universität wien

DIPLOMARBEIT

Relative Oscillation Theory for Sturm–Liouville Operators

Zur Erlangung des akademischen Grades

Magister der Naturwissenschaften (Mag. rer. nat.)

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Wien, am 19. 12. 2006

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CHAPTER 0

Introduction

We call an operator of the form:

$$\tau = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right) \tag{1}$$

for functions r, p, q a Sturm–Liouville operator. Sturm–Liouville operators arise for example when considering the radial part of the Laplacian of a rotation symmetric problem in any dimension. Sturm–Liouville equations of the type $-f''(x) + q(x)f(x) = \lambda f(x)$ arise in quantum mechanics and are called one-dimensional Schrödinger equations. Periodic Sturm–Liouville equations are for example used as one dimensional crystal models (e.g. the Kronig-Penney model).

The aim of oscillation theory is to relate the number of zeros of solutions of differential equations to spectral parameters of the associated self-adjoint operators. The research into oscillation theory started with Sturm's celebrated memoir [21] in the 19th century, and has continuously been extended since then. To name a result from it, the following holds. If we order the eigenvalues of a self-adjoint operator H associated with τ with separated boundary conditions as an increasing sequence: $\lambda_0 < \lambda_1 < \lambda_2 < \dots$, then the eigenfunction corresponding to λ_n has exactly n zeros. We will state this result in Lemma 1.13.

The aim of my master thesis is to develop a form of oscillation theory for Wronskians. This development was already started by Fritz Gesztesy, Barry Simon, and my advisor Gerald Teschl in [4]. The main difference of the approach taken here is to use solutions to different operators, instead of to different eigenvalues as is done in [4]. With the results given here Gerald Teschl and I were able to extend the known oscillation results for periodic Sturm–Liouville operators from [16].

The content of this thesis is composed as follows.

Chapter 1 summarizes some ideas from classical theory of Sturm–Liouville operators. Section 1.4 states some results of classical oscillation theory.

In Chapter 2, we mainly discuss properties of trace class operators. We will need this to obtain the convergence of approximating problems.

Chapter 3 gives an introduction to Krein's Spectral Shift function and its properties. Lemma 3.10 is at our best knowledge not found in literature.

Chapter 4 derives the main properties of the Wronskian. Most results of this chapter are new. They have been already investigated in [4] and [23] in the case of different eigenvalues. However, adapting these results to different potentials required new ideas. We wish to emphasise here that the results on Sturm-type theorems for Wronskians in Section 4.2 are of interest. Most other parts of this chapter are technical, and mainly used to proof our results.

The main new concept of the thesis will be presented in Chapter 5. Again at our best knowledge, all results are new. Also the method to obtain the result by the convergence of approximating potentials with compact support used in the proof of Theorem 5.7 has not been found in literature. The name of relative oscillation theory comes from the works of Karl-Michael Schmidt (e.g. [16]).

The applications in Chapter 6 are at best knowledge new in the generality.

Most of the work described in this thesis will be presented as joint work with Gerald Teschl in [11]. For simplicity I will restrict attention in my thesis to the case of relatively bounded perturbations and only treat the case using the spectral shift function. I further wish to point out that Remark 4.6 and Section 5.3 might be the starting point for further research in the field.

Thanks

I wish to thank my advisor Gerald Teschl for all the support I had, when writing this, and my colleague and friend Johanna Michor for proof reading. Finally, I thank my girlfriend Thérèse Tomiska for legal advice and being the person she is. Last but not least, I wish to thank my family.

This work was supported by the Austrian Science Fond (FWF) under grants P-17762 and Y-330 and the Faculty of Mathematics of the University of Vienna, which provided me with excellent working conditions.

I also have the pleasure to thank Dimitri Yafaev for helpful discussions on Lemma 3.10.

Errata

This version differs a bit from the one, I submitted as my master thesis, since I have corrected small mistakes in calculations. Furthermore, I wish to point out that the application of relative oscillation theory to periodic Sturm-Liouville operators in Section 6.2 has a mistake, since it assumes that $|D|'$ is one signed. A corrected version will appear in [11].

CHAPTER 1

A brief overview of Sturm–Liouville Operators

In this chapter, we review the classical theory of Sturm–Liouville operators.

1.1 Basic facts

For some interval $I = (a, b) \subseteq \mathbb{R}$ finite or infinite, let p, q, r be real valued functions with $p^{-1}, q, r \in L^1_{loc}(I)$ and $p, r > 0$. A Sturm–Liouville expression is a differential expression of the form:

$$\tau = \frac{1}{r(x)} \left(-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right). \tag{1.1}$$

We call f a solution of $\tau f = zf$, if f and pf' are absolutely continuous and satisfy the equation:

$$\frac{d}{dx} \begin{pmatrix} f \\ pf' \end{pmatrix} = \begin{pmatrix} 0 & 1/p \\ q - z & 0 \end{pmatrix} \begin{pmatrix} f \\ pf' \end{pmatrix}. \tag{1.2}$$

From the point of view of differential equations, the second chapter in [27] offers quite a few interesting results, as for example bounds and dependence results on parameters. We formulate the main results of interest to us in the following theorem:

Theorem 1.1. *If one gives an initial point (y_0, py'_0) and $x_0 \in I$, then the problem $\tau y = 0$ and $y(x_0) = y_0$, $py'(x_0) = py'_0$ has an unique solution defined almost everywhere. Furthermore, if q depends analytically (or continuously) on a parameter, the solution also does, in the sense that $y(x) \in \mathcal{H}(\mathbb{C})$ (or $y_n \to y$ in $L^1_{loc}(I)$). If the initial conditions are real, y is real.*

Let AC_{loc} be the space of all locally absolutely continuous functions. Then the domain of the maximal operator associated with τ is:

$$\mathfrak{D}(\tau) = \{f \in L^2(I, rdx), f, pf' \in AC_{loc}(I), \tau f \in L^2(I, rdx)\}. \tag{1.3}$$

The Wronskian W is given by

$$W_x(f, g) = f(x)(pg')(x) - (pf')(x)g(x). \tag{1.4}$$

We call an endpoint regular, if it is finite and q, p^{-1} are integrable in a neighborhood of it. If both endpoints are regular, τ is called regular. We call the endpoint a (resp. b) limit circle, if there exists $f, g \in \mathfrak{D}(\tau)$, which are linearly independent and satisfy $\lim_{x\to a} W_x(f,g) \neq 0$ (resp. $\lim_{x\to b} W_x(f,g) \neq 0$). If an endpoint is not limit circle, it is called limit point. We note that a regular implies a limit circle. We have the following theorem characterizing self-adjoint extensions of τ.

Theorem 1.2. *H is a self-adjoint extension of τ if we impose boundary conditions at limit circle endpoints. One has then that the domain of H is given by:*

$$\begin{aligned}\mathfrak{D}(H) = \{f \in \mathfrak{D}(\tau) | W_a(f, \psi_+) = 0 \text{ iff } \tau \text{ is l.c. at } a,\\ W_b(f, \psi_-) = 0 \text{ iff } \tau \text{ is l.c. at } b\}\end{aligned} \tag{1.5}$$

for some real functions $\psi_\pm \in \mathfrak{D}(\tau)$ with $W_{x_\pm}(\psi_\pm, f) \neq 0$ for some f and $x_+ = a$, $x_- = b$.

Proof. [22, Thm.9.6.]. □

These are not all self-adjoint extensions, since there exist coupled boundary conditions.

For the next theorem, we need the following notation. We call a function $f : (a,b) \to \mathbb{C}$ square integrable near a (or b), if for all $c \in (a,b)$ we have $f \in L^2((a,c))$ (or $f \in L^2((c,b))$).

Theorem 1.3 (Weyl alternative)**.** *τ is limit circle at a (resp. b) if and only if for one $z_0 \in \mathbb{C}$ all solutions of $(\tau - z_0)f = 0$ are square integrable near a (resp. b).*

Proof. [22, Thm.9.9.]. □

We will denote by $\psi_\pm(z)$ the solutions of $\tau\psi_\pm(z) = z\psi_\pm(z)$, where ψ_- (resp ψ_+) is square integrable near a (resp. b) if a (resp. b) is limit point or satisfies the appropriate boundary condition. By Theorem 1.1 and the choice of our boundary conditions, we have that $\psi_\pm(z)$ are real-valued for $z \in \mathbb{R}$. We note that we have an explicit formula for the resolvent:

Lemma 1.4. *For the resolvent of H, $R_H(z) = (H - z)^{-1}$, we have that:*

$$R_H(z)g(x) = \int_a^b G(z,x,y)g(y)r(y)dy \tag{1.6}$$

with the Green function G given by

$$G(z,x,y) = \frac{1}{W(\psi_+(z), \psi_-(z))} \begin{cases} \psi_+(z,x)\psi_-(z,y) & x \geq y \\ \psi_-(z,x)\psi_+(z,y) & x \leq y. \end{cases} \tag{1.7}$$

Proof. [22, Lem.9.7.]. □

For (a,b) a bounded subset of $\mathbb{R}$, we can estimate the L^2 of $G(z,x,y)$, since it is a bounded function. Then we get that $\|G(z,.,.)\|_{L^2((a,b)^2)} < \infty$, and by this that the resolvent is Hilbert-Schmidt. Since $R_H(z)$ is self-adjoint for real z, we obtain that $R_H(z)$ has purely discrete spectrum, and thus also H. This gives us the following result:

Theorem 1.5. *The spectrum of a Sturm–Liouville operator on a compact interval is purely discrete.*

1.2 Prüfer angles

Since we have that every non-zero real solution of $\tau u = \lambda u$ has $(u, pu') \neq (0,0)$ everywhere by the uniqueness of solutions, we can define Prüfer variables by:

$$u(x) = \rho(x) \sin \vartheta(x), \ \ (pu')(x) = \rho(x) \cos \vartheta(x). \tag{1.8}$$

The Prüfer angle ϑ is unique, if one fixes $\vartheta(x_0) \in [0, 2\pi)$ for some $x_0 \in (a,b)$ and the requirement $\vartheta \in C((a,b))$. It obeys the following differential equation

$$\vartheta'(x) = \frac{1}{p(x)} \cos^2(\vartheta(x)) - (q(x) - \lambda) \sin^2(\vartheta(x)), \tag{1.9}$$

which is independent of the Prüfer radius ρ.

Lemma 1.6. *We have that:*

$$W_x(\psi_-(\lambda), \partial_\lambda \psi_-(\lambda)) = -\int_a^x \psi_-^2(\lambda, t) r(t) dt \tag{1.10}$$

$$W_x(\psi_+(\lambda), \partial_\lambda \psi_+(\lambda)) = \int_x^b \psi_+^2(\lambda, t) r(t) dt. \tag{1.11}$$

Proof. We consider without restriction the case ψ_-. Differentiating (1.4), and integrating again, one finds:

$$W_x(\psi_-(\lambda), \psi_-(\tilde{\lambda})) = -(\tilde{\lambda} - \lambda) \int_a^x \psi_-(\lambda, t) \psi_-(\tilde{\lambda}, t) r(t) dt. \tag{1.12}$$

Now using this to evaluate the limit:

$$\lim_{\tilde{\lambda} \to \lambda} W_x \left(\psi_-(\lambda), \frac{\psi_-(\tilde{\lambda}, x) - \psi_-(\lambda, x)}{\tilde{\lambda} - \lambda} \right), \tag{1.13}$$

we obtain the result. □

Lemma 1.7. *We have, that at a fixed point the Prüfer angle of ψ_- (of ψ_+) is a strictly decreasing (increasing) function of λ.*

Proof. The derivative being increasing follows from:

$$\partial_\lambda \vartheta = -\frac{W(u, \partial_\lambda u)}{\rho^2} \tag{1.14}$$

and Lemma 1.6. □

Lemma 1.8. *At a zero of u the Prüfer angle ϑ is strictly increasing in x. We thus have, that the integer part of ϑ/π is an increasing function of x.*

Proof. Let x_0 be a zero of u, and assume without restriction, that $\vartheta(x_0) = 0$. It then follows

$$\lim_{x \to x_0} \frac{\rho(x) \sin \vartheta(x)}{x - x_0} = \frac{\rho(x_0) \cos \vartheta(x_0)}{p(x_0)} \Rightarrow \lim_{x \to x_0} \frac{\sin \vartheta(x)}{x - x_0} = \frac{1}{p(x_0)}, \tag{1.15}$$

where we used that $\lim_{x \to x_0} u(x)/(x - x_0) = u'(x_0)$. Now use this to obtain

$$\lim_{x \to x_0} \frac{\vartheta(x) - \vartheta(x_0)}{x - x_0} = \frac{1}{p(x_0)}, \tag{1.16}$$

which implies the result since $p > 0$. □

Let in the following denote by $\#(u)$ the number of sign changes of u in I. Since $u(x_0) \Rightarrow pu'(x_0) \neq 0$, we have that $\#(u)$ is equal to the number of zeros of u. A further description is given by the next lemma.

