# Fields with the Bogomolov Property

Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat.) der Fakultät für Mathematik der Universität Regensburg

vorgelegt von

Lukas Christopher Pottmeyer

aus Dortmund

im Jahr 2012

Promotionsgesuch eingereicht am: 25.10.2012
Die Arbeit wurde angeleitet von Prof. Dr. Walter Gubler.

Gutachter:

1. Prof. Dr. Walter Gubler (Universität Regensburg)
2. Prof. Dr. Philipp Habegger (Goethe-Universität Frankfurt)
3. Prof. Matthew Baker, Ph.D. (Georgia Institute of Technology)

weitere Mitglieder der Prüfungskomission:

Prof. Dr. Harald Garcke
Prof. Dr. Moritz Kerz
Prof. Dr. Klaus Künnemann

Tag der Verteidigung: 31.01.2013

# Contents

# Introduction

We will start this thesis with the purest and most fascinating objects in mathematics: primes. From Euclid's proof of the infinitude of primes to the Riemann hypothesis, the most famous open conjecture in number theory, primes have been the origin of almost all number theoretical problems. Apart from theoretical questions concerning their distribution among the integers or the infinity of certain special classes of primes, it has always been a competition to find explicit large primes. This latter problem was the motivation of Derrick Henry Lehmer to write his remarkable paper [Le33] of 1933. For a monic irreducible integer polynomial

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 = \prod_{i=1}^{d}(x - \alpha_i)$$

he studied the factorization of the integers $\Delta_n(f) = \prod_{i=1}^{d}(\alpha_i^n - 1)$ with regard to large prime factors of $\Delta_n(f)$. Of course, *large* heavily depends on the computing power at this time. Lehmer wrote his paper from a theoretical point of view, and his exemplary prime $\Delta_{127}(x^3 - x - 1) = 3233514251032733$ was not a new prime record in 1933. We remark that the largest known prime of today is the Mersenne prime $2^{43112609} - 1$ which is a 12978189-digit number. This prime was found in August 2008 by Edson Smith using the *Great Internet Mersenne Prime Search (GIMPS)* (see [GIMPS]). But Lehmer is also strongly related to this prime, as the website uses the Lucas-Lehmer test to decide whether a number of the form $2^p - 1$, $p$ prime, is a prime number. This test was developed by Lehmer in Section 5 of his Ph.D. thesis [Le30].
We see that $\Delta_n(f)$ is zero for some $n$ if and only if $f$ is a cyclotomic polynomial. In this case the set $\{\Delta_n(f)\}_{n\in\mathbb{N}}$ only consists of finitely many integers. Hence, one should exclude these polynomials in the search for large primes in $\Delta_n(f)$. The reason why the paper [Le33] became so famous is Lehmer's observation that $\Delta_n(f)$ is more likely to produce large primes if the measure

$$M(f) = \prod_{i=1}^{d} \max\{|\alpha_i|, 1\}$$

of $f$ is small. In 1857, Kronecker proved in [Kr57] that an algebraic integer with all its conjugates lying on the unit circle must be a root of unity. This implies that $M(f)$ is equal to 1 if and only if $f$ is a cyclotomic polynomial. Therefore, Lehmer searched for monic integer polynomials $f$ with small measure $M(f) > 1$. The measure $M(f)$ is called *Mahler measure*, after the paper [Ma62] of Kurt Mahler. Notice that Mahler introduced this measure in the form

$$M(f) = \exp(\int_0^1 \log|f(\mathbf{e}^{2i\pi t})|dt)$$

where $\mathbf{e}$ denotes the Euler-number. It is well known that these expressions for $M(f)$ are indeed equal (see for example [BG], Proposition 1.6.5).

Lehmer noticed that the Mahler measure $M(f)$ is especially small if the polynomial $f$ is *reciprocal*, i.e. $\pm f(x) = x^{\deg(f)} f(\frac{1}{x})$. Reciprocal polynomials are sometimes also called *symmetric*. Regarding the problem of finding polynomials with small Mahler measure, Lehmer pointed out:

> *We have not made an examination of all* 10*th degree symmetric polynomials but a rather intensive search has failed to reveal a better polynomial than*
> $$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

The above polynomial is called *Lehmer polynomial* and has a Mahler measure of approximately $1,176280821$. Meanwhile there has been an intensive examination. For example, all non-cyclotomic polynomials of degree at most 44, with Mahler measures less than $1,3$ are known (see [MRW08]). But still there has not been found a non-cyclotomic polynomial of smaller Mahler measure than the Lehmer polynomial. This is called *Lehmer's problem.* The observation of Lehmer that reciprocal polynomials are more likely to have small Mahler measure has been proved by Breusch in [Br51]. He proved that the Mahler measure of a non-reciprocal polynomial is $> 1,179$. Later, but independently, Smyth gave the sharp lower bound $M(x^3 - x - 1) = 1,324717\ldots$ for the Mahler measure of such polynomials (see [Sm71]).

Lehmer's problem is open in general, but for some classes of polynomials (like reciprocal polynomials) it has been proved. See the survey [Sm08] of Smyth for a summery of partial results regarding Lehmer's problem.

In modern language we use the *height* $h$ of an algebraic number to state Lehmer's problem. The height is a non-negative real valued function which behaves well under algebraic actions. This means that $h$ is invariant under galois-action, and we have $h(\alpha^d) = dh(\alpha)$ for all algebraic numbers $\alpha$ and all $d \in \mathbb{N}$. The height is one of the most powerful tools in Diophantine geometry. Using this notation we can reformulate Lehmer's problem in the following way, in which it is more common to speak of the Lehmer conjecture.

**Conjecture 1.** *There exists a positive constant $c$ such that for every $\alpha \in \overline{\mathbb{Q}}^*$ which is not a root of unity, we have $h(\alpha) \geq \frac{c}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$.*

This conjecture states that the Mahler measure of a non-cyclotomic polynomial cannot become arbitrary close to 1. For a rational number $\frac{a}{b}$, with coprime $a$ and $b$, the height is just $h(\frac{a}{b}) = \max\{\log|a|, \log|b|\}$. If we assume that $\frac{a}{b}$ is not in $\{-1, 0, 1\}$, then $h(\frac{a}{b})$ is not smaller than $\log 2$. The best general lower bound for a non root of unity $\alpha \in \overline{\mathbb{Q}}^*$ of degree $d \geq 2$ is due to Voutier ([Vo96]), shrinking the constant in a theorem of Dobrowolski ([Do79]). He gives the lower bound

$$h(\alpha) \geq \frac{1}{4d} \left( \frac{\log\log d}{\log d} \right)^3$$

which improves the constant $\frac{1}{1200}$ of Dobrowolski to $\frac{1}{4}$.

One of the most important properties of height functions is *Northcott's theorem*. It states that every set of points with bounded height and bounded degree is finite. Northcott's motivation for his theorem comes from the theory of algebraic dynamical systems. He proved this theorem in [No50] to show that any endomorphism $\varphi$ of an algebraic variety $X$ defined over a number field $K$ has only finitely many preperiodic points of bounded degree. A point $P \in X(\overline{K})$ is called *preperiodic* if the set of iterates

$$P,\ \varphi(P),\ \varphi(\varphi(P)),\ \dots$$

is finite.

As an example, we choose the doubling map [2] on an elliptic curve $E$ defined over $\mathbb{Q}$. Northcott's theorem implies that there are only finitely many points of bounded degree and bounded Néron-Tate height in $E(\overline{\mathbb{Q}})$. The preperiodic points of the map [2] are exactly the torsion points in $E(\overline{\mathbb{Q}})$, and the torsion points are exactly those points of height zero. In particular, there are only finitely many torsion points $P$ in $E(\overline{\mathbb{Q}})$ such that the degree $[\mathbb{Q}(P) : \mathbb{Q}]$ is at most $D$, for an arbitrary positive constant $D$.
In the last three decades the arithmetic side of dynamical systems has become its own part of mathematical research. Call and Silverman introduced a height function associated to a dynamical system in [CS93]. For a rational function $f \in \overline{\mathbb{Q}}(x)$ of degree at least 2, the associated dynamical height $\widehat{h}_f$ is uniquely determined by the properties $\deg(f)\widehat{h}_f = \widehat{h}_f \circ f$ and $|\widehat{h}_f - h| \leq C_f$, for a constant $C_f$. Vaguely speaking, we can say that this height behaves well under the dynamics of $f$ but still carries arithmetic information. The theorem of Call and Silverman is true in the much more general setting of any polarized algebraic dynamical system. They obtain these height functions in complete analogy to the construction of the Néron-Tate height on an elliptic curve. So it is not surprising that a dynamical height $\widehat{h}_f$ vanishes precisely at the preperiodic points of $f$.
With these height functions we can state a dynamical version of the Lehmer conjecture.

**Conjecture 2.** *For every rational function $f \in \overline{\mathbb{Q}}(x)$ of degree at least 2, there exists a positive constant $c_f$ such that for all $\alpha \in \overline{\mathbb{Q}}$ which are not preperiodic under $f$, we have $h(\alpha) \geq \frac{c_f}{[\mathbb{Q}(\alpha):\mathbb{Q}]}$.*

This conjecture implies Conjecture 1, as we have $\widehat{h}_{x^2} = h$. This can be easily deduced from the result of Call and Silverman explained above. Apart from special classes of rational functions $f$, there is not even a proof for a general lower bound of the type $c_f[\mathbb{Q}(\alpha) : \mathbb{Q}]^{-n}$, $n \in \mathbb{N}$, in Conjecture 2. One class of these special rational functions for which such a lower bound is known is given by Lattès maps. For a Lattès map $f$ there exists a positive constant $c_f$ such that we have

$$\widehat{h}_f(\alpha) \geq \frac{c_f}{[\mathbb{Q}(\alpha) : \mathbb{Q}]^3(\log[\mathbb{Q}(\alpha) : \mathbb{Q}])^2}$$

for all non-preperiodic $\alpha \in \overline{\mathbb{Q}}$ (see [Ma89], Corollary 1, and Lemma 1.23 below). These maps are associated to endomorphisms of an elliptic curve. If $E$ is an elliptic

curve defined over the number field $K$, $\Psi$ an endomorphism of $E$, and $\pi : E \to \mathbb{P}^1_K$ a finite covering, then the associated Lattès map is a rational function $f$ with the property $f \circ \pi = \pi \circ \Psi$. Lattès maps are named after Samuel Lattès, who studied these maps in [La18]. But Lattès was not the first to introduce this kind of maps. For an historical overview on Lattès maps, we refer to [Mi06], Sections 6 and 7.
Using Lattès maps, one can show that Conjecture 2 implies also an elliptic version of the Lehmer conjecture. We will explain this in Section 1 in detail.

As noticed above, there are classes of polynomials for which Lehmer's problem is solved. In analogy to this, we can search for classes of algebraic numbers satisfying the Lehmer conjecture. In a variation of this problem, one can search for lower bounds on the height of algebraic numbers, which do not depend on the degree of the algebraic number. We say that a field $L \subset \overline{\mathbb{Q}}$ has the *Bogomolov property* relative to the height $\widehat{h}_f$ if $\widehat{h}_f$ has a positive lower bound on the set of algebraic numbers $\alpha \in L$ with $\widehat{h}_f(\alpha) \neq 0$. This is a dynamical variation of a notion introduced by Bombieri and Zannier in [BZ01]. The name is given in analogy to the famous Bogomolov conjecture, yielding a lower bound of the Néron-Tate height on a certain set of algebraic points on an abelian variety (see [BG], Theorem 11.10.17).
By Northcott's theorem every number field has the Bogomolov property relative to every height function. Hence, interesting examples have infinite degrees over $\mathbb{Q}$.
Classically, we have $f = x^2$, and as noticed above $\widehat{h}_f = h$. Schinzel gave the first example of an infinite extension of the rationals with the Bogomolov property relative to $h$ in [Sch73], namely the maximal totally real field extension $\mathbb{Q}^{tr}$ of $\mathbb{Q}$. The next example came up almost 30 years after Schinzel's result. In 2000 Amoroso and Dvornicich proved that the maximal abelian field extension $\mathbb{Q}^{ab}$ of the rational numbers has the Bogomolov property relative to $h$ (see [AD00]). This result was generalized by Amoroso and Zannier ([AZ00], [AZ09]) to finite extensions of the maximal abelian field extension of any number field. In 2001, Bombieri and Zannier proved a $p$-adic version of Schinzels result in [BZ01]. They proved that for any rational prime $p$ the maximal totally $p$-adic field extension has the Bogomolov property relative to $h$; i.e. the maximal subfield $K$ of $\overline{\mathbb{Q}}$ such that $p$ splits completely in every subfield of $K$ of finite degree over $\mathbb{Q}$. Their result is even stronger, which we will explain in Section 2. Recently, Habegger gave in [Ha11] the example of a new field having the Bogomolov property relative to $h$. For an elliptic curve $E$ defined over $\mathbb{Q}$, let $E_{\text{tor}}$ denote the set of torsion points of $E$. Habbeger proved that there is a positive constant $c$ such that $h(\alpha)$ is either zero or $\geq c$ for all $\alpha \in \mathbb{Q}(E_{\text{tor}})$. This can be seen as an analogue of the result of Amoroso and Dvornicich, as the field $\mathbb{Q}^{ab}$ is generated by the torsion points of $\overline{\mathbb{Q}}^*$.
As there exists an elliptic version of the Lehmer conjecture, it is not surprising that there are examples of fields $L \subset \overline{\mathbb{Q}}$ such that the Néron-Tate height $\widehat{h}_E$ of an elliptic curve $E$ is bounded from below by a positive constant for all points $P \in E(L)$ with $\widehat{h}_E(P) \neq 0$. More generally, one can even assume $E$ to be an abelian variety and $\widehat{h}_E$ a canonical height on $E$ associated to an ample and even line bundle. The prescribed property of $L$ is again called Bogomolov property of $L$ relative to $\widehat{h}_E$. In this setting the field $L$ may depend on the elliptic curve - or the abelian variety - $E$.

Again the first example for such a field $L$ was $\mathbb{Q}^{tr}$. This was shown by Zhang in [Zh98], and this result is true for all abelian varieties $E$. If $E$ is an abelian variety defined over the number field $K$, then Baker and Silverman proved that the maximal abelian extension of $K$ has the Bogomolov property relative to the Néron-Tate height $\widehat{h}_E$ (see [BS04]). If $E$ is totally degenerate at a finite place $v$ of $K$, then $K^{nr,v}$, the maximal algebraic field extension of $K$ which is unramified at $v$, has the Bogomolov property relative to $\widehat{h}_E$. This result is due to Gubler (see [Gu07]) and will be discussed detailed in Chapter 3 of this thesis. The next two examples are only known to be true if $E$ is an elliptic curve. Baker and Petsche proved in [BP05] that any totally $p$-adic field extension of $\mathbb{Q}$, where $p$ is an odd rational prime, has the Bogomolov property relative to $\widehat{h}_E$ for all elliptic curves $E$. The result of Habegger is also true in the elliptic curve case; i.e. the field $\mathbb{Q}(E_{\mathrm{tor}})$ has the Bogomolov property relative to $\widehat{h}_E$ whenever $E$ is defined over the rational numbers (see [Ha11]).
In the dynamical case there are almost no examples of fields having the Bogomolov property relative to $\widehat{h}_f$, where $f$ is a rational function not of the form $x^d$. If $f$ is a Lattès map associated to an elliptic curve $E$, then one can use results of Baker and Petsche ([BP05]) on the Néron-Tate height of $E$ to deduce that $\mathbb{Q}^{tr}$ has the Bogomolov property relative to $\widehat{h}_f$ whenever $E$ is defined over $\mathbb{Q}^{tr}$, and that the maximal totally $p$-adic field has the Bogomolov property relative to $\widehat{h}_f$ for all rational primes $p \geq 3$. We can achieve these results, as the lower bounds for the Néron-Tate height of Baker and Petsche only depend on the $j$-invariant of the elliptic curve.
For an elliptic curve $E$ defined over a number field $K$ with non-archimedean absolute value $v$, and any $e \in \mathbb{N}$ denote by $M_e^E(v)$ the set of points $P \in E(\overline{\mathbb{Q}})$ such that the ramification index $e_{w|v}$ is bounded by $e$ for all extensions $w$ of $v$ to the field $K(P)$. One of the main results in this thesis is the following theorem.

**Theorem 1.** *Let $E$ be an elliptic curve defined over a number field $K$ with split-multiplicative reduction at a finite place $v$ on $K$. Then there are effective computable constants $c'$, $c'_T > 0$, depending on the degree of $K$, $e$, $v$ and the $j$-invariant of $E$, such that the Néron-Tate height $\widehat{h}_E(P) \geq c'$ for all $P \in M_e^E(v) \setminus E_{\mathrm{tor}}$, and such that there are less than $c'_T$ torsion points in $M_e^E(v)$.*

This has been proven in a non-effective version by Gubler in [Gu07] for abelian varieties which are totally degenerate at $v$. Theorem 1 first appears implicitly in Baker's paper [Ba03]. However, our argument will give explicit bounds $c'$ and $c'_T$ and it carries over to elliptic curves of bad reduction of any type in the case of $e = 1$. This leads to an example of a field $L$ such that $\widehat{h}_E(P) \geq c > 0$ for all non-torsion points $P \in E(L)$, but $L$ does not have the Bogomolov property relative to the canonical height $\widehat{h}_f$ of a Lattès map associated to $E$.
In complete analogy to the definition of $M_e^E(v)$ above we define $M_e(v)$ as the set of algebraic numbers $\alpha$ such that the ramification index $e_{w|v}$ is bounded by $e$ for all $w \mid v$ in $M_{K(\alpha)}$. Then we will obtain the following dynamical analogue of Theorem 1:

**Theorem 2.** *Let $E$ be an elliptic curve defined over a number field $K$ with split-multiplicative reduction at a finite place $v$ on $K$. Further let $f$ be a Lattès map* *associated to $E$. Then there are effective computable constants $c$, $c_P > 0$, depending on the degree of $K$, $e$, $v$ and the $j$-invariant of $E$, such that $\widehat{h}_f(\alpha) \geq c$ for all $\alpha \in M_e(v) \setminus \mathrm{PrePer}(f)$, and such that there are less than $c_P$ preperiodic points of $f$ in $M_e(v)$.*

In all examples of Bogomolov properties discussed above (including Theorem 2), the map $f$ was fixed and the task was to find fields, or more general subsets of $\overline{\mathbb{Q}}$, with the Bogomolov property relative to $\widehat{h}_f$. Conversely, we can fix a field $L$ and search for rational functions $f$ such that $L$ has the Bogomolov property relative to $\widehat{h}_f$. For $L = \mathbb{Q}^{tr}$ we can classify all these rational functions according to their Julia sets.

**Theorem 3.** *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree at least two. Then the following statements are equivalent:*

*i)* $\mathbb{Q}^{tr}$ *has the Bogomolov property relative to* $\widehat{h}_f$.

*ii) There is a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the Julia set of $\sigma(f)$ is not contained in the real line.*

*iii) The set* $\mathrm{PrePer}(f) \cap \mathbb{Q}^{tr}$ *is finite.*

*If $f$ is a polynomial, then these statements are equivalent to*

*iv)* $\mathrm{PrePer}(f) \nsubseteq \mathbb{Q}^{tr}$.

This result includes the Bogomolov property of $\mathbb{Q}^{tr}$ relative to $h$, as the Julia set of the map $x^2$ is just the unit circle.

The outline of this thesis is the following. Chapter 1 provides the basic results on heights and dynamical systems. One of the main results is an equidistribution theorem of Yuan which will be needed several times in this thesis. In particular, the proof of Theorem 3 relies heavily on this equidistribution theorem.
Chapter 2 consists of two parts. The first part is a summary of known results of Bogomolov properties relative to $h$, and results concerning lower bounds for the Néron-Tate height of an elliptic curve. In the second part of this chapter some of these results are transferred to a dynamical setting for Chebyshev polynomials or Lattès maps.
In Chapter 3 a non-effective version of Theorem 2 is proved, using Yuan's equidistribution theorem. This result is strengthened in the next chapter, but it was also the starting point of the author's research on this topic. At the end we will generalize a non-effective version of Theorem 1 to abelian varieties which are totally degenerate at $v$ and we will see that this also implies the non-effective version of Theorem 2.
As mentioned above, Chapter 4 provides proofs of Theorems 1 and 2, and variations of these results concerning changes of the reduction type of $E$ at $v$. The proof of Theorem 1 relies on discrete equidistribution results of the local heights on $E$. These results are due to Elkies for archimedean places and to Hindry and Silverman for non-archimedean places. Among other things, we will show that Theorems 1 and

2 are not true if we start with an absolute value $v$ at which $E$ has good reduction. This will follow from the criterion of Néron-Ogg-Shafarevich.
A proof of Theorem 3 is presented in Chapter 5. The main tools for this proof are Yuan's equidistribution theorem and a result of Eremenko and van Strien (see [EvS11]) on rational maps with Julia set lying in a circle on the Riemann sphere. This latter result states that if the Julia set of a rational function lies in the real line, then there exists a finite set of real intervals that is backward invariant under the action of this rational function. The first and the last section of this chapter provides detailed information on the behavior of $h$ on $\mathbb{Q}^{tr}$ and on finite extensions of this field. In particular, we will show how the Bogomolov property of $\mathbb{Q}^{tr}$ with respect to a polynomial $x^2 - c$ depends on the choice of parameter $c$. Moreover, we will see that the Bogomolov property relative to the classical height $h$ is not preserved under finite field extensions.

*Acknowledgment:* I would like to thank my advisor Walter Gubler for numerous answers, corrections and suggestions, Sinnou David for sharing his idea to prove Theorem 1 and Fabrizio Barroero, Sara Checcoli, Christian Christensen, Paul Fili, Philipp Habegger, Khoa Nguyen, Joseph Silverman, Adam Towsley, Tom Tucker, Emmanuel Ullmo and Martin Widmer for helpful discussions and remarks.
Moreover, I would like to thank the DFG Graduiertenkolleg GRK 1692 "Curvature, Cycles, and Cohomology" for financial support, and the Institute for computational and experimental research in mathematics (ICERM) in Providence for hospitality during February and March 2012.

# 1 Heights and dynamical systems

In this chapter we will prepare the main results of Chapters 3 and 5. The content of this chapter is standard, hence we will skip most of the proofs. For details we refer to the following books. See [Na], [Ne] and [La] for details on algebraic number theory and valuation theory. For information on the standard logarithmic-, the Weil- and the Néron-Tate height we refer to [BG]. Appendix A of this book contains all information on algebraic geometry we need. The standard references for elliptic curves are [Si09] and [Si94]. For details on the dynamics of rational functions on the Riemann sphere we refer to [Be] (see also [Mi] for general information on dynamical systems on the Riemann sphere). Our reference for the arithmetic side of dynamical systems is [Si07].
For the complete thesis we fix once and for all an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. More generally, the algebraic closure of any number field is always this fixed $\overline{\mathbb{Q}}$.

## 1.1 Heights

We will give a brief introduction to the theory of valuations on number fields. Mainly, this section is a collection of notations and we will skip all proofs.

**Definition.** An *absolute value* on a field $K$ is a multiplicative function $|.| : K \to \mathbb{R}_{\geq 0}$ such that

*i)* $|a| = 0 \Leftrightarrow a = 0$,

*ii)* (triangle inequality) $|a + b| \leq |a| + |b|$ for all $a, b \in K$.

Let $\mathcal{P}$ be the set of positive rational primes. We can write an arbitrary rational number $a$ as $\pm \prod_{p \in \mathcal{P}} p^{v_p(a)}$, with uniquely determined $v_p(a) \in \mathbb{Z}$. A complete set of pairwise not equivalent and non-trivial absolute values on $\mathbb{Q}$ is given by the set $M_{\mathbb{Q}} = \{|.|_p | p \in \mathcal{P} \cup \infty\}$. Here $|a|_\infty = \max\{a, -a\}$ is the standard absolute value and $|a|_p = p^{-v_p(a)}$ the $p$-adic absolute value for all $p \in \mathcal{P}$ (see [Ne], II Satz 3.7). Let $K$ be any number field and $|.|_v$ an absolute value on $K$. Then we can restrict $|.|_v$ to an absolute value on $\mathbb{Q}$. For any number field $K$ we define $M_K$ to be the complete set of pairwise not equivalent non-trivial absolute values such that the restriction of every element in $M_K$ to $\mathbb{Q}$ lies in $M_{\mathbb{Q}}$. An element in $M_K$ that restricts to $|.|_\infty$ is called *infinite* or *archimedean*, and an element in $M_K$ that restricts to $|.|_p$, $p \in \mathcal{P}$, is called *finite* or *non-archimedean*. For a non-archimedean absolute value $|.|_v$ we have a stronger inequality than the triangle inequality above. Namely, we have $|a + b|_v \leq \max\{|a|_v, |b|_v\}$ for all $a, b \in K$.
As the definition of the $v_p(a)$ indicates, there is a one to one correspondence between non-archimedean absolute values on $K$ and non-zero prime ideals in $\mathcal{O}_K$, the ring of integers of $K$. Every absolute value $v$ induces a *valuation* $v(.) = -\log |.|_v$. If $L/K$ is a finite field extension, then we write $w \mid v$, for $w \in M_L$ and $v \in M_K$, if and only if the restriction of $w$ to $K$ is $v$. Hence, the archimedean $v \in M_K$ are exactly those with $w \mid \infty$.

For a number field $K$ with absolute value $|.|_v$, we denote the *completion* of $K$ with respect to $|.|_v$ by $K_v$. Furthermore we set $k^\circ(v) := \{a \in K_v | |a|_v \leq 1\}$. This is a local ring with unique maximal ideal $k^{\circ\circ}(v) = \{a \in K_v | |a|_v < 1\}$. Now, the *residue field* of $K_v$ is defined as $k(v) := k^\circ(v)/k^{\circ\circ}(v)$.
Let again $L/K$ be a finite extension of the number field $K$, and $w \in M_L$, $v \in M_K$, with $w \mid v$. The *local degree of* $L/K$ *at* $w$ is the degree $[L_w : K_v]$. If $K$ is equal to $\mathbb{Q}$ we simply speak of the *local degree of* $w$ and denote it by $d_w$. From now on assume that $v \nmid \infty$. The *residue degree of* $w$ *over* $v$ is $f_{w|v} = [k(w) : k(v)]$. The (multiplicative) group $|K^*|_v = \{|a|_v | a \in K^*\}$ is called *value group of* $K$ *with respect to* $v$. It is a subgroup of $|L^*|_w$ of finite index. This index is denoted by $e_{w|v}$ and is called *ramification index of* $w$ *over* $v$. The extension $L/K$ is *unramified at* $w$ if $e_{w|v} = 1$, and *unramified at* $v$ if it is unramified at every $w \mid v$.
If $v \mid p$, we have $v(K^*) = \frac{\log(p)}{e_{v|p}}\mathbb{Z}$. Sometimes it is more convenient to use the *normalized valuation* $\mathrm{ord}_v(.) = \frac{e_{v|p}}{\log(p)} v(.)$.

