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1. Introduction

Mortality rates and fertility rates are declining in most industrialized countries leading to a significant change in the age structure of populations. Different economic issues such as fiscal policy or the consequences of population aging on pension systems have been studied extensively by economists using models of overlapping generations of households (OLG-Models). For many applications OLG models offer an attractive framework. The set-up of overlapping generations implies the presence of different generations of economic agents at every point in time which makes OLG models especially useful to study questions of intergenerational transfers. As Ludwig (2002) argues OLG models are very suitable for analysing issues of fiscal policy since here the costs and benefits of different generations can be well presented. In this context OLG-models have been used to study issues of Ricardian Debt Neutrality, i.e. the question, whether fiscal policy tends to redistribute between generations or not (see for example Buiter (1988)). In an OLG model the economy always consists of heterogeneous agents – for example young and old households in the basic two period model. This set-up is able to reflect the fact that people of different ages will not necessarily act economically the same way. This concerns for example consumption behaviour and thus saving decisions, which can plausibly differ between younger and older people, reflecting the fact that the younger tend to save for their retirement and the older to dissave what they have been saving for consumption during their retirement. So OLG models are able to take into account changes in the propensity to consume - or more generally, changing behaviour - as people grow older.

OLG models have been used to study very different topics. These range from issues concerning social security systems to very specific issues like the future development of Carbon Emissions (compare Dalton et al. 2005). Thus, the field of research is broad and in some of these studies the development of age-composition of the population plays a fundamental role. Introducing realistic demography to the framework of research can thus be very fruitful.

Starting with a simple two-period model, economists have made many extensions of the basic OLG model, extending it for example from two to three and N life periods or different mortality patterns. Extending the model from the basic two-period model to further periods has the advantage, that the more periods there are in the model, the better life-cycle aspects are represented. Life-cycle models with many generations can take account of detailed differences in wealth, marginal propensity to consume and in labour supply and earnings of different agents. But as Blanchard points out OLG models with more than two generations tend to be analytically intractable. This is due to the fact that generations at different stages of their life-cycle have systematically different propensities to consume and different levels of wealth, making aggregation difficult or even impossible (Blanchard (1989), page 115). But, concerning quantitative empirical work, a two period framework is not very realistic. Interpreted in real life time one time-period would cover about 30 years. In other words one period represents the whole working lifetime of an individual. Assuming that for example consumption behaviour does not change throughout the whole working life is certainly not very realistic. So, in order to depict realistic lifecycles, for empirical applications, numerically solved models which contain a large number of generations and detailed patterns of life-cycle earnings, savings and consumption must be relied on[1].

Many OLG models are based on a framework that is very restrictive or even based on unrealistic assumptions concerning demographic aspects. When it comes to empirical applications, ignoring demographic aspects can lead to significantly different results and as a consequence to false or imprecise conclusions (compare Bommier and Lee, page 138). Li and Tuljapurkar (2004) for example find in their simulations significant and dramatically different implications of aging on diverse economic variables when including an age-dependent death rate into their framework.

As an example for a very restrictive demographic set-up one can take a look at the basic two period OLG version as studied by Samuelson (1958) and Diamond (1965). These OLG models take the population growth rate as exogenous. Mortality is ignored and simply set to zero until the end of a predefined length of life which is the same for all households. In this narrow environment there is no place for proper demographic analysis, especially concerning the decline in mortality and fertility. And as long as there are such restrictive assumptions made upon mortality and other demographic variables investigated the effects of inherent changes in these and so the economic consequences of an aging society can not be studied using OLG models. For example, since the time of death is a predefined point in time in simple OLG models there is no uncertainty concerning an individual’s length of life. But it is this uncertainty of the time of death that is an important aspect for social security systems.

[1] Grafenhofer, Jaag, Keuschnigg and Keuschnigg (2006); page 1

Blanchard (1989) for example includes a form of insurance company to his framework in order to take into account the impact of uncertainty of lifetime on individual's behaviour.

2. Motivation and Outline

This section gives an outline of my thesis. First I present a short review on the literature concerning realistic demography in OLG models. This review on the literature serves as a motivation for the topic of my thesis and describes demographic aspects of diverse frameworks. A more detailed outline of my thesis concludes this section.

2.1 A short literature review

The standard OLG model as pioneered by Samuelson (1958) and Diamond (1965) is a simple two period model in which all economic agents live for exactly two periods and die at the end of the second. From a lifecycle point of view the first life period represents working life while the second represents retirement. Since in this simple approach death is fixed to a certain point in time, there is no individual lifetime uncertainty. Thus, having perfect foresight, agents know exactly how much to save for consumption during their retirement. And as they can not die unexpectedly bequests are not made, since it is assumed that agents do not consider their heirs and parents. In order to be able to represent life-cycle aspects in greater detail some effort has been put into extending the model to a greater number of periods (Auerbach and Kotlikoff (1987)). This is necessary in order to take altering behaviour at different stages of life into account. As Bommier and Lee (2001; page 137) point out, two periods are the minimum to encompass the fact that not all economic agents are the same. But a two period model, they argue, fails to represent the most basic feature of the human economic life cycle, namely that life not only ends but also begins with a period of dependency.

An initial step towards more realistic demography is Blanchard's (1989) continuous Model of Perpetual Youth which simply assumes an age-independent constant mortality rate. As there is a permanent risk of death economic agents face uncertainty concerning their lifespan. Since mortality is assumed to be constant and consequently age-independent everyone has the same life expectancy[2], regardless of her/his actual age. In other words, age doesn't alter the lifetime horizon and so, in some way, people have perpetual youth.

Facing the same probability to die implies that the propensity to consume is the same for all households. As acknowledged by Blanchard, this approach is unable to capture the life-cycle aspect of life, which is the essence of overlapping generation models[3]. The great advantage of Blanchard's model is its simple empirical application, which is the reason why it has become a widely used tool of quantitative analysis[4]. Its main drawback however is the absence of life-cycle detail concerning earnings, consumption and saving as well as its rigid assumption of an age-independent mortality rate. Concerning earnings for example Blanchard defines labour income to be less for older cohorts than for younger ones in order to be able to capture the effect of retirement on income. His way of defining the distribution of labour income between generations also implies that individual labour income decreases throughout the entire life time (if aggregate labour income is assumed to be constant through time). This certainly does not reflect realsitic income profiles.

Gertler (1999) extends Blanchard's framework distinguishing between workers and retirees. He assumes that workers do not face mortality but stochastically move into retirement. Once retired, they are confronted, as in Blanchard's model, with a constant risk of death. Keuschnigg and Keuschnigg (2003) examine the consequences of demographic change on diverse pension reform scenarios for Austria using Gertler's approach. They also assume that mortality occurs only when households enter their second and final life stage. Again retirees face a constant and so a quasi age-independent probability of death. In their work Gertler's model is extended by endogenizing labour supply. This extension is made in order to analyse the effects on the labour market resulting from population ageing. The transition from workers to retirees is modelled in a similar way as is mortality in the Perpetual Youth Model. Workers face a constant risk of being retired and thus face uncertainty concerning their future status in the life cycle while retired people face a permanent risk of death.

In addition to these age-independent approaches other authors model age-dependency of mortality. Bommier and Lee (2001) use a general mortality function which is able to capture the age of specific agents and so make allowance for different probabilities of death for people of different age groups. To represent the uncertainty of time of death they use a smooth survival function which they do not restrict to any specific form and thus develop a very general framework. They show that many results known from simpler models can be

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[2] This approach of modelling mortality implies an exponential distribution of the "probability of death" and the memorylessness of the exponential distribution implies constant expectation.
[3] Bommier, Lee page 136
[4] Grafenhofer et. al; page 2

extended to their more realistic framework. The motivation to develop such a model with realistic demography, they point out, is to avoid false conclusions in empirical applications.

Li and Tuljapurkar (2004) use a probability distribution of the *age at death* to achieve a more realistic representation of mortality. Alongside changes in the life expectancy, they argue, decreasing uncertainty in the timing of death is a very important aspect of demographic aging. Concerning individual decision making agents are not only confronted with an increase in their life expectancy but also with decreasing uncertainty of their lifespan. Indeed the relationship between increasing life expectancy and decreasing variance of age at death is almost linear[5]. Working directly with the distribution of death age, they argue, enriches the framework by allowing for a more realistic analysis.

As they demonstrate, declining mortality in the 20th century has lead to a tightening of the distribution of the *age at death* in all industrialized countries which can also be seen as a decrease in the variance of death age. This second aspect of aging can be taken into account by using a distribution function of age at death. And as Li and Tuljapurkar show, using an age-dependent death rate has dramatically different implications for the effect of aging on consumption, interest rate, wages and wealth, while their framework still remains analytically tractable.

Grafenhofer, Jaag, Keuschnigg and Keuschnigg (2006) develop an alternative age pattern. The core principle is an alternative way of thinking of “age”. Instead of defining age as “time since birth” they model only a few stages of age, where age itself captures diverse attributes such as earning potentials and taste which characterise a persons stage in his/her life-cycle. Agents move stochastically from one stage to the next, but not necessarily every period. Thus, not all agents age at the same speed. Some remain longer in a certain stage, while others move faster from one stage to the next. These few different stages of age, they argue, capture empirically realistic life-cycle differences. Since aging of people occurs stochastically, the framework is labelled “Probabilistic Aging” Model. It generalizes Gertler’s (1999) model of workers and retirees to more age groups and allows for mortality already in younger age groups. Concerning demographic realism, this is an improvement since in Gertler’s model workers – or more generally speaking, younger age groups – do not face any risk of death.

[5] See Li and Tuljapurkar (2004) page 4.

Additionally, since there are eight age groups modelled, the PA-model can take heterogeneity among workers and retirees into account, again enriching their framework's realism.

Hock and Weil (2006) pay particular interest to the influence of fertility on the population age structure. Changes in fertility change the population age structure and this again affects economic outcomes. But not only reduced fertility changes the population age structure but also the population age structure affects fertility. This interdependence is due to the dependency burden working-age people face. Built on three very simple equations, giving the laws of motion of three age-groups they study the interaction of fertility, population age structure and economic outcomes.

Reduced fertility leads in the short run to a "demographic dividend" meaning an unusually high ratio of potential workers to dependants – children and retirees. But in the long run - Hock and Weil argue - the increased old age dependency resulting from reduced fertility can prevail and more than recoup gains from reduced fertility. Workers can thus be confronted with an increase in dependants although the number of children they have to take care of is lower than before. Using the Probabilistic Aging Model, Hock and Weil study this dynamic interaction of fertility, age structure and economic consequences.

2.2 Outline of my thesis

As the overview on the literature shows, one main aspect of introducing more realistic demography to OLG models concerns different approaches of modelling mortality and ageing. Therefore the focus of my thesis is on different approaches to model mortality. In particular it concentrates on three different frameworks dealing with alternative perceptions of mortality. The following table gives an overview of these different mortality approaches and the outline of my thesis.

	mortality approach		
	age-independent	**age-dependent**	**Probabilistic Aging**
Section	3	4	5,6
papers discussed	**Diamond**: basic OLG model without life time uncertainty	**Li, Tuljapurkar**: age-dependency of mortality by introduction of *age at death* distribution function	**Grafenhofer et. al**: new definition of ageing and distinguishing real time from human age-characteristics
	Blanchard: introduction of a constant mortality rate generates individual life time uncertainty		**Hock and Weil**: feedback of population age structure on fertility
	Gertler: further heterogeneity by distinguishing between workers and retirees		

Table 2.1: outline of the thesis

The third section presents and analyzes alternative frameworks, dealing with *age-independent mortality* approaches covering the basic two period model of Samuelson (1958) and Diamond (1965), Blanchard's Model of Perpetual Youth (1989) and Gertler's (1999) extension of Blanchard's model to a model of retirees and workers.

The fourth section of the thesis makes a step closer to demographic realism by moving from age-independent to an *age-dependent mortality* framework. It deals with Li and Tuljapurkar's (2004) paper of an age-dependent mortality framework paying particular attention to the effect of decreasing uncertainty in the time of death, which comes along with increasing life expectancy.

The fifth section is dedicated to Grafenhofer's model of *Probabilistic Aging* (2006) which, by allowing for heterogeneous aging of different agents introduces a high degree of diversity among an economy's population and thus achieves a high level of diversity concerning individual's life cycles.

Finally Hock and Weil's (2006) analysis of the interaction of fertility, age structure and economic dependency will be presented in section 6, which is based on a simplified version of the Probabilistic Aging model.

Section 7 draws some conclusions.

The argument for considering these particular papers is that their analysis shows very well the necessity of paying attention to proper demographic representation in OLG models. These papers are of high relevance as they show the practicability and power of OLG models in analysing issues of intergenerational dependency or simply the impact of mortality and fertility changes on an economy. What shall be achieved is to underline the importance of demographic realism within economic analysis.

To summarize, this thesis presents and compares different approaches of achieving more realism in OLG models concerning demographics and life cycle details. The aim is to view these models under a demographic perspective and to analyze their quality in reflecting human life cycle details.

Thereby the main focus is on the implementation of different mortality and fertility assumptions to the models' framework and how these different assumptions affect individuals' behaviour. Another important aspect will be the resulting implications of changing individuals' behaviour on the economy as a whole.

3. Age-independent mortality

The following section presents age-independent mortality approaches, starting with the basic two period model of Samuelson (1958) and Diamond (1965). This basic approach simplifies mortality patterns by assuming perfect survivorship for all agents from the first stage of life to the second – and final – life period. Despite its very simple framework the model is able to reflect different consumption behaviour of workers and retirees. But of course it lacks realism concerning important demographic aspects, assuming perfect survivorship and thus absence of uncertainty in agent's life expectation.

The second model gives a more complex framework of modelling mortality. In Blanchard's Model of Perpetual Youth (1989) agents face a constant and thus age-independent risk of death. The main drawback of this model is that the constant mortality assumption yields also age-independent consumption behaviour. It therefore fails to reflect a very important aspect of the life-cycle, namely the changing propensities to consume that one would expect to occur as agents grow older.

Finally Gertler's model of retirees and workers combines both approaches, by adopting and extending Blanchard's mortality framework of constant risk of death. His extension of the model to two stages of life, one working life stage and a second representing retirement is able to capture the fact of different consumption behaviour of retirees and workers, as well as allowing for lifetime uncertainty among agents. Thus, concerning demographic realism, it makes good progress by combining the advantages of the two previously mentioned models.

3.1 The basic two period Model (Diamond)

In the Diamond model[6] the economy consists of two generations at any point in time, a young cohort of workers and an old cohort of retirees. Every period a new cohort is born and the former young generation moves into retirement. Consequently the former retirees die as they move out of retirement. Young people work to finance their consumption during the first life stage and to save for retirement. Retirees simply consume as much as they can afford from the savings they made during their youth and the interest earned on savings, as there is no altruistic behaviour assumed. Concerning the transition from workers to retirees and retirees to death, there is no uncertainty assumed. So workers in period t move with probability one from their working stage to retirement in t+1. Death occurs after the retirement period also with probability one. Thus there is no individual lifetime uncertainty.

The model's framework yields a population structure always consisting of two different types of agents. At every point in time a new generation of workers is born and at the same time former workers move into retirement. Thus, the economy consists of two different generations overlapping each other due to their transition through the life cycle. This is the core feature of overlapping generation models.

3.1.1 Structure of the model

Individuals

Individuals born at time t split their labour income w_t between consumption in period t, c_{1t}= (1-s_t) w_t, and savings for retirement $s_t w_t$ in period t+1. Second period consumption c_{2t+1} is financed by the savings and the interest earned on savings, $c_{2,t+1} = (1+r_{t+1})s_t w_t$. An individual seeks to maximize his/her utility arising from consumption in both periods. The individual's utility function is given by

$$u_t = \frac{c_{1t}^{1-\rho}}{1-\rho} + \frac{c_{2t+1}^{1-\rho}}{(1-\rho)(1+\theta)} \tag{3.1}$$

where ρ measures the household's willingness to shift consumption between the two periods and θ measures individual's time preference. A small ρ implies that marginal utility falls slowly as consumption rises and thus agents are more willing to shift consumption between the two periods. The value 1/ρ gives the intertemporal elasticity of substitution. The smaller ρ, the greater the intertemporal substitution. The budget constraint follows as

$$c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1} = w_t \, . \tag{3.2}$$

Solving the individual's maximization problem[7] yields an optimal savings rate as

$$s(r) = \frac{(1+r)^{(1-\rho)/\rho}}{(1+\theta)^{1/\rho} + (1+r)^{(1-\rho)/\rho}} \, . \tag{3.3}$$

<u>Proposition 3.1:</u>

When the interest rate in the Diamond economy increases agents will increase their savings as long as their intertemporal elasticity of substitution (1/ρ) is greater than one and decrease their savings otherwise. Under the assumption of logarithmic utility the savings rate is independent of the prevailing interest rate.

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[6] The model is presented following David Romer (1996). For comparability to other models presented in this thesis the notation is changed. In contrast to Romer technological progress is set to zero for simplicity.

Proof: see Appendix A.

Proposition 3.1 states that the relationship between the interest rate and agent's savings rate depend on the parameter ρ. Whether the savings rate increases with the interest rate or not depends on the strength of two opposing effects. Intuitively an increase in the interest rate has an income effect as well as a substitution effect. As the interest rate rises second period consumption becomes more attractive compared to first period consumption since the price for second period consumption decreases. Individuals therefore have an incentive to shift consumption from the first to the second period – this is the substitution effect. Opposing this effect, the higher interest rate also increases available income in both periods. The higher interest rate makes it possible to save less in the first period without reducing second period's consumption, this is the income effect. Intuitively it depends on the individual's willingness to shift consumption between the two periods which of these two effects dominates. A high value of ρ (ρ>1) indicates that individuals are not willing to shift consumption very much and so the income effect will dominate, while a low value of ρ (ρ<1) implies the dominance of the substitution effect[8]. If the utility function is assumed to be logarithmic the value of ρ is equal to one. Then the two effects balance and the interest rate does not affect saving decisions.

Firms

Firms have two factors of production - capital and labour – and produce output according to $Y_t = F(K_t,L_t)$. The production function is assumed to fulfil the Inada-conditions[9] and to have constant returns to scale. As the economy is assumed to be competitive, capital and labour earn their marginal products. This implies the interest rate to be $r=f'(k)$ and wage $w=f(k)-kf'(k)$, both expressed in units of labour (see Appendix A.2 at the end of the section for calculation).

3.1.2 Dynamics of the economy

Aggregate capital in period t+1 is the amount saved by the young cohort in period t. Thus, $K_{t+1}=s(r_{t+1})L_t w_t$, where L_t gives the number of individuals born at time t. Population grows at the exogenous rate n, thus $L_{t+1}=(1+n)L_t$. Expressed in units of labour – i.e. dividing by the

[7] The calculation of s is presented in Appendix A.1 at the end of section 3.

[8] This follows from the derivation of the savings rate with respect to r.

[9] Inada Conditions state that $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$ (see Romer, page 9).

size of the labour force in t+1 L_{t+1} yields $k_{t+1} = \frac{1}{1+n} s(r_{t+1}) w_t$. Considering the fact that capital and labour are paid their marginal products, capital accumulation follows

$$k_{t+1} = \frac{1}{1+n} s(f'(k_{t+1}))\left[f(k_t) - k_t f'(k_t)\right]. \qquad (3.4)$$

This equation implicitly defines k_{t+1} as a function of k_t. As soon as k_t takes a value such that $k_{t+1}=k_t$, the capital stock has reached its steady state. But as the right hand side of the equation also depends on k_{t+1}, the question if there is any unique value of k^* can not be answered easily for the general case. In the special case of logarithmic utility and with Cobb-Douglas production function the dynamic equation of capital accumulation takes the form[10] $k_{t+1} = \frac{1}{(1+n)(2+\theta)}(1-\alpha)k_t^{\alpha}$.

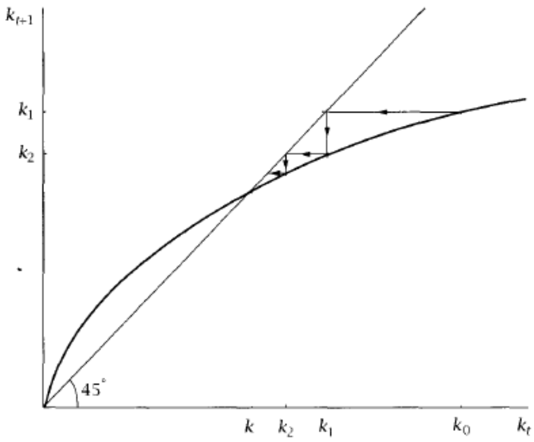

Fig. 3.1: dynamic behaviour of k (Romer, page 77)

The Figure plots k_{t+1} as a function of k_t. The 45° line represents all points where k_{t+1} equals k_t, thus all points possibly being an equilibrium. Setting $k_t=0$ implies k_{t+1} also being equal to zero. For small values of k_t the capital accumulation function lies above the 45°-degree line but crosses it eventually at the point k^* and stays below afterwards. Starting at point k_0 the capital stock converges to the point k^*, regardless of where the initial k_0 is set. Thus, in this special case, k^* is the unique, and globally stable, equilibrium value of the capital stock k.

In the general case the capital accumulation equation can yield different outcomes. For example there could be multiple values of k^* as in Figure 3.2a or the economy's capital stock could converge to zero, regardless of its initial value as in 3.2b. Figure 3.2c shows a case

[10] Logarithmic utility implies ρ equal to one which yields a savings rate r=1/(2+θ). The assumption of CD production function yields $F(K,L)=K^{\alpha}L^{1-\alpha}$. In units of labour $f(k)=k^{\alpha}$. Inserting this into (3.4) finally yields this expression.

where, depending on the initial value of capital, the economy converges to zero or to a strictly positive value of *k*.

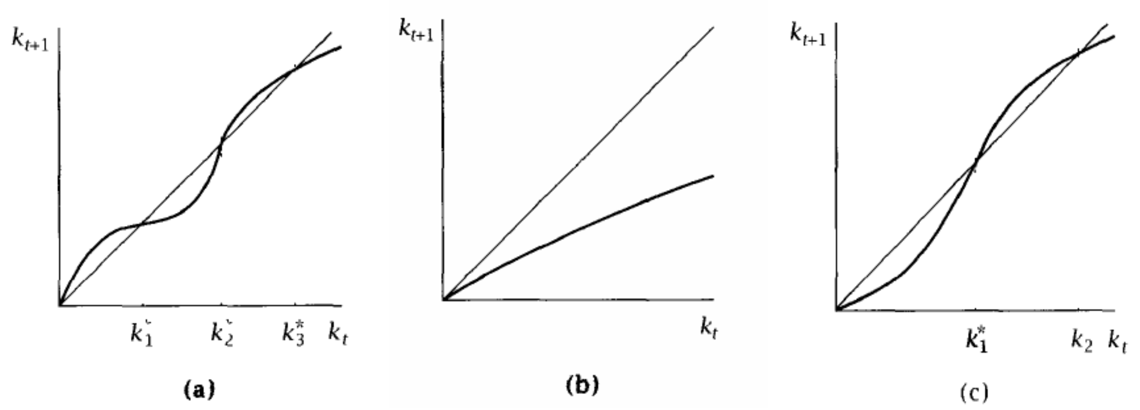

Fig. 3.2: possible relationships between k_t and k_{t+1} in the general case (Romer, page 80)

3.1.3 Lifecycle aspects of the Diamond model

In the Diamond model agents do not face any uncertainty concerning their length of life or their earning status. Thus they have perfect foresight and simply choose an optimal rate of saving in order to maximize their utility arising from consumption.

Of course the model's assumptions on the population's dynamics do not reflect realistic behaviour. But although the framework is very simple and restrictive concerning demographics, the Diamond model yields an important aspect of the human economic life cycle. As agents have two different stages of life, the model takes into account heterogeneity among individuals within the economy and allows for changing consumption behaviour as individuals change from one stage of life to another, i.e. "grow older". It thus reflects different behaviour among individuals living in the economy at the same time, as workers and retirees have different propensities to consume. But, as Bommier and Lee point out, this two-period framework fails to reflect a very important aspect of human life cycle, namely, that it also begins with a period of economic dependency (compare Bommier and Lee, page 136).

The assumption of two life periods has another disadvantage. As Hock and Weil remark, interpreted in real lifetime, both periods cover a length of 20 to 30 years (Hock and Weil, page4). Therefore, the model does not allow for detailed analysis within the two stages of life – work and retirement. Assuming unchanging behaviour of individuals throughout their whole working life or retirement does not seem very suitable to reflect human behaviour. Extending the framework to more stages of life would therefore allow for more detail in life cycle analysis.

3.2 A Model of Perpetual Youth (Blanchard)

The previous section showed that mortality patterns are modelled very simple in the Diamond model. One first improvement concerning the framework's realism is Blanchard's Perpetual Youth Model. Here life time uncertainty is introduced by the assumption of a permanent risk of death faced by individuals within the economy. One might expect that introducing more heterogeneity by allowing for different life spans of individuals leads to difficulties concerning the framework's analytical solvability. Blanchard avoids these potential difficulties by developing a model assuming *constant mortality rates*. Thus, each agent faces the same probability of death from period to period, regardless of his/her age. One can say that this approach leads to a "handy" model, but of course creates new sources for criticism because on the one hand Blanchard introduces life time uncertainty but on the other hand his approach creates an economy in which everyone has the same future life expectancy. As will be seen later in this section this again will have strong and unrealistic implications on individual's behaviour. Nevertheless the Perpetual Youth Model is able to reflect some realistic mechanisms concerning effects of retirement and changes in the probability of death. I discuss these features in later subsections as well as in the concluding subsection of this model (see section 3.2.3).

For now the following proposition states the core feature of the Perpetual Youth Model.

<u>*Remark 3.1:*</u>

Every agent in the Perpetual Youth Model faces the same future life expectancy – regardless of her/his individual age.

> <u>*Proof:*</u> The framework's setup implies an exponential distribution of the random variable "time until death" (see Remark 3.2). Therefore agents' future life expectancy is time invariant due the exponential distribution's property of mermorylessness. As a consequence individuals' age does not influence future life expectancy – it remains 1/p throughout time. And as p is assumed constant over time, each agent faces the same life expectancy.

As the framework is set in continuous time it is useful to compare the results obtained for the Perpetual Youth Model with those from the Ramsey(1928), Cass(1965) and Koopmans(1965)

Model of Infinitely Lived Agents to see the effects of the introduction of a constant probability of death on households' behaviour.

3.2.1 The model

Blanchard's framework rests on the assumption of a *constant probability of death*, defined as p. All agents within the economy face the same risk of death throughout their entire life. This assumption has important implications as already stated in remark 3.1. As Blanchard points out the constant death rate implies an exponential distribution for the random variable "time until death"[11].

<u>*Remark 3.2:*</u>

The assumption of constant mortality implies an exponential distribution of the random variable "time until death". Its density function is given by $f_x(t) = p\exp(-pt)$ with expected value $\mathrm{E}(X) = \int_0^\infty tp\exp(-pt)dt = p^{-1}$.

So the expected future lifespan of an agent is 1/p – the first moment of the exponential distribution. Clearly, a high probability of death, p, implies a short expected lifetime horizon. If p is set to zero the horizon becomes infinite similar to the Ramsey model of Infinitely Lived Agents. Therefore - as the Ramsey model can be seen as a special case of the Perpetual Youth Model - comparing these two models is convenient to see the implications of the presence of life time uncertainty.

At every instant of time a new cohort of agents is born, consisting of individuals who all have the same probability of death. As cohorts are assumed to be *"large enough"* they decrease over time with the constant rate of p[12]. Thus, although individuals face uncertainty concerning their individual length of life, cohorts decrease deterministically through time. Together with the normalization of cohort's size to p it follows that the size of a cohort s – s denoting time of birth - at time t is

$p\exp[-p(t-s)]$.

[11] The exponential distribution has the property of being „memoryless". This special property of the exponential distribution allows for constant, and thus age-independent, life expectancy.

[12] This is implied by the "Law of Large Numbers". "Large enough" ensures that the Law of large numbers can be applied.

This normalization ensures that at any point in time total population is equal to one[13].

3.2.1.1 Life Insurance as a consequence of uncertain life time

The probability of dying alters the maximization problem of agents in the Perpetual Youth Model. Life time uncertainty leads to a problem for individual's behaviour. Assuming that individuals maximize their total lifetime utility without caring about their relatives and prohibiting negative bequests the probability of dying gives rise to an optimality problem. Typically agents will die leaving positive or negative bequests behind. Leaving positive bequests implies that the individual has not consumed as much as would have been possible, meaning that s/he has not shown optimal behaviour. Blanchard therefore introduces a life insurance company, offering positive and negative life insurance. An individual who has accumulated wealth over time, facing the risk of leaving positive wealth behind, would be better off selling the claim on his or her wealth contingent on his/her death to this insurance company. In exchange for the claim on the estates of the individual, the insurance company makes premium payments to the living. This very important feature of the Perpetual Youth Model is stated in the following assumption.

Remark 3.3:

Facing life time uncertainty, rational individuals are going to contract all their wealth to a life insurance company to avoid suboptimal consumption behaviour.

The life insurance company is assumed not to make any profits implying an insurance premium of p per unit time. Rational agents will then contract to have all their wealth, v_t, go to the insurance company in case of their death, if negative bequests are assumed to be prohibited and in the absence of altruistic behaviour. In exchange individuals receive a premium pv_t per unit time.

The insurance company faces no uncertainty since the constant mortality rate p implies a non-stochastic death rate of p per unit time. Since population is normalized to one at any point in time, the insurance company receives pv_t from the dying and pays premia pv_t to those living.

