Math. Ann. 280, 207–245 (1988)

Mathematische Annalen

Continuous Étale Cohomology

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It is well known that the l-adic cohomology of a scheme X

$$H^i(X, \mathbb{Z}_l(j)) = \varprojlim_n H^i(X, \mathbb{Z}/l^n(j))$$

has good properties only if the étale cohomology groups $H^i(X, \mathbb{Z}/l^n(j))$ are finite (e.g. not for varieties over number fields). In particular, these groups do not arise as derived functors, which causes problems with functoriality, already for the usual long exact cohomology sequence associated to an exact sequence of l-adic sheaves. The situation is similar for continuous group cohomology of $\mathbb{Z}_l$-modules defined by continuous cochains. For example there does not always exist a Hochschild-Serre spectral sequence for a closed normal subgroup. The reason for these difficulties is the non-exactness of the inverse limit.

But the inverse limit is left exact, so we may try to define a good cohomology theory $H^*(X, (F_n))$, defined for inverse systems (F_n) of étale sheaves on X, by deriving the left exact functor

$$\text{(0.1)} \qquad \begin{array}{ccc} \left\{\begin{array}{l}\text{inverse systems } (F_n)\\ \text{of étale sheaves on } X\end{array}\right\} & \to & \{\text{abelian groups}\} \\ (F_n) & \rightsquigarrow & \varprojlim_n H^0(X, F_n). \end{array}$$

The aim of this paper is to show that this cohomology theory exists and has all the properties we want.

It turns out, that for locally constant sheaves F_n our groups coincide with the continuous étale cohomology groups $H^i_{\text{cont}}(X, (F_n))$ defined by Dwyer and Friedlander via the étale topological type of X (see [3], the definition there is more general, but compare (3.28) below). Our definition, being based on derived functors, applies to arbitrary sheaves [e.g., to $\mathbb{G}_m$ in the Kummer sequence (3.27)], and is particularly suited for calculations in derived categories. In fact, this paper, where we study the groups $H^i_{\text{cont}}(X, (F_n))$ and their properties for l-adic sheaves (F_n) by more elementary means, will be basic for a following one, in which we shall give a new approach to the derived category of $\mathbb{Z}_l$-sheaves.

The connection between continuous and usual l-adic cohomology is a follows. If we define the continuous l-adic cohomology by

$$H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(j)) = H^i(X, (\mathbb{Z}/l^n(j)))$$

(l invertible on X), the obvious spectral sequence induces short exact sequences

$$0 \to \varprojlim_n{}^1 H^{i-1}(X, \mathbb{Z}/l^n(j)) \to H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(j)) \to \varprojlim_n H^i(X, \mathbb{Z}/l^n(j)) \to 0, \tag{0.2}$$

where $\varprojlim_n{}^1$ is the derivative of the inverse limit [see (1.5) for the standard description].

Since $\varprojlim_n{}^1$ is zero for systems satisfying the Mittag-Leffler condition, one has

$$H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(j)) = H^i(X, \mathbb{Z}_l(j)),$$

if the $H^i(X, \mathbb{Z}/l^n(j))$ are finite, in particular, if X is a variety over an algebraically closed field (the "geometric" case). On the other hand we show

$$H^i_{\mathrm{cont}}(\operatorname{Spec} k, \mathbb{Z}_l(j)) = H^i_{\mathrm{cont}}(G_k, \mathbb{Z}_l(j))$$

(continuous group cohomology, see [14, Sect. 2]) for a field k with absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ (the "arithmetic" case). Connecting both cases, we show that for a variety X over k there is a Hochschild-Serre spectral sequence

$$H^p_{\mathrm{cont}}(G_k, H^q(X \times_k \bar{k}, \mathbb{Z}_l(j))) \Rightarrow H^{p+q}_{\mathrm{cont}}(X, \mathbb{Z}_l(j)), \tag{0.3}$$

which in general *does not exist* for the usual l-adic cohomology.

The paper is organized as follows. In Sect. 1 we collect the needed facts from homological algebra. In Sect. 2 we compare group cohomology of inverse systems with continuous group cohomology.

In Sect. 3 we introduce the étale cohomology groups $H^j(X, (F_n))$ of inverse systems (F_n) of sheaves on X_{et} and prove some of their major properties, for example the connection with continuous Galois cohomology for $X = \operatorname{Spec} k$, the existence of a Hochschild-Serre spectral sequence, of cohomology with supports and relative cohomology sequences, and of a Gysin morphism. There is also a cycle map

$$cl^X : CH^i(X) \to H^{2i}_{\mathrm{cont}}(X, \mathbb{Z}_l(i)) \tag{0.3}$$

for a smooth variety X over a field k which can be expected to have interesting properties, since it is far away from being trivial on cycles algebraically equivalent to zero, see (6.15).

In Sect. 4 we investigate some "l-adic" properties of abelian groups, like l-completeness or l-divisibility, and relate these to properties of inverse limits. We show that the groups $H^j(X, (F_n))$ are what we call "weakly l-complete" groups (called Ext-l-complete groups by Bousfield and Kan [1]), if $l^n F_n = 0$ for all n, e.g., for $(F_n) = \mathbb{Z}_l(i)$. Such groups behave very well, even if they are huge. For example, there holds a strong Nakayama lemma for them, whose application to the $H^j(X, (F_n))$ gives an extension and sharpening of Tate's results on the structure of continuous group cohomology in [14, Sect. 2].

In Sect. 5 we treat torsion-free l-adic sheaves F. We introduce their $\mathbb{Q}_l$-cohomology by

$$(0.4) \qquad H^j_{\text{cont}}(X, F\otimes\mathbb{Q}_l) := H^i_{\text{cont}}(X, F)\otimes\mathbb{Q}_l$$

and construct a long exact sequence

$$(0.5) \qquad \ldots \to H^j_{\text{cont}}(X,F) \to H^j_{\text{cont}}(X, F\otimes\mathbb{Q}_l) \xrightarrow{\beta} H^j(X, F\otimes\mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\delta} H^{j+1}_{\text{cont}}(X,F) \to \ldots$$

with a certain (true) sheaf $F\otimes\mathbb{Q}_l/\mathbb{Z}_l$ (e.g., $\mathbb{Z}_l(i)\otimes\mathbb{Q}_l/\mathbb{Z}_l = \mathbb{Q}_l/\mathbb{Z}_l(i)$), such that $\operatorname{Im}\beta$ is the maximal divisible subgroup of its target and $\operatorname{Im}\delta$ is the torsion group of $H^{j+1}_{\text{cont}}(X,F)$. This again generalizes results of Tate [14, Sect. 2].

In Sect. 6 we introduce a cupproduct for continuous étale cohomology, which often can be calculated in terms of (0.1) and allows the definition of Chern classes

$$c_i(E) \in H^{2i}_{\text{cont}}(X, \mathbb{Z}_l(i))$$

of vector bundles E on a scheme X (l invertible on X).

This paper was started by the search for a spectral sequence (0.3), the following investigation of Tate's sequence (2.2) in [14], and its explanation by construction (0.1) and the implied sequence (0.2). It is clear from the introduction that the mentioned paper of Tate also influenced this work later on. I also thank Chr. Deninger for bringing my attention to the derived functors of $\varprojlim$ and the papers by Roos [10, 11].

Notations and Conventions: For an object A in an abelian category and $m \in \mathbb{N}$ let ${}_mA$ be the kernel and A/m be the cokernel of $A \xrightarrow{m} A$. Functors are often only described on objects, when the functoriality is clear. The signs $\mathbb{Z}$, $\mathbb{Z}/l^n$ etc. denote the abelian groups as well as the associated constant sheaves, $\mathbb{Z}_l$ also denotes the inverse system of sheaves $(\mathbb{Z}/l^n)$ on a scheme X.

1. Inverse Systems and Right Derivatives

Let $\mathscr{A}$ be an abelian category and $\mathscr{A}^{\mathbb{N}}$ be the category of inverse systems in $\mathscr{A}$ indexed by the set $\mathbb{N}$ of natural numbers with the natural order. Thus objects in $\mathscr{A}^{\mathbb{N}}$ are inverse systems

$$\ldots \longrightarrow A_{n+1} \xrightarrow{d_n} A_n \longrightarrow \ldots \longrightarrow A_2 \xrightarrow{d_1} A_1$$

in $\mathscr{A}$ and morphism are commutative diagrams

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & A_{n+1} & \longrightarrow & A_n & & \ldots \longrightarrow & A_2 & \longrightarrow & A_1 \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\
\ldots \longrightarrow & B_{n+1} & \longrightarrow & B_1 & \longrightarrow & \ldots \longrightarrow & B_2 & \longrightarrow & B_1 .
\end{array}$$

Obviously $\mathscr{A}^{\mathbb{N}}$ is an abelian category, with kernels and cokernels taken "componentwise".

(1.1) **Proposition.** a) *$\mathscr{A}^{\mathbb{N}}$ has enough injectives if and only if $\mathscr{A}$ has enough injectives.*

b) *(A_n, d_n) is injective if and only if all A_n are injective and all d_n are split surjections.*

Proof. The forgetful functor

$$V_m : \mathscr{A}^{\mathbf{N}} \to \mathscr{A} \qquad (1.1.1)$$
$$(A_n, d_n) \rightsquigarrow A_m$$

has the exact, faithful left adjoint

$$U_m : \mathscr{A} \to \mathscr{A}^{\mathbf{N}} \qquad (1.1.2)$$
$$A \rightsquigarrow \ldots \to 0 \to 0 \to \underset{m\text{-th place}}{A} \xrightarrow{\mathrm{id}} A \xrightarrow{\mathrm{id}} \ldots \to A .$$

Therefore if $U_m A \hookrightarrow I$ is a monomorphism into an injective object of $\mathscr{A}^{\mathbf{N}}$, the composition $A \overset{\text{adj.}}{=\!=} V_m U_m A \hookrightarrow V_m I$ is a monomorphism into an injective object of $\mathscr{A}$.

The exact, faithful forgetful functor

$$V : \mathscr{A}^{\mathbf{N}} \to \mathscr{A}^{|\mathbf{N}|} \qquad (1.1.3)$$
$$(A_n, d_n) \rightsquigarrow (A_n)$$

into the product category $\mathscr{A}^{|\mathbf{N}|}$ is left adjoint to

$$P : \mathscr{A}^{|\mathbf{N}|} \to \mathscr{A}^{\mathbf{N}} \qquad (1.1.4)$$
$$(A_n) \rightsquigarrow \left(\prod_{i=1}^{n} A_i, p_n \right),$$

where $p_n : \prod_{i=1}^{n+1} A_i \to \prod_{i=1}^{n} A_i$ is the canonical projection. As above we conclude that P preserves monomorphisms and injective objects and that $\mathscr{A}^{\mathbf{N}}$ has enough injectives if $\mathscr{A}^{|\mathbf{N}|}$ has. Now an object (A_n) of $\mathscr{A}^{|\mathbf{N}|}$ is injective if and only if all A_n are injective in $\mathscr{A}$, so we get the other half of a).

If (A_n, d_n) is injective, the $A_m = V_m(A_n, d_n)$ are injective as remarked above, and the monomorphism $(A_n, d_n) \overset{\iota}{\hookrightarrow} PV(A_n, d_n)$ given by the adjunction has a left inverse, which immediately gives right inverses for the d_n by choosing such for the p_n. Conversely, if the d_n have right inverses s_n, then (A_n, d_n) is isomorphic to $P(\mathrm{Ker}\, d_{n-1})$, which is injective, if A_n and hence $\mathrm{Ker}\, d_{n-1}$ is injective for every $n \in \mathbf{N}$ (here $d_{-1} = 0$ by definition). This proves b); for an explicit description of ι see (1.5).

A left exact functor $h : \mathscr{A} \to \mathscr{B}$ into another abelian category induces a left exact functor $h^{\mathbf{N}} : \mathscr{A}^{\mathbf{N}} \to \mathscr{B}^{\mathbf{N}}$ in the obvious way. If $\mathscr{A}$ – and therefore $\mathscr{A}^{\mathbf{N}}$ – has enough injectives, we can define its right derivatives $R^i h^{\mathbf{N}}$.

(1.2) **Proposition.** *$R^i h^{\mathbf{N}} = (R^i h)^{\mathbf{N}}$ for $i \geqq 0$, i.e., $R^i h^{\mathbf{N}}(A_n, d_n) = (R^i h A_n, R^i h(d_n))$.*

Proof. If $(A_n, d_n) = A \hookrightarrow I^{\cdot}$ is an injective resolution in $\mathscr{A}^{\mathbf{N}}$, $A_n \hookrightarrow I_n^{\cdot}$ is an injective resolution for every n, and $R^i h(d_n)$ is induced by the transition map $I_{n+1}^i \to I_n^i$.

If inverse limits over $\mathbf{N}$ exist in $\mathscr{B}$, we can define the functor

$$\varprojlim_n h : \mathscr{A}^{\mathbf{N}} \to \mathscr{B}$$
$$(A_n, d_n) \rightsquigarrow \varprojlim_n (hA_n, h(d_n)),$$

which by definition is the composition of $h^{\mathbb{N}}$ with the limit functor $\varprojlim_n : \mathscr{B}^{\mathbb{N}} \to \mathscr{B}$. By the left exactness of the latter, $\varprojlim_n h$ is left exact if and only if h is, and we denote its i-th right derivative by $R^i\left(\varprojlim_n h\right)$. In the following, we often omit the transition maps d_n in the notation. If $\mathscr{A}$ and $\mathscr{B}$ have enough injectives, then if h is left exact and maps injectives to injectives, the same is true for $h^{\mathbb{N}}$ by 1.1, and we have a Grothendieck spectral sequence [6, 2.4.1]

$$(1.3) \qquad E_2^{p,q} = \varprojlim_n{}^p R^q h A_n \Rightarrow E^{p+q} = R^{p+q}\left(\varprojlim_n h\right)(A_n, d_n),$$

where $\varprojlim_n{}^p$ is the p-th right derivative of $\varprojlim_n$.

By Roos [10] the derivatives of $\varprojlim_n$ also exist – at least as universal δ-functors – if $\mathscr{B}$ has the property $(AB\,4^*)$, see [6]: infinite products exist and are exact functors (it was sufficient to consider products over $\mathbb{N}$ in our case). Moreover, in this case $\varprojlim_n{}^p = 0$ for $p \geqq 2$, and there is a functorial exact sequence

$$(1.4) \qquad 0 \to \varprojlim_n B_n \to \prod_n B_n \xrightarrow{\mathrm{id}-(d_n)} \prod_n B_n \to \varprojlim_n{}^1 B_n \to 0$$

for (B_n, d_n) in $\mathscr{B}^{\mathbb{N}}$, since the canonical exact sequence

$$(1.5) \qquad 0 \to (B_n, d_n) \xrightarrow{i} PV(B_n, d_n) \to PV(B_{n-1}, d_{n-1}) \to 0$$

$(B_{-1} = 0$ by definition), given by the commutative exact diagram,

$$\begin{array}{ccccccccc}
0 & \to & B_{n+1} & \xrightarrow{(\mathrm{id},\, d_n,\, d_{n-1}d_n,\, \dots)} & \prod\limits_{i=1}^{n+1} B_i & \xrightarrow{p_n - (d_n \mathrm{pr}_{n+1},\, d_{n-1}\mathrm{pr}_n,\, \dots)} & \prod\limits_{i=1}^{n} B_i & \to & 0 \\
 & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n-1}} & & \\
0 & \to & B_n & \xrightarrow{(\mathrm{id},\, d_{n-1},\, d_{n-2}d_{n-1},\, \dots)} & \prod\limits_{i=1}^{n} B_i & \xrightarrow{p_{n-1} - (d_{n-1}\mathrm{pr}_n,\, d_{n-2}\mathrm{pr}_{n-1},\, \dots)} & \prod\limits_{i=1}^{n-1} B_i & \to & 0
\end{array}$$

(pr_i the projections onto B_i), is a $\varprojlim_n$-acyclic resolution of (B_n, d_n), if $\mathscr{B}$ satisfies $(AB\,4^*)$, cf. Roos [10].

(1.6) **Proposition.** *If $\mathscr{A}$ has enough injectives, $h: \mathscr{A} \to \mathscr{B}$ is left exact and $\mathscr{B}$ satisfies $(AB\,4^*)$, there are functorial short exact sequences*

$$0 \to \varprojlim_n{}^1 R^{i-1} h A_n \to R^i\left(\varprojlim_n h\right)(A_n, d_n) \to \varprojlim_n R^i h A_n \to 0$$

for $i \geqq 0$ (where $R^{-1}h = 0$ by definition and the limits are taken via the maps $R^j h(d_n)$).

