PHYSICAL REVIEW B **67**, 035328 (2003)

Correlation energy, quantum phase transition, and bias potential effects in quantum Hall bilayers at $\nu=1$

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(Received 16 September 2002; revised manuscript received 31 October 2002; published 31 January 2003)

We study the correlation energy, the effective anisotropy parameter, and quantum fluctuations of the pseudospin magnetization in bilayer quantum Hall systems at total filling factor $\nu=1$ by means of exact diagonalizations of the Hamiltonian in the spherical geometry. We compare exact-diagonalization results for the ground-state energy with finite-size Hartree-Fock values. In the ordered ground-state phase at small layer separations the Hartree-Fock data compare reasonably with the exact results. Above the critical layer separation, however, the Hartree-Fock findings still predict an increase in the ground-state energy, while the exact ground-state energy is in this regime independent of the layer separation indicating the decoupling of layers and the loss of spontaneous phase coherence between them. We also find accurate values for the pseudospin anisotropy constant, whose dependence of the layer separation provides another very clear indication for the strong interlayer correlations in the ordered phase and shows an inflection point at the phase boundary. Finally, we discuss the possibility of interlayer correlations in biased systems even above the phase boundary for the balanced case. Certain features of our data for the pseudospin anisotropy constant as well as for quantum fluctuations of the pseudospin magnetization are not inconsistent with the occurrence of this effect. However, it appears to be rather weak at least in the limit of vanishing tunneling amplitude.

DOI: 10.1103/PhysRevB.67.035328 PACS number(s): 73.43.Cd, 73.43.Nq

I. INTRODUCTION

Quantum Hall ferromagnets are a rich and fascinating field of solid-state physics.[1–3] They can be realized in terms of the spins of electrons confined to layers in a strong perpendicular magnetic field, or in terms of a pseudospin given by some additional discrete degree of freedom such as the layer spin in bilayer systems.[4–11] Bilayer quantum Hall systems at total filling factor $\nu=1$ have attracted particular interest recently due to spectacular results by Spielman *et al.*,[12,13] who studied tunneling transport across the layers in samples with very small single-particle tunneling gap. These experiments have stimulated a large number of theoretical efforts toward their explanation, and also more general studies of such bilayer quantum Hall systems.[14–33]

The main finding of Ref. 12 is a pronounced peak in the differential tunneling conductance, which evolves if the layer separation d in units of the magnetic length ℓ is decreased below a critical value. This critical ratio d/ℓ agrees closely with the boundary between a ground-state phase supporting quantized Hall transport and a disordered phase as established in earlier experiments by Murphy *et al.*[11] using double well samples of similar geometry. Therefore, these two observations can be assumed to be manifestations of one and the same quantum phase transition. Moreover, recent exact-diagonalization studies[17] on bilayers at $\nu=1$ have revealed a quantum phase transition, very likely to be of first order, between a phase with strong interlayer correlations to a phase with weak interlayer correlations. The position of this transition agrees quantitatively with the critical value found by Spielman *et al.*

In the ordered phase at small d/ℓ the strong interlayer correlations are dominated by the *spontaneous interlayer phase coherence* between the layers. This keyword describes the fact that in the ground-state of such a system electrons predominantly occupy single-particle states in the lowest Landau level, which are symmetric linear combination of states in both layers. This type of single-particle states is preferred if a finite tunneling amplitude is present. However, by a large body of experimental and theoretical work,[4–33] this phenomenon is assumed to be a spontaneous symmetry breaking, i.e., it remains even in the limit of vanishing tunneling amplitude. The latter effect is clearly a many-body phenomenon.

In the present work, we report on further exact-diagonalization results in quantum Hall bilayers at total filling factor $\nu=1$. Our studies include the effective pseudospin anisotropy parameter, quantum fluctuations of the pseudospin magnetization, and the ground-state energy. Especially the last quantity shows very clearly the occurrence of the quantum phase transition and the decoupling of the layers above the critical d/ℓ, where the spontaneous interlayer phase coherence is lost. Moreover, we study the effects of a bias potential applied to the layers. In particular, we address the question of possible interlayer correlations in biased systems even above the phase boundary of the balanced case, an effect which was predicted recently by Hanna[20] and by Joglekar and MacDonald[23] based on time-dependent Hartree-Fock calculations. Some features of our data are not inconsistent with this prediction. However, this effect appears to be rather weak at least in the limit of vanishing tunneling amplitude and not too large biasing, consistent with the predictions of Refs. 20,23.