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[13] $\int_{-\infty}^{t} p \exp[-p(t-s)] ds = 1$

3.2.1.2 Consumption and Wealth

The following section solves the individual agent's maximization problem and derives aggregate magnitudes of consumption and wealth. The calculation will result in a system of equations which will be used to show the impact of mortality on the economy.

Individual Consumption

Individuals maximize their total life time utility facing a permanent risk of death. Thus, at time t an individual maximizes his/her expected total future utility arising from consumption,

$$E\left[\int_t^{\infty} u(c(z))\exp[-\theta(z-t)]dz \mid t\right] \tag{3.5}$$

where *u(c(z))* is the utility of consumption at time *z* and θ is a parameter measuring time preference. The higher θ, the less future consumption is valued compared to current consumption. Taking the expectation over the future utility reflects the uncertainty resulting from the risk of death. From the fact that the constant probability of death implies an exponential distribution of survivorship it follows that the probability for an individual - that survived until time t - to be still alive at a later point in time *z* is *exp[-p(z-t)]*. Blanchard assumes logarithmic utility of consumption yielding the individual's objective function

$$\int_t^{\infty} \log c(z)\exp[-(\theta+p)(z-t)]dz\,. \tag{3.6}$$

Here one can make the first comparison to the Ramsey model. Compared to the maximization problem of an infinitely lived agent in the Ramsey model the objective function has changed by increasing the rate at which future utility is discounted from θ to $\theta + p$[14].

After defining the objective function I now turn to the individual's budget constraint. Individuals have two sources of non-labour income (non-human wealth) $v(z)$[15]. First they receive income due to the interest *r(z)* earned on their savings and second they additionally receive the premium from the insurance company. As mentioned before the total premium received by the insured person is *pv(z)* as rational agents will contract their entire amount of assets to life insurance (see 3.2.1.1). So in total the effective rate of interest on assets equals *r*

---

[14] In the Ramey model the representative agent maximizes $U_t = \int_t^{\infty} u(c(z))\exp[-\theta(z-t)]dz$ (compare Blanchard(1989), page 48).

[15] Non human wealth describes the wealth accumulated by an individual over time while human wealth is the discounted future labor income.

+ *p*. Thus, compared to an infinitely lived agent in the Ramsey Model, the probability of death increases the effective interest rate of individual's assets by the death rate *p*. In total this yields individual's dynamic budget constraint as

$$\frac{dv(z)}{dz} = [r(z) + p]v(z) + y(z) - c(z) \tag{3.7}$$

Thus, the difference between labour income y(z) and consumption c(z) plus the effective interest on his/her wealth. To avoid individuals going into debt forever a no-Ponzi-game condition is implemented. It states that, in the limit, the present value of an individual's assets must be equal to zero.

$$\lim_{z\to\infty} \exp\left\{ -\int_t^z [r(\mu) + p]d\mu \right\} v(z) = 0 \tag{3.8}$$

Without this condition individuals could accumulate debt forever and, as leaving negative bequests behind is prohibited, could protect themselves by buying life insurance.

After the objective function and the dynamic budget constraint are defined one can now turn to solving the individual's maximization problem.

Optimum:

The first order condition for agent's consumption is given by

$$\frac{dc(z)}{dz} = \{[r(z) + p] - (\theta + p)\}c(z) = [r(z) - \theta]c(z)\,. \tag{3.9}$$

Let individual's human wealth be denoted by h(t) and the discount factor by R(t,z). An individual's consumption at time t in the Perpetual Youth economy is given by the following equations:

(i) $c(t) = (\theta + p)[v(t) + h(t)]$

(ii) $h(t) = \int_t^\infty y(z)R(t,z)dz$ (3.10)

(iii) $R(t,z) \equiv \exp\left\{ -\int_t^z [r(\mu) + p]d\mu \right\}$

Proof: see Appendix A.3.

The term *(θ+p)* gives the individual's propensity to consume. As is apparent from 3.10(i) the propensity to consume is independent of the interest rate *r* and of individual age. Independence from the interest rate follows from the assumption of logarithmic utility which implies unit elasticity of substitution between consumption over different periods (compare proposition 3.1). On the other hand age-independency of the individual's propensity to consume follows from the constant probability of death. Thus, all economic agents have the same propensity to consume, regardless of their age. This result is very problematic from a demographic point of view as one expects consumption behaviour to change with age. Further more it points out the perpetuity of agents' lifes in this model.

Aggregate Consumption

After considering individual behaviour I now turn to the aggregation of consumption. For this the aggregate magnitudes of human wealth and non-human wealth have to be derived. Aggregate consumption follows from summing up individual consumption over all living generations. Thus, aggregate consumption follows as

$$C(t) = \int_{-\infty}^{t} c(s,t)\, p \exp[-p(t-s)]ds \tag{3.11}$$

where c(s,t) denotes consumption at time t of a generation born at time s and *pexp[-p(t-s)]* is the size of the generation born t-s periods ago. For aggregate labour income Y(t), nonhuman wealth V(t) and human wealth H(t) definitions are analogue.

<u>*Remark 3.4:*</u>

Aggregate consumption at time t is given by equation 3.12

$$C(t) = (p+\theta)[H(t)+V(t)]. \tag{3.12}$$

<u>*Proof:*</u> see Appendix A

In remark 3.4 the aggregate expressions of human and non-human wealth are already used. These are derived in the following subsection.

Aggregate Human and Nonhuman Wealth

The dynamic behaviour of human wealth depends strongly on the assumptions made on labour income. In order to be able to take into account decreasing labour income with retirement and thus with age Blanchard assumes the distribution of labour income across different generations at a certain point in time t to follow

Remark 3.5:

The distribution of labour income across generations is given by the equation

$$y(s,t) = aY(t)\exp[-\alpha(t-s)], \ \alpha \geq 0. \tag{3.13}$$

The exponential term captures the decline of labour income with cohort's age. The parameter α measures the degree at which labour income decreases from younger to older cohorts and *a* is a constant[16]. Clearly α equal to zero implies constant and therefore age-independent labour income. This definition implies that labour income y(t) is smaller for members of older generations, as long as α is not zero. And, if aggregate labour income Y(t) is assumed to be constant over time, individual labour income decreases steadily over lifetime.

Remark 3.6:

Aggregate human wealth follows

$$\frac{H(t)}{dt} = \left[r(t) + p + \alpha\right]H(t) - Y(t) \tag{3.14}$$

where $\lim_{z\to\infty} H(z)\exp\left\{-\int_t^z [\alpha + p + r(\mu)]d\mu\right\} = 0$. (3.15)

Proof: see Appendix A

Remark 3.7:

Aggregate non-human wealth *V(t)* follows

$$\frac{dV(t)}{dt} = r(t)V(t) + Y(t) - C(t). \tag{3.16}$$

Proof: see Appendix A

Comparing equation (3.7) for the accumulation of individual nonhuman wealth with equation (3.16) shows a striking difference. Individual nonhuman wealth accumulates at rate *r*+*p* but aggregate wealth only at rate *r*. This is not surprising as the insurance premium payments – in

[16] The value of a is (α+*p*)/*p*. The calculation can be found in the appendix A.4.

aggregate a total of *pV(t)* – do not represent an increase in total wealth but a transfer payment from those who have died to those still living.

Aggregate Behaviour

Collecting previously derived equations, the following system of equations describes the Perpetual Youth economy in its aggregate

$$C(t) = (p+\theta)[H(t)+V(t)] \qquad (3.12)$$

$$\frac{dV(t)}{dt} = r(t)V(t) + Y(t) - C(t). \qquad (3.16)$$

$$\frac{H(t)}{dt} = \left[r(t) + p + \alpha\right]H(t) - Y(t) \qquad (3.14)$$

$$\lim_{z\to\infty} H(z)\exp\left\{-\int_t^z [\alpha + p + r(\mu)]d\mu\right\} = 0 \qquad (3.15)$$

How does the presence of mortality affect the economy according to these equations? Viewing equation 3.12 and 3.14 shows, that both, propensity to consume and the discount rate of human wealth (the term in brackets on the right hand side of equation 3.14) are increasing functions of the probability of death. Thus, an increase in p increases the propensity to consume and reduces the value of future income. Also an increase in α - the decline rate of labour income - increases the discount rate of aggregate human wealth. The faster labour income declines over time, the higher the discount rate of aggregate future labour income.

Notice: an alternative representation of aggregate consumption is given by equation 3.17. This will be more convenient for the following analysis (see Appendix A.5)

$$\frac{dC}{dt} = (r+\alpha-\theta)C - (p+\alpha)(p+\theta)V \qquad (3.17)$$

3.2.2 Characteristics of the general equilibrium

Blanchard assumes a simple production function $F(K,L)$ where output is produced by capital and labour. Population size – and thus labour force - is normalized to one. Both input factors are paid their marginal products.

The effect of mortality

To study the impact of mortality on the equilibrium, labour income is initially held constant. In the case of constant labour income - α is equal to zero – it follows from (3.17) and (3.16) that the dynamics of the economy are given by[17]

$$\frac{dC}{dt} = [F'(K) - \theta]C - p(p + \theta)K \tag{3.18}$$

$$\frac{dK}{dt} = F(K) - C\,. \tag{3.19}$$

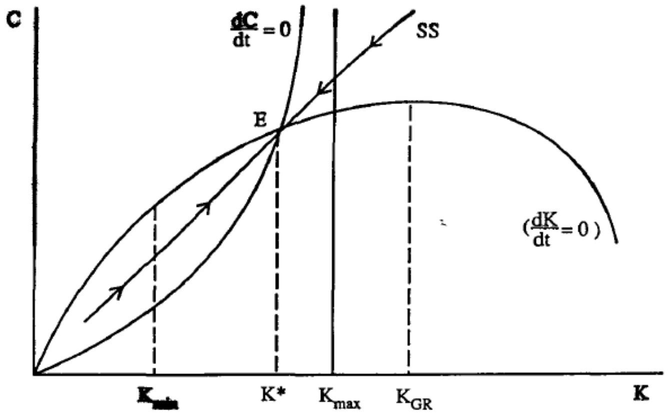

Figure 3.3: Dynamic adjustment with uncertain lifetime in the case of constant labour income (Blanchard, Fisher ; page 123)

Figure 3.3 shows the phase diagram of the dynamic system under the assumption of constant labour income. The two loci $\frac{dC}{dt}=0$ and $\frac{dK}{dt}=0$ describe the dynamic behaviour of consumption and capital. From equation (3.18) it follows that the first locus is an increasing function of K and approaches K_{max} asymptotically. The second locus is concave since F(K) is a concave function.

<u>*Proposition 3.2:*</u>

The steady state capital stock K^* lies between the boundaries $\theta < F'(K^*) < (\theta + p)$ where K_{min} denotes the level of capital for which $F'(K_{min}) = \theta + p$ and for K_{max} is such that $F'(K_{max}) = \theta$.

<u>Proof</u>: Appendix A

[17] Using r=F´(K) and V=K. For the second locus use Y=F(K)-KF´(K) from assumption of competitive markets and rV=F´(K)K.

The fact that $F'(K^*)$ is greater than θ implies that, under the assumption of constant labour income, the equilibrium is always dynamically efficient. This is due to the fact, that the upper boundary K_{max} is necessarily inferior to the level of capital that maximizes consumption, K_{GR} – the Golden Rule Level of capital - which satisfies $F'(K_{GR})=0$ [18]. A second striking result is stated in the next proposition.

Proposition 3.3:

K^* is a decreasing function of the probability of death - p. Thus, increasing mortality causes decreasing capital accumulation.

> *Proof:*
>
> An increase in p shifts the $\frac{dC}{dt}=0$ locus to the left. For a given level of K an increase in p requires C to increase in order to keep the relation $\frac{dC}{dt}=0$. This leads to the shift in the locus and a decreasing K*.

Since K^* is decreasing in p a higher probability of death increases the interest rate and decreases the capital stock. In the extreme case of r equal to θ there would be no capital accumulation at all. This is due to the assumption of constant labour income. Since individuals are born without nonhuman wealth (V) they would simply consume as much as they earn and neither save nor dissave. This can be seen by setting r equal to θ and V equal to zero in equation (3.17). Then the rate of change of consumption would be zero and thus constant through time. The assumption of constant labour income together with the fact that consumption doesn't change, implies that agents choose consumption equal to income. As a consequence an interest rate superior to θ is needed to allow for capital accumulation.

The effect of retirement

If labour income is assumed to decrease over time - $\alpha > 0$ - individuals have an incentive to save. As mentioned before this decrease in labour income is meant to capture the effect of retirement. Concerning the properties of the steady state the assumption of decreasing labour income can lead to dynamic inefficiency. In contrast to the case of constant labour income the steady state capital stock is no longer restricted to being inferior to the Golden Rule Level.

[18] From dK/dt=0 the maximal level of C follows as F´(K)=0.

Figure 3.4 shows the phase diagram for this scenario. In the case of declining labour income the two loci are given by

$$\frac{dC}{dt} = [F'(K) + \alpha - \theta]C - (p + \alpha)(p + \theta)K \quad (3.20)$$

$$\frac{dK}{dt} = F(K) - C\,. \quad (3.19)$$

K_{max} now is defined as $F'(K_{max})=\theta-\alpha$. As $\theta-\alpha$ can be negative or positive, the capital stock K can be superior to the Golden Rule Level of capital, which again satisfies $F'(K_{GR})=0$. Thus the steady state capital stock can be dynamically inefficient, as illustrated in figure 3.4.

Proposition 3.4:

The effect of retirement on income in this framework leads to increasing capital accumulation.

Proof:

Analysing the consequences of declining labour income (represented by an increase in α) shows that an increase in α shifts the $\frac{dC}{dt} = 0$ locus down leading to an increase in the capital stock as long as p+θ is assumed to be small (at least smaller than one). Then an increase in α increases both terms in equation 3.20. This increase is smaller on the ride hand side as long as the p+θ term is small. Thus, for a given level of C, K must increase to keep the equation $\frac{dC}{dt} = 0$. Blanchard (1985) proofs the proposition by deriving the derivative of the $\frac{dC}{dt} = 0$ locus with respect to α (see Blanchard (1985), page 238)[19].

The intuition behind this result is quite straightforward. Facing declining labour income with age individuals decide to save for later times (save for *"retirement"*). This leads to an increase in capital accumulation which decreases the interest rate. In total this can lead to capital over accumulation and thus to dynamical inefficiency.

[19] Blanchard's result of the derivation is either slightly incorrect (according to my opinion the last equation's denominator should be to the power of two) or not well comprehensible. I therefore only refer to his result but do not explicitly replicate his calculation. But as long as p+θ is assumed to be small enough the proposition is correct.

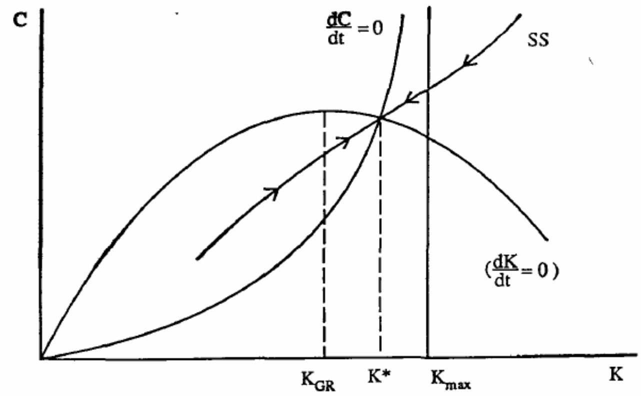

Figure 3.4: Dynamic adjustment with uncertain lifetime in the case of decreasing labour income (Blanchard, Fisher; page 126)

3.2.3 Lifecycle aspects of the model

In summary the presence of lifetime uncertainty in the Perpetual Youth Model yields two opposing dynamics as stated in propositions 3.3 and 3.4. First, the uncertain lifetime horizon leads to decreasing capital accumulation. The higher the probability of death, or equivalently the shorter the horizon, the lower is capital accumulation. Second, opposing this effect, reduced labour income during lifetime leads to increased savings and consequently to an increasing steady state capital stock. Depending on the size of these effects, the equilibrium of the economy can be dynamically inefficient (which would indicate capital over accumulation).

The model's framework introduces individual life time uncertainty. Thus, it makes good progress concerning demographic realism. But the unrealistic assumption of a constant probability of death also implies constant, and so age-independent, propensities to consume (compare equation (3.10)). Compared to the two period model of Diamond this is a clear drawback, as this is in sharp contrast to what one would expect to occur in reality. The assumptions made upon the distribution of labour income (compare equation (3.13)) also lack realism, as they imply that older cohorts have less labour income compared to younger ones. Again this is counterintuitive, as one would expect labour income to increase during working lifetime before decreasing before retirement. Nevertheless the dynamics implied by the model seem to capture some realistic dynamics, as the probability of death decreases capital accumulation and retirement increases it.

3.3 A model of retirees and workers (Gertler)

The previous section showed a quite simple approach of achieving more realism in OLG models. In his model Gertler extends Blanchard's model of Perpetual Youth by introducing two stages of life, one stage of working life and a second representing retirement. This extension allows for more heterogeneity among economic agents. In this set-up individuals are born as workers and face a permanent risk of being retired. Thus, they face uncertainty concerning their income status from one period to the next. The retirement risk is assumed to be constant through time so that workers do not differ concerning their individual retirement risk. Workers do not face any direct lifetime uncertainty, as mortality is assumed only to occur among retirees. Once retired, individuals are confronted with a constant risk of death, as in the Blanchard model. The transition from worker to retiree as well as the transition from retiree to death is modelled analogue to the mortality framework of the Perpetual Youth Model. The constant transition rates imply expected lifetime and working time in fashion of the Blanchard model.

In contrast to Blanchard's framework Gertler's model yields two sources of uncertainty. Both have implications on the model's framework. As in the Model of Perpetual Youth, Gertler assumes the existence of a life insurance company, redistributing wealth from dying retirees to those surviving (compare 3.2.1.1). The risk of falling out of the labour force implies that workers are potentially confronted with a loss in income – their wage. To address the problem of potential retirement agents' preferences are restricted to risk neutrality. Doing so yields certainty equivalent decision rules for workers facing income risk, as Gertler remarks (Gertler (1999), page 67).

3.3.1 Description of the model

The probability for a worker to remain working from one period to the next is defined as ω. As this probability is assumed to be independent of the agents working tenure, the expected length of working life follows to be $1/(1-\omega)$ using the same argument as in Blanchard's Model of Perpetual Youth[20]. Starting with retirement agents face a constant probability of death, defined as $1-\gamma$. Analogue to the working horizon, the average life expectancy, once retired, is $1/(1-\gamma)$.

[20] As in Blanchard's model, the constant probabilities of retirement and death imply exponential distributions for the variables "time until retirement" and "time until death". Thus the mentioned expected values follow as the first moments of the exponential distribution.

Individuals value their current state according to a CES non-expected utility function, which restricts workers to risk neutrality concerning income risk, but takes into account agents' desire to smooth consumption over different periods. Agents' value-function takes the form

$$V_t^z = \left[(C_t)^\rho + \beta^z E_t \{V_{t+1} \mid z\}^\rho \right]^{1/\rho} \tag{3.21}$$

where C denotes consumption and β is a discount factor, indicating that current consumption has a different value to agents than future consumption. The superscript (*z = w, r)* denotes whether the individual is a worker or a retiree. The discount parameter for retirees differs from the worker's discount factor, due to the probability of death. Thus, $\beta^w=\beta$ and $\beta^r=\beta*\gamma$. The expression $E_t \{V_{t+1} \mid z\}$ is the expectation of the value function next period, conditional on the individual's current status in the life cycle (r or w). Thus, for workers and retirees following expressions give the expected value for the next period

$$E_t \{V_{t+1} \mid w\} = \omega V_{t+1}^\omega + (1-\omega) V_{t+1}^r$$

$$E_t \{V_{t+1} \mid r\} = V_{t+1}^r$$

As Gertler states it, this way of modelling the preferences of individuals generates *certainty equivalent decision rules (Gertler, page 67).* Workers face the uncertainty of potential retirement, and consequently income loss, by averaging the value function over their two potential future states. Doing so, they only care about the expected value of their future income. The presence of the parameter ρ indicates agents' desire to smooth their consumption over time.

Concerning the risk of death for retirees, Gertler follows Blanchard's approach of assuming an insurance company redistributing from the dying to the surviving. It follows that each surviving retiree receives payments from the insurance company at a rate of R/γ on his/her wealth, if R denotes the gross return rate on assets[21].

Consumption by Retirees

In the following subsections I will derive retirees', workers' and aggregate consumption. As retirees are assumed not to have any labour income, their only source of income is out of assets. Assets and consumption of a retiree at time t are denoted by A_t^{rjk} and C_t^{rjk} where j is an index for the time of birth and k denotes the time, when the retiree left the labour force. Clearly, the amount of assets accumulated by an agent depends on how long an agent remains

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[21] As only the fraction γ of retirees survives, gross returns are spilt up equally among the surviving retirees. Thus each retiree receives a rate of R/γ on his/her assets.

part of the labour force. It is therefore necessary to capture the individual's length of working life in order to quantify his/her amount of assets. This is done by writing k as a superscript. The implicit assumption that agents with the same "biography" (same date of birth and same date of retirement) are economically the same (compare section 6.Probabilistic Aging) will make aggregation easy. Thus, a surviving retiree's assets accumulate at

$$A_{t+1}^{rjk} = (R_t / \gamma) A_t^{rjk} - C_t^{rjk} \qquad (3.22)$$

A retired agent chooses consumption so as to maximize his/her value function (3.21) subject to the asset accumulation equation (3.22).

Retirees' Optimality conditions:

Solving the maximization problem yields the consumption Euler equation for retirees as

$$C_{t+1}^{rjk} = (R_{t+1}\beta)^{\sigma} C_t^{rjk} . \qquad (3.23)$$

Thus retiree's change in consumption from period t to t+1 is determined by the interest rate and the subjective discount rate. Further calculation finally yields retirees' consumption at time t:

Let $\varepsilon_t \pi_t$ denote the retirees' marginal propensity to consume out of wealth (mpcw). Retirees' consumption at time t is given by

$$C_t^{rjk} = \varepsilon_t \pi_t (R_t / \gamma) A_t^{rjk} \qquad (3.24)$$

where the propensity to consume fulfils the following difference equation

$$\begin{aligned} \varepsilon_t \pi_t &= 1 - (R_{t+1}^{\sigma-1} \beta^{\sigma} \gamma) \frac{\varepsilon_t \pi_t}{\varepsilon_{t+1} \pi_{t+1}} \\ &= 1 - \beta\gamma, if \sigma = 1 \end{aligned} \qquad (3.25)$$

Proof: see Appendix A

In equation 3.24 and 3.25 π_t denotes the worker's mpcw. Consequently ε_t gives the factor, at which the retiree's mpcw exceeds the worker's mpcw.

Impact of uncertainties on retirees' consumption decision:

Viewing these equations gives a first insight on the impact of mortality on the economy. Notice first that in the case of logarithmic preferences (σ=1) the mpcw is constant. But for other values of σ the propensity to consume depends on the interest rate (see equation 3.25). The set-up of the model again implies that for all retirees the propensity to consume at time t is the same. As in the Perpetual Youth model the individual retiree's age does not influence consumption behaviour. In this sense retiree's mpcw is independent of the individual's age, but not of the individual's state in the life cycle (the PA model presented in section 6 extends this distinction between "age" and the position in the life cycle). This feature of consumption behaviour will make aggregation easy. As in Blanchard's model, an increasing probability of death (decrease in γ) raises the retiree's mpcw again reflecting what one expects to be the result from increased mortality.

Consumption by Workers

In contrast to retirees, workers earn labour income W_t. Their rate of return on assets is R_t and thus inferior to the retiree's rate of return, due to the fact that the insurance company only redistributes among retirees. For a worker, born at time j, the asset accumulation follows as

$$A_{t+1}^{wj} = R_t A_t^{wj} + W_t - C_t^{wj} \qquad (3.26)$$

The worker's maximization problem yields a more complicated decision rule for consumption than for retirees. The higher complexity results from the fact, that workers have to take into account the possibility of being retired and consequently the changing consumption behaviour implied by retirement (we expect increasing consumption as people move to retirement).

Workers' Optimality Conditions:

Let $\Lambda_{t+1} \equiv \frac{\partial V_{t+1}^r / \partial C_{t+1}^r}{\partial V_{t+1}^w / \partial C_{t+1}^w}$ denote the marginal rate of substitution of consumption across work and retirement. Workers' maximization problem yields the first order necessary condition

$$\omega C_{t+1}^{wj} + (1-\omega)\Lambda_{t+1} C_{t+1}^{rj(t+1)} = (R_{t+1}\Omega_{t+1}\beta)^{\sigma} C_t^{wj} \qquad (3.27)$$

and Ω_{t+1} is a factor that weights the gross return R_{t+1} and is given by

$$\Omega_{t+1} = \omega + (1-\omega)\varepsilon_{t+1}^{\frac{1}{1-\sigma}} . \qquad (3.28)$$

The Euler equation (3.27) states that a worker, knowing that s/he might be retired next period with a probability 1-ω, takes both possible future states into account for deriving his/her decision rule for consumption. It states, that the desired growth in consumption from period t to t+1 depends on the interest rate R relative to the subjective discount rate β.

Definition 3.1:

Workers' human wealth is defined as

$$H_t^j = \sum_{v=0}^{\infty} \frac{W_{t+v}}{\prod_{z=1}^{v} R_{t+z}\Omega_{t+z}/\omega}. \tag{3.29}$$

Let π denote the worker's mpcw and H_t^j his/her discounted future labour income. Worker's consumption at time t is given by

$$C_t^{wj} = \pi_t (R_t A^{wj} + H_t^j) \tag{3.30}$$

where the propensity to consume fulfils the following difference equation

$$\pi_t = 1 - (R_{t+1}\Omega_{t+1})^{\sigma-1}\beta^{\sigma}\frac{\pi_t}{\pi_{t+1}} \tag{3.31}$$

$= 1 - \beta$ if σ=1

Proof: Appendix A

Impact of uncertainties on workers consumption decision:

Considering workers' optimality conditions shows the impact of retirement risk and mortality on workers' consumption behaviour. As for retirees the propensity to consume is equal among workers. In contrast to retiree's mpcw the Ω term enters the worker's mpcw. This Ω factor is of high relevance in Gertler's framework. It is an increasing function in ε, the factor by which the retiree's mpcw exceeds the worker's mpcw and is higher than one, if ageing has a positive probability. The appearance of the term thus reflects the impact of ageing on the worker's decision process. The Λ term on the other hand indicates changing consumption behaviour as agents move from work to retirement and thus also reflects the impact of mortality on workers decisions.

In order to see the implications of the life cycle aspects on agents behaviour it is again convenient to make comparison to the case of infinite lives – just as done before in Blanchard's model. The presence of a life cycle implies a higher propensity to consume out of wealth than would be the case for an infinitely lived agent. This follows from the fact that Ω is greater than one[22]. Additionally the discount rate of future labour income increases, compared to infinite lives (compare equation (3.29)). While an infinite horizon implies a discount rate of R, this setting with a life cycle yields RΩ/ω as discount rate. The fact that R>1 and ω<1 implies that RΩ/ω>R and thus the discount rate is greater than the infinite horizon case. This increase comes on the one hand from the risk of retirement which makes workers put lower weight on future labour income – represented by ω in the discount factor.

On the other hand the Ω term reflects the fact of different propensities to consume out of wealth. As the propensity to consume is higher for retirees than for workers – or equivalently the consumption out of wealth is lower for workers - an additional unit of wealth has a higher value for workers than for retirees as workers can smooth their consumption over more periods. The marginal utility gain of an additional unit of wealth is lower for retirees than for workers[23]. This higher utility gain is reflected by the Ω term in the discount factor.

Aggregate consumption

The previous section showed that retirees and workers differ in their consumption behaviour and how both sources of uncertainty affected workers and retirees in their consumption behaviour. In this subsection I derive aggregate consumption and analyse the impact of retirement and mortality on the aggregated economy.

As the propensity to consume at time t is equal across retirees, the aggregate consumption of retirees C_t^r follows by summing up equation (3.24) across all individual retirees. A_t^r denotes aggregate assets of retirees in period t which in total earn an interest rate of R_t. Thus aggregate consumption of retirees is given by

$$C_t^r = \varepsilon_t \pi_t R_t A_t^r \tag{3.32}$$

Just as in the Model of Perpetual Youth, the aggregate consumption of retirees differs from the individual retiree's consumption by the factor of redistribution. Again the same argument holds. Insurance payments do not increase total wealth of retirees, and thus total consumption,

[22] This follows from the fact, that ε - the ratio of retiree's to worker's mpcw – is greater than one. Compare equation (3.28).

[23] Gertler page 71.

but redistributes from dying to surviving retirees. So in total there is no gain in non-human wealth from insurance.

From (3.30) for workers aggregate consumption is given by

$$C_t^w = \pi_t (R_t A_t^w + H_t) \qquad (3.33)$$

Definition 3.2:

Aggregate human wealth is defined as

$$H_t = \sum_{v=0}^{\infty} \frac{N_{t+v} W_{t+v}}{\prod_{z=1}^{v} (1+n) R_{t+z} \Omega_{t+z} / \omega} = \\ = N_t W_t + \frac{H_{t+1}}{(1+n) R_{t+1} \Omega_{t+1} / \omega} \qquad (3.34)$$

The discount rate for aggregate human wealth increases compared to individual worker's discount rate by the population growth rate (1+n). This is due to the fact, that the share of the total wage bill of those currently alive declines over time as the labour force grows (Gertler (1999), page 72).