**Lemma 1.1.** *Let $F/L/K$ be extensions of number fields, then we have*

*i) (product formula)* $\prod_{v \in M_K} |a|_v^{d_v} = 1$ *for all* $a \in K^*$,

*ii)* $[L : K] = \sum_{w|v} [L_w : K_v]$ *for all* $v \in M_K$,

*iii)* $[L_w : K_v] = e_{w|v} f_{w|v}$,

*iv)* $f_{u|v} = f_{u|w} f_{w|v}$ *and* $e_{u|v} = e_{u|w} e_{w|v}$ *for all* $u \in M_F$, $w \in M_L$, $v \in M_K$, *with* $u \mid w$ *and* $w \mid v$.

**Proof:** See [Ne], Chapter II § 6 and Chapter III § 1. □

Now we are prepared to define the standard logarithmic height on the algebraic numbers.

**Definition.** Let $\alpha$ be an arbitrary algebraic number in $\overline{\mathbb{Q}}$ and let $K$ be any number field containing $\alpha$. Then the *standard logarithmic height* of $\alpha$ is

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \max\{\log |\alpha|_v, 0\} \quad .$$

Lemma 1.1 *ii*) implies that this definition is well defined; i.e. independent of the choice of the number field $K$. Another very useful way to calculate the height $h$ is given by *Jensen's formula* ([BG], Proposition 1.6.5).

**Theorem 1.2.** *Let $\alpha$ be an arbitrary algebraic number in $\overline{\mathbb{Q}}$ with $d = \deg(\alpha)$. If $P(x) = \sum_{i=0}^d a_i x^i = a_d \prod_{i=1}^d (x - \alpha_i) \in \mathbb{Z}[x]$ is the minimal polynomial of $\alpha$, then we have*

$$h(\alpha) = \frac{1}{d} \left( \log |a_d| + \sum_{i=1}^{d} \max\{\log |\alpha_i|, 0\} \right) \quad .$$

**Proposition 1.3.** *Let $\boldsymbol{\mu}$ be the set of roots of unity. The height $h$ has the following important properties:*

*i)* $h(\alpha) \geq 0$ *for all* $\alpha \in \overline{\mathbb{Q}}$,

*ii)* $h(\alpha^n) = nh(\alpha)$ *for all* $\alpha \in \overline{\mathbb{Q}}^*$ *and all* $n \in \mathbb{Z}$,

*iii) (Kronecker's theorem)* $h(\alpha) = 0 \Leftrightarrow \alpha \in \{0\} \cup \boldsymbol{\mu}$,

*iv) (Northcott's theorem) The set* $\{\alpha \in \overline{\mathbb{Q}} | h(\alpha) \leq A, \deg(\alpha) \leq B\}$ *is finite for all constants* $A, B \in \mathbb{R}_+$.

**Proof:** The first two statements follow immediately from the definition of $h$. Proofs of Kronecker's and Northcott's theorem can be found in [BG], Theorem 1.5.9 and Theorem 1.6.8. □

We see, using the previous proposition, that the height of the elements $2^{1/n}$, $n \in \mathbb{N}$, is positive and tends to zero as $n$ increases. Hence, the best uniform lower bound for the height of an element in $\overline{\mathbb{Q}}^* \setminus \boldsymbol{\mu}$ must depend at least on the degree of the algebraic number.

**Conjecture 1.4** (Lehmer conjecture)**.** *There exists a positive constant $c$ such that* $h(\alpha) \geq \frac{c}{\deg(\alpha)}$ *for all* $\alpha \in \overline{\mathbb{Q}}^* \setminus \boldsymbol{\mu}$.

As a *strong Lehmer Conjecture* we can replace the unspecific $c$ by the logarithm of the Lehmer constant which is the largest real root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. This logarithm is approximately $0,162357612\ldots$.
Of course, $\overline{\mathbb{Q}}$ is not the only algebraic structure which is equipped with a height function. In a similar way as above we can define the height on $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ for all $n \in \mathbb{N}$. For arbitrary $\boldsymbol{\alpha} = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$ we have

$$h(\boldsymbol{\alpha}) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \max_i \{\log |\alpha_i|_v\} \quad , \tag{1.1}$$

where $K$ is any number field with $\alpha_i \in K$ for all $i \in \{0, \ldots, n\}$. This definition is again well defined by Lemma 1.1.
Height functions are also defined on algebraic varieties, as we will see in a moment. The main property of all these functions is Northcott's theorem which allows to count points of bounded height and bounded degree. The next theorem can be used to define the canonical height, also called *Néron-Tate height*, on an elliptic curve.

**Theorem 1.5.** *Let $E$ be an elliptic curve defined over the number field $K$ and let $f \in K(E)$ be a non-constant even function; i.e. $f = f \circ [-1]$. Then, for all $P \in E(\overline{K})$ the limit*

$$\widehat{h}_E(P) = \frac{1}{\deg(f)} \lim_{n \to \infty} 4^{-n} h(f([2]^n P)$$

*exists and is independent of the choice on $f$. Moreover, the function $\widehat{h}_E$ is the unique function such that for all $P \in E(\overline{K})$ we have* *i)* $\widehat{h}_E([m]P) = m^2\widehat{h}_E(P)$ *for one (all)* $m \in \mathbb{Z}$*, with* $|m| \geq 2$*,*

*ii)* $\deg(f)\widehat{h}_E = h \circ f + O(1)$*, for one (all) even function* $f \in K(E)$*, with* $\deg f \geq 2$*.*

For a proof and additional results we refer to [Si09], Chapter VIII.9. Especially we find that all statements of Proposition 1.3 can be translated to the Néron-Tate height in the obvious way. For the analogue of Kronecker's theorem one has to replace *roots of unity* by *torsion points*. Thus, we can recall the elliptic Lehmer conjecture.

**Conjecture 1.6** (Elliptic Lehmer conjecture)**.** *Let* $E$ *be an elliptic curve defined over a number field* $K$*. There exists a positive constant* $c$ *such that*

$$\widehat{h}_E(P) \geq \frac{c}{\deg(P)} \text{ for all } P \in E(\overline{\mathbb{Q}}) \setminus E_{\mathrm{tor}} \quad .$$

*Here* $\deg(P)$ *is the smallest degree of a number field over which* $P$ *is defined.*

**Remark 1.7.** More generally, let $A$ be an abelian variety defined over a number field $K$ and let $L$ be an even and ample line bundle on $A$. Then there exists a canonical non-negative height function $\widehat{h}_L : A \to \mathbb{R}$. For a construction and basic results we refer to [BG], Section 9.2. If $A$ is an elliptic curve we choose a Weierstrass equation $A : y^2 = x^3 + ax + b$ and denote the projection on the $x$-coordinate by $\pi$. By the addition law on an elliptic curve (see [BG], Proposition 8.3.8) we have $[-1](x, y) = (x, -y)$. Hence, $\pi$ is an even function and $L := \pi^*\mathcal{O}(1)$ is an even and ample line bundle on $A$. Notice that we have $\widehat{h}_L = 2\widehat{h}_A$, where $\widehat{h}_A$ is the Néron-Tate height from Theorem 1.5. To avoid confusion we remark that some authors refer to the height $\widehat{h}_L$ as the Néron-Tate height of an elliptic curve.

## 1.2 Dynamical systems

Let $S$ be a set and $f$ a self-map of $S$. The iteration of $f$ yields a dynamical system on $S$. We set $f^{(0)} = \mathrm{id}$ and $f^{(n)} = f \circ f^{(n-1)}$ for all $n \in \mathbb{N}$. The *(forward) orbit* of an element $a \in S$ under $f$ is given by the set $\{f^{(n)}(a)\}_{n \in \mathbb{N}_0}$. Moreover, for all $n \in \mathbb{N}$ we set $f^{-n}(a) = \{b \in S | f^{(n)}(b) = a\}$, and define the *backward orbit* of $a \in S$ as the set $\cup_{n\in\mathbb{N}} f^{-n}(a)$.

A classical aim in the theory of dynamical systems is to classify the points of $S$ according to the behavior of their orbits. A point $a \in S$ is called *periodic point of period* $n$ if $a = f^{(n)}(a)$. Periodic points of period one are called *fixed points*. If the orbit of $a$ is a finite set, we say that $a$ is a *preperiodic point* of $f$. The set of all preperiodic points of $f$ in $S$ is denoted by $\mathrm{PrePer}(f)$ and the subset of periodic points by $\mathrm{Per}(f)$. Notice that some authors exclude periodic points from the set of preperiodic points.

From now on we will reduce this setting to the case where $S$ is the Riemann sphere which we identify with $\mathbb{C} \cup \{\infty\}$, and $f$ is a rational function. On the Riemann sphere, we will always use the complex topology which is induced by the *chordal metric* $\rho$.

**Definition.** Let $f \in \mathbb{C}(x)$ be a rational function and $f'$ its derivative. The zeros of $f'$ are called *critical points of* $f$. If $\alpha \in \mathrm{Per}(f)$ such that $f^{(n)}(\alpha) = \alpha$, with $n \in \mathbb{N}$ minimal with this property, then the *multiplier* of $f$ at $\alpha$ is $\lambda_f(\alpha) := (f^{(n)})'(\alpha)$. Furthermore, $\alpha$ is called

$$\begin{array}{cl} \textit{superattracting} & \text{if } \lambda_f(\alpha) = 0, \\ \textit{attracting} & \text{if } |\lambda_f(\alpha)| < 1, \\ \textit{neutral} & \text{if } |\lambda_f(\alpha)| = 1, \\ \textit{repelling} & \text{if } |\lambda_f(\alpha)| > 1. \end{array}$$

A very important classification of points under the dynamics of $f$ is whether $f$ acts ´*stable*' or ´*chaotically*' on a small neighborhood of the point.

**Definition.** Let $f$ be a self map of the Riemann sphere. The *Fatou set* $F(f)$ of $f$ is the maximal open subset of the Riemann sphere, satisfying the condition: For all $\alpha \in F(f)$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho(\alpha, \beta) < \delta \Rightarrow \rho(f^{(n)}(\alpha), f^{(n)}(\beta)) < \varepsilon$$

for all $n \in \mathbb{N}$. The *Julia set* $J(f)$ of $f$ is the complement of $F(f)$.

Most of the rational functions considered in this thesis have very nice Julia sets. For example the Julia set of the map $x^2$ is just the unit circle, $J(x^2 - 2) = [-2, 2]$, and $J(\frac{x^4-8x}{4x^3+4})$ is the Riemann sphere. However, in most cases Julia sets are fractals and highly complicated. In Section 1.3 we will explain the special role of the examples above.

Some of the most important properties of the Julia set of a rational function are the following facts which can be found in [Be], Theorem 4.2.1, Theorem 3.2.4, Theorem 5.7.1 and Theorem 6.9.2.

**Facts 1.8.** *Let $f \in \mathbb{C}(x)$ be a rational function of degree at least two. Then we have*

a) *$J(f)$ is not empty,*

b) *$J(f)$ is completely invariant, i.e. $f(J(f)) = f^{-1}(J(f)) = J(f)$,*

c) *there are no isolated points in $J(f)$,*

d) *$J(f)$ is the closure of the repelling periodic points of $f$.*

Completely analog to the construction of the Néron-Tate height, Call and Silverman introduced a canonical height associated to the dynamical system defined by a rational function.

**Theorem 1.9.** *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree $\geq 2$. Then for all $\alpha \in \overline{\mathbb{Q}}$ the limit*

$$\widehat{h}_f(\alpha) = \lim_{n \to \infty} \frac{1}{\deg(f)^n} h(f^{(n)}(\alpha))$$

*exists. The height function $\widehat{h}_f$ is called the canonical height related to $f$, and it is the unique function fom $\overline{\mathbb{Q}}$ to $\mathbb{R}$ such that for all $\alpha \in \overline{\mathbb{Q}}$ we have*

$$i)\ \widehat{h}_f(f(\alpha)) = \deg(f)\widehat{h}_f(\alpha) \quad \textit{and} \quad ii)\ \widehat{h}_f = h + O(1).$$

For a proof and additional results we refer to [Si07], Section 3.4. Again, all statements of Proposition 1.3 can be translated to the canonical heights $\widehat{h}_f$ in the obvious way. Here, for the analogue of Kronecker's theorem one has to replace *roots of unity* by *preperiodic points*.

**Lemma 1.10.** *Let $f, g \in \overline{\mathbb{Q}}(x)$ be commuting rational functions of degree at least two. Then we have $\widehat{h}_f = \widehat{h}_g$.*

**Proof:** Take an arbitrary $\alpha \in \overline{\mathbb{Q}}$. Then for all $n \in \mathbb{N}$ we have

$$\widehat{h}_f(g(\alpha)) = \deg(f)^{-n}\widehat{h}_f(f^{(n)}(g(\alpha))) = \deg(f)^{-n}\widehat{h}_f(g(f^{(n)}(\alpha))) \quad .$$

Using the fact that $h(g(f^{(n)}(\alpha))) = \deg(g)h(f^{(n)}(\alpha)) + O(1)$ (see [Si07], Theorem 3.11), we see that this is equal to

$$\frac{\deg(g)}{\deg(f)^n}\left(h(f^{(n)}(\alpha)) + O(1)\right) = \deg(g)\left(\widehat{h}_f(\alpha) + \deg(f)^{-n}O(1)\right) \quad .$$

Now we take the limes $n \to \infty$ to obtain $\widehat{h}_f(g(\alpha)) = \deg(g)\widehat{h}_f(\alpha)$. With this relation the equality of $\widehat{h}_f$ and $\widehat{h}_g$ follows from the theorem above. □

**Definition.** Let $K$ be a field, and let $f \in K(x)$ be a rational function. A rational function $\varphi \in K(x)$ is called a *linear conjugate* of $f$ if there is a Möbius transformation $g \in K(x)$ such that $\varphi = g^{-1} \circ f \circ g$. Now let $K$ be a number field. We can extend every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ uniquely to an endomorphism of $\overline{\mathbb{Q}}(x)$, by setting $\sigma(x) = x$. A map $\sigma(f)$ is called *galois conjugate* of $f$.

**Lemma 1.11.** *Take a rational function $f \in \overline{\mathbb{Q}}(x)$ of degree $\geq 2$. Let $g^{-1} \circ f \circ g$ be a linear conjugate of $f$ and let $\sigma(f)$ be a galois conjugate of $f$. Then we have*

*i)* $\widehat{h}_{g^{-1}\circ f\circ g} \circ g^{-1} = \widehat{h}_f$,

*ii)* $\widehat{h}_{\sigma(f)} \circ \sigma = \widehat{h}_f$.

**Proof:** This follows immediately from Theorem 1.9 and the facts $\deg(g^{-1} \circ f \circ g) = \deg(f) = \deg(\sigma(f))$, $h \circ g = h + O(1)$ and $h \circ \sigma = h$. □

Now we are able to formulate a dynamical version of the Lehmer conjecture.

**Conjecture 1.12** (Dynamical Lehmer conjecture)**.** *For every rational function $f \in \overline{\mathbb{Q}}(x)$ with $\deg(f) \geq 2$ there exists a positive constant $c(f)$ such that*

$$h(\alpha) \geq \frac{c(f)}{\deg(\alpha)} \textit{ for all } \alpha \in \overline{\mathbb{Q}} \setminus \mathrm{PrePer}(f) \quad .$$

We will need the following Theorem due to Freire, Lopes, Mañé (see [FLM83]) and independently Lyubich.

**Theorem 1.13.** *Let $f \in \mathbb{C}(x)$ be a rational function of degree $\geq 2$. There exists a unique probability measure $\mu_f$ on $\mathbb{C}$ such that $\mu_f$ is $f$-invariant; i.e. $f^*\mu_f = \deg(f)\mu_f$ and $f_*\mu_f = \mu_f$. The support of $\mu_f$ is equal to the Julia set of $f$.*

**1.14.** We will very briefly introduce some notations from the theory of algebraic dynamical systems. For detailed information and proofs we refer to the expository article [Yu12] and the references therein.
Let $K$ be a number field, and let $X$ be a smooth projective variety of dimension $n$ with a morphism $f : X \to X$, both defined over $K$. Moreover, let $L$ be an ample line bundle on $X$. The triple $(X, L, f)$ is called *(polarized) algebraic dynamical system* if we have $f^*L \cong L^{\otimes q}$, for $q \geq 2$. We need to fix a line bundle $L$ to associate a canonical height and a canonical measure to the algebraic dynamical system. The canonical height $\widehat{h}_{X,L,f}$ for $(X, L, f)$ is uniquely determined by the properties given in Theorem 1.9. Namely,

$$\widehat{h}_{X,L,f}(f(P)) = q\widehat{h}_{X,L,f}(P) \ \forall P \in X(\overline{K}) \quad \text{and} \quad \widehat{h}_{X,L,f} = h_L + O(1),$$

where $h_L$ is any Weil height on $X$ (see [CS93]).
For a fixed non-archimedean $v \in M_K$ we write $\mathbb{C}_v$ to denote the completion of $\overline{K_v}$. This is a complete and algebraically closed field (see [BGR], Proposition 3.4.3). We consider $(X, L, f)$ as an algebraic dynamical system defined over $\mathbb{C}_v$. Due to Zhang ([Zh95]) there is a *canonical $f$-invariant $\mathbb{C}_v$-metric* $\|.\|_{f,v}$ on $L$; i.e.

$$f^*\|.\|_{f,v} = \|.\|_{f,v}^q \quad .$$

Let $c_1(L, \|.\|_{f,v})^{\wedge n}$ be the Chambert-Loir measure on the Berkovich space $X_v^{an}$ associated to $X/\mathbb{C}_v$ (see [Ch06] for a construction of this measure). Then, the measure $\mu_{f,v} := \deg_L(X)^{-1}c_1(L, \|.\|_{f,v})^{\wedge n}$ is the canonical probability measure satisfying

$$f^*\mu_{f,v} = q^n\mu_{f,v} \quad \text{and} \quad f_*\mu_{f,v} = \mu_{f,v} \quad .$$

We call $\mu_{f,v}$ the *$v$-adic canonical measure associated to* $(X, L, f)$. For the theory of Berkovich spaces we refer to [Ber] and [Te10].
For an archimedean place $v \in M_K$ we set $X_v^{an} := X(\mathbb{C})$ as a complex manifold. The construction of the canonical $f$-invariant measure $\mu_{f,v}$ on $X_v^{an}$ can be found in [Zh06], Chapter 3. We refer to the same reference for a construction of $\mu_{f,v}$ for arbitrary $v \in M_K$ using Tate's limit process.

Let $P \in X(\overline{\mathbb{Q}})$ be arbitrary and let $\delta_P$ be the Dirac measure at $P$. We denote the set $\{\sigma(P)|\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)\}$ of $K$-galois conjugates of $P$ by $G_K(P)$, and define the probability measure

$$\overline{\delta_P} := |G_K(P)|^{-1} \sum_{P' \in G_K(P)} \delta_{P'} \quad .$$

Now we can formulate Yuan's equidistribution theorem (see [Yu08], Theorem 3.7).

**Theorem 1.15** (Yuan). *Let $(X, L, f)$ be a polarized algebraic dynamical system defined over the number field $K$, and let $\{P_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise distinct points in $X(\overline{\mathbb{Q}})$ such that*

*i)* $\widehat{h}_f(P_i) \to 0$, *as* $i \to \infty$,

*ii) every infinite subsequence of $\{P_i\}_{i\in\mathbb{N}}$ is Zariski dense in $X$.*

*For any $v \in M_K$ the measures $\overline{\delta_i} := \overline{\delta_{P_i}}$ converge weakly to $\mu_{f,v}$. This means that for every continuous function $\varphi : X_v^{an} \to \mathbb{C}$ we have*

$$\int_{X_v^{an}} \varphi(x)\overline{\delta_i} = |G_K(P_i)|^{-1} \sum_{P_i' \in G_K(P_i)} \varphi(P_i) \to \int_{X_v^{an}} \varphi(x)\mu_{f,v} \quad ,$$

*as $i \to \infty$.*

Of course, the second requirement on the sequence $\{P_i\}_{i\in\mathbb{N}}$ in the above theorem is always true if $X = \mathbb{P}^1$.

**Remark 1.16.** Let $K$ be a number field and $f \in K(x)$ a rational function of degree at least 2. The canonical height associated to the algebraic dynamical system $(\mathbb{P}^1_K, \mathcal{O}(1), f)$ is the function $\widehat{h}_f$ from Theorem 1.9. The map $f$ extends uniquely to a continuous function on $\mathbb{P}^1(\mathbb{C}_v)$, $v \in M_K$. For a non-archimedean $v \in M_K$ we can define a $v$-adic chordal metric $\rho_v$ on $\mathbb{P}^1_K(\mathbb{C}_v)$, and define the *$v$-adic Julia set* $J_v(f)$ and *$v$-adic Fatou set* $F_v(f)$ of $f$ in complete analogy to the Definition of the Julia set of a rational function (we refer to [Si07], Chapter 5, for details). However, the space $\mathbb{P}^1(\mathbb{C}_v)$ has very unpleasant topological properties and it is more convenient to work in the associated Berkovich space $(\mathbb{P}^1)_v^{an}$. Following [BR] we define:

**Definition.** We use the notation from above. The *Berkovich Julia set* $J_v^{\mathcal{B}}(f)$ of $f$ is the support of the canonical measure $\mu_{f,v}$.

We see that $J_v^{\mathcal{B}}(f)$, in contrast to $J_v(f)$, is always non-empty. Moreover, the intersection $J_v^{\mathcal{B}}(f) \cap \mathbb{P}^1(\mathbb{C}_v)$ is exactly the $v$-adic Julia set $J_v(f)$. Of course, we work with the unique continuous extension of $f$ to $(\mathbb{P}^1)_v^{an}$. All this can be found in [BR], Chapter 10.

## 1.3 Dynamics associated with groups

In the last section we gave three examples of maps with a special Julia set. Namely, $x^2$, $x^2 - 2$ and $\frac{x^4-8x}{4x^3+4}$. The dynamics of each of these maps is defined by a group operation. For the rest of this section let $K$ be any field of characteristic 0.
Let $d \geq 2$ be an integer. The map $x^d : K \to K$ is defined by the $d$-th time multiplication in $K$. Hence, it is an endomorphism of the multiplicative group $K^*$. We call the map $x^d$ the *$d$-th power map.* Iterating $x^d$ is extremely easy, as the $n$-th iterate is just $x^{d^n}$. Moreover, it is not hard to check that the Julia set of a $d$-th power map, $d \geq 2$, is the unit circle.
The second class of rational functions in $K(x)$ again comes from the multiplication on $K$.

**Definition.** Set $\varphi(x) = x + x^{-1}$. The *d-th Chebyshev polynomial* is the unique map such that the diagram

$$\begin{tikzcd} K^* \arrow[r, "x \mapsto x^d"] \arrow[d, "\varphi"'] & K^* \arrow[d, "\varphi"] \\ K \arrow[r, "T_d"] & K \end{tikzcd} \tag{1.2}$$

commutes.

**1.17.** The Chebyshev polynomials are named after the Russian mathematician Чебышёв. We choose the transliteration *Chebyshev* which is common but far from unique.
For all $z \in K$ the representation $z = x + x^{-1}$ is unique. Hence, such a map $T_d$ exists and is unique. In fact, $T_d$ is a monic integer polynomial of degree $d$. Chebyshev polynomials are given by the recursive formula $T_1(x) = x$, $T_2(x) = x^2 - 2$ and $T_d(x) = xT_{d-1}(x) - T_{d-2}(x)$, for $d \geq 3$. See [Si07], Proposition 6.6, for proofs of these properties. The iteration of the $d$-th power map gives also rise to a simple iteration of $T_d$. We have $T_d^{(n)}(x) = T_{d^n}(x)$. Using the fact that $\varphi$ maps the unit circle onto the interval $[-2, 2]$ one can prove that the Julia set of every $T_d(x)$, $d \geq 2$, is $[-2, 2]$ (see also Remark 5.7).

The last class of rational functions we introduce in this chapter are Lattès maps. The construction of Lattès maps is quite similar to the construction of Chebyshev polynomials, although the algebraic structure is given by the addition on an elliptic curve.

**Definition.** Let $E$ be an elliptic curve defined over $K$ with given self-morphism $\Psi \neq [0]$ of degree greater than one, and let $\Gamma$ be a non-trivial subgroup of $\mathrm{Aut}(E)$. A rational function $f$ is called *Lattès map associated to $E$* if the diagram

$$\begin{tikzcd} E \arrow[r, "\Psi"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{P}^1_K \arrow[r, "f"] & \mathbb{P}^1_K \end{tikzcd} \tag{1.3}$$

commutes. Here, $\pi$ factors as $\pi : E \to {}^{E}/_{\Gamma} \xrightarrow{\sim} \mathbb{P}^1_K$. Notice that the quotient curve ${}^{E}/_{\Gamma}$ is indeed isomorphic to $\mathbb{P}^1_K$. If it is necessary to be more precise, we call such a Lattès map *associated to $E$, $\pi$ and $\Psi$*.

We talk of a Lattès map over a field $K$, if it is associated to an elliptic curve defined over $K$. Write $E : y^2 = x^3 + Ax + B$, then, up to change of coordinates on $\mathbb{P}^1$, there are the following possibilities for $\pi$ (see [Si07], Proposition 6.37):

$$\pi(x, y) = \begin{cases} x & , \text{ in any case} \\ x^2 & , \text{ if } j_E = 1728 \\ x^3 & , \text{ if } j_E = 0 \\ y & , \text{ if } j_E = 0 \end{cases} \tag{1.4}$$

This means that any $\pi$ in the diagram (1.3) is of the form $g \circ \pi'$, with $\pi'$ as above and $g \in K(x)$ of degree one.

**Remark 1.18.** *a*) Let $E$ be an elliptic curve defined over $K$ with homomorphism $\Psi : E \to E$ of degree $\geq 2$, and let $\Gamma$ be any non-trivial subgroup of $\mathrm{Aut}(E)$. Then there always exists a Lattès map $f$ associated to $E$, $\Psi$ and $\pi$, for any $\pi$ that factors as $\pi : E \to {}^{E}/_{\Gamma} \xrightarrow{\sim} \mathbb{P}^1_K$.
*b*) One can define Lattès maps by a commutative diagram (1.3), where $\pi$ is allowed to be any finite covering (as we have done in the introduction). Then $\pi$ might have arbitrary large degree. However, both definitions yield exactly the same class of rational functions. See [Mi06], Theorem 3.1, for the complex case and use the Lefschetz principle for the case of an arbitrary field of characteristic zero.

**1.19.** With the notation from (1.3) we have $\pi(E_{\mathrm{tors}}) = \mathrm{PrePer}(f)$ (see [Si07], Proposition 6.44, or Lemma 1.23 below). As all critical points of $f$ are preperiodic (see [Si07], Proposition 6.45, for a much stronger result) [Be], Theorem 4.3.1, implies that the Julia set of a Lattès map is the Riemann sphere. A nice article on these maps defined over $\mathbb{C}$, including historical information, is [Mi06].