Proof. If the d_n have right inverses s_n, the last map in the exact sequence

$$0 \to \varprojlim_n B_n \to \prod_n B_n \xrightarrow{\mathrm{id}-(d_n)} \prod_n B_n$$

is surjective: a section is (t_n) with $t_1 = 0$ and t_{n+1} for $n \geqq 1$ recursively defined by $t_{n+1} = s_n(t_n - \mathrm{pr}_n)$, where $\mathrm{pr}_m : \prod_n B_1 \to B_m$ is the canonical projection. In particular, this is the case for $(hI_n, h(d_n))$ with (I_n, e_n) injective in $\mathscr{A}^{\mathbb{N}}$. This shows that, under the assumption $(AB4^*)$ for $\mathscr{B}$, there is a functorial long exact sequence

$$\ldots \to \prod_n R^{i-1}hA_n \xrightarrow{\mathrm{id}-(R^{i-1}h(d_n))} \prod_n R^{i-1}hA_n \xrightarrow{\delta} R^i\left(\varprojlim_n h\right)(A_n, d_n)$$

$$\to \prod_n R^i hA_n \xrightarrow{\mathrm{id}-(R^i h(d_n))} \prod_n RihA_n \xrightarrow{\delta} \ldots$$

for (A_n, d_n) in $\mathscr{A}^{\mathbb{N}}$ inducing the claimed exact sequences.

(1.7) *Remark.* The exact sequences of (1.6) agree with those induced by the spectral sequence (1.3) (if it exists) and the vanishing of $\varprojlim_n^p$ for $p \geqq 2$, if above for δ one takes the *negative* of the usual connecting morphism. This is also necessary to reobtain (1.4) for h the identity functor.

For the rest of this section, fix an abelian category $\mathscr{A}$ with enough injectives and a left exact functor $h: \mathscr{A} \to \mathscr{B}$ into another abelian category $\mathscr{B}$, in which projective limits over $\mathbb{N}$ exist. For $A = (A_n, d_n)$ in $\mathscr{A}^{\mathbb{N}}$ define $A[m] = (A_{n+m}, d_{n+m})$ for each $m \geqq 0$; then there are canonical maps $A[m] \to A$ by the composition of the transition maps.

(1.8) **Lemma.** a) *The morphism $A[1] \to A$ induces isomorphisms*

$$R^i\left(\varprojlim_n h\right) A[1] \xrightarrow{\sim} R^i\left(\varprojlim_n h\right) A$$

for $i \geqq 0$.

b) *If $\mathscr{B}$ has enough injectives or satisfies $(AB4^*)$, the morphism $B[1] \to B$ induces isomorphisms*

$$\varprojlim_n^p B[1] \xrightarrow{\sim} \varprojlim_n^p B$$

for $p \geqq 0$ and $B \in Ob(\mathscr{B}^{\mathbb{N}})$.

Proof. a) If $A \hookrightarrow I^\cdot$ is an injective resolution, $A[1] \hookrightarrow I[1]^\cdot$ is one, too, and we obtain a commutative diagram

$$\begin{array}{ccc} A[1] & \hookrightarrow & I[1]^\cdot \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ A & \hookrightarrow & I \end{array}$$

The claim follows as $\varprojlim_n hI^i[1] \xrightarrow{\sim} \varprojlim_n hI^i$.

b) The first case follows from a) with h the identical functor. The second follows similarly, as $\varprojlim_n^1$ is effacable and the statement is obviously true for $\varprojlim_n$.

(1.9) **Corollary.** *If $A[m] \to A$ is zero for some $m \geqq 0$, i.e., A is AR-zero, cf.* [SGA 5, V 2.2], *then $R^i\left(\varprojlim_n h\right) A = 0$ for all $i \geqq 0$.*

We want a slight generalization of this.

(1.10) *Definition.* Call a system (A_n, d_n) ML-zero (Mittag-Leffler zero) if for any $n \geqq 1$ there is an $m = m(n) > 0$ such that the transition map $C_{n+m} \to C_n$ is zero.

Note that (A_n, d_n) is AR-zero, if one can take on m for all n.

(1.11) **Lemma.** *If $A = (A_n, d_n)$ is ML-zero, then $R^i\left(\varprojlim_n h\right)(A) = 0$ for all $i \geqq 0$.*

Proof. By assumption there is a cofinal set $J = \{n_1, n_2, n_3, \ldots\} \subseteq \mathbb{N}$ such that all transition maps in the inverse system $(A_{n_j}, d'_j)_{j \in \mathbb{N}}$, where the d' are obtained by composition of the d_n, are zero. Now $\varprojlim_n h$ is also the composition of the exact forgetful functor

$$V_J : \mathscr{A}^{\mathbb{N}} \to \mathscr{A}^{\mathbb{N}}, \quad (A_n, d_n) \rightsquigarrow (A_{n_j}, d'_j),$$

which preserves injective objects, with $\varprojlim_n h$. The spectral sequence therefore reduces the question to the case where all d_n are zero, which is solved by (1.9).

By similar arguments, the statement of Lemma (1.11) is also true for $\varprojlim_n$ and $\varprojlim_n^1$ on $\mathscr{B}$ satisfying just $(AB4^*)$.

(1.12) **Lemma.** *The objects in $\mathscr{A}^{\mathbb{N}}$ which are ML-zero, form a Serre subcategory of $\mathscr{A}^{\mathbb{N}}$.*

Proof. It is clear that subobjects and quotients inherite the property, and for an exact sequence $0 \to A \to B \to C \to 0$ with A and C ML-zero and $m_A(n)$ and $m_C(n)$ choosen for A and C as in Definition (1.10), we see that $m_A(n) + m_C(n + m_A(n))$ works for B.

We use the prefix ML, if something holds in the quotient category, i.e., up to systems being ML-zero. For example a complex $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathscr{A}^{\mathbb{N}}$ is ML-exact, if its homology $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ is ML-zero, and a system (A_n, d_n) is called (ML)-l-adic for a prime l, if it is ML-isomorphic to an l-adic system (A'_n, d'_n), i.e., one with $A'_{n+1}/l^n A'_n \xrightarrow[\sim]{d'_n} A'_n$. The functor $h^{\mathbb{N}}$ respects ML-zero systems, and by Lemma (1.10) the $R^i\left(\varprojlim_n h\right)$ factorize through the quotient category. In particular we note

(1.13) **Corollary.** *If $0 \to A \to B \to C \to 0$ is ML-exact in $\mathscr{A}^{\mathbb{N}}$, there is a long exact sequence*

$$\ldots \to R^i\left(\varprojlim_n h\right) A \to R^i\left(\varprojlim_n h\right) B \to R^i\left(\varprojlim_n h\right) C \to R^{i+1}\left(\varprojlim_n h\right) A \to \ldots$$

(1.14) *Remark.* Recall that a system (A_n, d_n) is said to satisfy the Mittag-Leffler condition, if for each n the image of the transition maps $A_{n+m} \to A_n$ is constant for $m \gg 0$. It is easy to see that this is the case if and only if (A_n, d_n) is ML-isomorphic to a system (A'_n, d'_n) with surjective d'_n. The next lemma is well known (see for example [10, Proposition 1]).

(1.15) **Lemma.** *If $\mathscr{B}$ satisfies $(AB4^*)$ and $B \in Ob(\mathscr{B}^{\mathbb{N}})$ satisfies the Mittag-Leffler condition, then $\varprojlim_n^1 B_n = 0$.*

Examples of categories with $(AB\,4^*)$ are the category Ab of abelian groups and the category Mod_R of R-modules over an arbitrary ring R – $(AB\,4^*)$ does not hold i.g. for the category of discrete G-modules for a profinite group G or – more generally – for the category of sheaves on a Grothendieck topology, see Roos [11].

(1.16) **Lemma.** *If for $A=(A_n, d_n)$ in $\mathscr{A}^{\mathbb{N}}$ all A_n have injective resolutions of length N, then A has an injective resolution of length $N+1$; in particular $R^i\left(\varprojlim_n h\right)A=0$ for $i>N+1$.*

Proof. If $V(A_n, d_n)\hookrightarrow I^\cdot$ is an injective resolution, then $PV(A_n, d_n)\hookrightarrow PI^\cdot$ is an injective resolution, so the claim follows with the exact sequence (1.5). Note that the indicated consequence holds even when neither (1.3) nor 1.6 can be applied.

(1.17) *Remark.* a) The functor $\varprojlim_n \mathscr{B}^{\mathbb{N}}\to\mathscr{B}$ preserves injectives, since it is right adjoint to the constant functor $\mathscr{B}\to\mathscr{B}^{\mathbb{N}}$, $B\rightsquigarrow(B)$, which is exact and faithful.

b) For any $m\in\mathbb{N}$ there is a unique morphism of δ-functors $\pi_m:\left(R^i\left(\varprojlim_n h\right),\delta\right)\to(R^ihV_m,\delta)$ such that

$$\pi_m^i: R^i\left(\varprojlim_n h\right)(A_n, d_n)\to R^ihV_m(A_n, d_n)=R^ihA_m$$

coincides with the canonical projection $\varprojlim_n hA_n\to hA_m$ for $i=0$. The π_m^i are compatible for different m, and the induced morphisms

$$R^i\left(\varprojlim_n h\right)(A_n, d_n)\to \varprojlim_n R^ihA_n$$

agree with those occurring in (1.3) or (1.6).

c) If $\mathscr{A}_1\xrightarrow{h}\mathscr{A}_2$ and $\mathscr{A}_2\xrightarrow{k}\mathscr{A}_1$ are adjoint functors, then $h^{\mathbb{N}}$ and $k^{\mathbb{N}}$ are adjoint in an obvious way: this holds in general for functor categories $\underline{\mathrm{Hom}}(I^0,\mathscr{A}_i)$, $\mathbb{N}$ being replaced by any small category I.

While the considerations of Mittag-Leffler properties are specific for the case $I=\mathbb{N}$ (made into a category as usual, with morphisms $j\leq i$), let us mention how the other results carry over to the case of arbitrary small categories I.

(1.18) **Proposition.** *Let $\mathscr{A}$ be an abelian category.*

a) *If $\mathscr{A}$ has enough injectives and possesses products indexed by the sets $I/i=\{\text{morphisms } j\to i \text{ in } I\}$ for all $i\in I$, then $\mathscr{A}^I=\underline{\mathrm{Hom}}(I^0,\mathscr{A})$ has enough injectives.*

b) *If $\mathscr{A}$ possesses arbitrary products, then $\varprojlim_I:\mathscr{A}^I\to\mathscr{A}$, the right adjoint of the constant functor $\mathscr{A}\to\mathscr{A}^I$, exists, is left exact, and maps injectives to injectives.*

The proofs are similar, the functor P in (1.1.4) now becomes

$$P:(A_i)_{i\in I}\rightsquigarrow\left\{i\to\prod_{j\to i}A_j\right\},$$

where for $i\to i'$ the morphism $\prod_{j\to i'}\to\prod_{j\to i}A_j$ has components $\mathrm{pr}_{j\to i\to i'}:\prod_{j\to i'}A_j\to A_j$ for $j\to i$. Proposition (1.2) holds similarly, and for $\mathscr{A}$, $\mathscr{B}$ and h with the properties assumed there, (1.3) becomes

$$E_2^{p,q} = \varprojlim_I{}^p R^q h A_i \Rightarrow R^{p+q}\left(\varprojlim_I h\right)(A_i),$$

where as before we have suppressed the transition maps in the notation. For the description of $\varprojlim_I{}^p$ in categories satisfying $(AB4^*)$ we refer to the paper of Roos [10].

2. Connections with Continuous Group Cohomology

Let G be a profinite group, then the category $M(G)$ of discrete G-modules is an abelian category with enough injectives, and we let

$$(M_n, d_n) \rightsquigarrow H^i(G, (M_n, d_n))$$

be the i-th right derived functor of the left exact functor

$$M(G)^{\mathbb{N}} \to \underline{Ab}$$
$$(M_n, d_n) \rightsquigarrow H^0(G, (M_n, d_n)) = \varprojlim_n H^0(G, M_n).$$

Then by 1.6 we have short exact sequences

$$(2.1) \qquad 0 \to \varprojlim_n{}^1 H^{i-1}(G, M_n) \to H^i(G, (M_n, d_n)) \to \varprojlim_n H^i(G, M_n) \to 0$$

for each $i \geqq 0$.

(2.2) **Theorem.** *Let (T_n, d_n) be an inverse system of discrete G-modules satisfying the Mittag-Leffler condition and let $T = \varprojlim_n T_n$ as a topological G-module. Then there are canonical isomorphisms*

$$H^i_{\text{cont}}(G, T) \cong H^i(G, (T_n, d_n))$$

for $i \geqq 0$, functorial in (T_n, d_n), where $H^i_{\text{cont}}(G, T)$ denote the continuous cohomology groups defined by Tate in [14]. *If $0 \to (R_n) \to (S_n) \to (T_n) \to 0$ is a ML-exact sequence of Mittag-Leffler systems with limits R, S and T, respectively, the above isomorphisms induce an isomorphism of long exact sequences*

$$\begin{array}{ccccccccc} \dots \to & H^i(G,(R_n)) & \to & H^i(G,(S_n)) & \to & H^i(G,(T_n)) & \to & H^{i+1}(G,(R_n)) & \to \dots \\ & \wr\| & & \wr\| & & \wr\| & & \wr\| & \\ \dots \to & H^i_{\text{cont}}(G,R) & \to & H^i_{\text{cont}}(G,S) & \to & H^i_{\text{cont}}(G,T) & \to & H^{i+1}_{\text{cont}}(G,R) & \to \dots, \end{array}$$

the lower one associated to the exact sequence $0 \to R \to S \to T \to 0$ (Note that Tate's existence criterion is fulfilled).

Proof. We may assume that all d_n are surjective. By definition $H^i_{\text{cont}}(G, T)$ is the i-th homology group of the complex

$$C^{\cdot}(G, T)$$

of continuous cochains in T, and by the definition of the topology of T, this is the inverse limit of the complexes $C^{\cdot}(G, T_n)$ of continuous cochains in T_n, where the T_n

have the discrete topology. It is well-known (using the homogeneous bar resolution, see [4, 1. and 2.]), that $C^{\cdot}(G, T_n)$ is the complex of fixed modules of a canonical complex

$$D^{\cdot}(G, T_n)$$

of discrete G-modules, which is an acyclic resolution of the discrete G-module T_n. Moreover, this resolution is functorial, and we get a resolution

$$(T_n, d_n) \hookrightarrow (D^1(G, T_n), d_n^1) \to (D^2(G, T_n), d_n^2) \to \dots$$

of (T_n, d_n) in $M(G)^{\mathbb{N}}$, from which $H^i_{\text{cont}}(G, T)$ is obtained by applying $\varprojlim_n H^0(G, -)$ and taking homology. So we only have to show that the systems $(D^i(G, T_n), d_n^i)$ are acyclic for $\varprojlim_n H^0(G, -)$, which follows at once from (2.1) and (1.15), since the $C^i(G, T_n) = H^0(G, D^i(G, T_n))$ form a Mittag-Leffler system. The functoriality is obvious.

(2.3) *Remark.* The theorem in particular applies to the case of finitely generated $\mathbb{Z}_l$-modules T with continuous G-action and the inverse system of the $T_n = T/l^n T$. Note that for an exact sequence $0 \to R \to S \to T \to 0$ of such modules, the sequence

$$0 \to (R/l^n R) \to (S/l^n S) \to (T/l^n T) \to 0$$

is ML-exact.

3. Continuous Étale Cohomology

Let X be a scheme and (F_n, d_n) be an inverse system of sheaves on the small étale site X_{et} of X, then we define

$$H^i(X, (F_n, d_n)) = R^i \left(\varprojlim_n \Gamma \right) (F_n, d_n),$$

for $i \geq 0$, *where* $\Gamma = H^0(X, -)$ is the section functor on the category $S(X_{et})$ of sheaves on X_{et}, i.e., $H^i(X, -)$ is the i-th derivative of the functor $S(X_{et})^{\mathbb{N}} \to \underline{Ab}$, $(F_n, d_n) \rightsquigarrow \varprojlim_n H^0(X, F_n)$. By (1.6) we have short exact sequences

$$(3.1) \qquad 0 \to \varprojlim_n{}^1 H^{i-1}(X, F_n) \to H^i(X, (F_n, d_n)) \to \varprojlim_n H^i(X, F_n) \to 0.$$

For example this applies to an étale l-adic sheaf $F = (F_n)$ on X for a prime l (mostly assumed to be invertible on X), and we call

$$H^i_{\text{cont}}(X, F) := H^i(X, (F_n))$$

the *continuous cohomology groups* of F. In particular we have *continuous l-adic cohomology groups* $H^i_{\text{cont}}(X, \mathbb{Z}_l(j))$ associated to the sheaves $\mathbb{Z}_l(j) = (\mathbb{Z}/l^n(j))$ (for a definition of these sheaves and the Tate twist $F(j)$ in general see for example [9, pp. 163, 164]). The name is justified by

(3.2) **Theorem.** *Let* $X = \operatorname{Spec} K$ *for a field* K *with separable closure* K_s *and let* $F = (F_n)$ *be a Mittag-Leffler system of sheaves on* X_{et} *(for example an l-adic sheaf).*

Let $F_{\bar{x}} = \varprojlim_n (F_n)_{\bar{x}}$ *be the inverse limit of the stalks at the geometric point* $\bar{x} = \operatorname{Spec} K_s$ *of* X, *considered as a topological group with continuous action of* $G_K = \operatorname{Gal}(K_s/K)$. *Then there are canonical isomorphisms for* $i \geqq 0$

$$H^i(X,(F_n)) \cong H^i_{\text{cont}}(G_K, F_{\bar{x}}),$$

functorial in F *and with respect to short ML-exact sequences.*

Proof. Since the functor $G \rightsquigarrow G_{\bar{x}}$ gives an equivalence between $S(X_{et})$ and $M(G_K)$, and $\Gamma(X,G) = G_{\bar{x}}^{G_K}$, the statement immediately follows from the definitions and Theorem 2.2.

