2.$$

Since

$$\frac{d}{dt}\frac{\big|\exp(\hat{\alpha}_{s,\pm}(\rho_j,t))\big|^2}{\gamma_{\pm,j}(t)} = \frac{d}{dt}\|\psi_{\pm}(\rho_j,n,0)\|^2 = 0,$$

we obtain that

$$\gamma_{\pm,j}(t) = \gamma_{\pm,j}(0)\big|\exp(\hat{\alpha}_{s,\pm}(\rho_j,t))\big|^2.$$

□

Therefore the scattering data for $H(t)$ can be expressed in terms of those for $H(0)$ and the function $\hat{\alpha}_s(z,t)$, which is completely determined by $H_q(t)$,

$$\begin{aligned} S_{\pm}(H(t)) = \Big\{&R_{\pm}(\lambda,0)\exp\Big(\pm 2R_{2r+2}^{1/2}(\lambda)\int_0^t \frac{\hat{G}_s(\lambda,0,x)}{G_r(\lambda,0,x)}dx\Big), \lambda\in\sigma(H_q); \\ &\big(\rho_j,\gamma_{\pm,j}(0)\big|\exp(\hat{\alpha}_{s,\pm}(\rho_j,t))\big|^2\big), 1\le j\le q\Big\}. \end{aligned} \tag{5.23}$$

The remaining step is to invoke the Gel'fand-Levitan-Marchenko theory which reconstructs $H(t)$ from its scattering data $S_{\pm}(H(t))$. The operator $H(t)$ is uniquely determined since $S_{\pm}(H(t))$ satisfy Hypothesis H.2.23 if $S_{\pm}(H(0))$ do. As the existence of a (unique) solution for the Toda equations was ensured from the outset (Lemma 5.5 and Theorem 5.6), the sequences $a(n,t)$, $b(n,t)$ constructed by this procedure satisfy the Toda equations.

Hence, in the short-range situation with asymptotically quasi-periodic initial condition $(a(0),b(0))$ one arrives at the following procedure for solving the initial value problem for the Toda equations (the arrows visualize dependencies):

(1) Compute the Jost solutions $\psi_{\pm}(z,n,0)$ from the Baker-Akhiezer functions by iterating the corresponding Volterra sum equation as in Lemma 2.3. This provides the scattering data for $H(0)$.

(2) Read off the scattering data of $H(t)$ from Theorem 5.6.

(3) Compute the Fourier coefficients of $R_\pm(\lambda, t)$ and use the solution $K_\pm(n, m, t)$ of the Gel'fand-Levitan-Marchenko equation to construct $a(n, t)$, $b(n, t)$.

This method is known as *inverse scattering transform*. The foundation for this method are Theorem 5.3, Lemma 5.5, Theorem 5.6, and Theorem 2.22.

The inverse scattering transform was first derived by Gardner et al. [23] in 1967 to solve the Korteweg-de Vries equation. Flaschka [21] established this method for the Toda equation in the case $r = 0$ and worked out the inverse procedure in the reflectionless case (i.e. $R_\pm(\lambda, t) = 0$). Further contributions were made by Case et al. [11] – [13]. Recently Teschl [46], [47] provided a rigorous treatment of the inverse scattering transform for the entire Toda hierarchy with asymptotically decaying initial condition.

Chapter 6

Trace formulas

A trivial example of a trace formula is

$$\mathrm{tr}(H - H_n) = b(n).$$

Here $H_n = H_{-,n} \oplus H_{+,n}$ and $H_{-,n}$, $H_{+,n}$ are restrictions of the Jacobi operator H to the subspaces $\ell^2(-\infty, n-1]$ and $\ell^2[n+1, \infty)$ obtained by imposing Dirichlet boundary conditions at n (i.e. $u(n) = 0$).

6.1 General trace formulas and the ξ-function

Trace formulas for bounded Jacobi operators are well studied objects (see for example [45], [48]). In this section we apply the theory to our time dependent Jacobi operators $H(t)$ which are short-range perturbations of quasi-periodic ones. We will follow the presentation in [45].

The time dependent Green function of $H(t)$ on the diagonal,

$$g(\lambda, n, t) = G(\lambda, n, n, t) = \langle \delta_n, (H - z)^{-1} \delta_n \rangle = \frac{\psi_-(\lambda, n, t)\psi_+(\lambda, n, t)}{W(\psi_-(\lambda), \psi_+(\lambda))},$$

is a Herglotz function. Its exponential representation is given by

$$g(z, n, t) = |g(i, n, t)| \exp\left(\int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \xi(\lambda, n, t) d\lambda \right), \qquad z \in \mathbb{C} \backslash \sigma(H_q),$$

where $\xi(\lambda, n, t)$ ([24]) is defined by

$$\xi(\lambda, n, t) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \arg g(\lambda + i\epsilon, n, t) \quad \text{for a.e. } \lambda \in \mathbb{R}, \quad \arg(.) \in (-\pi, \pi].$$

The function $\xi(\lambda, n, t)$ satisfies $0 \le \xi(\lambda, n, t) \le 1$, $\xi(\lambda, n, t) = 0$ for $\lambda < E_0$, $\xi(\lambda, n, t) = 1$ for $\lambda > E_{2g+1}$, and

$$\int_{\mathbb{R}} \frac{\xi(\lambda, n, t)}{1 + \lambda^2} d\lambda = \arg g(i, n, t).$$