**Example 1.20.** Take the elliptic curve $E : y^2 = x^3+1$ defined over $\mathbb{Q}$. The addition law for elliptic curves (see [BG], Proposition 8.3.8) yields

$$[2](x,y) = \left( \frac{x^4 - 8x}{4x^3+4}, \frac{x^6 + 20x^3 - 8}{8y^3} \right) \quad ,$$

for all $(x, y) \in E$ with $y \neq 0$, and $[2](x, y) = O$ if $y = 0$. The Lattès map $f$ associated to $E$, $\pi(x, y) = x$ and $[2]$ is determined by $f(\pi(x, y)) = \pi([2](x, y))$. Hence, we have $f(x) = \frac{x^4-8x}{4x^3+4}$.

**Proposition 1.21.** *Let $f$ be any Lattès map over $\overline{\mathbb{Q}}$, with $\deg(f) \geq 2$. There is a Lattès map $g \in \overline{\mathbb{Q}}(x)$ such that $\widehat{h}_f = \widehat{h}_g$, and $g \circ \pi = \pi \circ [m]$, for an integer $m \geq 2$.*

**Proof:** Any morphism $\Psi : E \to E$ of degree $\geq 2$ is of the form $\Psi = \tau_T \circ \Phi$, where $\tau_T$ is the translation by $T \in E(\overline{\mathbb{Q}})$ and $\Phi$ is an isogeny. As we assume that there exists a Lattès map associated to $E$ and $\Psi$, we know that $T$ is a torsion point (see [Si07], Corollary 6.58). Let $m-1$ be the order of $T$. Then, for every $P \in E(\overline{\mathbb{Q}})$ we have

$$[m] \circ \Psi(P) = [m]\Phi(P) + [m]T = \Phi([m]P) + T = \Psi \circ [m](P) \quad .$$

According to Remark 1.18 *a*) there exists a Lattès map $g$ associated to $E$, $\pi$ and $[m]$. As $\Psi$ and $[m]$ commute, the associated Lattès maps $f$ and $g$ commute also. By Lemma 1.10 we get $\widehat{h}_f = \widehat{h}_g$. $\square$

It is trivial to see that $\widehat{h}_{x^d}$ is just the standard logarithmic height $h$, for every $d \geq 2$. Now let $E$ and $f$ be as in (1.3). Looking at the diagrams (1.2) and (1.3), it is not surprising that the heights $h$ and $\widehat{h}_{T_d}$, respectively $\widehat{h}_E$ and $\widehat{h}_f$, are strongly related. The next two lemmas make these relations explicit.

**Lemma 1.22.** *Let $T_d(x) \in \mathbb{Z}[x]$, $d \geq 2$, be a Chebyshev polynomial. For all $z \in \overline{\mathbb{Q}}^*$ we have $\widehat{h}_{T_d}(z + z^{-1}) = 2h(z)$.*

**Proof:** As in (1.2) we define $\varphi(x) = x + x^{-1}$. We have to check that $\frac{1}{2}\widehat{h}_{T_d} \circ \varphi$ fulfills the two conditions given in Theorem 1.9 for the canonical height $\widehat{h}_{x^d} = h$. Using the commutativity of (1.2) we get

$$\frac{1}{2}\widehat{h}_{T_d}(\varphi(z^d)) = \frac{1}{2}\widehat{h}_{T_d}(T_d(\varphi(z))) = d\frac{1}{2}\widehat{h}_{T_d}(\varphi(z)) \quad .$$

As $\varphi$ has degree two, we also have $\frac{1}{2}\widehat{h}_{T_d} \circ \varphi = \frac{1}{2}h \circ \varphi + O(1) = h + O(1)$. □

**Lemma 1.23.** *Let $K$ be a subfield of $\overline{\mathbb{Q}}$, $E$ an elliptic curve defined over $K$ and $f$ a Lattès map associated to $E$ with diagram (1.3). Then we have*

$$\widehat{h}_f \circ \pi = \deg(\pi)\widehat{h}_E \quad .$$

**Proof:** If $\pi = g \circ \pi'$ with $g \in K(x)$, $\deg(g) = 1$, and $\pi'$ as in (1.4), then the equation

$$g^{-1} \circ f \circ g \circ \pi' = g^{-1} \circ f \circ \pi = g^{-1} \circ \pi \circ \Psi = \pi' \circ \Psi \tag{1.5}$$

shows that $g^{-1} \circ f \circ g$ is a Lattès map associated to $E$ and $\pi'$. Using Lemma 1.11 $i)$ we get $\widehat{h}_f \circ \pi = \widehat{h}_{g^{-1} \circ f \circ g} \circ \pi'$. Hence we may assume that $\pi$ is given as in (1.4). By Lemma 1.21 we can also assume that $\Psi = [m]$ in (1.3), for an integer $m \geq 2$. Then we know that $\deg(f) = \deg([m]) = m^2$. We will prove that $\deg(\pi)^{-1}\widehat{h}_f \circ \pi$ has the defining properties of $\widehat{h}_E$ given in Theorem 1.5. For any $P \in E(\overline{\mathbb{Q}})$ we have

$$\deg(\pi)^{-1}\widehat{h}_f(\pi([m]P)) = \deg(\pi)^{-1}\widehat{h}_f(f(\pi(P)) = m^2 \deg(\pi)^{-1}\widehat{h}_f(\pi(P)) \quad .$$

Thus property $i)$ holds. In all cases for $\pi$ the function $\pi^2$ is even. As above we use Theorem 1.9 and (1.3) to see

$$2\widehat{h}_f(\pi(P)) = 2(h(\pi(P)) + O(1)) = h(\pi^2(P)) + O(1) \quad ,$$

which proves property $ii)$ in Theorem 1.5 for the map $\pi^2$. □

## 1.4 Helpful calculations

We will use this section to state three technical lemmas. These results will be used in Section 3.

**Lemma 1.24.** *Let $p$ and $q$ be different rational primes. For all $n \in \mathbb{N}$ the prime $q$ is unramified in $K_n := \mathbb{Q}(p^{1/p^n})$.*

**Proof:** The discriminant $d$ of the $\mathbb{Q}$-basis $\{1, p^{1/p^n}, p^{2/p^n}, \ldots, p^{p^n-1/p^n}\}$ of $K_n$ is by definition the discriminant of the polynomial $f(x) = x^{p^n} - p$, which we denote by $\mathrm{disc}(f)$. We have $\mathrm{disc}(f) = \pm \mathrm{Res}(x^{p^n} - p, p^n x^{p^n-1})$, where Res denotes the *resultant*. For this fact and a definition of Res we refer to [Co], Section 3.3.2.

$\mathrm{Res}(x^{p^n} - p, p^n x^{p^n - 1})$ is (after permutation of the columns) just the determinant of a triangular matrix which has only powers of $p$ on the diagonal. We conclude that $d$ is a power of $p$.
The conductor of $\mathbb{Z}[p^{1/p^n}]$ is the ideal $\mathcal{F} := \{\alpha \in \mathcal{O}_{K_n} | \alpha\mathcal{O}_{K_n} \subseteq \mathbb{Z}[p^{1/p^n}]\}$. As $d \in \mathcal{F}$ and $d = p^k$, for some $k \in \mathbb{N}$, we see that $q\mathcal{O}_{K_n}$ and $\mathcal{F}$ are coprime. Hence [Ne], I. Satz 8.3, tells us that $q$ ramifies in $K_n$ if and only if $\overline{f}(x) = x^{p^n} - p \mod q$ has multiple roots. As $q \neq p$, this is not the case. □

**Lemma 1.25.** *Let $K$ be a field with discrete valuation $v$ and let $L/K$ be a finite and $K'/K$ any field extension. We choose any field which contains $L$ and $K'$ and build the compositum $LK'$ in this field. For all places $w' \mid v$ on $K'L$ define $v' = w'|_{K'}$ and $w = w'|_L$. If the residue field $k(v)$ is perfect, then we have $e_{w'|v'} \leq e_{w|v}$.*

**Proof:** Denote by $M$ the maximal unramified extension of $K_v$ inside $L_w$. Then $M/K_v$ is unramified and $L_w/M$ is totally ramified (see [La], II Proposition 10). Hence we have $e_{w|v} = [(KL)_w : M]$. See for example [Ne], II Satz 7.2, for the fact that $K'_{v'}M/K'_{v'}$ is also unramified. Thus we know $e_{w'|v'} \leq [(K'L)_{w'} : K'_{v'}M]$. Using the equation $(K'L)_{w'} = L_w K'_{v'}$ we get

$$e_{w|v} = [(KL)_w : M] \geq [(K'L)_{w'} : K'_{v'}M] \geq e_{w'|v'}$$

as desired. □

The real Lambert-$W$ function $W : [-\frac{1}{\mathbf{e}}, \infty) \to \mathbb{R}$ is given as the multivalued inverse map of $F(x) = x\mathbf{e}^x$, where $\mathbf{e}$ is the Euler constant. We have $W(-\frac{1}{\mathbf{e}}) = -1$, but elements in $(-\frac{1}{\mathbf{e}}, 0)$ have two pre-images under $F$. Thus $W$ has two branches in the interval $[-\frac{1}{\mathbf{e}}, 0)$. The upper branch $W_0(x)$ tends to 0 for $x \nearrow 0$ and the lower branch $W_{-1}(x)$ tends to $-\infty$ for $x \nearrow 0$. We do not need deep information on the Lambert-$W$ function and take it mainly as a useful notation. For more information on this function we refer to [CGHJK].
Similarly to [BP05], Lemma 15, we will use the following lemma.

**Lemma 1.26.** *Let $a, b > 0$ be positive constants with $b \geq a$ and let $r : \mathbb{R}_+ \to \mathbb{R}$ be given by $r(x) = ax - b - \log x$. Then $-\frac{1}{a}W_{-1}(-a\mathbf{e}^{-b})$ is the greatest root of $r(x)$ and $r(x)$ is positive for all $x > -\frac{1}{a}W_{-1}(-a\mathbf{e}^{-b})$. Moreover we have the inequalities*

$$\frac{5}{8} < -\frac{1}{a}W_{-1}(-a\mathbf{e}^{-b}) < \frac{8}{5a}(\log\frac{1}{a} + b) \quad .$$

**Proof:** The function $r(x)$ obviously tends to plus infinity, so we have to find the roots of $r(x)$ in order to prove the Lemma. We have

$$\begin{aligned} & ax - b - \log x = 0 \\ \Leftrightarrow\ & \mathbf{e}^{-ax}x = \mathbf{e}^{-b} \\ \Leftrightarrow\ & -ax\mathbf{e}^{-ax} = -a\mathbf{e}^{-b} \\ \Leftrightarrow\ & x \in \left\{-\tfrac{1}{a}W_0(-a\mathbf{e}^{-b}), -\tfrac{1}{a}W_{-1}(-a\mathbf{e}^{-b})\right\} \end{aligned}$$

Our assumption on $b$ provides that $W_0(-a\mathbf{e}^{-b})$ and $W_{-1}(-a\mathbf{e}^{-b})$ are defined. As we have $-\frac{1}{a}W_0(-a\mathbf{e}^{-b}) \leq -\frac{1}{a}W_{-1}(-a\mathbf{e}^{-b})$ we know that $r(x) \geq 0$, for all $x \geq$

$-\frac{1}{a}W_{-1}(-a\mathbf{e}^{-b})$. This proves the first part of the Lemma. Now let $y$ be in the interval $[-\frac{1}{\mathbf{e}}, 0)$. By definition we have $y = W_{-1}(y)\mathbf{e}^{W_{-1}(y)}$. Multiplying this equation by $-1$ and taking the logarithm yields

$$\log(-y) = \log(-W_{-1}(y)) + W_{-1}(y) = W_{-1}(y)\left(1 - \frac{\log(-W_{-1}(y))}{-W_{-1}(y)}\right) \leq \frac{\mathbf{e}-1}{\mathbf{e}} W_{-1}(y).$$

As $-W_{-1}(y) \geq 1$ this leads to the inequality

$$W_{-1}(y) \leq \log(-y) \leq \frac{\mathbf{e}-1}{\mathbf{e}} W_{-1}(y) \quad . \tag{1.6}$$

As we assume $b \geq a$, we know that $-a\mathbf{e}^{-b}$ is in the interval $[-\frac{1}{\mathbf{e}}, 0)$. We apply (1.6) to $y = -a\mathbf{e}^{-b}$ and multiply the inequalities by $-\frac{1}{a}$ to achieve

$$\frac{\mathbf{e}}{(\mathbf{e}-1)a}(\log\frac{1}{a} + b) \geq -\frac{1}{a}W_{-1}(-a\mathbf{e}^{-b}) \geq 1 - \frac{\log a}{a} \geq \frac{\mathbf{e}-1}{\mathbf{e}} \quad .$$

For the second inequality we again use the assumption $b \geq a$. The estimation $\frac{\mathbf{e}}{\mathbf{e}-1} < \frac{8}{5}$ concludes the proof. $\square$

# 2 The Bogomolov property

The main task in this thesis is to study lower bounds for the heights we discussed in the last chapter. Northcott's theorem implies that the height of an algebraic number of low degree cannot become an arbitrary small positive number. An interesting question is which other properties of an algebraic number yield a lower bound of its height. We adopt the notation of Bombieri and Zannier in [BZ01] for the following definition.

**Definition.** Let $L$ be a subfield of $\overline{\mathbb{Q}}$ and $f \in \overline{\mathbb{Q}}(x)$ with $\deg(f) \geq 2$. We say $L$ *has the Bogomolov property relative to* $\widehat{h}_f$ if and only if there exists a constant $c > 0$ such that $\widehat{h}_f(\alpha) \geq c$ for all $\alpha \in L \setminus \mathrm{PrePer}(f)$.
If $E$ is an elliptic curve defined over $\overline{\mathbb{Q}}$, we have a similar definition. We say $L$ *has the Bogomolov property relative to* $\widehat{h}_E$ if and only if there exists a constant $c > 0$ such that $\widehat{h}_E(P) \geq c$ for all $P \in E(L) \setminus E_{\mathrm{tor}}$.

By Proposition 1.3 this means that the height on $L$ is either zero or bounded from below by a positive constant. Classically $f$ is a $d$-th power map. In this case $\widehat{h}_f = h$ and $\mathrm{PrePer}(f) = \{0\} \cup \boldsymbol{\mu}$, the set of roots of unity. According to Northcott's theorem it is trivial to see that every number field has the Bogomolov property relative to every canonical height. There are some results regarding the case of power maps and elliptic curves. Notice that the Bogomolov property in general is not preserved under finite field extensions. A counterexample in the power map case can be found in [ADZ11] or in 5.16 below. Example 4.10 also provides a counterexample for the case of Néron-Tate heights on elliptic curves. These counterexamples are already obtained after a field extension of degree two.

## 2.1 Known results

We will summarize the known results of fields with the Bogomolov property. Let $\mathbb{Q}^{tr}$ be the maximal totally real subfield of $\overline{\mathbb{Q}}$. Furthermore, let $K$ be a number field with non-archimedean $v \in M_K$. Denote by $K^{ab}$ the maximal abelian field extension of $K$, and by $K^{nr,v}$ the maximal algebraic extension of $K$ which is unramified at $v$.

**Definition.** Let $p$ be a rational prime and $e, f \in \mathbb{N}$. We call a subfield $L$ of $\overline{\mathbb{Q}}$ *totally $p$-adic of type* $(e, f)$ if for every $\alpha \in L$ and all $w \in M_{\mathbb{Q}(\alpha)}$ with $w \mid p$, the ramification indices $e_{w|p}$ are bounded by $e$ and the residue degrees $f_{w|p}$ are bounded by $f$.

**Example 2.1.** Let $K$ be a number field and $K^{(d)}$ be the compositum of all fields $F$ with $[F : K] \leq d$. For a non-archimedean absolute value $v \mid p$ on $K$ there are only finitely many field extensions of $K_v$ of degree $d$. See for example [Na], Corollary 2 of Theorem 5.27. So, $[K(\alpha)_w : K_v]$ is uniformly bounded for all $\alpha \in K^{(d)}$ and $w \mid v$, $w \in M_{K(\alpha)}$. Especially $e_{w|p}$ and $f_{w|p}$ are uniformly bounded for all $\alpha \in K^{(d)}$. If we denote these bounds by $e$ and $f$, then $K^{(d)}$ is a totally $p$-adic field of type $(e, f)$. Recently Checcoli and Widmer proved that also the field $(\cdots(K^{(d)})^{(d)}\cdots)^{(d)}$ obtained after a finite iteration of this process is totally $p$-adic of type $(e', f')$ for some $e', f' \in \mathbb{N}$ (see [CW11]).

Now we can summarize the known positive results for fields with the Bogomolov property relative to $h$ in Table 1. The lower bounds for $h$ on $\mathbb{Q}^{tr}$ and $\mathbb{Q}^{ab}$ (see [AD00] and [IMPW]) are effective. For detailed results on the height $h$ on totally real numbers see Section 5.1.

| Field | Reference |
|---|---|
| $\mathbb{Q}^{tr}$ | Schinzel [Sch73] |
| finite extensions of $K^{ab}$ | Amoroso, Zannier [AZ00] |
| totally $p$-adic fields of any type | Bombieri, Zannier [BZ01] |
| $\mathbb{Q}(E_{\text{tor}})$, $E/\mathbb{Q}$ elliptic curve | Habegger [Ha11] |

Table 1: Fields with the Bogomolov property relative to $h$

The result of Bombieri and Zannier is semi-effective, as it provides a lower bound for the limes inferior in the set $\{h(\alpha)|\alpha \text{ in any totally } p\text{-adic field of type } (e, f)\}$. With little more effort the bound can be made effective, as we will show next.

**Proposition 2.2.** *Let $p$ be a prime, and let $L$ be any totally $p$-adic field of type $(e, f)$. There exists a positive constant $c_{p,e,f}$ only depending on $p$, $e$ and $f$ such that $h(\alpha) \geq c_{p,e,f}$ for all $\alpha \in L^* \setminus \boldsymbol{\mu}$. Moreover there are only finitely many roots of unity in $L$. If the extension $L/\mathbb{Q}$ is galois, we have $h(\alpha) > \frac{2}{M(\log(3M))^3}$ for all $\alpha \in L^* \setminus \boldsymbol{\mu}$, where $M := (2e(p^f + 1))^2$.*

**Proof:** Let $L/\mathbb{Q}$ be galois and let $\alpha \in L$ be of degree $d = [\mathbb{Q}(\alpha) : \mathbb{Q}] \geq p^f$. [BZ01], Theorem 3, implies the lower bound

$$h(\alpha) \geq -\frac{\log d}{2d-2} + \frac{d}{e(2d-2)}\left(\frac{1}{p^f+1} - \frac{1}{d}\right)\log p \quad . \tag{2.1}$$

In the formulation of Bombieri and Zannier there appears a so called normalized variance. We have just used the fact that this variance is always non-negative. In particular, (2.1) implies

$$h(\alpha) \geq -\frac{\log d}{2d-2} + \frac{1}{2e}\left(\frac{1}{p^f+1} - \frac{1}{d}\right)\log p \quad . \tag{2.2}$$

We claim that this is greater than $\frac{1}{18e(p^f+1)}$ whenever $d \geq M := (2e(p^f + 1))^2$. As the right hand side of (2.2) is a monotonically increasing function in terms of $d$ it suffices to verify the claim for $d = M$. Hence, we plug $M$ in (2.2), and get

$$\begin{aligned} h(\alpha) &> -\frac{\log(2e(p^f+1))}{(2e(p^f+1))^2-1} + \frac{\log p}{2e}\left(\frac{1}{p^f+1} - \frac{1}{(2e(p^f+1))^2}\right) \\ &= \frac{-2e(2e(p^f+1))^2\log(2e(p^f+1)) + (4e^2(p^f+1)-1)((2e(p^f+1))^2-1)\log p}{((2e(p^f+1))^2-1)(2e(2e(p^f+1))^2)}. \end{aligned}$$

In the following calculations we use several times the fact that $2e(p^f+1)$ is at least 6. Moreover, we use the rough estimates $\log(2e(p^f + 1)) < \frac{1}{3}(2e(p^f + 1))$ and $\log p > \frac{1}{2}$

to obtain

$$\begin{aligned} h(\alpha) &> \frac{-\frac{16}{3}e^4(p^f+1)^3 + 8e^4(p^f+1)^3 - 2e^2(p^f+1) - 2e^2(p^f+1)^2}{((2e(p^f+1))^2-1)(2e(2e(p^f+1))^2)} \\ &> \frac{-\frac{16}{3}e^4(p^f+1)^3 + 8e^4(p^f+1)^3 - \frac{2}{9}e^4(p^f+1)^3 - \frac{2}{3}e^4(p^f+1)^3}{2e(2e(p^f+1))^4} \\ &= \frac{1}{18e(p^f+1)} \quad , \end{aligned}$$

proving the claim. Until now we have not assumed that $\alpha$ is no root of unity. Therefore, the above estimate shows that there are only finitely many roots of unity in $L$. From now on let $\alpha \in L^*$ be no root of unity, and of degree $d \le M$. If $d = 1$ we have $h(\alpha) \ge \log 2$. For $d \ge 2$ we apply a general lower bound for the height of $\alpha$ due to Voutier (see [Vo96], Corollary 2). Namely, $h(\alpha) > \frac{2}{d^2(\log(3d))^3} \ge \frac{2}{M^2(\log(3M))^3}$. With this we can conclude

$$h(\alpha) > \min\{\log 2, \frac{2}{M^2(\log(3M))^3}, \frac{1}{18e(p^f+1)}\} = \frac{2}{M^2(\log(3M))^3} \quad , \tag{2.3}$$

for all $\alpha \in L^* \setminus \boldsymbol{\mu}$. If the extension $L/\mathbb{Q}$ is not galois, then we claim that the normal closure of $L$ is a totally $p$-adic field of type $(e', f')$, with $e', f' \in \mathbb{N}$ only depending on $p$, $e$ and $f$. Then we can apply the bound (2.3) with $e$ and $f$ replaced by $e'$ and $f'$.
We fix an algebraic closure of $\mathbb{Q}_p$ and define the compositum of all field extensions of $\mathbb{Q}_p$ of degree at most $ef$ in this algebraic closure by $\mathbb{Q}_p^{(ef)}$. As noticed in Example 2.1 there are only finitely many field extensions of $\mathbb{Q}_p$ of degree not greater than $ef$. Hence, the degree $[\mathbb{Q}_p^{(ef)} : \mathbb{Q}_p]$ is finite and it depends only on $p$, $e$, and $f$. Let $\alpha$ be an arbitrary element in $L$. For any conjugate $\alpha'$ of $\alpha$, and any $v \mid p$, $v \in M_{\mathbb{Q}(\alpha')}$, we have $[\mathbb{Q}(\alpha')_v : \mathbb{Q}_p] \le ef$. In particular, all conjugates of $\alpha$ lie in $\mathbb{Q}_p^{(ef)}$, proving the claim and the proposition. $\square$

The known results for Bogomolov properties of elliptic curves relative to Néron-Tate heights are summarized in Table 2. Of course, a given field can have the Bogomolov property relative to $\widehat{h}_E$ and not relative to $\widehat{h}_{E'}$, for elliptic curves $E$ and $E'$. The lower bounds for $\widehat{h}_E$ are effective in the cases $\mathbb{Q}^{tr}$ and $\mathbb{Q}^{tr}(i)$, for $E$ defined over $\mathbb{Q}^{tr}$, and in the case of a totally $p$-adic field of any type (see [BP05]). The result of Baker concerning finite extensions of $K^{nr,v}$ is a partial result in [Ba03], Section 5 (Case 1), and is actually valid for all fields with bounded ramification index over $v$.
Notice that the results of Zhang, Baker (see [Gu07] and 3.10) and Baker & Silverman (see [BS04]) are true for heights on abelian varieties. The generalization of a Tate curve in case of finite extensions of $K^{nr,v}$ is an abelian variety which is totally degenerate at $v$.

## 2.2 Applications to dynamical heights

Let $T_d$ again be a Chebyshev polynomial, $E$ an elliptic curve and $f$ a Lattès map associated to $E$. From Lemmas 1.22 and 1.23 it follows directly that the Lehmer

| Field | Restrictions | Reference |
|---|---|---|
| finite extensions of $\mathbb{Q}^{tr}$ | none | Zhang [Zh98] |
| $K^{ab}$ | $E/K$ | Baker, Silverman [Ba03],[Si04] |
| totally $p$-adic of any type | $p \geq 3$ | Baker, Petsche [BP05] |
| finite extensions of $K^{nr,v}$ | $E/K_v$ a Tate curve | Baker [Ba03] |
| $\mathbb{Q}(E_{\mathrm{tor}})$ | $E/\mathbb{Q}$ | Habegger [Ha11] |

Table 2: Fields with the Bogomolov property relative to $\widehat{h}_E$

conjecture 1.4 is equivalent to the dynamical Lehmer conjecture 1.12 for $\widehat{h}_{T_d}$, and that the elliptic Lehmer conjecture 1.6 for $\widehat{h}_E$ is equivalent to the dynamical Lehmer conjecture 1.12 for $\widehat{h}_f$.

It is also easy to see that the Bogomolov property relative to $\widehat{h}_{T_d}$ implies the Bogomolov property relative to $h$. However, the converse of this statement is false. In Section 5.3 we will see, that $\mathbb{Q}^{tr}$ does not have the Bogomolov property relative to $\widehat{h}_{T_d}$.

**Theorem 2.3.** *Let $K$ be a number field, $p$ a rational prime and $e, f \in \mathbb{N}$. Then the following fields have the Bogomolov property relative to $\widehat{h}_{T_d}$*

*i) $K^{ab}$ and*

*ii) totally $p$-adic fields of type $(e, f)$.*

*Moreover, $T_d$ has only finitely many preperiodic points in a totally $p$-adic field of type $(e, f)$.*

**Proof:** First, we notice that the preperiodic points of $T_d$ are exactly the images of roots of unity under the map $\varphi(z) = z + z^{-1}$. Use Lemma 1.22 and Kronecker's theorem to see this. Hence, a pre-image of an element in $\overline{\mathbb{Q}} \setminus \mathrm{PrePer}(T_d)$ under the map $\varphi$ is no root of unity.

*i)* Let $\alpha$ be an arbitrary element in $K^{ab} \setminus \mathrm{PrePer}(T_d)$. Take a pre-image $\beta$ of $\alpha$ under the map $\varphi$. Then we have $[K^{ab}(\beta) : K^{ab}] \leq 2$. By Lemma 1.22 and [AZ00], Theorem 1.1, we get

$$\widehat{h}_{T_d}(\alpha) = 2h(\beta) \geq c(K) \left( \frac{\log 4}{\log \log 10} \right)^{-13} \quad ,$$

for a constant $c(K) > 0$ only depending on the ground field $K$.