Our numerics are performed within the spherical geometry.[34] This geometry enables to obtain closed expressions for the Hartree-Fock ground-state energy even in finite systems. This quantity can be compared with exact-diagonalization results to infer the correlation energy. More-

over, since the sphere is free of boundaries, this geometry allows to take into account a neutralizing background in finite systems without any ambiguity.

This paper is organized as follows. In Sec. II, we describe our finite-size Hartree-Fock calculations of the ground-state energy in the spherical geometry. In Sec. III, we present our exact-diagonalization results, compare them with Hartree-Fock theory, and perform a detailed analysis of bias potential effects. We close with conclusions in Sec. IV.

II. FINITE-SIZE HARTREE-FOCK THEORY IN THE SPHERICAL GEOMETRY

In this section, we present details of our finite-size Hartree-Fock calculations in the spherical geometry.[34] Similar results for the case of bilayers at filling factor $\nu=2$ were already briefly discussed in Ref. 35. The notation follows the discussion of the $\nu=2$ system in Ref. 36. The technical advantage of the spherical geometry used here lies in the fact that it allows to obtain closed results for electron pair distribution function even in finite systems.

We consider a gas of Coulomb-interacting electrons in a quantum Hall bilayer system at total filling factor $\nu=1$. We assume a vanishing amplitude for electron tunneling between the layers, consistent with the experimental situation in Ref. 12. The layer degree of freedom is described in the usual pseudospin language,[1] where the pseudospin operator of each electron is given by $\vec{\tau}/2$ with $\vec{\tau}$ being the vector of Pauli matrices. The z component $\tau^z/2$ measures the difference in occupation between the two layers, while $\tau^x/2$ describes tunneling between them. The total pseudospin of all electrons is denoted by $\vec{T}$.

Differently from the pseudospin, the true electron spins are assumed to be fully aligned along the magnetic field perpendicular to the layers; therefore, an inessential Zeeman term in the Hamiltonian is, along with the constant cyclotron energy, neglected. In Ref. 17 a finite width of the quantum wells forming the bilayer system was taken into account in order to make quantitative contact to the experiments of Ref. 12. However, a finite well width mainly changes the position of the phase transition but does not alter any qualitative feature. In the present work, we therefore concentrate for simplicity on the case of zero well width. For this case, the critical layer separation in the limit of vanishing tunneling amplitude was found by exact-diagonalization calculations[17] to be $d=1.3\ell$. This value holds in the thermodynamic limit, but is remarkably rapidly approached in finite-size systems.[17] For instance, the phase boundary in a system of just 12 electrons deviates from the infinite-volume value by just a few percent.

In the gauge commonly used in the spherical geometry[34] the single-particle wave functions in the lowest Landau level have the form

$$\langle \vec{r}|m\rangle = \left[\frac{N_\phi+1}{2\pi\ell^2 N_\phi}\begin{pmatrix} N_\phi \\ \frac{N_\phi}{2}+m\end{pmatrix}\right]^{1/2} \times\left[\cos\left(\frac{\vartheta}{2}\right)e^{i\varphi/2}\right]^{N_\phi/2+m}\left[\sin\left(\frac{\vartheta}{2}\right)e^{-i\varphi/2}\right]^{N_\phi/2-m}, \tag{1}$$

where ϑ, φ are the usual angular coordinates of the location $\vec{r}$ on the sphere with radius $|\vec{r}|=\ell\sqrt{N_\phi/2}$. $m\in\{-N_\phi/2, -N_\phi/2+1,\ldots,N_\phi/2\}$ labels the different angular momentum states, and N_ϕ is the number of flux quanta penetrating the sphere. The Hartree-Fock ansatz for a spatially homogeneous state of $N=N_\phi+1$ electrons is