Lemma 1.9. *We have that:*

$$\#(u) = \lim_{x \uparrow b} \lceil \vartheta_u(x)/\pi \rceil - \lim_{x \downarrow a} \lfloor \vartheta_u(x)/\pi \rfloor - 1, \tag{1.17}$$

where $\lfloor x \rfloor = \sup\{n \in \mathbb{Z}, n \leq x\}$, $\lceil x \rceil = \inf\{n \in \mathbb{Z}, n \geq x\}$. *We can drop the limit if a (resp. b) is regular.*

Proof. The proof follows from the fact that the integer part of ϑ/π is an increasing function of x. □

1.3 Spectral theory

In this section, we recall a few facts about spectral theory. Let in the following $\mathfrak{H}$ denote a Hilbert space and $\mathfrak{L}(\mathfrak{H})$ the bounded linear operators from it into it. First in the general setting of $A : \mathfrak{H} \to \mathfrak{H}$ a self-adjoint operator, one defines the resolvent set as:

$$\rho(A) = \{z \in \mathbb{C} | (A - z)^{-1} \in \mathfrak{L}(\mathfrak{H})\}. \tag{1.18}$$

Then the resolvent is defined as $R_A : \rho(A) \to \mathfrak{L}(\mathfrak{H}), z \mapsto (A-z)^{-1}$. The resolvent turns then out to be a homomorphic function in $\rho(A)$. The complement of the resolvent set is called the spectrum $\sigma(A)$:

$$\sigma(A) = \mathbb{C} \setminus \rho(A). \tag{1.19}$$

For A self-adjoint we have that $\sigma(A) \subseteq \mathbb{R}$. To recall the notion, λ is an eigenvalue, if:

$$\exists \psi : \; A\psi = \lambda\psi. \tag{1.20}$$

For our purposes, we will split the spectrum into two parts, the discrete spectrum

$$\sigma_d(A) = \{\lambda \in \sigma(A), \exists \varepsilon > 0 : \dim \operatorname{Ran} P_{(\lambda-\varepsilon, \lambda+\varepsilon)}(A) < \infty\}, \tag{1.21}$$

and the essential spectrum $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A)$. Since our focus will be on the discrete spectrum, we give the following stability property of the essential spectrum:

Theorem 1.10 (Weyl). *For A, B self-adjoint and:*

$$R_A(a) - R_B(a) \in \mathfrak{C}(\mathfrak{H}), \tag{1.22}$$

where $\mathfrak{C}(\mathfrak{H})$ are the compact operators, for one $a \in \rho(A) \cap \rho(B)$, we have that:

$$\sigma_{ess}(A) = \sigma_{ess}(B). \tag{1.23}$$

Proof. [22, Thm.6.18], [14, Thm.XIII.14]. □

1.4 Oscillation theory

We first note Sturm's comparison theorem and a generalization making use of Wronskians.

Theorem 1.11 (Sturm [21]). *For u, v with $(pu')' = qu$ and $(\tilde{p}v')' = \tilde{q}v$ and $\tilde{q} \geq q$, $0 < p \leq \tilde{p}$, we have that between any two zeros of u, there is a zero of v.*

Proof. [27, Thm.2.6.3]. □

Theorem 1.12. *Suppose that u, v satisfy $(pu')' = qu$ and $(pv')' = \tilde{q}v$ and $\tilde{q} \geq q$, $W_c(u, v) = W_d(u, v) = 0$ for $c, d \in [a, b]$ possibly infinite. Then v has at least a zero in (c, d).*

Proof. [4, Cor.2.3.]. □

The next lemma relates the number of zeros of a solution to the corresponding eigenvalue. Unfortunately it only works below the essential spectrum. To find a similar result working inside gaps of the essential spectrum was the key idea behind my master thesis.

Lemma 1.13. *For $\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$ the eigenvalues below the essential spectrum of H, and ψ_n the eigenfunction corresponding to λ_n we have that $\#(\psi_n) = n$.*

Proof. The proof is done showing two inequalities. The first follows from Sturm's comparison Theorem 1.11, so we have $\#(\psi_n) \geq n$ (since we have $\#(\psi_n) > \#(\psi_{n-1})$). The other inequality is obtained by using test functions defined by:

$$\eta_j(x) = \begin{cases} \psi_n(x) & x_j < x < x_{j+1} \\ 0 & \text{else} \end{cases}, \quad j = 1, \ldots n. \tag{1.24}$$

Here $x_0 = a, x_1, \ldots, x_n$ are the zeros of ψ_n and $x_{n+1} = b$. Now since $\langle \eta_j, H\eta_j \rangle = \lambda_n \|\eta_j\|$, and the η_j being linearly independent, we get the result using [22, Thm.4.11.]. □

We call an endpoint of the interval (a, b) oscillatory, if a solution to $\tau f = 0$ has infinitely many zeros in every neighborhood of it. If this holds for one solution, then it holds for all solutions by Sturm's comparison theorem. Furthermore, we call an endpoint nonoscillatory if it only has finitely many zeros in some neighborhood of it.

Theorem 1.14 (Kneser [9]). *The operator on* $(1, \infty)$*:*

$$\tau = -\frac{d^2}{dx^2} + q(x) \tag{1.25}$$

is oscillatory if:

$$\liminf_{x \to \infty} x^2 q(x) > 1/4 \tag{1.26}$$

and nonoscillatory if

$$\limsup_{x \to \infty} x^2 q(x) < 1/4. \tag{1.27}$$

Proof. Proving this theorem is done in two steps. First note that for

$$\tau_0 = -\frac{d^2}{dx^2} + \frac{\kappa}{x^2} \tag{1.28}$$

$\tau_0 u = 0$ is explicitly solvable with the solution

$$u(x) = x^{\frac{1}{2} + \sqrt{\frac{1}{4} + \kappa}}. \tag{1.29}$$

Then u has infinitely many zeros in $(1, \infty)$ if $\kappa < -1/4$ and finitely many if $\kappa > 1/4$.

The result now follows using Sturm's Comparison Theorem.

□

This result has been generalized in [5], which yields a whole hierarchy of oscillation criteria. We note that here we leave plenty of room in the middle, since for example $q(x) = \sin(x)/x^2$, would yield $\limsup(q(x) \cdot x^2) = 1$ and $\liminf(q(x) \cdot x^2) = -1$.

CHAPTER 2

Ideals in the bounded linear operators

This chapter will give a brief tour of two-sided ideals in the algebra of bounded linear operators of a Hilbert space $\mathfrak{L}(\mathfrak{H})$. For this, we first recall the following general lemma about ideals in $\mathfrak{L}(\mathfrak{H})$.

Lemma 2.1 (Calkin [1])**.** *$\mathfrak{F}(\mathfrak{H})$ and $\mathfrak{C}(\mathfrak{H})$ are two sided ideals in $\mathfrak{L}(\mathfrak{H})$, and we have that the closure of the finite rank operators $\mathfrak{F}$ in the topology of $\mathfrak{L}$ is $\mathfrak{C}$. For every non trivial two-sided ideal $\mathfrak{I}$ in $\mathfrak{L}$ we have that:*

$$\mathfrak{F} \subseteq \mathfrak{I} \subseteq \mathfrak{C} \tag{2.1}$$

This implies, that $\mathfrak{C}$ is the only $\|.\|$-closed ideal, and that the closure of every other ideal is $\mathfrak{C}$.

This lemma shows that in order to understand these ideals, one should understand compact operators. We can represent a compact operator $K \in \mathfrak{C}$ as its Schmidt expansion being:

$$K = \sum_{n=0}^{\infty} s_n(K) \langle \varphi_n, \ . \ \rangle \psi_n, \tag{2.2}$$

where $s_n(K) \in [0, \infty)$ are called the singular values of K and φ_n, ψ_n are orthonormal bases. The proof can be sketched as K^*K being compact, positive and self-adjoint. Then, we apply the spectral theorem for compact operators. We thus have that the eigenvalues of K^*K can be ordered by decreasing modulus:

$$\lambda_0 \geq \lambda_1 \geq \dots \tag{2.3}$$

and converge to zero. Denote by ϕ_n the eigenfunction of K^*K corresponding to λ_n. We get that $s_n(K)^2 = \lambda_n(K^*K)$ and $\psi_n = K\varphi_n$.

Now, we come to the ideals, we will be interested in the **Schatten p-Classes**. First we recall the norms:

$$\|A\|_{\mathcal{J}^p} = \left(\sum_{n=0}^{\infty} s_n(A)^p \right)^{1/p}. \tag{2.4}$$

The Schatten p-Classes is then defined by:

$$\mathcal{J}^p(\mathfrak{H}) = \{A \in \mathfrak{C}(\mathfrak{H}) \mid \|A\|_{\mathcal{J}^p} < \infty\}. \tag{2.5}$$

Since $s_0(A) = \|A\|$, we have that $\|A\| \leq \|A\|_{\mathcal{J}^p}$. Also note that $\mathcal{J}^1$ are called trace class operators and $\mathcal{J}^2$ Hilbert-Schmidt operators. Further note that:

Lemma 2.2. *For $A \in \mathcal{J}^p$ and B some bounded operator, one has that:*

$$\|BA\|_{\mathcal{J}^p} \leq \|B\|\|A\|_{\mathcal{J}^p} \text{ and } \|AB\|_{\mathcal{J}^p} \leq \|B\|\|A\|_{\mathcal{J}^p}. \tag{2.6}$$

For $1/p + 1/q = 1$ and $A \in \mathcal{J}^p$, $B \in \mathcal{J}^q$, we have that:

$$\|AB\|_{\mathcal{J}^1} \leq \|A\|_{\mathcal{J}^p}\|B\|_{\mathcal{J}^q}. \tag{2.7}$$

Lemma 2.3. *For $1 \leq p < \infty$, $\mathcal{J}^p$ are two-sided $*$-ideals in $\mathfrak{L}$, and $(\mathcal{J}^p, \|.\|_{\mathcal{J}^p})$ are Banach spaces. The finite rank operators $\mathfrak{F}$ are $\|.\|_{\mathcal{J}^p}$ dense in $\mathcal{J}^p$.*

Proof. [6, Thm.11.1.] □

We have the following characterization of Hilbert-Schmidt operators.

Lemma 2.4. *For $\mathfrak{H} = L^2(M)$, we have that $\mathcal{J}^2 \cong L^2(M \times M)$. To be more explicit, we can represent $K \in \mathcal{J}^2$ as the following integral operator:*

$$Kf = \int_M \hat{K}(x, y) f(y) d\mu(y). \tag{2.8}$$

Then the isomorphism is simply mapping the operator to its kernel. Further note, that the norm $\|K\|_{\mathcal{J}^2}$ is exactly the L^2-norm $\|\hat{K}\|_{L^2(M\times M)}$ of the kernel.

Proof. [7, III.9.1.] □

2.1 Convergence in the Schatten-p-Norm

Lemma 2.5 (Grümm's Convergence Theorem)**.** *For $0 < p < \infty$, we have that if $A_n \to A$, $A_n^* \to A^*$ strongly and $\|A_n\|_{\mathcal{J}^p} \to \|A\|_{\mathcal{J}^p}$ then $\|A_n - A\|_{\mathcal{J}^p} \to 0$.*

Proof. [19, Thm.2.19.] □

Lemma 2.6. *For $1 \leq p < \infty$, we have that if $A_n \to A$ weakly and $\|A_n\|_{\mathcal{J}^p} \to \|A\|_{\mathcal{J}^p}$ then $\|A_n - A\|_{\mathcal{J}^p} \to 0$.*

Proof. [19, Thm.2.21.] □

Lemma 2.7 ([6, Thm.IV.11.3.])**.** *Let $p > 0$, $A \in \mathcal{J}^p$, $T_n \xrightarrow{s} T$, $S_n \xrightarrow{s} S$ sequences of strongly convergent bounded linear operators, then:*

$$\|T_n A S_n^* - T A S^*\|_{\mathcal{J}^p} \to 0. \tag{2.9}$$

Proof. First assume that $A = \langle \varphi, . \rangle \psi$ is a rank one operators. Then we have that

$$\|A\|_{\mathcal{J}^p} = \|\langle \varphi, . \rangle \psi\|_{\mathcal{J}^p} = \|\varphi\|\|\psi\|.$$

Now a bit of computation shows the result for rank one operators, which extends to finite rank operators.