(3.3) **Theorem.** *Let* $\pi: X' \to X$ *be a possibly infinite Galois covering of schemes with Galois group* G *and let* (F_n) *be a system of sheaves on* X_{et}. *Then there is a spectral sequence*

$$E_2^{p,q} = H^p(G,(H^q(X',\pi^* F_n))) \Rightarrow E^{p+q} = H^{p+q}(X,(F_n)).$$

Proof. This is just the Grothendieck spectral sequence for the composition of functors

$$\begin{array}{ccccc} S(X_{et})^{\mathbb{N}} & \to & M(G)^{\mathbb{N}} & \to & \underline{Ab} \\ (F_n) & \rightsquigarrow & (\Gamma(X',\pi^* F_n)) & & \\ & & (M_n) & \rightsquigarrow & \left(\varprojlim_n M_n\right)^G = \varprojlim_n M_n^G \end{array}$$

since $\Gamma(X',\pi^* F_n)^G = \Gamma(X, F_n)$.

(3.4) **Corollary.** *Let* X *be a scheme of finite type over a field* K *with separable closure* K_s, *let* $\bar{X} = X \times_K K_s$ *and* $G_K = \operatorname{Gal}(K_s/K)$. *If* (F_n) *is an inverse system of sheaves on* X_{et} *such that* $H^i(\bar{X}, F_n)$ *is finite for every* n *and* i, *then there is a spectral sequence*

$$E_2^{p,q} = H^p_{\text{cont}}\left(G_K, \varprojlim_n H^q(\bar{X}, F_n)\right) \Rightarrow E^{p+q} = H^{p+q}(X,(F_n)).$$

Proof. If the $H^i(\bar{X}, F_n)$ are finite, then the system $(H^i(\bar{X}, F_n))$ satisfies the Mittag-Leffler condition, and we can apply (2.2).

(3.5) *Remark.* a) The corollary applies, if X is proper over K and the F_n are constructible (by the proper base change theorem, see [9, VI 2.1]).

b) By Deligne's finiteness theorem, cf. [SGA 4½, p. 236], we get another example: if X is of finite type over K and $l \neq \operatorname{char} K$, then there are spectral sequences

$$H^p_{\text{cont}}(G_K, H^q(\bar{X}, \mathbb{Z}_l(j))) \Rightarrow H^{p+q}_{\text{cont}}(X, \mathbb{Z}_l(j))$$

where $H^q(\bar{X}, \mathbb{Z}_l(j)) = \varprojlim_n H^q(\bar{X}, \mathbb{Z}/l^n(j))$ and this group has the l-adic topology by (4.5) below and Jouanolou's theorem, cf. [SGA 5, V 5.3.1].

c) If moreover G_K has finite cohomology for all finite l-torsion G_K-modules (for example if K is finite or a local field), then there are isomorphisms

$$H^i_{\text{cont}}(G_K, H^q(\bar{X}, \mathbb{Z}_l(j))) \cong \varprojlim_n H^i(G_K, H^q(\bar{X}, \mathbb{Z}/l^n(j)))$$

$$H^i_{\text{cont}}(X, \mathbb{Z}_l(j)) \cong \varprojlim_n H^i(X, \mathbb{Z}/l^n(j))$$

by (1.15), (2.1), and (3.1), because the $H^i(X, \mathbb{Z}/l^n(j))$ are finite (by the finiteness of the $H^i(\bar{X}, \mathbb{Z}/l^n(j))$ and the Hochschild-Serre spectral sequence-note that this argument and especially Corollary 2.8 in [9, VI] is false without the assumption on G_K). Then the spectral sequence in b) is just the limit of the Hochschild-Serre spectral sequences for $\mathbb{Z}/l^n(j)$.

Much of the formalism of étale cohomology carries over to the groups $H^i(X, (F_n, d_n))$; on the other hand the cohomology $H^i(X, F)$ of a sheaf F can be identified with the cohomology of the constant system $\underline{F} = (F_n = F, d_n = \mathrm{id}_F)$.

For a closed subscheme $Z \hookrightarrow X$ and an inverse system (F_n) of sheaves on X_{et} we have the cohomology groups with support

$$H^i_Z(X, (F_n)) := R^i \Big(\varprojlim_n \Gamma_Z(X, -)\Big)(F_n),$$

where $\Gamma_Z(X, F) = \ker(\Gamma(X, F) \to \Gamma(U, F))$ for $U = X - Z$, and – since the sequence of functors

$$0 \to \varprojlim_n \Gamma_Z(X, -) \to \varprojlim_n \Gamma(X, -) \to \varprojlim_n \Gamma(U, -) \to 0$$

is exact on injectives – a long exact sequence

(3.6) $$\ldots \to H^{i-1}(U, ((F_n)) \to H^i_Z(X, (F_n)) \to H^i(X, (F_n)) \to H^i(U, (F_n)) \to \ldots.$$

There is also a cohomology with compact support for a scheme X separated and of finite type over a field k by setting

$$H_c(X, (F_n)) := H^i(\tilde{X}, (j_! F_n))$$

for a compactification $X \xrightarrow{j} \tilde{X}$, and a long exact sequence

(3.7) $$\ldots \to H^{i-1}_c(Z, (F_n)) \to H^i_c(U, (F_n)) \to H^i_c(X, (F_n)) \to H^i_c(Z, (F_n)) \to \ldots$$

for X, Z, U, and (F_n) as above (same argument as in [9, III 1.30]).

The main difference to usual étale cohomology is that $H^i(X, (F_n))$ in general does not commute with inverse limits in X and direct limits in (F_n). However, we still have

(3.8) **Proposition.** *Let z be a closed point of X and $\mathcal{O}^h_{X,z}$ be Henselization of the local ring at z. Then*

$$H^i_z(X, (F_n)) \xrightarrow{\sim} H^i_z(\operatorname{Spec} \mathcal{O}^h_{X,z}, (F_n))$$

for any inverse system (F_n) of sheaves on X_{et}.

Proof. The morphisms $\operatorname{Spec} \mathcal{O}^h_{X,z} \to X$ induces a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \to & \varprojlim_n^1 H^{i-1}_z(\operatorname{Spec} \mathcal{O}^h_{X,z}, F_n) & \to & H^i_z(\operatorname{Spec} \mathcal{O}^h_{X,z}, F_n) & \to & \varprojlim_n H^i_z(\operatorname{Spec} \mathcal{O}^h_{X,z}, F_n) & \to & 0 \\
 & & \alpha \downarrow \wr & & \downarrow & & \beta \downarrow \wr & & \\
0 & \to & \varprojlim_n^1 H^{i-1}_z(X, F_n) & \to & H^i_z(X, (F_n)) & \to & \varprojlim_n H^i_z(X, F_n) & \to & 0
\end{array}$$

in which α and β are isomorphisms [9, III 1.28].

Since inverse limits exist in $S(X_{et})$, we can define relative versions of all these functors and we get many Grothendieck spectral sequences by (1.17) and the fact that $H^0(X_{et}, -)$ and π_* for a morphism $\pi: X \to Y$ commute with inverse limits. For example, we already obtained (3.1) from $\varprojlim_n H^0(X, -) = \varprojlim_n \circ H^0(X, -)^{\mathbb{N}}$, and from $\varprojlim_n H^0(X, -) = H^0(X, -) \circ \varprojlim_n$ we get

(3.9) $$H^p\left(X, \varprojlim_n{}^q F_n\right) \Rightarrow H^{p+q}(X, (F_n)).$$

Note however, that $\varprojlim_n{}^q F_n$ does in general not vanish for $q \geqq 2$, see [11]. From $\varprojlim_n H^0(X, -) = \varprojlim_n H^0(Y, -) \circ \pi_*^{\mathbb{N}}$ and $\varprojlim_n \pi_* = \pi_* \circ \varprojlim_n$ we obtain the two spectral sequences

(3.10) $$H^p(Y, (R^q\pi_* F_n)) \Rightarrow H^{p+q}(X, (F_n)),$$

(3.11) $$R^p\pi_* \varprojlim_n{}^q (F_n) \Rightarrow R^{p+q}\left(\varprojlim_n \pi_*\right)(F_n).$$

As for the computation we have

(3.12) **Lemma.** *For a morphism $\pi: X \to Y$ and a system (F_n) of sheaves on X_{et}, $R^i\left(\varprojlim_n \pi_*\right)(F_n)$ is the sheaf associated to the pre-sheaf $V \mapsto H^i(V \times_Y X, (F_{n|V \times_Y X}))$ on Y_{et}. In particular, $\varprojlim_n{}^q F_n$ is the sheaf associated to the presheaf $U \mapsto H^q(U, (F_n))$ on X_{et}.*

Proof. Let $i: S(X_{et}) \hookrightarrow P(X_{et})$ be the embedding into the category of presheaves on X_{et}, and $a: P(X_{et}) \to S(X_{et})$ be the left-adjoint, $a(P)$ = associated sheaf. With the canonical map $\pi_P: P(X_{et}) \to P(Y_{et})$ we have $\pi_* = a\pi_P i$. Let $(F_n) = F \hookrightarrow I^\cdot$ be an injective resolution of F in $S(X_{et})$. Then

$$R^i\left(\varprojlim_n \pi_*\right)(F_n) = \mathscr{H}^i\left(\varprojlim_n \pi_*^{\mathbb{N}} I^\cdot\right) = \mathscr{H}^i\left(\pi_* \varprojlim_n I^\cdot\right) = a\pi_P \mathscr{H}^i\left(i \varprojlim_n I^\cdot\right),$$

as π_* commutes with $\varprojlim_n$, and by definition $\mathscr{H}^i\left(i \varprojlim_n I^\cdot\right)$ is the presheaf $U \rightsquigarrow H^p(U, (F_n))$ on X_{et}.

Like in any abelian category with enough injectives, the Ext-functors $\mathrm{Ext}^i_X((G_n, g_n), -)$ on $S(X_{et})^{\mathbb{N}}$ are the i-th right derivatives of the left exact functor $S(X_{et})^{\mathbb{N}} \to \underline{Ab}$ given by

$$(F_n, f_n) \rightsquigarrow \mathrm{Hom}_X((G_n, g_n), (F_n, f_n)).$$

(3.13) **Proposition.** *There is a long exact sequence*

$$\ldots \to \mathrm{Ext}^i_X((G_n, g_n), (F_n, f_n)) \to \prod_n \mathrm{Ext}^i_X(G_n, F_n) \xrightarrow{\alpha} \prod_n \mathrm{Ext}^i_X(G_{n+1}, F_n)$$
$$\to \mathrm{Ext}^{i+1}_X((G_n, g_n), (F_n, f_n)) \to \ldots,$$

$\alpha = (g_n^*) - (f_{n*})$, which is functorial in (G_n, g_n) and (F_n, f_n).

Proof. There is a functorial exact sequence

$$0 \to \mathrm{Hom}_X((G_n, g_n), (F_n, f_n)) \to \prod_n \mathrm{Hom}_X(G_n, F_n) \xrightarrow{(g_n^*)-(f_{n*})} \prod_n \mathrm{Hom}_X(G_{n+1}, F_n)$$

in which the last map is surjective for injective (F_n, f_n): if the f_n have right inverses s_n, a preimage for $(v_n) \in \prod_n \mathrm{Hom}_X((G_{n+1}, F_n)$ is (u_n) with $u_1 = 0$ and recursively $u_{n+1} = s_n(u_n g_n - u_n)$. This implies the claim by standard arguments in homological algebra, as $\underline{Ab}$ satisfies $(AB4^*)$.

(3.14) *Remark.* There are canonical isomorphism $\mathrm{Ext}^i_X(\mathbb{Z}, (F_n, f_n)) \cong H^i(X, (F_n, f_n))$, by extension of the obvious one for $i = 0$ to the derived functors. Via these and the analogous isomorphisms $\mathrm{Ext}^i_X(\mathbb{Z}, F_n) \cong H^i(X, F_n)$, the sequence in (3.13) for $(G_n, g_n) = \mathbb{Z}$ coincides with the sequence constructed in the proof of (1.6).

Whenever purity or semi-purity holds for a closed immersion $i: Z \hookrightarrow X$ (cf. [SGA 5, 3.1.4], [SGA 4½, (cycle) 2.2]), it can be extended to our setting – in the absolute case by the spectral sequence

(3.15) $$E_2^{p,q} = H^p(Z, (R^q i^! F_n)) \Rightarrow H_Z^{p+q}(X, (F_n))$$

and the short exact sequences

(3.16) $$0 \to \varprojlim_n{}^1 H_Z^{i-1}(X, F_n) \to H_Z^i(X, (F_n)) \to \varprojlim_n H_Z^i(X, F_n) \to 0$$

for a pro-sheaf (F_n) on X. The next theorem gives an example.

(3.17) **Theorem.** *Let S be a scheme and (Z, X) be a smooth S-pair of codimension c, i.e., a closed immersion $i: Z \hookrightarrow X$ of smooth S-schemes such that each fibre over S has codimension c. Let (F_n) be an inverse system of locally constant torsion sheaves on X_{et} with torsion prime to* char(S).

Then there are canonical isomorphisms for $j \in \mathbb{Z}$

(3.18) $$H^j(Z, (R^{2c} i^! F_n)) \xrightarrow{\sim} H_Z^{j+2c}(X, (F_n)).$$

If F_n is annihilated by l^n, l a prime invertible on S, there is a canonical isomorphism of inverse systems of sheaves

(3.19) $$(R^{2c} i^! F_n) \cong (i^* F_n \otimes_{\mathbb{Z}} R^{2c} i^! \mathbb{Z}/l^n).$$

Moreover, there is a canonical isomorphism of l-adic sheaves

(3.20) $$R^{2c} i^! \mathbb{Z}_l(c) = (R^{2c} i^! \mu_{l^n}^{\otimes c}) \xleftarrow{\sim} (\mathbb{Z}/l^n) = \mathbb{Z}_l$$

given by the local cycle class, thus by combining (3.18), (3.19), *and* (3.20) *one has the Gysin isomorphisms*

(3.21) $$H^j(Z, (i^* F_n(-c)) \xrightarrow{\sim} H_Z^{j+2c}(X, (F_n)), \quad j \in \mathbb{Z},$$

and, by composing with $H_Z^{j+2c}(X, (F_n)) \to H^{j+2c}(X, (F_n))$, the Gysin morphisms

(3.22) $$i_*: H^j(Z, (i^* F_n(-c))) \to H^{j+2c}(X, (F_n)), \quad j \in \mathbb{Z}.$$

Proof. (3.18) follows from (3.15), since $R^q i^! F_n = 0$ for $q \neq 2c$ see [SGA 4, XVI 3.7]. For (3.19) and (3.20) see [SGA 4, XV 3.8, 3.10], and [SGA 4½, (cycle) 2.2]; one only has to check that the isomorphisms there, of which the second one is given by the local cycle (or fundamental) class (loc. cit.), are compatible with the (obvious) transition maps. The noetherian assumption in [SGA $4\frac{1}{2}$] can be avoided by using [SGA 4, XVI 3.8.1], the separatedness by localization.

As usual there are analogs of (3.15), (3.18), (3.21), and (3.22) with supports in a closed subscheme W of Z.

We end this section by defining a *global cycle class* in the continuous l-adic cohomology. It will be clear from the construction that this can be done whenever one has a cycle class in the usual l-adic cohomology, but to fix ideas we stick to varieties over a field k.