$$|\Psi\rangle=\prod_{m=-\frac{N_\phi}{2}}^{N_\phi} 2\left(\sum_{\sigma\in\{T,B\}} z_\sigma c^+_{m\sigma}\right)|0\rangle, \tag{2}$$

where $|0\rangle$ is the fermionic vacuum. $c^+_{m\sigma}$, $\sigma\in\{T,B\}$, creates an electron in the top/bottom layer in angular momentum state m, and z_σ are the components of a normalized two-spinor describing the layer degree of freedom. From this state we obtain the pair distribution functions

$$g(\vec{r}_1-\vec{r}_2)=\langle\Psi|\sum_{i\neq j}[\,\delta(\vec{r}_1-\hat{\vec{r}}_i)\,\delta(\vec{r}_2-\hat{\vec{r}}_j)]|\Psi\rangle = \left(\frac{N_\phi+1}{2\pi\ell^2 N_\phi}\right)^2\left[1-\left(1-\frac{|\vec{r}_1-\vec{r}_2|^2}{2\ell^2 N_\phi}\right)^{N_\phi}\right], \tag{3}$$

$$h(\vec{r}_1-\vec{r}_2)=\langle\Psi|\sum_{i\neq j}[\,\delta(\vec{r}_1-\hat{\vec{r}}_i)\,\tau_i^z\,\delta(\vec{r}_2-\hat{\vec{r}}_j)\,\tau_j^z]|\Psi\rangle = \left(\frac{N_\phi+1}{2\pi\ell^2 N_\phi}\right)^2(\langle z|\tau^z|z\rangle)^2 \times\left[1-\left(1-\frac{|\vec{r}_1-\vec{r}_2|^2}{2\ell^2 N_\phi}\right)^{N_\phi}\right]. \tag{4}$$

Here, the indices i,j refer to electrons and the Pauli matrices τ^z act on the layer spins. The expression $|\vec{r}_1-\vec{r}_2|$ is the chord distance on the sphere. Note that in the limit of large numbers of flux quanta N_ϕ one obtains from Eq. (3) the well-known expression for the infinite system in planar geometry,

$$\lim_{N_\phi\to\infty} g(r)=\left(\frac{1}{2\pi\ell^2}\right)^2(1-e^{-r^2/2\ell^2}). \tag{5}$$

To calculate the energy of the Coulomb interaction it is convenient to consider the linear combination $V_\pm=(V_S\pm V_D)/2$ of the interactions V_S and V_D between electrons in the same layer and different layers, respectively.[36] Using the above pair distribution functions one obtains for the energy per particle

$$\varepsilon^{HF}=\varepsilon^{HF}_{el}-\frac{1}{2}B=\frac{1}{2}[-F_+ +(\langle z|\tau^z|z\rangle)^2(H-F_-)]. \tag{6}$$

Here, ε^{HF}_{el} is the Hartree-Fock energy of the interaction between electrons. The quantity

$$B=\frac{e^2}{\epsilon\ell}\,\frac{N_\phi+1}{2\sqrt{N_\phi/2}}\left[1-\frac{1}{2\sqrt{N_\phi/2}}\,\frac{d}{\ell}+\left(1+\frac{1}{N_\phi}\,\frac{d^2}{2\ell^2}\right)^{1/2}\right] \tag{7}$$

arises from the direct (Hartree) contribution of V_+ and cancels against a neutralizing homogeneous background of half the total electron charge which is present in each layer and ensures charge neutrality. In this work we have always subtracted this term from the ground-state energies considered here. The quantity

$$H=\frac{e^2}{\epsilon\ell}\frac{N_\phi+1}{2\sqrt{N_\phi/2}}\left[1+\frac{1}{2\sqrt{N_\phi/2}}\frac{d}{\ell}-\left(1+\frac{1}{N_\phi}\frac{d^2}{2\ell^2}\right)^{1/2}\right] \tag{8}$$

stems from the direct term of V_-, and

$$F_\pm=\frac{e^2}{\epsilon\ell}\frac{N_\phi+1}{2\sqrt{2N_\phi}}\left[I(1)\pm\left(\frac{1}{\alpha}\right)^{N_\phi+1/2}I(\alpha)\right] \tag{9}$$

represent the exchange (Fock) contributions from $V_\pm$ with

$$I(\alpha)=\int_0^\alpha dx\,\frac{x^{N_\phi}}{\sqrt{1-x}}\quad \alpha=\frac{1}{1+\frac{1}{N_\phi}\frac{d^2}{2\ell^2}}. \tag{10}$$

In the above equations, $e^2/(\epsilon\ell)$ is the Coulomb energy scale with $(-e)$ being the electron charge and ϵ the dielectric constant of the semiconductor material. Note that all the above contributions to ε^{HF} depend on the layer separation d/ℓ as well as on the number of flux quanta N_ϕ, i.e., on the system size.