*ii)* Let $L$ be a totally $p$-adic field of type $(e, f)$, and let $\alpha \in L \setminus \mathrm{PrePer}(T_d)$ be arbitrary and $\beta$ a pre-image of $\alpha$ under the map $\varphi$. Then $L(\beta)$ has degree at most two over $L$, and the multiplicativity of the ramification index and the residue degree yields that $L(\beta)$ is a totally $p$-adic field of type $(2e, 2f)$. Proposition 2.2 yields a lower bound for $h(\beta)$ depending only on $e$, $f$ and $p$. Hence, Lemma 1.22 tells us that the positive lower bound for $\widehat{h}_{T_d}(\alpha)$ also just depends on $e$, $f$ and $p$. Proposition 2.2 also provides finiteness of roots of unity in the union of totally $p$-adic fields of type $(2e, 2f)$. This yields the finiteness of preperiodic points of $T_d$ in $L$. □

Let $E$ again be an elliptic curve defined over a number field and $f$ a Lattès map associated to $E$. The Bogomolov property relative to $\widehat{h}_f$ implies the Bogomolov property relative to $\widehat{h}_E$. But again the converse is not true in general (see Example 4.10). Nevertheless, we can give a condition in which case the converse is true.

**2.4.** Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve defined over a field $K$ of characteristic $\neq 2, 3$ and let $\gamma \in \overline{K}^*$. The elliptic curve

$$E_\gamma : y^2 = x^3 + \gamma^2 Ax + \gamma^3 B$$

defined over $K(\gamma)$ is the *twist of $E$ by $\gamma$*. Notice that this is a Weierstrass equation of the curve given by $y^2 = \gamma(x^3 + Ax + B)$. If $A$, respectively $B$, is equal to zero, then $E_\gamma$ is defined over $K(\gamma^3)$, respectively $K(\gamma^2)$. $E$ and $E_\gamma$ are isomorphic over $\overline{K}$ and an isomorphism is given by

$$g_\gamma : E \tilde{\to} E_\gamma \quad ; \quad (x, y) \mapsto (\gamma x, \gamma\sqrt{\gamma} y) \quad .$$

As $g_\gamma$ is an isomorphism, it commutes with multiplication by $m \in \mathbb{Z}$. This gives a simple relation between the canonical heights on $E$ and $E_\gamma$. For any $P \in E$ we have

$$\widehat{h}_E(P) = \frac{1}{2} \lim_{n\to\infty} \frac{1}{4^n} h(x([2]^n P)) = \frac{1}{2} \lim_{n\to\infty} \frac{1}{4^n} h(\gamma x([2]^n g_\gamma(P))) = \widehat{h}_{E_\gamma}(g_\gamma(P)) \quad . (2.4)$$

**Proposition 2.5.** *Consider a field $\mathbb{Q} \subseteq K \subseteq \overline{\mathbb{Q}}$. Let $E$ be an elliptic curve defined over $K$ and $f$ a Lattès map related to the diagram* (1.3). *If there is a positive constant $c > 0$ such that for every elliptic curve $E'$ defined over $K$, which is $\overline{K}$-isomorphic to $E$, $\widehat{h}_{E'}(P) \geq c$ is true for all $P \in E'(K) \setminus E'_{\mathrm{tor}}$, then we have*

$$\widehat{h}_f(\alpha) \geq \deg(\pi)c \textit{ for all } \alpha \in K \setminus \mathrm{PrePer}(f) \quad .$$

*In particular this relation is true if $c$ only depends on the $j$-invariant of $E$.*

**Proof:** We write again $E : y^2 = x^3 + Ax + B$. As in the proof of Lemma 1.23 we use (1.3) and Lemma 1.11 *i)* to reduce to the case where $\pi$ is of the form given in (1.4). Take an arbitrary $\alpha \in K \setminus \mathrm{PrePer}(f)$ and a point $P \in E(\overline{K})$ with $\pi(P) = \alpha$. As $\alpha$ is non-preperiodic we know that $P$ is not a torsion point. Below we will prove that there exists a $\gamma \in K$ such that $g_\gamma(P) \in E_\gamma(K)$. Then we can use Lemma 1.23, (2.4) and our assumption to conclude

$$\widehat{h}_f(\alpha) = \deg(\pi)\widehat{h}_E(P) = \deg(\pi)\widehat{h}_{E_\gamma}(g_\gamma(P)) \geq \deg(\pi)c \quad .$$

In order to prove the existence of such a $\gamma$ we have to consider four different cases depending on the representation of $\pi$.

First, let $\pi$ be the projection on the $x$-coordinate. Define $\beta := \alpha^3 + A\alpha + B$. If $\beta$ is a square in $K$, then $P$ is in $E(K)$ and there is nothing more to prove. Let

$\beta$ be no square in $K$ and fix a square root $\sqrt{\beta}$ in the algebraic closure such that $P = (\alpha, \sqrt{\beta})$. Then $g_\beta(P) = (\alpha\beta, \beta^2) \in E_\beta(K)$.
In the second case we take $\pi(x,y) = x^2$. Notice that this can only occur if $j_E = 1728$, so we can assume $E : y^2 = x^3 + Ax$. This follows directly from the formula for $j_E$, see [Si09], page 45. For fixed roots we have $P = (\sqrt{\alpha}, \sqrt[4]{\alpha}\sqrt{\alpha + A})$. If $\alpha$ is in $K^2$, we get $g_{\sqrt{\alpha}(\alpha+A)}(P) \in E_{\sqrt{\alpha}(\alpha+A)}(K)$ similar to the first case. If $\alpha$ is no square in $K$, then we first twist $E$ by $\sqrt{\alpha}$ and see that $g_{\sqrt{\alpha}}(P) = (\alpha, \alpha\sqrt{\alpha + A}) \in E_{\sqrt{\alpha}}(\overline{K})$. As in the first case $g_{\sqrt{\alpha}}(P)$ is either in $E_{\sqrt{\alpha}}(K)$ or can be seen as an element of $(E_{\sqrt{\alpha}})_{\alpha+A}(K)$. Both yields the desired result. The elliptic curves $(E_{\sqrt{\alpha}})_{\alpha+A}$ and $E_{\sqrt{\alpha}}$ are both defined over $K$, since $B = 0$ and $\sqrt{\alpha} \in K^{1/2}$ (see 2.4).
The cases $\pi_1(x,y) = x^3$ and $\pi_2(x,y) = y$ are only possible if $j_E = 0$, so we have $E : y^2 = x^3 + B$ (see again [Si09], page 45). As seen in 2.4 we can twist $E$ by an element in $K^{1/3}$ and get an elliptic curve which is still defined over $K$. For fixed roots $P_1 = (\sqrt[3]{\alpha}, \sqrt{\alpha + B})$ and $P_2 = (\sqrt[3]{\alpha^2 - B}, \alpha)$ are pre-images of $\alpha$ under $\pi_1$, respectively $\pi_2$. If a twist is necessary, then for $\pi_1$ twist $E$ by $\gamma_1 := \sqrt[3]{\alpha}^{-1}$ and for $\pi_2$ twist $E$ by $\gamma_2 := \sqrt[3]{\alpha^2 - B}^{-1}$. Each case yields $x(g_{\gamma_i}(P)) = 1 \in K$, $y(g_{\gamma_i}(P)) \in K^{1/2}$ and $E_{\gamma_i}$ is defined over $K$. Thus we can proceed as in the first case. □

**Corollary 2.6.** *Let $\mathbb{Q}^{tr}$ be the maximal algebraic totally real field extension of $\mathbb{Q}$ and let $f$ be a Lattès map associated to an elliptic curve $E$ over $\mathbb{Q}^{tr}$, with $j$-invariant $j_E$. Then we have*

$$\widehat{h}_f(\alpha) \geq \frac{1}{108(h(j_E) + 10)^5} \text{ for all } \alpha \in \mathbb{Q}^{tr} \setminus \operatorname{PrePer}(f) \quad .$$

**Proof:** See [BP05], Theorem 17, for the result concerning $\widehat{h}_E$. The corollary then follows immediately from Proposition 2.5. □

**Corollary 2.7.** *Now let $K$ be a number field, $p$ an odd prime and $E$ an elliptic curve defined over $K$ having no additive reduction at all places of $K$ lying above $p$. If $L/K$ is a totally $p$-adic field of type $(e, f)$, for $e, f \in \mathbb{N}$, and $f$ a Lattès map associated to $E$ with diagram* (1.3)*, then we have:*

*i)* $\widehat{h}_f(\alpha) \geq \frac{25}{256}(\frac{\log p}{6eM})^3(\log(6eM) + \frac{\log p}{3e} + \frac{1}{6}h(j_E) + \frac{32}{5})^{-2}$ *for all* $\alpha \in L \setminus \operatorname{PrePer}(f)$

*ii)* $|\operatorname{PrePer}(f) \cap L| \leq \frac{24eM}{5\log p}(\log(6eM) + \frac{\log p}{3e} + \frac{1}{6}h(j_E) + \frac{32}{5}) + 2,$

*where* $M = \max\{p^{6f} + 1 + 2p^{3f}, 72e\nu\}$ *, with $\nu$ the maximum of* $0$ *and* $-\operatorname{ord}_w(j_E)$ *for all places $w \in M_K$ lying above $p$.*

**Proof:** The results concerning $\widehat{h}_E$ can be found in [BP05], Theorem 21. In general, the reduction type of an elliptic curve over a place $v$ is not preserved under $\overline{K}$-isomorphisms. So we cannot apply Proposition 2.5 to prove *i)*. By the multiplicativity of the ramification index and the residue degree (Lemma 1.1 *iv)*), every extension of $L$ of degree $n$ is a $p$-adic field of type $(ne, nf)$. Let $\alpha \in L$ be arbitrary and $P \in E(\overline{K})$ with $\pi(P) = \alpha$. Then $P$ is defined over a $p$-adic field of type $(6e, 6f)$,

since the degree of $\pi$ is at most 6. Moreover, $P$ is a torsion point if and only if $\alpha$ is preperiodic (see 1.19).
If $\alpha$ is not preperiodic, then Lemma 1.23 applied for [BP05], Theorem 21, yields statement $i$).
If $\alpha$ is preperiodic, then $P$ is a torsion point. Applying [BP05], Theorem 21, we find that there are at most

$$c := \frac{48eM}{5\log p}(\log(6eM) + \frac{\log p}{3e} + \frac{1}{6}h(j_E) + \frac{32}{5})$$

choices for $P$. Denote the set of critical values of $\pi$ by $\mathrm{CritVal}(\pi)$. By the Riemann-Hurwitz formula we get $|\,\mathrm{CritVal}(\pi)| \leq 2\deg(\pi)$. Apart from the critical values, every $\alpha \in L$ has exactly $\deg(\pi)$ pre-images under $\pi$. This implies

$$\begin{aligned} |\,\mathrm{PrePer}(f) \cap L| = |\pi(E_{\mathrm{tor}}) \cap L| &\leq \frac{c - |\,\mathrm{CritVal}(\pi)|}{\deg(\pi)} + |\,\mathrm{CritVal}(\pi)| \\ &\leq \frac{c}{\deg(\pi)} + 2\deg\pi - 2 \quad . \end{aligned}$$

Clearly, we have $c \geq 24$. With this and the fact $2 \leq \deg(\pi) \leq 6$ we conclude that the maximum of the right hand side is attained by $\deg\pi = 2$, which proves statement $ii$). □

We have already noticed that the set of preperiodic points of a Chebyshev polynomial $T_d$ is given by $\{\zeta + \zeta^{-1} | \zeta \in \boldsymbol{\mu}\}$. In particular, $\mathbb{Q}(\mathrm{PrePer}(T_d))$ is an abelian extension of $\mathbb{Q}$. The preperiodic points of the map $x^d$ are exactly the roots of unity. Hence, the theorem of Amoroso and Dvornicich in [AD00] and Theorem 2.3 imply that the field $\mathbb{Q}(\mathrm{PrePer}(f))$ has the Bogomolov property relative to $\widehat{h}_f$, whenever $f$ is a power map or a Chebyshev polynomial.

**Question 2.8.** *For which rational functions $f \in \overline{\mathbb{Q}}(x)$, with $\deg(f) \geq 2$, does the field $\mathbb{Q}(\mathrm{PrePer}(f))$ have the Bogomolov property relative to $\widehat{h}_f$?*

According to the results of Habegger (see Tables 1 and 2) it seems to be very likely that a Lattès map over $\mathbb{Q}$ yields a positive answer to Question 2.8.

# 3 Heights and ramification / Non-effective results

## 3.1 Introduction

We use the notations from Chapter 1. The starting point of our research is the following theorem.

**Theorem 3.1** (Gubler). *Let $K$ be a number field and let $A$ be an abelian variety defined over $K$ which is totally degenerate at a finite place $v \in M_K$. Further, let $L$ be an ample even line bundle and $K'$ be a finite extension of $K^{nr,v}$. Then there is an $\varepsilon > 0$ such that $\widehat{h}_L(P) \geq \varepsilon$ for all non-torsion points $P \in A(K')$. Moreover, there are only finitely many torsion points in $A(K')$.*

**Proof:** See [Gu07], Corollary 6.7. □

An abelian variety $A$ of dimension $n$ is called totally degenerate at $v$ if the analytification in the sense of Berkovich $A_v^{an}$ is isomorphic to ${}^{(\mathbb{G}_m^n)_v^{an}}/_M$, where $M$ is a lattice in $\mathbb{G}_m^n(\mathbb{C}_v)$. In case that $A$ is an elliptic curve this is precisely the definition of a Tate curve defined over $\mathbb{C}_v$; i.e. $A_v^{an} \cong {}^{(\mathbb{G}_m)_v^{an}}/_{q^{\mathbb{Z}}}$, $|q|_v < 1$. Note that an elliptic curve is a Tate curve over $\mathbb{C}_v$ if and only if $E$ has potential multiplicative reduction at $v$. For a proof and details on this topic we refer to [Si94], Chapter V §5, and [Si09], Section VII.5.
Let $K$ be a number field and let $v \in M_K$ be finite. Moreover, let $E$ be an elliptic curve with potential multiplicative reduction at $v$ and $f$ a Lattès map associated to $E$. The $j$-invariant of $E$ is no algebraic integer (see [Si09], Proposition VII.5.5), hence (1.4) tells us that $\deg(\pi) = 2$. The same argument yields that $E$ does not have complex multiplication (see [Si94], Theorem II.6.1). The goal of this section is to prove that $K^{nr,v}$, the maximal algebraic extension of $K$ which is unramified at $v$, has the Bogomolov property relative to $\widehat{h}_f$. The bound $c > 0$ in Theorem 3.1 follows from an equidistribution theorem and is non-effective. We have no information how $c$ changes under $\overline{K}$-isomorphisms and hence we cannot use Proposition 2.5.
In Section 3.2 we will show that the Bogomolov property for $K^{nr,v}$ relative to $\widehat{h}_f$ does not follow directly from Theorem 3.1. In Section 3.3 we use the dynamical equidistribution Theorem 1.15 to give a direct non-effective proof of the desired result. In the last section we generalize Theorem 3.1 such that the Bogomolov property of $K^{nr,v}$ relative to $\widehat{h}_f$ will be an easy corollary.

## 3.2 A naive approach

We fix a number field $K$ and a non-archimedean $v \in M_K$. Let $E$ be an elliptic curve with potential multiplicative reduction at $v$ and let $f$ be a Lattès map with diagram (1.3).
With Lemma 1.23 the Bogomolov property for $K^{nr,v}$ relative to $\widehat{h}_f$ would be a direct consequence of Theorem 3.1 if $[K^{nr,v}(\pi^{-1}(K^{nr,v})) : K^{nr,v}]$ is finite. We will give an example to show that this is generally not the case.

**Lemma 3.2.** *Let $\alpha$ be an algebraic integer with prime ideal decomposition $\alpha\mathcal{O}_{\mathbb{Q}(\alpha)} = P_1^{e_1}\cdots P_r^{e_r}$. If $e_i$, $i \in \{1,\dots,r\}$, is odd, then $P_i$ ramifies in $\mathbb{Q}(\sqrt{\alpha})$. Especially, the rational prime $p$ ramifies in $\mathbb{Q}(\sqrt{\alpha})$ if $p \mid N(\alpha)$ and $p^2 \nmid N(\alpha)$, where $N$ denotes the norm of an algebraic number.*

**Proof:** Since we have $\alpha\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})} = (\sqrt{\alpha}\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})})^2$, we see that every exponent of a prime ideal in

$$\alpha\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})} = (P_1\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})})^{e_1}\cdots(P_r\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})})^{e_r}$$

is even. This shows that $P_i$ ramifies in $\mathbb{Q}(\sqrt{\alpha})$ whenever $e_i$ is odd. □

**Lemma 3.3.** *Let $p$ and $q$ be rational primes congruent to 1 modulo 36. Then $3\mathcal{O}_{\mathbb{Q}(\sqrt{p})} = P_1P_2$ and $3\mathcal{O}_{\mathbb{Q}(\sqrt{q})} = Q_1Q_2$, for distinct prime ideals $P_1$ and $P_2$ (resp. $Q_1$ and $Q_2$) in $\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ (resp. $\mathcal{O}_{\mathbb{Q}(\sqrt{q})}$). Further we have*

*i) 3 splits completely in $\mathbb{Q}(\sqrt{p},\sqrt{q})$*

*ii) $P_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$ and $Q_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$ have exactly one common prime factor.*

**Proof:** It is well known that 3 splits completely in $\mathbb{Q}(\sqrt{l})$ for every $l \equiv 1 \mod 3$ (see [Ne] I, Satz 8.5). Using valuation theory it is easy to see that 3 also splits completely in the compositum $\mathbb{Q}(\sqrt{p})\mathbb{Q}(\sqrt{q}) = \mathbb{Q}(\sqrt{p},\sqrt{q})$. A proof of this fact which uses only ideal theoretical methods can be found in [Sch], Theorem 3.2.4.
It remains to prove statement *ii*). Using the first statement we know that every ideal in $\{P_1,P_2,Q_1,Q_2\}$ decomposes in a product of exactly two distinct prime ideals over $\mathbb{Q}(\sqrt{p},\sqrt{q})$. Let us assume we have $P_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})} = Q_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})} = \mathfrak{P}_1\mathfrak{P}_2$. Intersection of both sides with $\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ yields

$$P_1 = (Q_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}) \cap \mathcal{O}_{\mathbb{Q}(\sqrt{p})} \quad .$$

We want to show that the right hand side of this equation is equal to $3\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$. As $p$ and $q$ are congruent to 1 modulo 4, we know that

$$\left\{1, \frac{1+\sqrt{p}}{2}, \frac{1+\sqrt{q}}{2}, \frac{(1+\sqrt{p})(1+\sqrt{q})}{4}\right\}$$

is a $\mathbb{Z}$-basis of $\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$ (see [Zi06], Satz 2.9). We can write an arbitrary element in $(Q_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}) \cap \mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ in the form

$$\begin{aligned} &\left(a + b(\frac{1+\sqrt{q}}{2})\right)\left(c + d(\frac{1+\sqrt{p}}{2}) + e(\frac{1+\sqrt{q}}{2}) + f(\frac{(1+\sqrt{p})(1+\sqrt{q})}{4})\right) \\ =&\alpha_1 + \alpha_2\sqrt{p} + \alpha_3\sqrt{q} + \alpha_4\sqrt{pq} \quad , \end{aligned}$$

with $a$, $b$, $c$, $d$, $e$, $f \in \mathbb{Z}$ and

$$\begin{aligned} \alpha_1 &= (a + \frac{b}{2})(c + \frac{d}{2} + \frac{e}{2} + \frac{f}{4}) + (\frac{b}{2}(\frac{e}{2} + \frac{f}{4})q) \\ \alpha_2 &= (a + \frac{b}{2})(\frac{d}{2} + \frac{f}{4}) + q\frac{bf}{8} \\ \alpha_3 &= (a + \frac{b}{2})(\frac{e}{2} + \frac{f}{4}) + \frac{b}{2}(c + \frac{d}{2} + \frac{e}{2} + \frac{f}{4}) \\ \alpha_4 &= (a + \frac{b}{2})\frac{f}{4} + \frac{b}{2}(\frac{d}{2} + \frac{f}{4}) \quad . \end{aligned}$$

We have the following properties:

I) $3 \mid N(a+b(\frac{1+\sqrt{q}}{2})) = a^2+ab+(1-q)\frac{b^2}{4}$

II) $\alpha_3 = \frac{ae}{2}+\frac{af}{4}+\frac{be}{2}+\frac{bf}{4}+\frac{bc}{2}+\frac{bd}{4} = 0$

III) $\alpha_4 = \frac{af}{4}+\frac{bf}{4}+\frac{bd}{4} = 0$

Using II) and III) we can write our chosen element as

$$\underbrace{ac+\frac{ad}{2}+(q-1)\frac{be}{4}+(q-1)\frac{bf}{8}}_{:=\alpha_5}+\underbrace{(\frac{ad}{2}+(q-1)\frac{bf}{8})}_{:=\alpha_6}\sqrt{p}.$$

In order to prove that this element lies in $3\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$, it suffices to show that 3 divides $2\alpha_5$ and $2\alpha_6$. It is clear that 3 divides $q-1$, hence we are done if 3 also divides $a$. So let $3 \nmid a$. Then, by I), we have $a \equiv -b \mod 3$. Putting this into II) leads us to $2\alpha_5 \equiv 2ac+ad \equiv -2bc-bd \equiv 0 \mod 3$. As $3 \nmid a$, III) tells us $bd \equiv 0 \mod 3$. Thus we have $2\alpha_6 \equiv ad \equiv -bd \equiv 0 \mod 3$.
Now we have $P_1 = (Q_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}) \cap \mathcal{O}_{\mathbb{Q}(\sqrt{p})} = 3\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$, which clearly is not possible. This shows that $P_1$ and $Q_1$ cannot have the same prime ideal decomposition in $\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$. If we assume that they have no common prime ideal factor, then $P_1 = Q_2$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$, which again leads to a contradiction. Hence, $P_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$ and $Q_1\mathcal{O}_{\mathbb{Q}(\sqrt{p},\sqrt{q})}$ have exactly one common prime ideal factor. $\square$

**Example 3.4.** Take the Tate curve $E : y^2 = x^3 + \frac{1}{4}x^2 + 6$ defined over $\mathbb{Q}_3$ and let $\pi : E \to \overline{\mathbb{Q}_3}$ be the projection on the $x$-coordinate. We claim that the degree of $\mathbb{Q}^{nr,3}(\pi^{-1}(\mathbb{Q}^{nr,3}))$ over $\mathbb{Q}^{nr,3}$ is infinite. Notice that

$$\mathbb{Q}^{nr,3}(\pi^{-1}(\mathbb{Q}^{nr,3})) = \mathbb{Q}^{nr,3}(\sqrt{4x^3+x^2+24} | x \in \mathbb{Q}^{nr,3}) \quad .$$

Lemma 3.3 tells us that $\sqrt{p} \in \mathbb{Q}^{nr,3}$ for all primes $p \equiv 1 \mod 36$. Let $p$ be such a prime. Then $N(4\sqrt{p}^3+p+24) = -16p^3+p^2+48p+24^2$ is divisible by 3 but not by 9. Lemma 3.2 tells us that $\sqrt{4\sqrt{p}^3+p+24}$ is not in $\mathbb{Q}^{nr,3}$. Let $q \equiv 1 \mod 36$, $q \neq p$, be another such prime and set $\alpha := (4\sqrt{p}^3+p+24)(4\sqrt{q}^3+q+24)$. We have $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{p},\sqrt{q})$ and 3 splits completely in $\mathbb{Q}(\alpha)$ and every subfield (see Lemma 3.3). Denote by $\mathfrak{P}_1,\ldots,\mathfrak{P}_4$ the prime ideals in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ above 3, and let $P, Q \mid 3$ be the prime ideals in $\mathcal{O}_{\mathbb{Q}(\sqrt{p})}$ (resp. $\mathcal{O}_{\mathbb{Q}(\sqrt{q})}$) with $4\sqrt{p}^3+p+24 \in P$ and $4\sqrt{q}^3+q+24 \in Q$. According to Lemma 3.3 we have, without loss of generality, $P\mathcal{O}_{\mathbb{Q}(\alpha)} = \mathfrak{P}_1\mathfrak{P}_2$ and $Q\mathcal{O}_{\mathbb{Q}(\alpha)} = \mathfrak{P}_2\mathfrak{P}_3$. Hence, we have

$$\alpha\mathcal{O}_{\mathbb{Q}(\alpha)} = \mathfrak{P}_1{\mathfrak{P}_2}^2\mathfrak{P}_3 I \text{ , with } I \nmid 3.$$

Applying Lemma 3.2, $\mathfrak{P}_1$ and $\mathfrak{P}_3$ ramify in $\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})}$. We can conclude that 3 ramifies in $\mathcal{O}_{\mathbb{Q}(\sqrt{\alpha})}$. This is equivalent to the fact $\sqrt{\alpha} \notin \mathbb{Q}^{nr,3}$.
Now we will use a theorem from Kummer theory, for a proof of the following result we refer to [Ro], Theorem 15.3.1.

**Theorem 3.5.** *Let $F$ be a field with $\mathrm{char}(F) \neq 2$ and denote by $(F^*)^2$ the subgroup of $F^*$ consisting of all squares in $F^*$. For any group $H$ with $(F^*)^2 \subseteq H \subseteq F^*$ we define $H^{1/2} := \{a \in \overline{F} | a^2 \in H\}$. Then we have*

$$[F(H^{1/2}) : F] = \left|H/(F^*)^2\right| \quad .$$

Set $H = \langle K^{*2}, \{4\sqrt{p}^3 + p + 24 | p \equiv 1 \mod 36 \text{ prim}\}\rangle$. In order to estimate $|H/K^{*2}|$ we notice that for all $a, b \in H$ we have

$$\begin{array}{llll} a \equiv b \mod K^{*2} & \Leftrightarrow & ab^{-1} \equiv 1 \mod K^{*2} & \\ & \Leftrightarrow & \sqrt{ab^{-1}} \in K & \Leftrightarrow \sqrt{ab} \in K \end{array}$$

But we have shown above that two different elements $4\sqrt{p}^3+p+24 \in H$ cannot fulfill this equivalence. From Dirichlet's prime theorem we know that there are infinitely many primes congruent 1 modulo 36. Hence, $[K(H^{1/2}) : K] = |H/K^{*2}| = \infty$, and using Theorem 3.5

$$[\mathbb{Q}^{nr,3}(\pi^{-1}(\mathbb{Q}^{nr,3})) : \mathbb{Q}^{nr,3}] = [\mathbb{Q}^{nr,3}(\sqrt{4x^3 + x^2 + 24} | x \in \mathbb{Q}^{nr,3}) : \mathbb{Q}^{nr,3}] = \infty \quad .$$

This example shows that the Bogomolov property for $K^{nr,v}$ relative to $\widehat{h}_f$ does not follow directly from Theorem 3.1.