Next note that the uniform boundedness principle implies, that $\exists\gamma > 0$ such that $\|T_n\| \leq \gamma$ and $\|S_n\| \leq \gamma$. Next we note that for a finite rank operator F we obtain:

$$\begin{aligned}\|T_n A S_n^* - T A S^*\|_{\mathcal{J}^p} \leq & \|T_n\| \|A - F\|_{\mathcal{J}^p} \|S_n^*\| \\ & + \|T_n F S_n^* - T F S^*\|_{\mathcal{J}^p} \\ & + \|T\| \|A - F\|_{\mathcal{J}^p} \|S^*\|,\end{aligned}$$

where all terms can be made arbitrarily small, which shows the lemma. □

Taking the adjoint here is important as the example from [19], $T_n = T = 1$, $A = \langle\varphi_0, .\rangle\varphi_0$ and $S_n = \langle\varphi_n, .\rangle\varphi_0$ shows.

2.2 The Trace and the Determinant

An important property of trace class operators is that one can extend the well-known functionals trace and determinant from linear algebra to them. To define them, let ψ_n be an orthonormal basis. We can define the continuous linear functional $\mathrm{tr} : \mathcal{J}^1 \to \mathbb{C}$ called the trace by

$$\mathrm{tr}(A) = \sum_{n=0}^{\infty} \langle\psi_n, A\psi_n\rangle. \tag{2.10}$$

Lemma 2.8 (Properties of the trace)**.** *We have that:*

- tr *is independent of the choice of orthonormal basis.*
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ *for* $A, B \in \mathfrak{L}$ *such that* $AB, BA \in \mathcal{J}^1$.
- *For* U *unitary and* $A \in \mathcal{J}^1$*, we have that* $\mathrm{tr}(A) = \mathrm{tr}(UAU^*)$.

Proof. [25, Prop.1.7.1.] □

Lemma 2.9. *If* P *is an orthonormal projection, we have that:*

$$\dim \mathrm{Ran}(P) = \mathrm{tr}(P) \tag{2.11}$$

where one sets $\mathrm{tr}(P) = \infty$ *if* P *is not trace class.*

And we can also define the following functional:

$$\det : 1 + \mathcal{J}^1 \to \mathbb{C} \tag{2.12}$$

$$\det(1 + A) = \prod_{n=0}^{\infty} (1 + \langle\psi_n, A\psi_n\rangle), \tag{2.13}$$

which is called the determinant.

Lemma 2.10 (Properties of the determinant)**.** *We have that:*

- det *is independent of the choice of orthonormal basis.*
- $\det(1 + AB) = \det(1 + BA)$ *for* $A, B \in \mathfrak{L}$ *such that* $AB, BA \in \mathcal{J}^1$.

- *For U unitary and $U \in \mathcal{J}^1$:* $\det(1+A) = \det(1+UAU^*)$.
- $\det((1+A)(1+B)) = \det(1+A)\det(1+B)$ *for $A, B \in \mathcal{J}^1$.*

Proof. [7, Chap.IV.1.5-7.], [25, (1.7.11)]. □

Note that the $\|.\|_{\mathcal{J}^1}$-continuity of these two functionals and $\mathfrak{F} \subseteq \mathcal{J}^1$ implies, that these functionals can be obtained as limits from the ones we know from linear algebra. Further note the following theorem relating the trace and determinant to the eigenvalues of the operator.

Lemma 2.11 (Lidskii Trace Theorem [12])**.** *For A a trace class operator and $\lambda_1, \lambda_2, \ldots$ the sequence of non-zero eigenvalues with multiplicity of A, we have that:*

$$\operatorname{tr}(A) = \sum_j \lambda_j \quad \textit{and} \quad \det(1+A) = \prod_j (1+\lambda_j). \tag{2.14}$$

Proof. [6, Thm.IV.6.1.] □

CHAPTER 3

Krein's spectral shift function

In this chapter, we will discuss some aspects of Krein's spectral shift function (SSF). For a discussion on this in the literature see [17] for the case of rank one perturbations (and also for trace class in a remark), and [25] for the more general case. For historical purposes also see the paper by Krein [10].
For self-adjoint operators H_1, H_0, we call $\xi(\lambda) = \xi(\lambda, H_1, H_0)$ a *spectral shift function*, if it satisfies:

$$\operatorname{tr}(f(H_1) - f(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda, H_1, H_0) f'(\lambda) d\lambda \tag{3.1}$$

for all smooth functions with comptact support $f \in C_0^\infty(\mathbb{R})$.

Assume that H_0, H_1 are bounded from below and $\lambda < \inf \sigma_{ess}(H_j)$, $j = 0, 1$, then using a function for f with compact support in $(-\infty, \lambda)$ and which is 1 on $(\sigma(H_1) \cup \sigma(H_0)) \cap (-\infty, \lambda)$, we find that

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_0) = \xi(\lambda) - \lim_{\lambda \to -\infty} \xi(\lambda). \tag{3.2}$$

This formula justifies the name spectral shift function. Especially note that, if the two operators are bounded from below, one can choose ξ such that the last term disappears.

A lot of texts on the SSF make the assumption $H_1 - H_0 \in \mathcal{J}^1$. Since in our case, $H_1 - H_0$ will be a multiplication operator (and hence cannot be trace class), we need the version for resolvent comparable operators. We will first introduce these, quote the existence result, and then derive some properties.

3.1 Construction of the SSF

We call two operators H_1 and H_0 resolvent comparable, if

$$R_{H_1}(z) - R_{H_0}(z) \in \mathcal{J}^1. \tag{3.3}$$

Theorem 3.1 (Krein [10])**.** *Let H_1 and H_0 be two resolvent comparable self-adjoint operators, then there exists $\xi \in L^1(\mathbb{R}, (\lambda^2+1)^{-1})$ such that (3.1) is satisfied.*

Proof. [25, Thm.8.7.1.] □

As already noted, equation (3.1) only fixes ξ up to an arbitrary constant. The next lemma will give a condition on how to fix the unknown constant.

Lemma 3.2. *Suppose* $[0,1] \ni \varepsilon \mapsto H(\varepsilon)$ *a family of operators, which is continuous in the norm:*

$$\rho(A,B) = 2|\operatorname{Im} a| \|R_A(a) - R_B(a)\|_{\mathcal{J}^1}. \tag{3.4}$$

Furthermore abbreviate $\xi_\varepsilon = \xi(H(\varepsilon), H(0))$ *and fix* $\xi_0 = 0$. *Then there exists an unique choice of* ξ_ε *such that* $s \mapsto \xi_s$ *is continuous* $[0,1] \to L^1((\lambda^2+1)^{-1})$.

Proof. [25, Lem.8.7.5.] □

We further note the following result on how to construct Krein's spectral shift function. For $a \in \mathbb{C}_+$ fixed, denote by $\tilde{D}_a : \rho(H_0) \to \mathbb{C}$ the generalized perturbation determinant given by:

$$\tilde{D}_a(z) = \det(1 + (z - a^*) R_{H_1}(a^*) V R_{H_0}(z)). \tag{3.5}$$

We then have that a spectral shift function ξ_a is given by:

$$\xi_a(\lambda) = \frac{1}{2\pi}\left(\lim_{\varepsilon\to 0} \arg \tilde{D}_a(\lambda + i\varepsilon) - \lim_{\varepsilon \to 0} \arg \tilde{D}_a(\lambda - i\varepsilon)\right). \tag{3.6}$$

The multivaluedness arises here, by choosing the branch of the argument. We have only required the trace formula to hold for $f \in C_0^\infty(\mathbb{R})$. There is a larger known class, given by the next lemma:

Lemma 3.3. *For* $f : \mathbb{R} \to \mathbb{R}$ *with two locally bounded derivatives satisfying:*

$$\exists \varepsilon > 0 : (\lambda^2 f'(\lambda))' = O(|\lambda|^{-1-\varepsilon}), |\lambda| \to \infty, \tag{3.7}$$

and:

$$\lim_{\lambda\to-\infty} f(\lambda) = \lim_{\lambda\to+\infty} f(\lambda), \quad \lim_{\lambda\to-\infty} \lambda^2 f'(\lambda) = \lim_{\lambda\to+\infty} \lambda^2 f'(\lambda), \tag{3.8}$$

the trace formula holds.

Proof. [25, Thm.8.7.1.] □

We remark that this lemma coverts resolvents, since the function $f(\lambda) = (\lambda - z)^{-1}$ satisfies these assumptions for $\operatorname{Im} z \neq 0$. This lemma does not capture all possible f yet, see the remark after Theorem I.10 in [17].

Remark 3.4. *At least Theorem 3.1 also holds under the weaker condition* $R^n_{H_1}(z) - R^n_{H_0}(z) \in \mathcal{J}^1$ *([26]).*

3.2 Properties of the SSF

In this section, we assume, that we have a function ξ satisfying the trace formula, and then derive properties of it.

Lemma 3.5. *For* ξ *a SSF also* $\xi + c$ *for a fixed* $c \in \mathbb{R}$ *is a SSF.*

Proof. For the proof just note, that for $f \in C_0^\infty(\mathbb{R})$, we have that: $\int_{-\infty}^{\infty} f'(\lambda)d\lambda = 0$, so the $+c$ doesn't change anything in the trace formula. □

One has to note here that $1 \in L^1((\lambda^2+1)^{-1})$ and thus we cannot fix ξ by this requirement.

Lemma 3.6. *Let $\lambda_0, \lambda_1 \notin \sigma(H_0) \cup \sigma(H_1)$, $\Omega = (\lambda_0, \lambda_1)$ an open interval, and $\dim \operatorname{Ran} P_\Omega(H_0) < \infty$, $\dim \operatorname{Ran} P_\Omega(H_1) < \infty$. Then we have that:*

$$\dim \operatorname{Ran} P_\Omega(H_1) - \dim \operatorname{Ran} P_\Omega(H_0) = \xi(\lambda_1) - \xi(\lambda_0). \tag{3.9}$$

Proof. Use the trace formula for a function f with compact support in Ω and which is 1 on the points in the spectra. □

Lemma 3.7. *For H_i, $i = 1, 2, 3$, all resolvent comparable, we have that:*

$$\xi(H_1, H_3) = \xi(H_1, H_2) + \xi(H_2, H_3). \tag{3.10}$$

Proof. This follows from the linearity of the trace and integral. □

Lemma 3.8. *We have that:*

$$\xi(A_1 \oplus A_2, B_1 \oplus B_2) = \xi(A_1, B_1) + \xi(A_2, B_2). \tag{3.11}$$

Proof. For $f \in C_0^\infty(\mathbb{R})$, we have that:

$$\begin{aligned} &\operatorname{tr}(f(A_1 \oplus A_2) - f(B_1 \oplus B_2)) \\ &\quad = \operatorname{tr}(f(A_1) - f(B_1)) + \operatorname{tr}(f(A_2) - f(B_2)) \\ &\quad = \int_{-\infty}^{\infty} \xi(\lambda, A_1, B_1) f'(\lambda) d\lambda + \int_{-\infty}^{\infty} \xi(\lambda, A_2, B_2) f'(\lambda) d\lambda, \end{aligned} \tag{3.12}$$

which shows the result. □

For the next lemma, we will need the following result from perturbation theory. First we call a family of operators $A(s)$ an analytic family in the sense of Kato if and only if:

- A is defined on a domain $G \subseteq \mathbb{C}$.
- For all $s \in G$, $A(s)$ is a closed operator with nonempty resolvent set.
- For every $s_0 \in G$, there is a $z_0 \in \rho(A(s_0))$ so that $z_0 \in \rho(A(s))$ for s near s_0 and $(A(s) - z_0)^{-1}$ is an analytic operator-valued function of s near s_0.

hold.