Remark 3.8:

Defining the share of assets held by retirees as λ_t, the aggregate consumption function is given by

$$C_t = \pi_t \{ [1 + (\varepsilon_t - 1)\lambda_t] R_t A_t + H_t \}. \qquad (3.35)$$

Proof: Appendix A

Remark 3.9:

The retirees' share of assets develops according to

$$\lambda_{t+1} = \omega(1 - \varepsilon_t \pi_t) \lambda_t R_t \frac{A_t}{A_{t+1}} + (1 - \omega). \qquad (3.36)$$

Proof: Appendix A

Impact of uncertainties on aggregate consumption:

The life cycle feature of the framework affects aggregate human wealth in three ways. First, as for individual human wealth, workers know that their expected working tenure is finite. This leads to an increasing discount rate on future labour income, as the worker can not be sure of how long s/he will be remaining part of the labour force. This fact is represented by the presence of the ω in the discount term. Second, as agents wish to smooth their consumption over different periods, the utility gain from additional income is higher the earlier it is earned. This is represented by the term Ω. And finally, the population growth rate 1+n increases the discount rate of aggregate human wealth.

Aggregate consumption depends on the distribution of wealth between retirees and workers (λ_t) reflecting the fact that the different consumption behaviour of retirees and workers has a bearing on aggregate consumption.

3.3.2 Characteristics of the general equilibrium

The economy's output is produced following a Cobb-Douglas production function

$$Y_t = (X_t N_t)^{\alpha} K_t^{1-\alpha} \tag{3.37}$$

where X_t denotes the state of technology and K_t equals total assets A_t. Productivity grows at the exogenous rate 1+x, thus $X_{t+1}=(1+x)X_t$. Capital depreciates at rate δ. Assuming competitive markets wage and the interest rate follow as the marginal products of labour and capital. Thus

$$W_t = \alpha \frac{Y_t}{N_t} \tag{3.38}$$

and

$$R_t = (1-\alpha)\frac{Y_t}{K_t} + (1-\delta). \tag{3.39}$$

Capital accumulates at

$$K_{t+1} = Y_t - C_t + (1-\delta)K_t. \tag{3.40}$$

To study the effects of mortality and ageing it is convenient to view the economy in its steady state.

Remark 3.10:

The steady state of the economy can be expressed as a system of seven non-linear equations

(i) $(x+n+\delta)k = 1-\pi\{[1+(\varepsilon-1)\lambda]Rk+h\}$

(ii) $\lambda = \psi \dfrac{(1+x+n)(1+n-\gamma)}{1+x+n-\gamma\omega(R\beta)^{\sigma}}$

(iii) $h = \Gamma(R,\Omega)\dfrac{R\alpha}{R-(1+x+n)}$

(iv) $R = (1-\alpha)k^{-1}+1-\delta$ (3.41)

(v) $\pi = 1-(R\Omega)^{\sigma-1}\beta^{\sigma}$

(vi) $\varepsilon\pi = 1-R^{\sigma-1}\beta^{\sigma}\gamma$

(vii) $\Omega = \omega+(1-\omega)\varepsilon^{\frac{1}{1-\sigma}}$

In remark 3.10 k and h denote assets and human capital after normalizing to output and Ψ is the ratio of retirees to workers. Γ(R,Ω) is the fraction of human wealth H_t^{ss} to the level that would result in the steady state of the representative agent model H_t^{*ss} [24]. These seven equations are the steady state versions of the previously defined variables k, λ, h, R, π, ε and Ω. The following section will explain the impact of ageing and mortality on the economy using this system of equations.

The effects of mortality and income risk

Mortality and income risk affect the economy in three ways. The first striking result is, that the capital stock depends negatively on λ, the retiree's share of non-human wealth. This share again depends on Ψ, the ratio of retirees to workers. A rise in the retiree's asset's share λ lowers the capital stock, since retiree's mpcw is superior to the worker's mpcw. Thus a rise in λ rises total consumption and thus reduces the economy's steady state savings which equal the economy's investment.

[24] H_t grows at rate (1+x)(1+n)≈1+x+n in the steady state, so $H_t^{ss} = W_t + H_t^{ss}(1+x)(1+n)/[(1+n)R\Omega/\omega]$ and $H*_t^{ss} = W_t + H_t^{*ss}(1+x)(1+n)/R$. Thus $H_t^{ss} = W_t R/[R-(1+x)\omega/\Omega]$ and $H_t^{*ss} = W_t R/[R-(1+x+n)]$. The fraction Γ(R,Ω) results as $\Gamma(R,\Omega) \equiv \dfrac{H_t^{ss}}{H_t^{*ss}} = \dfrac{R-(1+x+n)}{R-(1+x)\omega/\Omega}$.

Second, the presence of life cycle increases both, retiree's and worker's propensity to consume - $\varepsilon\pi$ and ε - compared to infinite lives. Thus, the probability of dying, or retiring, raises the propensity to consume which again reduces k.

The third effect is the reduction of human wealth h. Compared to infinite lives human wealth is reduced to the fraction $\Gamma(R,\Omega)$ of human wealth that would result in the steady state of the representative agent model. This is due to the higher discount rate arising in the life cycle economy.

The life cycle economy's equilibrium thus depends on the dynamics of the labour force and the population.

3.3.3 Life cycle aspects of the model

Life cycle aspects affect the economy through different channels. As in Blanchard's model, the propensity to consume is an increasing function of the probability of death for retirees (compare equation (3.25)). Thus, the higher the probability of death, the faster retirees consume out of wealth. But this is also true for workers, even though they do not directly face mortality. Their risk of being retired induces them to consume at a higher rate than they would, if they were infinitely lived (compare equation (3.29)).

The second effect of the life cycle on the economy is that the worker's risk of retirement leads to an increase in the discount rate on human wealth. Additionally, retiree's higher propensity to consume leads to a further increase in this discount rate.

Concerning the aggregate consumption function (3.33), a striking modification is the appearance of the retiree's share of assets. The distribution of wealth between retirees and workers matters, as propensities to consume are different. This difference in the consumption behaviour on the individual's level is reflected in the aggregate consumption function by the appearance of λ, which is an increasing function of the retiree's population share (compare equation (3.35)).

Compared to Blanchard's model, Gertler achieves improvements concerning demographic realism. As aggregate consumption (3.33) depends on the retirees share on assets, a change in the population structure will affect consumption. On the other hand he introduces heterogeneity among agents of the economy by modelling two different stages of life. This is a clear improvement, as the main drawback of Blanchard's model is the implication of age-independent consumption behaviour (compare equation (3.10)).

Appendix A

The Diamond Model

A.1 Maximization problem in the Diamond Model

To solve the individual's maximization problem one has to maximize the utility function

$$u_t = \frac{c_{1t}^{1-\rho}}{1-\rho} + \frac{c_{2t+1}^{1-\rho}}{(1-\rho)(1+\theta)} \qquad \text{A.1}$$

with respect to the budget constraint

$$c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1} = w_t \,. \qquad \text{A.2}$$

To calculate the first order conditions one has to set up the Lagrange function, yielding

$$L = \frac{c_{1t}^{1-\rho}}{1-\rho} + \frac{1}{1+\theta}\frac{c_{2t+1}^{1-\rho}}{1-\rho} + \lambda\left[w_t - \left(c_{1t} + \frac{1}{1+r_{t+1}} c_{2t+1}\right)\right]. \qquad \text{A.3}$$

Differentiating the Lagrangian with respect to consumption in both periods c_{1t} and c_{2t+1} results in the expressions

$$\frac{\partial L}{\partial c_{1t}} = c_{1t}^{-\rho} - \lambda \qquad \text{A.4}$$

and

$$\frac{\partial L}{\partial c_{2t+1}} = \frac{1}{1+\theta} c_{2t+1}^{-\rho} - \lambda \frac{1}{1+r_{t+1}} \,. \qquad \text{A.5}$$

The first order conditions for c_{1t} and c_{2t+1} follow by setting these expressions to zero

$$c_{1t}^{-\rho} = \lambda \qquad \text{A.6}$$

$$\frac{1}{1+\theta} c_{2t+1}^{-\rho} = \lambda \frac{1}{1+r_{t+1}} \,. \qquad \text{A.7}$$

Inserting A.6 into A.7 and expressing in terms of c_{2t+1}

$$c_{2t+1} = \left(\frac{1+r}{1+\theta}\right)^{1/\rho} c_{1t} \qquad \text{A.8}$$

and further inserting into the budget constraint (A.2) yields

$$c_{1t} + \frac{(1+r_{t+1})^{(1-\rho)/\rho}}{(1+\theta)^{1/\rho}} c_{1t} = w_t \qquad \text{A.9}$$

and rearranging gives

$$c_{1t} = \frac{(1+\theta)^{1/\rho}}{(1+\theta)^{1/\rho} + (1+r)^{\frac{1-\rho}{\rho}}} w_t \,. \qquad \text{A.10}$$

Recalling the fact that $c_{1t}=(1-s_t)\, w_t$ yields an expression for the optimal saving rate s(r) as

$$s(r) = \frac{(1+r)^{(1-\rho)/\rho}}{(1+\theta)^{1/\rho} + (1+r)^{(1-\rho)/\rho}} \,. \qquad \text{A.11}$$

Proof of Proposition 3.1:

To understand the influence of the interest rate r on the savings rate s one has to derive the derivative of s with respect to r. This yields

$$\frac{\partial s(r)}{\partial r} = \frac{\frac{\partial (1+r)^{(1-\rho)/\rho}}{\partial r}\left[(1+\theta)^{1/\rho} + (1+r)^{(1-\rho)/\rho}\right] - (1+r)^{(1-\rho)/\rho}\left[\frac{\partial (1+r)^{(1-\rho)/\rho}}{\partial r}\right]}{\left[(1+\theta)^{1/\rho} + (1+r)^{(1-\rho)/\rho}\right]^2} \qquad \text{A.12}$$

which, by neglecting the denominator, simplifies to

$$\frac{\partial (1+r)^{(1-\rho)/\rho}}{\partial r}(1+\theta)^{1/\rho} \,. \qquad \text{A.13}$$

Thus the question, if the savings rate is increasing in r or not, depends on whether this last expression is increasing in r or not. Calculating its derivative yields

$$\frac{1-\rho}{\rho}(1+r)^{(1-2\rho)/\rho} \,. \qquad \text{A.14}$$

This is greater than zero if, and only if, $\rho < 1$. Of course the expression is smaller than zero if, and only if $\rho > 1$. Thus the savings rate is an increasing function of the interest rate if $\rho < 1$ and vice versa.

A.2 Marginal products of capital and labour

To derive the marginal products of capital and labour divide the aggregate capital stock K through the size of the labour force L. This gives the capital per unit of labour, denoted by k. Taking the derivatives with respect to K respectively L then yields the expressions for wage and interest rate:

$$\frac{\partial F(K,L)}{\partial K} = \frac{\partial}{\partial K}\left[Lf\left(\frac{K}{L}\right)\right] = Lf'(k)\frac{1}{L} = f'(k) \qquad \text{A.15}$$

$$\frac{\partial F(K,L)}{\partial L} = \frac{\partial}{\partial L}\left[Lf\left(\frac{K}{L}\right)\right] = f(k) + Lf'\left(\frac{K}{L}\right)K\left(-L^{-2}\right) = f(k) - kf'(k) \qquad \text{A.16}$$

Perpetual Youth Model

A.3 Optimum:

The agent maximizes his/her utility function

$$\int_t^{\infty} \log c(z) \exp[-(\theta + p)(z - t)]dz \qquad \text{A.17}$$

with respect to the dynamic budget constraint

$$\frac{dv(z)}{dz} = [r(z) + p]v(z) + y(z) - c(z) \qquad \text{A.18}$$

and the No-Ponzi-Game condition

$$\lim_{z\to\infty} \exp\left\{-\int_t^z [r(\mu) + p]d\mu\right\} v(z) = 0 \qquad \text{A.19}$$

To derive the first order condition use to maximum principle (see Sydsaeter et.al., page 321). Therefore derive the corresponding Hamiltonian which is given by

$$H = \exp(-(\theta + p)(z - t))\log(c(z)) + \mu(z)[(r(z) + p)v(z) + y(z) - c(z)] \qquad \text{A.20}$$

Applying the maximum principle requires the derivation of the Hamiltonian with respect to c and v (see Barro and Sala-I-Martin, page 508).

$$H_c = \exp(-(\theta + p)(z - t))\frac{1}{c(z)} - \mu(z) = 0 \qquad \text{A.21}$$

$$H_v = \mu(z)(r(z) + p) = -\frac{d\mu(z)}{dz} \qquad \text{A.22}$$

taking logs of A.21

$$-(\theta + p)(z - t) - \log(c(z)) = \log(\mu(z)) \qquad \text{A.23}$$

and differentiating with respect to time (z)

$$-(\theta + p) - \frac{dc(z)}{dz} \cdot \frac{1}{c(z)} = \frac{d\mu(z)}{dz} \cdot \frac{1}{\mu(z)} \qquad \text{A.24}$$

inserting from A.22 on the right hand side of A.24 finally yields the first order condition

$$\frac{dc(z)}{dz} = \{[r(z) + p] - (\theta + p)\}c(z) = [r(z) - \theta]c(z)\,. \qquad \text{A.25}$$

To derive equation (3.10i) one has to integrate the first order condition (3.9) to express c(z) as a function of c(t). Thus,

$$c(z) = c(t)\exp\int_t^z [r(\mu) - \theta]d\mu \qquad \text{A.26}$$

The intertemporal form of the budget constraint is given by integrating A.18 forward to some time T,

$$\begin{aligned} v(T) &= v(t)\exp\left[\int_t^T (r(\mu) + p)d\mu\right] + \int_t^T y(z)\exp\left[\int_z^T (r(\mu) + p)d\mu\right]dz \\ &- \int_t^T c(z)\exp\left[\int_z^T (r(\mu) + p)d\mu\right]dz \end{aligned} \qquad \text{A.27}$$

multiplying with

$$R(t,T) \equiv \exp\left\{-\int_t^T [r(\mu) + p]d\mu\right\} \qquad \text{A.28}$$

as defined in (3.10.iii) results in

$$\begin{aligned} v(T)R(t,T) &= v(t) + \int_t^T y(z)\exp\left[-\int_t^z (r(\mu) + p)d\mu\right]dz \\ &- \int_t^T c(z)\exp\left[-\int_t^z (r(\mu) + p)d\mu\right]dz \end{aligned} \qquad \text{A.29}$$

Letting T go to infinity it follows from the no-Ponzi game condition that the left hand side of the equation is zero. Defining the present value of labour income as in (3.10.ii) as h(t) yields

$$\int_t^\infty c(z)R(t,z) = v(t) + h(t) \qquad \text{A.30}$$

Replacing A.26 into the intertemporal budget constraint (A.30) yields

$$\int_t^\infty c(t)\exp\int_t^z [r(\mu) - \theta]\exp\left\{-\int_t^z [r(\mu) + p]d\mu\right\}dz = v(t) + h(t)\,. \qquad \text{A.31}$$

This simplifies to

$$\int_t^\infty c(t)\exp\left\{-\int_t^z [\theta + p]d\mu\right\}dz = c(t)\int_t^\infty \exp[-(z - t)(\theta + p)]dz = v(t) + h(t)\,. \qquad \text{A.32}$$

Solving the last integral finally yields

$$c(t)\frac{1}{\theta + p} = v(t) + h(t) \qquad \text{A.33}$$

and so equation (3.10i) follows.

<u>Proof of Remark 3.4:</u>

Using the definition for aggregate consumption $C(t) = \int_{-\infty}^{t} c(s,t)p\exp[-p(t-s)]ds$ and inserting equation (3.10i) yields

$$C(t) = \int_{-\infty}^{t} (\theta + p)[v(t) + h(t)]p\exp[-p(t-s)]ds\,. \qquad \text{A.34}$$

Since the propensity to consume is independent of age – thus independent of s – it follows

$$C(t) = (\theta + p)\int_{-\infty}^{t} v(t)p\exp[-p(t-s)] + h(t)p\exp[-p(t-s)]ds \qquad \text{A.35}$$

which yields equation (3.12)

$$C(t) = (p + \theta)[H(t) + V(t)] \qquad \text{A.36}$$

using definitions for H(t) and V(t) in fashion of C(t).

<u>A.4 Derivation of a:</u>

Inserting the labour income distribution equation from Definition 3.1 into the definition of aggregate labour income $Y(t) = \int_{-\infty}^{t} y(s,t)p\exp[-p(t-s)]ds$ yields

$$Y(t) = \int_{-\infty}^{t} aY(t)\exp[-\alpha(t-s)]p\exp[-p(t-s)]ds \qquad \text{A.37}$$

$$Y(t) = aY(t)p\int_{-\infty}^{t} \exp[-(s-t)(\alpha + p)]ds \qquad \text{A.38}$$

$$1 = ap\frac{1}{\alpha + p} \qquad \text{A.39}$$

<u>Proof of Remark 3.6</u>:

Recall that individual human wealth is denoted by

$$h(s,t) = \int_{t}^{\infty} y(z)R(t,z)dz \qquad \text{A.40}$$

Including Definition (3.1) of the labour income distribution in the definition of individual human wealth (A.40) yields

$$h(s,t) = \int_t^{\infty} aY(z)\exp[-\alpha(z-s)]R(t,z)dz \qquad \text{A.41}$$

and rearranging gives

$$h(s,t) = a\left\{\underbrace{\int_t^{\infty} Y(z)\exp[-\alpha(z-t)]R(t,z)dz}_{A}\right\}\exp[-\alpha(t-s)] \qquad \text{A.42}$$

The term in brackets (A) can be expressed as

$$\int_t^{\infty} Y(z)\exp[-\alpha(z-t)]\exp\left\{-\int_t^{z}[r(\mu)+p]dz\right\} = \int_t^{\infty} Y(z)\exp\left\{-\int_t^{z}[\alpha + r(\mu)+p]dz\right\} \qquad \text{A.43}$$

since $\exp[-\alpha(z-t)] = \exp\int_t^{z} -\alpha dz$.

For aggregate human wealth $H(t) = \int_{-\infty}^{t} h(s,t)p\exp[-p(t-s)]ds$ it follows that

$$H(t) = \int_{-\infty}^{t} ap\left[\int_t^{\infty} Y(z)\exp\left\{-\int_t^{z}[\alpha + r(\mu)+p]d\mu\right\}dz\right]\exp[-\alpha(t-s)]\exp[-p(t-s)]ds \qquad \text{A.44}$$

using the result obtained for *a* (A.39) and the fact that only the last two exp-functions depend on s yields

$$H(t) = (\alpha+p)\int_t^{\infty} Y(z)\exp\left\{-\int_t^{z}[\alpha + r(\mu)+p]d\mu\right\}dz\int_{-\infty}^{t}\exp[-(\alpha+p)(t-s)]ds \qquad \text{A.45}$$

The last integral yields $(1-0)\dfrac{1}{\alpha+p}$ so that the expression simplifies to

$$H(t) = \int_t^{\infty} Y(z)\exp\left\{-\int_t^{z}[\alpha + r(\mu)+p]d\mu\right\}dz \qquad \text{A.46}$$

Leibniz's Formula (see Sydsaeter et al., page 154) implies that the differential of H(t) with respect to time is given by

$$\frac{\partial H(t)}{\partial t} = Y(t)0 - Y(t)1\exp\left\{\underbrace{-\int_t^t [\alpha + p + r(\mu)]d\mu}_{=0}\right\} \qquad \text{A.47}$$

$$+\underbrace{\int_t^\infty Y(z)[\alpha + p + r(t)]\exp\left\{-\int_t^z [\alpha + p + r(\mu)]d\mu\right\}dz}_{B}$$

The last expression (B) simplifies to $[\alpha + p + r(t)]H(t)$ yielding in total the expression

$$\frac{\partial H(t)}{\partial t} = [r(\mu) + \alpha + p]H(t) - Y(t). \qquad \text{A.48}$$

<u>Proof of Remark 3.7:</u>

Applying Leibniz's Formula for the derivative of aggregate non-human wealth with respect to time yields

$$\frac{\partial V(t)}{\partial t} = v(t,t)p \cdot \underbrace{\exp[-p(t-t)]}_{=1} - v(s,t)p\underbrace{\exp[-p(t+\infty)]}_{=0} \qquad \text{A.49}$$

$$+\int_{-\infty}^{t}\left\{\frac{\partial v(s,t)}{\partial t}p\exp[-p(t-s)] + v(s,t)p\exp[-p(t-s)](-p)\right\}ds$$

which simplifies to

$$\frac{\partial V(t)}{\partial t} = v(t,t)p - pV(t) + \int_{-\infty}^{t}\frac{\partial v(s,t)}{\partial t}p\exp[-p(t-s)]ds \qquad \text{A.50}$$

Non-human wealth at birth v(t,t) is assumed to be zero. Inserting the budget constraint A.18 finally yields

$$\frac{\partial V(t)}{\partial t} = r(t)V(t) + Y(t) - C(t) \qquad \text{A.51}$$

<u>A.5 Derivation of the first locus equation (3.17)</u>:

Differentiating $C(t) = (p+\theta)[H(t) + V(t)]$ with respect to time and replacing the derivatives of H(t) and V(t) by their expression from A.48 and A.51 yields

$$\frac{\partial C(t)}{\partial t} = (p+\theta)[(r(t)V(t) + Y(t) - C(t)) + (r + p + \alpha)H(t) - Y(t)] \qquad \text{A.52}$$

$$\frac{\partial C(t)}{\partial t} = -(p+\theta)C(t) + \underbrace{(p+\theta)[rV(t) + (r+p+\alpha)H(t)]}_{A}. \qquad \text{A.53}$$

The last term (A) can be rearranged to

$$\underbrace{(p+\theta)r[V(t)+H(t)]}_{B} + (p+\theta)(p+\alpha)H(t), \qquad \text{A.54}$$

where the first part (B) equals rC(t). Augmenting by αC(t) - αC(t) leads to

$$\frac{\partial C(t)}{\partial t} = (r+\alpha+\theta)C(t) - \underbrace{(p+\alpha)C(t) + (p+\alpha)(p+\theta)H(t)}_{C}. \qquad \text{A.55}$$

Term C yields

$$(p+\alpha)\left[\underbrace{(p+\theta)H(t) - C(t)}_{D}\right] \qquad \text{A.56}$$

where term D equals (p+θ)V(t). Thus equation A.57 follows.

$$\frac{dC}{dt} = (r+\alpha-\theta)C - (p+\alpha)(p+\theta)V \qquad \text{A.57}$$

Proof of Proposition 3.2:

The first inequality follows from the first locus (3.18) if dC/dt is set to zero. Then F´(K) must be superior to θ in order to keep the relation without having the capital stock taking a negative value.

The second inequality is proved through contradiction. Notice first that F(K) is a concave function. Suppose that contrary to the second inequality

$$F' = \theta + p(1+\varepsilon) \qquad \text{A.58}$$

Setting dC/dt to zero, it follows

$$(1+\varepsilon)C = (p+\theta)K \qquad \text{A.59}$$

using dK/dt = 0 implies F(K)=C and rearranging A.58 to $(\theta + p) = F' - p\varepsilon$. Inserting both to A.59 yields

$$(1+\varepsilon)F(K) = (F' - p\varepsilon)K \qquad \text{A.60}$$

but this again would imply

$$F(K) < (1+\varepsilon)F(K) = (F' - p\varepsilon)K < F'(K)K \Rightarrow F(K) < F'(K)K \qquad \text{A.61}$$

This is impossible because of concavity of F(K) implying $F(K) > F'(K)K$.

Gertler's model

***Proof of Retiree's Optimality conditions*:**

A retiree maximizes the value function

$$V_t^r = \left[(C_t^r)^\rho + \beta\gamma (V_{t+1}^r)^\rho \right]^{1/\rho} \qquad \text{A.62}$$

subject to the asset accumulation equation

$$A_{t+1}^{rjk} = (R_t / \gamma) A_t^{rjk} - C_t^{rjk} . \qquad \text{A.63}$$

The corresponding Bellmann equation (see Grafenhofer et al., page 15) is given by

$$V(A_t^r) = \max_{C_t^r} \left[(C_t^r)^\rho + \beta\gamma (V_{t+1}^r)^\rho \right]^{1/\rho} \qquad \text{A.64}$$

The first order necessary condition follows as the derivative of the value function with respect to current consumption C_t. This yields

$$\frac{\partial V_t}{\partial C_t} = \frac{1}{\rho} \left[(C_t)^\rho + \beta\gamma (V_{t+1})^\rho \right]^{\frac{1}{\rho}-1} \left[\rho (C_t)^{\rho-1} + \beta\gamma\rho \frac{\partial V_{t+1}}{\partial A_{t+1}} (V_{t+1})^{\rho-1} (-1) \right] = 0 \qquad \text{A.65}$$

and consequently

$$(C_t)^{\rho-1} = \beta\gamma \frac{\partial V_{t+1}}{\partial A_{t+1}} (V_{t+1})^{\rho-1} \qquad \text{A.66}$$

Notice that V_{t+1} depends on $A_{t+1} = (R_t / \gamma) A_t - C_t$. Therefore the derivative is given by

$$\frac{\partial V_{t+1}}{\partial C_t} = \beta\gamma\rho \frac{\partial V_{t+1}}{\partial A_{t+1}} (V_{t+1})^{\rho-1} (-1). \qquad \text{A.67}$$

The envelope theorem implies the reaction of the value function with respect to changes in assets as the partial derivative of V with respect to A and evaluating at C (see Sydsaeter et al., page 109). To derive the derivative of V_{t+1} w.r.t. A_{t+1} first notice that

$$C_{t+1} = (R_{t+1} / \gamma) A_{t+1} - A_{t+2} \qquad \text{A.68}$$

and clearly

$$V_{t+1} = \left[(C_{t+1})^\rho + \beta\gamma (V_{t+2})^\rho \right]^{1/\rho} \qquad \text{A.69}$$

Then the derivative follows as

$$\frac{\partial V_{t+1}}{\partial A_{t+1}} = \frac{1}{\rho} \underbrace{\left[(C_{t+1})^\rho + \beta\gamma (V_{t+2})^\rho \right]^{\frac{1}{\rho}(1-\rho)}}_{(V_{t+1})^{1-\rho}} \rho (C_{t+1})^{\rho-1} \frac{R_{t+1}}{\gamma} \qquad \text{A.70}$$

or equivalently

$$\frac{\partial V_{t+1}}{\partial A_{t+1}} = \left(V_{t+1}\right)^{1-\rho}\left(C_{t+1}\right)^{\rho-1}\frac{R_{t+1}}{\gamma} \qquad \text{A.71}$$

inserting the last expression into A.66 and using $\sigma = 1/(1-\rho)$ finally yields the Euler equation

$$C_{t+1} = (R_{t+1}\beta)^{\sigma} C_t \;. \qquad \text{A.72}$$

Denoting $\varepsilon_t \pi_t$ as the retirees' marginal propensity to consume out of wealth (mpcw) consumption in period t is given by

$$C_t^{rjk} = \varepsilon_t \pi_t (R_t / \gamma) A_t^{rjk} \qquad \text{A.73}$$

To find a solution for the value function (and thus for optimal consumption!), conjecture that

$$V_t^r = \Delta_t^r C_t^r \qquad \text{A.74}$$

Substitute this in the objective function resulting in

$$\Delta_t^r C_t^r = \left[\left(C_t^r\right)^{\rho} + \beta\gamma\left(\Delta_{t+1}^r C_{t+1}^r\right)^{\rho}\right]^{1/\rho} \qquad \text{A.75}$$

inserting the Euler equation A.72 yields

$$\Delta_t^r C_t^r = \left[\left(C_t^r\right)^{\rho} + \beta\gamma\left(\Delta_{t+1}^r (R_{t+1}\beta)^{\sigma} C_t^r\right)^{\rho}\right]^{1/\rho} \qquad \text{A.76}$$

solving for Δ gives

$$\left(\Delta_t^r\right)^{\rho} = 1 + \beta^{\sigma} R_{t+1}^{\sigma-1}\gamma\left(\Delta_{t+1}^r\right)^{\rho} \qquad \text{A.77}$$

rearranging gives

$$\frac{1}{\left(\Delta_t^r\right)^{\rho}} = 1 - \beta^{\sigma} R_{t+1}^{\sigma-1}\gamma\left(\frac{\Delta_{t+1}^r}{\Delta_t^r}\right)^{\rho} \qquad \text{A.78}$$

denoting $\varepsilon_t \pi_t = \left(\frac{1}{\Delta_t}\right)^{\rho}$ and $\varepsilon_{t+1}\pi_{t+1} = \left(\frac{1}{\Delta_{t+1}}\right)^{\rho}$ respectively finally results in

$$\varepsilon_t \pi_t = 1 - (R_{t+1}^{\sigma-1}\beta^{\sigma}\gamma)\frac{\varepsilon_t \pi_t}{\varepsilon_{t+1}\pi_{t+1}} \qquad \text{A.79}$$

<u>Proof of Workers' Optimality Conditions</u>**:**

A worker maximizes his/her value function

$$V_t^w = \left[\left(C_t^w\right)^{\rho} + \beta\left[\omega V_{t+1}^w + (1-\omega)V_{t+1}^r\right]^{\rho}\right]^{1/\rho} \qquad \text{A.80}$$

subject to

$$A_{t+1}^{wj} = R_t A_t^{wj} + W_t - C_t^{wj} \qquad \text{A.81}$$

The first order necessary condition is given by the derivative of the value function A.80 wrt. C_t

$$\frac{\partial V_t}{\partial C_t} = \frac{1}{\rho}(V_t)^{1-\rho}\left[\rho(C_t)^{\rho-1} + \beta\rho\left[\omega V_{t+1}^w + (1-\omega)V_{t+1}^r\right]^{\rho-1}\left[\omega\frac{\partial V_{t+1}^w}{\partial A_{t+1}^w}(-1) + (1-\omega)\frac{\partial V_{t+1}^r}{\partial A_{t+1}^r}(-1)\right]\right] = 0 \qquad \text{A.82}$$

or equivalently

$$(C_t)^{\rho-1} = \beta\left[\omega V_{t+1}^w + (1-\omega)V_{t+1}^r\right]^{\rho-1}\left[\omega\frac{\partial V_{t+1}^w}{\partial A_{t+1}^w} + (1-\omega)\frac{\partial V_{t+1}^r}{\partial A_{t+1}^r}\right] \qquad \text{A.83}$$

Derivatives of $\frac{\partial V_{t+1}^w}{\partial A_{t+1}^w}$ and $\frac{\partial V_{t+1}^r}{\partial A_{t+1}^r}$ are similar to the retirees' case yielding

$$\frac{\partial V_{t+1}^w}{\partial A_{t+1}^w} = \frac{1}{\rho}\left(V_{t+1}^w\right)^{1-\rho}\left[\rho\left(C_{t+1}^w\right)^{\rho-1} R_{t+1}\right] \qquad \text{A.84}$$

and

$$\frac{\partial V_{t+1}^r}{\partial A_{t+1}^r} = \frac{1}{\rho}\left(V_{t+1}^r\right)^{1-\rho}\left[\rho\left(C_{t+1}^r\right)^{\rho-1} R_{t+1}\right] \qquad \text{A.85}$$

Notice the difference in the change in retirees' value function w.r.t. A_{t+1}. This results from the fact that the agent retires in period t+1 and therefore received an interest rate of R on his/her wealth.