(3.23) **Theorem.** *Let X be a smooth variety over a field k and let $l \neq \mathrm{char}(k)$ be a prime. If Z is a closed subscheme of codimension c, then*

$$H^i_{Z,\mathrm{cont}}(X, \mathbb{Z}_l(j)) = 0 \quad \text{for} \quad i < 2c \quad \text{and} \quad j \in \mathbb{Z},$$

$$H^{2c}_{Z,\mathrm{cont}}(X, \mathbb{Z}_l(j)) \xrightarrow{\sim} \varprojlim_n H^{2c}_Z(X, \mathbb{Z}/l^n(j))$$

In particular, for Z irreducible and reduced there is a canonical cycle class (with support)

$$\mathrm{cl}(Z) = \mathrm{cl}^X_Z(Z) \in H^{2c}_{Z,\mathrm{cont}}(X, \mathbb{Z}_l(c)),$$

which is the "limit" of the cycle classes $\mathrm{cl}(Z) \in H^{2c}_Z(X, \mathbb{Z}/l^n(c))$ *defined in* [SGA $4\frac{1}{2}$, (cycle) 2.2.10]. *The global cycle glass* $\mathrm{cl}^X(Z)$ *is defined as the image of* $\mathrm{cl}^X_Z(Z)$ *in* $H^{2c}_{\mathrm{cont}}(X, \mathbb{Z}_l(c))$.

Proof. The first statement follows from (3.16) and the vanishing of $H^i_Z(X, \mathbb{Z}/l^n(j))$ for $i < 2c$, see [SGA $4\frac{1}{2}$, (cycle) 2.2.8]. The rest is clear, since $\mathrm{cl}_{n+1}(Z)$ is mapped to $\mathrm{cl}_n(Z)$ under the canonical morphism $H^{2c}_Z(X, \mathbb{Z}/l^{n+1}(c)) \to H^{2c}_Z(X, \mathbb{Z}/l^n(c))$.

(3.24) *Remark.* If Z is a smooth prime cycle, then, directly by definition, $\mathrm{cl}^X_Z(Z)$ and $\mathrm{cl}^X(Z)$ are the images of $1_Z \in H^0(Z, \mathbb{Z}_l)$ under the Gysin isomorphism and morphism, respectively. In particular, if k is perfect, then for any prime cycle Z of codimension c there is a canonical isomorphism

$$H^{2c}_{Z,\mathrm{cont}}(X, \mathbb{Z}_l(c)) \cong \mathbb{Z}_l$$

such that $\mathrm{cl}(Z)$ is the image of 1 (use excision as in [6, VI, pp. 268, 269]).

For a cycle $Z = \sum n_j Z_j$ of codimension c, with irreducible and reduced Z_j, we let as usual

$$\mathrm{cl}(Z) = \sum_j n_j \,\mathrm{cl}(Z_j) \in H^{2c}_{|Z|,\mathrm{cont}}(X, \mathbb{Z}_l(c)) \cong \bigoplus_j H^{2c}_{Z_j,\mathrm{cont}}(X, \mathbb{Z}_l(c))$$

and

$$\mathrm{cl}^X(Z) = \sum_j n_j \,\mathrm{cl}^X(Z_j).$$

Then we have the following functorial properties [see also (6.13) below].

(3.25) **Proposition.** *Let $f: Y \to X$ be a morphism of smooth varieties over the field k.*

a) *If Z is a cycle on X and f^*Z is defined, then* $\mathrm{cl}^Y(f^*Z) = f^*\mathrm{cl}^X(Z)$.

b) *If f is a closed immersion and W is a cycle on Y, then* $\mathrm{cl}^X(W) = f_*\mathrm{cl}^Y(W)$, *where f_* is the Gysin map.*

Proof. This is implied by the corresponding statements for the cycle classes with support, which follow by passing to the limit from the properties of the classes $\mathrm{cl}_n(Z)$ proved in [SGA 4½, (cycle) 2.3].

(3.26) **Lemma.** *For any scheme X on which l is invertible let*

$$\delta=\delta_X:\mathrm{Pic}(X)=H^1(X,\mathbb{G}_m)\to H^2_{\mathrm{cont}}(X,\mathbb{Z}_l(1))$$

be the connecting morphism for the exact sequence

$$\text{(3.27)}\qquad \begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & \mu_{l^{n+1}} & \to & \mathbb{G}_m & \xrightarrow{l^{n+1}} & \mathbb{G}_m & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \to & \mu_{l^n} & \to & \mathbb{G}_m & \xrightarrow{l^n} & \mathbb{G}_m & \to & 0 \\ & & \vdots & & \vdots & & \vdots & & \end{array}\qquad (*)$$

of inverse systems of sheaves on X (called the Kummer sequence).

a) *If $i:Z\hookrightarrow X$ is a smooth S-pair of codimension 1 for some scheme S, defined by the invertible ideal $J\subsetneq\mathcal{O}_X$, then $\delta_X(-[J])=i_*(1_Z)$, where $[J]$ is the class of J in* $\mathrm{Pic}(X)$ *and $i_*:H^0_{\mathrm{cont}}(X,\mathbb{Z}_l)\to H^2_{\mathrm{cont}}(X,\mathbb{Z}_l(1))$ is the Gysin morphism.*

b) *If X is smooth over a field and D is a divisor, then $\delta_X([\mathcal{O}_X(D)])=\mathrm{cl}^X(D)$.*

Proof. Since $J=\mathcal{O}(-D)$ for the associated Cartier divisor D in a), this follows from [SGA 4½, (cycle) 2.1.2] and the commutative diagram

$$\begin{array}{cccccc} \mathrm{cl}(D) & H^1_D(X,\mathbb{G}_m) & \xrightarrow{\delta} & H^2_D(X,\mu_{l^n}) & \mathrm{cl}_n(D) \\ \Big| & \| & 1.17\mathrm{b)} & \uparrow & \\ \Big| & H^1_D(X,\mathbb{G}_m) & \xrightarrow{\delta} & H^2_{D,\mathrm{cont}}(X,\mathbb{Z}_l(1)) & \\ \downarrow & \downarrow & & \downarrow & \\ [\mathcal{O}(D)] & H^1(X,\mathbb{G}_m) & \xrightarrow{\delta} & H^2_{\mathrm{cont}}(X,\mathbb{Z}_l(1)) & \end{array}$$

for any Cartier divisor D in X, where the δ above is the connecting morphism for the sequence (*).

(3.28) *Generalization.* Let I, J be small categories, and let $X:I\to\mathit{Schemes}$ be a functor into the category of schemes, i.e., a diagram of schemes. A sheaf F on X is given by sheaves F_i on X_i for each $i\in I$ and morphisms $F_i\to X(\alpha)_*F_j$ for each $\alpha:j\to i$ in I, functorial in the obvious sense. Then the category $S(X)$ of sheaves on X is an abelian category with enough injectives and arbitrary products (in fact, a topos). The groups $H^0(X_i,F_i)$ form an I^0-diagram of abelian groups via the transition maps $H^0(X_i,F_i)\to H^0(X_i,X(\alpha)_*F_j)=H^0(X_j,F_j)$ for $\alpha:j\to i$, and we define the cohomology $H^*(X,F)$ by deriving the left exact functor

$$F\rightsquigarrow \varprojlim_I H^0(X_i,F_i)$$

on $S(X)$. We get a spectral sequence

$$\text{(3.29)}\qquad E_2^{p,q}=\varprojlim_I{}^p\, H^q(X_i,F_i)\Rightarrow H^{p+q}(X,F).$$

For J^0-diagrams of sheaves on X, $(F_j) \in S(X)^J$, we define the cohomology by deriving the functor

$$(F_j) \rightsquigarrow \varprojlim_J H^0(X, F_j) = \varprojlim_J \varprojlim_I H^0(X_i, F_{ij}),$$

compare 1.18. We obtain the same result, if we extend X to a diagram $X^J: I \times J \to$ *Schemes* ("*constant in* J-direction"), regard (F_j) as sheaf on this, and take the cohomology (note the equivalence $S(X)^J \xrightarrow{\sim} S(X^J)$).

The case treated above is $I = pt$, $J = \mathbb{N}$; for I = arbitrary and $J = pt$ one obtains usual étale cohomology on diagrams of schemes, and for I = arbitrary and $J = \mathbb{N}$ we get, for example, continuous l-adic cohomology $H^*(X, \mathbb{Z}_l(j))$ of diagrams X as above, e.g., of simplicial schemes ($I = \Delta^0$). For relative cohomology $H^*_{\mathrm{cont}}(f, \mathbb{Z}_l(j))$ of morphisms $f: X_1 \to X_2$ $(I = 1 \to 2)$ one rather derives the functor $\varprojlim_n \mathrm{Ker}(H^0(X_2, -) \to H^0(X_1, -))$ to get an exact sequence

$$\ldots \to H^i_{\mathrm{cont}}(f, \mathbb{Z}_l(j)) \to H^i_{\mathrm{cont}}(X_2, \mathbb{Z}_l(j)) \xrightarrow{f^*} H^i_{\mathrm{cont}}(X_1, \mathbb{Z}_l(j)) \to H^{i+1}_{\mathrm{cont}}(f, \mathbb{Z}_l(j)) \to \ldots.$$

All results of this paper have counterparts for diagrams of schemes.

(3.30) **Lemma.** *Let X be a locally noetherian simplical scheme and let $(F_j)_{j \in J}$ be a diagram of locally constant étale sheaves on X. Then for $r \geq 0$ there are canonical isomorphisms between the continuous étale cohomology groups $H^r_{\mathrm{cont}}(X, (F_j))$ defined by Dwyer and Friedlander* [3, 2.8] *and the groups defined above, functorial in X and (F_j).*

Proof. With the notations in [3, 2.8], one has for $r \leq m$

$$H^r_{\mathrm{cont}}(X, (F_j)) = \pi_{m-r} \operatorname*{holim}_{\leftarrow n} \operatorname*{holim}_{\leftarrow J} \varinjlim_U{}' \mathrm{Hom}(\pi \cdot U, K(\mathcal{M}_j, m) \langle n \rangle)_{NG_j},$$

where $\mathcal{M}_j = (M_j, G_j)$ is the coefficient system determined by F_j, and $\varinjlim'$ indicates limit only over those hypercoverings U of X for which F_j is trivial on U_{00}. Since $K(\mathcal{M}_j, m) \to K(\mathcal{M}_j, m)\langle n \rangle$ is a weak equivalence for $n > m$ we may omit $\operatorname*{holim}_{\leftarrow n}$ and $\langle n \rangle$.

There are natural homomorphisms of cochain complexes

$$\bar{N}\,\mathrm{Hom}(\pi \cdot U, K(\mathcal{M}_j, m))_{NG_j} \leftarrow \tau_{\leq m} C^{\cdot}(\pi \cdot U, L_j)[m] \hookleftarrow \tau_{\leq m} F_j(\mathrm{diag}\, U)[m],$$

functorial in F_j, where $\bar{N}$ is the normalization functor giving an equivalence between simplicial abelian groups and cochain complexes, $\tau_{\leq m}$ is the canonical truncation, and $C^{\cdot}(\pi \cdot U, L_j)$ is the usual cochain complex for the local system L_j associated to F_j on $\pi \cdot U$. The first map is a quasi-isomorphism.

Let $(F_j) \hookrightarrow (I_j^{\cdot})$ be an injective resolution in $S(X)^J$, then there are canonical quasi-isomorphisms of J-diagrams of complexes

$$\varinjlim_U (F_j(\mathrm{diag}\, U))_j \to \mathrm{tot} \varinjlim_U (I_j^{\cdot}(\mathrm{diag}\, U))_j \leftarrow (H^0(X, I_j^{\cdot}))_j,$$

since the left functor is defined on all systems (F_j) of sheaves and its homology gives a universal δ-functor computing the cohomology of (F_j). Since holim corresponds to $R\varprojlim$ and π_i to H^{-i} via $\bar{N}$, the result follows – the functoriality is clear.

4. Inverse Limits and Properties of Abelian Groups

Let A be an abelian group and l a prime number, then we have a short exact sequence of inverse systems

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & {}_{l^{n+1}}A & \longrightarrow & A & \xrightarrow{l^{n+1}} & l^{n+1}A & \longrightarrow & 0 \\ & & \downarrow & & l\downarrow & & \cap| & & \\ 0 & \longrightarrow & {}_{l^n}A & \longrightarrow & A & \xrightarrow{l^n} & l^nA & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \end{array} \rightarrow \tag{4.1}$$

By definition, $\varprojlim_n {}_{l^n}A = T_l(A)$ is the l-Tate module of A, while $\varprojlim_n l^nA = \bigcap_n l^nA = l\text{-div}(A)$ is the subgroup of l-divisible elements. Denote by (A, l) the system $\ldots \xrightarrow{l} A \xrightarrow{l} A \xrightarrow{l} \ldots$, then $\varprojlim_n (A, l)$ and $\varprojlim_n^1 (A, l)$ are uniquely l-divisible by 1.8. Finally, by the exact sequence

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & l^{n+1}A & \longrightarrow & A & \longrightarrow & A/l^{n+1}A & \longrightarrow & 0 \\ & & \cap| & & \| & & \downarrow & & \\ 0 & \longrightarrow & l^nA & \longrightarrow & A & \longrightarrow & A/l^nA & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

we get the exact sequence $\left(\text{note that } \varprojlim_n^1 (A, \mathrm{id}) = 0\right)$

$$0 \to \bigcap_n l^nA \to A \to \hat{A} \to \varprojlim_n^1 l^nA \to 0,$$

where $\hat{A} = \varprojlim_n A/l^nA$ is the l-adic completion of A, and therefore a canonical isomorphism $\varprojlim_n^1 l^nA \cong \mathrm{coker}(A \to \hat{A})$. So by (4.1) we get a "long" exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \varprojlim_n {}_{l^n}A & \longrightarrow & \varprojlim_n (A, l) & \longrightarrow & \bigcap_n l^nA & \longrightarrow & \varprojlim_n^1 {}_{l^n}A \\ & \| & & & & \| & & \\ & T_l(A) & & \begin{array}{c}\text{uniquely}\\ l\text{-divisible}\end{array} & & l\text{-div}(A) & & \end{array}$$

$$\begin{array}{ccccc} \longrightarrow \varprojlim_n^1 (A, l) & \longrightarrow & \varprojlim_n^1 l^nA & \longrightarrow 0 \\ & & \| & \\ \begin{array}{c}\text{uniquely}\\ l\text{-divisible}\end{array} & & \mathrm{coker}(A \to \hat{A}) & \end{array} \tag{4.2}$$

(4.3) **Lemma.** a) *The image of* $\varprojlim_n (A, l)$ *in* l-div(A) *is* l-Div(A), *the maximal* l*-divisible subgroup of* A.

b) $\mathrm{Tor}\left(\varprojlim_n^1 {}_{l^n}A\right) = l\text{-}D(\mathrm{Tor}(A)) \cong \mathrm{Tor}(l\text{-}D(A))$, *where* Tor$(B)$ *is the torsion subgroup and* $l\text{-}D(B) = l\text{-div}(B)/l\text{-Div}(B)$ *for an abelian group* B.

c) $l\text{-Div}\left(\varprojlim_n{}^1 {}_{l^n}A\right)=0.$

Proof. a) can be checked directly, and b) follows from a) and the commutative diagram

$$\begin{array}{ccccccc} 0\to & T_l(A) & \to & \varprojlim_n (A,l) & \to & l\text{-div(A)} \\ & \wr| & & \circlearrowleft & & \circlearrowleft \\ 0\to & T_l(\mathrm{Tor}(A)) & \to & \varprojlim_n (\mathrm{Tor}(A),l) & \to & l\text{-div}(\mathrm{Tor}(A)) \end{array}$$

$$\begin{array}{ccccc} \to & \varprojlim_n {}_{l^n}A & \to & \varprojlim_n{}^1 (A,l) \\ & \wr| & & \uparrow \\ \to & \varprojlim_n {}_{l^n}(\mathrm{Tor}(A)) & \to & \varprojlim_n{}^1 (\mathrm{Tor}(A),l) \end{array}$$

since $\varprojlim_n{}^1 (\mathrm{Tor}(A),l)$ is l-torsion free and $l\text{-div}(\mathrm{Tor}(A))=\mathrm{Tor}(l\text{-div}(A))$ is l-torsion. Note that A is the direct sum of $l\text{-Div}(A)$ and another group A' with $l\text{-Div}(A')=0$.

For c) consider the exact sequence

$$0\to \varprojlim_n {}_{l^n}A\to \prod_n {}_{l^n}A \xrightarrow{\alpha} \prod_n {}_{l^n}A\to \varprojlim_n{}^1 {}_{l^n}A\to 0$$

with $\alpha=\mathrm{id}-(d_n)$, where ${}_{l^{n+1}}A\xrightarrow{d_n}{}_{l^n}A$ is the canonical map. Suppose given $a^{(m)}=(a_n^{(m)})\in\prod_n {}_{l^n}A$ with $a^{(m)}\equiv la^{(m+1)} \bmod \operatorname{Im}\alpha$, i.e.,

$$a_n^{(m)}=la_n^{(m+1)}+u_n^{(m)}-d_n u_{n+1}^{(m)}$$

for $(u_n^{(m)})\in\prod_n {}_{l^n}A$. Then

$$a_n^{(0)}=\sum_{m=0}^{\infty} l^m(u_n^{(m)}-d_n u_{n+1}^{(m)})=\sum_{m=0}^{\infty} l^m u_n^{(m)}-d_n\sum_{m=0}^{\infty} l^m u_{n+1}^{(m)},$$

i.e., $a^{(0)}=\alpha(v_n)$ with $v_n=\sum_{m=0}^{\infty} l^m u_n^{(m)}$ (note that these sums really are finite, since $l^n u_n^{(m)}=0$). q.e.d.