## 3.3 A direct proof for Lattès maps

For this section we fix the following situation. Let $K$ be a number field and $v \in M_K$ be non-archimedean with $v \mid p$. Further, let $E : y^2 + xy = x^3 + Ax + B$ be an elliptic curve defined over $K$ such that $E$ regarded as a curve defined over $\mathbb{C}_v$ is a Tate curve, and let $f$ be a Lattès map associated to $E$ given by a commutative diagram (1.3). As we have seen in Section 3.1, we can assume that $\pi$ is the projection on the $x$-coordinate. With Proposition 1.21 we can also assume that $\Psi = [m]$, $m \in \mathbb{Z}$ with $|m| \geq 2$. The valuation $v$ extends uniquely to $K_v$, from there it extends uniquely to $\overline{K_v}$ and finally to $\mathbb{C}_v$. We denote this valuation on $\mathbb{C}_v$ again by $v$. We prove in this section the following theorem. In Chapter 4 we will obtain an effective version of this theorem.

**Theorem 3.6.** *The field $K^{nr,v}$ has the Bogomolov property relative to $\widehat{h}_f$. In other words: There is a constant $C > 0$ such that for all $\alpha \in K^{nr,v} \setminus \mathrm{PrePer}(f)$ we have $\widehat{h}_f(\alpha) > C$. Moreover, the set $\mathrm{PrePer}(f) \cap K^{nr,v}$ is finite.*

The GAGA-functor on Berkovich spaces (see [Ber], Section 3.4) transfers (1.3) into a commutative diagram

$$\begin{tikzcd} E_v^{an} \arrow[r, "{[m]^{an}}"] \arrow[d, "\pi^{an}"'] & E_v^{an} \arrow[d, "\pi^{an}"] \\ (\mathbb{P}^1)_v^{an} \arrow[r, "f^{an}"] & (\mathbb{P}^1)_v^{an} \end{tikzcd}$$

where $E_v^{an}$ and $(\mathbb{P}^1)_v^{an}$ are the Berkovich spaces associated to $E$ and $\mathbb{P}^1$. GAGA stands for *Géométrie Algébrique et Géométrie Analytique*, as it is a non-archimedean version of the famous paper of Serre with this title. The valuation function on $(\mathbb{P}^1)_v^{an}$ is defined as

$$\mathrm{val} : (\mathbb{P}^1)_v^{an} \to \mathbb{R} \cup \{\pm\infty\} \quad ; \quad y \mapsto -\log |X|_y .$$

Here $X$ is the variable in the ring of polynomials $\mathbb{C}_v[X]$. We have $E \cong {}^{(\mathbb{G}_m)_v^{an}}/_{q^{\mathbb{Z}}}$, $q \in \mathbb{C}_v$ with $|q|_v < 1$. So there is a canonical valuation function $\overline{\mathrm{val}}$ on $E_v^{an}$ given by

$$\overline{\mathrm{val}} : {}^{(\mathbb{G}_m)_v^{an}}/_{q^{\mathbb{Z}}} \to {}^{\mathbb{R}}/_{v(q)\mathbb{Z}} \quad ; \quad \overline{y} \mapsto \overline{-\log |X|_y} .$$

Obviously we have $\overline{\mathrm{val}}(E_v^{an}) = {}^{\mathbb{R}}/_{v(q)\mathbb{Z}}$.
We have to compare the algebraic dynamical systems $(E, L, [m])$ and $(\mathbb{P}^1, \mathcal{O}(1), f)$, where $L := \pi^*\mathcal{O}(1)$ on $E$ is ample and even (see Remark 1.7). As $\deg(f) = m^2$, we have $\mathcal{O}(1)^{m^2} \cong f^*\mathcal{O}(1)$. The theorem of the cube tells us $[m]^*L \cong L^{m^2}$. Notice that, by the commutativity of (1.3), we have $[m]^*L = [m]^*\pi^*\mathcal{O}(1) = \pi^*f^*\mathcal{O}(1)$. The canonical metrics $\|.\|_f$ and $\|.\|_{[m]}$ on $\mathcal{O}(1)$, respectively $L$, have the properties

$$(f^{an})^*\|.\|_f = \|.\|_f^{m^2} \text{ and } ([m]^{an})^*\|.\|_{[m]} = \|.\|_{[m]}^{m^2} .$$

For details we refer to [Zh95], Section 2, and [Gu10], Section 3. Just using the definitions of the different maps we get

$$([m]^{an})^*(\pi^{an})^*\|.\|_f = ((\pi^{an})^*\|.\|_f)^{m^2} .$$

This implies the equation

$$(\pi^{an})^*\|.\|_f = \|.\|_{[m]} . \tag{3.1}$$

We recall from 1.14 that we have the $v$-adic canonical measures $\mu_{[m],v} = c_1(L, \|.\|_{[m]})$ and $\mu_{f,v} = c_1(\mathcal{O}(1), \|.\|_f)$ associated to $(E, L, [m])$, respectively $(\mathbb{P}^1, \mathcal{O}(1), f)$. Using the projection formula (for example [Gu07a], Corollary 3.9 b)) and (3.1) we deduce

$$(\pi^{an})_*\mu_{[m],v} = \deg(\pi)\mu_{f,v} .$$

This leads us to $\mathrm{supp}((\pi^{an})_*\mu_{[m],v}) = \mathrm{supp}(\mu_{f,v})$. As $\mu_{[m],v}$ is a positive measure we get

$$\pi^{an}(\mathrm{supp}(\mu_{[m],v})) = \mathrm{supp}((\pi^{an})_*\mu_{[m],v}) = \mathrm{supp}(\mu_{f,v}) . \tag{3.2}$$

**Remark 3.7.** Every disk $(a, r)$ around $a \in \mathbb{C}_v$ with radius $r \in \mathbb{R}$, $|q|_v \leq r \leq 1$, gives us a multiplicative seminorm on the Tate algebra $\mathbb{C}_v\{X, qX^{-1}\}$ (this is the completion of $\mathbb{C}_v[X, qX^{-1}]$ with respect to the gauß-norm induced by $v$), and hence a point of $E_v^{an}$. Explicitly, $|.|_{(a,r)}$ is given by

$$|\sum_{n\in\mathbb{Z}} a_n X^n|_{(a,r)} = |\sum_{n\in\mathbb{Z}} b_n (X-a)^n|_{(a,r)} = \max_{n\in\mathbb{Z}} |b_n|_v r^n .$$

The subdomain of $E_v^{an}$ consisting of all points $(0, r)$, with $|q|_v < r < 1$, is called the *skeleton* of $E$. We denote the skeleton of $E$ by $S(E)$. It is easy to see that $\overline{\text{val}}$ maps $S(E)$ homeomorphic onto ${}^{\mathbb{R}}/_{v(q)\mathbb{Z}}$. For the general theory of skeletons we refer to [Ber], Section 6.5, and for more information on our special case we refer to [Gu10], Example 7.2.

**Proposition 3.8.** *With the same notations as above, we have* $\operatorname{supp}(\mu_{[m],v}) = S(E)$.

**Proof:** See [Gu10], Corollary 7.3. □

To prove Theorem 3.6, we assume that $\widehat{h}_f$ has no positive lower bound on $K^{nr,v} \setminus \operatorname{PrePer}(f)$ or that there are infinitely many preperiodic points of $f$ in $K^{nr,v}$. In both cases we obtain a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of pairwise distinct elements in $K^{nr,v}$ such that

$$\lim_{n\to\infty} \widehat{h}_f(\alpha_n) \longrightarrow 0 \text{ , as } n \to \infty.$$

We will show that this contradicts (3.2) and Proposition 3.8.

**Lemma 3.9.** *If there is a sequence* $\{\alpha_n\}_{n\in\mathbb{N}}$ *as above, then*

$$\operatorname{supp}(\mu_{f,v}) \subseteq \text{val}^{-1}\left(\frac{\log p}{e_{v|p}}\mathbb{Z} \cup \{\pm\infty\}\right) \quad ,$$

*where* $e_{v|p}$ *is the ramification index of* $v$ *over* $p$.

**Proof:** Take $y \in (\mathbb{P}^1)_v^{an}$ with $\text{val}(y) \notin \frac{\log p}{e_{v|p}}\mathbb{Z} \cup \{\pm\infty\}$. Choose an open neighborhood $I$ of $\text{val}(y)$ such that $I$ does not contain an element of $\frac{\log p}{e_{v|p}}\mathbb{Z}$. The value group of $v$ on $K^{nr,v}$ is $\frac{\log p}{e_{v|p}}\mathbb{Z}$ and val is continuous. So the open neighborhood $U_y := \text{val}^{-1}(I)$ of $y$ does not contain a rational point of $(\mathbb{P}^1)_v^{an}(K^{nr,v})$. With Theorem 1.15 we get

$$\mu_{f,v}(U_y) = \lim_{n\to\infty} [K(\alpha_n) : K]^{-1} \sum_{\alpha' \in G_K(\alpha_n)} \delta_{\alpha'}(U_y) = 0.$$

So $y$ is no point of $\operatorname{supp}(\mu_{f,v})$, proving the lemma. □

With Proposition 3.8 and (3.2) we conclude

$$\pi^{an}(S(E)) \subseteq \text{val}^{-1}\left(\frac{\log p}{e_{v|p}}\mathbb{Z} \cup \{\pm\infty\}\right) \quad . \tag{3.3}$$

As $\pi^{an}$ and val are continuous, $\frac{\log p}{e_{v|p}}\mathbb{Z} \cup \{\pm\infty\}$ is discrete and $S(E)$ is not, this is very likely to be impossible. But to prove this we need a better understanding of the map $\pi^{an}$.

In rigid geometry, Tate has described the isomorphism between ${}^{\mathbb{G}_m}/_{q^{\mathbb{Z}}}$ and $E(\mathbb{C}_v)$. The $x$ and $y$ coordinate in $E(\mathbb{C}_v)$ of an element $\zeta \in {}^{\mathbb{G}_m}/_{q^{\mathbb{Z}}}$ are explicitly given by

$$x(\zeta) = \sum_{n=-\infty}^{\infty} \frac{q^n\zeta}{(1-q^n\zeta)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}$$

$$y(\zeta) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}\zeta^2}{(1-q^n\zeta)^3} + \sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}.$$

For a proof and further information on this isomorphism we refer to [Si94], V.3 and V.4. Thus, $\pi^{an}$ is defined on rational points of ${}^{(\mathbb{G}_m)_v^{an}}/_{q^{\mathbb{Z}}}$ by

$$\pi^{an}(\zeta) = \sum_{n=-\infty}^{\infty} \frac{q^n\zeta}{(1-q^n\zeta)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}.$$

As a morphism of strict $\mathbb{C}_v$-affinoid spaces, $\pi^{an}$ is induced by a homomorphism $(\pi^{an})^\sharp : \mathbb{C}_v[X] \to \mathbb{C}_v\{X, qX^{-1}\}$ of the related $\mathbb{C}_v$-affinoid algebras (see [Ber], Chapter 2 and [Bo77]). With Tate's isomorphism we know

$$(\pi^{an})^\sharp(X) = \sum_{n=-\infty}^{\infty} \frac{q^nX}{(1-q^nX)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)}.$$

Thus, for any $f(X) \in \mathbb{C}_v[X]$ and any $y \in E_v^{an}$ we have

$$|f(X)|_{\pi^{an}(y)} = \left| f\left( \sum_{n=-\infty}^{\infty} \frac{q^nX}{(1-q^nX)^2} - 2\sum_{n=1}^{\infty} \frac{nq^n}{(1-q^n)} \right) \right|_y.$$

In order to compute $\mathrm{val}(\pi^{an}(0,r)) = -\log|X|_{\pi^{an}(0,r)}$ for an element $(0,r) \in S(E)$ we have to compute $|(\pi^{an})^\sharp(X)|_{(0,r)}$. We have $|q^nX|_{(0,r)} = |q^n|_v r < 1$ for all $n \geq 0$ and hence

$$\left| \frac{q^nX}{(1-q^nX)^2} \right|_{(0,r)} = |q^nX|_{(0,r)}$$

for all $n \geq 0$. Obviously we also have

$$r = |X|_{(0,r)} = |q^0X|_{(0,r)} > |q^1X|_{(0,r)} > \cdots$$

leading to

$$\left| \sum_{n=0}^{\infty} \frac{q^nX}{(1-q^nX)^2} \right|_{(0,r)} = r. \tag{3.4}$$

For all negative integers $n$ we have $|q^nX|_{(0,r)} > 1$, and hence

$$\left| \frac{q^nX}{(1-q^nX)^2} \right|_{(0,r)} = \left| \frac{1}{q^nX} \right|_{(0,r)}$$

for all $n < 0$. With the trivial inequalities

$$\left| \frac{1}{q^{-1}X} \right|_{(0,r)} > \left| \frac{1}{q^{-2}X} \right|_{(0,r)} > \cdots$$

we conclude

$$\left|\sum_{n=1}^{\infty}\frac{q^{-n}X}{(1-q^{-n}X)^2}\right|_{(0,r)} = \left|\frac{1}{q^{-1}X}\right|_{(0,r)} = |q|_v r^{-1}. \tag{3.5}$$

The equation

$$\left|2\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}\right|_{(0,r)} = \left|2\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}\right|_v = |2q|_v \tag{3.6}$$

similarly follows with elementary properties of non-archimedean absolute values. Since $(0,r)$ is an element of the skeleton, we know $|q|_v < r < 1$. So (3.4), (3.5) and (3.6) are leading us to

$$|X|_{\pi^{an}(0,r)} = \left|\sum_{n=-\infty}^{\infty}\frac{q^nX}{(1-q^nX)^2} - 2\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}\right|_{(0,r)} \leq \max\{r, |q|_v r^{-1}\}. \tag{3.7}$$

If we choose $(0,r) \in S(E)$ with $1 < r^2 < |q|_v$ and $\log r \notin \frac{\log p}{e_{v|p}}\mathbb{Z}$, then the value in (3.7) is equal to $r$. So there is an element with

$$\mathrm{val}(\pi^{an}((0,r))) = -\log|X|_{\pi^{an}(0,r)} = -\log r \notin \frac{\log p}{e_{v|p}}\mathbb{Z}.$$

This contradicts (3.3) and proves Theorem 3.6. □

## 3.4 A generalization of the starting point

The proof of Theorem 3.1 actually shows that a stronger formulation than the one Gubler gave in his paper is also valid. We will reformulate this theorem such that Theorem 3.6 will be an easy corollary. The generalization is the following. Let $K$ be a number field and let $A$ be an abelian variety defined over $K$ which is totally degenerate at a finite place $v \in M_K$ with $v \mid p$. For $e \in \mathbb{N}$ we define

$$M_e^A(v) := \{P \in A(\overline{\mathbb{Q}}) | e_{w|v} \leq e \text{ for all } w \in M_{K(P)},\ w \mid v\} \quad .$$

The proof of the following theorem is essentially the same as the proof given in [Gu07]. Nevertheless, we will recall the complete proof here.

**Theorem 3.10.** *Let $L$ be an ample even line bundle on $A$. For all $e \in \mathbb{N}$ there exists an $\varepsilon > 0$ such that $\widehat{h}_L(P) \geq \varepsilon$ for all non-torsion points $P \in M_e^A(v)$. Moreover, for every $e \in \mathbb{N}$ the number of torsion points in $M_e^A(v)$ is finite.*

**Proof:** Assume for the sake of contradiction that the theorem is wrong, and let $A$ be an abelian variety which is a counterexample of minimal dimension $n$. By assumption there exists an integer $e$ and a sequence $\{P_i\}_{\in\mathbb{N}}$ of pairwise distinct points in $M_e^A(v)$ such that $\widehat{h}_L(P_i) \to 0$ as $i \to \infty$.

Analogue to Section 3.3 we have a continuous valuation function $\overline{\mathrm{val}} : A_v^{an} \to \mathbb{R}^n/v(M)$. There is a finite extension $F/K_v$ such that $A(F') = {(F'^*)^n}/M$ for every algebraic extension $F'$ of $F$. Denote by $d$ the degree of $F$ over $K_v$ and set $F' = F(P_i)$ for some $i \in \mathbb{N}$. Let $w' \mid v$ be an arbitrary valuation on $F'$ and set $w = w'|_{K_v(P_i)}$. Then, by the definition of $M_e^A(v)$, we have $e_{w'|p} = e_{w'|w}e_{w|v}e_{v|p} \leq dee_{v|p}$. We can conclude that $e_{w'|p}$ divides $e' := (dee_{v|p})!$. In particular, $\overline{\mathrm{val}}(P_i)$ lies in the discrete subset $\frac{\log p}{e'}\mathbb{Z}/v(M)$ of $\mathbb{R}/v(M)$ for all $i \in \mathbb{N}$. The set $M_e^A(v)$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-invariant. Hence, the sequence $\{\overline{\mathrm{val}}(G_K(P_i))\}_{i\in\mathbb{N}}$ cannot be equidistributed in $\mathbb{R}^n/v(M)$. We can apply [Gu07], Corollary 6.6, which follows from the Bogomolov conjecture and the tropical equidistribution theorem, to conclude that there is an infinite subsequence $\{P_i\}_{i\in I}$ contained in $t + B$, where $t$ is a torsion point in $A(\overline{\mathbb{Q}})$, and $B$ is a proper abelian subvariety of $A$. Define $Q_i := P_i - t$ for all $i \in I$. If $d'$ is the degree of $K(t)$ over $K$, then we can conclude as above that for all $i \in I$ we have $e_{w|v} \leq d'e$, for all $w \in M_{K(Q_i)}$, $w \mid v$. Hence, the sequence $\{Q_i\}_{i\in I}$ consists of pairwise distinct points in $M_{d'e}^B(v)$. Let $m$ be the order of $t$, then we have

$$\widehat{h}_L(Q_i) = m^{-2}\widehat{h}_L([m]Q_i) = m^{-2}\widehat{h}_L([m]P_i) = \widehat{h}_L(P_i) \to 0 \quad ,$$

as $i \to \infty$. This contradicts the minimality of $A$, which proves the theorem. □

**Remark 3.11.** Theorem 3.6 follows easily from Theorem 3.10. Let $E$ be an elliptic curve for which Theorem 3.10 applies, and let $f$ be an associated Lattès map with diagram (1.3). We have noticed in Section 3.1 that $\pi$ has degree two. Hence, every pre-image of an element in $K^{nr,v}$ under $\pi$ lies in $M_2^E(v)$. As we have $\mathrm{PrePer}(f) = \pi(E_{\mathrm{tor}})$ (see 1.19), Theorem 3.10 immediately implies that there are only finitely many preperiodic points of $f$ in $K^{nr,v}$. For all $\alpha \in K^{nr,v} \setminus \mathrm{PrePer}(f)$ we choose a $P \in M_2^E(v) \setminus E_{\mathrm{tor}}$ with $\alpha = \pi(P)$. By Lemma 1.23 we can conclude $\widehat{h}_f(\alpha) = 2\widehat{h}_E(P) \geq 2\varepsilon$, for the positive constant $\varepsilon$ from Theorem 3.10.
For simplicity we have restricted the argumentation to the field $K^{nr,v}$. More generally, our argumentation is valid for all sets of points in $\overline{\mathbb{Q}}$ with bounded ramification over $v$. We refer to the next chapter for more details.

# 4 Heights and ramification / Effective results

## 4.1 Introduction

In this chapter we will prove a generalized effective version of Theorem 3.6. For the complete chapter we fix the following notations. Let $K$ be a number field with non-archimedean absolute value $v \mid p$, and let $E$ be an elliptic curve defined over $K$ with $j$-invariant $j$. By $d$ we denote the degree $[K : \mathbb{Q}]$, and by $d_v$ the local degree $[K_v : \mathbb{Q}_p]$. Let further $f$ be a Lattès map associated to $E$. As in Section 3.4 we define

$$M_e(v) := \{\alpha \in \overline{\mathbb{Q}} | e_{w|v} \leq e \text{ for all } w \in M_{K(\alpha)},\ w \mid v\} \text{ and}$$
$$M_e^E(v) := \{P \in E(\overline{\mathbb{Q}}) | e_{w|v} \leq e \text{ for all } w \in M_{K(P)},\ w \mid v\},$$

for a fixed $e \in \mathbb{N}$.
In his proof of Theorem 1.4 in [Ba03], Baker proved that for all $e \in \mathbb{N}$ there is a positive constant $c$, only depending on $E$, $K$, $v$ and $e$, such that $\widehat{h}_E(P) \geq c$ for all non-torsion points $P \in M_e^E(v)$. This bound $c$ is non-effective, but as stated in [Ba03], Remark 5.1, it could be made effective by the methods he used. The result of Baker is not stated explicitly, it can be found in [Ba03], Section 5, Case 1. Based on an idea of Sinnou David we will use in Section 4.2 a different proof than Baker to obtain the following completely explicit theorem.

**Theorem 4.1.** *If $E$ has split-multiplicative reduction at $v \mid p$, then there are effective computable constants $c'(j,d,e,v)$, $c'_T(j,d,e,v) > 0$ only depending on $j$, $d$, $e$ and $v$, with $\widehat{h}_E(P) \geq c'(j,d,e,v)$ for all $P \in M_e^E(v) \setminus E_{\mathrm{tor}}$ and such that there are less than $c'_T(j,d,e,v)$ torsion points in $M_e^E(v)$. More precisely, we have*

*i)* $\widehat{h}_E(P) \geq \frac{\frac{\log p}{2d}\mathfrak{c} - 3\log 2}{(8\mathfrak{c}^3 - 2\mathfrak{c})(e!\operatorname{ord}_v(j^{-1}))^2} > 0$ *for all* $P \in M_e^E(v) \setminus E_{\mathrm{tor}}$

*ii)* $|E_{\mathrm{tor}} \cap M_e^E(v)| < \frac{1}{3}\left(e!\mathfrak{c}\operatorname{ord}_v(j^{-1})\right)^3 + \frac{1}{2}\left(e!\mathfrak{c}\operatorname{ord}_v(j^{-1})\right)^2$,

*where* $\mathfrak{c} := \left\lceil \frac{10d}{\log p}\left(\log(\frac{6d}{\log p}) + \frac{1}{6}h(j) + \frac{32}{5}\right)\right\rceil$.

From this result the (effective version) of the Bogomolov property of $K^{nr,v}$ relative to $\widehat{h}_f$ will be an easy corollary as we have seen in Section 3.4. Precisely, the statement reads as follows.

**Theorem 4.2.** *If $E$ has split-multiplicative reduction at $v$, then there are effective computable constants $c(j,d,e,v)$, $c_P(j,d,e,v) > 0$ only depending on $j$, $d$, $e$ and $v$, with $\widehat{h}_f(\alpha) \geq c(j,d,e,v)$ for all $\alpha \in M_e(v) \setminus \mathrm{PrePer}(f)$ and such that there are less than $c_P(j,d,e,v)$ preperiodic points in $M_e(v)$. With the notation of Theorem 4.1, we have*

*i)* $\widehat{h}_f(\alpha) \geq \frac{\frac{\log p}{2d}\mathfrak{c} - 3\log 2}{2(8\mathfrak{c}^3 - 2\mathfrak{c})(e!\operatorname{ord}_v(j^{-1}))^2} > 0$ *for all* $\alpha \in M_e(v) \setminus \mathrm{PrePer}(f)$

*ii)* $|\mathrm{PrePer}(f) \cap M_e(v)| < \frac{4}{3}\left(e!\mathfrak{c}\operatorname{ord}_v(j^{-1})\right)^3 + \left(e!\mathfrak{c}\operatorname{ord}_v(j^{-1})\right)^2 + 2.$

In the last section of this chapter we study our results in the case of an elliptic curve that has any reduction type at $v$. In contrast to Baker's proof, we can adopt the proof from Section 4.2 to show that the field $K^{nr,v}$ has the Bogomolov property relative to $\widehat{h}_E$ for all $E$ with bad reduction (of any type) at $v$. This result gives rise to an example which shows that the Bogomolov property relative to $\widehat{h}_E$ is not equivalent to the Bogomolov property relative to $\widehat{h}_f$, where $f$ is a Lattès map associated to $E$.