This implies together with the asymptotic behavior of $g(., n, t)$ that

$$g(z, n, t) = \frac{1}{E_{2g+1} - z} \exp\left(\int_{E_0}^{E_{2g+1}} \frac{\xi(\lambda, n, t) d\lambda}{\lambda - z} \right). \tag{6.1}$$

In order to compute $\xi(\lambda,n,t)$ we set $\lim_{\epsilon\to 0} g(\lambda+\mathrm{i}\epsilon,.) = g(\lambda+\mathrm{i}0,.)$. The scattering relations imply for $\lambda\in\sigma(H_q)$

$$\begin{aligned} g(\lambda+\mathrm{i}0,n,t) &= T(\lambda)R_{2g+2}^{-1/2}(\lambda)\Big(\prod_{j=1}^{g}(\lambda-\mu_j(t))\Big)\psi_-(\lambda,n,t)\psi_+(\lambda,n,t) \\ &= |\psi_\pm(\lambda,n,t)|^2\left(1+R_\pm(\lambda,t)\frac{\psi_\pm(\lambda,n,t)}{\overline{\psi_\pm(\lambda,n,t)}}\right)\frac{\prod_{j=1}^{g}(\lambda-\mu_j(t))}{R_{2g+2}^{1/2}(\lambda)}, \end{aligned}$$

therefore

$$\xi(\lambda,n,t)=\frac{1}{\pi}\arg\left(1+R_\pm(\lambda,t)\frac{\psi_\pm(\lambda,n,t)}{\overline{\psi_\pm(\lambda,n,t)}}\right)+\frac{1}{2},\qquad \lambda\in\sigma(H_q).$$

Since $g(\lambda,n,t)$ is real valued and monotonic for λ in the spectral gaps υ_j, we define the Dirichlet eigenvalues associated with $\tau(t)$ corresponding to a Dirichlet boundary condition at $n\in\mathbb{Z}$ as follows

$$\mu_j^\tau(n,t)=\sup\Big(\{E_{2j-1}\}\cup\{\lambda\in\upsilon_j \mid g(\lambda,n,t)<0\}\Big)\in\overline{\upsilon_j},\quad 1\le j\le g. \tag{6.2}$$

This yields

$$\begin{aligned} \xi(\lambda,n,t) &= \frac{1}{2}\chi_{\sigma(H_q)}(\lambda)+\sum_{j=1}^{g}\chi_{(E_{2j-1},\mu_j^\tau(n,t))}(\lambda)+\chi_{(E_{2g+1},\infty)}(\lambda) \\ &\quad +\frac{1}{\pi}\arg\left(1+R_\pm(\lambda,t)\frac{\psi_\pm(\lambda,n,t)}{\overline{\psi_\pm(\lambda,n,t)}}\right)\chi_{\sigma(H_q)}(\lambda), \end{aligned} \tag{6.3}$$

where $\chi_\Omega(.)$ denotes the characteristic function of the set $\Omega\subset\mathbb{R}$.

Lemma 6.1. *Assume (5.20). Then we have the following trace formula*

$$\begin{aligned} b^{(l)}(n,t) &= \mathrm{tr}(H(t)^l - H_n(t)^l) \\ &= \frac{1}{2}\sum_{j=0}^{2g+1}E_j^l-\sum_{j=0}^{g}\mu_j^\tau(n,t)^l+\int_{\sigma(H_q)}\frac{l\lambda^{l-1}}{\pi}\arg\left(1+R_\pm(\lambda,t)\frac{\psi_\pm(\lambda,n,t)}{\overline{\psi_\pm(\lambda,n,t)}}\right)d\lambda, \end{aligned}$$

where $\mu_j^\tau(n,t)$ is defined in (6.2) and

$$b^{(1)}(n,t)=b(n,t),$$

$$b^{(l)}(n,t)=l g_l(n,t)-\sum_{j=1}^{l-1}g_{l-j}(n,t)b^{(j)}(n,t),\qquad l\ge 2,$$

with $g_j(n,t)=\langle\delta_n,H(t)^j\delta_n\rangle$.

Proof. We expand both sides of (6.1) and compare coefficients to obtain ([45], Theorem 5.1)

$$b^{(l)}(n,t)=E_{2g+1}^l-l\int_{E_0}^{E_{2g+1}}\lambda^{l-1}\xi(\lambda,n,t)d\lambda. \tag{6.4}$$

Now insert the explicit representation (6.3) of $\xi(\lambda,n,t)$. □

Remark 6.2. *In the periodic case this result is due to Teschl [45].*

In the special case $l = 1$,

$$b(n,t) = \frac{1}{2}\sum_{j=0}^{2g+1} E_j - \sum_{j=0}^{g} \mu_j^\tau(n,t) + \frac{1}{\pi}\int_{\sigma(H_q)} \arg\left(1 + R_\pm(\lambda,t)\frac{\psi_\pm(\lambda,n,t)}{\overline{\psi_\pm(\lambda,n,t)}}\right) d\lambda.$$

6.2 Perturbation determinants, Krein's ξ-function

Finally, we want to derive the connection between the transmission coefficient $T(z)$ and Krein's spectral shift theory [30]. Therefore we investigate the derivative of $\alpha(z) = T^{-1}(z)$. The same proof as for Lemma 2.11 yields

$$\begin{aligned} \frac{d}{dz}\alpha(z) &= -\alpha(z)\sum_{n\in\mathbb{Z}} \big(G(z,n,n,t) - G_q(z,n,n,t)\big) \\ &= -\alpha(z)\mathrm{tr}\big((H(t)-z)^{-1} - (H_q(t)-z)^{-1}\big), \end{aligned} \tag{6.5}$$

where $G(\lambda,.,.,t)$, $G_q(\lambda,.,.,t)$ are the Green functions of $H(t)$, $H_q(t)$.