In the Hartree-Fock ground-state of an unbiased system all spins lie in the xy plane of the pseudospin space, i.e., $\langle z|\tau^z|z\rangle=0$, and we end up with

$$\varepsilon_0^{HF}=-\frac{1}{2}F_+. \tag{11}$$

III. RESULTS

In this section, we report on our results from exact numerical diagonalizations of the many-body Coulomb Hamiltonian in the spherical geometry.[34] In such a system the ground-state has vanishing total angular momentum[34] and, for unbiased bilayers, the smallest possible value of the z component of the total pseudospin $\vec{T}$, i.e., $T^z=0$ for an even number of electrons and $|T^z|=1/2$, otherwise.

A. Ground-state and correlation energy in the unbiased system

Figure 1 shows the exact and the Hartree-Fock ground-state energy [both in units of the Coulomb energy scale $e^2/(\epsilon\ell)$] as a function of d/ℓ for several numbers of electrons N. In both cases, the contribution from the neutralizing background (7) is subtracted. At zero layer separation, we recover the case of a quantum Hall monolayer with the layer spin playing the role of the electron spin. Here, the ground-state is the well-known spin-polarized $\nu=1$ monolayer ground-state described exactly by Hartree-Fock theory. In the spherical geometry, the finite-size ground-state energy per particle is given by

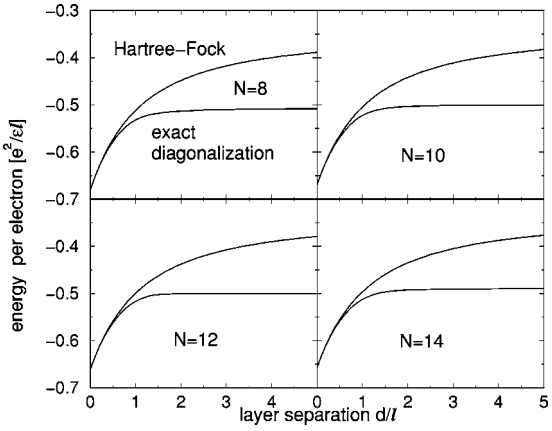

FIG. 1. The ground-state energy as a function of the layer separation in units of the magnetic length for different numbers N of electrons. The exact diagonalization data is compared with finite-size Hartree-Fock results. In both cases, the contribution from the neutralizing background has been subtracted. In the ordered phase below the critical value of d/ℓ the results agree reasonably and coincide for vanishing layer separation. Above the critical layer separation the exact ground-state energy is independent of d/ℓ corresponding to uncoupled $\nu=1/2$ monolayers, while Hartree-Fock theory still predicts an artificial increase in ground-state energy.

$$\varepsilon^{HF}=-\frac{e^2}{\epsilon\ell}\frac{2^{2N_\phi}}{\sqrt{N_\phi/2}\binom{2N_\phi+2}{N_\phi+1}}\overset{\rightarrow}{N_\phi\to\infty}-\frac{e^2}{\varepsilon\ell}\sqrt{\frac{\pi}{8}}, \tag{12}$$

with $N_\phi=N-1$ being the number of flux quanta.