## 4.2 Proof of the main results

**Proof of Theorem 4.1:** Let $P$ be a point in $M_e^E(v)$, $w \mid v$ a valuation on $K(P)$ and $k_P(w)$ the residue field of $K(P)_w$. We choose a minimal Weierstrass equation for $E$ over $K(P)_w$. Denote by $\widetilde{E}$ the reduction of $E$ modulo $w$ and by $\widetilde{E}_{ns}$ the set of all non-singular points in $\widetilde{E}$. We set $E_0(K(P)_w) := \{P \in E(K(P)_w) | \widetilde{P} \in \widetilde{E}(k_P(w))_{ns}\}$. $E_0(K(P)_w)$ is a subgroup of $E(K(P)_w)$ of index $\operatorname{ord}_w(j^{-1}) = e_{w|v} \operatorname{ord}_v(j^{-1})$ (see [Si94], Corolarry IV.9.2). So we have $e_{w|v} \operatorname{ord}_v(j^{-1})P \in E_0(K(P)_w)$. From the choice of $P$ it is clear that $Q := e! \operatorname{ord}_v(j^{-1})P \in E_0(K(P)_w)$ for all $w \mid v$.
We take the local heights (also called *Néron functions*) $\lambda_w$ on $E(K(P)_w) \setminus O$ normalized such that we have the equation

$$\widehat{h}_E(P) = \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \in M_{K(P)}} d_w \lambda_w(P) \quad \forall P \in E(\overline{K}) \setminus O \quad .$$

We refer to [Si94], Chapter VI, for the details. For all elements $P'$ in $E_0(K(P)_w)$ and all $w \mid v$, $w \in M_{K(P)}$, we have

$$\lambda_w(P') = \frac{1}{2}\max\{w(x(P')^{-1}), 0\} + \frac{1}{12} w(\Delta) \tag{4.1}$$

$$\geq \frac{1}{12} w(j^{-1}) = \frac{1}{12} v(j^{-1}) \tag{4.2}$$

([Si94], Theorem VI.4.1). We recall that a valuation $w$ is defined as $w(.) = -\log |.|_w$. Notice that $E$ has split-multiplicative reduction over $K(P)$ at every $w \mid v$. This allows us to use the equation $w(\Delta) = w(j^{-1})$ (see [Si09], Proposition VII.5.1) to obtain (4.2).
We define the set $\Lambda_s = \{iQ | i \in \mathbb{N}, i \leq s\}$ for all $s \in \mathbb{N}$ such that $\Lambda_s$ consists of exactly $s$ points. Now we will estimate $\widehat{h}_E(P)$, respectively $\widehat{h}_E(Q)$, using bounds for the local heights. For any given absolute value $w$ we set $w^+$ to be the maximum of $w$ and 0.
If $w$ is archimedean, then we can use a theorem of Elkies improved by Baker and Petsche (see [BP05], Appendix A, and [Hr87]), namely

$$\sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \lambda_w(R - R') \geq -\frac{s}{2}\log s - \frac{16}{5}s - \frac{1}{12} w^+(j^{-1})s. \tag{4.3}$$

For a non-archimedean $w \in M_{K(P)}$, with $w(j^{-1}) \leq 0$, $E$ has potential good reduction at $w$ (see for example [Si09], Proposition VII.5.5). Let $K' \mid K(P)$ be a finite

extension such that $E$ has good reduction over $K'$ at a place $w' \mid w$. Then equation (4.1) shows that $\lambda_{w'}$ is non-negative on $E(K'_{w'}) \setminus O$. As $\lambda_{w'}$ and $\lambda_w$ coincide on $E(K(P)_w) \setminus O$, $\lambda_w$ is a non-negative function.
For non-archimedean $w \in M_{K(P)}$ with $w(j^{-1}) > 0$, [HS90], Proposition 1.2, gives the inequality

$$\sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \lambda_w(R - R') \geq \frac{1}{12} \left( \frac{s}{\operatorname{ord}_w(j^{-1})} \right)^2 w(j^{-1}) - \frac{s}{12} w(j^{-1}).$$

Thus for an arbitrary non-archimedean valuation $w$ we will use the estimation

$$\sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \lambda_w(R - R') \geq -\frac{s}{12} w^+(j^{-1}). \tag{4.4}$$

In the following calculations we will use Lemma 1.1 *ii)* several times. With (4.2), (4.3), (4.4) we get:

$$\sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \widehat{h}_E(R - R') = \sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \in M_{K(P)}} d_w \lambda_w(R - R')$$

$$\geq \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \mid \infty} d_w \left(-\frac{1}{2} s \log s - \frac{16}{5} s\right) - \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \mid \infty} d_w w^+(j^{-1}) \frac{1}{12} s \quad \text{(with (4.3))}$$

$$- \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \nmid \infty, w \nmid v} d_w \frac{s}{12} w^+(j^{-1}) \quad \text{(with (4.4))}$$

$$+ \sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \mid v} d_w \frac{1}{12} v(j^{-1}) \quad \text{(with (4.2))}$$

$$= -\frac{1}{2} s \log s - \frac{16}{5} s - \frac{1}{[K(P):\mathbb{Q}]} \sum_{w \nmid v} \left( d_w w^+(j^{-1}) \right) \frac{s}{12} + \frac{s^2 - s}{[K(P):\mathbb{Q}]} \sum_{w \mid v} d_w \frac{1}{12} v(j^{-1})$$

We know that for all $w \mid v$ we have $v(j^{-1}) = w(j^{-1}) = w^+(j^{-1})$. Thus we can use the definition of the standard logarithmic height $h$ to obtain

$$\begin{aligned} \sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \widehat{h}_E(R - R') &\geq \tfrac{d_v v(j^{-1})}{12d} s^2 - \left( \tfrac{1}{12} h(j) + \tfrac{16}{5} \right) s - \tfrac{1}{2} s \log s \\ &\geq \tfrac{\log p}{12d} s^2 - \left( \tfrac{1}{12} h(j) + \tfrac{16}{5} \right) s - \tfrac{1}{2} s \log s. \end{aligned} \tag{4.5}$$

If $P$ is a torsion point, then the left hand side is equal to zero. Clearly, $h(j)$ is greater than or equal to $\frac{d_v v(j^{-1})}{d} \geq \frac{\log p}{d}$, so we can apply Lemma 1.26 to deduce that the right hand side is greater than zero if $s \geq \frac{48d}{5\log p}\left(\log(\frac{6d}{\log p}) + \frac{1}{6}h(j) + \frac{32}{5}\right)$. As $s$ is a natural number, we get a contradiction to our choice of $P$ as a torsion point for $s = \mathfrak{c}$. This shows that there cannot exist a torsion point $P \in M_e^E(v)$ such that the order of $e!\operatorname{ord}_v(j^{-1})P$ is greater than or equal to $\mathfrak{c}$. Hence, there cannot exist a torsion point $P \in M_e^E(v)$ of order greater than or equal to $\mathfrak{c}\operatorname{ord}_v(j^{-1})(e!)$. Using $|E[k]| = k^2$ and the well known formula $\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$, for all $n \in \mathbb{N}$, we get that there are less than

$$\frac{1}{6}\mathfrak{c}\operatorname{ord}_v(j^{-1})(e!)(\mathfrak{c}\operatorname{ord}_v(j^{-1})(e!)+1)(2\mathfrak{c}\operatorname{ord}_v(j^{-1})(e!)+1)$$

torsion points in $M_e^E(v)$. Part *ii)* of Theorem 4.1 follows from the fact that $O$ lies in $E[k]$ for all $k \in \mathbb{N}$.

From now on we assume that $P$ is no torsion point. Then $\Lambda_s$ is defined for all $s \in \mathbb{N}$ and so (4.5) is valid for all $s \in \mathbb{N}$. We can combine the formula for the sum of the first $n$ squares from above with the elementary formula $\sum_{k=1}^n k^3 = \frac{1}{4}n^2(n+1)^2$ to achieve $\sum_{k=1}^n k^2(n+1-k) = \frac{1}{12}(n+1)^4 - \frac{1}{12}(n+1)^2$. Now, the definition of $\Lambda_s$ and the property $\widehat{h}_E(kQ) = k^2\widehat{h}_E(Q)$ for all $k \in \mathbb{Z}$ lead us to

$$\sum_{\substack{R,R'\in\Lambda_s\\ R\neq R'}} \widehat{h}_E(R-R') = \left(2\sum_{i=1}^{s-1} i^2(s-i)\right)\widehat{h}_E(Q) = \left(\frac{1}{6}s^4 - \frac{1}{6}s^2\right)\widehat{h}_E(Q) \quad . \tag{4.6}$$

If we further use (4.5) and the definition of $Q$, we find that the height $\widehat{h}_E(P)$ is bounded from below by

$$C'(j,d,e,v) := \max_{s\in\mathbb{N}} \frac{\frac{\log p}{2d}s - \left(\frac{1}{2}h(j) + \frac{96}{5}\right) - 3\log s}{(s^3-s)(e!\operatorname{ord}_v(j^{-1}))^2} \quad .$$

The value $C'(j,d,e,v)$ is obviously positive. In what follows we will calculate a lower bound for $C'(j,d,e,v)$. Let $H(j)$ be the multiplicative height of $j$ and let $\mathfrak{c}_W := -\frac{6d}{\log p}W_{-1}\left(-\frac{\log p}{6d}H(j)^{-1/6}\mathbf{e}^{-32/5}\right)$ be the greatest root of the real function

$$r(x) = \frac{\log p}{2d}x - \left(\frac{1}{2}h(j) + \frac{96}{5}\right) - 3\log x$$

(see Lemma 1.26). Then we know that this function is strictly positive for all $x > \mathfrak{c}_W$. Especially we have $r(2x) \geq \frac{\log p}{2d}x - 3\log 2 > 0$ for all $x \geq \mathfrak{c}_W$, with equality if and only if $x = \mathfrak{c}_W$. Again by Lemma 1.26 we have $1 < 2\mathfrak{c}_W < 2\mathfrak{c}$. With these inequalities we finally deduce

$$C'(j,d,e,v) \geq \frac{r(2\mathfrak{c})}{(8\mathfrak{c}^3 - 2\mathfrak{c})(e!\operatorname{ord}_v(j^{-1}))^2} \geq \frac{\frac{\log p}{2d}\mathfrak{c} - 3\log 2}{(8\mathfrak{c}^3 - 2\mathfrak{c})(e!\operatorname{ord}_v(j^{-1}))^2} > 0,$$

which concludes the proof. □

**Proof of Theorem 4.2:** We will combine Lemma 1.23 and the proof of Theorem 4.1. $E$ is assumed to have split-multiplicative reduction at $v$, and hence $\pi$ has degree two (see Section 3.1). Let $\alpha$ be an algebraic number in $M_e(v)$ and take a point $P \in E(\overline{K})$ with $\pi(P) = \alpha$. Then for all $w \mid v$ in $M_{K(P)}$ we have either $e_{w|v} \leq e$ or $e_{w|v} = 2e'$ with $e' \leq e$. Now we can start exactly the above proof with $Q := e!2\,\mathrm{ord}_v(j^{-1})P$ instead of $e!\,\mathrm{ord}_v(j^{-1})P$.
If $\alpha$ is preperiodic, then $P$ is a torsion point of order less than $e!2\mathfrak{c}\,\mathrm{ord}_v(j^{-1})$. So there are less than $\frac{8}{3}\mathfrak{c}\left(e!\,\mathrm{ord}_v(j^{-1})\right)^3 + 2\left(e!\mathfrak{c}\,\mathrm{ord}_v(j^{-1})\right)^2$ choices for $P$. As every $\alpha$, that is no critical value of $\pi$, has exactly two pre-images under $\pi$, and there are exactly 4 critical values of $\pi$ (see [Si07], Lemma 6.38), we get analogue to the proof of Corollary 2.7

$$|\,\mathrm{PrePer}(f) \cap M_e(v)| < \frac{4}{3}\left(e!\mathfrak{c}\,\mathrm{ord}_v(j^{-1})\right)^3 + \left(e!\mathfrak{c}\,\mathrm{ord}_v(j^{-1})\right) + 2.$$

If $\alpha$ is no preperiodic point, then $P$ is not a torsion point and we have

$$\widehat{h}_f(\alpha) = 2\widehat{h}_E(P) \geq \frac{\frac{\log p}{2d}\mathfrak{c} - 3\log 2}{2(8\mathfrak{c}^3 - 2\mathfrak{c})(e!\,\mathrm{ord}_v(j^{-1}))^2} > 0 \quad .$$

This concludes the proof. □

## 4.3 Corollaries and additional results

Now we want to study the behavior of the canonical height $\widehat{h}_E$ on the set $M_e^E(v)$ if $E$ has not split-multiplicative reduction at $v$. If $E$ has multiplicative or potential multiplicative reduction at $v$, then it has split-multiplicative reduction after a finite field extension. So, Theorem 4.1 will be true in this case after a small adjustment of the constants. In the case of good reduction of $E$ at $v$ the criterion of Néron-Ogg-Shafarevich will show that there are infinitely many torsion points and points of arbitrarily small positive height in $M_e^E(v)$ for all $e \in \mathbb{N}$. If $E$ has additive potential good reduction at $v$, then we will prove that Theorem 4.1 is true for $e = 1$, i.e. for the field $K^{nr,v}$.

**Corollary 4.3.** *Let $E$ be an elliptic curve with potential multiplicative reduction at $v \mid p$. Then we have*

*i)* $\widehat{h}_E(P) \geq \frac{\frac{\log(p)}{2dk} - 3\log 2}{(8\mathfrak{c}_k^3 - 2\mathfrak{c}_k)(e!\,\mathrm{ord}_v(j^{-1}))^2} > 0$ *for all* $P \in M_e^E(v) \setminus E_{\mathrm{tor}}$

*ii)* $|E_{\mathrm{tor}} \cap M_e^E(v)| < \frac{1}{3}\left(e!\mathfrak{c}_k\,\mathrm{ord}_v(j^{-1})\right)^3 + \frac{1}{2}\left(e!\mathfrak{c}_k\,\mathrm{ord}_v(j^{-1})\right)^2.$

*Here* $\mathfrak{c}_k := \left\lceil \frac{10dk}{\log p}\left(\log(\frac{6dk}{\log p}) + \frac{1}{6}h(j) + \frac{32}{5}\right)\right\rceil$, *and* $k \leq 48$ *is the smallest degree of a field extension* $K'/K$ *such that* $E$ *over* $K'$ *has split-multiplicative reduction at a* $v' \mid v$, $v' \in M_{K'}$.

**Proof:** By assumption there is a field extension $K'/K$ such that $E$ over $K'$ has split-multiplicative reduction at a place $v' \mid v$ of $K'$. Assume furthermore that the degree of $K'/K$ is minimal with this property and denote this degree by $k$. If $E$ is given in Legendre normal form and $p \neq 2$, then $E$ admits split-multiplicative reduction at a place $v' \mid v$ on an extension field $K'/K$ of degree at most 4 (see [Si09], proof of Proposition VII 5.4 c)). The fact that $E$ is isomorphic to an elliptic curve in Legendre normal form over a field extension of degree at most 12 shows that $k \leq 48$ whenever $p \neq 2$. If $p = 2$ we can write $E$ in Deuring normal form. The following facts regarding this normal form can be found in the proofs of [Si09], Proposition A 1.3 and Corollary A 1.4. We obtain such a Deuring normal form of $E$ after a field extension of degree at most 12. Now $E$ admits split-multiplicative reduction at a place $v' \mid v$ at least after a quadratic extension. This shows that $k \leq 48$ in any case. Of course we have $k = 2$ in case of multiplicative reduction at $v$.
Let $P$ be a point in $M_e^E(v)$ and $w' \mid v'$ an extension of $v'$ to $K'(P)$. Denote the restriction of $w'$ to $K(P)$ by $w$. Then by assumption and Lemma 1.25 we have $e \geq e_{w|v} \geq e_{w'|v'}$. Thus we can apply Theorem 4.1 with $d$ replaced by $dk$. □

We have proven a lower bound for the canonical heights $\widehat{h}_E$ and $\widehat{h}_f$, where $f$ is a Lattès map associated to $E$, on sets. Our main interest concerns lower bounds on fields. For a field $L$ lying inside $M_e(v)$ for some $e \in \mathbb{N}$, Theorem 4.1 and Theorem 4.2 give us lower bounds for the canonical heights on $E(L) \setminus E_{\text{tor}}$ and $L \setminus \text{PrePer}(f)$. But we can achieve much nicer bounds if we additionally assume that $L/K$ is normal. In this case the term $e!$ in our bound can be replaced by $e$. Formally:

**Corollary 4.4.** *Let $E$ be an elliptic curve defined over $K$ with split-multiplicative reduction at $v \mid p$ and let $\mathfrak{c}$ be as in Theorem 4.1. Let further $L/K$ be a galois extension with $L \subset M_e(v)$, for a fixed $e \in \mathbb{N}$. Then we have*

*i)* $\widehat{h}_E(P) \geq \frac{\frac{\log p}{2d}\mathfrak{c} - 3\log 2}{(8\mathfrak{c}^3 - 2\mathfrak{c})(e\,\text{ord}_v(j^{-1}))^2} > 0$ *for all* $P \in E(L) \setminus E_{\text{tor}}$

*ii)* $|E_{\text{tor}} \cap L| < (\mathfrak{c}\,\text{ord}_v(j^{-1})e)^2$

**Proof:** In the proof of Theorem 4.1, $e!$ was an upper bound for the lowest common multiple of all $e_{w|v}$, $w \in M_{K(P)}$, which does not depend on $P$. Now let $K'(P)$ be the normal closure of $K(P)$. From our assumption we know that $K'(P)$ is contained in $L \subset M_e$. Thus we have $e_{w'|v} \leq e$ for all $w' \in M_{K'(P)}$ lying above $v$. Exactly as in the proof of Theorem 4.1, we conclude that for all these $w'$ we have $e_{w'|v}\,\text{ord}_v(j^{-1})P \in E_0(K'(P)_{w'})$. But as $K'(P)/K$ is galois, we know that all these $e_{w'|v} =: e_v$ are equal. Hence we set $Q := e_v\,\text{ord}_v(j^{-1})P$, where $e_v \leq e$. Exactly as in the proof of Theorem 4.1 we achieve that there is no torsion point in $E(L)$ of order greater than $\mathfrak{c}\,\text{ord}_v(j^{-1})e$. Let $P$ be a torsion point in $E(L)$ of maximal order $k \leq \mathfrak{c}\,\text{ord}_v(j^{-1})e$. We claim that all torsion points in $E(L)$ lie in $E[k]$. Assume this is not the case, then there exists a torsion point $P' \in E(L)$ of order not dividing $k$. The order of the point $P + P' \in E(L)$ is exactly the smallest common multiple of the orders of $P$ and $P'$, and hence it is greater than $k$. This is a contradiction to

the maximality of $k$, proving the claim. We can conclude that there are less than $k^2 \leq (\mathfrak{c}\operatorname{ord}_v(j^{-1})e)^2$ torsion points in $E(L)$.
Statement $i)$ follows exactly as in the proof of Theorem 4.1. □

**Remark 4.5.** Obviously, Corollary 4.4 similarly holds for places $v$ where $E$ has potential split-multiplicative reduction. Moreover, Corollaries 4.3 and 4.4 have dynamical analogues. Let $f$ be a Lattès map associated to the elliptic curve $E$. In the setting of Corollary 4.3 we can apply the proof of Theorem 4.2 to achieve

$$\widehat{h}_f(\alpha) \geq \frac{\frac{\log(p)}{2dk} - 3\log 2}{2(8\mathfrak{c}_k^3 - 2\mathfrak{c}_k)(e!\operatorname{ord}_v(j^{-1}))^2} > 0 \text{ for all } \alpha \in M_e(v) \setminus \operatorname{PrePer}(f) \text{ , and}$$
$$|\operatorname{PrePer}(f) \cap M_e(v)| < \frac{4}{3}(e!\mathfrak{c}_k \operatorname{ord}_v(j^{-1}))^3 + (e!\mathfrak{c}_k \operatorname{ord}_v(j^{-1}))^2 + 2 \quad .$$

In the setting of Corollary 4.4 we can similarly combine the proofs of Theorem 4.2 and Corollary 4.4 to get

$$\widehat{h}_f(\alpha) \geq \frac{\frac{\log p}{2d}\mathfrak{c} - 3\log 2}{2(8\mathfrak{c}^3 - 2\mathfrak{c})(e\operatorname{ord}_v(j^{-1}))^2} > 0 \text{ for all } \alpha \in L \setminus \operatorname{PrePer}(f) \text{ , and}$$
$$|\operatorname{PrePer}(f) \cap L| < \frac{4}{3}(e\mathfrak{c}\operatorname{ord}_v(j^{-1}))^3 + (e\mathfrak{c}\operatorname{ord}_v(j^{-1}))^2 + 2 \quad .$$

In the case where $E$ has additive reduction at $v$ and $e = 1$ we can also use our computation from Theorem 4.1. The following proposition is actually a remark of Joseph Silverman in an email to the author.

**Proposition 4.6.** *Let $v \mid p$ be a finite place of $K$ where $E$ has additive reduction. With $\mathfrak{c}$ as in Theorem 4.1, we have*

*i)* $\widehat{h}_E(P) \geq \frac{\frac{\log p}{2d}(\mathfrak{c}+2) - 3\log 2}{(8(\mathfrak{c}+2)^3 - 2(\mathfrak{c}+2))144} > 0$ *for all* $P \in M_1^E(v) \setminus E_{\text{tor}}$

*ii)* $|E_{\text{tor}} \cap M_1^E(v)| < (12\mathfrak{c} + 24)^2$.

**Proof:** The proof is almost the same as in the case of split-multiplicative reduction. Let $P$ be in $M_1^E(v)$ and $w \mid v$ a place of $K(P)$. By assumption $v$ is unramified in the extension $K(P)/K$. Hence, $E$ has still additive reduction at $w$ over $K(P)$. Fix a minimal Weierstrass equation of $E$ over $K(P)_w$, and denote the discriminant of $E$ by $\Delta$. Then $E_0(K(P)_w)$ is a subgroup of $E(K(P)_w)$ of order at most 4 (see [Si94], Corollary IV 9.2). Thus we have $Q := 12P \in E_0(K(P)_w)$ for all $w \mid v$ on $K(P)$. The explicit formula (4.1) for the local height $\lambda_w$ on $E_0(K(P)_w)$ gives us

$$\lambda_w(Q) \geq \frac{1}{12}w(\Delta) \geq \frac{1}{12e_{v|p}}\log p > 0 \quad \text{for all } w \mid v \text{ on } K(P) \quad .$$

This follows from the fact that $w(\Delta) = 0$ if and only if $E$ has good reduction at $w$ ([Si09], Proposition VII.5.1), which is not the case as we have noticed above. Define the set $\Lambda_s := \{iQ | i \in \mathbb{N}, i \leq s\}$ for all $s \in \mathbb{N}$ such that $\Lambda_s$ consists of exactly $s$ points. Word-for-word as before we get the lower bound

$$\sum_{\substack{R,R' \in \Lambda_s \\ R \neq R'}} \widehat{h}_E(R - R') \geq \frac{\log p}{12d} s^2 - \left( \frac{1}{12} h(j) + \frac{\log p}{12d} + \frac{16}{5} \right) s - \frac{1}{2} s \log s \quad .$$

Using Lemma 1.26 and the definition of $\mathfrak{c}$ we find that the right hand side is greater 0 for $s$ greater than $\mathfrak{c} + \frac{8}{5} < \mathfrak{c} + 2$. Thus there cannot exist a torsion point of order greater than $12\mathfrak{c} + 24$. Notice that $M_1^E(v) = E(K^{nr,v})$, and therefore $M_1^E$ is an abelian group. As in the proof of Corolarry 4.4, we conclude that all torsion points in $M_1^E$ lie in the set $E[k]$, for a $k \leq 12\mathfrak{c} + 24$. This implies part $ii$). If $P$ is not a torsion point we conclude

$$\widehat{h}_E(P) \geq \frac{\frac{\log p}{2d}(\mathfrak{c} + 2) - 3 \log 2}{(8(\mathfrak{c} + 2)^3 - 2(\mathfrak{c} + 2))144} > 0 \quad .$$

This again follows exactly as in the proof of Theorem 4.1. □

**Remark 4.7.** Remark 4.5 obviously implies the desired dynamical analogue of Theorem 3.1. Let $E$ be an elliptic curve defined over the number field $K$ with potential multiplicative reduction at $v$, and let $f$ be an associated Lattès map. Then every finite extension of $K^{nr,v}$ has the Bogomolov property relative to $\widehat{h}_f$. The next lemma shows that there are also infinite extensions of $K^{nr,v}$ having the Bogomolov property relative to $\widehat{h}_f$, for example the compositum of $K^{nr,v}$ and $K^{(d)}$. One can use the results in Section 3.2 to see that this is indeed an infinite extension of $K^{nr,v}$.

**Lemma 4.8.** *Let $L/K$ and $M/K$ be field extensions such that $L \subset M_e(v)$ and $M \subset M_{e'}(v)$ for a non-archimedean valuation $v$ on $K$. Then $LM \subset M_{ee'}(v)$.*

**Proof:** Without loss of generality we assume that $L$ and $M$ are extension fields of $K$. Let $F \subset M$ be any subfield with $[F : K] \leq \infty$. Moreover we choose any $w' \in M_{LM}$, with $w' \mid v$, and define $w = w'|_L$ and $v' = w'|_F$. With Lemma 1.25 we conclude $e_{w'|v'} \leq e_{w|v} \leq e$. This leads us to $e_{w'|v} = e_{w'|v'} e_{v'|v} \leq ee'$. The fact that for every $\alpha \in LM$ there exists such a finite extension $F/K$ with $\alpha \in LF$ concludes the proof. □

**Proposition 4.9.** *If $E$ has good reduction at $v \mid p$ and $f$ is an associated Lattès map, then neither of the statements in Theorem 4.1 and 4.2 is true.*

**Proof:** Lets start with Theorem 4.1. By the criterion of Néron-Ogg-Shafarevich all points of order $m$ with $p \nmid m$ are unramified over $v$ (see [Si09], Theorem VII 7.1). In particular, there are infinitely many torsion points of $E$ in $K^{nr,v}$ and hence in $M_e^E(v)$ for all $e \in \mathbb{N}$.
Take an arbitrary point $P_0 \in E(K^{nr,v}) \setminus E_{\mathrm{tor}}$ and an integer $m \geq 2$ coprime to $p$. Choose a sequence $\{P_n\}_{n \in \mathbb{N}}$ of points in $E(\overline{K})$, satisfying $[m]P_n = P_{n-1}$. By [Si09], Proposition VIII 1.5 b), all $P_n$ are elements in $E(K^{nr,v}) \setminus E_{\mathrm{tor}}$ and we have

$$\widehat{h}_E(P_n) = \frac{1}{m^2} \widehat{h}_E(P_{n-1}) = \frac{1}{m^{2n}} \widehat{h}_E(P_0) \to 0 \quad ,$$

as $n \to \infty$. Thus there are points of arbitrarily small height in $M_e^E(v)$ for all $e \in \mathbb{N}$. Contradictions for the statements in Theorem 4.2 follow quickly. For all $P$ in $E(K^{nr,v})$ the value $\pi(P)$ is also an element of $K^{nr,v}$. The degree of $\pi$ is finite, hence the equation $\mathrm{PrePer}(f) = \pi(E_{\mathrm{tor}})$ (see 1.19) shows that $K^{nr,v}$ contains infinitely many preperiodic points of $f$. We apply Lemma 1.23 to the sequence $\{P_n\}_{n\in\mathbb{N}}$ from above to see that $\widehat{h}_f$ can get arbitrarily small on $K^{nr,v} \setminus \mathrm{PrePer}(f)$. □

**Example 4.10.** As usual let $K$ be a number field, $E : y^2 = x^3 + Ax + B$ an elliptic curve defined over $K$, and let $f$ be a Lattès map associated to $E$ and $\pi(x, y) = x$. The following example shows two things. Firstly, the Bogomolov property relative to $\widehat{h}_E$ is in general not preserved under finite field extensions. Secondly, a field can have the Bogomolov property relative to $\widehat{h}_E$ but not relative to $\widehat{h}_f$.
We will use the theory of twists that we have introduced in 2.4. Assume that $E$ has additive reduction at a finite $v \in M_K$, and that there is an element $\gamma \in K$ such that the twist $E_\gamma$ has good reduction at $v \in M_K$. (One might choose $K = \mathbb{Q}$, $v = p \geq 3$, $E : y^2 = x^3 + p^2x$ and $\gamma = p^{-1}$). Let $g_{\gamma^{-1}} : E_\gamma \to E$ be the isomorphism from 2.4. Notice that $\sqrt{\gamma}$ cannot be an element in $K^{nr,v}$, since the reduction type of $E$ at $v$ changes if we extend $K$ to $K(\sqrt{\gamma})$. As $E_\gamma$ has good reduction at $v$, Proposition 4.9 yields a sequence $\{P_i\}_{i\in\mathbb{N}}$ in $E_\gamma(K^{nr,v})$ such that $\widehat{h}_{E_\gamma}(P_i) \to 0$, as $i \to \infty$. The elements $g_{\gamma^{-1}}(P_i)$ all lie in $E(K^{nr,v}(\sqrt{\gamma}))$, and from the definition of $g_{\gamma^{-1}}$ we know that $\alpha_i := \pi(g_{\gamma^{-1}}(P_i))$ is in $K^{nr,v}$. Using Lemma 1.23 and (2.4) we get

$$\widehat{h}_f(\underbrace{\alpha_i}_{\in K^{nr,v}}) = 2\widehat{h}_E(\underbrace{g_{\gamma^{-1}}(P_i)}_{\in K^{nr,v}(\sqrt{\gamma})}) = 2\widehat{h}_{E_\gamma}(P_i) \to 0 \quad ,$$

as $i \to \infty$. But $K^{nr,v}$ has the Bogomolov property relative to $\widehat{h}_E$, as $E$ has bad reduction at $v$ (see Proposition 4.6).