Theorem 3.9. *Let $A(s)$ be an analytic family in the sense of Kato for s near 0 that is self-adjoint for s real. Let λ be a discrete eigenvalue of multiplicity n. Then, there are n not necessarily distinct single-valued functions, analytic near $s = 0$, $\lambda_1(s), \ldots, \lambda_n(s)$, with $\lambda_k(0) = \lambda$, such that $\lambda_1(s), \ldots, \lambda_n(s)$ are eigenvalues of $A(s)$ for s near 0. Further these are the only eigenvalues near λ.*

Proof. [14, Thm XII.13] □

Lemma 3.10. *Let $V \geq 0$ be relatively bounded with respect to H_0 such that $H_1^{\pm} = H_0 \pm V$ and H_0 are resolvent comparable. Set $H_\varepsilon^{\pm} = H_0 \pm \varepsilon V$. Then*

$$\xi(\lambda, H_1^{\pm}, H_0) = \mp \sum_{\varepsilon \in [0,1]} \dim \ker(H_\varepsilon^{\pm} - \lambda), \tag{3.13}$$

for all $\lambda \in \rho(H_0) \cap \rho(H_1^{\pm}) \cap \mathbb{R}$.

Proof. We just do the proof in the + case. First of all observe that H_ε^+ is an analytic family in the sense of Kato and satisfies the assumptions of Lemma 3.2.

Furthermore, by Weyl's theorem there is a $\delta > 0$ such that $\sigma_{ess}(H_\varepsilon^+) \cap (\lambda - \delta, \lambda + \delta) = \emptyset$. Hence, by Theorem 3.9, for every ε there is a neighborhood such that H_ε has precisely n eigenvalues (counting multiplicities) inside $(\lambda - \delta, \lambda + \delta)$. Moreover, by compactness of $[0,1]$ we can find n analytic functions $\lambda_j(\varepsilon)$, $1 \leq j \leq n$, which describe the eigenvalues (counting multiplicity).

These functions will not be defined for all $\varepsilon \in [0,1]$ since the eigenvalues will enter at $\lambda - \delta$ for some ε and could leave at $\lambda + \delta$ for some ε (they are nondecreasing by our assumption $V \geq 0$). However, we can define them for all $\varepsilon \in [0,1]$ by setting them equal to $\lambda \pm \delta$ in these cases. Furthermore, we can assume that $\lambda_j(\varepsilon)$ crosses λ for precisely one ε (get rid of those which stay below λ by decreasing δ).

Now within $(\lambda - \delta, \lambda + \delta)$ the spectral shift function ξ_ε is a step function which decreases by one at every $\lambda_j(\varepsilon)$. Hence the result follows since $\xi_0 = 0$ and $\xi_1 = -n$, for at $\varepsilon = 1$ all $\lambda_j(\varepsilon)$ have crossed λ. □

3.3 An Example of the SSF

In this section, we give a basic example of the SSF. Let (a, b) be a finite interval, $q \in L^1((a, b))$, then τ given by

$$\tau = -\frac{d^2}{dx^2} + q(x) \tag{3.14}$$

is a regular Strum-Liouville operator. We define for $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$ the self-adjoint extensions with separated boundary conditions $A_{\alpha\beta}$ of τ given by the domain:

$$\begin{aligned}\mathfrak{D}(A_{\alpha\beta}) = \{f \in L^2(a,b) | f, f' \in AC_{loc}(a,b), \tau f \in L^2(a,b) \\ \cos(\alpha) f(a) - \sin(\alpha) f'(a) = 0 \\ \cos(\beta) f(b) - \sin(\beta) f'(b) = 0\}.\end{aligned} \tag{3.15}$$

Choose, $\alpha, \alpha_0, \beta, \beta_0$ such that:

$$0 \leq \alpha_0 \leq \alpha < \pi \quad \text{and} \quad 0 < \beta \leq \beta_0 \leq \pi$$

where either $\alpha_0 < \alpha$ or $\beta_0 > \beta$. Since the monotonicity of the Prüfer angle, we have then that $\lambda_n^0 > \lambda_n$, where λ_n denotes the n-th eigenvalue of $A_{\alpha\beta}$ (and we assume, they are ordered such that $\lambda_n < \lambda_{n+1}$).

With this we can explicitly compute Krein's spectral shift function ξ with Lemma 3.6 to be:

$$\xi(\lambda, A_{\alpha\beta}, A_{\alpha_0\beta_0}) = \begin{cases} 1 & \lambda_n \leq \lambda < \lambda_n^0 \\ 0 & \text{else.} \end{cases} \tag{3.16}$$

All this also works, if one endpoint is limit point. For a detailed discussion see [3, Sec.2.].

CHAPTER 4

Wronskians

We have already encountered the Wronskian, when specifying boundary conditions. In this chapter, we investigate properties of the Wronskian of solutions to different Sturm–Liouville equations. Except for Section 4.2, the results in this chapter are technical and will be used to proof other results.

To begin the investigation, let (a,b) be a possibly infinite interval. For $p^{-1}, r, q_0, q_1 \in L^1_{loc}((a,b))$, real-valued, $p, r > 0$, define the differential expressions:

$$\tau_0 = \frac{1}{r(x)}\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q_0(x)\right), \tag{4.1}$$

$$\tau_1 = \frac{1}{r(x)}\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q_1(x)\right). \tag{4.2}$$

We will often take the point of viewing $\tau_1 = \tau_0 + (q_1 - q_0)$ as a perturbation of τ_0. In this section, we will furthermore assume that all solutions are real valued. This is necessary, since we will heavily rely on Prüfer variables, and wish to speak about the zeros of the Wronskian. The Wronskian is defined as

$$W(u,v) = u(pv') - (pu')v = \det\begin{pmatrix} u & pu' \\ v & pv' \end{pmatrix}. \tag{4.3}$$

Recall the Prüfer variables from Section 1.2

$$u = \rho_u \sin\vartheta_u,\ pu' = \rho_u \cos\vartheta_u,\ v = \rho_v \sin\vartheta_v,\ pv' = \rho_v \cos\vartheta_v. \tag{4.4}$$

With this, we obtain the identity:

$$W(u,v) = \rho_u\rho_v \sin(\vartheta_u - \vartheta_v) = \rho_u\rho_v \sin\Theta_{u,v}, \tag{4.5}$$

where we used $\Theta_{u,v} = \vartheta_u - \vartheta_v$. From this we have that $\Theta_{u,v} \in AC_{loc}(I)$. For $\tau_j u_j = 0$, $j = 0,1$, we have that $W_x(u_0,u_1)$ is absolutely continuous and satisfies

$$W_x'(u_0,u_1) = (q_1(x) - q_0(x))u_0(x)u_1(x). \tag{4.6}$$

Denote by ϑ_j the Prüfer angle of u_j, $j = 0,1$, and $\Theta_{0,1} = \vartheta_0 - \vartheta_1$.

Lemma 4.1. *Let $q_0 \geq q_1$, then $\Theta_{0,1}(x_0) = 0 \pmod{\pi}$ implies that $(\Theta_{0,1}(x) - \Theta_{0,1}(x_0))/(x - x_0) \geq 0$ for $|x - x_0| > 0$. In other words this means, that the integer part of $\Theta_{0,1}/\pi$ is increasing.*

If we have $q_0 - q_1 \geq \varepsilon > 0$, we even have $(\Theta_{0,1}(x) - \Theta_{0,1}(x_0))/(x - x_0) > 0$ for $|x - x_0| > 0$.

Proof. By (4.6) we have

$$\begin{aligned} W_x(u_0, u_1) &= \rho_0(x)\rho_1(x)\sin(\Theta_{0,1}(x)) \\ &= \int_{x_0}^{x} (q_1(t) - q_0(t))u_0(t)u_1(t)dt \end{aligned} \tag{4.7}$$

and there are two cases to distinguish: Either $u_0(x_0)$, $u_1(x_0)$ are both different from zero or both equal to zero. If both are equal to zero, they must change sign at x_0 and hence in both cases the integrand is of one sign near x_0. Thus the result follows.

□

For technical reasons, we will denote by $\#(u_0, u_1)$ the number of sign flips of $W(u_0, u_1)$ in the interval I. The previous lemma asserts that this is equivalent to the number of zeros of $W(u_0, u_1)$ as long as $q_0 > q_1$. If $q_0 = q_1$ the case of $W(u_0, u_1) = 0$ on some open set can arise.

Lemma 4.2. *For $q_0 \geq q_1$, $\tau_j u_j = 0$, $j = 0, 1$, we have that:*

$$\#(u_0, u_1) = \lim_{x \uparrow b} \lceil \Theta_{0,1}(x)/\pi \rceil - \lim_{x \downarrow a} \lfloor \Theta_{0,1}(x)/\pi \rfloor - 1, \tag{4.8}$$

where $\lfloor x \rfloor = \sup\{n \in \mathbb{Z}, n \leq x\}$, $\lceil x \rceil = \inf\{n \in \mathbb{Z}, n \geq x\}$. We can drop the limit if a (resp. b) is regular.

Proof. Without loss of generality, we can assume that we have an $x_0 \in (a, b)$ with $W_{x_0}(u_0, u_1) \neq 0$ and

$$\lfloor \Theta_{0,1}(x_0)/\pi \rfloor = \lceil \Theta_{0,1}(x_0)/\pi \rceil - 1.$$

Now let $[x_1, x_2]$ a maximal interval on which $W_x(u_0, u_1) = 0$. Then we have for all sufficiently small ε, that:

$$\lceil \Theta_{0,1}(x_2 + \varepsilon)/\pi \rceil = \lceil \Theta_{0,1}(x)/\pi \rceil + 1, \forall x \in [x_1 - \varepsilon, x_2]$$

and

$$\lfloor \Theta_{0,1}(x_1 - \varepsilon)/\pi \rfloor = \lfloor \Theta_{0,1}(x)/\pi \rfloor + 1, \forall x \in [x_1, x_2 + \varepsilon].$$

Thus we obtain the result. □

We will now look at the interpolating differential expressions $\tau_\varepsilon = \tau_0 + \varepsilon(\tau_1 - \tau_0)$. We will denote by u_ε any solution of $\tau_\varepsilon u_\varepsilon = 0$, and by $\psi_{\varepsilon,\pm}$ the solutions satisfying the appropriate conditions at a for $-$ and b for $+$. This is meant in the sense that the appropriate conditions are the boundary conditions of self-adjoint extensions, where we assume that all self-adjoint extensions have the same boundary conditions.

Lemma 4.3. *For $a < c < d < b$ we have that:*

$$W_d(u_\varepsilon, v_{\tilde{\varepsilon}}) - W_c(u_\varepsilon, v_{\tilde{\varepsilon}}) = \int_c^d (\varepsilon - \tilde{\varepsilon})(q_0(t) - q_1(t))u_\varepsilon(t)v_{\tilde{\varepsilon}}(t)dt. \tag{4.9}$$

Proof. Integrate 4.6. ☐

Lemma 4.4. *For w with $w, pw' \in AC_{loc}$ and $W_x(w, \psi_{\varepsilon,\pm}) = 0$, we have that:*

$$\partial_\varepsilon W_x(w, \psi_{\varepsilon,+}) = \kappa \int_x^b \psi_{\varepsilon,+}^2(t)(q_1(t) - q_0(t))dt \tag{4.10}$$

$$\partial_\varepsilon W_x(w, \psi_{\varepsilon,-}) = -\kappa \int_a^x \psi_{\varepsilon,-}^2(t)(q_1(t) - q_0(t))dt, \tag{4.11}$$

where κ satisfies $\psi_{\varepsilon,\pm}(x) = \kappa w(x)$.

Proof. We just prove the result in the "$\psi_{\varepsilon,-}$" case. The other case works similarly. Note that $W_x(w, \psi_-) = 0$ implies that $w(x) = \kappa \cdot \psi_-(x)$ and $w'(x) = \kappa \cdot \psi'_-(x)$. The derivate in x implies: For the second one, using $W_a(\psi_{\varepsilon,-}, \psi_{\tilde{\varepsilon},-}) = 0$ (since, they have the same boundary condition at a):

$$\begin{aligned}
\partial_\varepsilon W_x(w, \psi_{\varepsilon,-}) &= \lim_{\tilde{\varepsilon}\to\varepsilon} \frac{W_x(w, \psi_{\varepsilon,-}) - W_x(w, \psi_{\varepsilon,-})}{\tilde{\varepsilon} - \varepsilon} \\
&= \lim_{\tilde{\varepsilon}\to\varepsilon} \frac{W_x(\kappa \cdot \psi_{\varepsilon,-}, \psi_{\tilde{\varepsilon},-})}{\tilde{\varepsilon} - \varepsilon} \\
&= \lim_{\tilde{\varepsilon}\to\varepsilon} \frac{-(\tilde{\varepsilon} - \varepsilon) \cdot \int_a^x \kappa \cdot \psi_{\varepsilon,-}(t)\psi_{\tilde{\varepsilon},-}(t)(q_1(t) - q_0(t))dt}{\tilde{\varepsilon} - \varepsilon} \\
&= -\kappa \int_a^x \psi_{\varepsilon,-}^2(t)(q_1(t) - q_0(t))dt.
\end{aligned} \tag{4.12}$$

☐

The next lemma is similar to Lemma 1.8.