At finite layer separation the Hartree-Fock ground-state becomes unexact, but provides still a reasonable approximation to the exact ground-state energy if d/ℓ is smaller than the critical value of $d/\ell=1.3$. In other words, the correlation energy given by the difference between the exact ground-state energy and the Hartree-Fock value is small. For larger layer separations $d/\ell\gtrsim 1.3$ Hartree-Fock theory still predicts an increase of the ground-state energy with increasing layer separation, while the exact ground-state energy becomes independent of d/ℓ. The latter result is again a particulary clear signature of the decoupling of the two layers and the loss of spontaneous phase coherence between them above the critical layer separation. The discrepancy between the exact ground-state energy and the Hartree-Fock result in the disordered phase, i.e., the large correlation energy, shows that this quantum phase transition is a correlation phenomenon that cannot be described within simple Hartree-Fock theory. In the Hartree-Fock ansatz used here all electrons are in the same pseudospin state implementing phase coherence between the layers. This coherence is lost above the critical d/ℓ, and the system behaves, at least in terms of its ground-state energy, like two decoupled monolayers with filling factor $\nu=1/2$. Therefore, the failure of the Hartree-Fock theory might appear as a consequence of the artificial phase coherence. However, as it is well known, the Hartree-Fock approach is generally inadequate to describe quantum Hall monolayers at $\nu=1/2$, which have a very peculiar and highly correlated ground-state.

B. The pseudospin anisotropy parameter and bias potential effects

The difference in the Coulomb interaction for electrons in the same layer and in different layers provides a strong mechanism balancing the charges between the layers. In the pseudospin language this can be expressed approximately by an effective easy-plane anisotropy contribution[9] to the energy per particle,

$$\varepsilon_a = 8\pi\ell^2\beta\frac{\langle T^z\rangle^2}{N^2}, \tag{13}$$

introducing an anisotropy parameter β, and $\langle T^z\rangle$ denotes the expectation value of the z component of the total pseudospin.[37] For vanishing tunneling between the layers as considered here this operator represents a good quantum number, and eigenstates can be labeled using their value of T^z. In this case, the above energy contribution can be viewed just as a charging energy of a capacitor formed by the two isolated layers. In the absence of quantum correlations, and for a large system, the anisotropy parameter takes the value

$$8\pi\ell^2\beta_{cl} = \frac{e^2}{\epsilon\ell}\frac{d}{\ell}, \tag{14}$$

corresponding to the classical total charging energy of $E_c = N\varepsilon_a = Q^2/(2C)$ with $Q=-eT^z$ being the charge of the capacitor, $C=\epsilon A/(4\pi d)$ its capacity, and $A=2\pi\ell^2 N$ its area. In the presence of quantum correlations the effective anisotropy parameter will deviate from this value for two different reasons: (i) *Interlayer correlations* can modify the value of β, and (ii) even in the absence of correlations *between* the layers, *intralayer correlations* can have an impact on β if the ground states of the two mutually uncorrelated layers change nontrivially if electrons are transferred from one layer to the other, i.e., if T^z is changed. The latter effect is independent of the layer separation. Therefore, in the absence of interlayer correlations and for a given value of T^z, the contribution to the ground-state energy which depends on the layer separation is just given by a simple classical electrostatic expression,[38] which can be derived similarly as Eq. (7),

$$\varepsilon_a^{cl} = 8\pi\ell^2\beta_{cl}\frac{\langle T^z\rangle^2}{N^2}, \tag{15}$$

with

$$8\pi\ell^2\beta_{cl} = \frac{e^2}{\epsilon\ell}\frac{N_\phi+1}{\sqrt{N_\phi/2}}\left[1+\frac{1}{2\sqrt{N_\phi/2}}\frac{d}{\ell}-\left(1+\frac{1}{N_\phi}\frac{d^2}{2\ell^2}\right)^{1/2}\right], \tag{16}$$

which converges to the expression (14) for $N=N_\phi+1\to\infty$. Thus, if no interlayer correlations are present, the contribution to the effective anisotropy parameter with a nontrivial dependence on the layer separation is given by the above classical expression, with a possible additional contribution independent of the layer separation which arises from intralayer quantum effects.