Recall that an abelian group is called l-complete, if the canonical map $A\to\varprojlim_n A/l^nA=\hat{A}$ is an isomorphism.

(4.4) **Proposition.** *For an abelian group A are equivalent:*

a) *A is l-complete.*

b) *$A=\varprojlim_n A_n$, where the A_n are $\mathbb{Z}_l$-modules with finite exponents.*

c) *All groups in the sequence* (4.2) *are zero.*

Proof. Clearly a) implies b), and for A like in b) we have $l\text{-div}(A)=0$, as $\varprojlim_n A_n\subsetneqq\prod_n A_n$; so we only have to show that $A\to\hat{A}$ is surjective. Take a sequence

$a^{(m)}=(a_n^{(m)})\in\varprojlim_n A_n\subseteq\prod_n A_n$ with $a^{(m+1)}\equiv a^{(m)}\bmod l^m\left(\varprojlim_n A_n\right)$. Then $a_n^{(m+1)} - a_n^{(m)}\in l^m A_n$ and thus $a_n^{(m)}=a_n^{(m(n))}$ for $m\geqq m(n):=\max(m(n-1)$, exponent of $A_n)$.

Let $a=(a_n^{(m(n))})\in\prod_n A_n$. Then $a\in\varprojlim_n A_n$, as

$$d_n a_{n+1}^{(m(n+1))}=a_n^{(m(n+1))}=a_n^{(m(n))}$$

for the transition maps $d_n: A_{n+1}\to A_n$. Furthermore we have

$$a_n^{(m(n))}-a_n^{(m)}\begin{cases}=0 & m\geqq m(n)\\ \in l^m A_n & m\leqq m(n),\end{cases}$$

i.e., $a\mapsto(a^{(n)})\in\varprojlim_n A/l^n A$.

It remains to show that a) implies c). Since $l\text{-div}(A)=0=\varprojlim_n{}^1 l^n A$, we get $\varprojlim_n{}^1\, {}_{l^n}A \xrightarrow{\sim} \varprojlim_n{}^1 (A,l)$. But as $\varprojlim_n{}^1 (A,l)$ is l-divisible, both groups must be zero by (4.3c). Finally it is clear that $\varprojlim_n (A,l)=0$ for $l\text{-Div}(A)=0$. q.e.d.

The inverse limit $A=\varprojlim_n A_n$ carries a natural topology, namely the one induced by the product topology on $\prod_n A_n$, where the A_n are endowed with the discrete topology. In the situation of (4.4b), this "limit topology" is always coarser than the l-adic topology of A. It might be different, as the example $A=\prod_n \mathbb{Z}/l = \varprojlim_n \prod_{m=1}^{n} \mathbb{Z}/l$ shows. However we have

(4.5) **Lemma.** *If (A_n) is ML-l-adic, the limit topology of $A=\varprojlim_n A_n$ is the l-adic one.*

Proof. Because the topology does not change under a ML-isomorphism, we only have to consider an l-adic system (A_n). But then

$$A\cap\left(\prod_{n>m} A_n\times\prod_{n=1}^{m}\{1\}\right)\subseteq l^m\cdot\prod_n A_n.$$

(4.6) *Definition* (cf. [1, VI, Sect. 3]). Call an abelian group A weakly l-complete or Ext-l-complete, if $\varprojlim_n (A,l)=0=\varprojlim_n{}^1 (A,l)$.

(4.7) *Remark.* This property can be expressed without inverse limits, because for an abelian group (resp. $\mathbb{Z}_l$-module) A there are canonical isomorphisms (compare 5.5)

$$\varprojlim_n (A,l)\xrightarrow{\sim}\operatorname{Hom}\left(\mathbb{Z}\left[\frac{1}{l}\right],A\right)\qquad(\xleftarrow{\sim}\operatorname{Hom}_{\mathbb{Z}_l}(\mathbb{Q}_l,A)),$$

$$\varprojlim_n{}^1 (A,l)\xrightarrow{\sim}\operatorname{Ext}^1\left(\mathbb{Z}\left[\frac{1}{l}\right],A\right)\qquad(\xleftarrow{\sim}\operatorname{Ext}^1_{\mathbb{Z}_l}(\mathbb{Q}_l,A)).$$

The vanishing of the first group is equivalent to $l\text{-Div}(A)=0$.

(4.8) **Proposition.** a) *If two groups in an exact sequence* $0 \to A \to B \to C \to 0$ *are weakly l-complete, then also the third.*

b) *The weakly l-complete groups form a full abelian subcategory of* $\underline{Ab}$.

c) *If* (A_n) *is an inverse system of abelian groups such that each* A_n *has a finite l-power exponent, then* $\varprojlim_n^1 A_n$ *is weakly l-complete.*

Proof. a) The exact sequence $0 \to (A, l) \to (B, l) \to (C, l) \to 0$ induces an exact sequence

$$0 \to \varprojlim_n (A, l) \to \varprojlim_n (B, l) \to \varprojlim_n (C, l) \to \varprojlim_n^1 (A, l)$$
$$\to \varprojlim_n^1 (B, l) \to \varprojlim_n^1 (C, l) \to 0.$$

b) We have to show that for a morphism $f: A \to B$ of weakly l-complete groups $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are again weakly l-complete. But from the short exact sequences

$$0 \to \operatorname{Ker} f \to A \to \operatorname{Im} f \to 0$$
$$0 \to \operatorname{Im} f \to B \to \operatorname{Coker} f \to 0$$

we obviously have $\varprojlim_n (\operatorname{Im} f, l) = 0 = \varprojlim_n^1 (\operatorname{Im} f, l)$. Thus $\operatorname{Im} f$ is weakly l-complete, and we can proceed by a).

c) follows by applying a) to the sequences

$$0 \to \varprojlim_n A_n \to \prod_n A_n \to X \to 0$$
$$0 \to X \to \prod_n A_n \to \varprojlim_n^1 A_n \to 0,$$

because $\varprojlim_n A_n$ and $\prod_n A_n$ are l-complete by (4.4).

We apply this to extend Tate's results for continuous group cohomology in [14, Sect. 2] to continuous étale cohomology.

(4.9) **Corollary.** *Let X be a scheme and* (F_n) *be an inverse system of sheaves on* X_{et} *such that each* F_n *is killed by some finite power of l. Then* $H^i(X, (F_n))$ *is weakly l-complete for each* $i \geq 0$*; in particular* $l\text{-Div}(H^i(X, (F_n))) = 0$.

Proof. This follows by applying (4.4) and (4.8a) and (4.8c) to the sequences (3.1).

(4.10) **Corollary.** *In the situation of* (4.9), $H^i(X, (F_n))$ *is a finitely generated* $\mathbb{Z}_l$*-module if and only if* $H^i(X, (F_n))/lH^i(X, (F_n))$ *is finite.*

Proof. This holds for any $\mathbb{Z}_l$-module M with $l\text{-Div}(M) = 0$: Let $Y \subset M$ be a finitely generated $\mathbb{Z}_l$-module with $Y + lM = M$. Then M/Y is l-divisible, hence zero by the exact sequence

$$0 = \varprojlim_n (M, l) \to \varprojlim_n (M/Y, l) \to \varprojlim_n^1 (Y, l) = 0.$$

Since the vanishing of $\varprojlim_n^1 (A, l)$ carries over to quotients, we get the following "Nakayama lemma" by the same arguments.

(4.11) **Lemma.** *Let* $\varphi: Y \to M$ *be a homomorphism of abelian groups, which induces a surjection* $Y/lY \twoheadrightarrow M/lM$. *If* $\varprojlim_n^1 (Y, l) = 0$ and $l\text{-Div}(M) = 0$, *then* φ *is surjective.*

(4.12) **Corollary.** *Let $\varphi: Y \to M$ be a homomorphism of weakly l-complete groups. If φ induces isomorphisms $Y/lY \xrightarrow{\sim} M/lM$ and ${}_lY \xrightarrow{\sim} {}_lM$, it is an isomorphism itself.*

Proof. By (4.11) there is an exact sequence $0 \to X \to Y \xrightarrow{\varphi} M \to 0$, and the induced exact sequence

$$0 \to {}_lX \to {}_lY \xrightarrow{\sim} {}_lM \to X/lX \to Y/lY \xrightarrow{\sim} M/lM \to 0$$

shows $X/lX = 0$, i.e., X is l-divisible. But X is weakly l-complete by (4.8a), hence $X = 0$.

In (4.10), the $H^i(X, (F_n))$ have a (canonical) $\mathbb{Z}_l$-structure, since we may derive in the category of $\mathbb{Z}_l$-modules. It is not clear a priori that a weakly l-complete group admits a $\mathbb{Z}_l$-structure, especially not a unique one; however we have

(4.13) **Lemma.** a) *The weakly l-complete groups have a unique structure of* $\mathbb{Z}_l$-modules such that any homomorphism between them is a *$\mathbb{Z}_l$-morphism.*

b) *If A is weakly l-complete, there is an exact sequence*

$$0 \to T_1 \to T_0 \to A \to 0$$

with torsion-free l-complete groups T_0 and T_1.

Proof. a) Give any weakly l-complete group A the $\mathbb{Z}_l$-structure induced by the $\mathbb{Z}_l$-structure of $\mathbb{Q}_l/\mathbb{Z}_l$ and the isomorphism

$$A = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \xrightarrow[\sim]{\delta} \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, A)$$

coming from the Ext-sequence for $0 \to \mathbb{Z} \to \mathbb{Z}\left[\frac{1}{l}\right] \to \mathbb{Q}_l/\mathbb{Z}_l \to 0$, then the functoriality is clear. The uniqueness follows from b), since any l-complete group has a unique $\mathbb{Z}_l$-structure (the l-adic topology is Hausdorff).

b) Choosing an exact sequence $0 \to X_1 \to X_0 \to A \to 0$ with torsion-free abelian groups X_0 and X_1, the sequence

$$0 \to \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, X_1) \to \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, X_0) \to \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, A) \cong A \to 0$$

is exact since $\mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, A) = 0$, and $T_i := \mathrm{Ext}^1(\mathbb{Q}_l/\mathbb{Z}_l, X_i) = \varprojlim_n X_i/l^n X_i$ [compare (5.3) below] is torsion-free l-adic.

(4.14) *Remark.* a) From (4.13b) we see that the category of weakly l-complete groups is the smallest abelian subcategory of $\underline{Ab}$ containing the l-complete groups.

b) If A is weakly l-complete, the connected component of the unit element w.r.t. the l-adic topology is $\mathrm{Ker}(A \twoheadrightarrow \hat{A}) = l\text{-div}(A)$, which is also isomorphic to $\varprojlim_n^1 {}_{l^n}A$ by (4.2). This is compatible with the fact that for any inverse system (A_n) of topological abelian groups the topology of $\varprojlim_n^1 A_n$ induced by the canonical surjection $\prod_n A_n \twoheadrightarrow \varprojlim_n^1 A_n$ is the indiscrete one.

c) There is an interesting structure theory for weakly (Ext-)l-complete groups, see [8] and [1, Sect. 2–4], also for further literature.

When is $\varprojlim_n^1 H^i(X, F_n) = 0$ for a scheme X and an inverse system of sheaves (F_n) on X_{et}? In view of the fact that "very often" the cohomology groups $H^i(X, F)$ are countable, e.g. if F is constructible and X is of finite type over a field k with separable absolute Galois group $G_k = \mathrm{Gal}(k_s/k)$, the answer is in many cases given by the following result of Gray [5, Proposition]:

(4.15) **Lemma.** *Let (A_n) be an inverse system of countable abelian groups. Then $\varprojlim_n^1 A_n = 0$ if and only if (A_n) satisfies the Mittag-Leffler condition.*

(4.16) **Corollary.** *Let (A_n) be an inverse system of countable abelian groups. Then $\varprojlim_n A_n = 0 = \varprojlim_n^1 A_n$ if and only if (A_n) is ML-zero.*

Proof. A Mittag-Leffler system (A_n) with $\varprojlim_n A_n = 0$ is ML-zero, compare (1.14).

(4.17) **Corollary.** *A countable group A is weakly l-complete if and only if it has finite exponent l^m for some $m \in \mathbb{N}$.*

5. Torsion-Free l-Adic Sheaves and $\mathbb{Q}_l$-Cohomology

Often l-adic sheaves arise by the following construction.

(5.1) **Definition and Proposition.** *For a scheme X define the left exact functor*

$$\underline{T_l}: S(X_{et}) \to S(X_{et})^{\mathbb{N}}$$

by

$$F \rightsquigarrow ({}_{l^n}F),$$

where the transition maps ${}_{l^{n+1}}F \to {}_{l^n}F$ are induced by the multiplication with l.

a) *$R^i \underline{T_l} F = 0$ for $i \geqq 2$ and $R^1 \underline{T_l} F = \underline{T_l}^1 F := (F/l^n F)$, where the transition maps $F/l^{n+1}F \to F/l^n F$ are the canonical surjections.*

b) *$\underline{T_l} F$ is l-adic if and only if $\varinjlim_n {}_{l^n}F = l\text{-}\mathrm{Tor}(F)$ is l-divisible.*

Proof. We use the following fact.

(5.2) **Lemma.** *Let I be an injective sheaf on X_{et}. Then for every $m \in \mathbb{N}$*

a) *I is m-divisible,*

b) *$H^0(X, I)$ is m-divisible,*

c) *$H^i(X, {}_m I) = 0$ for $i \geqq 1$.*

Proof. a) follows from the statement b) for all schemes (étale) over X, b) follows from the Ext-sequence

$$0 \to \mathrm{Hom}_X(\mathbb{Z}/m, I) \to \mathrm{Hom}_X(\mathbb{Z}, I) \xrightarrow{m} \mathrm{Hom}_X(\mathbb{Z}, I) \to \mathrm{Ext}^1_X(\mathbb{Z}/m, I)$$

associated to $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$, and c) follows from a), b and the cohomology sequence for $0 \to {}_m I \xrightarrow{m} I \to I \to 0$.

For the proof of (5.1a) note that $(\underline{T_l}, \underline{T_l}^1, 0, \dots)$ is an exact δ-functor on $S(X_{et})$, and $T_l^1(I) = 0$ for injective I by (5.2a). The statement (5.1b) is clear, as both assertions are equivalent to the equality ${}_{l^n}F = l_{l^{n+1}}F$ for all $n \in \mathbb{N}$.

(5.3) **Corollary.** *For a sheaf F on X_{et} there is a long exact sequence*

$$0 \to H^1(X, ({}_{l^n}F)) \to \operatorname{Ext}^1_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to H^0(X, (F/l^nF))$$
$$\to H^2(X, ({}_{l^n}F)) \to \operatorname{Ext}^2_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to H^1(X, (F/l^nF)) \to \dots,$$

functorial in F and X.

Proof. The functor $S(X_{et}) \to \underline{Ab}$, $F \rightsquigarrow \operatorname{Hom}_X(\mathbb{Q}_l/\mathbb{Z}_l, F) = \varprojlim_n \operatorname{Hom}_X(\mathbb{Z}/l^n\mathbb{Z}, F)$ $= \varprojlim_n H^0(X, {}_{l^n}F)$ is the composition of T_l with $\varprojlim_n H^0(X, -)$, and $\underline{T_l}$ carries injectives to $\varprojlim_n H^0(X, -)$-acyclics by (5.2c) and (3.1); note that $H^0(X, {}_{l^{n+1}}I)$ $\to H^0(X, {}_{l^n}I)$ is surjective by (5.2b). The statement thus follows from (5.1a) and the Grothendieck spectral sequence

$$H^p(X, R^q\underline{T_l}F) \Rightarrow \operatorname{Ext}^{p+q}_X(\mathbb{Q}_l/\mathbb{Z}_l, F).$$

The functoriality of the latter in F and X is clear.

(5.4) **Corollary.** *If F is l-divisible, there are canonical isomorphisms*

$$H^i(X, ({}_{l^n}F)) \xrightarrow{\sim} \operatorname{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l, F).$$

(5.5) **Definition and Proposition.** *For a sheaf F on X_{et} denote by $\underline{V_l}F$ or (F, l) the inverse system $\dots \xrightarrow{l} F \xrightarrow{l} F \xrightarrow{l} F \to \dots$. Then there are canonical isomorphisms*

$$H^i(X, (F, l)) \xrightarrow{\sim} \operatorname{Ext}^i_X\left(\mathbb{Z}\left[\frac{1}{l}\right], F\right),$$

functorial in F and X.