Lemma 4.5. *We have for the Prüfer angle $\vartheta_{\varepsilon,\pm}$ of $\psi_\pm$ that:*

$$\frac{\partial}{\partial\varepsilon}\vartheta_{\varepsilon,+}(x) = \frac{\int_x^b (q_1(t) - q_0(t))\psi_{\varepsilon,+}(t)^2 dt}{\rho_{\varepsilon,+}(x)^2} \leq 0, \tag{4.13}$$

$$\frac{\partial}{\partial\varepsilon}\vartheta_{\varepsilon,-}(x) = -\frac{\int_a^x (q_1(t) - q_0(t))\psi_{\varepsilon,-}(t)^2 dt}{\rho_{\varepsilon,-}(x)^2} \geq 0. \tag{4.14}$$

Proof. First note that: $\partial_\varepsilon \vartheta_u = -W(u, \partial_\varepsilon u)/\rho_u^2$ and then use the last lemma. ☐

Remark 4.6. *Suppose v_j satisfy $(p_j v_j')'(x) = q(x)v_j(x)$, $j = 0, 1$. Define the again modified Wronskian by:*

$$W_x(v_0, v_1) = v_0(x)p_1(x)v_1'(x) - p_0(x)v_0'(x)v_1(x), \tag{4.15}$$

Then one finds in exactly the same way:

$$W_x'(v_0, v_1) = -(p_0(x) - p_1(x))v_0'(x)v_1'(x). \tag{4.16}$$

Thus we can also apply the results in the case of different p.

4.1 Derivatives and Zeros

Now assume $q_0 - q_1 > 0$, so we can talk about the zeros of the Wronskians. This will allow us to state results on how the zeros of the Wronskian behave as curves in the (x, ε)-plane.

Lemma 4.7. *The zeros in* ε *of* $W(\varphi, \psi_{\varepsilon,-})$ *with* $\tau_{\varepsilon_0}\varphi = 0$ *satisfy the differential:*

$$\varepsilon'(x) = -\frac{W'(\varphi, \psi_{\varepsilon,-})}{\partial_\varepsilon W(\varphi, \psi_{\varepsilon,-})} = \frac{(\varepsilon_0 - \varepsilon(x))(q_1(x) - q_0(x))\psi^2_{\varepsilon(x),-}(x)}{\int_x^b \psi^2_{\varepsilon(x),-}(t)(q_1(t) - q_0(t))dt}. \tag{4.17}$$

For $W(\varphi, \psi_{\varepsilon,+})$*, we have that:*

$$\varepsilon'(x) = -\frac{W'(\varphi, \psi_{\varepsilon,+})}{\partial_\varepsilon W(\varphi, \psi_{\varepsilon,+})} = -\frac{(\varepsilon_0 - \varepsilon(x))(q_1(x) - q_0(x))\psi^2_{\varepsilon(x),-}(x)}{\int_a^x \psi^2_{\varepsilon(x),-}(t)(q_1(t) - q_0(t))dt}. \tag{4.18}$$

Proof. The proof consists of applying the implicit function theorem. □

It follows from Lemma 4.7, that in our case the zeros of $W(\varphi, \psi_{\varepsilon,-})$ (resp. $W(\varphi, \psi_{\varepsilon,+})$) move to the right (resp. left) as ε increases.

Lemma 4.8. *Different zeros of* $W_x(\varphi, \psi_{\varepsilon,\pm})$ *don't collide.*

Proof. This follows from the right hand sides of (4.17) and (4.18) being locally Lipschitz, and the uniqueness theorem of solutions. □

4.2 Sturm-type theorems

In this section, we give analog results to Sturm's comparison theorem, Theorem 1.11, for Wronskians.

Theorem 4.9 (Comparison theorem for Wronskians)**.** *Suppose* u_j *satisfies* $\tau_j u_j = 0$, $j = 0, 1, 2$.

Then, if $q_0 \geq q_1 \geq q_2$*, between two zeros of* $W_x(u_0, u_1)$ *there is at least one zero of* $W_x(u_0, u_2)$ *and between two zeros of* $W_x(u_1, u_2)$ *there is at least one zero of* $W_x(u_0, u_2)$*.*

Proof. Let c, d be two consecutive zeros of $W_x(u_0, u_1)$. We first assume, that $W_c(u_0, u_2) = 0$ and consider $\tau_\varepsilon = (2 - \varepsilon)\tau_1 + (\varepsilon - 1)\tau_2$, $\varepsilon \in [1, 2]$, restricted to (c, d) with boundary condition generated by the Prüfer angle of u_0 at c. Set $u_\varepsilon = \psi_{\varepsilon,-}$, then we have $\Theta_{u_\varepsilon,u_0}(c) = 0$ for all ε and $\Theta_{u_\varepsilon,u}(d)$ is increasing, implying that $W_x(u, u_\varepsilon)$ has at least one zero in (c, d) for $\varepsilon > 1$.

To finish our proof, let $\tilde{u}_2$ be a second linearly independent solution. Then, since $W(u_2, \tilde{u}_2)$ is constant, we can assume $0 < \Theta_{u_2,\tilde{u}_2}(x) < \pi$. This implies $\Theta_{u_0,\tilde{u}_2}(c) = \Theta_{u_2,\tilde{u}_2}(c) < \pi$ and $\Theta_{u_0,\tilde{u}_2}(d) = \Theta_{u_0,u_2}(d) + \Theta_{u_2,\tilde{u}_2}(d) > \pi$. Consequently $W_x(u_0, \tilde{u}_2)$ also has at least one zero in (c, d).

The second claim is proven analogous. □

We note that by Remark 4.6 the theorem also holds, if we have that $-(p_j u_j')' + q_j u_j = \lambda_j u_j$, $j = 0, 1, 2$ and $q_0 \geq q_1 \geq q_2$, $p_2 \geq p_1 \geq p_0$.

Corollary 4.10. *Suppose u_j satisfies $\tau_j u_j = 0$, $j = 0, 1, 2$.*
Then, if $q_0 \geq q_1 \geq q_2$, we have that:

$$\#(u_0, u_1) \leq \#(u_0, u_2) + 1, \qquad \#(u_1, u_2) \leq \#(u_0, u_2) + 1. \tag{4.19}$$

As a further consequence we obtain the following generalization of the results [4, Thm.7.1,7.2.].

Corollary 4.11. *Suppose u_j, v_j satisfy $\tau_j u = 0$, $j = 0, 1$, with $q_0 \geq q_1$.*
Then the zeros of $W_x(u_0, u_1)$ interlace the zeros of $W_x(u_0, v_1)$ and vice versa (in the sense that there is exactly one zero of one function in between two zeros of the other). In particular,

$$|\#(u_0, u_1) - \#(u_0, v_1)| \leq 1, \qquad |\#(u_0, u_1) - \#(v_0, u_1)| \leq 1, \tag{4.20}$$

and

$$|\#(u_0, u_1) - \#(v_0, v_1)| \leq 2. \tag{4.21}$$

Lemma 4.12. *Let $\tau_j u_j = 0$, $j = 0, 1$ and $q_1 \geq q_0$. Further suppose that h is a function with h, ph' absolutely continuous and $(h(x), p(x)h'(x)) \neq (0, 0)$ for all x. Now let c, d be distinct sign flips of $W_x(u_0, u_1)$. Furthermore assume that at no $x \in [c, d]$, there is a sign flip of $W_x(u_0, h)$, then $W(u_1, h)$ has a sign flip in (c, d). If either c (or d) is a sign flip of $W_x(u_0, h)$, then c (or d) is also a sign flip of $W_x(u_1, h)$.*

Proof. The second assertion follows from the Wronskian being a determinant, so we restrict ourself to the first case.

Without loss of generality $\Theta_{1,0}(a) = 0$, $\Theta_{1,0}(b) = \pi$, and $\Theta_{h,0} \in (\delta, \pi - \delta)$, for some $\delta > 0$. Now:

$$\Theta_{1,0}(a) - \Theta_{h,0}(a) < 0 \quad \text{and} \quad \Theta_{1,0}(b) - \Theta_{h,0}(b) > 0. \tag{4.22}$$

Thus there exists $x \in (a, b)$ such that $0 = \Theta_{1,0}(x) - \Theta_{h,0}(x) = \vartheta_1(x) - \vartheta_h(x) = \Theta_{1,h}(x)$, which is equivalent to $W_x(u_1, h) = 0$. □

The condition $(h(x), p(x)h'(x)) \neq (0, 0)$ for all x is important, since then ϑ_h and $\Theta_{h,0}$ are continuous functions.

Theorem 4.13 (Triangle inequality for Wronskians)**.** *For $\tau_j u_j = \lambda_j u_j$, $j = 0, 1$, and $q_0 \geq q_1$ (or $q_0 \leq q_1$) in some interval (c, d), we have for any function h with h, ph' absolutely continuous and $(h(x), p(x)h'(x)) \neq (0, 0)$ for all x, that:*

$$\#(u_0, u_1) \leq \#(u_0, h) + \#(h, u_1) + 1. \tag{4.23}$$

Proof. We first prove the case $q_0 \geq q_1$. Assume that $N = \#(u_0, u_1)$ is finite, if not, the following works for arbitrary large N. Denote by $x_1, \dots, x_N$ points inside the sign flips of $W(u_0, u_1)$. If $W_{x_i}(u_0, h) = 0$, we have that $W_{x_i}(u_1, h) = 0$. Thus it is sufficient to show that the formula stays valid on (c, x_i) and (x_i, d) to show the result. So we assume without loss of generality that $W_{x_i}(u_0, h) \neq 0$. Then the theorem follows by the last lemma, since there is either a sign flip of $W(u_0, h)$ or $W(u_1, h)$ in any interval (x_i, x_{i+1}).

The case $q_0 \leq q_1$ follows by interchanging the role of u_0 and u_1. □

Take $p(x) = 1$, $q_0(x) = 0$, and $q_1(x) = -1$ on $I = (0, \pi)$. Then the simple example $u_0(x) = 1$, $u_1(x) = \sin(x)$, and $h(x) = \cos(x)$ shows that our inequality is optimal.

Lemma 4.14. *Let $p, u, v : (c, d) \to \mathbb{R}$, be functions with $p > 0$, $u, v, pu', pv' \in AC((c, d))$ such that $W_x(u, v) \neq 0, x \in (c, d)$, then the zeros of u and v alternate in (c, d).*

Proof. First note, that $W_x(u, v) \neq 0$ implies $(u(x), p(x)u'(x)) \neq (0, 0)$ and $(v(x), p(x)v'(x)) \neq (0, 0)$ for all $x \in (c, d)$. Furthermore, it implies that u and v have no common zero. Next assume the contrary, namely that we have $e, f \in (c, d)$ such that $u(e) = u(f) = 0$, and $u(x) > 0, v(x) > 0, x \in (e, f)$. It follows, that $p(e)u'(e) > 0$, $p(f)u'(f) < 0$, implying $W_e(u, v) = 0 \cdot p(e)v'(e) - p(e)u'(e)v(e) > 0$ and $W_f(u, v) = 0 \cdot p(f)v'(f) - p(f)u'(f)v(f) < 0$, which would imply that the Wronskian had a zero in (e, f) by the mean value theorem: A contradiction. □

4.3 Higher Derivates and Differential equations

In this section, we will make the assumption $\Delta q = q_1 - q_0 \in C^2$ and take a look at higher derivates of the Wronskian and possible differential equations for it. Let $\tau_0 u = 0$, $\tau_1 v = 0$, we can compute the following higher derivates:

$$W'(u, v) = \Delta q uv \tag{4.24}$$

$$W''(u, v) = \Delta q' uv + \Delta q\,(uv' + u'v) \tag{4.25}$$

$$W'''(u, v) = \left(\Delta q'' + \frac{\Delta q(2q_0 + \Delta q)}{p}\right) uv + 2\Delta q u'v' + 2\Delta q'\,(uv' + u'v)\,. \tag{4.26}$$

For shortness, we will use the notation $W = W(u, v)$. Using these derivates, we can obtain an alternative proof for Lemma 4.1.