A bounded linear operator A is called *trace class* if and only if

$$\mathrm{tr}|A| := \mathrm{tr}\sqrt{A^* A} < \infty,$$

where tr denotes the trace. The operator $H - H_q$ is trace class since the multiplication operators by $a(n) - a_q(n)$, $b(n) - b_q(n)$ are trace class due to Hypothesis H.2.1 and trace class operators form an ideal in the Banach space of bounded linear operators. Therefore we can rephrase (6.5) as

$$-\frac{d}{dz}\ln\alpha(z) = \mathrm{tr}\big((H(t)-z)^{-1} - (H_q(t)-z)^{-1}\big), \tag{6.6}$$

which shows that $\alpha(z)$ is the *perturbation determinant* of the pair $H(t)$, $H_q(t)$ in the sense of Krein [30] up to a constant, that is,

$$\begin{aligned} C\alpha(z) &= \det\big[(H(t)-z)(H_q(t)-z)^{-1}\big] \\ &= \det\big[I + (H(t)-H_q(t))(H_q(t)-z)^{-1}\big]. \end{aligned}$$

The asymptotic behavior of $\alpha(z)$ which is given by (2.42)

$$\lim_{z\to\pm\infty} A\alpha(z) = 1, \qquad A = \prod_{j\in\mathbb{Z}} \frac{a(j)}{a_q(j)},$$

and the asymptotic behavior of the Green functions $G(z,n,n,t)$, $G_q(z,n,n,t)$ imply that $\lim_{z\to\pm\infty} C\alpha(z) = \det[I] = 1$ and hence $C = A$. By [30], Theorem 1, the perturbation determinant $A\alpha(z)$ has the following representation

$$\alpha(z) = \frac{1}{A}\exp\Big(\int_{\mathbb{R}} \frac{\xi_\alpha(\lambda)d\lambda}{\lambda - z}\Big),$$

where

$$\xi_\alpha(\lambda) = \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\arg\alpha(\lambda + i\epsilon).$$

Using Neumann's expansion we infer

$$\operatorname{tr}\big((H(t)-z)^{-1}-(H_q(t)-z)^{-1}\big) = -\sum_{j=0}^{\infty} \frac{\operatorname{tr}(H(t)^j - H_q(t)^j)}{z^{j+1}}$$

and

$$\ln A\alpha(z) = -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_{\mathbb{R}} \lambda^j \xi_\alpha(\lambda) d\lambda.$$

Hence expanding both sides of (6.6) yields the trace formula

$$\operatorname{tr}(H(t)^j - H_q(t)^j) = j \int_{\mathbb{R}} \lambda^{j-1} \xi_\alpha(\lambda) d\lambda.$$

The function $\xi_\alpha(\lambda)$ is called *spectral shift function* of the pair $(H(t), H_q(t))$.

Appendix A

Periodic Jacobi operators

A.1 Jacobi operators with periodic coefficients

For the convenience of the reader we recall some facts from Floquet theory and give explicit formulas for the fundamental solutions, the Green function, et cetera. For a detailed treatment see [7], Appendix B, and [48], Chapter 7.

To the assumption a_p, $b_p \in \ell^\infty(\mathbb{Z},\mathbb{R})$, $a(n) \neq 0$ for $n \in \mathbb{Z}$, in (1.1), (1.2) we now add the periodicity condition.

Hypothesis H. A.1. *Suppose there is an $N \in \mathbb{N}$ such that*

$$a_p(n+N) = a_p(n), \qquad b_p(n+N) = b_p(n). \tag{A.1}$$

Associated with a_p, b_p is the *periodic* Jacobi operator H_p. We abbreviate

$$A_p = \prod_{j=1}^{N} a_p(n_0+j) = \prod_{j=1}^{N} a_p(j).$$

We introduce the following *fundamental solutions* (also called sin- and cos- solutions) c, $s \in \ell(\mathbb{Z})$ of the periodic Jacobi difference equation

$$\tau_p c_p(z,.\,,n_0) = z c_p(z,.\,,n_0), \qquad \tau_p s_p(z,.\,,n_0) = z s_p(z,.\,,n_0), \tag{A.2}$$

fulfilling the initial conditions

$$c_p(z,n_0,n_0) = s_p(z,n_0+1,n_0) = 1, \qquad c_p(z,n_0+1) = s_p(z,n_0,n_0) = 0.$$

For $n > n_0$, $s_p(z,n,n_0)$ is a polynomial in z of degree $n-1-n_0$ ([48], (1.68)) and

$$\begin{aligned} s_p(z,n_0+N,n_0) &= \frac{a_p(n_0)}{A_p}\prod_{j=1}^{N-1}(z-\mu_j(n_0)), \\ c_p(z,n_0+N+1,n_0) &= \frac{-a_p(n_0)}{A_p}\prod_{j=1}^{N-1}(z-\mu_j(n_0+1)). \end{aligned} \tag{A.3}$$