Proof. $\operatorname{Hom}_X\left(\mathbb{Z}\left[\frac{1}{l}\right], F\right) = \operatorname{Hom}_X\left(\varinjlim_n l^n\mathbb{Z}, F\right) = \varprojlim_n (H^0(X, F), l)$ is the composition of $\varprojlim_n H^0(X, -)$ with the exact functor $F \rightsquigarrow (F, l)$, which carries injectives to $\varprojlim_n H^0(X, -)$ acyclics by (5.2b) and (3.1).

(5.6) **Corollary.** *For any l-divisible sheaf F on X_{et} the isomorphisms above induce an isomorphism of long exact sequences*

$$\begin{array}{ccccccc}
\dots \to & \operatorname{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l, F) & \to & \operatorname{Ext}^i_X\left(\mathbb{Z}\left[\frac{1}{l}\right], F\right) & \to & \operatorname{Ext}^i_X(\mathbb{Z}, F) \\
& \uparrow\wr & & \uparrow\wr & & \uparrow\wr \\
\dots \to & H^i(X, \underline{T_l}F) & \to & H^i(X, (F, l)) & \to & H^i(X, F)
\end{array}$$

$$\begin{array}{cc}
\to \operatorname{Ext}^{i+1}_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to \dots \\
\uparrow\wr \\
\to H^{i+1}(X, \underline{T_l}F) \to \dots,
\end{array}$$

coming from the short exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z}\left[\frac{1}{l}\right] \to \mathbb{Q}_l/\mathbb{Z}_l \to 0$$

and

$$0 \to \underline{T_l F} \to (F, l) \to \underline{F} \to 0,$$

respectively.

Proof. Let $F \hookrightarrow I^{\cdot}$ be an injective resolution, then by (5.2a) there is an exact sequence

$$0 \to \underline{T_l I^{\cdot}} \to (I, l)^{\cdot} \to \underline{I^{\cdot}} \to 0,$$

and the claimed isomorphism is obtained by taking homology in the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_X(\mathbb{Q}_l/\mathbb{Z}_l, I^{\cdot}) & \to & \mathrm{Hom}_X\left(\mathbb{Z}\left[\frac{1}{l}\right], I^{\cdot}\right) & \to & \mathrm{Hom}_X(\mathbb{Z}, I^{\cdot}) & \to 0 \\ & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & \\ 0 \to & H^0(X, \underline{T_l I^{\cdot}}) & \to & H^0(X, (I, l)^{\cdot}) & \to & H^0(X, I^{\cdot}) & \to 0, \end{array}$$

as $\underline{T_l F} \hookrightarrow \underline{T_l I^{\cdot}}$ is an $\varprojlim_n H^0(X, -)$-acyclic resolution for l-divisible F. It follows from the definitions that the induced isomorphisms are those of (5.4) and (5.5).

(5.7) *Remark.* If F is an l-primary torsion sheaf [e.g. the sheaves $\mathbb{Q}_l/\mathbb{Z}_l(j)$ on X_{et}], one may replace $\mathbb{Z}\left[\frac{1}{l}\right]$ by $\mathbb{Z}_l\left[\frac{1}{l}\right] = \mathbb{Q}_l$ and $\mathbb{Z}$ by $\mathbb{Z}_l$ in the statements of (5.5) and (5.6), as $\mathbb{Z}_l/\mathbb{Z}$ and $\mathbb{Q}_l/\mathbb{Z}\left[\frac{1}{l}\right]$ are uniquely l-divisible.

(5.8) **Proposition.** *For l-torsion sheaves F on X_{et}, the homomorphisms* $\mathrm{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \to \mathrm{Ext}^i_X(\mathbb{Q}_l, X)$ *induce functorial isomorphisms*

$$\mathrm{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \xrightarrow{\sim} \mathrm{Ext}^i_X(\mathbb{Q}_l, F).$$

Proof. As $\mathbb{Q}_l$ is flat over $\mathbb{Z}_l$, this is implied by the case $i=0$, which follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathbb{Q}_l/\mathbb{Z}_l, F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \to & \mathrm{Hom}_X(\mathbb{Q}_l, F) \\ \downarrow\wr & & \downarrow\wr \\ T_l H^0(X, F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \xrightarrow{\sim} & \varprojlim_n (H^0(X, F), l), \end{array}$$

in which the lower maps is an isomorphism by the following lemma, whose proof is left to the reader.

(5.9) **Lemma.** *If A is an abelian l-torsion group, the map*

$$T_l A \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to \varprojlim_n (A, l)$$

$$(a_n) \otimes \frac{1}{l^m} \mapsto (a_{n+m}),$$

is an isomorphism. If A is l-divisible, it is a topological isomorphism.

For ML-isomorphic inverse systems (F_n) and (G_n) on X_{et} and an $m \in \mathbb{N}$, the systems $(F_n/l^m F_n)$ and $(G_n/l^m G_n)$ are also ML-isomorphic, so $\varprojlim_n F_n/l^m F_n$ and $\varprojlim_n G_n/l^m G_n$ are isomorphic. In particular, for a ML-l-adic system $F=(F_n)$ the system $(F_n/l^m F_n)$ is ML-constant, and we define the *sheaf* $F \otimes \mathbb{Z}/l^m$ on X_{et} by

$$F \otimes \mathbb{Z}/l^m = \varprojlim_n F_n/l^m F_n$$

$(\cong F'_n/l^m F'_n$ for $n \gg m$ where $(F'_n) \subseteqq (F_n)$ is the subsystem of universal images).

(5.10) **Definition and Proposition.** *For any ML-l-adic system $F=(F_n) \in Ob(S(X_{et})^{\mathbb{N}})$ define the sheaf $F \otimes \mathbb{Q}_l/\mathbb{Z}_l \in Ob(S(X_{et}))$ by*

$$F \otimes \mathbb{Q}_l/\mathbb{Z}_l = \varinjlim_m F \otimes \mathbb{Z}/l^m,$$

where $F \otimes \mathbb{Z}/l^m \xrightarrow{\varphi_m} F \otimes \mathbb{Z}/l^{m+1}$ is induced by $F \xrightarrow{l} F$.

a) *The functor $F \rightsquigarrow F \otimes \mathbb{Q}_l/\mathbb{Z}_l$ only depends on the ML*-isomorphism classes of objects and morphisms.

b) *$F \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is l-divisible.*

Proof. a) is clear from the remarks above, and b) follows from the exact commutative diagrams

$$\begin{array}{ccccccc} F \otimes \mathbb{Z}/l^{m+1} & \xrightarrow{l} & F \otimes \mathbb{Z}/l^{m+1} & \longrightarrow & F \otimes \mathbb{Z}/l & \longrightarrow & 0 \\ {\scriptstyle \varphi_m}\uparrow & & {\scriptstyle \varphi_m}\uparrow & & {\scriptstyle 0}\uparrow & & \\ F \otimes \mathbb{Z}/l^m & \xrightarrow{l} & F \otimes \mathbb{Z}/l^m & \longrightarrow & F \otimes \mathbb{Z}/l & \longrightarrow & 0, \end{array}$$

as the inductive limit is an exact functor for étale sheaves.

(5.11) **Lemma.** *Let $F=(F_n)$ be an l-adic sheaf on X_{et}.*

a) *F is ML-torsion-free if and only if ${}_lF_{n+1} = l^n F_{n+1}$ for all $n \in \mathbb{N}$.*

b) *If F is ML-torsion-free, there is a canonical isomorphism*

$$F \xrightarrow{\sim} \underline{T_l}(F \otimes \mathbb{Q}_l/\mathbb{Z}_l).$$

Proof. a) By definition, F is ML-torsion-free if and only if $({}_l\mathrm{F}_n)$ is ML-zero. If this is the case, then for any $n \in \mathbb{N}$ there is an $m=m(n)>0$ such that ${}_lF_{n+m} \subseteqq \mathrm{Ker}(F_{n+m} \to F_n) = l^n F_{n+m}$. By the commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & {}_lF_{n+m} + l^n F_{n+m}/l^n F_{n+m} & \longrightarrow & F_{n+m}/l^n F_{n+m} & \xrightarrow{l} & F_{n+m}/l_{n+1} F_{n+m} \\ & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 \longrightarrow & {}_lF_{n+1}/l^n F_{n+1} & \longrightarrow & F_{n+1}/l^n F_{n+1} & \xrightarrow{l} & F_{n+1} \end{array}$$

this is the case if and only if ${}_lF_{n+1} = l^n F_{n+1}$, i.e., if *all* maps in $({}_lF_n)$ are zero.

b) By induction on r we get ${}_{l^r}F_n = l^{n-r} F_n$ for all $1 \leqq r \leqq n$, and from the commutative diagrams

$$\begin{array}{ccccc} F \otimes \mathbb{Z}/l^{n+1} & \xrightarrow{\sim} & F_{n+1} & = & F_{n+1} \\ {\scriptstyle \varphi_n}\uparrow & & \uparrow{\scriptstyle l} & & \\ F \otimes \mathbb{Z}/l^n & \xrightarrow{\sim} & F_{n+1}/l^n F_{n+1} & \xrightarrow{\sim} & F_n \end{array}$$

we see that the φ_n are injective and identify $F\otimes\mathbb{Z}/l^n\mathbb{Z}$ with ${}_{l^n}(F\otimes\mathbb{Z}/l^{n+1})$. As this holds for all $n\in\mathbb{N}$, we get canonical isomorphisms

$$ {}_{l^n}(F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\overset{\sim}{\leftarrow} F\otimes\mathbb{Z}/l^n \overset{\sim}{\rightarrow} F_n, $$

and via these the transition maps $F_{n+1}\to F_n$ coincide with the maps ${}_{l^{n+1}}(F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\to{}_{l^n}(F\otimes\mathbb{Q}_l/\mathbb{Z}_l)$ induced by l-multiplications.

(5.12) *Remark.* a) If $F=(F_n)$ is ML-l-adic, the projections $F\otimes\mathbb{Z}/l^n\to F_n$ induce a canonical morphism of the l-adicsystem $(F\otimes\mathbb{Z}/l^n)$ – with obvious transition maps – into F. This morphism is a ML-isomorphism and an AR-isomorphism if F is AR-l-adic. So by (5.10a) and (5.11b) we get a canonical ML-isomorphism

$$ T_l(F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\to F, $$

if F is ML-torsion-free, which is an AR-isomorphism if F is AR-l-adic.

b) In particular, AR-l-adic and ML-torsion-free implies AR-torsion-free, so that we may simply talk of torsion-freeness.

(5.13) *Definition.* For a ML-l-adic sheaf $F=(F_n)$ on X_{et} define

$$ H^i_{\mathrm{cont}}(X,F\otimes\mathbb{Q}_l):=H^i(X,(F_n))\otimes_{\mathbb{Z}_l}\mathbb{Q}_l\,. $$

(5.14) **Theorem.** a) *If F is a torsion-free ML-l-adic sheaf on X_{et}, there is a canonical exact sequence*

$$ \ldots\to H^i_{\mathrm{cont}}(X,F)\to H^i_{\mathrm{cont}}(X,F\otimes\mathbb{Q}_l)\to H^i(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\to H^{i+1}_{\mathrm{cont}}(X,F)\to\ldots, $$

functorial in F and X.

b) *In this sequence one has*

$$ \mathrm{Tor}(H^i_{\mathrm{cont}}(X,F))=\mathrm{Im}(H^{i-1}(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\to H^i_{\mathrm{cont}}(X,F)), $$
$$ l\text{-}\mathrm{Div}(H^i(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l))=\mathrm{Im}(H^i_{\mathrm{cont}}(X,F\otimes\mathbb{Q}_l)\quad\to H^i(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l)). $$

Proof. a) Using the ML-isomorphism $F\leftarrow T_l(F\otimes\mathbb{Q}_l/\mathbb{Z}_l)$, this is the sequence from (5.6) for $F\otimes\mathbb{Q}_l/\mathbb{Z}_l$:

$$ \ldots\to H^i_{\mathrm{cont}}(X,\underline{T_l}(F\otimes\mathbb{Q}_l/\mathbb{Z}_l))\to H^i(X,(F\otimes\mathbb{Q}_l/\mathbb{Z}_l,l))\to H^i(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\to\ldots $$

combined with the isomorphism

$$ H^i_{\mathrm{cont}}(X,\underline{T_l}(F\otimes\mathbb{Q}_l/\mathbb{Z}_l))\otimes_{\mathbb{Z}_l}\mathbb{Q}_l\overset{\sim}{\rightarrow}H^i(X,(F\otimes\mathbb{Q}_l/\mathbb{Z}_l,l)) $$

implied by (5.8) and the commutative diagram

$$ \begin{array}{ccc} H^i_{\mathrm{cont}}(X,\underline{T_l}(F\otimes\mathbb{Q}_l/\mathbb{Z}_l))\otimes_{\mathbb{Z}_l}\mathbb{Q}_l & \longrightarrow & H^i(X,(F\otimes\mathbb{Q}_l/\mathbb{Z}_l,l)) \\ {\scriptstyle 5.4}\downarrow\wr & & \downarrow\wr{\scriptstyle 5.5} \\ \mathrm{Ext}^i_X(\mathbb{Q}_l/\mathbb{Z}_l,F\otimes\mathbb{Q}_l/\mathbb{Z}_l)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l & \xrightarrow[\sim]{5.8} & \mathrm{Ext}^i_X(\mathbb{Q}_l,F\otimes\mathbb{Q}_l/\mathbb{Z}_l) \end{array} $$

b) This is clear, since the $H^i_{\mathrm{cont}}(X,F\otimes\mathbb{Q}_l)$ are uniquely divisible, the $H^i(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l)$ are torsion groups, and l-$\mathrm{Div}(H^i_{\mathrm{cont}}(X,F))=0$ for all $i\geqq 0$ by (4.9).

(5.15) **Theorem.** *Let $X=\mathrm{Spec}\,k$ for a field k with separable closure k_s and $F=(F_n)$ be a ML-l-adic sheaf on X_{et}. Let $T=\varprojlim_n(F_n)_{\bar{x}}$ be its "stalk" at the geometric point*

$\bar{x}=\mathrm{Spec}\,k_s$ *with the l-adic topology and give* $T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$ *the induced topology* w.r.t. $T\to T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$.

a) *The stalk of* $F\otimes\mathbb{Q}_l/\mathbb{Z}_l$ *at* $\bar{x}$ *is isomorphic to* $T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l$ *as module under* $G_k=\mathrm{Gal}(k_s/k)$.

b) G_k *acts continuously on* T *and* $T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$, *and there are functorial isomorphisms for* $i\geqq 0$

$$H^i_{\mathrm{cont}}(X,F)\xrightarrow{\sim}H^i_{\mathrm{cont}}(G_k,T).$$

c) *If* $\mathrm{Tor}(T)$ *has a finite exponent, these induce isomorphisms for* $i\geqq 0$

$$H^i_{\mathrm{cont}}(X,F\otimes\mathbb{Q}_l)\xrightarrow{\sim}H^i_{\mathrm{cont}}(G_k,T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l).$$

d) *If* F *is torsion free, these induce isomorphisms of the long exact sequences*

$$\begin{array}{ccccccc}
\ldots\to & H^i_{\mathrm{cont}}(X,F) & \to & H^i_{\mathrm{cont}}(X,F\otimes\mathbb{Q}_l) & \to & H^i(X,F\otimes\mathbb{Q}_l/\mathbb{Z}_l) \\
 & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\ldots\to & H^i_{\mathrm{cont}}(G_k,T) & \to & H^i_{\mathrm{cont}}(G_k,T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l) & \to & H^i(G_k,T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l)
\end{array}$$

$$\begin{array}{ccc}
\to & H^{i+1}_{\mathrm{cont}}(X,F) & \to\ldots \\
 & \downarrow\wr & \\
\to & H^{i+1}_{\mathrm{cont}}(G_k,T) & \to\ldots
\end{array}$$

given by (5.14) *and the short exact sequence*

$$0\to T\to T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l\to T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l\to 0,$$

respectively.