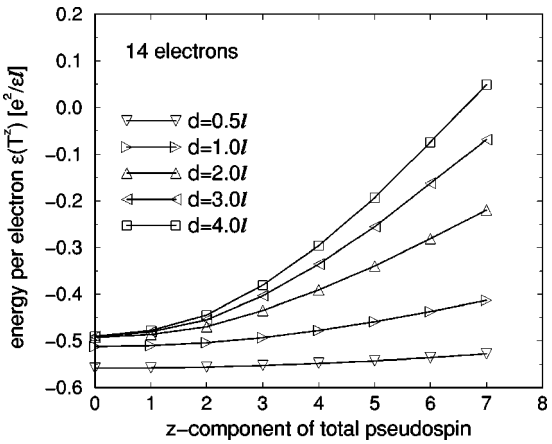

FIG. 2. The energy of the lowest state having a given quantum number T^z as a function of this quantity for various layer separation in a system of $N=14$ electrons. $T^z=0$ corresponds to the ground-state of the balanced system at a given layer separation, and each curve is for not too large T^z well described by a parabola.

Let us now analyze the anisotropy parameter in terms of exact-diagonalization results. The lowest states with a given value of T^z have vanishing total angular momentum on the sphere[34], i.e., they are spatially homogeneous. Figure 2 shows the energy of these lowest state in the sector of a given value of T^z as a function of T^z for $N=14$ electrons and several layer separations. At all layer separations, in the ordered as well as in the disordered phase, the dependence of the energy on T^z is, for not too large T^z, parabolic, validating the phenomenological ansatz (13).

Figure 3 shows values for $8\pi\ell^2\beta$ obtained from parabolic fits of $\varepsilon_a(T^z)$ using $T^z\in\{0,1,2,3\}$ for $N=12$ and $N=14$ electrons as a function of d/ℓ. If higher values of T^z are included the quality of the fits considerably decreases.

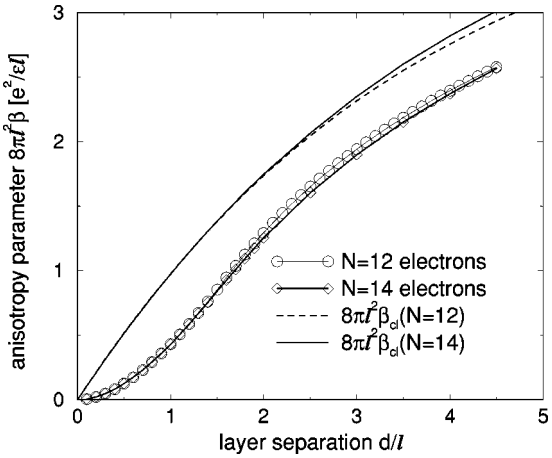

FIG. 3. The anisotropy parameter $8\pi\ell^2\beta$ obtained from exact-diagonalization data as a function of the layer separation for $N=12$ and $N=14$ electrons. Both data sets for this (in the bulk limit) intensive quantity agree very well and show an inflection point near the phase transition at $d/\ell\approx 1.3$. The corresponding values for $8\pi\ell^2\beta_{cl}$ [cf. Eq. (16)] are also shown which describe (up to a constant) the expected behavior in the absence of interlayer correlations.

We therefore concentrate on the system sizes $N=12$ and $N=14$, where a sufficient number of moderate values for T^z (as compared to its maximum $N/2$) are available. We have also plotted in Fig. 3 the classical electrostatic expression (16) for both systems sizes.

The anisotropy parameter β is in the bulk limit an intensive quantity. Both data sets for $8\pi\ell^2\beta$ shown in Fig. 3 are nearly identical establishing that β is only very weakly dependent on the system size for already quite small systems which are accessible via exact diagonalization techniques. As to be expected β increases with increasing layer separation. Moreover, it shows an inflection point near the critical value $d/\ell=1.3$, which we interpret as a further signature of the quantum phase transition. Above the inflection point the anisotropy parameter β as obtained from exact-diagonalization data has the same curvature as β_{cl}. Below the inflection point at $d/\ell\approx 1.3$ both data sets differ clearly, in particular, in curvature, which indicates the presence of strong interlayer correlations in this regime. However, we should stress that the concrete form of these deviations from the classical behavior, namely, the occurence of an inflection point and a change in curvature, is the result of the present numerical study and has not been predicted on other theoretical grounds.