Inserting A.84, A.85 and using the conjectures

$$V_t^r = \Delta_t^r C_t^r,\ V_t^w = \Delta_t^w C_t^w \text{ and } \Delta_t^w = \pi_t^{-1/\rho} \qquad \text{A.86}$$

in A.83 yields

$$(C_t)^{\rho-1} = \beta R_{t+1}\left[\omega + (1-\omega)\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho}\right]\left[\omega C_{t+1}^w + (1-\omega)\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}C_{t+1}^r\right]^{\rho-1} \qquad \text{A.87}$$

This becomes

$$C_t = \beta^{1-\rho} R_{t+1}^{1-\rho}\Omega_{t+1}^{1-\rho}\left[\omega C_{t+1}^w + (1-\omega)\Lambda_{t+1}C_{t+1}^r\right]^{\rho-1} \qquad \text{A.88}$$

using the definitions

$$\Lambda_{t+1} \equiv \frac{\Delta_{t+1}^r}{\Delta_{t+1}^w} \qquad \text{A.89}$$

and

$$\Omega_{t+1} = \omega + (1-\omega)\left(\frac{\Delta^r_{t+1}}{\Delta^w_{t+1}}\right)^{1-\rho} = \omega + (1-\omega)\varepsilon_{t+1}^{\frac{1}{1-\sigma}} \qquad \text{A.90}$$

rearranging finally gives

$$\omega C^{wj}_{t+1} + (1-\omega)\Lambda_{t+1} C^{rj(t+1)}_{t+1} = (R_{t+1}\Omega_{t+1}\beta)^{\sigma} C^{wj}_t \qquad \text{A.91}$$

As in retirees' optimality problem, insert the conjectured solution for V^w_t and V^r_t to the value function,

$$\Delta^w_t C^w_t = \left[(C^w_t)^{\rho} + \beta\left[\omega\Delta^w_{t+1}C^w_{t+1} + (1-\omega)\Delta^r_{t+1}C^r_{t+1}\right]^{\rho}\right]^{1/\rho} \qquad \text{A.92}$$

substituting from the first order condition A.91 gives then

$$\Delta^w_t C^w_t = \left[(C^w_t)^{\rho} + \beta\left[\Delta^w_{t+1}(R_{t+1}\Omega_{t+1}\beta)^{\sigma} C^w_t\right]^{\rho}\right]^{1/\rho} \qquad \text{A.93}$$

or equivalently

$$(\Delta^w_t)^{\rho} = 1 + \beta^{\sigma}(R_{t+1}\Omega_{t+1})^{\sigma-1}(\Delta^w_{t+1})^{\rho} \qquad \text{A.94}$$

rearranging gives

$$\frac{1}{(\Delta^w_t)^{\rho}} = 1 - \beta^{\sigma}(R_{t+1}\Omega_{t+1})^{\sigma-1}\left(\frac{\Delta^w_{t+1}}{\Delta^w_t}\right)^{\rho} \qquad \text{A.95}$$

denoting $\pi_t = \left(\frac{1}{\Delta_t}\right)^{\rho}$ and $\pi_{t+1} = \left(\frac{1}{\Delta_{t+1}}\right)^{\rho}$ respectively finally results in

$$\pi_t = 1 - (R_{t+1}\Omega_{t+1})^{\sigma-1}\beta^{\sigma}\frac{\pi_t}{\pi_{t+1}} \qquad \text{A.96}$$

***Proof of Remark 3.8*:**

Recall $C^r_t = \varepsilon_t \pi_t R_t A^r_t$ and $C^w_t = \pi_t(R_t A^w_t + H_t)$ and define $\lambda_t \equiv A^r_t / A_t$ and $1-\lambda_t \equiv A^w_t / A_t$.

Summing up C^r_t and C^w_t gives

$$C_t = \pi_t\left[\varepsilon_t R_t A_t \lambda_t + R_t A_t(1-\lambda_t) + H_t\right] \qquad \text{A.97}$$

and rearranging finally yields

$$C_t = \pi_t\left\{\left[1 + (\varepsilon_t - 1)\lambda_t\right]R_t A_t + H_t\right\} \qquad \text{A.98}$$

***Proof of Remark 3.9*:**

Retirees' assets accumulate by the savings of current retirees at time t as well as by the savings of those workers who retire from t to t+1. Thus

$$\lambda_{t+1}A_{t+1} = \lambda_t R_t A_t - C^r_t + (1-\omega)\left[(1-\lambda_t)R_t A_t + W_t - C^w_t\right] \qquad \text{A.99}$$

The amount of assets held by workers at time t+1 equals the amount of assets carried by workers from t to t+1 time the fraction of those staying in the labour force ω. This implies

$$\omega\left(A_t^w R_t + W_t - C_t^w\right) = A_{t+1}^w = \left(1-\lambda_{t+1}\right)A_{t+1} \qquad \text{A.100}$$

Thus

$$\left(A_t^w R_t + W_t - C_t^w\right) = \frac{1}{\omega}\left(1-\lambda_{t+1}\right)A_{t+1} \qquad \text{A.101}$$

Recall also that $C_t^r = \varepsilon_t \pi_t R_t A_t^r$

$$\lambda_{t+1}A_{t+1} = \left(1-\varepsilon\pi\right)\lambda_t R_t A_t + (1-\omega)\left(1-\lambda_t\right)A_{t+1}\frac{1}{\omega} \qquad \text{A.102}$$

rearranging gives finally

$$\lambda_{t+1} = \omega(1-\varepsilon_t\pi_t)\lambda_t R_t \frac{A_t}{A_{t+1}} + (1-\omega) \qquad \text{A.103}$$

<u>Proof of Remark 3.10</u>:

In the steady state the all quantitative variables grow at the exogenously given growth rate of the effective labour force X_tN_t which equals (1+x)(1+n) which is approximately equal to (1+x+n) (see Gertler, page 74). Therefore the normalization to output is convenient. Denote $k_t = \frac{K_t}{Y_t}$ and $h_t = \frac{H_t}{Y_t}$.

(i) For the first equation use

$$K_{t+1} = Y_t - C_t + (1-\delta)K_t \qquad \text{A.104}$$

dividing by $Y_{t+1} = \left(1+x+n\right)Y_t$ and using the definition for C_t and K_t=A_t

$$k_{t+1} = \frac{1}{\left(1+x+n\right)} - \frac{\pi_t\left\{\left[1+\left(\varepsilon_t-1\right)\lambda_t\right]R_tK_t + H_t\right\}}{\left(1+x+n\right)Y_t} + \frac{(1-\delta)}{\left(1+x+n\right)}k_t \qquad \text{A.105}$$

rearranging and using the fact that in the steady state k_t=k_{t+1}

$$\left(x+n+\delta\right)k = 1 - \pi\left\{\left[1+\left(\varepsilon-1\right)\lambda\right]Rk + h\right\} \qquad \text{A.106}$$

(ii) Use K_t=A_t and the result from 3.25 in

$$\lambda_{t+1} = \omega(1-\varepsilon_t\pi_t)\lambda_t R_t \frac{A_t}{A_{t+1}} + (1-\omega) \qquad \text{A.107}$$

yielding

$$\lambda_{t+1} = \omega(R_{t+1}^{\sigma-1}\beta^{\sigma}\gamma)\lambda_t R_t \frac{K_t}{K_{t+1}} + (1-\omega) \qquad \text{A.108}$$

expressing K in terms of k gives

$$\lambda_{t+1} = \omega(R_{t+1}^{\sigma-1}\beta^{\sigma}\gamma)\lambda_t R_t \frac{1}{(1+x+n)} + (1-\omega) \qquad \text{A.109}$$

As the economy is in its steady state

$$\lambda\left(1-\omega\gamma(R^{\sigma}\beta^{\sigma})\frac{1}{(1+x+n)}\right) = (1-\omega) \qquad \text{A.110}$$

augmenting by the ratio of retirees to workers $\Psi = \frac{1-\omega}{1+n-\gamma}$ and rearranging gives finally

$$\lambda = \psi \frac{(1+x+n)(1+n-\gamma)}{1+x+n-\gamma\omega(R\beta)^{\sigma}} \qquad \text{A.111}$$

(iii) Using Definition 3.3

$$H_t = N_t W_t + \frac{H_{t+1}}{(1+n)R_{t+1}\Omega_{t+1}/\omega} \qquad \text{A.112}$$

and dividing by Y_{t+1}

$$\frac{h_t}{1+x+n} = \frac{\alpha}{1+x+n} + \frac{h_{t+1}}{(1+n)R\Omega/\omega} \qquad \text{A.113}$$

rearranging

$$h_t = \alpha + \frac{(1+x+n)h_{t+1}}{(1+n)R\Omega/\omega} = \alpha + \frac{(1+x)h_{t+1}}{R\Omega/\omega} \qquad \text{A.114}$$

setting $h_t = h_{t+1}$

$$h\left(1-\frac{(1+x)}{R\Omega/\omega}\right) = \alpha \qquad \text{A.115}$$

solving for h gives

$$h = \frac{\alpha R\Omega/\omega}{R\Omega/\omega-(1+x)} = \alpha R \frac{\Omega/\omega}{R\Omega/\omega-(1+x)} \underbrace{\frac{R\Omega/\omega-(1+x)}{R\Omega/\omega-(1+n+x)\Omega/\omega}}_{\Gamma(R,\Omega)^{-1}} \Gamma(R,\Omega) \qquad \text{A.116}$$

which finally gives

$$h = \frac{\alpha R}{R-(1+n+x)}\Gamma(R,\Omega) \qquad \text{A.118}$$

Equations (iv)-(vii) follow by simply setting t=t+1 in the corresponding equations 3.39, 3.31, 3.25, and 3.28.

4. Age-dependent mortality (Li and Tuljapurkar)

That mortality and fertility rates in industrialized countries are declining and thereby leading to significant changes in the population's age composition is a well known fact. As an example Figure 4.1 shows the increasing cohort life expectancy for the US between 1900 and 2060. Decreasing old-age mortality leads to a shift in the expected lifespan while the reduction of fertility rates causes a further reduction of the relative share of young people in the population. In total these two effects cause *demographic population ageing*. This so far is straightforward. For economic analysis demographic aging can have strong implications. Viewing the models presented in the thesis so far, one has to admit, that their approach of age-independent mortality lacks realism. Constant death rates, implying age-independency of mortality, can simply not take account of changes in mortality and thus do not allow for demographic aging to enter the framework. That taking account of demographic aging in economic analysis has important implications is shown by Li and Tuljapurkar (2004). They find significant effects on several economic variables when including age-dependent mortality into their framework.

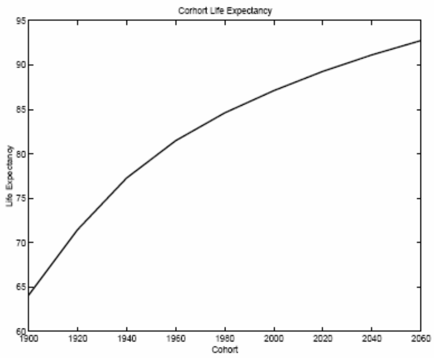

Figure 4.1 Cohort life expectancy in the US (Li, Tuljapurkar; page 1)

4.1 Incorporating the distribution function of age at death

The following section discusses the age-dependent mortality approach of Li and Tuljapurkar (2004). The main feature of their framework is the implementation of a *distribution function of age at death* to a continuous time OLG model. Doing so, the authors find significant implications of changes in life expectancy on economic outcomes. Besides increasing life expectancy, they pay particular attention to the effect of *decreasing uncertainty in the timing of death* which they find to have important impacts on resulting economic magnitudes. The observed relationship between those two effects is almost linear (Li and Tuljapurkar (2004), page 6). Thus, as life expectancy increases the uncertainty of lifetime decreases almost linearly. Decreasing life time uncertainty has again strong implications on agents' behaviour within the economy. Li and Tuljapurkar try to account for both effects by incorporating a *distribution function* of the *age at death* to the framework of a continuous time OLG model. Changes in life expectancy can then be represented by a shift in the first moment (the mean) of the *distribution function of age at death*, whereas a decrease in life time uncertainty can be seen as decreasing the variance of this distribution function. In this sense demographic ageing in industrialized countries can be seen as leading to a tightening of the distribution function of *death age*. Figure 4.2 shows this tightening of the "*death age" distribution* for the US between 1950 and 2010.

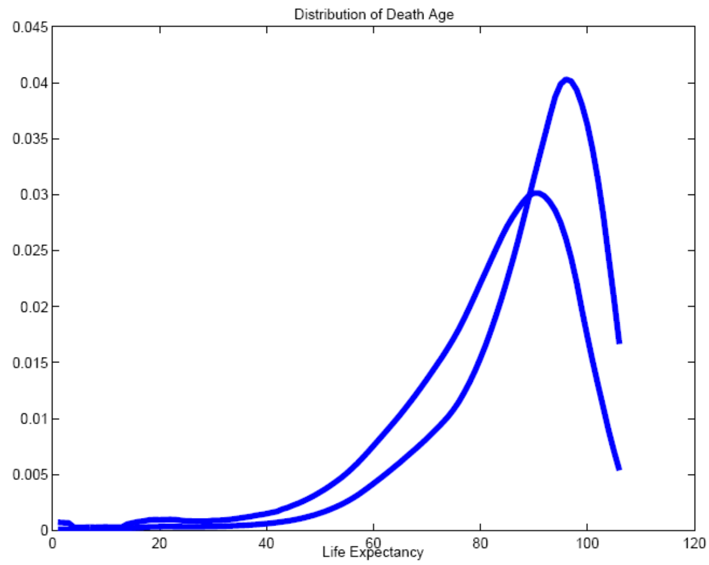

Figure 4.2 Distribution function of age at death in 1950 and 2010 for the US (Li, Tuljapurkar; page 5)

Both effects are clearly apparent in the graph – increased life expectancy shifts the distribution's mean to the right whereas reduced life time uncertainty can be seen as the reduction of the distribution's standard deviation. The distribution's shape also suggests that using a normal distribution will attain a close approximation of real mortality.

Li and Tuljapurkar also examine the effects of increasing life expectancy on schooling and retirement decisions (see section 4.3). The argument for paying attention to impact of changes in life expectancy on schooling tenure is that an increased life time horizon is likely to affect individuals' decisions to invest in their own human capital in order to receive higher future wages. To account for the effect of schooling on individuals' human capital, wages are defined as an increasing function of the schooling tenure (see Definition 4.1). The intuition

behind this definition is, that the longer the schooling tenure, the higher the individual's labour productivity and therefore the higher the individual's wage.
In contrast to previous models Li and Tuljapurkar work with a distribution function of the random variable *death-age*. This is defined as follows

Remark 4.1:
At time z let $\mu(s,z)$ be the instantaneous death-rate at age s and survivorship rate $l(s,z)$ be the probability that an individual born at time s will be alive at time z. The random variable *death-age* is denoted by T. Then the density function of T - $\phi(s,z)$ - is given by the product $\mu(s,z)\cdot l(s,z)$.

One important feature of OLG models lies in aggregation patterns. Even though aggregation seems to become a difficult task when using a distribution function instead of simpler approaches (as in Blanchard's framework) the following result simplifies matters a great deal.

Proposition 4.1:
Aggregate variables can be derived as the expectation over the distribution function of the random variable "*death age*" T as defined in definition 4.1.

Proof: see Appendix B

Based on Blanchard's OLG framework (compare section 3.2), Li and Tuljapurkar incorporate the distribution of death age into a continuous time OLG model. Compared to Blanchard's model the constant probability of death is replaced by the age-dependent survival probability. As in Blanchard's model agents maximize their expected utility from consumption in the presence of life time uncertainty

$$\int_t^\infty l(z-t)\cdot u(c(z))\cdot e^{-\theta(z-t)}dz. \tag{4.1}$$

The utility function is assumed as a Constant Relative Risk Aversion (CRRA) function with the relative risk aversion coefficient γ. In equation 4.1 θ denotes individual's rate of time preference. The economy is assumed to follow a Cobb-Douglas production function with production factors capital und human capital

$$Y = AK^{\alpha}H^{1-\alpha}. \tag{4.2}$$

Definition 4.1:

Relative wages are defined as

$$y(a_s) = we^{f(a_s)} \tag{4.3}$$

Relative wages depend on the years of schooling (a_s) chosen by individuals. As in Blanchard's model, a redistributing life insurance company solves the optimal behaviour problems implied by uncertain life time (compare section 3.2.1.1).

Solving the utility maximization problem yields very unhandy and complicated expressions. Therefore the results are not given in the text. As I will not discuss the results of the maximization problem I present only a short description of the derivation process in Appendix B. The following section summarizes Li and Tuljaprukar's findings of changes in the mean and the variance of death-age on various economic variables. For their analysis the authors make the following assumption

Assumption 4.1:

The random variable *death-age* is normally distributed and θ=0.03, α=0.03, γ=1 and maximum age is set to T_{max}=120[25]. Further, total human capital is constant.

4.2 Effects of changing life expectancy and life time uncertainty

Effect of changing life expectancy

To analyze the effects of changing life expectancy on the economy, the variance of death age is initially held constant. At this stage of the analysis, neither retirement aspects nor schooling decisions are considered. Thus, agents work for their entire life. In this set-up Li and Tuljapurkar find, that increasing life expectancy will increase aggregate wealth. This result rests on

Assumption 4.2:

Individual net assets increase with high age and thus people of high age posses more assets than very young people.

This can be seen as follows. Individual wealth does not necessarily increase monotonically through life, but in the long run the existence of life insurance implies a tendency for individual assets to increase with age. At high age, they argue, individual assets will eventually increase due to the existence of the insurance company (see Li and Tuljapurkar, page 10). Now, as population ages - due to increased life expectancy and the fact that fertility is adjusted in order to keep the population size constant (see assumption 4.1) – there are more people of old age who each posses more assets than very young people. Thus, as a consequence economy wide assets – wealth – increase with increasing life expectancy. Figure B4.1 in Appendix B at the end of section 4 shows surface plots indicating the behaviour of diverse economic variables with respect to changes in life expectancy and death-age variance. Wage changes in the same direction as life expectancy while the interest rate develops in the opposite direction.

Consumption behaviour is quite similar to the development of wealth. Individual consumption increases with life expectancy as well as individual life-time labour income. Again, as population ages due to increased life expectancy, aggregate consumption and labour income increase. The first line in table 4.1 (below) summarizes these findings.

Effects of changing death age variance

To study the "pure" effect of changing death-age variance, life expectancy is now held constant. How does a changing death-age variance affect the economy, conditional on constant life expectancy? Following proposition is fundamental for analyzing the effects of changing death-age variance

Remark 4.2:

As the variance of death-age increases more people reach very high ages while more people die at young ages. Thus, increasing variance of death-age leads to a higher share of "extreme" death-age cases in the population.

As with changes in life expectancy, Li and Tuljapurkar find that aggregate wealth will change in the same direction as the variance of death-age. This can be seen clearly by considering the previous proposition. Increasing variance implies an increasing probability of death at young

[25] Li and Tuljapurkar choose these parameter values according to the work of Kalemli-Ozcan, Ryder and Weil (2000) for comparability.

age. As more people die at young age, less young people contribute to total wealth. On the other hand increasing variance also implies that more people reach a very high age. As those very old people possess much more assets (see assumption 4.2) compared to the very young likewise facing high mortality, the "gain" in total wealth by the increase in very old people offsets the loss by the very young. Thus, total wealth will increase as variance of death age increases.

Consumption shows the same response to increasing variance as in the case of increasing life expectancy. As the variance increases, wage increases and the interest rate decreases. To keep the balance between life-time income and life-time consumption, individual consumption must rise. Increasing variance increases the percentage of old people in the economy, leading to a total increase in consumption. Line 2 of Table 4.1 summarizes the effects of changing variance on diverse variables according to Li and Tuljapurkar's findings (see also Figure B4.1 in the Appendix B).

-	Initial cons.	agg. cons.	agg. wealth	interest rate	wage
increasing life expectancy	Inc.	inc.	incr.	dec.	incr.
Decreasing variance of death age	dec.	dec.	dec.	inc	dec.

Table 4.1: Changes in variables when life expectancy or the variance of death age change

Joint effects

The previous analysis shows the behaviour of diverse economic variables when affected by changes either in life expectancy or the variance of death-age. To see the implications of joint changes in both – life expectancy and the variance of death-age – Li and Tuljapurkar first take a look at the behaviour of the interest rate. The previous sections showed that the interest rate is an increasing function of life expectancy and a decreasing function of the variance of death-age. Note, that

Remark 4.3:

The assumption of a constant and therefore age-independent life expectancy implies that the variance of death-age $v_0 = e_0^2$.

Remark 4.3 again implies that in the case of constant mortality the variance of death-age increases with life expectancy which stands in contrast to what is observed for countries like the US where increasing life expectancy is accompanied by decreasing variance of death-age. As shown in table 4.1 decreasing life time uncertainty works as an opposing force to the effects of increasing life expectancy. Taking both effects into account results in a much higher interest rate than in the case of constant death rate (see table 4.2 and figure 4.3). Following the same argument, initial and total consumption are lower when using age-dependent mortality than using a constant death rate. Here decreasing variance in the death-age works to reduce the magnitudes (see table 4.2 and figure 4.3).

Comparing different modellations of mortality

This section compares the implications of the age-dependent mortality framework on diverse economic variables with those implied by other mortality assumptions. These alternative assumptions are: a constant mortality approach as in Blanchard's Model, fixed death-age as in the Diamond Model, normal distribution of mortality as assumed by Li and Tuljapurkar and a distribution fitted to real data.

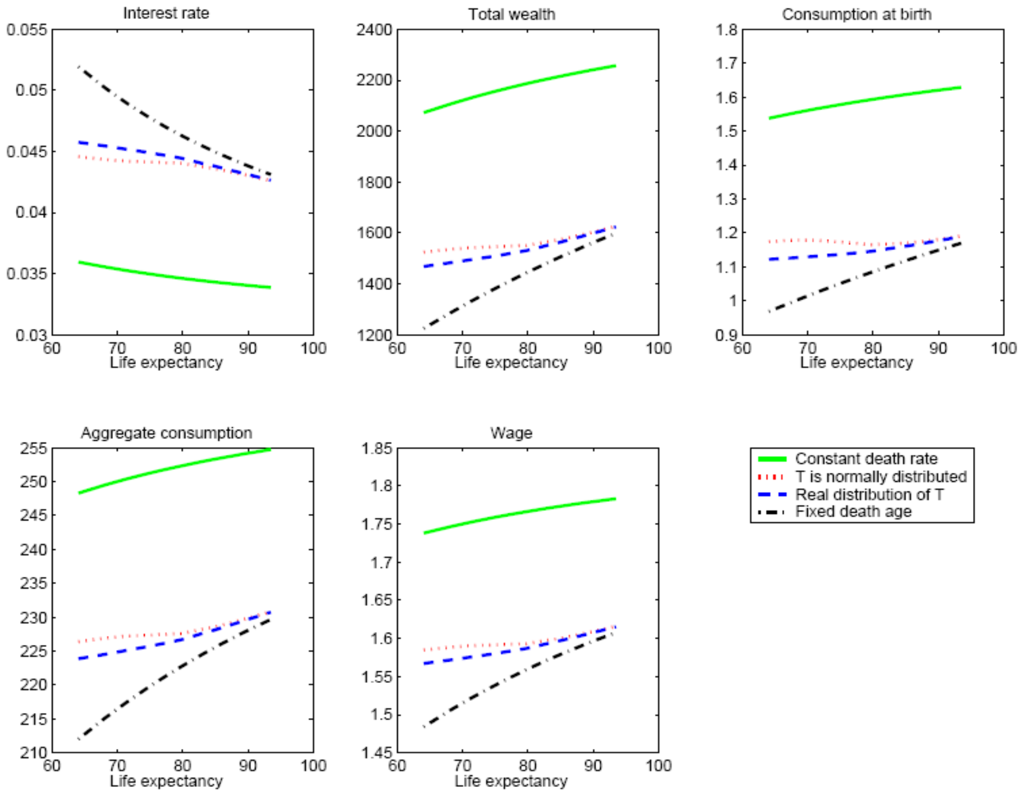

Figure 4.3 Effects from increasing life expectancy on diverse economic variables

(Li, Tuljapurkar; page 14)

Figure 4.3 plots the comparative statics of the interest rate, initial consumption, consumption, wage and wealth when life expectancy changes under different mortality assumptions. Table

4.2[26] lists the numerical results obtained by Li and Tuljapurkar. As can be seen in the figure and the table the normality assumption gives a good image of the real data, whereas the constant death rate clearly overestimates wealth, consumption and wage while underestimating the interest rate.

Name	*Constant Death Rate*	*Fixed Death Age*	*Normal Distribution*	*Fitted Distribution*
H	100	100	100	100
r	0.0346	0.0463	0.044	0.0444
w	1.7662	1.5592	1.593	1.5868
K	2186.6	1443.4	1550.3	1530.3
$c(0)$	1.5932	1.0841	1.1648	1.1456
C	252.308	222.7485	227.5745	226.6899

Table 4.2 Comparing results from different mortality assumptions (Li, Tuljapurkar; page 15)

Li and Tuljapurkar argue that these differences come from the different assumptions made on the variance of death-age (compare Li and Tuljapurkar, page 14). As shown in table 4.1 decreasing variance of death age causes an increase of the interest rate. Therefore the resulting interest rate is much higher in the case of normal distributed mortality compared to constant death rates (compare Remark 4.3). Table 4.2 confirms the impression of figure 4.3 that the normal distribution assumption yields a very close approximation to the fitted data and thus a close estimate on real data.

4.3 The impact of changing life expectancy on schooling and the effect of compulsory retirement

Li and Tuljapurkar examine the effects of increasing life expectancy on schooling decisions. The argument for paying attention to the impact of changes in life expectancy on schooling tenure is that an increased life time horizon is likely to affect individuals' decisions to invest in their own human capital in order to receive higher future wages. To take account of the effect of schooling on individuals' human capital, wages are defined as an increasing function of the schooling tenure (compare definition 4.1). The intuition behind this definition is, that the longer the schooling tenure is, the higher is the individual's labour productivity and therefore the higher the individual's wage. Additionally attention is paid to the impact of retirement age on individuals' schooling decisions.

[26] In the table H denotes aggregate human capital, r the interest rate, w the wage, K aggregate capital, c(0) the initial consumption and C aggregate consumption. H is assumed constant and therefore set to 100. Life expectancy (e0) is set to 79,83 years (see Li and Tuljapurkar, page 15).

Schooling without retirement

To study the effect of changes in life expectancy on an agent's schooling decision, one first has to specify the effect of the schooling tenure on income. Labour income is defined as an increasing function of schooling years a_s (see definition 4.1). Thus, as agents choose to increase their schooling tenure, they will receive higher wages in future. The productivity term $f(a_s)$ is defined as increasing function of schooling years

Definition 4.2:

The *productivity term* determining the relationship between schooling years and wage is defined as

$$f(a_s) = \frac{\Theta}{1-\Psi} a_s^{1-\Psi}$$

where Θ=0,32 and Ψ=0,58[27].

Effect from introducing schooling

Introducing schooling into the framework affects human capital. In contrast to the former analysis total human capital is not constant anymore.

On the one hand the introduction of *schooling* affects human capital directly by reducing the working tenure from the whole life span by several years[28]. On the other hand the length of schooling-life increases labour productivity and thereby efficient labour (the more years of schooling are chosen, the higher the function $f(a_s)$ implying higher labour productivity and wages). The more efficient labour is, the higher is total human capital. Thus, there are two opposing effects of schooling on human capital. These two effects oppose each other, and thus it is not clear if increasing a_s yields an increase or a decrease of total human capital in the first place. Yet, Li and Tuljaprukar find in their calibration that the overall effect is an increase in total human capital (Li and Tuljapurkar, page 19).

Effect of increasing life expectancy with schooling

Li and Tuljapurkar find a positive relationship between increasing life expectancy and schooling tenure – being in line with other researchers' results (e.g. the studies of Boucekkine, de la Croix and Licandro (2002) and of Kalemli-Ozcan, Ryder and Weil (2000); see Li and

[27] These values are chosen to allow for comparison with results of Kalemli-Ozcan, Ryder and Weil(2000).
[28] Notice, that at this point, retirement is not yet considered. Thus, before introducing schooling agents where working for their entire life. The next subsection will examine the effects from introducing retirement to the framework.