Proof. a) By passing to an l-adic sheaf we easily see that $(F\otimes\mathbb{Z}/l^m)_{\bar{x}}$ is canonically isomorphic to T/l^mT as G_k-module and that these isomorphism induce an isomorphism

$$(F\otimes\mathbb{Q}_l/\mathbb{Z}_l)_{\bar{x}}=\varinjlim_m(F\otimes\mathbb{Z}/l^m)_{\bar{x}}\xrightarrow{\sim}\varinjlim_m T\otimes_{\mathbb{Z}_l}\mathbb{Z}/l^m=T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l/\mathbb{Z}_l.$$

b) By (4.5) the limit topology on T is the l-adic one, which implies the continuity of the action and the claimed isomorphisms by (3.2).

c) We have to show that the canonical map $H^i_{\mathrm{cont}}(G_k,T)\to H^i_{\mathrm{cont}}(G_k,T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l)$ induces an isomorphism

$$H^i_{\mathrm{cont}}(G_k,T)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l\xrightarrow{\sim}H^i_{\mathrm{cont}}(G_k,T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l).$$

As $\mathbb{Q}_l$ is flat over $\mathbb{Z}_l$, it suffices to show that we have isomorphisms

$$C^i(G_k,T)\otimes\mathbb{Q}_l\xrightarrow{\sim}C^i(G_k,T\otimes\mathbb{Q}_l)$$

for the groups of continuous cochains. But if $\varphi:G_k^i\to T$ has its image in $\mathrm{Tor}(T)=\mathrm{Ker}(T\to T\otimes\mathbb{Q}_l)$, it is by assumption annihilated by l^r for some $r\in\mathbb{N}$, which shows the injectivity of the above maps. The surjectivity is implied by the fact that every cochain $\psi:G_k^i\to T\otimes\mathbb{Q}_l$ has a compact image, which therefore is contained in $l^{-s}\mathrm{Im}(T\to T\otimes\mathbb{Q}_l)$ for some $s\in\mathbb{N}$.

d) In view of (3.2) and the construction of the upper sequence we have to show that the exact sequence

$$0\to(T/l^mT)\to(T\otimes\mathbb{Q}_l/\mathbb{Z}_l,l)\to(T\otimes\mathbb{Q}_l/\mathbb{Z}_l,\mathrm{id})\to 0$$

induces a commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & \varprojlim_n T/l^n T & \to & \varprojlim_n (T\otimes \mathbb{Q}_l/\mathbb{Z}_l, l) & \to & T\otimes\mathbb{Q}_l/\mathbb{Z}_l & \to 0 \\ & \wr\uparrow & & \wr\uparrow & & \| & \\ 0 \to & T & \to & T\otimes_{\mathbb{Z}_l}\mathbb{Q}_l & \to & T\otimes\mathbb{Q}_l/\mathbb{Z}_l & \to 0 \end{array}$$

in which the vertical arrows are topological isomorphisms. This follows from (4.5) and (5.9). q.e.d.

We leave it to reader to formulate and prove the $\mathbb{Q}_l$-analogues of the results in Sect. 2 and Sect. 3, by "tensoring everything with $\mathbb{Q}_l$" over $\mathbb{Z}_l$ or with $\mathbb{Q}$ over $\mathbb{Z}$. I am indebted to the referee for pointing out a possible source of confusion: Like for any abelian category the above means to work modulo (the Serre subcategory of) objects of *finite exponent*, not modulo torsion objects. Note that

$$H^n_{\mathrm{cont}}(X, F\otimes\mathbb{Q}_l) := H^n_{\mathrm{cont}}(X, F)\otimes\mathbb{Q}_l$$

for $F=(F_n)$ in $S(X_{et})^{\mathbb{Z}_l}$ [see (6.9)] in fact only depends on $F\otimes\mathbb{Q}_l$ read as "F up to systems of finite exponent", while for $F=\left(\bigoplus_{i=1}^{n}\mathbb{Z}/l^i\right)=\varinjlim_m {}_{l^m}F$ in $\underline{Ab}^{\mathbb{Z}_l}$ one has $\mathbb{Q}_l\otimes\varprojlim_n F_n \neq 0$. The spectral sequences (3.10) etc. carry over since only finitely many terms are involved in the limit groups.

(5.16) **Proposition.** *Let $F=(F_n)$ be a torsion-free ML-l-adic sheaf on X_{et} and $Z \subseteq X$ be a closed subscheme. Then the exact sequence* (3.16)

$$0\to \varprojlim_n{}^1 H^{i-1}_Z(X, F_n) \to H^i_{Z,\mathrm{cont}}(X, F) \to \varprojlim_n H^i_Z(X, F_n) \to 0$$

identifies $\varprojlim_n^1 H^{i-1}_Z(X, F_n)$ *with* $l\text{-div}(H^i_{Z,\mathrm{cont}}(X, F))$.

Proof. We may assume F to be l-adic; then the ML-exact sequences

$$0\to F \xrightarrow{l^n} F \to F_n \to 0$$

induce sequences

$$0\to H^i_{Z,\mathrm{cont}}(X, F)/l^n \to H^i_Z(X, F_n) \to {}_{l^n}H^i_{Z,\mathrm{cont}}(X, F) \to 0,$$

and, by passing to the inverse limit, an isomorphism

$$\varprojlim_n H^i_{Z,\mathrm{cont}}(X, F)/l^n \xrightarrow{\sim} \varprojlim_n H^i_Z(X, F_n)$$

since $T_l(H^i_{Z,\mathrm{cont}}(X, F))=0$. This implies that in the sequence (3.16) the cokernel can be identified with the l-adic completion of $H^i_{Z,\mathrm{cont}}(X, F)$, hence the statement for the kernel.

6. Cupproducts and Chern Classes

It will be useful and clearer to introduce cupproducts in the general setting of derived functors. This is certainly well-known and included only for lack of a suitable reference.

(6.1) *Definition.* Let $f:\mathscr{A}\to\mathscr{B}$ be a left exact functor between abelian categories and assume that $\mathscr{A}$ possesses enough injectives and a tensor product.

a) Call an exact sequence $0\to A'\to A\to A''\to 0$ in $\mathscr{A}$ admissible, if $0\to A'\otimes B \to A\otimes B\to A''\otimes B\to 0$ is exact for each object B of $\mathscr{A}$.

b) Say that $\mathscr{A}$ has the property $C1(f)$, if any object A of $\mathscr{A}$ can be embedded into an f-acyclic object A^0 such that $0\to A\to A^0\to A/A^0\to 0$ is admissible.

c) Say that $\mathscr{A}$ has the property $C2(f)$, if there is an exact functor $C^{\cdot}$ associating to each object A of $\mathscr{A}$ an f-acyclic resolution

$$C^{\cdot}(A): C^0(A)\xrightarrow{d^0} C^1(A)\to\ldots\to C^i(A)\xrightarrow{d^i} C^{i+1}(A)\to\ldots$$

of A, which is admissible, i.e., such that $0\to \operatorname{Ker} d^i\to C^i(A)\to \operatorname{Im} d^i\to 0$ is admissible for every $i\geqq 0$.

(6.2) **Proposition.** *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories with tensor products. Assume that $\mathscr{A}$ has enough injectives, and let f_1, f_2, and f_3 be left exact functors $\mathscr{A}\to\mathscr{B}$.*

a) *If $\mathscr{A}$ has the property $C1(f_i)$ for $i=1,2$, then every morphism of bi-functors*

$$f_1(A)\otimes f_2(B)\to f_3(A\otimes B)$$

can uniquely be extended to a family of morphisms of bi-functors

$$R^pf_1A\otimes R^qf_2B\xrightarrow{\cup} R^{p+q}f_3(A\otimes B)$$

(called cupproduct) such that

i) *for $p=0=q$ it is the given morphism;*

ii) *if $0\to A'\to A\to A''\to 0$ is admissible and B is an object of $\mathscr{A}$, then the diagram*

$$\begin{array}{ccc} R^pf_1A''\otimes R^qf_2B & \xrightarrow{\cup} & R^{p+q}f_3(A''\otimes B)\\ \downarrow{\scriptstyle \delta\otimes\mathrm{id}} & & \downarrow{\scriptstyle\delta}\\ R^{p+1}f_1A'\otimes R^qf_2B & \xrightarrow{\cup} & R^{p+q+1}f_3(A'\otimes B)\end{array}$$

is commutative;

iii) *if $0\to B'\to B\to B''\to 0$ is admissible, then for each object A of $\mathscr{A}$ the diagram*

$$\begin{array}{ccc} R^pf_1A\otimes R^qf_2B'' & \xrightarrow{\cup} & R^{p+q}f_3(A\otimes B'')\\ \downarrow{\scriptstyle (-1)^p\mathrm{id}\otimes\delta} & & \downarrow{\scriptstyle\delta}\\ R^pf_1A\otimes R^{q+1}f_2B' & \xrightarrow{\cup} & R^{p+q+1}f_3(A\otimes B')\end{array}$$

is commutative.

b) *If $A^{\cdot}$ and $B^{\cdot}$ are f_1-acyclic and f_2-acyclic admissible resolutions of A and B, respectively, there is a morphism*

$$A^{\cdot}\otimes B^{\cdot}\to J^{\cdot}$$

into an f_3-acyclic resolution $J^{\cdot}$ of $A\otimes B$ inducing the identity on $A\otimes B$. For any such the morphisms induced by

$$f_1A^{\cdot}\otimes f_2B^{\cdot}\to f_3(A^{\cdot}\otimes B^{\cdot})\to f_3J^{\cdot}$$

in the homology give the above cupproduct (here as usual, $A^{\cdot}\otimes B^{\cdot}$ is the complex with $(A^{\cdot}\otimes B^{\cdot})^n=\bigoplus_{p+q=n} A^p\otimes B^q$ and differential $d=d_A\otimes\mathrm{id}+(-1)^p\,\mathrm{id}\otimes d_B$ on $A^p\otimes B^q$).

c) *Let g_1, g_2, and g_3 be left exact functors from $\mathscr{B}$ to another abelian category $\mathscr{C}$ with tensor products. Assume that $\mathscr{B}$ has enough injectives and that f_i carries injectives to g_i-acyclics for $i=1,2,3$. Let $\mathscr{A}$ have the property $C1(f_i)$ for $i=1,2$ and $\mathscr{B}$ have the property $C2(g_i)$ for $i=1,2$. Then for couplings*

$$f_1(A_1)\otimes f_2(A_2)\to f_3(A_1\otimes A_2),$$
$$g_1(B_1)\otimes g_2(B_2)\to g_3(B_1\otimes B_2),$$

inducing cupproducts $\cup_f$ and $\cup_g$ respectively, there is a functorial cupproduct

$${}_1E_r^{p,q}\otimes {}_2E_r^{s,t}\to {}_3E_r^{p+s,q+t}, \quad r\geqq 2,$$

of the Grothendieck spectral sequences

$${}_iE_2^{\mu,\nu}=R^\mu g_i R^\nu f_i A_i \Rightarrow {}_iE^{\mu+\nu}=R^{\mu+\nu} g_i f_i A_i, \quad i=1,2,3,$$

(where $A_3=A_1\otimes A_2$), which coincides with

$$R^p g_1 R^q f_1 A_1\otimes R^s g_2 R^t f_2 A_2 \xrightarrow{(-1)^{sq}\cup_g} R^{p+s} g_3 (R^q f_1 A_1\otimes R^t f_2 A_2)$$
$$\xrightarrow{R^{p+s}g(\cup_f)} R^{p+s} g_3 R^{q+t} f_3 (A_1\otimes A_2)$$

for $r=2$ and is compatible on the limit terms ${}_iE_\infty^{\mu,\nu}$ with the cupproduct

$$R^m(g_1 f_1)A_1\otimes R^n(g_2 f_2)A_2 \xrightarrow{\cup_{gf}} R^{m+n}(g_3 f_3)(A_1\otimes A_2)$$

induced by

$$g_1 f_1 A_1\otimes g_2 f_2 A_2\to g_3(f_1 A_1\otimes f_2 A_2)\to g_3 f_3 (A_1\otimes A_2).$$

Proof. a) This is proved by induction and dimension shifting in a standard way, compare Bredon [2, II 6.2].

b) Since $A^\cdot$ and $B^\cdot$ are admissible, $A^\cdot\otimes B^\cdot$ is a resolution of $A\otimes B$, as easily follows from the first or second spectral sequence for the double complex $(A^p\otimes B^q$, $d_A\otimes \mathrm{id}, (-1)^p \mathrm{id}\otimes d_B)$, whose total complex is $A^\cdot\otimes B^\cdot$. Therefore we get the wanted morphism into an injective resolution $I^\cdot$ of $A\otimes B$. The second claim again follows by induction and dimension shifting; note that $A^\cdot\otimes B^\cdot\to J^\cdot$ induces $A^\cdot[1]\otimes B^\cdot \to J^\cdot[1]$ by taking the same morphisms and $A^\cdot\otimes B^\cdot[1]\to J^\cdot[1]$ by multiplying $A^p\otimes B^{q+1}\to J^{p+q+1}$ with $(-1)^p$, and that canonically $H^p(A^\cdot[1])=H^{p+1}(A^\cdot)$ etc. (here $(A^\cdot[1])^p=A^{p+1}$ and $d_{A[1]}=-d_A$ as usual).

c) Let $A_i^\cdot$ be admissible f_i-acyclic resolutions of A_i for $i=1,2$ and $C_i^\cdot$ be the functorial admissible g_i-acyclic resolutions for $i=1,2$. Let $J^\cdot$ be an injective resolution of $A_1\otimes A_2$ and let $f_3 J^\cdot \hookrightarrow I^{\cdot\cdot}$ be an injective resolution of the complex $f_3 J^\cdot$.

By b) we have a morphism $A_1^\cdot\otimes A_2^\cdot\to J^\cdot$ such that

$$f_1 A_1^\cdot\otimes f_2 A_2^\cdot\to f_3 J^\cdot$$

induces $\cup_f$. Similarly, $C_1^\cdot(f_1 A_1^\cdot)\otimes C_2^\cdot(f_2 A_2^\cdot)$ is a resolution of the complex $f_1 A_1^\cdot\otimes f_2 A_2^\cdot$, therefore we get a morphism into an injective resolution of $f_1 A_1^\cdot\otimes f_2 A_2^\cdot$ and by composition a morphism into $I^{\cdot\cdot}$. This means we have pairings

$$C_1^p(f_1 A_1^q)\otimes C_2^s(f_2 A_2^t)\xrightarrow{h} I^{p+s,q+t}$$

with

$$_Id \circ h = h \circ (c_1 \otimes \mathrm{id}) + (-1)^p h \circ (\mathrm{id} \otimes c_2)$$

and

$$_{II}d \circ h = h \circ (d_{1*} \otimes \mathrm{id}) + (-1)^q h \circ (\mathrm{id} \otimes d_{2*}).$$

Here $_Id$ and $_{II}d$ are the first and second differentials of $I^{\cdot\cdot}$, respectively, c_i is the differential of $C_i^{\cdot}$, and d_{i*} the differential induced by the differential d_i of $A_i^{\cdot}$ $(i=1,2)$. Taking homology w.r.t. the second differentials we get pairings

$$C_1^p(R^q f_1 A_1) \otimes C_2^s(R^t f_2 A_2) \xrightarrow{h} {}_{II}H^{q+t}(I^{p+s,\cdot}),$$

as the $C_i^{\cdot}$ are exact functors. By assumption on $I^{\cdot\cdot}$ the complex $_{II}H^{q+t}(I^{\cdot})$ is an injective resolution of $H^{q+t}(f_3 J^{\cdot}) = \mathrm{R}^{q+t} f_3(A_1 \otimes A_2)$, so by b) we see that $(-1)^{qs}h$ induces the claimed pairing on the E_2-terms.

On the other hand $(-1)^{qs}h$ induces pairings of the associated double complexes (with second differentials $(-1)^{p+s}{}_{II}d$, $(-1)^q d_{1*}$ and $(-1)^s \mathrm{d}_{2*}$ respectively), whose first spectral sequences (after applying g_1, g_2, and g_3 respectively) are just the Leray spectral sequences. So we get the pairing on these in an obvious way, compare Bredon [2, Appendix 3]. To see the compatibility with the cupproduct on the limit term, one may argue with the second spectral sequences like in [2, IV 6.5].

If $\mathscr{A}$ is an abelian category with tensor product, then $\mathscr{A}^{\mathbb{N}}$ has the tensor product given by

$$(A_n, d_n) \otimes (B_n, e_n) = (A_n \otimes B_n, d_n \otimes e_n).$$

(6.3) **Proposition.** a) *If $\mathscr{A}$ has the property $C1(f)$ (resp. $C2(f)$) for $f: \mathscr{A} \to \mathscr{B}$, then $\mathscr{A}^{\mathbb{N}}$ has the property $C1(f^{\mathbb{N}})$ (resp. $C2(f^{\mathbb{N}})$), and $\mathscr{A}^{\mathbb{N}}$ has the property $C1\left(\varprojlim_n f\right)$ $\left(\text{resp. } C2\left(\varprojlim_n f\right)\right)$, if $\mathscr{B}$ satisfies $(AB4^*)$.*

b) *If G is a profinite group, then $M(G)$ has the property $C2(H^0(G, -))$.*

c) *If $\mathscr{A} = S(X_{et})$ for a scheme X and every induced sheaf is f-acyclic (e.g., $f = H_Z^0(X, -)$ for a closed subscheme $Z \subsetneq X$), then $\mathscr{A}$ has the property $C2(f)$. If moreover $\mathscr{B} = S(Y_{et})$ for a scheme Y and f carries induced sheaves to $H^0(Y, -)$-acyclics (e.g., $f = \pi_*$ for a morphism $\pi: X \to Y$ or $f = i^!$ for a closed immersion $i: Y \to X$), then $\mathscr{A}^{\mathbb{N}}$ has the property $C2\left(\varprojlim_n f\right)$.*

Proof. b) follows with the resolutions by induced modules, cf. Serre [12, I 2.5], and the first claim of c) with the Godement resolution, cf. Milne [9, V 1.15].