The results of Sec. III A have established the absence of interlayer correlations in an unbiased system above the critical d/ℓ. If interlayer correlations vanish also in a biased system (with finite T^z) the anisotropy parameter β found by exact numerical diagonalizations should be the same as β_{cl} up to a rigid shift (being independent of the layer separation) arising from intralayer effects. As seen in Fig. 3 this is for $d\gtrsim 1.3\ell$ to a quite good degree of approximation, but not perfectly, the case. In particular, β_{cl} increases with increasing system size, while the exact-diagonalization values appear to decrease. The small discrepancy between β and β_{cl} (after subtracting a rigid shift) might, therefore, be seen as an indication for the presence of interlayer correlations in biased systems even above the critical layer separation of the balanced system, as predicted recently in Ref. 23. However, if so, this effect appears to be rather small, at least in the limit of vanishing tunneling and not too large biasing, consistent with the predictions of Refs. 20,23.

The value for β at $d/\ell=1$ is by a factor of about 2 larger than the effective anisotropy parameter found recently from exact diagonalization studies of a vertical pair of parabolically confined quantum dots in the quantum Hall regime.[25] In the latter case, this effective anisotropy parameter agrees quite reasonably with results from numerical Hartree-Fock calculations. On the other hand, the values for β shown in Fig. 3 agree very reasonably within a discrepancy of less than 10% with data reported in Ref. 9 for an infinite system. Those values were obtained from an approximate effective-field theory neglecting correlation effects beyond Hartree-Fock exchange. Therefore, the data of Ref. 9 does not show an inflection point signalling a ground-state phase transition.

C. Quantum fluctuations of the pseudospin magnetization

In Ref. 17 the quantum phase transition between a pseudospin-polarized phase-coherent state and a disordered ground-state was analyzed by studying the pseudospin magnetization $\langle T^x\rangle$ along with their fluctuation $(\Delta T^x)^2=\langle (T^x)^2\rangle-\langle T^x\rangle^2$ as a function of the tunneling gap. Here, we report on results for ΔT^x at zero tunneling as a function of d/ℓ in the ground-state within various sectors of T^z. These states are the absolute ground-state of the system at an appropriate bias voltage between the layers.

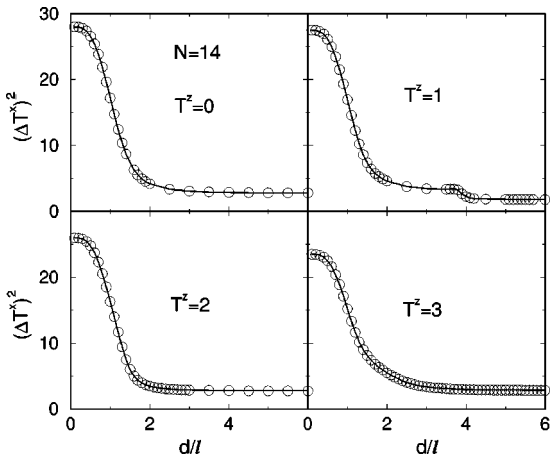

FIG. 4. The pseudospin fluctuation $(\Delta T^x)^2$ as a function of the layer separation for different sectors of T^z in a system of $N=14$ electrons.

The ordered phase at small layer separations is characterized by large fluctuations of the pseudospin magnetization and, therefore, by a large susceptibility of this quantity with respect to interlayer tunneling.[17] At zero tunneling T^z is a good quantum number, while $\langle T^x\rangle=\langle T^y\rangle=0$, and for the fluctuations it holds $\Delta T^x=\Delta T^y$ with $\Delta T^z=0$. Figure 4 shows $(\Delta T^x)^2$ within the ground-state of several sectors of T^z as a function of d/ℓ for $N=14$ electrons. At zero layer separation one has

$$(\Delta T^x)^2=\frac{1}{2}\left[\frac{N}{2}\left(\frac{N}{2}+1\right)-(T^z)^2\right], \tag{17}$$

and for finite layer separation $(\Delta T^x)^2$ decreases for all values of T^z with increasing d/ℓ to rather small values. This decay mainly occurs in the vicinity of the critical value $d\approx 1.3\ell$. In the upper right panel $(T^z=1)$ yet another transition occurs at larger layer separations, which appears to be a peculiarity of this system size. Note that the quantity ΔT^x is, on the other hand, bounded from below by the standard uncertainty relation $\Delta T^x\Delta T^y=(\Delta T^x)^2\geq T^z/2$.