Tuljapurkar, page 19). As they point out, the difference to the study of Kalemli-Ozcan, Ryder and Weil (assuming a constant death rate!) is that the steady schooling tenure is much lower (6 or 7 years). This, they argue is due to the negative effects of the variance of death age (see again page 19). As for other magnitudes (compare table 4.1) the decrease in life time uncertainty opposes the dynamics implied by increasing life expectancy.

Most apparent are the effects of introducing schooling to the framework on the behaviour of the interest rate. As figure 4.4 shows, the interest rate first increases and then starts to decrease as life expectancy increases – contrasting the previously discussed case without schooling. The interest rate increases until life expectancy reaches a median level but starts to decrease as life expectancy reaches very high levels. The argument for this initial increase is, that "*at median life expectancy [...] the variance of death-age decreases dramatically*" and "*it seems that schooling increases the effect of the variance of death-age" (Li and Tuljapurkar, p. 20)*.

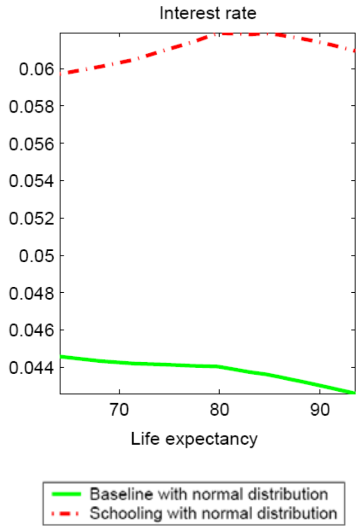

Figure 4.4 Effect of increasing life expectancy on the interest rate with schooling

(Li, Tuljapurkar; page 18)

4.4 Effect of retirement

To examine the effects of changing life expectancy in the framework including schooling and retirement aspects, first just the *pure retirement effect* is considered. This means, that schooling tenure is first held constant and only the age of retirement is changed. The second part of this subsection will then examine the joint effects, when agents are able to choose their optimal schooling tenure - also facing compulsory retirement age.

The pure retirement effect:

If schooling tenure is held constant, the introduction of retirement works as a reduction in total human capital. This reduction in human capital again reflects the reduction in the total labour force that occurs due to the shortening of agents' working life time by the introduction of a compulsory retirement age. Figure 4.5 shows the behaviour of diverse economic variables with different retirement ages. The interest rate, total wealth, total consumption and total labour (human capital) decreases as retirement age decreases. Wages and initial consumption on the other hand are found to be increasing with a reduction in the retirement age.

Retirement and schooling:

Combining both life cycle aspects – schooling and retirement – the following figure shows the results of Li and Tuljapurkars simulations. The effect of changing retirement age on schooling decisions is found to be negative – the higher the retirement age is set the lower agents choose their schooling tenure (see figure 4.5). This reflects the desire to earn higher wages, the longer the expected retirement period is. Confronted with long retirement periods individuals choose to invest more into their human capital in order to earn higher wages and so to finance their consumption during retirement.

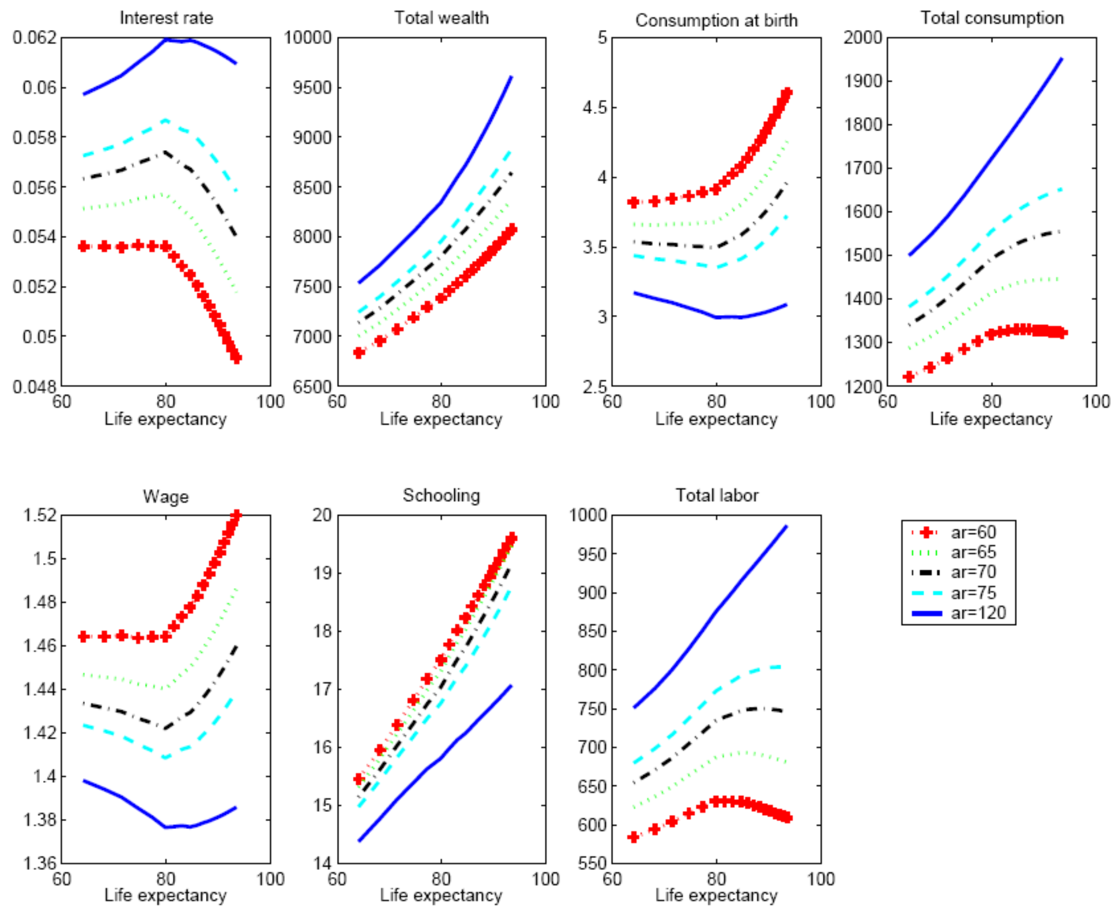

Figure 4.5 Effects from increasing life expectancy with schooling and retirement (Li, Tujlapurkar; page 23)

The figure shows that the reduction of retirement age increases schooling, wage and initial consumption while reducing the interest rate, total wealth, total consumption and of course total human capital. It seems that as life expectancy increases the effect of reducing the retirement age becomes more significant for the interest rate, consumption (initial and total), wage and human capital. Intuitively, increasing life expectancy makes retirement effects more severe, as more people are concerned.

4.5 Life cycle aspects of the model

By incorporating the distribution function of death-age Li and Tuljapurkar create a model yielding a high degree of demographic realism. This age-dependent mortality framework is able to take account of the changing age structure of the economy's population which is a clear improvement compared to age-independent mortality frameworks previously discussed. As has been shown before, assuming mortality to be constant through time and thus independent of age leads to age-independent consumption behaviour. Contrasting this unsatisfying result this framework yields changing consumption behaviour as people grow older.

In total the results drawn from the simulations based on their model Li and Tuljapurkar find realistic and important impacts of changing life expectancy and decreasing life time uncertainty on consumption behaviour.

In paying attention also to the impact of increasing life expectancy on individual's schooling decisions the authors achieve an even higher degree of demographic realism. Concerning the economy as a whole extended schooling tenure due to increasing life expectancy increases total human capital available to the economy. On the other hand the impact of compulsory retirement age is to reduce available total human capital. Therefore studying the response of schooling decisions when agents face increasing life expectancy and decreasing life time uncertainty is an important issue. As Li and Tuljapurkar show, paying attention to altering schooling tenure together with the consideration of the effect of compulsory retirement age has significant impacts on resulting economic magnitudes.

Appendix B

Proof of Proposition 4.1(taken from Li and Tuljapurkar, page 4):

Let $\phi(x)$ be the distribution function of death age T. Age-dependent survival curve l(a) is

$$l(a) = \int_a^\infty \phi(t)dt \tag{B1}$$

The aggregate of some function j(a) of age *a* is defined with respect to survivorship

$$J = \int_0^\infty j(x)l(x)dx = \int_0^\infty j(x)\int_x^\infty \phi(t)dtdx \tag{B2}$$

Changing the order of integration this turns into

$$J = \int_0^\infty\int_0^t j(x)dx\phi(t)dt = E_T\left[\int_0^T j(x)dx\right] \tag{B3}$$

Thus, the aggregate of a variable j(a) can be expressed in terms of the expectation over the distribution of death-age T.

Agents' maximization problem (taken from Li and Tuljapurkar, p. 7):

As the concrete maximization problem is not discussed in my thesis I just give a short overview of the derivation process. First notice useful definitions:

Definition B.1:

$$g(z) \equiv E_T\left[e^{zT}\right] \tag{B4}$$

$$P(z) \equiv E_T\left[e^{z(T\wedge a_s)}\right] \tag{B5}$$

$$Q(z) \equiv E_T\left[e^{z(T\wedge a_r)}\right] \tag{B6}$$

$$\lambda(a) \equiv E_T[T \wedge a] \tag{B7}$$

Consumption:

The standard optimality conditions yield an optimal individual lifetime consumption path c(a) (see Li and Tuljapurkar, page 8)

$$c(a) = c_0 \exp(ka) \tag{B8}$$

where

$$k \equiv \frac{r-\theta}{\gamma} \tag{B9}$$

The household's budget constraint requires that the present value of consumption has to be equal to the present value of lifetime earnings (see Romer, page 41), taking into account lifetime uncertainty. Reconsidering the fact that the aggregate of a variable can be expressed in terms of the expectation over the random variable death-age (see Proposition 4.1) this yields

$$E_T\left[\int_0^T c(a)\exp(-ra)da\right] = E_T\left[\int_0^T w\exp(f(as))\exp(-r(T\wedge a))da\right] \tag{B10}$$

The left hand side of the equation gives

$$E_T\left[\int_0^T c(a)\exp(-ra)da\right] = c_0E_T\left[\int_0^T \exp((k-r)a)da\right] = c_0E_T[\exp((k-r)T)-1]\frac{1}{k-r} \tag{B11}$$

For the right hand side notice that the expression can be separated (see Li and Tuljapurkar, page 26) in the sense that

$$E_T[\exp(z(T\wedge a_r))] = E_T\left[\exp(zT)1_{T<a_r}\right] + E_T\left[\exp(za_r)1_{T\geq a_r}\right] \tag{B12}$$

Using this yields for the right hand side of equation B.10 yields

$$\begin{aligned} &w\exp(f(a_s))E_T\left[\int_{T\wedge a_s}^{T\wedge a_r} \exp(-ra)da\right] = \\ &w\exp(f(a_s))\left\{\underbrace{E_T[\exp(-r(T\wedge a_r))]}_{Q(-r)} - \underbrace{E_T[\exp(-r(T\wedge a_s))]}_{P(-r)}\right\}\frac{1}{-r} \end{aligned} \tag{B13}$$

Equating these results (B.11 and B.13) gives finally **consumption at birth** as

$$c_0 = \frac{(k-r)w\exp(f(a_s))}{r}\frac{P(-r)-Q(-r)}{g(k-r)-1} \tag{B14}$$

C_0 thus depends on a_s, the amount of years of schooling. **The optimal schooling tenure** follows as the derivative of (B14) w.r.t. a_s as

$$f'(a_s)[P(-r)-Q(-r)] + \frac{dP(-r)}{da_s} = 0 \tag{B15}$$

Aggregate consumption follows by using the expectation for aggregation as

$$
\begin{aligned}
C(t) &= bNE_T\left[\int_0^T c(a)da\right] = bNE_T\left[\int_0^T c_0 \exp(ka)da\right] \\
&= bNc_0E_T\left[\int_0^T \exp(ka)da\right] = bNc_0E_T\left[\exp(kT)-1\right]\frac{1}{k}
\end{aligned}
\tag{B16}
$$

where b refers to the birth rate and N to the size of the population. This finally gives

$$
C(t) = bNc_0\frac{1}{k}\left(g(k)-1\right) \tag{B17}
$$

Capital and human capital

To derive aggregate wealth first define to evolution of individuals' net assets as

$$
\frac{dv(a)}{da} = [r+\mu(a)]v(a) + y(a) - c(a) \tag{B18}
$$

Notice that v(0)=0. Further, from equation 4.3,

$$
y(a) = wh(a) = w\exp(f(a_s)) \tag{B19}
$$

Solving this first order differential equation yields (see Li and Tuljapurkar, page 28)

$$
v(a)\exp\left[\int_0^a -(r+\mu(m)dm)\right] = \int_0^a [y(x)-c(x)]\exp\left(\int_0^x -(r+\mu(m)dm)\right)dx \tag{B20}
$$

$$
v(a) = \frac{\exp(ra)}{l(a)}\int_0^a [y(x)-c(x)]\exp(-rx)l(x)dx \tag{B21}
$$

Therefore aggregate wealth K(t) follows as

$$
K(t) = bNE_T\left[\int_0^T v(x)dx\right] = bNE_T\left[\int_0^T \frac{\exp(rx)}{l(x)}\int_0^x [y(a)-c(a)]\exp(-ra)l(a)dadx\right] \tag{B22}
$$

By the definition of expectation this becomes

$$
K(t) = bN\int_0^{T\max} \exp(rx)\int_0^x [y(a)-c(a)]\exp(-ra)l(a)dadx \tag{B23}
$$

exchanging the order of integration yields

$$
K(t) = bN\int_0^{T\max} [y(a)-c(a)]\exp(-ra)l(a)\int_a^{T\max} \exp(rx)dxda \tag{B24}
$$

and further

$$K(t)=\frac{bN}{r}\int_{0}^{T\max}[y(a)-c(a)]l(a)[\exp(r(T_{\max}-a))-1]da \tag{B25}$$

We can also write this in expectation form

$$K(t)=\frac{bN}{r}E_T\left\{\int_{0}^{T}[y(a)-c(a)][\exp(r(T_{\max}-a))-1]da\right\} \tag{B26}$$

Defining

$$\phi(r,e_0,\sigma_0)\equiv\frac{1}{w}E_T\left\{\int_{0}^{T}[y(a)-c(a)][\exp(r(T_{\max}-a))-1]da\right\} \tag{B27}$$

this reduces to a simpler expression for aggregate wealth as

$$K(t)=\frac{bN}{r}w\phi(r,e_0,\sigma_0) \tag{B28}$$

Substituting the income y(a) into B27 gives

$$\begin{aligned}\phi(r,e_0,\sigma_0)\equiv{}&\frac{1}{w}E_T\left\{\int_{0}^{T\wedge a_s}[-c(a)][\exp(r(T_{\max}-a))-1]da\right\}\\&+\frac{1}{w}E_T\left\{\int_{T\wedge a_s}^{T\wedge a_r}[y(a)-c(a)][\exp(r(T_{\max}-a))-1]da\right\}\\&+\frac{1}{w}E_T\left\{\int_{T\wedge a_r}^{T}[y(a)-c(a)][\exp(r(T_{\max}-a))-1]da\right\}\end{aligned} \tag{B29}$$

Further calculation yields an expression for ϕ as (see Li and Tuljapurkar, page 29)

$$\phi(r,e_0,\sigma_0)=\exp(f(a_s))\varphi(r,e_0,\sigma_0) \tag{B30}$$

where

$$\varphi(r,e_0,\sigma_0)=\frac{r-k}{kr}\frac{(Q(-r)-P(-r))(g(k)-1)}{g(k-r)-1}-\lambda(a_r)+\lambda(a_s) \tag{B31}$$

Finally it results that aggregate capital (aggregate wealth) can be expressed as

$$K(t) = \frac{bN}{r} w \exp(f(a_s))\varphi(r, e_0, \sigma_0) \tag{B32}$$

Aggregate human capital follows from the definition of relative wages simply as

$$\exp(f(a_s))L \tag{B33}$$

where L refers to the size of the workforce between ages a_s and a_r.

Expressed alternatively, human capital follows as

$$H(t) = bN \exp(f(a_s))[\lambda(a_r) - \lambda(a_s)] \tag{B34}$$

Dividing B32 by B34 and using the first order conditions of the production function (equation 4.2) yields the equation

$$\frac{r-k}{kr} \frac{(Q(-r) - P(-r))(g(k)-1)}{g(k-r)-1} = \frac{\lambda(a_r) + \lambda(a_s)}{1-\alpha} \tag{B35}$$

By solving equations B35 and B15 we can find age of schooling and equilibrium interest rate r. All other variables can be found accordingly (see Li and Tuljaprukar, page 9).

Surface plots

Of the interest rate, consumption, wage and wealth with respect to changes in life expectancy and the variance of death-age.

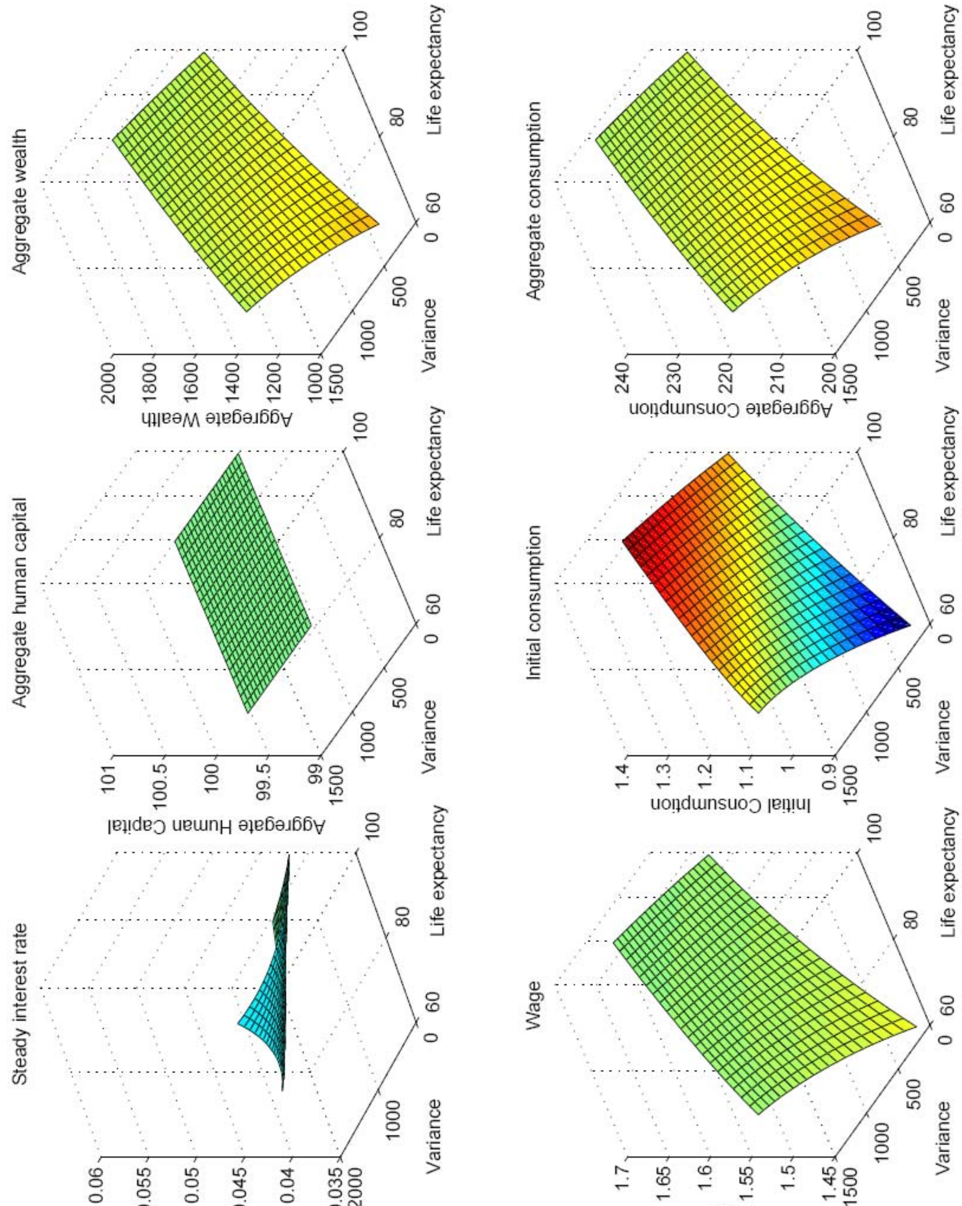

Figure B4.1 Surface plots of changing life expectancy and life time uncertainty on diverse economic variables (Li, Tuljapurkar; page 11)

5. Probabilistic Aging (Grafenhofer et al.)

The core principle of the PA model is its alternative way of thinking about ageing. Instead of defining age in the usual way simply as time that has passed since birth, ageing is viewed as a stochastic process, moving agents from one stage of life to the next. Life stages differ in diverse characteristics such as labour productivity, health and mortality defining individual's economic abilities. Agents within the same life stage – i.e. age-group or age-class - face the same characteristics. The transition from one stage to the next can be seen as an ageing-shock. Just as in Gertler´s model of retirees and workers the randomness of transition implies that people within the same stage of life can differ in actual age, i.e. their date of birth.

By modelling ageing as a stochastic transition through several stages of life, the authors try to take into account that in real life people seem to "age" at different speed. For example through their individual history of illness or accidents people of same age differ in health, which in turn affects their mortality and labour productivity. Thus, people of same "age" can have a quite different status of health and vitality, due to their personal life cycle history.

In the PA framework ageing is no longer viewed as a monotonous and smooth process, but as discrete events, ageing shocks, that occur stochastically and change individual's life cycle characteristics. While some people seem to stay younger for a longer period of time, others are hit by ageing shocks more frequently and thus age at higher speed.

To be able to capture the different speed of ageing among individuals, a discrete number of A states of increasing age are defined. Ageing in this context means, that people move from one state of life to the next. As the transition from one state to the next occurs stochastically, the framework allows for different people staying in a specific state of life for varying length of time. In this set-up people born at the same time can reach different stages of their life cycle at different points in time. People within the same stage of their lifecycle – or equivalently of same "age" – on the other side, can differ concerning the time when they were born. As a result the PA framework yields a very high degree of heterogeneity among agents within the economy.

In the PA model diverse characteristic variables like an agent's earnings potential as well as mortality risk differ across different age-groups and are thus age-group specific. Contrasting age-group dependent characteristics like mortality, some other variables depend on the individual's personal life cycle. For example the amount of assets, or more generally his/her wealth, an agent has accumulated over time, is dependent on the agents individual lifecycle history. If, for example, an agent managed to stay in a very productive state of life for a long time s/he will probably have accumulated more capital than an agent who aged very fast and so could not earn – and thus save - as much. Therefore a concept is needed, that can capture these agent-dependent issues as well as those entirely determined by the agent's position in the life cycle.

5.1 The concept of lifecycle histories

In order to be able to make allowance for heterogeneous ageing the initial task is to distinguish real time from the ageing-process. The PA's framework rests on a discrete number of A ageing states. As previously mentioned, diverse characteristics are solely determined by the agent's age-group affiliation, such as his/her earnings potentials. When an agent of age-group *a* is hit by an ageing shock s/he moves from this state *a* to *a+1*, implying a change in these characteristics.

Definition 5.1:

The life cycle history (life cycle biography) is the collection of the dates of ageing shocks that already hit an agent and is represented by a vector α. The entries of this vector are the dates at which the ageing shocks occurred.

Besides their date of birth, agents differ by their individual *life cycle history*, as they "age" at different speed. For a household of age-group *a* the number of possible lifecycle histories thus is given by

$$\mathrm{B}_t^a \equiv \{(\alpha_1,...,\alpha_a) : \alpha_1 < ... < \alpha_a \leq t\}$$

As soon as an agent experiences an ageing shock his/her lifecycle biography is updated by the entry of the date of ageing.

Assumption 5.1:

All economic agents with the same life cycle history α are viewed as being identical.

Having the same life cycle history implies that agents have experienced exactly the same age-group specific characteristics for exactly the same length of time. Thus, they have been earning the same amount of money and saved the same proportion of their income (because this economic behaviour is determined by age-group affiliation!). Thus, concerning economic relevance, they represent the same type of agents and therefore can be viewed as being identical. This feature will make aggregation quite easy.

5.2 Demographics of the PA Model

Agents face the risk of ageing as well as the risk of dying. Thus, from time-period *t* to *t+1*, there are three possible events for each agent. With probability 1-γ^a s/he dies (note, that the superscript *a* indicates age-group specifity of a parameter). With probability $\gamma^a\omega^a$ s/he survives without ageing. With probability γ^a (1- ω^a) s/he survives but ages and thus belongs to the age-group a+1 in the next period. This of course is different for agents of the last age-group. They can survive with probability γ^A or die with probability 1-γ^A. These probabilities, for survivorship and death, differ between different age-groups. Thus mortality and ageing are modelled age-group specific.

Concerning issues of savings and other variables the individual lifecycle history is important. In order to be able to aggregate identical agents one must collect agents who are at time t in the same age-group *a* and also have the same lifecycle history α. Such a group of identical agents is then denoted by $N^a_{\alpha,t}$. From time *t* to *t+1* an age-group *a* is divided into three subgroups (again due to the "law of large numbers"), reflecting the three scenarios mentioned above (see Remark 5.1): (i) into a group of those dying, (ii)those surviving without ageing and (iii) those who age from time *t* to *t+1*. Since ageing in the last case implies a change in the life cycle history (the agents are hit by an ageing shock, thus their life cycle biographies must be updated), the new lifecycle history for the corresponding group is then denoted by α´.

<u>*Remark 5.1*</u>

From time t to t+1 age-group a is divided into the following three subgroups:

(i) $N^*_{\alpha^*} = N^a_{\alpha,t} * (1-\gamma^a)$ dying (denoted by the superscript *)

(ii) $N^a_{\alpha,t+1} = N^a_{\alpha,t} * \gamma^a \omega^a$ no aging (5.1)

(iii) $N^{a+1}_{\alpha',t+1} = N^a_{\alpha,t} * \gamma^a (1-\omega^a)$ aging

<u>*Proof*</u>: *see Appendix C*

As agents with the same lifecycle biography α are viewed as being identical, a particular age-group consists of $N^a_{\alpha,t}$ identical agents at time *t*. Summing up over all possible lifecycle histories at time t within a specific age-group *a* yields the total number of agents within age-group *a* as

$$N^a_t \equiv \sum_{\alpha \in \mathrm{B}^a_t} N^a_{\alpha,t} \ . \qquad (5.2)$$

Thus, this aggregation formula collects all agents that have ended up at time *t* to be in age-group *a*, regardless of their individual date of birth. As mortality and ageing probabilities are identical for all agents within the same age-group, the law of large numbers implies how age-groups develop over time. In particular the law of large numbers implies a deterministic behaviour for the evolution of age-groups. The number of newborns in period *t+1* is denoted by $N^1_{(t+1),t+1}$. The evolution of age-groups is given by

Remark 5.2:

Age-groups develop according to following three equations:

(i) $N^a_{t+1} = \gamma^a \omega^a * N^a_t + \gamma^{a-1}(1-\omega^{a-1}) * N^{a-1}_t$, $\omega^A = 1$,

(ii) $N^1_{t+1} = \gamma^1 \omega^1 * N^1_t + N^1_{(t+1),t+1}$ (5.3)

(iii) $N_{t+1} = N_t + N^1_{(t+1),t+1} - \sum_{a=1}^{A}(1-\gamma^a)N^a_t$, $N_t \equiv \sum_{a=1}^{A} N^a_t$

Proof: *see Appendix C*

5.3 Consumption and Saving

As in Gertler's model agents value their current state according to a CES – non expected utility function (compare equation (3.21) in section 3). Again the argument for this is that assuming this kind of preferences restricts agents to risk-neutrality in the presence of income risk but allows for an arbitrary intertemporal elasticity of substitution. Thus, agents are able to value current consumption at a different level than future consumption. Income risk occurs here, as the risk of switching from one age-group to the next also implies a change in labour income. This is due to the fact, that earnings potentials are assumed to be age-group specific. The difference to Gertler's model is that every age-group faces the risk of dying.

5.3.1 Individual consumption

An individual maximizes his/her expected future utility arising from consumption. The maximization problem yields the following Bellman equation

$$V(A^a_{\alpha,t}) = \max_{C^a_{\alpha,t}} \left[\left(C^a_{\alpha,t}\right)^\rho + \gamma^a \beta \left(\overline{V}^a_{\alpha,t+1}\right)^\rho \right] \quad (5.4)$$

Thus, the value function – dependent on current assets held by the agent – is defined as the maximum over a function of current consumption and discounted future expected utility $\overline{V}^a_{\alpha,t+1}$. The discount factor consists of an exogenous factor β augmented by γ^a, the probability of death for the age-group. Conditional on surviving, an agent's expected utility next period is

$$\overline{V}^{a}_{\alpha,t+1} = \omega^{a} V^{a}_{\alpha,t+1} + (1-\omega^{a}) V^{a+1}_{\alpha',t+1} \tag{5.5}$$

Thus, equation (5.5) is the mean over the value function in both possible future states. As agent's consumption possibilities depend not only on his/her current wage but also on the assets accumulated by the agent, one has to take a look at how assets evolve over time.

5.3.1.1 Individual's wage and assets

Clearly the accumulation of capital depends on the individual agent's life cycle history, as mentioned before. The longer an agent remains in highly productive states of life, the more labour income s/he earns, and is therefore able to accumulate more capital. Labour income is assumed to be group specific and equal for all agents within the same age-group.