We recall the *fundamental matrix*

$$\Phi(z,n,n_0) = \begin{pmatrix} c_p(z,n,n_0) & s_p(z,n,n_0) \\ c_p(z,n+1,n_0) & s_p(z,n+1,n_0) \end{pmatrix}$$

and investigate what happens if we move on N steps, that is, we look at the *monodromy matrix* $M(z, n_0) = \Phi(z, n_0 + N, n_0)$. Using periodicity (A.1) one can find a periodic matrix $Q(z, n_0)$ such that

$$M(z, n_0) = \exp(\mathrm{i}NQ(z, n_0)), \qquad \operatorname{tr} Q(z, n_0) = 0.$$

The *Hill discriminant*

$$\Delta(z) = \frac{1}{2}\operatorname{tr} M(z, n_0) = \frac{1}{2}\Big(c_p(z, n_0 + N, n_0) + s_p(z, n_0 + N + 1, n_0)\Big) \tag{A.4}$$

and the eigenvalues of $M(z, n_0)$

$$m^{\pm}(z) = \exp(\pm \mathrm{i}Nq(z)) = \Delta(z) \pm (\Delta(z)^2 - 1)^{1/2} \tag{A.5}$$

are independent of n_0. The eigenvalues $m^{\pm}(z)$ and the function $q(z)$ are called *Floquet multipliers* and *Floquet momentum*, respectively. The branch in the above root is fixed as

$$(\Delta(z)^2 - 1)^{1/2} = -\frac{1}{2A_p}\prod_{j=0}^{2N-1}\sqrt{z - E_j},$$

where $\{E_j\}_{j=0}^{2N-1}$ are the zeros of $\Delta(z)^2 - 1$. Note that

$$\begin{aligned} m^+(z)m^-(z) &= 1, \\ m^+(z) - m^-(z) &= 2\mathrm{i}\sin(Nq(z)) = 2(\Delta(z)^2 - 1)^{1/2}, \\ m^+(z) + m^-(z) &= 2\cos(Nq(z)) = 2\Delta(z), \end{aligned} \tag{A.6}$$

therefore $q(z) = N^{-1}\arccos\Delta(z)$. The spectrum of H_p is characterized by

$$\sigma(H_p) = \{\lambda \in \mathbb{R} \mid |\Delta(\lambda)| \le 1\} = \bigcup_{j=0}^{N-1}[E_{2j}, E_{2j+1}]. \tag{A.7}$$

It is purely absolutely continuous, that is, $\sigma_p(H_p) = \sigma_{sc}(H_p) = \emptyset$, where $\sigma_p(.)$ and $\sigma_{sc}(.)$ denote the point (set of eigenvalues) and singular continuous spectrum. The sets

$$\overline{\upsilon}_0 = (-\infty, E_0], \qquad \overline{\upsilon}_j = [E_{2j-1}, E_{2j}], \qquad \overline{\upsilon}_N = [E_{2N-1}, \infty), \qquad 1 \le j \le N-1,$$

are called spectral gaps. Using the Herglotz property (cf. [48], Chapter 7) we infer for the zeros $\mu_j(n)$ of the fundamental solutions (A.3) that

$$\mu_j(n) \in \overline{\upsilon}_j, \qquad 1 \le j \le N-1.$$

If all spectral gaps are open (as we assumed in the quasi-periodic setting, i.e. (1.4)), we have

$$g = N - 1, \qquad R_{2g+2}^{1/2}(z) = 2A_p(\Delta(z)^2 - 1)^{1/2}.$$

In the case where some spectral gaps are closed, the index sets

$$\begin{aligned} J' &= \{1 \le j' \le N-1 \mid E_{2j'-1} = E_{2j'}\}, \\ J &= \{0, 1, \dots, 2N-1\}\backslash\{j', j'+1 \mid j' \in J'\}, \end{aligned}$$

are introduced and we define

$$Q(z) = \prod_{j' \in J'} (z - E_{2j'-1}), \qquad R_{2g+2}(z) = \prod_{j \in J} (z - E_j). \tag{A.8}$$

Then one infers

$$g = N - 1 - |J'| = N - 1 - \deg(Q) = (|J| - 2)/2,$$
$$2A_p(\Delta(z)^2 - 1)^{1/2} = R_{2g+2}^{1/2}(z)Q(z).$$

The solutions of the periodic Jacobi equation are completely characterized by Floquet's theorem. In what follows, we set $n_0 = 0$ and omit it.

Theorem A.2. *The solutions of $\tau_p u = zu$ can be characterized as follows.*

(i). *If $\Delta(z)^2 \neq 1$, there exist two* Floquet solutions

$$\psi_{p,\pm}(z,n) = c_p(z,n) + m_\pm(z) s_p(z,n),$$

satisfying

$$\psi_{p,\pm}(z,n) = p_\pm(z,n) e^{\pm \mathrm{i} q(z) n}, \qquad p_\pm(z, n+N) = p_\pm(z,n).$$

(ii). *If $\Delta(z)^2 = \pm 1$, then either all solutions satisfy $\psi_p(z, n+N) = \pm\psi_p(z,n)$ or there are two solutions satisfying*

$$\psi_p(z,n) = p(z,n), \qquad \hat{\psi}_p(z,n) = \hat{p}(z,n) + np(z,n)$$

with $p(z,n+N) = \pm p(z,n)$ and $\hat{p}(z,n+N) = \pm\hat{p}(z,n)$.