Lemma 4.15. *Let $q_0 > q_1$, $\Delta q \in C^2(I)$ and $\tau_0 u = 0$, $\tau_1 v = 0$, then the Wronskian changes sign at each of its zeros. If we denote by x_0 a zero of W_x, we either have $W'_{x_0} \neq 0$ or $u(x_0) = v(x_0) = 0$, $W_{x_0} = W'_{x_0} = W''_{x_0} = 0$ and $W'''_{x_0} \neq 0$.*

Proof. If $W'_{x_0} = 0$, (4.24) implies that $u(x_0) = 0$ or $v(x_0) = 0$. Then by $W_{x_0} = 0$, we have that both are zero and by this $W''_{x_0} = 0$. From u, v being nonzero, we have that $u'(x_0) \neq 0 \neq v'(x_0)$ and thus $W'''_{x_0} \neq 0$. Integrating W''' shows $C_1(x - x_0) \leq W''(x_0) \leq C_2(x - x_0)$ where $C_1 C_2 > 0$ in a neighborhood of x_0. This shows that W changes sign, so we obtain the result. □

The Wronskian obeys the following third order non-linear differential equation:

$$\begin{aligned} &\left(W'' - \frac{\Delta q'}{\Delta q} W'\right)^2 - \Delta q^2 W^2 \\ = 2W' &\left[W''' - \frac{2\Delta q'}{\Delta q} W'' + \left(\frac{\Delta q' - \Delta q''}{\Delta q} + \frac{2 \cdot q_0 + \Delta q}{p}\right) W'\right]. \end{aligned} \tag{4.27}$$

However, we don't see any way of drawing useful conclusions from this differential equation. In the case of Δq constant, we find:

$$(W'')^2 - \Delta q^2 W^2 = 2W' \left[W''' + \left(\frac{2 \cdot q_0 + \Delta q}{p} \right) W' \right]. \tag{4.28}$$

Even though this already looks simpler, it is still non-linear. The last equation, was first found in [23].

Assuming we know a lot about τ_0, it might be a good idea to try to eliminate v from the equation. Doing this one finds:

$$- u\Delta q W'' + (u\Delta q' + 2u'\Delta q) W' + \frac{u\Delta q^2}{p} W = 0. \tag{4.29}$$

This equation gives us the hope to be able to transform it to standard Sturm–Liouville form. This would give us an equation, we can say a lot about. Unfortunately, we find:

$$- \left(\frac{1}{u^2 |\Delta q|} W' \right)' + \frac{\operatorname{sgn}(\Delta q)}{u^2 p} W = 0 \tag{4.30}$$

This differential equation leads to problems, since near any zero of u the potential term is not locally integrable.

If Equation (4.30) was well posed, one could use classical oscillation theory to determine if the Wronskian has finitely or infinitely many zeros. However, since we are mainly interested in this question in the case of u having infinitely many zeros, this hope fails.

CHAPTER 5

Relative Oscillation Theory

In this chapter, we give an overview of relative oscillation theory using the spectral shift function. Further note, that an extension of this theory using approximation by regular problems will appear in [11].

We will again look at some finite or infinite interval (a,b), $r, p^{-1}, q_0, q_1 \in L^1_{loc}((a,b))$, $r, p > 0$, and the two differential expressions:

$$\tau_0 = \frac{1}{r(x)}\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q_0(x)\right) \tag{5.1}$$

$$\tau_1 = \frac{1}{r(x)}\left(-\frac{d}{dx}p(x)\frac{d}{dx} + q_1(x)\right). \tag{5.2}$$

Then denote by H_0 a self-adjoint extension of τ_0 and by H_1 a self-adjoint extension of τ_1 with the same boundary conditions as H_0 at limit-circle endpoints. In order to extend the notion of oscillation theory to comparing different differential expression, we define:

Definition 5.1. *Suppose $q_1 \le q_0$ (or $q_1 \ge q_0$) and let u_j be solutions of $\tau_j u_j = 0$, j =0,1.*

We call τ_1 relatively oscillatory with respect to τ_0 if $\#(u_0, u_1)$ is infinite and relatively nonoscillatory if $\#(u_0, u_1)$ is finite.

Our requirement in this definition of $q_1 \le q_0$ might seem too strong. However, the next example will show us that it is needed. Take $p = r = 1$ and $(a,b) = \mathbb{R}$. Furthermore define the functions q_0 and q_1 by the following:

$$q_0(x) = \begin{cases} 0 & \lfloor x \rfloor \ \text{even} \\ -4\pi^2 & \lfloor x \rfloor \ \text{odd} \end{cases}, \quad q_1(x) = q_0(x+1). \tag{5.3}$$

Since τ_1 is just a shifted version of τ_0, it is obvious that H_0 and H_1 are unitairly equivalent and thus have the same spectrum. However, for $\tau_j u_j = 0$, $j = 0, 1$, given by:

$$u_0(x) = \begin{cases} 1 & \lfloor x \rfloor \ \text{even} \\ \cos(2\pi x) & \lfloor x \rfloor \ \text{odd} \end{cases}, \quad u_1(x) = \begin{cases} \cos(2\pi x) & \lfloor x \rfloor \ \text{even} \\ 1 & \lfloor x \rfloor \ \text{odd} \end{cases}, \tag{5.4}$$

we find that the Wronskian is given by $W_x(u_0, u_1) = (-1)^{\lfloor x \rfloor} \sin(2\pi x)$ and has infinitely many zeros in $\mathbb{R}$.

By Corollary 4.11, Definition 5.1 is independent of the chosen solutions. To demonstrate its usefulness, we establish its connection with the spectra of self-adjoint operators associated with τ_j.

Lemma 5.2. *Suppose $q_1 \le q_0$, $\lambda_0 \le \lambda_1$ and let H_j be self-adjoint operators associated with τ_j, $j = 1, 2$. Then the following holds*

1. *$\tau_0 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_1$ if and only if* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$.
2. *Suppose* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_j) < \infty$, $j = 0, 1$. *If $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$ for one $\lambda \in (\lambda_0, \lambda_1)$ then it is relatively nonoscillatory for all $\lambda \in (\lambda_0, \lambda_1)$.*

Proof. The first claim follows from Theorem 5.4 below. For the second claim, fix $\lambda, \tilde{\lambda} \in (\lambda_0, \lambda_1)$, and $(\tau_j - \lambda)u_j = 0, (\tau_j - \tilde{\lambda})\tilde{u}_j = 0$, $j = 0, 1$. Then apply our triangle inequality twice to find

$$\#(u, v) \le \#(u, \tilde{u}) + \#(v, \tilde{v}) + \#(\tilde{u}, \tilde{v}) + 2.$$

The result now follows using again Theorem 5.4 below. □

Furthermore,

Theorem 5.3. *Suppose $q_1 \le q_0$ and let H_j be self-adjoint operators associated with τ_j, $j = 1, 2$.*

Suppose $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty$ *and $\tau_1 - \lambda_0$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_0$. Then* $\dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_1) < \infty$ *if and only if $\tau_1 - \lambda_1$ is relatively nonoscillatory with respect to $\tau_0 - \lambda_1$.*

Proof. Again a simple consequence of Theorem 5.4 and our triangle inequality. □

Theorem 5.4 (Gesztesy, Simon, Teschl [4])**.** *Suppose H_0 be a self-adjoint operator associated with τ_0. Then for $\lambda_0 < \lambda_1$ and $(\tau_0 - \lambda_j)u_j = 0$, $j = 0, 1$, we have that:*

$$\#(u_0, u_1) < \infty \Leftrightarrow \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1)}(H_0) < \infty. \tag{5.5}$$

We will now first give, the regular case of a result on how to connect the spectra of two different operators to the number of zeros of the Wronskian. Then we will present the singular case. The advantage of the regular case is that we do not have to use Krein's Spectral Shift Function.

5.1 The regular case

In this section, we will assume (a, b) to be a finite interval and $r, p^{-1}, q_0, q_1 \in L^1((a, b))$. This means that τ_0 and τ_1 will be regular Sturm–Liouville operators.

Theorem 5.5. *For H_0, H_1 self-adjoint extensions of τ_0 and τ_1 with the same boundary conditions at a and b, we have that*

$$\dim \operatorname{Ran} P_{(-\infty,\lambda)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda]}(H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)). \tag{5.6}$$

Proof. Without restriction, we only proof the $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ case. As in Chapter 4, we will look at the interpolating operator H_ε being a self-adjoint extension of τ_ε with the same boundary conditions. Denote by $\Theta_{\varepsilon,0}$ the Prüfer angle of $W(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$. By Lemma 4.2, we have that

$$\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)) = \lim_{x\uparrow b} \lceil \Theta_{1,0}(x)/\pi \rceil - \lim_{x\downarrow a} \lfloor \Theta_{1,0}(x)/\pi \rfloor - 1. \tag{5.7}$$

Since $\psi_{\varepsilon,-}$ has a boundary condtion at a and $\psi_{0,+}$ is constant, $\lfloor \Theta_{\varepsilon,0}(a) \rfloor$ is constant and can without restriction be assumed to be equal 0. Therefore $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ corresponds to the number of times $\Theta_{\varepsilon,0}(b)$ hits a multiple of π and thus to $\#\{\varepsilon \in (0,1) | \lambda \in \sigma(H_\varepsilon)\}$. Since as in the proof of Lemma 3.10 the eigenvalues are decreasing functions in ε, we obtain the result. □

Note here, that we have $(-\infty, \lambda]$ at the projector connected to H_0, since if λ is an eigenvalue of H_0 the Wronskian has already lost a zero.

5.2 The singular case

In order to show the singular case, we will require the following hypothesis:

Hypothesis 5.6. *Suppose H_0 and H_1 are self-adjoint operators associated with τ_0 and τ_1 and separated boundary conditions such that:*

(i) $r^{-1}(q_0 - q_1)$ is relatively bounded with respect to H_0 with H_0-bound less then one.

(ii) $\sqrt{r^{-1}|q_0 - q_1|} R_{H_0}(z)$ is Hilbert-Schmidt for one (and hence for all) $z \in \rho(H_0)$.

Next we proof the following theorem:

Theorem 5.7. *Let H_0, H_1 satisfy Hypothesis 5.6 and $q_0 - q_1 \geq 0$. Then for every $\lambda \in \rho(H_0) \cap \rho(H_1) \cap \mathbb{R}$ we have*

$$\xi(\lambda, H_1, H_0) = \#(\psi_{0,\pm}(\lambda), \psi_{1,\mp}(\lambda)). \tag{5.8}$$

To proof this, we will first derive consequences of Hypothesis 5.6 and then derive the theorem using an approximation argument by compactly supported functions.

We will now look at what Theorem 5.7 tells us about the operators associated with the differential expression. First note that Hypothesis 5.6 implies, that by Weyl's theorem our operators have the same essential spectrum. Suppose that $(\lambda_0, \lambda_1) \cap \sigma_{ess}(H_0) = \emptyset$. Then we know that the eigenvalues inside (λ_0, λ_1) can only accumulate at λ_0 and λ_1. So we obtain from Theorem 5.7 that $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)) < \infty$ for all $\lambda \in (\lambda_0, \lambda_1)$ and thus: $\tau_1 - \lambda$ is relatively nonoscillatory with respect to $\tau_0 - \lambda$. We further note that Theorem 5.7 also implies that λ_j $j = 0, 1$ is an accumulation point of eigenvalues of H_1 in (λ_0, λ_1) if and only if $\tau_1 - \lambda_j$ is relatively oscillatory with respect to $\tau_0 - \lambda_j$.

These considerations show that the notion of relative oscillation is the correct generalization of oscillation theory to the case inside spectral gaps.

5.2.1 Consequences of Hypothesis 5.6

In this section, we draw consequences from Hypothesis 5.6 showing how to derive convergence in the norm:

$$\rho(A, B) = 2|\mathrm{Im}\, z| \|R_A(z) - R_B(z)\|_{\mathcal{J}^1} \tag{5.9}$$

from it. To do this, we derive some properties of relatively bounded operators multiplied by strongly continuous families of operators in Lemma 5.9. The key example for these operators will be multiplication operators by characteristic functions strongly converging to unity.

For convenience, we rewrite Hypothesis 5.6 in form of operators. Here we use $V^2 = r^{-1}(q_0 - q_1)$.