<u>*Definition 5.2:*</u>

Wage is defined by the age-group specific productivity characteristic θ^{a} and takes the form

$$y^{a}_{t} = \begin{cases} (1-\tau) w_{t} \theta^{a}_{t} & : a \in \{1,\ldots,a^{R}-1\} \\ p_{t} & : a \in \{a^{R},\ldots A\}. \end{cases} \tag{5.6}$$

This formulation captures the different wage profiles for different age-groups. Group specific labour income depends on the factor of productivity θ^{a}_{t} which differs among different age-groups, leading to changing earnings between different groups. Labour income is earned as long as people remain part of the labour force. Thus, they receive labour income until they move to the first stage of retirement a^{R}. Workers finance the social security system by paying taxes on labour income τ. As soon as people retire they receive pension payments p.

As in Blanchard's and Gertler's models the possibility of dying leads to an optimality problem (compare section 3.2.1.1). Again the introduction of a redistributing life insurance company helps to solve the problem of uncertain lifetime. Agents receive insurance payments from the company contingent on the bequests they could leave behind unintended when dying. This leads in total to

<u>*Remark 5.3:*</u>

Let γ denote age-group specific insurance payments. The asset accumulation equation for agents is given by

$$\gamma^{a} A^{a}_{\alpha,t+1} = R_{t+1}\left[A^{a}_{\alpha,t} + y^{a}_{t} - C^{a}_{\alpha,t}\right], \qquad A^{a}_{\alpha,t+1} = A^{a+1}_{\alpha',t+1}. \tag{5.7}$$

<u>Proof:</u> *see Appendix C*

The equation on the right hand side states simply, that the value of a persons assets is the same in period t+1 whether she ages (represented by the right hand side of the equation – in this case the life cycle biography is updated to α' and age-group affiliation is then a+1) or not (represented by the left hand side of the equation - life cycle biography remains α and age group index remains a).

5.3.1.2 Optimality conditions

Optimality Conditions:

Solving the agent's maximization problem yields necessary optimality conditions

$$\left(C^{a}_{\alpha,t}\right)^{\rho-1} = \beta R_{t+1}\overline{\eta}^{a}_{\alpha,t+1}, \qquad \text{and} \qquad \eta^{a}_{\alpha,t} = \beta R_{t+1}\overline{\eta}^{a}_{\alpha,t+1} \tag{5.8}$$

where

$$\eta^{a}_{\alpha,t} \equiv \frac{dV^{a}_{\alpha,t}}{dA^{a}_{\alpha,t}}\left(V^{a}_{\alpha,t}\right)^{\rho-1} \text{ and } \overline{\eta}^{a}_{\alpha,t+1} \equiv \left[\omega^{a}\frac{dV^{a}_{\alpha,t+1}}{dA^{a}_{\alpha,t+1}} + (1-\omega^{a})\frac{dV^{a+1}_{\alpha',t+1}}{dA^{a+1}_{\alpha',t+1}}\right]\left(\overline{V}^{a}_{\alpha,t+1}\right)^{\rho-1}. \tag{5.9}$$

The Euler equation for agent's consumption follows as

$$\omega^{a}C^{a}_{\alpha,t+1} + (1-\omega^{a})\Lambda^{a}_{\alpha,t+1}C^{a+1}_{\alpha',t+1} = \left(\beta R_{t+1}\Omega^{a}_{\alpha,t+1}\right)^{\sigma}C^{a}_{\alpha,t} \tag{5.10}$$

Proof: Appendix C

When the agent decides how much to consume each period, s/he compares the expected utility gain from saving with the current loss in utility that arises when s/he postpones consumption for further asset accumulation. The $\overline{\eta}$ term gives expected utility next period times the weighted shadow price of next period's assets (the term in the square bracket). Asset's weighted shadow price itself yields the change in the objective function (expected utility) with respect to changes in the amount of assets held, taking into account both possible future states an agent could end up in – conditional on surviving. In particular $\overline{\eta}$ represents the expected utility change from saving, considering the uncertainty of agent's future state in the life cycle and taking into account agent's desire to smooth consumption (as indicated by the appearance of the ρ term in the exponent).

The Euler equation states the relationship between current consumption and next period's consumption. Desired consumption next period is a function of the interest rate *R* and the discount rate *β*. The *Ω* term augments the interest rate and reflects the impact of ageing and thus dying on consumption decisions. Since agent's characteristics change as they grow older, additional units of money will be valued differently in the two possible future states (staying in group *a* or ageing to *a+1*). The *Ω* term captures this change as it can be expressed as a function of the transition probability ω^a and the marginal rate of substitution between two age states *a* and *a+1*[29]. It is easy to see that the term is always greater than one as long as ageing has a positive probability, and so the possibility of ageing leads to a higher weight on the interest rate.

5.3.1.3 Impact of life-time uncertainty on individual's behaviour

Proposition 5.1:

Let Δ denote the inverse of agent's marginal propensity to consume. Agent's behaviour can be described by the following system of dynamic equations.

$$\text{(i)}\, C^a_{\alpha,t} = (1/\Delta^a_t)\left(A^a_{\alpha,t} + H^a_{\alpha,t}\right)$$

$$\text{(ii)}\, V^a_{\alpha,t} = \left(\Delta^a_t\right)^{1/\rho} C^a_{\alpha,t}$$

$$\text{(iii)}\, \Delta^a_t = 1 + \gamma^a \beta^a \left(\Omega^a_{t+1} R_{t+1}\right)^{\sigma-1} \Delta^a_{t+1}$$

$$\text{(iv)}\, \Omega^a_{t+1} = \omega^a + (1-\omega^a)\left(\Lambda^a_{t+1}\right)^{1-\rho},\ \Lambda^a_{t+1} = \left(\Delta^{a+1}_{t+1} / \Delta^a_{t+1}\right)^{1/\rho} \qquad (5.11)$$

$$\text{(v)}\, H^a_{\alpha,t} = y^a_t + \gamma^a \overline{H}^a_{\alpha,t+1} / \left(\Omega^a_{t+1} R_{t+1}\right)$$

$$\text{(vi)}\, \overline{H}^a_{\alpha,t+1} = \omega^a H^a_{\alpha,t+1} + (1-\omega^a)\left(\Lambda^a_{t+1}\right)^{1-\rho} H^{a+1}_{\alpha',t+1}$$

Proof: see appendix C

Equation *(5.11)(i)* states that an agent's consumption at time period *t* equals the age-group specific marginal propensity to consume times the amount of assets held by the agent plus the present value of the agent's total future labour income (her/his human wealth). Accordingly, as stated by *(5.11)(v)*, the agent's level of human wealth at time period *t* equals his/her current labour income *y* (which corresponds to pension income for the groups of retirees) plus the expected discounted level of his/her future human wealth ($\overline{H}$). In equation *(5.11)(v)* the expected human wealth next period ($\overline{H}$) is discounted at an interest rate that is augmented by

[29] $\Omega^a_{t+1} = \omega^a + (1-\omega^a)\left(\Lambda^a_{t+1}\right)^{1-\rho} = \omega^a\left[1 + MRS^a\right]$; Grafenhofer et. al. page 16.

the probability of death γ^a and the term Ω^a. As previously explained, the augmentation of the interest rate by the Ω term reflects the effects of the possibility of ageing. Additionally the probability of surviving enters the discount factor. Not surprisingly agents take into account the possibility that they might not be alive next period and thus increase their discount rate on future labour income (just as in Gertler's model). This means that agents value future labour income less the higher their probability of death. In total mortality leads to a higher discount factor on future labour income, reflecting the fact that – in the presence of life time uncertainty – future income is valued less than if one would live forever. This stands in perfect analogy to Gertler's results.

Analyzing equation *(5.11)(iii)* shows that the only source of changing propensities to consume is the rate of mortality $1-\gamma^a$ (Grafenhofer et al., page 18). If the survival rate were the same for all age-classes the marginal propensity to consume would be the same for all age-groups. Thus, a change in the propensity to consume from one age-group to the next reflects exclusively a change in mortality rates. This can be understood by considering equation *(5.11)(iii)*.

If mortality rates increase from one age-class to the next, both values Λ and Ω exceed one, implying that decreasing survivorship rates increase the marginal propensity to consume. An increase in Ω can be seen as an increase in the discount rate (as explained above). Thus, an increasing probability of dying increases the rate at which agents value their future income.

As it becomes more probable to die, the marginal rate of substitution increases and the less willing agents are to postpone consumption leading to an increasing propensity to consume.

5.3.2 Aggregate Consumption

To derive aggregate variables one has to recall that some magnitudes depend on the individual life cycle (like human wealth) whereas other variables are solely determined by age-group characteristics (such as wage). As economic agents with the same life-cycle history are viewed as being identical, aggregation becomes quite easy. At date *t* an age-group *a* contains a number of $N^a_{\alpha,t}$ agents with the same life cycle history α. As these agents are assumed to be identical, they behave the same way, meaning that they show e.g. the same consumption behaviour. Thus, these $N^a_{\alpha,t}$ agents consume $C^a_{\alpha,t}$ each. Simply aggregating over all age-groups including all possible life cycle histories α leads to the aggregation formulas

$$C^a_t \equiv \sum_{\alpha \in \mathrm{B}^a_t} C^a_{\alpha,t} N^a_{\alpha,t} \text{ and } C_t \equiv \sum_{a=1}^{A} C^a_t \,. \tag{5.12}$$

Using the fact that the variables Δ_t^a, Ω_t^a and Λ_t^a are the same for all agents within the same age-group (as they are determined by group affiliation) yields total age-group consumption as

$$C_t^a = (1/\Delta_t^a)\left(A_t^a + H_t^a\right). \tag{5.13}$$

As labour income is the same for all agents within the same age-group, the total age-group labour income simply follows as $y_t^a N_t^a$.

Concerning economy wide labour patterns, one fact to be considered is the changing labour productivity between different age-classes.

<u>*Remark 5.4:*</u>

Denoting labour supply in efficiency units by L^s and the number of retirees by N^R aggregate labour income at time t follows as

$$Y_t = (1-\tau)w_t L_t^s + p_t N_t^R$$

where

$$L_t^s \equiv \sum_{a=1}^{a^R-1} \theta_t^a N_t^a \text{ and } N_t^R \equiv \sum_{a^R}^{A} N_t^a .$$

For the aggregation of human wealth and assets one has to take into account the dependency on the individual life cycle and age-group affiliation. Very clearly assets are determined by the life cycle history of earnings as well as by current labour income and consumption behaviour which is entirely determined by age-group affiliation. Considering the aggregate expression of human wealth is simpler, as wage related income is solely determined by age-group affiliation and thus is the same for all agents within the same age-class.

<u>*Remark 5.5:*</u>

Aggregate expressions for human capital follow as

$$H_t^a = h_t^a N_t^a \quad h_t^a \equiv H_{\alpha,t}^a \qquad h_t^a = y_t^a + \gamma^a \frac{\omega^a h_{t+1}^a + (1-\omega^a)\left(\Lambda_{t+1}^a\right)^{1-\rho} h_{t+1}^{a+1}}{\Omega_{t+1}^a R_{t+1}} \tag{5.14}$$

<u>*Proof:*</u> see Appendix C

The variable h_t^a denotes per capita human capital, thus the expression for total age-group human capital H_t^a is straightforward. As the last age-group does not face any transition probability the expression for this group simplifies by setting Ω and ω equal to one. Turning to asset accumulation, aggregate equations follow as (see Appendix C)

<u>*Remark 5.6:*</u>

Aggregate assets evolve according to the three equations

(i) $A^1_{t+1} = R_{t+1}\omega^1 S^1_t \qquad S^a_t \equiv A^a_t + Y^a_t - C^a_t$

(ii) $A^a_{t+1} = R_{t+1}\left[\omega^a S^a_t + \left(1-\omega^{a-1}\right)S^{a-1}_t\right]$ (5.15)

(iii) $A_{t+1} = R_{t+1}\left[A_t + Y_t - C_t\right]$

<u>Proof:</u> see Appendix C

Comparing the last equation with the equation for individual asset accumulation shows that the factor of redistribution cancels out, just as in Blanchard's and Gertler's models. Thus in the aggregate there is no net-effect from the existence of the insurance company, as this only redistributes from the dying to the living.

5.4 Life cycle aspects of the model

A special feature of the PA model is that it contains a number of intertemporal models as special cases. Choosing certain values for the parameters of the model replicates these models. Studying the different implications of these models can help to understand the impact of uncertainty in lifetime and earnings and how they are represented in the PA model.

5.4.1 A synthesis of models

Setting the parameters $\omega^a = \gamma^a = 1$ yields a model of infinitely lived agents. As mortality and ageing are set to zero, age characteristics do not play any role. Marginal propensity to consume and human wealth are independent of age as can be seen by setting $\omega^a = \gamma^a = 1$ in equations *(5.11)(i)-(vi)*. The formula describing human wealth reduces to $H_t = y_t + H_{t+1} / R_{t+1}$. Compared to a model with positive mortality the discount rate on human capital is thus lower, reflecting the absence of life time uncertainty. The Euler equation of consumption then takes the form $C_{t+1} = \left(\beta R_{t+1}\right)^{\sigma} C_t$. Again the absence of life cycle aspects simplifies matters. Consumption decisions only depend on the relation between the preference factor β and the interest rate *R*.

Blanchard's Perpetual Youth Model can be reproduced by setting $\gamma^a = \gamma < 1$ and $\omega^a=1$. This setting implies age independent consumption behaviour. Setting $\omega^a=1$ implies that Ω^a is also equal to one. Using this in equation *(5.11.iii)* giving the inverse of the marginal propensity to consume together with the assumption of age-independent mortality shows the independency of consumption behaviour and age. Thus, the marginal propensity to consume is equal across agents, regardless of their individual time of birth. But compared to the case of infinitely lived agents the constant probability of death increases the propensity to consume. Human wealth is affected in the same way, as mortality increases the discount factor of future earnings (compare equation *(5.11.v)*). Uncertainty of life time thus results in higher consumption and less weight on potential future earnings though the age-independentness of mortality yields equal behaviour for all agents, regardless of their individual age.

Gertler's model of retirees and workers is replicated by defining two age-groups and setting the parameters to $\gamma^1 =1$, $\gamma^2 < 1$, $\omega^1<1$ and $\omega^2=1$. Thus, the first age-group faces no risk of dying but the risk of moving into the second stage of life ($\omega^1<1$), retirement. Death only occurs within the second age-group (among retirees). Gertler's model generates heterogeneity between age-groups concerning consumption behaviour and human wealth and thus makes a good step towards proper representation of life cycle aspects.

Finally, Diamond's basic two period model can be reproduced by setting $\gamma^1 =1$, $\gamma^2 =0$, $\omega^1=0$ $\omega^2=1$ and $y_{t+1}^2=0$. Thus, there is no life time uncertainty as people live for exactly two periods. In the second life period there is no income earned and agents only consume out of their wealth.

5.4.2 Impact of uncertainties

This comparison shows very well the impact of mortality and income risk on the economy. First, positive probabilities of death and changing life-stages imply changes in the marginal propensities to consume and an increase in the discount rate on future labour income. Second, as these probabilities vary between age-groups also the consumption behaviour varies between different stages of life. Also the discount rate on human capital changes due to different transition probabilities between age-groups.

In general the PA Model implies similar impacts of ageing and dying as Gertler's model. Just as in Gertler's model and Blanchard's Model of Perpetual Youth the marginal propensity to consume is an increasing function of the probability of death for each age-group, again reflecting realistic behaviour. The improvement that the PA model brings about lies in the extension of Gertler's model to more periods of life and to allow for young-age mortality. As shown in the next subsection the PA framework is therefore able to replicate life-cycle details at a high level.

Aggregate consumption and savings depend strongly on the population dynamics (see equations 5.12, 5.13, 5.14 and 5.15). The distribution of wealth between age-groups determines economy wide consumption, due to altering propensities to consume with increasing age (as in Gertler's model). Therefore the population structure has strong impacts on aggregate magnitudes and changes in mortality or transition rates will directly affect the economy as a whole. Thus, not only life cycle aspects are modelled realistically in the PA Model but also the impact of demographic changes on the economy can be studied well. Grafenhofer et al. conclude that *the PA Model is a much more powerful tool of policy analysis as compared to the Perpetual Youth Model as well as the […] extension by Gertler* (p 31).

5.5 The PA model and real data

How well is the PA model able to represent life cycle details? In order to answer this question, Grafenhofer et al. apply the model to real data on mortality rates and wage profiles. To attach real data to the PA framework one has to calculate the age-group specific characteristics such as labour productivity and survival rates from the data. Grafenhofer et. al. assume eight age-groups as shown in Table 1. The first life period thus starts with an age-group corresponding to young adults "aged" 20 to 29, and so on. Table 1 also shows the age-group characteristics for each age-class.

Table 1: Demographic and Life-Cycle Parameters

1. Age groups	1	2	3	4	5	6	7	8
2. Cohorts	20-29	30-39	40-49	50-59	60-69	70-79	80-84	85-89
3. Data N^a/N	0.168	0.222	0.192	0.168	0.120	0.089	0.025	0.016
4. Model N^a/N	0.179	0.177	0.175	0.168	0.148	0.107	0.031	0.016
5. Labor prod. θ^a	1.000	1.362	1.561	1.582	1.295	0.000	0.000	0.000
6. Prob. $1-\gamma^a$	0.001	0.001	0.004	0.012	0.028	0.042	0.096	0.200
7. Prob. $1-\omega^a$	0.099	0.099	0.096	0.089	0.074	0.061	0.115	0.000
8. Propens. $1/\Delta^a$	0.047	0.052	0.059	0.069	0.086	0.110	0.168	0.230

Notes: θ^a life-cycle labor productivity determines wage $w^a = w\theta^a$, $1-\gamma^a$ probability of dying, $1-\omega^a$ probability of aging, $1/\Delta^a$ marginal propensity to consume. Data sources: BFS (2004), and own calculations.

Table 5.1 Demographic and Life-Cycle Parameters (Grafenhofer et al., page 10)

Thus, age-group two has a 36% higher labour productivity than group one implying higher wage and so on, as can be seen in line 5. The last line shows the changing consumption behaviour of agents as they move from one stage of life to the next. As can be seen, individual's propensity to consume increases throughout their entire life time and becomes considerably high within the last two age-groups. The following figures show the PA model's implications on life cycle earnings and survivorship compared to real data.

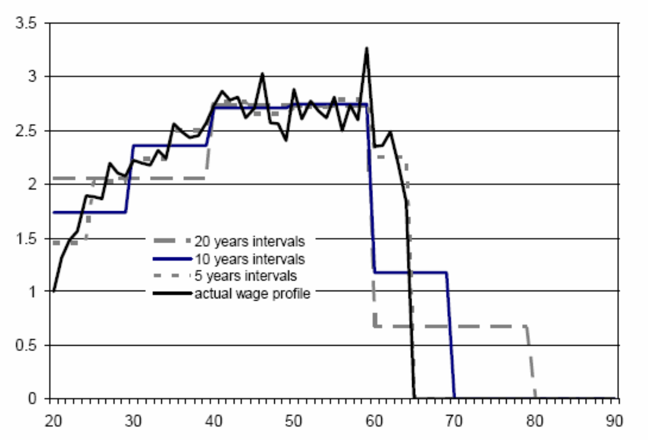

Figure 5.1: Life cycle wages with different time intervals (Grafenhofer et al., page 12)

As can be seen in Figure 5.1, the higher the number of age-groups, the better life cycle earnings can be represented with the PA model. Choosing five-year intervals gives a very close approximation to the real data, but the ten year interval does very well too.

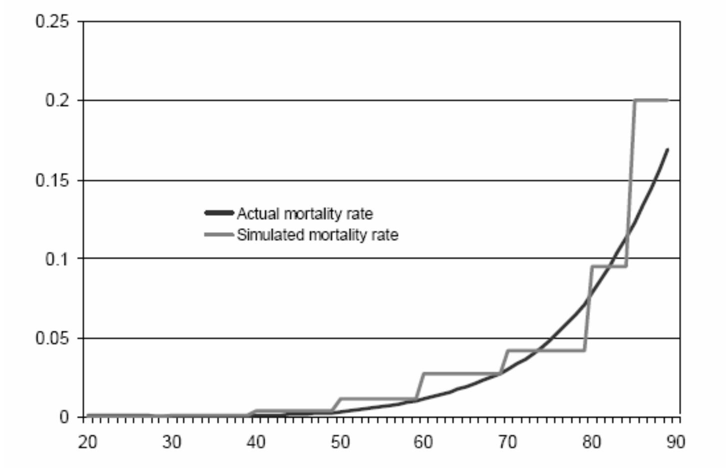

Figure 5.2: simulated and actual mortality rates (Grafenhofer et al., page 13)

Concerning the representation of mortality patterns the model gives a very detailed image of the increasing probability of death with increasing age. This approximation closely reflects real data and thus gives a quite realistic image of mortality.

In total the PA model is able to capture life cycle details at a high level, as shown in Figure 5.1 and 5.2. As Table 5.1 shows the PA model is able to reflect changing consumption behaviour with increasing age as well as changing wage related income. Thus, the PA model does well concerning the issues of demographic realism.

Appendix C

Proof of Remark 5.1:

This follows from the law of large numbers. Therefore each age-group is divided into three subgroups, according to the transition probabilities. Thus, the fraction 1-γ dies, while the fraction γ survives. From the surviving 1-ω age, while ω remain in the age-group.

***Proof of Remark 5.2*:**

(i)

This equation states, that the number of people in age-group a at time t+1 consists of those, who already where in age-group a at time t and survived to t+1 but did not age (the fraction ωγ of age-group a at time t). Additionally those who where in age-group a-1at time t and survived to t+1 but aged to age a enter the age-group at t+1 (a fraction of (1-ω)γ of age-group a-1).

(ii)

The youngest age-group consists at time t+1 of those who where in this age-group at time t, survived but did not age (a fraction of ωγ) and the number of new-borns $N^1_{(t+1),t+1}$.

(iii)

Total population at time t+1 equals the number of people already alive at time t (N_t) plus the number of new-borns $N^1_{(t+1),t+1}$ minus those of each group that did not survive from t to t+1 (captured by the last term in the expression)

***Proof of Remark 5.3*:**

It is again assumed, that the insurance company pays an actuarially fair group specific premium. Age-group *a* possesses a total amount of S^a savings at the end of period t. As a fraction of $1-\gamma^a$ dies from t to t+1 the insurance company collects a total amount of $(1-\gamma^a)S^a$ savings and pays premiums, in total $\pi^a\gamma^aS^a$. Assuming zero profits for the insurance company, this yields a premium rate of $\pi^a=(1-\gamma^a)/\gamma^a$ or $1+\pi^a=1/\gamma^a$. Therefore the agent's assets accumulate at $A^a_{\alpha,t+1} = R_{t+1}\left[(1+\pi^a)S^a\right]$ which is equivalent to

$$\gamma^a A^a_{\alpha,t+1} = R_{t+1}\left[A^a_{\alpha,t} + y^a_t - C^a_{\alpha,t}\right]. \qquad \text{(C.1)}$$

***Proof of Optimality Conditions*:**

This proof is quite similar to the proof of the optimality conditions in Gertler's model. The first order necessary conditions follow as the derivative of the value function with respect to consumption as

$$\frac{dV_t}{dC_t} = \frac{1}{\rho}(V_t)^{(1-\rho)/\rho}\left[\rho C_t^{\rho-1} + \gamma\beta\rho\overline{\eta}\right]\underbrace{\frac{1}{\gamma}R_{t+1}(-1)}_{A} = 0 \tag{C.2}$$

where the last part (A) is the derivative of the value function next period with respect to A_{t+1} (as A_{t+1} depends on C_t)

This simplifies to the necessary condition in equation 5.8

$$C_t^{\rho-1} = \beta\overline{\eta}R_{t+1} \tag{C.3}$$

where

$$\overline{\eta}_{t+1}^{a} \equiv \left[\omega^a \frac{dV_{t+1}^{a}}{dA_{t+1}^{a}} + (1-\omega^a)\frac{dV_{t+1}^{a+1}}{dA_{t+1}^{a+1}}\right]\left(\overline{V}_{t+1}^{a}\right)^{\rho-1} \tag{C.4}$$

The second equation (equation 5.9) follows by defining

$$\eta \equiv \frac{dV_t^a}{dA_t^a}\left(V_t^a\right)^{\rho-1} = \left(C_t^a\right)^{\rho-1} \tag{C.5}$$

The last equality is simply the derivation of the value function with respect to A_t after rearranging. Therefore the equation

$$\eta_{\alpha,t}^{a} = \beta R_{t+1}\overline{\eta}_{\alpha,t+1}^{a} \tag{C.6}$$

holds. Notice that rearranging (C.5) results in

$$dV^a / dA^a = \left(V^a / C^a\right)^{1-\rho} \tag{C.7}$$

Defining the marginal rate of substitution between two age-states as

$$MRS^a \equiv dA^a / dA^{a+1}\,|_{d\overline{V}=0} = \frac{1-\omega^a}{\omega^a}\frac{dV^{a+1}/dA^{a+1}}{dV^a/dA^a} = \frac{1-\omega^a}{\omega^a}\left(\Lambda^a\right)^{1-\rho} \tag{C.8}$$

where

$$\Lambda^a \equiv \frac{V^{a+1}/C^{a+1}}{V^a/C^a} \tag{C.9}$$

comes from equation (C.7).

Now define

$$\Omega^a \equiv \omega^a\left[1+MRS^a\right]=\omega^a+\left(1-\omega^a\right)\left(\Lambda^a\right)^{1-\rho} \qquad \text{(C.10)}$$

Now take out ω^a and $dV^a/dA^a=\left(V^a/C^a\right)^{1-\rho}$ out of the equation (C.4) to obtain

$$\overline{\eta}^a=\Omega^a\left[\overline{V}^a/\left(V^a/C^a\right)\right]^{\rho-1} \qquad \text{(C.11)}$$

Rewriting expected utility as

$$\overline{V}^a=\left[\omega^a C^a+\left(1-\omega^a\right)\Lambda^a C^{a+1}\right]\left(V^a/C^a\right) \qquad \text{(C.12)}$$

results by substitution in (C.11) as

$$\overline{\eta}^a=\Omega^a\left[\omega^a C^a+\left(1-\omega^a\right)\Lambda^a C^{a+1}\right]^{\rho-1} \qquad \text{(C.13)}$$

using this and σ=1/(1-ρ)in equation (C.3) finally gives the Euler equation

$$\omega^a C^a_{\alpha,t+1}+(1-\omega^a)\Lambda^a_{\alpha,t+1}C^{a+1}_{\alpha',t+1}=\left(\beta R_{t+1}\Omega^a_{\alpha,t+1}\right)^{\sigma}C^a_{\alpha,t} \qquad \text{(C.14)}$$

as stated in equation 5.10.

<u>Proof of Proposition 5.1 (taken from Grafenhofer et al.)</u>**:**

Grafenhofer et al. show (Grafenhofer et al., page 34) that the consumption function fulfils the Euler equation and therefore is the optimal policy. First they insert 5.11(i) into the left hand term of (C.14) and use $A^a_{\alpha,t+1}=A^{a+1}_{\alpha',t+1}$ and definitions 5.11(iv) and 5.11(vi) and get,

$$\left[\Omega^a_{t+1}A^a_{\alpha,t+1}+\overline{H}^a_{\alpha,t+1}\right]/\Delta^a_{t+1}=\left(\beta\Omega^a_{t+1}R_{t+1}\right)^{\sigma}C^a_{\alpha,t} \qquad \text{(C.15)}$$

Multiplying by

$$\gamma^a\Delta^a_{t+1}/\left(\Omega^a_{t+1}R_{t+1}\right) \qquad \text{(C.16)}$$

and using 5.11(v) and equation (C.1) on the left hand side

$$A^a_{\alpha,t}+H^a_{\alpha,t}-C^a_{\alpha,t}=\gamma^a\beta^{\sigma}\left(\Omega^a_{\alpha,t}R_{t+1}\right)^{\sigma-1}\Delta^a_{t+1}C^a_{\alpha,t} \qquad \text{(C.17)}$$

substituting again 5.11(i) on the left side and cancel the C-terms yields a result corresponding to 5.11(iii). Hence the stated policy is optimal since it fulfils (C.14), a reformulation of the necessary conditions in Proposition 5.4.

To proof the second equation (5.11(ii)) Grafenhofer et al. show that the indirect utility as stated in 5.11(ii) fulfils the Bellman equation (equation 5.4) indirectly (see Grafenhofer et al., page 35). Therefore insert 5.11(ii) into the Euler equation (C.14), multiply by $\left(\Delta_{t+1}^{a}\right)^{1/\rho}$, use the definition of Λ_{t+1}^{a} and $\overline{V}_{\alpha,t+1}^{a}$. This gets

$$\overline{V}_{\alpha,t+1}^{a} = \left(\beta\Omega_{t+1}^{a} R_{t+1}\right)^{\sigma} V_{\alpha,t}^{a} \left(\Delta_{t+1}^{a} / \Delta_{t}^{a}\right)^{1/\rho} \tag{C.18}$$

Taking to the power of ρ and multiplying by $\gamma^{a}\beta$ and using 5.11(iii) to substitute

$$\gamma^{a}\beta\left(\Omega_{t+1}^{a} R_{t+1}\right)^{\sigma-1} \Delta_{t+1}^{a} = \Delta_{t}^{a} - 1 \tag{C.19}$$

The result is

$$\gamma^{a}\beta\left(\overline{V}_{\alpha,t+1}^{a}\right)^{\rho} = \left(\Delta_{t}^{a} - 1\right)\left(V_{\alpha,t}^{a}\right)^{\rho} / \Delta_{t}^{a} \tag{C.20}$$

Remember that 5.11(ii) states that

$$V_{\alpha,t}^{a} = \left(\Delta_{t}^{a}\right)^{1/\rho} C_{\alpha,t}^{a} \tag{C.21}$$

which is used on the right hand side. A minor rearrangement shows that the Bellmann equation (5.4) is fulfilled.