**Remark 4.11.** We claim that $K^{nr,v}$ does not have the Bogomolov property relative to $h$. Let $q$ be the prime number with $v \mid q$, and let $p$ be any other prime. The field $\mathbb{Q}^{nr,q}$ is a subfield of $K^{nr,v}$. By Lemma 1.24, the elements $p^{1/p^n}$ lie in $\mathbb{Q}^{nr,q}$, for all $n \in \mathbb{N}$. As seen before, the height $h$ of the elements $p^{1/p^n}$ tends to zero. Thus, $K^{nr,v}$ does not have the Bogomolov property relative to $h$.
For some number fields $K$ there is an infinite extension $L/K$ which is everywhere unramified. For example those number fields with infinite class field tower (see [Ne], VI, §6 and §7, and the references therein). It is an interesting (and open) question whether or not such a field $L \subset \overline{\mathbb{Q}}$ has the Bogomolov property relative to $h$. This is a question of Ellenberg, brought to the author in a conversation with Martin Widmer.

# 5 Heights and totally real numbers

## 5.1 Introduction

Let us first recall the basic definitions of totally real algebraic numbers. An algebraic number is called *totally real* if all its conjugates are real. Similarly, a number field $K$ is called *totally real* if all its embeddings into $\overline{\mathbb{Q}}$ are real. In the same way, a number field $K$ is called *totally imaginary* if no embedding of $K$ is real. A *complex multiplication field*, short *CM-field*, is a totally imaginary quadratic extension of a totally real field. Sometimes the union of totally real fields and CM-fields is denoted by *J-fields*.
In this chapter we will study dynamical heights on the compositum of all totally real number fields which we denote by $\mathbb{Q}^{tr}$. In [Sch73] Schinzel proved the following.

**Theorem 5.1** (Schinzel)**.** *The standard logarithmic height $h$ on $\mathbb{Q}^{tr} \setminus \{-1, 0, 1\}$ is bounded from below by $\frac{1}{2}\log(\frac{1+\sqrt{5}}{2})$.*

This bound is sharp, as it is attained by $\frac{1+\sqrt{5}}{2}$. Actually, Schinzel proved that this bound is valid for all algebraic numbers $\alpha \neq 0$ lying in a CM-field, with $|\alpha| \neq 1$. The latter condition is necessary for the existence of a lower bound. This was proven in [AN07] by Amoroso and Nuccio. In Section 5.5 we will give a very short proof of this result.
A one-page proof of Theorem 5.1 was later given by Höhn and Skoruppa (see [HS93]). Garza, Ishak and Pinner ([GIP10]) generalized Schinzels result by showing that it suffices to have *´enough'* real conjugates in order to have a lower bound for the height $h$. Formally they proved:

**Theorem 5.2.** *Let $\alpha_1, \ldots, \alpha_t \in \overline{\mathbb{Q}}^*$ be algebraic numbers such that $\boldsymbol{\alpha} := \alpha_1 + \cdots + \alpha_t \neq \alpha_1^{-1} + \cdots + \alpha_t^{-1}$. Denote the degree of $\boldsymbol{\alpha}$ by $d$, the number of real conjugates of $\boldsymbol{\alpha}$ with $r$ and set $R := \frac{r}{d}$. If $r$ is at least one we have*

$$\sum_{i=1}^{t} h(\alpha_i) \geq \frac{R}{2}\log\left(\frac{2^{1-1/R} + \sqrt{4^{1-1/R} + 4}}{2}\right) .$$

Table 3 summarizes briefly partial results of this theorem. With the notation from above we see that the point $(\alpha_1, \ldots, \alpha_t)$ lies on the curve $x_1 + \cdots + x_t = \boldsymbol{\alpha}$ in $\mathbb{G}_m^n(\overline{\mathbb{Q}}^*)$. From a deep theorem of Zhang (see [Zh92], Theorem 6.2) it follows that there exists a positive lower bound for $\sum_{i=1}^{t} h(\alpha_i)$ which depends on $\boldsymbol{\alpha}$.
Smyth studied in [Sm80] the set $\mathcal{M} = \{h(\alpha) | \alpha \text{ an integer in } \mathbb{Q}^{tr}\}$ and proved that $\frac{1}{2}\log(\frac{1+\sqrt{5}}{2})$ is an isolated point in $\mathcal{M}$. Furthermore, he could prove that every point of the interval $[\lambda, \infty)$ is a limit point of elements in $\mathcal{M}$. Here $\lambda$ is given as follows. Define the sequence $\alpha_0 = 1$ and $\alpha_{n+1} - \alpha_{n+1}^{-1} = \alpha_n$. It is not hard to check that all $\alpha_n$ are totally real and that $h(\alpha_n)$ is independent of choices. We have

$$\lambda = \lim_{n\to\infty} h(\alpha_n) = 0.2732\ldots \quad .$$

| $t$ | $\boldsymbol{\alpha}$ | Reference |
|---|---|---|
| 1 | totally real | Schinzel [Sch73] |
| 2 | 1 | Zagier [Za93] |
| arbitrary | natural number | Beukers, Zagier [BZ97] |
| arbitrary | totally real | Samuels [Sa06] |
| 1 | at least one real conjugate | Garza [Ga07] (see also [Ho11]) |

Table 3: Partial results of Theorem 5.2

Moreover, Flammang ([Fl96], extending works of Smyth [Sm81]) gave the six smallest values in $\mathcal{M}$. They are $h(\alpha_1)$, $h(\alpha_2)$, $h(\alpha_3)$, $h(\zeta_7+\zeta_7^{-1})$, $h(\zeta_{60}+\zeta_{60}^{-1})$ and $h(\alpha_4)$, and these six values are the only elements in $\mathcal{M}\cap[0,0.2713]$. Here $\zeta_n$ denotes a primitive $n$-th root of unity.

We want to study Schinzel's theorem from a dynamical sight. The height $h$ is equal to the dynamical height $\widehat{h}_{x^2}$. Hence, with loss of the explicit bound, we can state Theorem 5.1 as:

$$\mathbb{Q}^{tr} \textit{ has the Bogomolov property relative to } \widehat{h}_f, \textit{ where } f(x)=x^2.$$

In this formulation it is natural to ask for which other rational functions $f\in\overline{\mathbb{Q}}(x)$ the above statement is true. In Section 5.2 we will show that this statement is true whenever at least one galois conjugate of $f$ has a Julia set not contained in the real line. This result follows directly from the dynamical equidistribution Theorem 1.15, and hence it is non-effective. We will also give a class of quadratic polynomials $f_c$ such that $\mathbb{Q}^{tr}$ does not have the Bogomolov property relative to $\widehat{h}_{f_c}$. In Section 5.3 we will prove the converse; i.e. $\mathbb{Q}^{tr}$ has the Bogomolov property relative to $\widehat{h}_f$ if and only if at least one galois conjugate of $f$ has non-real Julia set. Roughly speaking, this emphasizes the strength of the equidistribution theorem, as we have a lower bound for the canonical height on $\mathbb{Q}^{tr}$ if and only if we can apply the equidistribution theorem. A possible connection to Salem numbers is mentioned in the short Section 5.4. The last section of this chapter is independent from the previous results. There we study the surprising behavior of $h$ on finite extensions of $\mathbb{Q}^{tr}$.

## 5.2 A first example

To prepare the first Theorem in this section we recall some notations from the end of Section 1.2. For an algebraic number $\alpha$ and any number field $K$, we set $G_K(\alpha):=\{\sigma(\alpha)|\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/K)\}$. For every $v\in M_K$ we denote by $\mathbb{C}_v$ the completion of $\overline{K_v}$. For all $v\in M_K$ we fix once and for all an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_v$. Let $(X,L,f)$ be an algebraic dynamical system defined over $K$. Then there exists a canonical $f$-invariant probability measure $\mu_{f,v}$ on the Berkovich space $X_v^{an}$. If $f(x)\in K(x)$ is a rational function of degree at least two, then $\mu_{f,v}$ is the canonical measure associated to $(\mathbb{P}^1,\mathcal{O}(1),f)$. In case where $v$ is archimedean this is exactly the canonical measure from Therem 1.13. The Berkovich Julia set $J_v^{\mathcal{B}}(f)$ is defined as the support of the

canonical measure $\mu_{f,v}$. This includes the archimedean case, where the Berkovich Julia set coincides with the usual Julia set $J(f)$.
We want to study canonical heights $\widehat{h}_f$ on the field $\mathbb{Q}^{tr}$. This was our main motivation for the next theorem and a first version only covered Corollary 5.4. Paul Fili pointed out that the same proof applies in a more general setting (see his preprint [FM12] with Zachary Miner).

**Theorem 5.3.** *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree $\geq 2$, and let $K$ be a number field with valuation $v \in M_K$ such that the Berkovich Julia set $J_v^{\mathcal{B}}(f)$ is not contained in the closure of $(\mathbb{P}^1)_v^{an}(K)$. If $L/K$ is a galois extension lying in $K_v$, then $L$ has the Bogomolov property relative to $\widehat{h}_f$. Furthermore, there are only finitely many preperiodic points of $f$ in $L$.*

**Proof:** Let $F$ be a number field such that $f \in F(x)$ and $K \subseteq F$. Assume there is a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ in $L$ of pairwise distinct elements satisfying $\widehat{h}_f(\alpha_i) \to 0$ for $i \to \infty$. Denote by $\overline{\delta_i}$ the equidistributed probability measures on the set $G_F(\alpha_i)$. The support of $\overline{\delta_i}$ lies in $K_v$ for all $i \in \mathbb{N}$, as $L/K$ was assumed to be galois. Notice that the choice of $F$ implies $G_F(\alpha_i) \subseteq G_K(\alpha_i)$, for all $i \in \mathbb{N}$.
By assumption, there exists an $\alpha \in J_v^{\mathcal{B}}(f) = \mathrm{supp}(\mu_{f,v})$ which is not contained in the closure of $K_v$ in $(\mathbb{P}^1)_v^{an}$. As $(\mathbb{P}^1)_v^{an}$ is a Hausdorff space, there is an open neighborhood $U$ of $\alpha$ such that $U \cap K_v = \emptyset$. By Theorem 1.15, the measures $\overline{\delta_i}$ converge weakly to $\mu_{f,v}$ and hence

$$0 = \lim_{i\to\infty} \overline{\delta_i}(U) = \mu_{f,v}(U) \neq 0 \quad .$$

This is a contradiction, and hence there cannot exist such a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$. $\square$

The case $K = \mathbb{Q}$, $L = \mathbb{Q}^{tr}$ and $v = \infty$ yields the following result.

**Corollary 5.4.** *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree $\geq 2$ such that the Julia set $J(f)$ of $f$ is not contained in the real line. Then $\mathbb{Q}^{tr}$ has the Bogomolov property relative to $\widehat{h}_f$. Furthermore, there are only finitely many preperiodic points of $f$ in $\mathbb{Q}^{tr}$.*

Some examples of rational functions with real Julia set are Chebyshev polynomials (see 1.17), polynomials of the form $x^2 - c$ with real $c \geq 2$ (see Remark 5.7), and linear conjugates of Blaschke products by a Möbius transformation that maps the unit circle onto the real line. Recall that a Blaschke product is a map in $\mathbb{C}(x)$ of the form

$$B(x) = a \prod_{i=1}^{t} \frac{x - a_i}{1 - \overline{a_i}x}, \quad |a| = 1 \text{ , and } |a_i| < 1 \quad \forall i = 1, \ldots, t \quad .$$

It is not hard to check that $|B(z)| = 1$ if and only if $|z| = 1$. Hence, the unit circle is a closed completely invariant set. It follows that the Julia set of a Blaschke product lies on the unit circle (see [Be], Theorem 4.4.2). Let $g$ be any Möbius transformation which maps the unit circle onto the real line. Then the map $g \circ B \circ g^{-1}$ has Julia set $g(J(B)) \subseteq \mathbb{R}$ (see [Be], Theorem 3.1.4).

**Remark 5.5.** Apart from a non-effective version of Schinzels result, Corollary 5.4 also includes a special case of a theorem of Zhang ([Zh98], Corollary 2). As we have noticed in 1.19, the Julia set of a Lattès map $f$ over a number field $K$ is the Riemann sphere. Hence, Corollary 5.4 applies for Lattès maps. Let $E$ be the elliptic curve associated to $f$. Lemma 1.23 tells us that there also exists a positive constant $c$ such that $\widehat{h}_E(P) \geq c$ for all non-torsion points $P \in E(\mathbb{Q}^{tr})$ and there are only finitely many torsion points in $E(\mathbb{Q}^{tr})$. Notice that there is an effective constant $c$ in the case where $K$ is totally real (see [BP05], Theorem 17).

The next proposition provides a class of polynomials $f$ such that $\mathbb{Q}^{tr}$ does not have the Bogomolov property relative to $\widehat{h}_f$.

**Proposition 5.6.** *For every $c \in \overline{\mathbb{Q}}$ we set $f_c(x) := x^2 - c$. If $c$ is a rational number with $c \geq 2$, then $\mathbb{Q}^{tr}$ does not have the Bogomolov property relative to $\widehat{h}_{f_c}$.*

**Proof:** Take an $\epsilon \in (-c, c) \cap \mathbb{Q}$ such that $\epsilon$ is no preperiodic point of $f_c$. This is possible by Northcott's theorem and the fact that $\widehat{h}_{f_c}$ vanishes precisely at the preperiodic points of $f_c$ (see Theorem 1.9). We will prove that for each $n \in \mathbb{N}$ the set $f_c^{-n}(\epsilon)$ is contained in $\mathbb{Q}^{tr}$. Then for all $n$ we take an arbitrary $\gamma_n$ in $f_c^{-n}(\epsilon)$ and obtain a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in $\mathbb{Q}^{tr}$, with $\widehat{h}_{f_c}(\gamma_n) = \frac{1}{2^n}\widehat{h}_{f_c}(\epsilon)$. This tends to zero, which proves the claim.

Since $c$ is in $\mathbb{Q}$, we have $\sigma(f_c^{(n)}(\gamma)) = f_c^{(n)}(\sigma(\gamma))$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all algebraic numbers $\gamma$. Thus we know that the set $f_c^{-n}(\epsilon)$ is galois invariant, as also $\epsilon$ was chosen to lie in $\mathbb{Q}$. Hence it suffices to show that $f_c^{-n}(\epsilon)$ is contained in the real line in order to prove the proposition.
We will prove this by induction over $n$. For $n = 1$ we have $f_c^{-1}(\epsilon) = \pm\sqrt{\epsilon + c}$ which is real by the choice of $\epsilon$. Moreover we have $|\pm\sqrt{\epsilon + c}| < c$, since $c \geq 2$ and $|\epsilon| < c$. Now assume that for a given $n \in \mathbb{N}$ we have $f_c^{-n}(\epsilon) \subset \mathbb{R}$ and $|\gamma| < c$ for all $\gamma \in f_c^{-n}(\epsilon)$. Every $\beta \in f_c^{-(n+1)}(\epsilon)$ is an element of $f_c^{-1}(\gamma)$ for a $\gamma \in f_c^{-n}(\epsilon)$. So, it is of the form $\beta = \pm\sqrt{\gamma + c}$. We conclude exactly as in the case $n = 1$ that we have $\beta \in \mathbb{R}$ and $|\beta| < c$. □

**Remark 5.7.** Indeed we claim that the above argumentation proves that $J(f_c)$ is contained in the real line for all real $c \geq 2$. We see that such a $f_c$ has the repelling fixed points $\pm\sqrt{c + \frac{1}{4}} + \frac{1}{2}$. By Fact 1.8 *d*), $-\sqrt{c + \frac{1}{4}} + \frac{1}{2}$ is an element of $J(f_c)$ contained in the interval $(-c, c)$. With the same proof as above we conclude that the backward orbit of this repelling fixed point lies in $\mathbb{R}$. As the backward orbit of every element in $J(f_c)$ is dense in $J(f_c)$ (see [Be], Theorem 4.2.7 ii)), we get $J(f_c) \subseteq \mathbb{R}$.
If $c$ is in $\overline{\mathbb{Q}} \cap \mathbb{R}$, $c \geq 2$, the same proof shows that $\mathbb{Q}(c)^{tr} := \{\alpha \in \overline{\mathbb{Q}} | \sigma(\alpha) \in \mathbb{R}\ , \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(c))\}$ does not have the Bogomolov property relative to $\widehat{h}_{f_c}$.

The proof of our main theorem in this chapter is similar to the one above, but uses some abstract results on Julia sets that lie in a circle on the Riemann sphere. One of these results is the following Lemma.

**Lemma 5.8.** *Let $f \in \mathbb{C}(x)$ be a rational map of degree $\geq 2$. If the Julia set $J(f)$ of $f$ is contained in the real line, then $f$ has a representative with only real coefficients.*

**Proof:** Choose the representative of $f$ with monic enumerator. Then we can write $f(x) = \frac{A(x)+iB(x)}{C(x)+iD(x)}$ with real valued polynomials $A$, $B$, $C$ and $D$ such that $A(x)+iB(x)$ and $C(x)+iD(x)$ have no common zeros. As $A(x)+iB(x)$ was assumed to be monic, we know that $A$ is not equal to zero and $\deg(B) < \deg(A)$. For $r \in \mathbb{R}$ we see that $f(r)$ is real if and only if $A(r)D(r) - B(r)C(r) = 0$. Applying the Facts 1.8 to our assumption $J(f) \subset \mathbb{R}$ we know that $f$ maps infinitely many real elements to real elements. Hence we have the equation $A(x)D(x) = B(x)C(x)$. We want to show that this can only occur if $D(x) = B(x) = 0$, this means only if $f \in \mathbb{R}(x)$.
Assume that $B(x) \neq 0$. Then also $D(x) \neq 0$ and we have $0 \leq \deg(B) < \deg(A)$. Thus $A(x)$ cannot be a divisor of $B(x)$. This shows that a greatest common divisor $R_1(x)$ of $A(x)$ and $C(x)$ is not constant. Write $A(x) = R_1(x)R_2(x)$ and $C(x) = R_1(x)R_3(x)$. By the maximality of $R_1(x)$ we see that $R_2(x)$ is a divisor of $B(x)$, and $R_3(x)$ is a divisor of $D(x)$. The equation $A(x)D(x) = B(x)C(x)$ gives us a polynomial $R_4(x)$ such that $B(x) = R_2(x)R_4(x)$ and $D(x) = R_3(x)R_4(x)$. This leads to the equations

$$A(x) + iB(x) = R_1(x)R_2(x) + iR_2(x)R_4(x) = R_2(x)\left(R_1(x) + iR_4(x)\right)$$

$$C(x) + iD(x) = R_1(x)R_3(x) + iR_3(x)R_4(x) = R_3(x)\left(R_1(x) + iR_4(x)\right).$$

$R_1(x)$ is not constant and so both polynomials have a common zero, which was excluded by assumption. Hence $B(x) = D(x) = 0$ and $f \in \mathbb{R}(x)$. □

## 5.3 Proof of the main result

Now we are prepared to proof our main theorem.

**Theorem 5.9.** *As usual let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree at least two. Then the following statements are equivalent:*

*i)* $\mathbb{Q}^{tr}$ *has the Bogomolov property relative to* $\widehat{h}_f$.

*ii)* *There is a* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *such that the Julia set* $J(\sigma(f))$ *is not contained in* $\mathbb{R}$.

*iii)* *The set* $\mathrm{PrePer}(f) \cap \mathbb{Q}^{tr}$ *is finite.*

**Proof:** Notice again that $J(f)$ cannot be empty, see Fact 1.8 *a*). With Corollary 5.4, we will conclude easily that *ii*) yields *i*) and *iii*). Assume there is a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $J(\sigma(f))$ is not contained in the real line. If *i*) or *iii*) are wrong, then there is a sequence of pairwise distinct elements $\{\alpha_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}^{tr}$ with $\widehat{h}_f(\alpha_n) \to 0$ as $n \to \infty$. Then, by Lemma 1.11 *ii*), $\{\sigma(\alpha_n)\}_{n\in\mathbb{N}}$ is an infinite sequence in $\mathbb{Q}^{tr}$ with canonical height $\widehat{h}_{\sigma(f)}(\sigma(\alpha_n))$ tending to zero. This is not possible due to Corollary 5.4.

The implication *iii)* $\Rightarrow$ *ii)* is not hard either. Assume $J(\sigma(f))$ is contained in the real line for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Using Facts 1.8, we see that $J(f)$ contains the infinite set of repelling periodic points of $f$. For all maps $\sigma(f)$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there are only finitely many non-repelling periodic points (see [Be], §9.6). Hence there are infinitely many points $\alpha \in \overline{\mathbb{Q}}$ such that $\sigma(\alpha)$ is a repelling periodic point of $\sigma(f)$, for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It follows from our assumption that all these $\alpha$ are totally real. In particular we get $|\,\mathrm{PrePer}(f) \cap \mathbb{Q}^{tr}| = \infty$.

Finally, we prove that *i)* implies *ii)*. Assume again that $J(\sigma(f))$ is contained in the real line for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then Lemma 5.8 tells us that $f \in K(x)$ for a totally real number field $K$. Let $\sigma_1, \ldots, \sigma_d$ be a complete set of embeddings of $K$ into $\overline{\mathbb{Q}}$. For each $\sigma_i(f)$ there exists a finite set of intervals such that all backward orbits of these intervals again lie in this finite set of intervals. This interesting result can be found in [EvS11], Theorem 2 and in the discussion afterwards. Thus, for all $\sigma_i$ we can choose a real interval $(a_i, b_i)$ such that for all $c \in (a_i, b_i)$ the backward orbit of $c$ is contained in the real line. For all $\sigma_i$ take a $c_i \in (a_i, b_i) \cap \mathbb{Q}$ and choose a global $\varepsilon > 0$ such that $(c_i - \varepsilon, c_i + \varepsilon) \subset (a_i, b_i)$ for all $1 \leq i \leq d$. All the $\sigma_i$ give rise to non-equivalent absolute values on $K$. By the approximation theorem of Artin and Whaples (see [La], Chapter II, §1), there exists a $c \in K$ such that $|\sigma_i(c - c_i)| = |\sigma_i(c) - c_i| < \varepsilon$. This implies that $\sigma_i(c)$ lies in the interval $(a_i, b_i)$ for all $\sigma_i$. Here we have used that $c$ is totally real. There are infinitely many points $c$ with this property in $K$, but as a number field $K$ contains only finitely many preperiodic points of $f$. This again follows from Northcott's theorem and Theorem 1.9. Thus we can assume that $c$ is not a preperiodic point of $f$.
For every $\gamma$ with $f^{(n)}(\gamma) = c$ we have $\sigma(f)^{(n)}(\sigma(\gamma)) = \sigma(c)$, $n \in \mathbb{N}$. From the choice of our intervals it follows that all conjugates of $\gamma$ are in the real line, and hence we can conclude $f^{-n}(c) \subset \mathbb{Q}^{tr}$. For all $n \in \mathbb{N}$ we choose a $\gamma_n$ in $f^{-n}(c)$. This leads to a sequence $\{\gamma_n\}$ in $\mathbb{Q}^{tr}$ such that

$$\widehat{h}_f(\gamma_n) = \frac{1}{\deg(f)^n} \widehat{h}_f(c) \to 0 \quad .$$

Since $c$ is not preperiodic, the $\gamma_n$ form an infinite sequence of non-preperiodic points. This shows that $\mathbb{Q}^{tr}$ cannot have the Bogomolov property relative to $\widehat{h}_f$. $\square$

In the case where $f \in \overline{\mathbb{Q}}[x]$ is a polynomial we can give one more equivalence.

**Theorem 5.10.** *Let $f \in \overline{\mathbb{Q}}[x]$ be a polynomial. Then the following statements are equivalent:*

*i)* $\mathbb{Q}^{tr}$ *does not have the Bogomolov property relative to* $\widehat{h}_f$.

*ii)* $J(\sigma(f)) \subset \mathbb{R}$ *for all* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

*iii)* $\mathrm{PrePer}(f) \subset \mathbb{Q}^{tr}$.

*iv)* $\widehat{h}_f(\alpha) > 0$ *for all* $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}^{tr}$.

**Proof:** *i*) and *ii*) are equivalent by Theorem 5.9 and the equivalence of *iii*) and *iv*) follows immediately from Northcott's theorem and Theorem 1.9. As we have a polynomial, the Julia set of $f$ is the boundary of the compact set

$$\{y \in \mathbb{C} | |f^{(n)}(y)| \nrightarrow \infty, \text{ as } n \to \infty\} \quad .$$

See [Mi], Lemma 9.4. This set is called the *filled Julia set* of $f$. Every preperiodic point of $f$ (except $\infty$)is contained in the filled Julia set of $f$. For a polynomial it follows from the definitions of the Julia set and the filled Julia set that $\infty$ is in neither of both sets. Since both sets are closed, they are bounded. Now assume that *ii*) is true. Then the Julia set of every $\sigma(f)$ is a closed subset of a closed interval $I$. The only bounded subset of the Riemann sphere with such a boundary is the set itself. This means that all these $J(\sigma(f))$ coincide with their filled Julia set and hence they contain $\mathrm{PrePer}(\sigma(f)) = \sigma(\mathrm{PrePer}(f))$. This shows that $\mathrm{PrePer}(f)$ is contained in $\mathbb{Q}^{tr}$, proving *iii*).
Now we assume *iii*). Then $\mathrm{PrePer}(\sigma(f)) = \sigma(\mathrm{PrePer}(f))$ lies in the real line for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence, the closure of the repelling periodic points of $\sigma(f)$ lies in the real line. By Fact 1.8 *d*), the closure of the repelling periodic points is the Julia set. This proves *ii*) and concludes the proof. □

## 5.4 Some remarks

Let's go back to the quadratic polynomials $f_c = x^2 - c \in \overline{\mathbb{Q}}[x]$. We have seen in Proposition 5.6 that the canonical heights $h_{f_c}$ can get arbitrarily small on $\mathbb{Q}^{tr}$ for every rational $c \geq 2$. This behavior may change completely for non-rational $c$. In order to state this explicitly, we will prove a well known classification result.