Hypothesis 5.8. *Suppose H_0 is self-adjoint and V symmetric such that:*

(i) V^2 is relatively bounded with respect to H_0 with H_0 bound less then one.

(ii) $VR_{H_0}(z)$ is Hilbert-Schmidt for one (and hence for all) $z \in \rho(H_0)$.

We recall that (i) means that: $\mathfrak{D}(V^2) \supseteq \mathfrak{D}(H_0)$ and that there exists $0 \leq a < 1, 0 \leq b$ such that:

$$\|V^2\psi\| \leq a\|H_0\psi\| + b\|\psi\|, \; \forall \psi \in \mathfrak{D}(H_0). \tag{5.10}$$

(i) implies, that we have an analytic family of type (A). This implies having an analytic family in the sense of Kato, which we use in the proof of Lemma 3.10.

If we think of $R_{H_0}(z)V$ as the adjoint of $VR_{H_0}(z^*)$, it is Hilbert-Schmidt. It is clear that this operator will be equal to the product of $R_{H_0}(z)$ and V on a core of V.

Lemma 5.9. *Assume Hypothesis 5.8 (i) and let $K : [0,1] \to \mathfrak{L}$ be a strongly continuous family of self-adjoint bounded operators which commute with V and $\|K(\varepsilon)\| \leq 1$. Then also the following are true:*

1. *$K(\varepsilon)V^2$ relatively bounded with respect to H_0 with H_0 bound less then one.*
2. *$H(\varepsilon) = H_0 + K(\varepsilon)V^2$ are self-adjoint operators on $\mathfrak{D}(H_0)$.*
3. *$V^2 R_{H(\varepsilon)}(z)$ is uniformly bounded.*

Proof. For 1.: For fixed $\psi \in \mathfrak{D}(H_0)$, we have

$$\|K(\varepsilon)V^2\psi\| \leq \|K(\varepsilon)\|\|V^2\psi\| \leq \|V^2\psi\|. \tag{5.11}$$

Thus $K(\varepsilon)V^2$ is relatively bounded.

For 2.: Kato-Rellich Theorem, [13, Thm.X.12].

For 3.: A calculation using (5.10) shows for $\psi \in \mathfrak{H}$, that:

$$\|V^2 R_{H(\varepsilon)}(z)\psi\| \leq \frac{1}{1-a}\left(a + \frac{b + a|z|}{|\mathrm{Im}\, z|}\right)\|\psi\| \tag{5.12}$$

and thus $V^2 R_{H(\varepsilon)}(z)$ is bounded. □

Lemma 5.10. *Suppose H_0 and V satisfy Hypothesis 5.8 (i), (ii). Let $K : [0,1] \to \mathfrak{L}$ be a strongly continuous family of self-adjoint bounded operators which commute with V abd $\|K(\varepsilon)\| \le 1$. Then we have that $H(\varepsilon) = H_0 + K(\varepsilon)V^2$, and where H_0 satisfies the assumptions of Lemma 3.2.*

Proof. Without loss of generality, we will only show continuity at 1. Then it is sufficient by the second resolvent equation to show $R_{H(0)}(z_0)VK(\varepsilon) \to R_{H(0)}(z_0)VK(1)$ and $VR_{H(\varepsilon)}(z_0) \to VR_{H(1)}(z_0)$ in the Hilbert-Schmidt topology. The first statement follows from Lemma 2.7.

For the second statement first note that by the second resolvent equality and Lemma 5.9,

$$VR_{H(1)}(z_0) = VR_{H_0}(z_0)(1 - V^2 R_{H(1)}(z_0)) \tag{5.13}$$

is Hilbert-Schmidt and

$$VR_{H(\varepsilon)}(z_0) = VR_{H(1)}(z_0)(1 + (K(1) - K(\varepsilon)))V^2 R_{H(\varepsilon)}(z_0)). \tag{5.14}$$

By Lemma 2.7, it is sufficient to show $(K(1) - K(\varepsilon))V^2 R_{H(\varepsilon)}(z_0) \to 0$ strongly and $V^2 R_{H(\varepsilon)}(z_0))$ is uniformly bounded in ε. Now since $K(\varepsilon) \to K(1)$ strongly and Lemma 5.9, 3, we are done. □

5.2.2 Proof of Theorem 5.7

Now we begin with our first step towards singular operators by proving the case where $q_1 - q_0$ has compact support.

Lemma 5.11. *Let H_j, $j = 0, 1$, be Sturm–Liouville operators on (a, b) associated with τ_j and suppose that $r^{-1}(q_1 - q_0)$ has support in a compact interval $[c, d] \subseteq (a, b)$, where $a < c$ if a is singular and $d < b$ if b is singular. Moreover, suppose theorem H_0 and H_1 have the same boundary conditions (if any).*

Suppose $\lambda_0 < \inf \sigma_{ess}(H_0)$. Then

$$\dim \operatorname{Ran} P_{(-\infty,\lambda_0)}(H_1) - \dim \operatorname{Ran} P_{(-\infty,\lambda_0]}(H_0) = \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \tag{5.15}$$

Suppose $\sigma_{ess}(H_0) \cap [\lambda_0, \lambda_1] = \emptyset$. Then

$$\begin{aligned} &\dim \operatorname{Ran} P_{[\lambda_0,\lambda_1)}(H_1) - \dim \operatorname{Ran} P_{(\lambda_0,\lambda_1]}(H_0) \\ &\qquad = \#(\psi_{1,\mp}(\lambda_1), \psi_{0,\pm}(\lambda_1)) - \#(\psi_{1,\mp}(\lambda_0), \psi_{0,\pm}(\lambda_0)). \end{aligned} \tag{5.16}$$

Proof. Define $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon)H_0$ as usual and observe that $\psi_{\varepsilon,-}(z, x) = \psi_{0,-}(z, x)$ for $x \le c$ respectively $\psi_{\varepsilon,+}(z, x) = \psi_{0,+}(z, x)$ for $x \ge d$. Furthermore, $\psi_{\varepsilon,\pm}(z, x)$ is analytic with respect to ε and $\lambda \in \sigma_d(H_\varepsilon)$ if and only if $W_d(\psi_{0,+}(\lambda), \psi_{\varepsilon,-}(\lambda)) = 0$. Now the proof can be done as in the regular case, Theorem 5.5 . □

Lemma 5.12. *Suppose H_0, H_1 satisfy the same assumptions as in the previous lemma and set $H_\varepsilon = \varepsilon H_1 + (1 - \varepsilon)H_0$. Then*

$$\|\sqrt{r^{-1}(q_1 - q_0)} R_{H_\varepsilon}(z)\|_{\mathcal{J}_2} \le C(z), \qquad \varepsilon \in [0, 1]. \tag{5.17}$$

In particular, H_0 and H_1 are resolvent comparable and

$$\xi(\lambda, H_1, H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)) \tag{5.18}$$

for every $\lambda \in \mathbb{R}\backslash(\sigma(H_0) \cup \sigma(H_1))$. *Here* $\xi = \xi_1$ *is assumed to be constructed such that* $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ *is a continuous mapping* $[0,1] \to L^1((\lambda^2+1)^{-1})$.

Proof. Denote by $G_\varepsilon(z,x,y)$ the Green function of H_ε. A simple estimate shows

$$\int_a^b \int_a^b |G_\varepsilon(z,x,y)|^2 |r(y)^{-1}(q_1(y) - q_0(y))| r(x) dx\, r(y) dy \le C(z)$$

for $\varepsilon \in [0,1]$, which establishes the first claim.

Furthermore, a straightforward calculation (using (4.6)) shows

$$G_{\varepsilon'}(z,x,y) = G_\varepsilon(z,x,y) + (\varepsilon - \varepsilon') \int_a^b G_{\varepsilon'}(z,x,t)(q_1(t) - q_0(t)) G_\varepsilon(z,t,y) dt.$$

(Note that this does not follow from the second resolvent identity unless $r^{-1}(q_1 - q_0)$ is relatively bounded with respect to H_0.) Hence

$$\|R_{H_{\varepsilon'}}(z) - R_{H_\varepsilon}(z)\|_{\mathcal{J}_1} \le |\varepsilon' - \varepsilon| C(z)^2 \tag{5.19}$$

and thus $\varepsilon \mapsto \xi(H_\varepsilon, H_0)$ is continuous. The rest follows from (3.9). □

Using the variational characterization for the SSF from Lemma 3.10, we can directly prove the result for the SSF:

Lemma 5.13. *Let* H_j, $j = 0,1$, *be Sturm–Liouville operators on* (a,b) *associated with* τ_j *and suppose that* $V^2 = r^{-1}(q_1 - q_0)$ *has support in a compact interval* $[c,d] \subseteq (a,b)$, *where* $a < c$ *if* a *is singular and* $d < b$ *if* b *is singular,* $q_0 - q_1 \ge 0$ *and* V *satisfies Hypothesis 5.8. Then*

$$\xi(\lambda, H_1, H_0) = \#(\psi_{1,\mp}(\lambda), \psi_{0,\pm}(\lambda)) \tag{5.20}$$

for almost every $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H_0)$. *Here* $\xi = \xi_1$ *is assumed to be constructed such that* $\varepsilon \mapsto \xi(H_0 - \varepsilon V^2, H_0)$ *is a continuous mapping* $[0,1] \to L^1((\lambda^2+1)^{-1})$, *where* $V^2 = H_1 - H_0$.

Proof. By Lemma 3.10, it is sufficient to show that for $\lambda \in \mathrm{Ker}(H_0 - \varepsilon V^2)$ that the number of zeros of the Wronskian increases by one. Without loss of generality, we will only show the formula for $\psi_{1,+}$ and $\psi_{0,-}$.

Since both functions solve the same differential equation for $x \in (c,b)$, we have that $W_x(\psi_{\varepsilon,+}(\lambda), \psi_{0,-}(\lambda))$ is constant for $x \in (c,b)$. Furthermore since the Wronskian being zero would imply that the solutions are linearly dependent and thus $\psi_{0,+}$ being in L^2 near b and λ thus an eigenvalue, we can assume it to be non-zero, and so no zero can escape to b. Denoting by $\vartheta_{i,\pm}$ the Prüfer angle of $\psi_{i,\pm}$, we now have that $\lambda \in \mathrm{Ker}(H_0 - \varepsilon V^2)$ is equivalent to

$$W_a(\psi_{\varepsilon,+}, \psi_{0,-}) = 0 \Leftrightarrow (\vartheta_{\varepsilon,+}(a) - \vartheta_{0,-}(a)) \pmod{\pi} = 0. \tag{5.21}$$

Now since $\vartheta_{\varepsilon,+}(a)$ is strictly monotonic in ε and Lemma 4.2, the result follows. □

Proof. (of Theorem 5.7) We first assume that we have compact support near one endpoint, say a.

Define by K_ε the multiplication operator by $\chi_{(a,b_\varepsilon]}$ with $b_\varepsilon \uparrow b$. Then K_ε satisfies the assumptions of Lemma 5.10. Denote by $H_\varepsilon = H_0 - K_\varepsilon V^2$, and by

$\psi_{\varepsilon,-}$ the corresponding solutions satisfying the boundary condition at a. By Lemma 5.10 we have that $\xi(H_\varepsilon, H_0) \to \xi(H_1, H_0)$ locally in L^1 and it remains to control the Wronskians.

We first show the $(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ case. Observe that

$$W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = W_x(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)) \tag{5.22}$$

for $x \leq b_\varepsilon$ and $W_x(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$ is constant for $x \geq b_\varepsilon$. Hence the value $\#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda))$ is increasing to $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ and constant for large ε. By monotone convergence, we have that $\#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) \to \#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda))$ locally in L^1.

Since Hypothesis 5.8 is satisfied with V^2, it is clear that it is also satisfied for $\chi_{(a,b_\varepsilon]}V^2$ and we can apply Lemma 5.11. Thus we have that $\#(\psi_{\varepsilon,-}(\lambda), \psi_{0,+}(\lambda)) = \xi(\lambda, H_\varepsilon, H_0)$, which converges to $\xi(H_1, H_0)$ locally in L^1. We thus have that $\#(\psi_{1,-}(\lambda), \psi_{0,+}(\lambda)) = \xi(\lambda, H_1, H_0)$ for almost every $\lambda \in \mathbb{R}\backslash\sigma_{ess}(H_0)$. For the $\#(\psi_{1,+}(\lambda), \psi_{0,-}(\lambda))$ case simply exchange the roles of H_0 and H_1.