<u>Proof of Remark 5.5 (see Grafenhofer et al., page 20)</u>**:**

Human capital is quite simple to asses as wage related income, and so per capita human wealth, is the same for all people within the same age-group. To show the validity of the proposition Grafenhofer et al. first take a look at the last age group denoted by A. All retirees of this age-group have the same present value of future income, discounted at a common rate for all retirees (Grafenhofer et al., page 20). This gives,

$$H_{\alpha,t}^{A} = y_{t}^{A} + \gamma^{A} H_{\alpha,t+1}^{A} / R_{t+1} \tag{C.22}$$

Writing the per capita value as

$$h_{t}^{A} \equiv H_{\alpha,t}^{A} \tag{C.23}$$

aggregate human capital follows simply as $H_t^A = h_t^A N_t^A$

The same holds for all other age-groups, which follows from 5.11(v) and 5.11(vi).

<u>Proof of Remark 5.5 (see Grafenhofer et al, page 35)</u>:

Multiplying the equation (C.1) with $N_{\alpha,t}^a$ and sum over all biographies α yields

$$X_t \equiv \sum_{\alpha \in B_t^a} A_{\alpha,t+1}^a \gamma^a N_{\alpha,t}^a = R_{t+1} S_t^a \quad \text{where } S_t^a \equiv A_t^a + Y_t^a - C_t^a \tag{C.24}$$

multiplying by ω^a

$$\omega^a X = \sum_{\alpha \in B_t^a} A_{\alpha,t+1}^a N_{\alpha,t+1}^a = A_{t+1}^a - \sum_{\alpha \in B_t^{a-1} \times (t+1)} A_{\alpha,t+1}^a N_{\alpha,t+1}^a = A_{t+1}^a - \gamma^{a-1}\left(1-\omega^a\right) \sum_{\alpha \in B_t^{a-1}} A_{\alpha,t+1}^{a-1} N_{\alpha,t+1}^{a-1} \tag{C.25}$$

The first equality in (C.25) uses from 5.1(ii)

$$N_{\alpha,t+1}^a = N_{\alpha,t}^a * \gamma^a \omega^a \tag{C.26}$$

Then the N-term from (C.24) can be replaced when multiplying (C.24) with ω^a. This represents the fact that only the fraction $\gamma^a \omega^a$ from the age-group survives without ageing and therefore remains part of the age-group.

The second equality in (C.25) states that assets of age-group a at time t+1 consists of the assets held by members of the age-group who remain in a at time t+1, $\omega^a X$, plus the sum of inflowing assets of members belonging to age-group a-1 at time t who were hit by an ageing-shock at time t. Notice that ageing is indicated by the term x(t+1), denoting the entry of the date of ageing to the biographies of aged agents.

The last equality in (C.25) gives a closer look at the entry of newcomers. From equation 5.1(iii) we know that the mass of newcomers is equal to

$$N_{\alpha',t+1}^a = N_{\alpha,t}^{a-1} * \gamma^{a-1}(1-\omega^{a-1}) \tag{C.27}$$

Since everyone in age-group a-1 has the same probability of moving to group a, the law of large numbers implies that an equal fraction out of each class of biographies α is moving into group a. This is expressed in the last sum-term. This sums over the entire set of possible histories (B), but takes only the common fraction of each biography group. Each of the movers possesses assets equal to $A_{\alpha,t+1}^{a-1} = A_{\alpha',t+1}^a$. The last aggregation yields

$$\sum_{\alpha \in B_t^{a-1}} A_{\alpha,t+1}^{a-1} N_{\alpha,t+1}^{a-1} = \left(R_{t+1} / \gamma^{a-1}\right) S_t^{a-1} \tag{C.28}$$

by applying (C.24) to group a-1. Inserting this into the equation giving $\omega^a X$ yields

$$\omega^a X = A^a_{t+1} - \gamma^{a-1}\left(1-\omega^a\right)\left(R_{t+1}/\gamma^{a-1}\right)S^{a-1}_t \tag{C.29}$$

Recalling that

$$X_t \equiv R_{t+1}S^a_t \tag{C.30}$$

gives finally equation (ii) from the Proposition as

$$A^a_{t+1} = R_{t+1}\left[\omega^a S^a_t + \left(1-\omega^{a-1}\right)S^{a-1}_t\right] \tag{C.31}$$

Equation (i) simply follows from the fact that newborns do not possess any assets.

Equation (iii) follows as the sum over all groups.

6. Interaction of age structure and fertility (Hock and Weil)

Hock and Weil examine the interaction of the fertility rate and a population's age structure. The fact that changes in fertility lead to changes in the age composition of a society is straight forward and well studied. In their analysis Hock and Weil go a step further by focusing on the feedback of the changing age structure of the population on fertility. The argument for this is that working age people in a society are confronted with a large number of dependents which they have to support through a system of social security. As the population grows older, the number of old age dependents grows relatively to the number of potential workers. This increase in old age dependency again is associated with higher costs for social security and thus lower consumption possibilities for the working. Confronted with increasing costs due to an increasing number of dependents, they argue, that agent's decision of having children may be affected. Agents seek to keep the standard of living they have known from their parents and could thus decide to reduce the amount of children they wish to have. As a consequence these individual choices of reducing fertility can lead to an economy wide reduction in fertility. Seen this way, fertility is no longer an exogenously given parameter that changes due to shocks or factors outside the economy. It becomes an endogenous variable, which depends strongly on economic surroundings.

Thus, the change in the age composition of the society is not only a result of changing fertility but fertility itself is affected by the population's age structure via the channel of dependency. This in total works of course as a multiplier and leads to a highly dynamic problem.

Hock and Weil develop a rather simple OLG model of an economy consisting of only three age groups, one representing the working live-stage and two stages of life, where agents are dependents – youth and retirement. The transition between these stages is modelled similar to Blanchard's Perpetual Youth Model. Thus, agents of each age-group face a constant risk of changing their states. Based on three equations giving the *laws of motion* for each age-group, they define old-age and youth dependency ratios and analyse the consequences of changes in fertility and mortality under two scenarios. First, they assume fertility to be independent of the old age dependency ratio. Second, they assume fertility decisions to depend on the age structure of the population and thus being an endogenous variable.

6.1 Basic model equations

The age structure of the economy is defined by the triple (A_y, A_m, A_o) where A_y refers to the stock of youths, A_m to the number of working age people and A_o gives the number of retirees.

Following Blanchard's approach of modelling mortality, the transition between different age states is assumed to be given by a constant probability for each age-group. These probabilities are denoted by λ_y, λ_m and λ_o respectively.

Definition 6.1:

The dynamics of the different age-groups are given by following equations:

(i) $\dot{A}_y(t) = N(t) - \lambda_Y A_y(t)$

(ii) $\dot{A}_M(t) = \lambda_Y A_y(t) - \lambda_M A_M(t)$ (6.1)

(iii) $\dot{A}_O(t) = \lambda_M A_M(t) - \lambda_O A_O(t).$

N(t) here refers to the number of new-borns at time t. The laws of motion for the different age groups are simply the difference between inflow and outflow into each group, whereby the transition between these groups is modelled similar to models previously discussed. Hock and Weil simplify matters by assuming mortality to occur only among retirees. Thus, λ_o gives the probability of dying for retirees, whereas the other values λ_y and λ_m give the rate of transition for young and working people. Compared to the PA Model this simplification is a clear regress concerning demographic realism.

Dependency ratios

To analyse interdependencies between the age structure of an economy and its fertility rates Hock and Weil take a closer look at how changes in fertility and mortality affect economy wide dependencies. For this they first define

Definition 6.2:

Old age dependency ratio is defined as

$$o(t) = \frac{A_O(t)}{A_M(t)} \quad (6.2)$$

Definition 6.3:

Youth dependency ratio is defined as

$$y(t) = \frac{A_y(t)}{A_M(t)}. \quad (6.3)$$

Total economic dependency is given by the needs-weighted sum of youth dependency and old age dependency (instead of simply summing up both ratios). The reason for introducing different weights on consumption needs is that these needs are assumed to differ across old and young people implying that changes in the two ratios do not affect the economy wide consumption possibilities the same way. Increasing old age dependency (relative to youth dependency) is likely to be associated with higher costs for social security compared to increasing youth dependency. This results from the fact, that elder are supported through social security at a higher level than youths. Thus, total dependency is given by

$$e(t) = \rho_Y y(t) + \rho_O o(t) \tag{6.4}$$

where the ρ terms indicate the different weights of dependency due to differing needs of older and younger agents. Further, Hock and Weil assume the economy's output to be produced solely out of labour which yields total output as $\Omega(t) = W(t)A_M(t)$ where W(t) denotes wage at time t. Consumption of all age-groups is indexed to wages, thus $c_M(t)=\eta W(t)$, $c_Y=\rho_Y c_M(t)$ and $c_O=\rho_O c_M(t)$. This yields the consumption index η to satisfy

$$\eta(t) = \frac{1}{1+e(t)}. \tag{6.5}$$

under the aggregate resource constraint that total consumption must equal the total output of the economy[30]. This consumption index can also be interpreted as the support ratio (as discussed by Cutler et al. (1990)) and thus as the ratio between the production capacity of the economy and the consumption needs of the population (see Hock and Weil, page 11).

Dynamics of the old age dependency ratio

Using the motion laws previously defined in equation (6.1) yields

<u>*Remark 6.1:*</u>

The motion of the old age dependency ratio is given by

$$\dot{o}(t) = \lambda_M - (\lambda_O - \lambda_M)o(t) - \lambda_Y y(t)o(t) \tag{6.6}$$

with corresponding zero motion locus

$$o(t)\,|_{\dot{o}=0} = \frac{\lambda_M}{(\lambda_O - \lambda_M) + \lambda_y y(t)} \equiv Z_O(y(t)) \,. \tag{6.7}$$

[30] $c_Y(t)A_y(t) + c_M(t)A_M(t) + c_O(t)A_O(t) = \Omega(t)$. As output is produced solely by labour, it follows that $\Omega(t) = W(t)A_M(t)$

Thus, for a given level of the youth dependency ratio y there exists exactly one equilibrium level of old age dependency o (Hock and Weil, page 8).

Dynamics of the youth dependency ratio

Remark 6.2:

The motion of the youth dependency ratio is given by

$$\dot{y}(t) = \phi n(t) - (\lambda_Y - \lambda_M) y(t) - \lambda_Y [y(t)]^2 \tag{6.8}$$

The motion of youth dependency ratio does not explicitly depend on the old age dependency ratio o(t). The expression ϕn(t) gives the flow of births per worker where n(t) is the average fertility rate per fertile and ϕ is the proportion of fertile workers. Under this setting changes in old age dependency do not directly affect y as the zero motion locus does not depend on the level of old age dependency. Thus, so far, there is no feedback effect from the age structure of the population on the fertility rate. The following section analyses the effects of changing fertility and old age mortality under the assumption of exogenous fertility.

6.2 Exogenous fertility

In the simple case of exogenous fertility the analysis of the dynamic system is straightforward. As the zero motion locus of the youth dependency ratio does not depend on the old age dependency ratio o(t) one can simply solve the motion equation (6.8) for y(t) - given a certain level of fertility.

Remark 6.3:

The equilibrium level of youth dependency is given by

$$\bar{y} = -\frac{1}{2}\left(1 - \frac{T_Y}{T_M}\right) + \sqrt{\left(\left[\frac{1}{2}\left(1 - \frac{T_Y}{T_M}\right)\right]^2 + T_Y \phi n\right)} \tag{6.9}$$

Given this level of youth dependency one can use equation (6.7) to derive the corresponding equilibrium value of the old age dependency as

Remark 6.4:

The equilibrium level of old-age dependency is given by

$$\bar{o} = Z_o(\bar{y}) = \frac{1}{(T_M / T_O - 1) + (T_M / T_Y)\bar{y}} . \tag{6.10}$$

Dynamic system with exogenous fertility

Figure 6.1 shows the dynamic system for the case of exogenous fertility. As can be seen in the phase diagram the economy converges to the equilibrium point $(\bar{y}, \bar{o})$ which is globally stable.

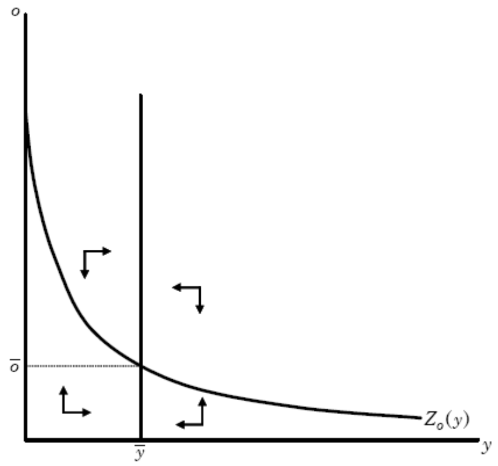

Figure 6.1. Dynamic System with exogenous fertility (Hock and Weil, page 12)

In order to be able to analyze the consequences of changing fertility it is convenient to consider the flow of births per worker in terms of the gross reproductive rate (GRR), as Hock and Weil point out.

Definition 6.5:

Let G denote the gross reproductive rate and T_M the expected time spent in the working age age-group. The flow of births per worker is denoted by

$$\phi n = \frac{G}{T_M} \tag{6.11}$$

Further analysis will consider the effect of a decrease in the reproductive rate G in order to study the effect of declining fertility. Notice, that a value of G equal to one would imply replacement fertility (as Hock and Weil consider individuals and not e.g. females as reproductive units). Using the new definition of fertility yields the equilibrium level of youth dependency as

$$\bar{y} = -\frac{1}{2}\left(1 - \frac{T_Y}{T_M}\right) + \sqrt{\left(\left[\frac{1}{2}\left(1 - \frac{T_Y}{T_M}\right)\right]^2 + \frac{T_Y}{T_M}G\right)} \qquad (6.12)$$

The effect of declining fertility with exogenous fertility

A decrease in fertility can be seen as a decrease in the gross reproductive rate G. Inserting the new definition for the flow of births into the previous result for youth dependency shows, that a decrease in G reduces the equilibrium value of $\bar{y}$ (compare equation (6.12)). Figure 6.2 shows the effect graphically. The decrease in $\bar{y}$ shifts the y-locus to the left, leading in the long run to a decrease in the equilibrium level of the youth dependency ratio but to an increase in the level of old age dependency. As can also be seen in Figure 6.2, the decrease in G leads in the first place to an increase in the consumption index – i.e. the support ratio. This is due to the fact, that declining fertility reduces total economic dependency as fewer children are born. This effect is the so called *demographic dividend*. But as old age dependency increases through time – due to the reduced fertility rate less youths enter the workforce over time reducing total labour force in the long run– this demographic dividend effect declines. Thus, the reduction in total dependency through a drop in the fertility rate has only a temporary beneficial effect on the economy wide consumption possibilities. As time passes the increased old-age dependency ratio increases total dependency again and can even more than offset the former beneficial effect from reduced fertility. Figure 6.2 shows one possible outcome, where the economy ends up with a level of economic dependency higher than the initial one.

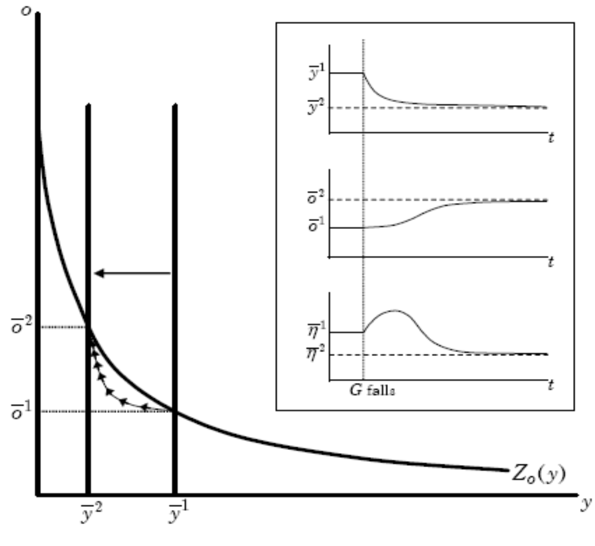

Figure 6.2 One possible outcome from declining fertility (Hock and Weil, page 15)

Dependency minimizing fertility rate

Whether an economy ends up with a higher total dependency rate after fertility has changed or not depends on how high the initial level of fertility has been. This is shown in Figure 6.3.

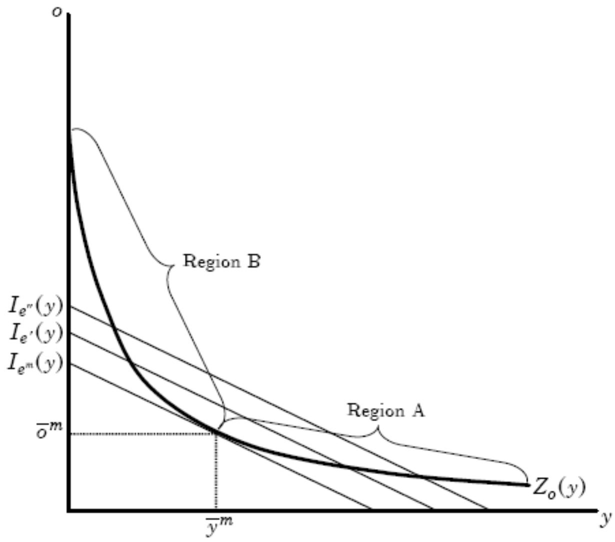

Figure 6.3 Dependency Minimizing Fertility (Hock and Weil, page 16)

Figure 6.3 illustrates how the initial level of fertility affects the outcome of decreasing fertility. The pair $\left(\bar{y}^m, \bar{o}^m\right)$ gives the level of o and y minimizing total economic dependency. As can be seen in the diagram, economies with initially high levels of fertility lie in Region A. For these economies a reduction in fertility is associated with a lower level of total dependency (the new equilibrium point lies on a lower iso-dependency line). For economies with initially high levels of fertility – lying in region B – a reduction in fertility leads to an increase in total dependency.

Remark 6.5:

At the dependency minimizing point $\left(\bar{y}^m, \bar{o}^m\right)$ the iso-dependency line has the same slope as the Z_O locus implying that following equation holds:

$$\bar{o}\left(G, \frac{T_Y}{T_M}, \frac{T_O}{T_M}\right) = \sqrt{\left(\frac{\rho_Y}{\rho_O}\right)\frac{T_Y}{T_M}} \tag{6.13}$$

There exists a unique level of G – denoted by G^m - that yields the dependency minimizing point $\left(\bar{y}^m, \bar{o}^m\right)$. While an economy lying in Region A (with fertility exceeding G^m) can easily reduce fertility in order to maximize consumption and reduce total dependency, economies in Region B face difficulties in achieving optimal fertility. At the beginning of the transition to higher fertility, the economy lying in Region B faces rising total dependency and thus a reduction in consumption possibilities. The transition is costly. Figure 6.4 illustrates the different situations for the two cases.

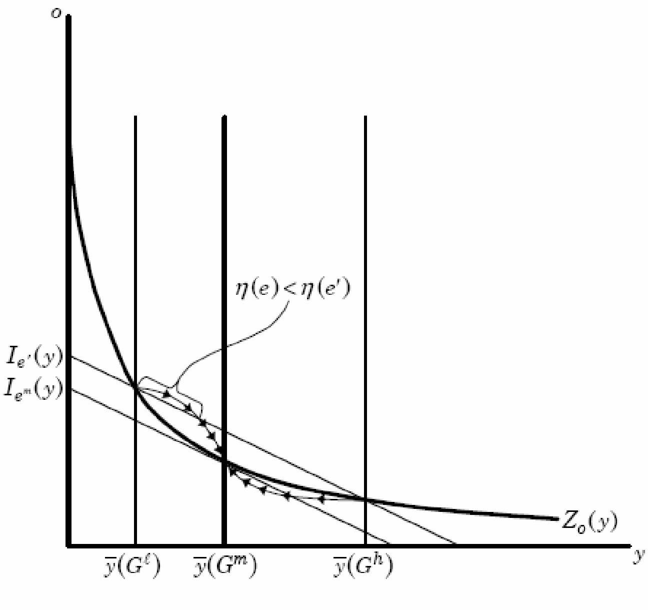

Figure 6.4 Transition to Consumption-Maximizing Equilibrium (Hock and Weil, page 17)

As the diagram shows, an increase in fertility for economies in Region B raises dependency in the economy and thus lowers the consumption index η. Only after some time has passed the rise in the youth dependency ratio starts to have beneficial effects and total dependency begins to fall as more workers enter the labour market, increasing the number of workers relative to total dependents. From then on the economy starts to profit from increased fertility. The case is much easier for the economy in Region A. Simply reducing fertility has an immediate beneficial effect on the consumption index via reduced youth dependency. Even though old age dependency raises total dependency is reduced leading to a more efficient positioning of the economy.

Effect of declining old age mortality with exogenous fertility

Besides reduced fertility rates, the second driving force leading to an ageing society is increasing life expectancy due to a reduction in old age mortality. In terms of the presented framework a reduction in old age mortality implies a shift in the $Z_O(y)$ locus upwards as illustrated in Figure 6.5. As explained before, the motion equation of the youth dependency ratio does not directly depend on the old age dependency ratio. And as long as there is no direct link from old age dependency to fertility modelled, this implies that the youth dependency locus remains unchanged.

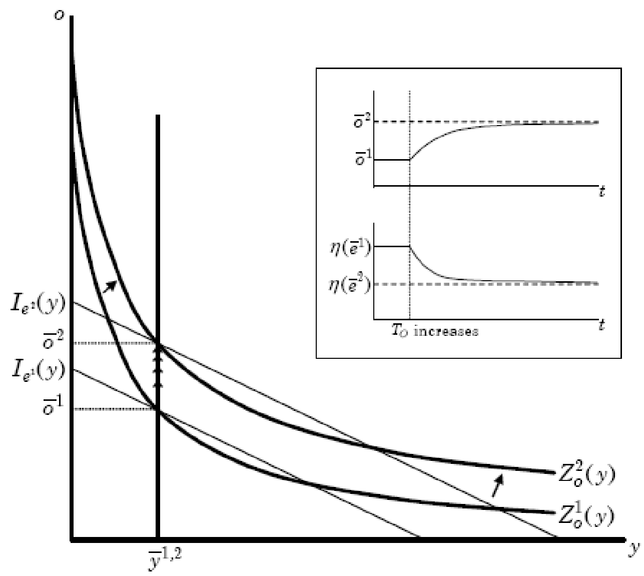

Figure 6.5 Effects of a Decline in Old-Age Mortality with constant Fertility (Hock and Weil, page 20)

As indicated by the diagram the rise in old age dependency shifts the economy to a higher iso-dependency level. The consequence of increasing dependency is a reduction in consumption possibilities and thus a reduction in the consumption index. As shown in the box of Figure 6.5 old age dependency begins to rise due to the decrease in mortality, reducing the consumption index.

Impact of declining mortality on optimal fertility

Considering optimality patterns for the economy, one has to notice that changing old age mortality also induces a change in the consumption-maximizing fertility rate G^m. An increase in the old age dependency ration o(t) leads, ceteris paribus, to a rise in G^m as this is an increasing function of retiree's life expectancy (compare Hock and Weil, page 20).

Why does decreasing mortality call for increasing optimal fertility? In the first place rising fertility would lead of course to an increase in youth dependency and therefore increase total economic dependency. But as old age dependency is more consumption intensive compared to y the future beneficial effect from higher fertility prevails. Thus, optimal fertility must rise in response to decreasing old age mortality. Higher fertility today implies a larger labour force in future leading to a reduction in old age dependency in the long run. The increase in fertility increases youth dependency y in equilibrium, as shown in Figure 6.6.

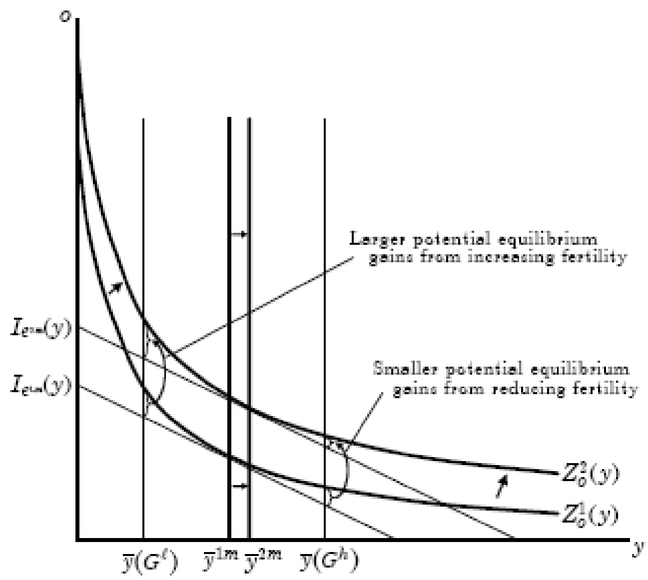

Figure 6.6. Effect of Decline in Old-Age Mortality on Consumption Maximizing Equilibrium (Hock and Weil, page 21)

As illustrated in the diagram, the increase in the optimal level of fertility leads to a shift in the optimal y-locus to the right (indicated by y^m). Depending on the current level of fertility this shift implies that the optimal rate of fertility moves either nearer to actual fertility rate or further away. Particularly for countries with high fertility (such as G^h in the diagram) the increase in y^m moves the locus closer to actual level of youth dependency. This again implies that besides a decrease in the highest attainable consumption the potential gain from adjusted fertility is reduced (the scope for adjustment is reduced due to the increase in y^m). For countries with lower levels of fertility (smaller than G^m) the opposite holds. The fact that the optimal level of fertility moves further away from current fertility increases the potential gain from fertility adjustment but again the highest attainable level of consumption is reduced.

6.3 Endogenous fertility

In this section fertility patterns are reconsidered. The previous sections assumed independency of fertility from old age dependency. Contrasting this perspective fertility is here seen to result from individual decision making. Thus, agents within an economy are assumed to decide themselves how many children to have. No longer can fertility then be viewed as an exogenously given factor, independent of economic surroundings. If people are able to make their own decisions concerning fertility it is very likely that prevailing dependencies influence these decisions. The thought is that working-age agents within an economy face a certain level of dependency. Obligated to finance the public pension system workers have no choice in supporting the old and thus increasing old age dependency necessarily raises dependency costs. Thus, rising dependency implies increasing costs for those obligated to support dependents. Confronted with increasing old age dependency potential parents could therefore choose to adjust their fertility in order to reduce total dependency and this way increase their consumption possibilities.

Asymmetric Costs and Tax Rate

Hock and Weil set up a framework in order to analyse this interaction between prevailing dependency and the fertility rate. They assume workers to finance the public pension system as well as transfers to the young through taxes paid on labour income. One important aspect of their analysis is the asymmetric cost scheme between retirees and children. Retirees are supported entirely through the pension system, while children receive mainly privately financed support. Each retiree receives pension payments defined as a fraction β of after tax wages. Per capita transfers to youths is assumed as a fixed proportion π of the transfers to the elders, thus each child receives transfers as the fraction $\alpha=\beta\pi$ of after tax income.

Remark 6.6:

The balanced-budget tax rate is given by

$$\tau(t)=\frac{\alpha y(t)+\beta o(t)}{1+\alpha y(t)+\beta o(t)}. \tag{6.14}$$

This again implies that tax payments rise as youth dependency or old age-dependency ratios increase. But as payments to the young are only a fraction of pension payments, increases in both dependency ratios have quantitatively different effects. Increasing taxes again reduce transfer payments to elders and youths as they receive a certain proportion of after-tax income.

Optimal Reproductive Rate with Endogenous Fertility

One can again ask the question what the “optimal” level of fertility would be for workers to choose. As will become clear, the level of fertility chosen by agents privately would be, in general, very different from the “optimal” level a social planner would choose. Similar to the previous section dealing with exogenous fertility (compare equation (6.13)), there exists a tax-rate minimizing gross reproductive rate G_τ^m

Optimal reproductive rate:

The tax-rate minimizing gross reproductive rate G_{τ}^{m} is given by the implicit solution to (Hock and Weil, page 24)

$$\bar{o}\left(G_{\tau}^{m}, \frac{T_Y}{T_M}, \frac{T_O}{T_M}\right) = \sqrt{\frac{\alpha}{\beta}\frac{T_Y}{T_M}} \tag{6.15}$$

Agent's private choice

To capture the influence of the age structure on fertility decisions Hock and Weil set up a framework in which fertile workers include their utility from rearing children as well as the costs induced by having children into their maximization problem. After tax wage is split up between consumption and child rearing. In this quite simple model workers maximize their log utility function

$$\max_{c(t),n(t)} \ln[c(t)] + \theta \ln[n(t)] \tag{6.17}$$

subject to their budget constraint

$$c(t) + \xi W(t) n(t) = w(t) \tag{6.18}$$

The term $\xi W(t)$ in the budget constraint captures the "price of children" – thus the costs of child rearing[31].

Individual Optimum:

Agents' optimal private choice of fertility is given by

$$\tilde{n}(t) = \psi \frac{w(t)}{W(t)} = \frac{\psi}{1 + \alpha y(t) + \beta o(t)} \tag{6.19}$$

where $\psi = \theta / (\xi(1+\theta))$ and θ indicates the relative preference for children.