**Lemma 5.11.** *The Julia set of $f_c(x) = x^2 - c \in \mathbb{C}[x]$ is contained in the real line if and only if $c \in \mathbb{R}$ and $c \geq 2$.*

**Proof:** We have seen in Remark 5.7 that $J(f_c)$ is contained in the real line whenever $c$ is real and $c \geq 2$. Moreover, the proof of Lemma 5.8 shows that $J(f_c)$ is not contained in the real line for all non-real $c$. Hence, from now on we assume that $c$ is real.
It remains to show that $J(f_c) \nsubseteq \mathbb{R}$ for all $c < 2$. Recall from the proof of Theorem 5.10 that all preperiodic points of $f_c$ are real if $J(f_c)$ is contained in the real line. One fixed point of $f_c$ is $x_c := \sqrt{c + \frac{1}{4}} + \frac{1}{2}$. This is not real for all $c < -\frac{1}{4}$, and hence we can conclude $J(f_c) \nsubseteq \mathbb{R}$ for all $c < -\frac{1}{4}$.
If $c \in [-\frac{1}{4}, 0]$, then the pre-image of $x_c$ which is different from $x_c$ is given by $-x_c$ which is a negative real number. Hence, at least one element in $f_c^{-2}(x_c)$ is non-real. As pre-images of fixed points are preperiodic, we find $J(f_c) \nsubseteq \mathbb{R}$ for all $c \leq 0$.
In the last case, where $c$ lies in the interval $(0, 2)$, we see that the pre-image $-x_c$ of $x_c$ is smaller than $-c$. Again we conclude that $f_c^{-2}(x_c)$, and hence $J(f_c)$, cannot lie in the real line. □

Let $q > 4$ be an element in $\mathbb{Q} \setminus \mathbb{Q}^2$. Then the Julia set of $f_{\sqrt{q}}$ is real. However, we claim that $\mathbb{Q}^{tr}$ does not have the Bogomolov property relative to $\widehat{h}_{f_{\sqrt{q}}}$. This is due to the facts that $f_{-\sqrt{q}}$ is a galois conjugate of $f_{\sqrt{q}}$, and that $J(f_{-\sqrt{q}})$ is not contained in the real line (see Lemma 5.11). Now Theorem 5.9 proves the claim.

**Definition.** *A Salem number is a real algebraic integer $\alpha > 1$ such that all conjugates of $\alpha$ have absolute value $\leq 1$ and at least one conjugate has absolute value equal to* 1*.*

As one conjugate of the Salem number $\alpha$ has absolute value 1, the inverse of a conjugate is again a conjugate of $\alpha$. This implies, using the definition of a Salem number, that $\alpha^{-1}$ is the only real conjugate of $\alpha$ and all other conjugates lie on the unit circle. Hence $\alpha+\alpha^{-1}$ is a totally real number. Since Vijayaraghavan introduced these numbers in [Vi41] there is the following conjecture.

**Conjecture 5.12.** *There is a constant $C > 1$ such that $\alpha > C$ for all Salem numbers $\alpha$.*

Using Jensen's formula (Theorem 1.2) we have $\deg(\alpha)h(\alpha) = \log(\alpha)$ for all Salem numbers $\alpha$. Hence, the Lehmer conjecture 1.4 implies Conjecture 5.12. On the other hand, the three smallest known values of $\deg(\alpha)h(\alpha)$, $\alpha \in \overline{\mathbb{Q}}$, are taken by Salem numbers (we refer to the website [Mo] for tables of small Salem numbers and small values of the measure $\deg(.)h(.)$).
Denote by $T_d$ the $d$-th Chebyshev polynomial. We know from 1.17 that the Julia set of each $T_d$ is the real interval $[-2, 2]$. Thus, Theorem 5.9 provides that the Bogomolov property for $\mathbb{Q}^{tr}$ does not hold relative to $\widehat{h}_{T_d}$, $d \geq 2$. The next best bound one can ask for is a bound of *Lehmer strength*. This means, one can ask whether there exists a positive constant $c$ such that $\deg(\alpha)\widehat{h}_{T_d}(\alpha) \geq c$ for all $\alpha$ in $\mathbb{Q}^{tr} \setminus \mathrm{PrePer}(T_d)$. This would be quite a strong result, because it would imply Conjecture 5.12. This follows from Lemma 1.22 and the fact that $\alpha + \alpha^{-1}$ is totally real for all Salem numbers $\alpha$. On the other hand, the existence of such a bound $c$ seems to be very likely, as $\mathbb{Q}^{tr}$ has the Bogomolov property relative to all $\widehat{h}_{f_{2-\varepsilon}}$ for every algebraic $\epsilon > 0$ (see Lemma 5.11).

Although we cannot prove a higher dimensional analogue of Theorem 5.9, we will state a possible generalization as a question.

**Question 5.13.** *Let $(X, L, f)$ be a polarized algebraic dynamical system defined over a totally real number field $K$. Which of the following statements are equivalent?*

*i) There exists a positive constant c such that $\widehat{h}_{X,L,f}$ on $X(\mathbb{Q}^{tr})$ is either zero or bounded from below by a positive constant.*

*ii) There is a $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the Julia set $J(\sigma(f))$ is not contained in $X(\mathbb{R})$.*

*iii) The set $\mathrm{PrePer}(f) \cap X(\mathbb{Q}^{tr})$ is not Zariski dense in $X$.*

## 5.5 Finite extensions of $\mathbb{Q}^{tr}$

We will study the surprising behavior of $h$ in finite extensions of $\mathbb{Q}^{tr}$. In 2007 Amoroso and Nuccio proved that there are elements with arbitrarily small height in the union of all CM-fields. As in [ADZ11] we will give a very short direct proof of this theorem. The proof of this theorem is independent of the results from the previous sections of this paper.

**Theorem 5.14.** *There are algebraic numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\mathbb{Q}(\alpha_n)$ is a CM-field, none of these elements is a root of unity and $h(\alpha_n)$ tends to zero.*

Like in [AN07] we will use the following characterization of CM-fields.

**Lemma 5.15.** *A number field $K$ is a CM-field if and only if there exists an element $\alpha$ with $K = \mathbb{Q}(\alpha)$ and $|\sigma(\alpha)| = 1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*

See [AN07], Proposition 2.3, or [BL78], Theorem 1.

**Proof of Theorem 5.14:** Take an algebraic number $\alpha$ that is no root of unity and such that $|\sigma(\alpha)| = 1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (for example, set $\alpha = \frac{\sqrt{-15}+1}{4}$). For all $n \in \mathbb{N}_0$ we fix a $2^n$-th root of $\alpha$ and denote it by $\alpha^{1/2^n}$. For all $\sigma$ we have $1 = |\sigma(\alpha)| = |\sigma(\alpha^{1/2^n})|^{2^n}$. By Lemma 5.15, $\mathbb{Q}(\alpha^{1/2^n})$ is a CM-field and we have $0 \neq h(\alpha^{1/2^n}) = \frac{1}{2^n}h(\alpha) \to 0$, as $n \to \infty$. □

The next immediate corollary is a simple example of the fact that the Bogomolov property is not preserved under finite field extensions. See also [ADZ11] for the same result.

**Corollary 5.16.** $\mathbb{Q}^{tr}(i)$ *does not have the Bogomolov property relative to $h$.*

**Proof:** This follows from Theorem 5.14 and the fact that $\mathbb{Q}^{tr}(i)$ is the compositum of all CM-fields. To prove the fact we have to show that every CM-field is contained in $\mathbb{Q}^{tr}(i)$. Let therefore $\mathbb{Q}(\alpha)$ be a CM-field. By Lemma 5.15 we can assume that $|\sigma(\alpha)| = 1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence $\alpha + \alpha^{-1}$ and $i(\alpha - \alpha^{-1})$ are totally real. Now we see $\alpha = \frac{1}{2}(\alpha + \alpha^{-1} + \alpha - \alpha^{-1}) \in \mathbb{Q}^{tr}(i)$ . □

**Remark 5.17.** A field $K \subseteq \overline{\mathbb{Q}}$ is said to have the Northcott property if and only if the set $\{\alpha \in K | h(\alpha) \leq c\}$ is finite for every constant $c$. We refer to [BZ01], [DZ08] and [Wi11] for information on this property. As we have $\widehat{h}_f = h + O(1)$ for every dynamical height $\widehat{h}_f$, it does not make any difference to define the Northcott property relative to a dynamical height. Notice, that the Northcott property is preserved under finite extensions. The field $\mathbb{Q}^{tr}$ is far from having the Northcott property, as we have noticed in Section 5.1 that there is a constant $\lambda$ such that $\{h(\alpha) \geq \lambda | \alpha \in \mathbb{Q}^{tr}\}$ is dense in the interval $[\lambda, \infty)$.

Schinzel's original theorem from [Sch73] is much stronger than the formulation of Theorem 5.1. For example, he proved that the height of every non-zero element in $\mathbb{Q}^{tr}(i)$ that does not lie on the unit circle is a least $\frac{1}{2}\log\left(\frac{\sqrt{5}+1}{2}\right)$. Moreover, [Sch73], Corollary 1, states the following.

**Theorem 5.18** (Schinzel)**.** *Let $\alpha \neq 0$ be an algebraic number with $D = [\mathbb{Q}^{tr}(\alpha) : \mathbb{Q}^{tr}]$ and such that the minimal polynomial $P$ of $\alpha$ over $\mathbb{Q}^{tr}$ is not reciprocal; i.e. $P(\alpha^{-1}) \neq 0$. Then we have*

$$h(\alpha) > \frac{1}{2D} \log \left( \frac{\sqrt{17}+1}{4} \right) \quad .$$

*In particular this inequality is true for all elements in an extension field of $\mathbb{Q}^{tr}$ of finite and odd degree $D$.*

**Proof:** Let $K$ be the field generated by the coefficients of $P$ and then use [Sch73], Corollary 1, and Jensen's formula (Theorem 1.2). Notice that an irreducible reciprocal polynomial different from $x \pm 1$, must have even degree. $\square$

A natural question arising from Corollary 5.16 is: Does a finite extension $K$ of $\mathbb{Q}^{tr}$ has the Bogomolov property relative to $h$ if $i \notin K$? This question remains unanswered in general, but we can use Theorem 5.2 for a positive answer if $K$ is not totally imaginary.

**Proposition 5.19.** *Let $K = \mathbb{Q}^{tr}(\alpha)$ such that $\alpha$ has at least one real conjugate. Denote the number of real conjugates of $\alpha$ by $r_\alpha$ and set $R_\alpha = \frac{r_\alpha}{\deg(\alpha)}$. Then we have*

$$h(\beta) \geq \frac{R_\alpha}{2} \log \left( \frac{2^{1-1/R_\alpha} + \sqrt{4^{1-1/R_\alpha}+4}}{2} \right) \quad ,$$

*for all $\beta \in K^* \setminus \{\pm 1\}$.*

**Proof:** Let $\beta$ be an arbitrary element in $K^* \setminus \{\pm 1\}$, and let $L$ be a totally real number field with $\beta \in L(\alpha)$. We have $[L(\alpha) : \mathbb{Q}] = \deg(\alpha)[L(\alpha) : \mathbb{Q}(\alpha)]$. Furthermore, an embedding of $L(\alpha)$ is real if and only if it is an extension of a real embedding of $\mathbb{Q}(\alpha)$. Hence, the number of real embeddings of $L(\alpha)$ is $r_\alpha[L(\alpha) : \mathbb{Q}(\alpha)]$. Since $\beta$ is an element of $L(\alpha)$, we also have $[L(\alpha) : \mathbb{Q}] = \deg(\beta)[L(\alpha) : \mathbb{Q}(\beta)]$, and a real embedding of $L(\alpha)$ must be an extension of a real embedding of $\mathbb{Q}(\beta)$. This means that the number of real embeddings of $L(\alpha)$ is at most $r_\beta[L(\alpha) : \mathbb{Q}(\beta)]$, with $r_\beta$ defined as above. Together this yields

$$R_\beta = \frac{r_\beta[L(\alpha) : \mathbb{Q}(\beta)]}{[L(\alpha) : \mathbb{Q}]} \geq \frac{r_\alpha[L(\alpha) : \mathbb{Q}(\alpha)]}{[L(\alpha) : \mathbb{Q}]} = R_\alpha \quad .$$

Now we can use Theorem 5.2, with $t = 1$, to achieve

$$h(\beta) \geq \frac{R_\alpha}{2} \log \left( \frac{2^{1-1/R_\alpha} + \sqrt{4^{1-1/R_\alpha}+4}}{2} \right)$$

which is the claimed result. $\square$

**Remark 5.20.** We use the setting of Proposition 5.19. If $\beta$ is another generator of the field extension $K/\mathbb{Q}^{tr}$, then a similar argumentation as in the above proof shows $R_\alpha = R_\beta$. Hence, the lower bound for the height on $K^* \setminus \{\pm 1\}$ is actually independent of the choice of the generator of $K/\mathbb{Q}^{tr}$.

**Remark 5.21.** There is also a proof of Corollary 5.16 using dynamical methods and Theorem 5.9. The Möbius transformation $g(x) = \frac{x+i}{x-i}$ maps the real line onto the unit circle. Take the map $g^{-1} \circ x^2 \circ g$. By [Be], Theorem 3.1.4, we have $J(g^{-1} \circ x^2 \circ g) = g^{-1}(J(x^2)) = \mathbb{R}$. The same is true for the only galois conjugate $\frac{x-i}{x+i}$ of $g$. Furthermore, we have $\widehat{h}_{g^{-1} \circ x^2 \circ g} = h \circ g$ by Lemma 1.11. Now Theorem 5.9 tells us that there are pairwise distinct totally real algebraic numbers $\{\alpha_n\}_{n \in \mathbb{N}}$ such that

$$0 \neq \widehat{h}_{g^{-1} \circ x^2 \circ g}(\alpha_n) = h(g(\alpha_n)) \to 0 \quad .$$

As $g(\alpha_n)$ is in $\mathbb{Q}^{tr}(i)$ for all $n \in \mathbb{N}$, we get the corollary.

Lukas Christopher Pottmeyer Regensburg, February 2012

# References

[ADZ11] Amoroso, F.; David, S.; Zannier, U.: *Fields with the property (B)*, hal-00649954 (2011)

[AD00] Amoroso, F.; Dvornicich, R.: *A lower bound for the height in abelian extensions*, Journal of Number Theory **80**, No.2, 260-272 (2000)

[AN07] Amoroso, F.; Nuccio, F. A. E.: *Algebraic numbers of small Weil's height in CM-fields: On a theorem of Schinzel*, Journal of Number Theory **122**, No.1, 247-260 (2007)

[AZ00] Amoroso, F.; Zannier, U.: *A relative Dobrowolski lower bound over abelian extensions*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV **29**, No.3, 711-727 (2000)

[AZ09] Amoroso, F.; Zannier, U.: *A uniform relative Dobrowolski's lower bound over abelian extensions*, Bulletin of the Londen Mathematical Society **42**, No.3, 489-498 (2009)

[Ba03] Baker, M.: *Lower bounds for the canonical height on elliptic curves over abelian extensions*, International Mathematics Research Notices, No.29, 1571-1589 (2003)

[BP05] Baker, M.; Petsche, C.: *Global discrepancy and small points on elliptic curves*, International Mathematics Research Notices, No.61, 3791-3834 (2005)

[BR] Baker, M.; Rumely, R.: *Potential Theory and Dynamics on the Berkovich Projective Line*, Mathematical Surveys and Monographs **159**, AMS (2010)

[BS04] Baker, M.; Silverman, J.H.: *A lower bound for the canonical height on abelian varieties over abelian extensions*, Mathematical Research Letters **11**, No.2-3, 377-396 (2004)

[Be] Beardon, A. F.: *Iteration of Rational Functions. Complex analytic dynamical systems*, Graduate Texts in Mathematics **132**; Springer (1991)

[Ber] Berkovich, V.: *Spectral theory and analytic geometry over non-archimedean fields*, Mathematical Surveys and Monographs **33**; AMS (1990)

[BZ97] Beukers, F.; Zagier, D.: *Lower bounds of heights of points on hypersurfaces*, Acta Arithmetica **79**, No.2, 103-111 (1997)

[BL78] Blanksby, P. E.; Loxton, J. H.: *A Note on the Characterization of CM-fields*, Journal of the Australian Mathematical Society (Series A) **26**, 26-30 (1978)

[BG] Bombieri, E.; Gubler, W.: *Heights in Diophantine geometry*, New Mathematical Monographs **4**; Cambridge University Press (2006)

[BZ01] Bombieri, E.; Zannier, U.: *A note on heights in certain infinite extensions of* $\mathbb{Q}$, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Serie IX. Rendiconti Lincei. Matematica e Applicazioni **12**, No.1, 5-14 (2001)

[Bo77] Bosch, S.: *Zur Kohomologietheorie rigid analytischer Räume*, Manuscripta Mathematica **20**, 1-27 (1977)

[BGR] Bosch S., Güntzer U., Remmert R.: *Non-Archimedean analysis. A systematic approach to rigid analytic geometry*, Grundlehren der Mathematischen Wissenschaften **261**; Springer (1984)

[Br51] Breusch, R.: *On the distribution of the roots of a polynomial with integral coefficients*, Proceedings of the American Mathematical Society **2**, 939-941 (1951)

[CS93] Call, G. S.; Silverman, J. H.: *Canonical heights on varieties with morphisms*, Compositio Mathematica **89**, No.2, 163-205 (1993)

[Ch06] Chambert-Loir, A.: *Mesures et équidistribution sur les espaces de Berkovich*, Journal für die Reine und Angewandte Mathematik **595**, 215-235 (2006)

[CW11] Checcoli, S.; Widmer, M.: *On the Northcott property and other properties related to polynomial mappings*, arXiv:1111.5060v1; (2011)

[Co] Cohen, H.: *A course in computational algebraic number theory*, Graduate Texts in Mathematics **138**; Springer (1993)

[CGHJK] Corless, R.; Gonnet, G.; Hare, D.; Jeffrey, D.; Knuth, D.: *On the Lambert W Function*, Advances in Computational Mathematics **5**, No. 4, 329-359 (1996)

[Do79] Dobrowolski, E.: *On a question of Lehmer and the number of irreducible factors of a polynomial* Acta Arithmetica **34**, 391-401 (1979)

[DZ08] Dvornicich, R.; Zannier, U.: *On the properties of Northcott and of Narkiewicz for fields of algebraic numbers*, Functiones et Approximatio. Commentarii Mathematici **39**, Part 1, 163-173 (2008)

[EvS11] Eremenko, A.; van Strien, S.: *Rational maps with real multipliers*, Transaction of the American Mathematical Society **363**, No.12, 6453-6463 (2011)

[FM12] Fili, P.; Miner, Z.: *Equidistribution and the heights of totally real and totally* $p$*-adic numbers*, preprint (2012)

[Fl96] Flammang, V.: *Two new points in the spectrum of the absolute Mahler measure of totally positive algebraic integers*, Mathematics of Computation **65**, No.213, 307-311 (1996)

[FLM83] Freire, A.; Lopes, A.; Mañé, R.: *An invariant measure for rational maps*, Boletim da Sociedade Brasileira de Matemática **14**, No.1, 45-62 (1983)

[Ga07] Garza, J.: *On the heights of algebraic numbers with real conjugates*, Acta Arithmetica **128**, 385-389 (2007)

[GIP10] Garza, J.; Ishak, M. I. M; Pinner, C.: *On the products of heights summing to real numbers*, Acta Arithmetica **142**, No.1, 51-58 (2010)

[GIMPS] *Great Internet Mersenne Prime Search*, `http://www.mersenne.org/`

[Gu07] Gubler, W.: *The Bogomolov conjecture for totally degenerate abelian varieties*, Inventiones Mathematicae **169**, No.2, 377-400 (2007)

[Gu07a] Gubler, W.: *Tropical varieties for non-archimedean analytic spaces*, Inventiones Mathematicae **169**, No.2, 321-376 (2007)

[Gu10] Gubler, W.: *Non-archimedean canonical measures on abelian varieties*, Compositio Mathematica **146**, No.3, 683-730 (2010)

[Ha11] Habegger, P.: *Small Height and Infinite Non-Abelian Extensions*, arXive:1109.5859 (2011)

[HS90] Hindry, M.; Silverman, J. H.: *On Lehmer's conjecture for elliptic curves*, Séminaire de Théorie des Nombres, Paris 1988/89, Progress in Mathematics **91**, 103-116 (1990)

[Ho11] Höhn, G.: *On a theorem of Garza regarding algebraic numbers with real conjugates*, International Journal of Number Theory **7**, No.4, 943-945 (2011)

[HS93] Höhn, G.; Skoruppa, N.-P.: *Un résultat de Schinzel*, Journal de Théorie des Nombres de Bordeaux **5**, No.1, 185 (1993)

[Hr87] Hriljac, P.: *Splitting fields of principal homogeneous spaces*, Number Theory Seminar New York 1984/85, Lecture Notes in Mathematics **1240**, 214-229 (1987)

[IMPW] Ishak, M. I. M.; Mossinghoff, M.; Pinner, C.; Wiles, B.: *Lower bounds for the height in cyclotomic extensions*, Journal of Number Theory **130**, No.6, 1408-1424 (2010)

[Kr57] Kronecker, L.: *Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten*, Journal für die Reine und Angewandte Mathematik **53**, 173-175 (1857)

[La] Lang, S.: *Algebraic Number Theory*, Graduate Texts in Mathematics **110**, Second Edition; Springer (1994)

[La18] Lattès, S.: *Sur l'irération des substitutions rationnelles et les fonctions de Poincaré*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Siences, Paris **166**, 26-28 (1918)

[La83] Laurent, M.: *Minoration de la hauteur de Néron-Tate*, Séminaire de Théorie des Nombres, Paris 1981/82, Progress in Mathematics **38**, 137-151 (1983)

[Le30] Lehmer, D. H.: *An extended theory of Lucas functions*, Annals of Mathematics, 2nd Series **31**, 419-448 (1930)

[Le33] Lehmer, D. H.: *Factorization of certain cyclotomic functions*, Annals of Mathematics, 2nd Series **34**, 461-479 (1933)

[Ma62] Mahler, K.: *On some inequalities for polynomials in several variables*, Journal of the London Mathematical Society **37**, 341-344 (1962)

[Ma89] Masser, D. W.: *Counting points of small height on elliptic curves*, Bulletin de la Société Mathématique de France **117**, No.2, 247-265 (1989)

[Mi] Milnor, J.: *Dynamics in One Complex Variable*, Princeton Annals in Mathematics **160**, Third Edition; Princeton University Press (2006)

[Mi06] Milnor, J.: *On Lattès maps* ; Dynamics on the Riemann sphere. A Bodil Branner Festschrift. Zürich: European Mathematical Society Publishing House, 9-43 (2006)

[Mo] Mossinghoff, M.: *Lehmer's Problem*; `http://www.cecm.sfu.ca/~mjm/Lehmer/`

[MRW08] Mossinghoff, M.; Rhin, G.; Wu, Q.: *Minimal Mahler measures*, Experimental Mathematics **17**, No.4, 451-458 (2008)

[Na] Narkiewicz, W.: *Elementary and Analytic Theory of Algebraic Numbers*, Second Edition; PWN-Polish Scientific Publishers and Springer-Verlag (1990)

[Ne] Neukirch, J.: *Algebraische Zahlentheorie*; Springer (2007)

[No50] Northcott, D. G.: *Periodic points on an algebraic variety*, Annals of Mathematics. Second Series **51**, 167-177 (1950)

[Ro] Roman, S.: *Field Theory*, Graduate Texts in Mathematics **158**, Second Edition; Springer (2006)

[Sa06] Samuels, C.: *Lower bounds on the projective heights of algebraic points*, Acta Arithmetica **125**, No.1, 41-50 (2006)

[Sch] Schertz, R.: *Complex Multiplication*, New Mathematical Monographs **15**; Cambridge University Press (2010)

[Sch73] Schinzel, A.: *On the product of the conjugates outside the unit circle of an algebraic number*, Acta Arithmetica **24**, 385-399 (1973)

[Si94] Silverman, J. H.: *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **151**; Springer (1994)

[Si04] Silverman, J. H.: *A lower bound for the canonical height on elliptic curves over abelian extensions*, Journal of Number Theory **104**, No.2, 353-372 (2004)

[Si07] Silverman, J. H.: *The Arithmetic of Dynamical Systems*, Graduate Texts in Mathematics **241**; Springer (2007)

[Si09] Silverman, J. H.: *The arithmetic of elliptic curves*, Graduate Texts in Mathematics **106**, Second Edition; Springer (2009)

[Sm71] Smyth, C.: *On the product of the conjugates outside the unit circle of an algebraic integer*, Bulletin of the London Mathematical Society **3**, 169-175 (1971)

[Sm80] Smyth, C.: *On the measure of totally real algebraic integers*, Journal of the Australian Mathematical Society (Series A) **30**, 137-149 (1980)

[Sm81] Smyth, C.: *On the measures of totally real algebraic integers. II*, Mathematics of Computation **37**, 205-208 (1981)

[Sm08] Smyth, C.: *The Mahler measure of algebraic numbers: A survey*, Number Theory and Polynomials, LMS Lecture Notes Series **352**, 322-349 (2008)

[Te10] Temkin, M.: *Introduction to Berkovich analytic spaces*, preprint available at `http://www.math.huji.ac.il/~temkin/papers/Lecture_Notes_Berkovich_Analytic_Spaces.pdf` (2010)

[Vi41] Vijayaraghavan, T.: *On the fractional parts of the powers of a number. II.*, Proceedings of the Cambridge Philosophical Society **37**, 349-357 (1941)

[Vo96] Voutier, P.: *An effective lower bound for the height of algebraic numbers*, Acta Arithmetica **74**, No.1, 81-95 (1996)

[Wi11] Widmer, M.: *On certain infinite extensions of the rationals with Northcott property*, Monatshefte für Mathematik **162**, No.3, 341-353 (2011)

[Yu08] Yuan, X.: *Big line bundles over arithmetic varieties*, Inventines Mathematicae **173**, No.3, 603-649 (2008)

[Yu12] Yuan, X.: *Algebraic Dynamics, Canonical Heights and Arakelov Geometry*, AMS/IP Studies in Advanced Mathematics **51**, Part 2, 893-929 (2012)

[Za93] Zagier, D.: *Algebraic Numbers Close to Both* 0 *and* 1, Mathematics of Computation **61**, No.203, 485-491 (1993)

[Zh92] Zhang, S.: *Positive line bundles on arithmetic surfaces*, Annals of Mathematics. Second Series **136**, No.3, 569-587 (1992)

[Zh95] Zhang, S.: *Small points and adelic metrics*, Journal of Algebraic Geometry **4**, No.2, 281-300 (1995)

[Zh98] Zhang, S.: *Equidistribution of small points on abelian varieties*, Annals of Mathematics. Second Series **147**, No.1, 159-165 (1998)

[Zh06] Zhang, S.: *Distributions in Algebraic dynamics*, Essays in geometry in memory of S. S. Chern, Surveys in Differential Geometry **10**, 381-430 (2006)

[Zi06] Zimmermann, N.: *Die Normeinsgruppen biquadratischer Zahlkörpererweiterungen* ; Dissertation an der Westfälischen Wilhelms-Universität Münster (2006)