Hence the result holds if we have compact support near one endpoint. Now repeat the argument to remove the compact support assumption near the other endpoint as well. □

5.3 Remarks

One might ask oneself, why we need the condition $q_0 - q_1 \geq 0$ in Theorem 5.7. the value $\#(u, v)$ is always positive, so choosing $q_0 - q_1 < 0$ and assuming Theorem 5.7 would lead to a contradiction. Therefore, we cannot completely drop the condition $q_0 - q_1 \geq 0$ in our setting.

However, one should note that $\lim_{x\to\infty}(\#_{(-x,x)}(u) - \#_{(-x,x)}(v))$ would be a signed analog for $\#(u, v)$, so an investigation might be possible ... Here $\#_{(-x,x)}(u)$ denotes the number of zeros of u in $(-x, x)$. So it might be interesting to investigate generalizations of Hartman's Theorem [8], [4, Thm.1.2.], in our setting.

In [11], we give a further extension to describe changes in the number of eigenvalues in spectral gaps. The proof of these results doesn't use Krein's spectral shift function, but an approximation technique developed by Günther Stolz and Joachim Weidmann in [20]. This method also allows us to obtain the results from [4] using relative oscillation theory. Furthermore we also treat the case of H_0 bounded from below and V^2 relatively form bounded with respect to H_0 in [11].

We also believe that the results can be extended to the case of Sturm–Liouville operators with different p.

CHAPTER 6

Applications

In this chapter we first recall basic properties of periodic Sturm–Liouville operators. After these preparations, we state our extensions of the oscillation results of those in spectral gaps.

6.1 Periodic Sturm–Liouville operators

For this section, let $(a, b) = \mathbb{R}$, $r = 1$, $\alpha > 0$ and p, q α-periodic functions. The last conditions mean:

$$p(x + \alpha) = p(x), \quad q(x + \alpha) = q(x). \tag{6.1}$$

For u a solution to τ also $\tilde{u}(x) = u(x + \alpha)$ will be a solution to τ. We will now be interested in the relation between u and $\tilde{u}$. Let (c, s) be a fundamental system for τ, that is $c(0) = 1$, $pc'(0) = 0$, $s(0) = 0$ and $ps'(0) = 1$. We then have the principal solution matrix given by:

$$\Pi(x) = \begin{pmatrix} c(x) & s(x) \\ pc'(x) & ps'(x) \end{pmatrix}. \tag{6.2}$$

The matrix M:

$$M = \begin{pmatrix} c(\alpha) & s(\alpha) \\ pc'(\alpha) & ps'(\alpha) \end{pmatrix} \tag{6.3}$$

is called the monodromy matrix. Now a calculation shows that:

$$\Pi(x + l\alpha) = \Pi(\alpha)\Pi(x + (l-1)\alpha) = M\Pi(x + (l-1)\alpha) = \cdots = M^l \Pi(x). \tag{6.4}$$

From this, we see that investigating M, will answer our questions about the relation between a solution and the solution shifted by α. We call the trace of M the discriminant $D = \mathrm{tr}(M)$. Since $\det M = 1$, we find the the eigenvalues of M are given by:

$$\mu_\pm = D/2 \pm \sqrt{D^2/4 - 1}. \tag{6.5}$$

Then one defines the floquet exponents as:

$$\gamma_\pm = \frac{1}{\alpha} \ln \mu_\pm. \tag{6.6}$$

We note that $\det M = 1$ implies that $\mu_+\mu_- = 1$, and thus $\gamma_+ + \gamma_- = 0$. Now since $\Pi(x)M^{x/\alpha}$ will be periodic in x, there exists α-periodic functions $p_\pm$ such that:

$$u_\pm(x) = p_\pm(x) \cdot e^{\gamma_\pm x} \tag{6.7}$$

or

$$u_-(x) = (p_-(x) + xp_+(x)) \cdot e^{\gamma_\pm x} \tag{6.8}$$

are a basis of the solution space. (6.8) can arise, if $\gamma_+ = \gamma_- = 0$.

Now, we will look at the self-adjoint extension H of τ and its spectrum. To do this, we will let all quantities depend on the eigenvalue parameter λ. We can determine the spectrum using D. For $|D| > 2$ the square root is positive, and we get two solutions, where one is square integrable near $+\infty$ and one near $-\infty$. This implies that we are in the resolvent set. For $|D| \leq 2$, we find that our solutions are bounded, but nowhere decaying, we are thus in the essential spectrum. So we have, that:

$$\sigma(H) = \sigma_{ess}(H) = \{\lambda \mid |D(\lambda)| \leq 2\}\,. \tag{6.9}$$

We call the maximal intervals $(\lambda, \tilde{\lambda})$ with $|\Delta(\hat{\lambda})| > 1, \hat{\lambda} \in (\lambda, \tilde{\lambda})$, spectral gap. By [2, Thm.2.3.1], we can order the spectral gaps $(-\infty, \lambda_0), (\mu_0, \mu_1), (\lambda_1, \lambda_2), \ldots$ as an increasing sequence, with λ_n corresponding to the n-th periodic eigenvalue and μ_n to the n-th anti-periodic eigenvalue. Denote by ψ_n the eigenfunction to λ_n and by φ_n the eigenfunction to μ_n, then we have:

Theorem 6.1. *1. ψ_0 has no zero in $[0, \alpha]$.*

2. ψ_{2m+1} and ψ_{2m+2} have exactly $2m+2$ zeros in $[0, \alpha)$.

3. φ_{2m} and φ_{2m+1} have exactly $2m+1$ zeros in $[0, \alpha)$.

Proof. [2, Thm.3.1.2]. □

Lemma 6.2. *For the derivate in λ of the discriminant we have:*

$$\frac{d}{d\lambda}D(\lambda) = \operatorname{tr}\left(\begin{pmatrix} u(\alpha) & v(\alpha) \\ pu'(\alpha) & pv'(\alpha) \end{pmatrix} \cdot \begin{pmatrix} \int uv & \int v^2 \\ -\int u^2 & -\int uv \end{pmatrix}\right). \tag{6.10}$$

Here all integrals are evaluated over one period α and (u, v) forms a fundamental system for $\tau - \lambda$.

Proof. [16, After prop. 3], [24, Lem.16.5:(†)]. □

One can note, that for u with period α and $u(0) = 1$, $u'(0) = 0$, and v such that $W(u, v) = 1$, that:

$$\frac{d}{d\lambda}D(\lambda) = v(\alpha) \cdot \int_0^\alpha u^2(t)dt. \tag{6.11}$$

Note that we have for u the endpoint of the n-th spectral gap:

$$D(\lambda) = \begin{pmatrix} (-1)^n & v(\alpha) \\ 0 & (-1)^n \end{pmatrix}. \tag{6.12}$$

This gives us the following formula to calculate $|D|'$:

$$\frac{d}{d\lambda}|D|(\lambda) = (-1)^n v(\alpha) \int_0^\alpha u^2(t)dt. \tag{6.13}$$

We will now turn our attention to the self-adjoint extension of the periodic Sturm–Liouville operator H_0. For $z \in \rho(H_0)$, we find by (6.7) for $\psi_\pm(z,x)$ the following estimate

$$|\psi_\pm(z,x)| \leq C(z)\exp(\gamma_\pm x) \tag{6.14}$$

with $\pm\Re\gamma_\pm < 0$ and $\gamma_+ + \gamma_- = 0$. Here $C(z)$ is some constant, which can depend on z. We thus note the following estimate on the Green function:

$$|G(z,x,y)| \leq |C(z)|^2 \begin{cases} \exp(\gamma_+ x + \gamma_- y) & x \geq y \\ \exp(\gamma_- x + \gamma_+ y) & x \leq y. \end{cases} \tag{6.15}$$

Now integrating $|\sqrt{|\tilde{q}(x)|}G(z,x,y)|^2$ over $\mathbb{R}^2$, one sees that:

$$\int_{\mathbb{R}^2} |\sqrt{|\tilde{q}(x)|}G(z,x,y)|^2 d(x,y) \leq const. \cdot \int_{\mathbb{R}} |\tilde{q}(x)|dx. \tag{6.16}$$

One thus finds, that Hypothesis 5.6 is satisfied for $\tilde{q} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \tilde{q} \leq 0$. We need to require $\tilde{q} \in L^2(\mathbb{R})$ to have that the multiplication operator by $\tilde{q}$ is relatively bounded.

6.2 An oscillation criteria for periodic Sturm–Liouville operators

The following theorem extends the results from [16].

Theorem 6.3. *For p, q_0 α-periodic, D the discriminant of the periodic differential equation, and q_1 a perturbed potential, such that $q_0 - q_1 \in L^1((1,\infty)) \cap L^2((1,\infty))$ and $q_0 - q_1$ has one sign near infinity. We define differential expressions by:*

$$\tau_0 = -\frac{d}{dx}p(x)\frac{d}{dx} + q_0(x), \quad \tau_1 = -\frac{d}{dx}p(x)\frac{d}{dx} + q_1(x) \tag{6.17}$$

and let H_0, H_1 be self-adjoint extensions of τ_0, τ_1 on $(1,\infty)$. Furthermore, let λ_0 be an endpoint of a gap in the essential spectrum of H_0. Then λ_0 is an accumulation point of eigenvalues of H_1 if

$$\liminf_{x\to\infty} x^2(q_1(x) - q_0(x)) > \frac{\alpha^2}{4|D|'(\lambda_0)} \tag{6.18}$$

and λ_0 is not an accumulation point of eigenvalues if

$$\limsup_{x\to\infty} x^2(q_1(x) - q_0(x)) < \frac{\alpha^2}{4|D|'(\lambda_0)}. \tag{6.19}$$

Proof. First note that by standard arguments, we can restrict our attention to interval near infinity, where $q_0 - q_1$ is one signed. Then the result was shown in [16, Thm.1] in the case, where the limit exists. The result now follows using Theorem 4.9. □

We note that even the second order term was also computed in [16]. So we obtain:

Theorem 6.4. *Under the same assumptions as the last theorem, and:*

$$\lim_{x\to\infty}(q_1(x)-q_0(x))x^2 = \frac{\alpha^2}{4|D|'(\lambda_0)}. \tag{6.20}$$

λ_0 *is an accumulation point of eigenvalues, if*

$$\liminf_{x\to\infty}\left(\left((q_1(x)-q_0(x))x^2 - \frac{\alpha^2}{4|D|'(\lambda_0)}\right)(\log x)^2\right) > \frac{\alpha^2}{4|D|'(\lambda_0)}. \tag{6.21}$$

And λ_0 *is not an accumulation point of eigenvalues, if*

$$\limsup_{x\to\infty}\left(\left((q_1(x)-q_0(x))x^2 - \frac{\alpha^2}{4|D|'(\lambda_0)}\right)(\log x)^2\right) < \frac{\alpha^2}{4|D|'(\lambda_0)}. \tag{6.22}$$

Proof. As the last theorem, but use [16, Thm.2.]. □

The main argument in [16] is a perturbation argument for the difference of Prüfer angles for the solution to the unperturbed equation and to the perturbed equation. Since this exactly corresponds to calculating the asymptotic of the Prüfer angle of the Wronskian. The name of relative oscillation theory also already arises in this paper

APPENDIX A

Notation

L^1	the space of all integrable functions
L^2	the space of all square integrable functions
$\mathfrak{C}$	the compact operators
$\mathfrak{F}$	the finite rank operators
$\mathfrak{L}$	the bounded linear operators
$\mathcal{J}^p$	the Schatten-p-class
$\mathcal{J}^2$	the Hilbert-Schmidt operators
$\mathcal{J}^1$	the trace class operators
$A_n \xrightarrow{s} A$	A_n converges strongly to A
$R_A(z)$	the resolvent of A at z
$\sigma(A)$	the spectrum of A
$\sigma_d(A)$	the discrete spectrum of A
$\sigma_{ess}(A)$	the essential spectrum of A
$\rho(A)$	the resolvent set of A
$\mathbb{C}$	the complex numbers
$\mathbb{R}$	the real numbers
$L^1_{loc}(I)$	the set of all functions $I \to \mathbb{C}$, which are locally integrable
$AC_{loc}(I)$	the set of all functions $I \to \mathbb{C}$, which are locally absolutely continuous
$W(u,v)$	the Wronskian
f'	the derivative of f in x
tr	the trace
det	the determinant
z^*	the complex conjugate of z

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Curriculum Vitae

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