We recall the zeros of $s(z,N) := s(z,N,0)$, $\mu_j := \mu_j(0)$, $1 \le j \le N-1$, and we denote the zeros of $dq(z)/dz$ by λ_j, $1 \le j \le N-1$. By (A.6),

$$\frac{d\Delta(z)}{dz} = -N \sin(Nq(z)) \frac{dq(z)}{dz},$$

so the numbers λ_j are also the zeros of the derivative of the Hill discriminant $d\Delta(z)/dz$. They are indeed the analogous quantities to the constants λ_j defined in (1.17) since

$$\begin{aligned} \Big| \int_{a_j} \omega_{\infty_+,\infty_-} \Big| &= \frac{2}{N} \Big| \int_{E_{2j-1}}^{E_{2j}} \frac{\Delta'(z)}{(\Delta^2(z)-1)^{1/2}} \Big| \\ &= \frac{2}{N} \Big| \ln\big(\Delta(z) + (\Delta^2(z)-1)^{1/2}\big) \Big|_{E_{2j-1}}^{E_{2j}} = 0, \qquad 1 \le j \le g. \end{aligned}$$

The leading coefficient of the polynom $\Delta'(z)$ is derived from (A.4)

$$\Delta'(z) = \frac{1}{2} \Big(\frac{z^N}{A_p} + O(z^{N-1}) \Big)' = \frac{N}{2A_p} \prod_{j=1}^{N-1} (z - \lambda_j)$$

and one infers for the norm (cf. (1.14))

$$\sum_{n=1}^{N} |\psi_{p,\pm}(z,n)|^2 = \frac{2a_p(0)\Delta'(z)}{s_p(z,N)} = N \prod_{j=1}^{N-1} \frac{z - \lambda_j}{z - \mu_j}, \qquad z \in \sigma(H_p). \tag{A.9}$$

In addition, we have

$$\begin{aligned}\phi_\pm(z,n_0) = \phi_\pm(z,n_0+N) &= \frac{m^\pm(z) - c_p(z,n_0+N,n_0)}{s_p(z,n_0+N,n_0)}\\ &= \frac{c_p(z,n_0+N+1,n_0)}{m^\pm(z) - s_p(z,n_0+N+1,n_0)}\\ &= \frac{\frac{1}{2}(s_p(z,n_0+N+1,n_0) - c_p(z,n_0+N,n_0)) \pm (\Delta(z)^2-1)^{1/2}}{s_p(z,n_0+N,n_0)}\end{aligned}$$

and

$$\begin{aligned}W(\psi_{p,\pm}(z,.,n_0),\psi_{p,\mp}(z,.,n_0)) &= \mp\frac{2a_p(n_0)(\Delta(z)^2-1)^{1/2}}{s_p(z,n_0+N,n_0)}\\ &= \pm\frac{\prod_{j=0}^{2N-1}\sqrt{z-E_j}}{\prod_{j=1}^{N-1}(z-\mu_j(n_0))}.\end{aligned}$$

The Green function on the diagonal is given by

$$G(z,n,n) = \frac{s_p(z,n+N,n)}{2a_p(n)(\Delta(z)^2-1)^{1/2}} = -\frac{\prod_{j=1}^{N-1}(z-\mu_j(n))}{\prod_{j=0}^{2N-1}\sqrt{z-E_j}}.$$

Moreover,

$$\psi_+(z,n,n_0)\psi_-(z,n,n_0) = \frac{a(n_0)s_p(z,n+N,n)}{a(n)s_p(z,n_0+N,n_0)} = \prod_{j=1}^{N-1}\frac{z-\mu_j(n)}{z-\mu_j(n_0)}.$$

A.2 A transformation operator

In this section we want to advocate a different approach to the transformation operator for perturbations of periodic Jacobi operators. It follows the method presented in [48], Section 10.1, for Jacobi operators with constant background. Unfortunately, we did not find an estimate on the kernel of this transformation operator which would allow us to pursue this approach further. Nevertheless, the formulas might be of interest on their own.

Again we study short-range perturbations H of periodic Jacobi operators H_p. The associated coefficients a, b satisfy $a(n) \to a_p(n)$ and $b(n) \to b_p(n)$ as $|n| \to \infty$ with the following rate of convergence.

Hypothesis H. A.3. *Suppose a_p, b_p are given periodic sequences and H_p is the corresponding Jacobi operator. Let H be a perturbation of H_p such that*

$$\sum_{n\in\mathbb{Z}} |n|\Big(|a(n)-a_p(n)| + |b(n)-b_p(n)|\Big) < \infty. \tag{A.10}$$

The existence of solutions $u_\pm(z,n)$ of the perturbated system $\tau u = zu$ which asymptotically look like the periodic ones follows from Lemma 2.3. This was first derived in [48], Lemma 7.10. Now assume the following ansatz for the solution $u(n)$ of $\tau u = zu$ (we only consider the "+" case here)

$$u(n) = \sum_{j=n}^{\infty}\frac{1}{A(n,j)}K(n,j)\psi_p(j), \tag{A.11}$$

where $\psi_p(n)$ are the solutions of the periodic system $\tau_p u = zu$, $K(n,m)=0$ if $m<n$, and

$$A(n,n+j) = A_j(n) = \prod_{m=n}^{\infty} \frac{a(m)}{a_p(m+j)} = \prod_{m=n}^{n+j-1} a_p(m) \prod_{m=n}^{\infty} \frac{a(m)}{a_p(m)}.$$