Social planner's choice

Contrasting individual worker's decision on fertility, a social planner would take into account the effects of fertility decisions on total dependency. As a consequence the optimal solution for fertility derived by the social planner will differ from worker's private fertility choice. Planner's optimality problem yields the objective function

$$\max_{\hat{n}} \ln\left[\frac{1}{1+\alpha y(\hat{n})+\beta(\hat{n})} - \xi\hat{n}\right] + \theta\ln[\hat{n}]. \qquad (6.20)$$

<u>*Social Optimum:*</u>

Social planner's first order condition yields

$$\hat{n} = \tilde{n}(\hat{n}) + \frac{1}{\xi(1+\theta)} \frac{\left[-\alpha y'(\hat{n}) - \beta o'(\hat{n})\right]\hat{n}}{\left(1+\alpha y(\hat{n})+\beta o(\hat{n})\right)^2} \qquad (6.21)$$

The term $\tilde{n}(\hat{n})$ denotes the level of fertility that workers would choose given the values of $(y(\hat{n}), o(\hat{n}))$. Then the equilibrium equations (6.9) and (6.10) imply that the term in brackets on the right hand side of equation (6.21) satisfies

$$\left[-\alpha y'(\hat{n}) - \beta o'(\hat{n})\right] = \left[-\alpha + \beta\left[o(\hat{n})\right]^2 \left(T_M / T_Y\right)\right] y'(\hat{n}). \qquad (6.22)$$

The fertility rate chosen by a social planner $(\hat{n})$ is higher than agent's private choice on fertility based on the age structure implied by $\hat{n}$ if the relation

$$\beta[o(\hat{n})]^2 (T_M / T_Y) - \alpha < 0 \qquad (6.23)$$

holds and lower otherwise. Both levels of fertility are equal if (6.15) holds. Then $\hat{n}$ equals the tax minimizing rate of fertility n_τ^m. Thus, the social planner optimizing fertile workers' equilibrium utility would set the fertility rate equal to that which minimizes the tax rate. It is important to notice, that decreasing old age mortality will lead to an increasing divergence between the private choice and the social planner's optimum, which minimizes the tax-rate (see Hock and Weil, p 27). Thus the optimal response to decreasing mortality differs from the desired response of individuals.

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[31] Assuming constant wage growth equal to g and denoting childrens' consumption needs by χ it follows that $\xi = \chi T_Y / (1 - gT_Y)$.

The effect of a decline in old age mortality with endogenous fertility

Through the channel of taxes the age structure of the population now affects fertility decisions. As old age dependency rises due to decreasing old age mortality, taxes rise due to higher costs for social security. This income effect affects fertility decisions. Necessarily the y-locus has now another shape than in previous analysis in section 6.2 because the old age dependency now affects the y-locus. By including the flow of births per worker $\tilde{n}$ into the framework the locus takes the form

$$o\,|_{\dot{y}=0} = \frac{1}{\beta}\left[\frac{\phi\psi T_Y}{(1-T_Y/T_M)y+y^2} - (1+\alpha y)\right] \equiv Z_y(y). \tag{6.24}$$

As shown in Figure 6.7 the locus is no longer a vertical line but downward slopping and convex. There is one single crossing point of the two loci which implies that the equilibrium is globally stable.

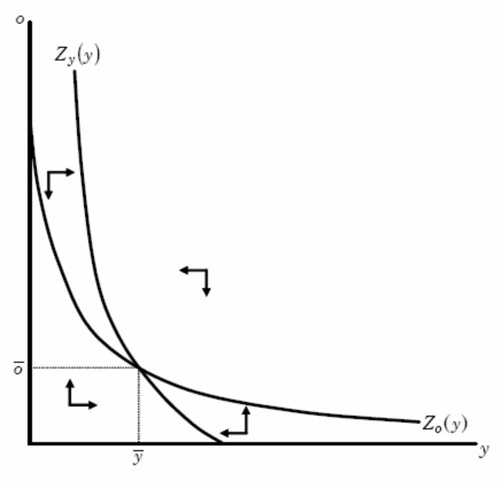

Figure 6.7 Global Dynamics with Endogenous Fertility (Hock and Weil, page 28)

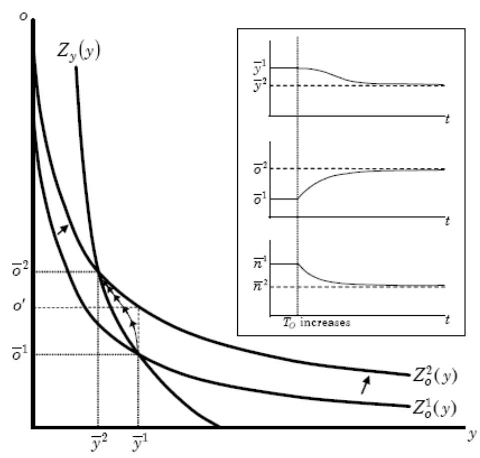

Figure 6.8 Effects of Decline in Old-Age Mortality with Endogenous Fertility (Hock and Weil, page 28)

As old age mortality decreases the Zo locus shifts upwards, representing increasing old age dependency due to increased life expectancy. If fertility were exogenous as in the previous analysis in section 6.2 the new level of old age dependency would increase to o'. Independency of fertility decisions and old age dependency would imply that there would be no shift in the youth dependency locus. But things are very different now, as fertility decisions are affected by the age structure of the population. As increasing old age dependency drives up the tax rate workers' disposable income is reduced – this is the income effect previously mentioned. This induces fertile workers to adjust their fertility rate. Decreasing fertility again leads to increasing old age dependency as shown in a previous section. Thus, there is a multiplier effect leading to further increase in old age dependency. The economy in Figure 6.8 thus converges to a higher level of old age dependency.

6.4 Life cycle aspects of the model

Hock and Weil analyze the impact of changing mortality and fertility on the economy. Concerning individual's life cycle the framework generates uncertainties as the transition between different life stages is modelled in a probabilistic fashion. But, compared to the PA model, this framework assumes mortality to occur only among retirees. This is a clear drawback concerning demographic realism.

In general the analysis of the interaction of fertility, population's age structure and economic outcomes is interesting from a demographic point of view. Reduced fertility leads to a reduction of the future labour force which again implies an increase in old age dependency. Increasing life expectancy due to a reduction in old age mortality also increases old age dependency. Thus, the model yields the dynamics one would expect. Taking into account the feedback effect from the population's age structure on fertility, leads to an even greater effect of declining old age mortality on old age dependency. This is due to a multiplier effect induced by the agents' reaction on increased old age dependency again reflecting quite realistic behaviour.

But the framework itself rests on very simplifying assumptions, e.g. assuming that there exists no capital in the economy. Thus, this analysis only shows the "pure" dynamics of changing fertility and mortality, which can in reality be superposed by other effects. For example reduced fertility could be associated with increasing female labour force participation. This model does not allow for such effects.

Appendix D

***Proof of Remark 6.1*:**

Combining equations 6.1(ii) and 6.1(iii) and using definitions 6.2 and 6.3 yields the motion equation

$$\dot{o}(t) = \lambda_M - (\lambda_O - \lambda_M)o(t) - \lambda_Y y(t)o(t)$$

The zero-motion equation follows by simply setting the motion equation to zero and rearranging.

***Proof of Remark 6.2*:**

Combining equations 6.1(i) and 6.1(ii) and using definitions 6.2 and 6.3 yields the motion equation

$$\dot{y}(t) = \phi n(t) - (\lambda_Y - \lambda_M)y(t) - \lambda_Y [y(t)]^2$$

***Proof of Remark 6.3*:**

Equation 6.9 follows by setting

$$\dot{y}(t) = \phi n(t) - (\lambda_Y - \lambda_M)y(t) - \lambda_Y [y(t)]^2$$

equal to zero and solving for y. Denoting Tj as the inverse of λj finally gives the expression

$$\bar{y} = -\frac{1}{2}\left(1 - \frac{T_Y}{T_M}\right) + \sqrt{\left(\left[\frac{1}{2}\left(1 - \frac{T_Y}{T_M}\right)\right]^2 + T_Y \phi n\right)}$$

***Proof of Remark 6.4*:**

Follows by simply exchanging Tj for λj.

***Proof of Remark 6.6*:**

Aggregate resource equation is given by

$$A_Y \alpha W(t)(1 - \tau(t)) + A_M(t)W(t)(1 - \tau(t)) + A_O \beta W(t)(1 - \tau(t)) = \Omega(t) = W(t)A_M(t)$$

Dividing by A_M and using the definitions for the old-age dependency ratio and the youth dependency ratio it follows that

$$(\alpha y + \beta o + 1)(1-\tau)W(t) = W(t)$$

Rearranging gives

$$\tau(t) = \frac{\alpha y(t) + \beta o(t)}{1 + \alpha y(t) + \beta o(t)}$$

***<u>Proof of Individual Optimum</u>*:**

Inserting the budget constraint

$$c(t) + \xi W(t) n(t) = w(t)$$

into the objective function

$$\max_{c(t),n(t)} \ \ln[c(t)] + \theta \ln[n(t)]$$

gives the first order condition as

$$\frac{\partial}{\partial n} = \frac{1}{(w(t) - \xi W(t) n(t))}(-\xi W(t)) + \theta \frac{1}{n} = 0$$

solving for n gives

$$n = \frac{\theta w(t)}{\xi W(1+\theta)} = \frac{w(t)}{W(t)} \frac{\theta}{\xi(1+\theta)} = \frac{(1-\tau)W(t)}{W(t)} \frac{\theta}{\xi(1+\theta)}$$

Defining $\psi = \theta / (\xi(1+\theta))$ and using the result from Proposition 6.5? gives finally

$$n = \frac{1}{(1 + \alpha y + \beta o)} \psi \ .$$

<u>Proof of Social Optimum</u>:

Inserting the budget constraint again into the objective function yields

$$\ln[w - \xi W n] + \theta \ln(n)$$

which can be rewritten as

$$\ln\left[\frac{1}{\left(1+\alpha y(n)+\beta o(n)\right)}-\xi n\right]+\theta\ln(n)$$

the first order condition follows as

$$\frac{\partial}{\partial\hat{n}}=\left[\frac{1}{1+\alpha y(\hat{n})+\beta o(\hat{n})}-\xi\hat{n}\right]^{-1}\left((-1)\left(1+\alpha y(\hat{n})+\beta o(\hat{n})\right)^{-2}\left(\alpha y'(\hat{n})+\beta o'(\hat{n})\right)-\xi\right)+\theta\frac{1}{n}$$

the first order condition can be written as

$$\hat{n}=\tilde{n}(\hat{n})+\frac{1}{\xi(1+\theta)}\frac{\left[-\alpha y'(\hat{n})-\beta o'(\hat{n})\right]\hat{n}}{\left(1+\alpha y(\hat{n})+\beta o(\hat{n})\right)^{2}}$$

where $\tilde{n}$ refers to workers decision on fertility.

7. Conclusion

To conclude I will first briefly summarize the models presented in my thesis and highlight the main findings. I will also present a final comparison and discussion of the differences between the models concerning their degree of demographic realism and their ability of representing life cycle details.

7.1 Summary of findings

This thesis presented and compared different approaches of achieving a higher degree of demographic realism in Models of Overlapping Generations. The main difference between the models presented in this work lay within the modelling of mortality patterns, with the last section dealing with some fertility aspects too. First models with a quite unrealistic view on mortality as being independent of age were presented. The next section moved to age-dependent mortality patterns followed by the alternative view on ageing in the PA Model and finally a discussion of the interaction of fertility and the populations' age-structure.

Within the age-independent mortality approaches the models discussed covered the simple but also very fundamental model by Samuelson (1958) and Diamond (1965), Blanchard's (1989) Perpetual Youth Model and Gertler's (1999) extension of Blanchard's Model to a two life-stage model with retirees and workers. As a starting point of the analysis the Diamond Model showed surely not the highest degree of demographic realism, assuming only two stages of life and perfect survivorship between those two life-stages. One could say that this certainly did not reflect real-life demographics very well. Nevertheless the framework allowed for different propensities to consume in the two different stages of life which is an important feature of the human economic life-cycle. Furthermore it reflected a very important aspect of the human economic life cycle, namely the existence of different „stages" of life, working live and retirement.

Using the Diamond model as a benchmark, the improvement achieved by Blanchard's Perpetual Youth Model certainly lay within the introduction of individual life-time uncertainty. Even though the assumption of a constant, and therefore age-independent, mortality rate seemed not to really improve demographic realism to a very high degree, it still showed two very important and realistic impacts of the presence of a life-cycle under life-time uncertainty.

First, the economy's capital stock was a decreasing function of the probability of death (see Proposition 3.10). Second, retirement caused the capital stock to increase, reflecting the effect of retirement on saving decisions (see Proposition 3.11). Therefore increasing mortality caused capital accumulation to decrease, meaning that the higher the probability of death, the less people tended to save. The more severe retirement affected income negatively on the other hand (represented by a high value of α in Definition 3.1) the more people saved. Thus, these dynamics reflected quite realistic dynamics. The drawback of Blanchard's model was that the assumption of a constant mortality rate implied not only age-independentness of mortality, but also of future life expectancy and consumption behaviour.

Gertler created a model of retirees and workers which was able to combine the benefits of both models, by developing a model capturing both, individual life-time uncertainty and altering consumption behaviour with age. Though again consumption behaviour within each life stage (work or retirement) resulted to be independent of agents' literal age (meaning the time that has passed since his/her birth) but not independent of the agents' state in the life cycle. Otherwise effects similar to those already found for Blanchard's model could be observed concerning the impact of life-time uncertainty on retirees' consumption behaviour (notice that mortality was only assumed for retirees). Again it turned out that the propensity to consume was an increasing function of the probability of death (see equation 3.25). Considering workers, the effect of "life-stage uncertainty" was to increase the discount rate on his/her human capital. The lower the probability of remaining part of the workforce was, the lower potential future labour income was valued by the agent. Viewing the Gertler economy in its aggregate showed that the dynamics influencing capital accumulation were quite similar to the Blanchard economy. Again the higher the probability of death (or retirement) was, the lower was the resulting capital stock (see Section 3.3.2.1). Capital accumulation itself also depended on the retirees' share of total assets. This simply reflected the fact, that retirees had a different propensity to consume due to their risk of death. Concerning total human wealth in the economy, it was shown, that the presence of life-time uncertainty led to a reduction of human wealth via an increased discount rate (see equation 3.41 (iii)). Thus Gertler's model achieved a high degree of demographic realism combining the advantages of Diamond's Model (different consumption behaviour in different stages of life) with a life-time uncertainty pattern.

Li and Tuljapurkar achieved an even higher degree of demographic realism by breaking up the restrictive assumption of age-independent mortality. In their framework they could take account of changing life expectancy and life-time uncertainty on agents' behaviour directly by incorporating a distribution function of a random variable called "death-age" into Blanchard's framework. As Li and Tuljaprukar stated, *this approach naturally allows us to think about individual decision making in response to life extension […] and changes in the uncertainty of the timing of death […] (Li and Tuljapurkar, p.2).* These changes in life expectancy and life time uncertainty were represented by changes in the average death-age (mean of the distribution) and changes in the variance of death-age (variance of the distribution).

Assuming first constant human capital (see Assumption 4.1) they found, that the pure effect of increasing life expectancy (thus, leaving the death-age variance unchanged) was to increase total wealth, consumption and wages while decreasing the interest rate in the economy. The pure effect of decreasing death-age variance on the other hand was to decrease wealth, consumption and wages while increasing the interest rate. Thus, as the death-age variance is likely to be decreasing over time an age-independent mortality approach (implying that the variance of death-age moves in the same direction as life-expectancy (see Remark 4.2)) underestimates the effect of increasing life expectancy on the interest rate while leading to overestimated magnitudes for wealth, consumption and wages (see section 4.2.4). In comparing the implications of their model with a fixed death-age scenario and a constant mortality scenario, they found a much better fit to real data when assuming death-age to be normally distributed (see section 4.2.4).

A more sophisticated analysis followed by letting the changing life-time horizon affect agents' decision of investing in their human capital (increasing the years of schooling increases future wages; see Definition 4.3) and by introducing retirement. Doing so, they achieved an even higher degree of demographic realism. These extensions affected human capital available to the economy directly and indirectly, by reducing the size of the workforce on the one hand and by improving the productivity of workers on the other hand. Thus, in contrast to the simpler setting, human capital was not assumed to be constant anymore, improving the framework's realism.

This impact of increasing life expectancy on individually chosen schooling tenure again reflects realistic behaviour, as people - confronted with increasing life-expectancy and lower life-time uncertainty - are likely to increase years of schooling in order to gain higher wages in future. This only makes sense, if income earned in future is valued relatively high, which again is more likely to be the case, the lower the probability of death (as already seen in

Blanchard's model, an increasing probability of death increases the discount rate on future human capital).

The effect of the introduction of schooling to their framework was most apparent when viewing the behaviour of the interest rate (see section 4.3.1.2). In contrast to the results obtained in the simpler case with constant human capital, the interest rate did not decrease monotonically as life expectancy increased, but increased for some time before decreasing with high life-expectation. This led again to strong impacts on other economic variables (Li and Tuljapurkar, page 25). On the other hand retirement worked as a reduction in available human capital (see Equation 4.5). While decreasing retirement age decreased the interest rate, total wealth, total consumption and total labour (human capital), it increased wages. Li and Tuljapurkar concluded that the implementation of an age-dependent mortality had strong and significant impacts on resulting economic magnitudes (Li and Tuljapurkar, page 25) and should therefore be taken into account in economic analysis.

The Probabilistic Aging Model allowed for a very high degree of heterogeneity among agents. In distinguishing between time passing and ageing as a process altering peoples' life cycle characteristics Grafenhofer et al. gave a very realistic image of differences between people within an economy. This approach allowed for ageing at different speed, which reflected the fact that people in real life seem not to age synchronically (some individuals experience illness or accidents affecting their health which in turn affects their mortality risk and labour productivity). Here, the individual life cycle history determined an agent's economically relevant characteristics, such as earnings potential and labour productivity. The consequence was that agents profoundly differed from each other which yielded a high degree of demographic realism and life cycle detail. Compared to Gertler's model Grafenhofer et al. allowed not only for life-stage uncertainty but also for mortality in younger age-stages.
Taking a look at the model's implications of life-time uncertainty showed that the resulting dynamics were similar to those already obtained by Gertler. Not surprisingly, as the PA Model is an extension of Gertler's framework. Thus, the same implications hold. First, different age-groups had different propensities to consume which were again equal for all agents within the same age-group. The group-specific propensities to consume again depended negatively on the group's mortality risk. So the extension to more age-groups yielded more diversity between agents but still implied equality of consumption behaviour within each group.

Second, the discount rate on future labour income depended negatively on the group-specific mortality risk and a term that was positively correlated with the risk of ageing (or changing states in the life cycle). This again reflected the fact, that future labour income was valued more the higher the probability of remaining part of the labour force and the higher the probability of surviving was.

Also, as in Gertler's model, the distribution of wealth mattered again. As age-groups differed in consumption behaviour, the population structure had strong impacts on economic magnitudes. As the group-specific consumption behaviour was strongly influenced by the mortality and transition probabilities, demographic changes resulted in aggregate changes. Thus, the PA Model allowed for demographic changes to alter the economic surrounding.

As the PA model is an extension of Gertler's model it was not surprising to find similar implications of the presence of life time and life stage uncertainties. The difference was that the PA model offered a richer and more detailed framework, allowing for mortality in early life stages and dealing with a higher number of life stages. In total the PA model was therefore able to represent life cycle details to a higher degree and thus gave a very high degree of demographic realism.

The last section of the thesis presented a framework by Hock and Weil, paying particular attention to the potential feedback effect of the age-structure of a population on its fertility rates. The reasoning behind the assumption of a feedback effect was that working-age people are usually obliged to finance a social security system by paying taxes. As the population aged, the old-age dependency ratio increased, leading to an increase in total costs for social security. Facing this increase in costs, Hock and Weil argued, that people could decide to reduce their own fertility in order to reduce total dependency costs (the taxes they were due to pay plus the costs involved in child-rearing). Therefore the age structure of a population was likely to affect peoples' private choice on fertility.

To study this interaction between fertility and the age-structure of a population Hock and Weil set up a rather simple OLG model, consisting of only three age-groups. In the first part of their study they assumed fertility to be exogenously given, and thus independent of the age structure. Results showed, that in the short run a decline in fertility lead to an initial decline in total dependency due to a reduction in the youth dependency ratio, but resulted in an increased old-age dependency ratio in the long run (see 6.2.2). The cause of this increased old-age dependency was that the reduction in fertility led to smaller cohorts entering the labour force over time, reducing the ratio of working-age people to old-age dependents (see 6.2.2). Depending on the initial level of fertility the increase in old-age dependency could even more than offset the initial benefit from reduced youth-dependency resulting in an increase in total dependency (see 6.2.2.1). Decreasing old-age mortality was found to directly result in an increase in total dependency (see 6.2.3).

Endogenizing fertility lead to the interesting result, that decreasing old-age mortality led to an increasing divergence between the optimal fertility rate (social planner's choice minimized the tax rate) and agents' desired fertility rate (see 6.3.2.2). This divergence was due to fact that decreasing mortality increased the level of tax-minimizing fertility, thus increased the optimal response-level of fertility. As decreasing old-age mortality came along with increasing taxes in order to finance the social security system, agents' reaction was likely to be exactly the opposite, namely to reduce fertility in order to reduce their own dependency related costs (see 6.3.2.1 and 6.3.2.2). The consequence of this divergence was a multiplier effect, leading to an even stronger impact of reduced old-age mortality on old-age dependency.

Thus, the model by Hock and Weil showed interesting dynamics and certainly covered an issue so far not studied in this thesis. But on the other hand the framework rested on very simplifying assumptions, e.g. absence of capital. Looking at the life-cycle patterns of the model, one has to admit that the simplifying assumptions on mortality (which occurred only in the third age-group) and transition rates (again constant for each member of the same age-group) did not yield any improvement concerning life cycle realism. This analysis only showed the "pure" dynamics of changing fertility and mortality, which can in reality be superposed by other effects such as increasing female labour participation. Therefore one had to be careful in drawing too strong conclusions form the framework's implications.

To summarize, the attempts of improving demographic realism of the presented models were mainly based on different mortality assumptions. First, Blanchard created individual life time uncertainty assuming a constant risk of death. This approach was improved by Gertler, extending the model to two life stages and therefore combining the advantages of the Diamond model with Blanchard's life time uncertainty aspect. Grafenhofer et al. finally extended the model to more stages of life and by introducing life time uncertainty among all age-groups. All these models assumed constant transition rates between life stages (in case of Blanchard the transition was directly to death) implying constant consumption behaviour within each life-stage.

A different approach was the incorporation of a probability distribution of death into Blanchard's framework by Li and Tuljarurkar. Doing so Li and Tuljapurkar created the only really age-dependent mortality framework presented in this thesis (Grafenhofer et al. and Gertler create life-stage dependent mortality).
The main impacts of the introduction of life time uncertainty were mainly to affect capital accumulation, the discount rate on human capital (future labour income) and consumption behaviour. Mortality worked as a decreasing force on capital accumulation while the introduction of retirement opposed this dynamic. Life time uncertainty further worked to increase the discount rate on future labour income and on the propensity to consume.

7.2 Concluding remarks

The structure of the thesis was meant to show a tendency of increasing realism concerning demographics and the representation of life cycle details. What became clear was that a distinct ranking of the models concerning their degree of demographic realism and quality in representing life cycle details was only possible to a limited extent. Within the age-independent mortality approaches the path from the Diamond model to the Perpetual Youth Model and finally to Gertler's model of two life stages, showed a quite clear path of increasing realism, first by Blanchard's introduction of individual life-time uncertainty and then by Gertler's distinction between retirees and workers - creating heterogeneity in agents' consumption behaviour. The PA model can be seen as an extension of Gertler's model to more life stages and introducing mortality for younger age-groups. Therefore it yielded a higher degree of demographic realism and life cycle details, as compared to the age-independent mortality frameworks of Diamond, Blanchard and Gertler. Their framework can be seen as creating a "life-stage dependentness" of mortality.

Clearly, also Li and Tuljapurkar's age-dependent mortality framework improved Blanchard's approach of modelling life-time uncertainty as a constant probability of death, but it is hard to state whether it did this to a higher degree than the PA model or not. Both models created an age-related difference in mortality risk between agents, whereby Li and Tuljapurkar assumed a distribution function of death-age (thus creating age-dependent mortality) while Grafenhofer et al. defined different mortality risks for different age-groups (creating life-stage dependent mortality). As both models were able to replicate real mortality patterns well and as a ranking of the models is not necessary one can simply state that both offered a clearly more realistic image of mortality than the age-independent frameworks. So the structure of my thesis still gives the tendency of increasing realism, even though I do not claim one of the two models (PA Model or Li and Tuljapurkars model) to be more realistic than the other.

The exception of this tendency of increasing realism is the last section dealing with the interaction of fertility and the population's age-structure. This model simply deals with another aspect of demographics and therefore enriches the thesis as a whole but does not offer a better representation of life cycle aspects or mortality than the models discussed in section 5 and 6. Nevertheless it has its eligibility among the other models in this thesis, as it examines the dynamics underlying demographic changes (the change in the age-composition on fertility).

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Abstract

This thesis deals with the issue of realistic demographic structures in Models of Overlapping Generations (OLG). Basic OLG models are often built on very restrictive demographic assumptions and are not able to represent realistic human economic life-cycle details to a high degree. Facing changing population age structures in most industrialized countries, the issue of proper demographic analysis becomes very important. Leaving these changes aside economic analysis can lead to false or imprecise conclusions. Therefore attention should be paid to realistic representation of demographics and life cycle details in models used for economic analysis (Bommier and Lee, page 138).

In this thesis I present and compare different OLG frameworks focusing on their representation of demographics and life cycle details. Starting with the basic OLG model developed by Samuelson (1958) and Diamond (1965) I show different approaches of modelling realistic mortality patterns. The first step towards more realism in OLG models is the introduction of individual life time uncertainty by Blanchard (1989) and Yaari (1965) assuming a constant risk of death throughout life. Blanchard's somewhat unsatisfying result of age-independent life-expectancy and consumption behaviour is improved by Gertler (1999) combining individual life time uncertainty with the introduction of two different life-stages. In their Probabilistic Aging model (PA Model) Grafenhofer et al. (2006) further improve Gertler's framework to a higher number of life stages and therefore generating a higher degree of heterogeneity among people within the economy. Li and Tuljapurkar (2004) develop an age-dependent mortality approach by incorporating a probability distribution of age at death to Blanchard's framework and are so able to let changing life expectancy and changing life time uncertainty affect individuals' behaviour. Finally Hock and Weil (2006) analyze the interaction of fertility and the population age-structure using a very simple OLG model with probabilistic transition between different life stages. I show that the main consequences of introducing life time uncertainty are to affect capital accumulation, consumption behaviour and the discount rate on future labour income. Capital accumulation is affected negatively by increasing risk of death while retirement opposes this dynamic by increasing capital accumulation. Consumption behaviour is affected by increasing the marginal propensity to consume while an increasing probability of death increases the discount rate on future labour income.

Abstract (Deutsch)

Diese Diplomarbeit beschäftigt sich mit der Implementierung realistischer Demographie in Modellen überlappender Generationen (OLG-Modelle). Einfache OLG-Modelle basieren oft auf sehr restriktiven Annahmen bezüglich ihrer demographischen Struktur und können daher keine hinreichend realistischen Lebenszyklen abbilden. Angesichts der Veränderungen in der Altersstruktur der Bevölkerung industrialisierter Länder gewinnt die Berücksichtigung demographischer Analysen an Bedeutung. Lässt man diese Veränderungen außer Acht, kann eine ökonomische Analyse zu falschen bzw. unpräzisen Schlussfolgerungen führen. Deswegen sollten demographischer Realismus und detaillierte Lebenszyklen in Modellen berücksichtigt werden, die zur ökonomischen Analyse verwendet werden (Bommier und Lee, Seite 138).

In dieser Diplomarbeit präsentiere und vergleiche ich verschieden OLG-Modelle im Hinblick auf deren Fähigkeit, realistische Demographie und Lebenszyklen abzubilden. Am Anfang steht dabei das einfache OLG Modell von Samuelson (1958) und Diamond (1965). Davon ausgehend präsentiere ich verschiedene Ansätze realistischere Demographie zu modellieren. Der erste Schritt zu mehr Realismus ist die Implementierung individueller Lebensunsicherheit im Modell von Blanchard (1989) und Yaari (1965), in dem eine konstante Sterbewahrscheinlichkeit angenommen wird. Die wenig zufrieden stellenden Ergebnisse einer altersunabhängigen Lebenserwartung und Konsumneigung werden im Modell von Gertler (1999) verbessert, in dem individuelle Lebensunsicherheit mit der Modellierung zweier Lebensabschnitte kombiniert wird. In ihrem Probabilistic Aging Modell (PA-Modell) verbessern Grafenhofer et al. (2006) Gertlers Modell, indem sie weitere Lebensabschnitte einführen und damit den Grad an Diversität zwischen den Menschen der Ökonomie erhöhen. Li und Tuljapurkar (2004) entwickeln ein Modell mit altersabhängigem Sterberisiko, indem sie die Wahrscheinlichkeitsverteilung des „Sterbealters“ in Blanchards Modell implementieren. Dadurch können sich Veränderungen in Lebenserwartung und Überlebensunsicherheit auf das Verhalten der Individuen auswirken. Schließlich analysieren Hock und Weil (2006) die Wechselwirkung zwischen Fertilität und der Altersstruktur der Bevölkerung in einem einfachen OLG Modell, in dem der Übergang zwischen den Lebensabschnitten analog zum PA-Modell modelliert ist. Ich zeige, dass sich die Einführung von Lebensunsicherheit auf Kapitalakkumulation, Konsumverhalten und die Diskontierungsrate auf zukünftiges Arbeitseinkommen auswirken. Während höheres Sterberisiko mit geringerer Kapitalakkumulation verbunden ist, führt Pensionierung zu einem

Kapitalanstieg. Die Konsumneigung steigt mit höherer Sterbewahrscheinlichkeit, während zukünftiges Arbeitseinkommen höher abdiskontiert wird.

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