As seen in Fig. 4 the phase transition seems to occur rather at the same region of d/ℓ in all sectors of T^z, with apparently a slight tendency to move to larger layer separations with increasing T^z. Therefore, in the case of vanishing tunneling gap, the critical layer separation depends only very weakly on a bias voltage between the layers. Thus, if there is an increase of the critical layer separation in biased systems as predicted in Refs. 20,23, this effect is rather small. This is consistent with the results of the preceding section, and with Refs. 20,23.

Recently, Nomura, and Yoshioka[30] have introduced a parameter S defined by $\langle \vec{T}^2\rangle=S(S+1)$ to describe the ''effec-

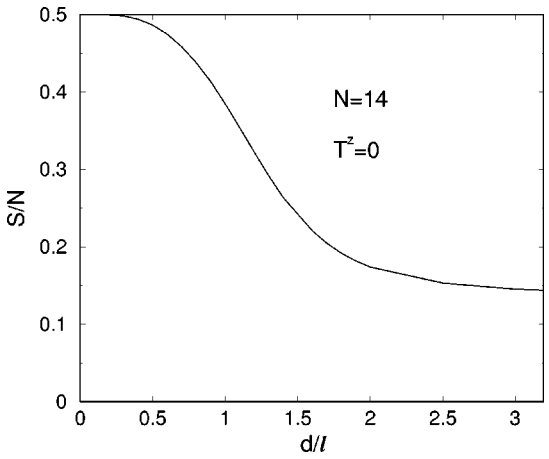

FIG. 5. The effective pseudospin length S per particle as a function of d/ℓ. This data obtained in the spherical geometry agrees very well with recent results for a toroidal system Ref. 30.

tive length'' of the pseudospin in a given state. Figure 5 shows S divided by the number of particles for $N=14$ electrons and $T^z=0$ (corresponding to the upper left panel of Fig. 4). This plot can be compared directly with data of Ref. 30 obtained in the toroidal geometry, establishing a very good agreement between exact-diagonalization results on the sphere and on the torus.

IV. CONCLUSIONS

We have investigated ground-state properties of bilayer quantum Hall systems at total filling factor $\nu=1$ and vanishing single-particle tunneling gap by means of exact numerical diagonalizations in finite systems. Specifically, the ground-state energy, the pseudospin anisotropy parameter, and the quantum fluctuations of the pseudospin magnetization are studied as functions of the layer separation in units of the magnetic length.

The exact ground-state energies are compared with results of finite-size Hartree-Fock calculations described in Sec. II. The availability of closed expressions for pair distribution functions and Hartree-Fock energies even in finite systems is a specific property of the spherical system geometry used here. The exact ground-state energies (with a contribution from a neutralizing background being subtracted) is independent of d/ℓ above the critical layer separation. This demonstrates the decoupling of layers and the loss of spontaneous phase coherence between them in the disordered phase.

We have also performed a very detailed analysis of the effective pseudospin anisotropy parameter. We have found accurate numerical values for this quantity as a function of the layer separation, and compared it with a classical electrostatic expression valid in the absence of interlayer correlations. This comparison establishes the strong interlayer correlations in the ordered phase at small layer separations, and the quantum phase transition is signaled by an inflection point of the anisotropy parameter at the phase boundary. Moreover, we have analyzed the possibility of interlayer correlations in biased systems even above the phase boundary of the unbiased case. Certain features of our data are not inconsistent with the occurrence of this effect, which, however, appears to be quite small at least in the limit of vanishing tunneling amplitude.

In summary, our results show that the quantum phase transition in quantum Hall bilayers at total filling factor $\nu=1$ shows its signatures in various physical quantities and represents a subtle correlation effect.

ACKNOWLEDGMENTS

I thank Anton A. Burkov, Steven M. Girvin, Yogesh N. Joglekar, Allan H. MacDonald, and, in particular, Charles B. Hanna for many useful discussions.

[1] For a review on experimental research see J.P. Eisenstein in *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997); for a review on theoretical research see S.M. Girvin and A.H. MacDonald, in the same volume.

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