Inserting the ansatz (A.11) into the Jacobi difference equation

$$a(n)u(n+1) + b(n)u(n) + a(n-1)u(n-1) = zu(n)$$

we obtain

$$\begin{aligned}
& a(n) \sum_{j=n+1}^{\infty} \frac{K(n+1,j)}{A(n+1,j)} \psi_p(j) + b(n) \sum_{j=n}^{\infty} \frac{K(n,j)}{A(n,j)} \psi_p(j) \\
& \quad + a(n-1) \sum_{j=n-1}^{\infty} \frac{K(n-1,j)}{A(n-1,j)} \psi_p(j) = \sum_{j=n}^{\infty} \frac{K(n,j)}{A(n,j)} z\psi_p(j) \\
& = \sum_{j=n}^{\infty} \frac{K(n,j)}{A(n,j)} \big(a_p(j)\psi_p(j+1) + b_p(j)\psi_p(j) + a_p(j-1)\psi_p(j-1)\big) \\
& = \sum_{j=n}^{\infty} \Big(\frac{K(n,j-1)}{A(n,j-1)} a_p(j-1) + \frac{K(n,j)}{A(n,j)} b_p(j) \\
& \quad + \frac{K(n,j+1)}{A(n,j+1)} a_p(j) \Big) \psi_p(j) + \frac{K(n,n)}{A(n,n)} a_p(n-1)\psi_p(n-1).
\end{aligned}$$

Set $K(n,n+i) = K_i(n)$ and $A(n,n+i) = A_i(n)$. We compare coefficients of $\psi_p(.)$ and obtain for $\psi_p(n-1)$ that $K_0(n) := K(n,n) = 1$ and for $\psi_p(j)$ that

$$\begin{aligned}
& a(n) \frac{K_{j-1}(n+1)}{A_{j-1}(n+1)} + b(n) \frac{K_j(n)}{A_j(n)} + a(n-1) \frac{K_{j+1}(n-1)}{A_{j+1}(n-1)} \\
& = a_p(n+j-1) \frac{K_{j-1}(n)}{A_{j-1}(n)} + b_p(n+j) \frac{K_j(n)}{A_j(n)} + a_p(n+j) \frac{K_{j+1}(n)}{A_{j+1}(n)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
K_{j+1}(n) - K_{j+1}(n-1) \quad = \quad & (b(n) - b_p(n+j))K_j(n) \\
& - a_p(n+j-1)^2 K_{j-1}(n) + a(n)^2 K_{j-1}(n+1).
\end{aligned} \tag{A.12}$$

Since $\lim_{n\to\infty} K_j(n) = 0$, $j \in \mathbb{N}$, summing up (A.12) with respect to n yields

$$\begin{aligned}
K_1(n) \quad &= \quad - \sum_{m=n+1}^{\infty} (b(m) - b_p(m)) \\
K_2(n) \quad &= \quad - \sum_{m=n+1}^{\infty} \big((b(m) - b_p(m+1))K_1(m) + (a(m)^2 - a_p(m)^2) \big) \\
K_{j+1}(n) \quad &= \quad a_p(n+j)^2 K_{j-1}(n+1) - \sum_{m=n+1}^{\infty} \Big((b(m) - b_p(m+j))K_j(m) \\
& \qquad + \big(a(m)^2 - a_p(m+j)^2\big) K_{j-1}(m+1) \Big).
\end{aligned} \tag{A.13}$$

These are the generalizations of the formulas (10.9) in [48]. However, the terms $b(m)-b_p(m+j)$, $a(m)^2-a_p(m+j)^2$ pose additional difficulties in estimating $K_j(n)$.

Associated with $K_j(n)=K(n,n+j)$ is the operator

$$(\mathcal{K}f)(n)=\sum_{m=n}^{\infty}\frac{1}{A(n,m)}K(n,m)f(m), \qquad f\in\ell_+^\infty(\mathbb{Z}).$$

which acts as a *transformation operator* for the pair τ_p, τ,

$$\tau\mathcal{K}f=\mathcal{K}\tau_p f, \qquad f\in\ell_+^\infty(\mathbb{Z}).$$

Finally, we note the following result for the Volterra sum equation.

Lemma A.4. *Consider the Volterra sum equation*

$$f(n)=g(n)+\sum_{m=n+1}^{\infty}K(n,m)f(m). \qquad \text{(A.14)}$$

Then

$$f(n)=\sum_{j=0}^{\infty}L_j(n)g(n+j),$$

where the coefficients $L_j(n)$ *of* $g(n+j)$ *are given by*

$$\begin{aligned} L_0(n) &= 1,\\ L_j(n) &= \sum_{m=0}^{j-1}L_m(n)K_{j-m}(n+m), \qquad j\geq 1, \end{aligned} \qquad \text{(A.15)}$$

with $K_j(n)=K(n,n+j)$.

Proof. Using the standart iteration trick the solution of (A.14) is formally given by

$$f(n)=\sum_{j=0}^{\infty}f_j(n)$$

with

$$f_0(n)=g(n), \qquad f_{j+1}(n)=\sum_{m=n+1}^{\infty}K(n,m)f_j(m).$$

We prove (A.15) by induction. We have

$$f(n)=g(n)+f_1(n)+\dots=g(n)+K(n,n+1)g(n+1)+O(g(n+2)),$$

hence

$$L_1(n)=K_1(n).$$

The term $g(n+j)$ only appears in the first j summands, therefore its coefficient $L_j(n)$ is completely determined by $f_1(n),\dots,f_j(n)$

$$\begin{aligned} \sum_{k=0}^{j}f_k(n) &= \sum_{k=0}^{j}L_k(n)g(n+k)+O(g(n+j+1))\\ &= f_0(n)+K_1(n)\sum_{l=0}^{j-1}f_l(n+1)+K_2(n)\sum_{l=0}^{j-1}f_l(n+2)+\dots\\ &= f_0(n)+K_1(n)\Big(\sum_{l=0}^{j-1}L_l(n+1)g(n+1+l)+O(g(n+j+1))\Big)+\dots \end{aligned}$$

Comparing the coefficients of $g(n+.)$ yields

$$L_0(n) = 1, \qquad L_1(n) = K_1(n),$$

$$L_j(n) = K_1(n)L_{j-1}(n+1) + K_2(n)L_{j-2}(n+2) + \cdots + K_j(n)L_0(n+j).$$

By induction, we replace $L_{j-k}(n+k)$ with (A.15) and collect the terms according to $K_{j-k}(n+k)$. This leads to the desired result. □

Bibliography

[1] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory*, Akademie-Verlag Berlin, 1983.

[2] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, no. 3, 425–442 (2003).

[3] Yu.M. Berezanskii, *Expansions in Eigenfunctions of Self-Adjoint Operators*, Trans. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, Rhode Island, 1968.

[4] A. Boutet de Monvel, I. Egorova, and E. Khruslov, *Soliton asymptotics of the Cauchy problem solution for the Toda lattice*, Inverse Problems **13**, 223–237 (1997).

[5] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, 955–977 (2000).

[6] A. Boutet de Monvel and I. Egorova, *Transformation Operator for Jacobi Matrices with Asymptotically Periodic Coefficients*, J. Difference Equ. and Appl. **10/8**, 711–727 (2004).

[7] W. Bulla, F. Gesztesy, H. Holden and G. Teschl, *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, Memoirs of the Amer. Math. Soc. **135/641**, 1998.

[8] K.M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[9] K.M. Case, *The discrete inverse scattering problem in one dimension*, J. Math. Phys. **15**, 143–146 (1974).

[10] K.M. Case, *Orthogonal polynomials II*, J. Math. Phys. **16**, 1435–1440 (1975).

[11] K.M. Case, *On discrete inverse scattering problems. II*, J. Math. Phys. **14**, 916–920 (1973).

[12] K.M. Case and S.C. Chiu *The discrete version of the Marchenko equations in the inverse scattering problem*, J. Math. Phys. **14**, 1643–1647 (1973).

[13] K.M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14**, 594–603 (1973).

[14] P. Deift, L.C. Li, and C. Tomei, *Toda flows with infinitely many variables*, J. Func. Anal. **64**, 358–402 (1985).

[15] I. Egorova, *On the asymptotically periodic solutions of the Toda lattice*, Preprint.

[16] I. Egorova, J. Michor, and G. Teschl, *Scattering theory for Jacobi operators with quasi-periodic background*, in preparation.

[17] H. Farkas and I. Kra, *Riemann Surfaces*, GTM 71, Springer, New York, 1980.

[18] E. Fermi, J. Pasta, S. Ulam, *Studies of Nonlinear Problems*, Collected Works of Enrico Fermi, University of Chicago Press, Vol. II, 978–988, 1965. Theory, Methods, and Applications, 2nd ed., Marcel Dekker, New York, 2000.

[19] N.E. Firsova, *The direct and inverse scattering problems for the one-dimensional perturbed Hill operator*, Math. USSR, Sb. **58**, 351–388 (1987).

[20] H. Flaschka, *On the Toda lattice. I*, Phys. Rev. B **9**, 1924–1925 (1974).

[21] H. Flaschka, *On the Toda lattice. II*, Progr. Theoret. Phys. **51**, 703–716 (1974).

[22] O. Forster, *Riemannsche Flächen*, HTB 184, Springer, Heidelberg, 1977.

[23] C.S. Gardner, J.M. Green, M.D. Kruskal, and R. M. Miura, *A method for solving the Korteweg-de Vries equation*, Phys. Rev. Letters **19**, 1095–1097 (1967).

[24] F. Gesztesy and B. Simon, *The xi function*, Acta Math. **176**, 49–71 (1996).

[25] F. Gesztesy, R. Nowell, and W. Pötz, *One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics*, Differ. Integral Equ. **10**, No.3, 521–546 (1997).

[26] J.S. Geronimo and W. Van Assche, *Orthogonal polynomials with asymptotically periodic recurrence coefficients*, J. App. Th. **46**, 251–283 (1986).

[27] G.S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl. **17**, 1684–1688 (1976).

[28] G.S. Guseinov, *The determination of an infinite Jacobi matrix from the scattering data*, Soviet Math. Dokl. **17**, 596–600 (1976).

[29] G.S. Guseinov, *Scattering problem for the infinite Jacobi matrix*, Izv. Akad. Nauk Arm. SSR, Mat. **12**, 365–379 (1977).

[30] M.G. Krein, *Perturbation determinants and a formula for the traces of unitary and self-adjoint operators*, Soviet. Math. Dokl. **3**, 707–710 (1962).

[31] I.M. Krichever, *Nonlinear equations and elliptic curves*, Revs. Sci. Tech. **23**, 51–90 (1983).

[32] E.A. Kuznetsov and A.V. Mikhaĭlov, *Stability of stationary waves in nonlinear weakly dispersive media*, Ž. Èksper. Teoret. Fiz. **67** no. 5, 1717–1727 (1974), in Russian.

[33] P. Lax, *Scattering Theory*, Academic Press, New York, 1967.

[34] B.M. Levitan, *Inverse Sturm-Liouville Problems*, VNU, Utrecht, 1987.

[35] B.M. Levitan and I.S. Sargsjan, *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators*, Amer. Math. Soc., Providence, Rhode Island, 1975.

[36] V.A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[37] R. Narasimhan, *Complex Analysis in One Variable*, Birkhäuser, Boston, 1985.

[38] T. Parthasarathy, *On Global Univalence Theorems*, LNM 577, Springer, Berlin, 1983.

[39] L. Percolab, *The inverse problem for the periodic Jacobi matrix*, Theor. funk., funk. an., pril. **42**, 107–121 (1984), in Russian.

[40] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, rev. and enl. edition, Academic Press, San Diego, 1980.

[41] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, Academic Press, San Diego, 1975.

[42] M. Reed and B. Simon, *Methods of Modern Mathematical Physics III. Scattering Theory*, Academic Press, San Diego, 1979.

[43] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV. Analysis of Operators*, Academic Press, San Diego, 1978.

[44] G. Teschl, *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Diff. Eqs. **129**, 532–558 (1996).

[45] G. Teschl, *Trace formulas and inverse spectral theory for Jacobi operators*, Comm. Math. Phys. **196-1**, 175–202 (1998).

[46] G. Teschl, *Inverse Scattering Transform for the Toda Hierarchy*, Math. Nach. **202**, 163–171 (1999).

[47] G. Teschl, *On the initial value problem of the Toda and Kac-van Moerbeke hierarchies*, AMS/IP Studies in Advanced Mathematics **16**, 375–384 (2000).

[48] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. **72**, Amer. Math. Soc., Rhode Island, 2000.

[49] G. Teschl, *Almost Everything You Always Wanted to Know About the Toda Equation*, Jahresbericht der DMV 2001.

[50] M. Toda, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.

[51] M. Toda, *Theory of Nonlinear Waves and Solitons*, Kluwer, Dordrecht, 1989.

[52] M. Tsuji, *Potential Theory in modern Functional Analysis*, Maruzen, Tokyo, 1959.

[53] V. Voichick and L. Zalcman, *Inner and outer functions on Riemann surfaces*, Proc. Amer. Math. Soc. **16**, 1200–1204 (1965).

[54] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite systems of intervals or a Cantor set of positive length*, Commun. Math. Phys. **226**, 567–605 (2002).

[55] G.N. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.

[56] D.R. Yafaev, *Mathematical Scattering Theory: General Theory*, Amer. Math. Soc., Rhode Island, 1992.

[57] D.R. Yafaev, *Scattering Theory: Some Old and New Problems*, Springer-Verlag, Berlin, 2000.

[58] N.J. Zabusky and M.D. Kruskal, *Interaction of solitons in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett. **15**, 240–243 (1963).

Curriculum Vitae

Born on December 24, 1979, in Vienna, Austria, as the eldest daughter of Elli and Dr. Peter Michor, I grew up in Kritzendorf, Lower Austria, where I also went to primary school. From 1990 to 1998 I attended the Bundesgymnasium Klosterneuburg.

Academic career. In October 1998 I began to study mathematics and physics at the University of Vienna. For the academic year 2001/02 I received a grant (*Leistungsstipendium*) of the Faculty of Natural Science and Mathematics. I finished my mathematical studies with distinction in July 2002, obtaining the academic degree *Magister der Naturwissenschaften* (MSc). For my diploma thesis supervised by Prof. G. Teschl I was awarded the *Studienpreis 2003* of the *Austrian Mathematical Society*.

Since September 2002 I am working on my doctoral thesis under the guidance of Prof. G. Teschl. For the academic year 2003/04 I was awarded a Ph.D. scholarship of the *Austrian Academy of Sciences (ÖAW)* under DOC-21388. I spent September 2004 at Stanford University, California. Since December 2004 I have a research position at the University of Vienna (project P-17762 of the *Austrian Science Fund (FWF)*, 'Streutheorie für Jacobioperatoren', Prof. G. Teschl).

Since March 2002 I am a teaching assistant (tutor) at the Faculty of Mathematics, University of Vienna (SS02, WS02, SS03: Literaturstudium, WS02: Workshop (Einführung in das mathematische Arbeiten), WS03: Repetitorium zu Analysis 1, SS04: Repetitorium zu Analysis 2, WS04: Übungen zur technischen Praxis der Computersysteme).

Other employments. From April to September 2003 I worked part time at B&C Consulting, Prof. W. Ettl, in the field of financial mathematics.

List of Publications.

1. *Trace formulas and inverse spectral theory for finite Jacobi operators*, diploma thesis, 2002.
2. *Reconstructing Jacobi matrices from three spectra*, with G. Teschl, in Spectral Methods for Operators of Mathematical Physics, J. Janas, P. Kurasov, and S. Naboko (eds.), 151-154, Oper. Theory Adv. Appl. **154**, Birkhäuser, Basel, 2004.
3. *Scattering theory for Jacobi operators and applications to completely integrable systems*, doctoral thesis, 2005.
4. *Scattering theory for Jacobi operators with quasi-periodic background*, with I. Egorova and G. Teschl, in preparation.

Scientific talks.

1. April 15, 2005: *"Scattering theory for Jacobi operators with quasi-periodic background"*, Austrian Academy of Sciences, Vienna.
2. June 18, 2005: *"Scattering theory for Jacobi operators with quasi-periodic background"*, AMS/DMV/ÖMG meeting in Mainz, Germany, June 16-19, 2005.