In a), the statements for $f^{\mathbb{N}}$ are clear by (1.2). For $\varprojlim_n f$ let (A_n, d_n) be an object of $\mathscr{A}^{\mathbb{N}}$ and let $0 \to A_n \to A_n^0 \to A_n^0/A_n \to 0$ be admissible sequences for $n \in \mathbb{N}$, with f-acyclic objects A_n^0. Then

$$0 \to (A_n, d_n) \xrightarrow{\iota} P(A_n^0) \to P(A^0)/(A_n, d_n) \to 0,$$

with ι induced by the adjunction of P and V [see the proof of (1.1)], is an admissible sequence, and functorial, if the A_n^0 can be defined functorially. So it suffices to show that $P(A_n^0)$ is acyclic for $\varprojlim_n f$. If $\mathscr{B}$ satisfies $(AB4^*)$, this follows from (1.3), since

$f^{\mathbf{N}}P(A_n^0)=P(fA_n^0)$ is acyclic for $\varprojlim_n$. Now assume that $\mathscr{A}=S(X_{et})$, $\mathscr{B}=S(Y_{et})$, and that the A_n^0 are induced sheaves on X_{et}. If such sheaves are carried to $H^0(Y,-)$-acyclic sheaves, the same is true for injective sheaves, so there is a spectral sequence

$$\varprojlim_n{}^p R^q f^{\mathbf{N}} P(A_n^0) \Rightarrow R^{p+q}\left(\varprojlim_n f\right) P(A_n^0)$$

by the lemma below. Therefore it suffices to show that $P(fA_n^0)$ is acyclic for $\varprojlim_n$, which again follows from the lemma.

The examples given in c) are well-known, note that $i^!$ carries induced sheaves to induced sheaves.

(6.4) **Lemma.** *If X is a scheme and (F_n) is a family of $H^0(X,-)$-acyclic sheaves on X_{et}, then $P(F_n)$ is acyclic for $\varprojlim_n$ and $\varprojlim_n H^0(X,-)$.*

Proof. By (3.12) we only have to show the second statement, which follows with (3.1) and (1.15).

If there is a natural cupproduct for functors f_1, f_2, and f_3 on $\mathscr{A}$, we always take the "componentwise" cupproduct for $f_1^{\mathbf{N}}$, $f_2^{\mathbf{N}}$, and $f_3^{\mathbf{N}}$ on $\mathscr{A}^{\mathbf{N}}$. If $\varprojlim_n f_i$ exists for $i=1,2,3$, we always take the cupproduct for these functors that is induced by the functorial morphisms

$$\varprojlim_n f_1 A_n \otimes \varprojlim_n f_2 B_n \to \varprojlim_n f_3(A_n \otimes B_n)$$

which are the limit of the bilinear maps $f_1 A_n \times f_2 B_n \to f_3(A_n \otimes B_n)$. In particular, there are canonical cupproducts (compare [6, V 1.17(c)]

(6.5) $$H^p_{Z_1}(X,(F_n)) \otimes H^q_{Z_2}(X,(G_n)) \to H^{p+q}_{Z_1 \cap Z_2}(X,(F_n \otimes G_n))$$

for étale pro-sheaves (F_n) and (G_n) on a scheme X, Z_1 and Z_2 being closed subschemes, and a canonical cupproduct

(6.6) $$H^p(G,(M_n)) \otimes H^q(G,(N_n)) \to H^{p+q}(G,(M_n \otimes N_n))$$

for inverse systems of discrete G-modules (M_n) and (N_n) for a profinite group G.

(6.7) **Proposition.** *Via the isomorphism in* (2.2), *the cupproduct* (6.6) *is compatible with the cupproduct for continuous group cohomology defined by Tate in* [14, Sect. 2].

Proof. The cupproduct for continuous group cohomology, w.r.t. the continuous pairing $\varprojlim_n M_n \times \varprojlim_n N_n \to \varprojlim_n (M_n \otimes N_n)$, is induced by a morphism of complexes

$$C^{\cdot}\left(G, \varprojlim_n M_n\right) \otimes C^{\cdot}\left(G, \varprojlim_n N_n\right) \to C^{\cdot}\left(G, \varprojlim_n (M_n \otimes N_n)\right)$$

[notations as in the proof of (2.2)], which is obtained by passing to the limit over n from morphisms

$$C^{\cdot}(G, M_n) \otimes C^{\cdot}(G, N_n) \to C^{\cdot}(G, M_n \otimes N_n).$$

These are in turn induced by morphisms

$$D^{\cdot}(G, M_n)\otimes D^{\cdot}(G, N_n)\to D^{\cdot}(G, M_n\otimes N_n)$$

via taking fixed modules under G, compare [4, 4]. Since $D^{\cdot}(G, (M_n))$ is an admissible $\varprojlim_n H^0(G, -)$-acyclic resolution for (M_n), for any (M_n) satisfying the Mittag-Leffler condition, the claim follows from (6.2b).

(6.8) **Proposition.** *If* $\mathscr{B} = \underline{Ab}$ *in the situation of* (6.2a), *then there are commutative diagrams for systems* (A_n) *and* (B_n) *in* $\mathscr{A}^{\mathbb{N}}$

$$\begin{array}{ccccc}
R^p\left(\varprojlim_n f_1\right)(A_n) & \times & R^q\left(\varprojlim_n f_2\right)(B_n) & \to & R^{p+q}\left(\varprojlim_n f_3\right)(A_n\otimes B_n) \\
\uparrow & & \downarrow & & \uparrow \\
\varprojlim_n^1 R^{p-1}f_1 A_n & \times & \varprojlim_n R^q f_1 B_n & \to & \varprojlim_n^1 R^{p+q-1}f_3(A_n\otimes B_n), \\
\overline{(x_n)} & \times & (y_n) & \mapsto & \overline{(x_n\cup y_n)}
\end{array}$$

where $\overline{(z_n)}$ *is the image of* $(z_n)\in\prod_n C_n$ *in* $\varprojlim_n^1 C_n$ *via the canonical sequence* (1.4), *and* $(t_n)\in\varprojlim_n C_n$ *is regarded as element of* $\prod_n C_n$ *via the same sequence, and where the vertical arrows are those from* (1.6). *In particular, the cupproduct vanishes on* $\varprojlim_n^1 R^{p-1}f_1A_n\times\varprojlim_n^1 R^{q-1}f_2B_n$.

Proof. By Remark (1.7), the vertical maps are the edge morphisms of the Leray spectral sequence for $\varprojlim_n\circ f_i$, $i=1,2,3$. The claim then follows from (6.2c) (case $s=0$), once we have checked that the cupproduct

$$\varprojlim_n^1 C_n\times\varprojlim_n D_n\to\varprojlim_n^1 (C_n\otimes D_n)$$

for inverse systems (C_n) and (D_n) of abelian groups is given by mapping $(\overline{(c_n)}, (d_n))$ to $\overline{(c_n\otimes d_n)}$. This follows at once from (6.2b) and the standard resolution for $\varprojlim_n$ on $\underline{Ab}^{\mathbb{N}}$, see (1.5).

Remark. If one interchanges the terms on the left side, one has to add a sign $(-1)^q$ below to have a commutative diagram, as we change by $(-1)^{pq}$ above and $(-1)^{(p-1)q}$ below. This agrees perfectly with the sign $(-1)^{qs}$ in (6.2c).

We now want to compare the cupproduct with the Yoneda-pairing. For later applications it is convenient to consider the Ext-groups not in $S(X_{et})^{\mathbb{N}}$, but in a certain subcategory.

(6.9) *Definition.* For an abelian category $\mathscr{A}$ and a prime l, denote by $\mathscr{A}^{\mathbb{Z}_l}$ the full subcategory of $\mathscr{A}^{\mathbb{N}}$ consisting of the systems (A_n, d_n) with $l^nA_n=0$ for all $n\in\mathbb{N}$ (called pro-l-systems).

(6.10) **Proposition.** *Let* X *be a scheme and* Z *be a closed subscheme.*

a) $S(X_{et})^{\mathbb{Z}_l}$ *has enough injectives.*

b) *The cohomology functors* $H^i_Z(X, -)$ *are the same, whether computed as derived functors of* $H^0_Z(X, -)$ *in* $S(X_{et})^{\mathbb{N}}$ *or* $S(X_{et})^{\mathbb{Z}_l}$.

c) *Let* $\mathrm{Ext}^i_{l-X}((F_n),(G_n)) := \mathrm{Ext}^i_{S(X_{et})^{\mathbb{Z}_l}}((F_n),(G_n))$ *for objects* $F=(F_n)$ *and* $G=(G_n)$ *of* $S(X_{et})^{\mathbb{Z}_l}$.

i) *There are canonical isomorphisms of δ-functors*

$$\mathrm{Ext}^i_{l-X}(\mathbb{Z}_l, G) \xrightarrow[\sim]{\varphi^i} H^i(X,G), \qquad i \geq 0,$$

where $\mathbb{Z}_l$ *stands for the system* $(\mathbb{Z}/l^n)$ *with obvious transition maps.*

ii) *There are canonical functorial homomorphisms*

$$H^i(X,G) \xrightarrow{\psi^i} \mathrm{Ext}^i_{l-X}(F, F\otimes G)$$

such that the diagram

$$\begin{array}{ccccc} H^r(X,F) & \times & H^s(X,G) & \longrightarrow & H^{r+s}(X,F\otimes G) \\ \varphi^r\downarrow\wr & & \psi^s\downarrow & & \wr\downarrow\varphi^{r+s} \\ \mathrm{Ext}^r_{l-X}(\mathbb{Z}_l,F) & \times & \mathrm{Ext}^s_{l-X}(F,F\otimes G) & \longrightarrow & \mathrm{Ext}^{r+s}_{l-X}(\mathbb{Z}_l,F\otimes G) \end{array}$$

commutes, where the upper map is the cupproduct and the lower the Yoneda pairing.

Proof. a) This follows as in the proof of (1.1a), since $S(X_{et}, \mathbb{Z}/l^n\mathbb{Z})$ has enough injectives and for a family (I_n), with I_n injective in $S(X_{et}, \mathbb{Z}/l^n\mathbb{Z})$, the inverse system $P(I_n)$ is an injective object of $S(X_{et})^{\mathbb{Z}_l}$.

b) For (I_n) as above, the object $P(I_n)$ is acyclic for $H^0_{\mathbb{Z}}(X, -)$ on $S(X_{et})^{\mathbb{N}}$, since the inclusion $S(X_{et}, \mathbb{Z}/l^n\mathbb{Z}) \to S(X_{et})$ maps injective sheaves to flabby sheaves.

c) i) The isomorphisms are obtained by deriving the obvious one for $i=0$.

ii) Let $G \hookrightarrow C^{\cdot}(G)$ be the "canonical resolution" obtained by repeated embedding $G \xrightarrow{\varepsilon} C^0(G) \xrightarrow{\iota} PVC^0(G)$, where C^0 is the Godement functor (cf. [9, III 120(c)] and the proof of (6.3c) above), and let $(F\otimes G)^{\cdot}$ be an injective resolution of $F\otimes G$. Then there is a morphism $F\otimes C^{\cdot}(G) \to (F\otimes G)^{\cdot}$ of resolutions of $F\otimes G$, unique up to homotopy, and the ψ^i are induced by the obvious morphisms

$$\mathrm{Hom}_{l-X}(\mathbb{Z}_l, C^{\cdot}(G)) \to \mathrm{Hom}_{l-X}(F, F\otimes C^{\cdot}(G)) \to \mathrm{Hom}_{l-X}(F, (F\otimes G)^{\cdot}).$$

With this, the commutavitiy is proved literally as Proposition 1.20 in V Sect. 1 of Milne's book [9].

Using (6.10c) ii), one now can proceed exactly as Milne in [9, VI 6.5] to prove the wanted projection formula for the cupproduct:

(6.11) **Proposition.** *Let* $i: Z \to X$ *be a smooth S-pair of co-dimension c for a scheme S, and let l be a prime invertible on S. Then*

$$\begin{array}{ccccc} H^r_{\mathrm{cont}}(Z,\mathbb{Z}_l) & \times & H^s_{\mathrm{cont}}(Z,\mathbb{Z}_l) & \xrightarrow{\cup} & H^{r+s}_{\mathrm{cont}}(Z,\mathbb{Z}_l) \\ i^*\uparrow & & \downarrow i_* & & \uparrow i_* \\ H^r_{\mathrm{cont}}(X,\mathbb{Z}_l) & \times & H^{s+2c}_{\mathrm{cont}}(X,\mathbb{Z}_l(c)) & \xrightarrow{\cup} & H^{r+s+2c}_{\mathrm{cont}}(X,\mathbb{Z}_l(c)) \end{array}$$

is commutative, where i_* *is the Gysin map.*

We now can prove the main result of this section.

(6.12) **Theorem.** *Let l be a prime. Then there is a unique theory of Chern classes*

$$c_i(E) \in H^{2i}_{\mathrm{cont}}(S, \mathbb{Z}_l(i)), \qquad i \geq 0,$$

for locally free $\mathcal{O}_S$*-modules E on schemes S on which l is invertible such that for* $c(E) = (c_i(E)) \in \prod_i H^{2i}_{\mathrm{cont}}(S, \mathbb{Z}_l(i))$

a) $c(L)=(1,\delta_S([L]),0,\ldots)$ *for* L *invertible,*

b) $c(E)=c(E')\cdot c(E'')$ *for an exact sequence of locally fre* $\mathcal{O}_S$*-modules* $0\to E'\to E\to E''\to 0$.

c) $f^*c_i(E)=c_i(f^*E)$ *for morphisms* $f: S'\to S$.

Proof. This is provided by the following data and axioms, see Grothendieck [7] and also Jouanolou [SGA 5, VII 3].

i) Via the cupproduct, $H^*(S)=\bigoplus_{i\geq 0} H^{2i}_{\text{cont}}(S,\mathbb{Z}_l(i))$ is a graded ring, functorially associated to S in a contravariant way.

ii) $\delta_S: \text{Pic}(S)\to H^2_{\text{cont}}(S,\mathbb{Z}_l(1))$, defined in (3.26) via the Kummer sequence $0\to\mathbb{Z}_l(1)\to(G_m,l)\to\underline{G_m}\to 0$, is a morphism of functors.

iii) For any smooth S-pair $i: Z\hookrightarrow X$ of codimension c, S a scheme over $\mathbb{Z}[1/l]$, the Gysin morphism $i_*: H^*(Z)\to H^*(X)$ is a group morphism raising degrees by $2c$.

A1) For any vector bundle E of rank m on $S/\mathbb{Z}[1/l]$ there is a Dold-Thom isomorphism $H^*(\mathbb{P}_S(E))\cong\bigoplus_{\nu=0}^{m-1} H^*(S)\xi_E^{m-\nu}$, which holds more generally by the proposition below.

A2) If $i: Z\hookrightarrow X$ is a smooth S-pair of codimension 1, then $\delta_X([\mathcal{O}(Z)])=i_*(1_Z)$, see (3.26a).

A3) The Gysin morphisms behave transitively. This is immediately clear from the description of the local cycle class in [SGA 4½, (cycle) 2.2.2c].

A4) For a smooth S-pair $i: Z\hookrightarrow X$ the projection formula holds by 6.11: $i_*(i^*(x)\cup z)=x\cup i_*(z)$ for $x\in H^*(X)$, $z\in H^*(Z)$.

In fact, then the Chern classes $c_i(E)$ of a vector bundle E of rank m on S are the unique elements in $H^{2i}_{\text{cont}}(S,\mathbb{Z}_l(i))$ with

$$\sum_{i=0}^{m} c_i(E)\cup\xi_{\check{E}}^{m-1}=0 \quad\text{in}\quad H^{2m}_{\text{cont}}(P_S(\check{E}),\mathbb{Z}_l(m))$$
$$c_0(E)=1,\qquad c_i(E)=0\quad\text{for}\quad i>m,$$

where $\check{E}$ is the dual bundle and the notations are as in (6.13). (In [9, p. 273] one either should take $\check{E}$ as above or ξ the class of $\mathcal{O}(-1)$.) For general locally free $\mathcal{O}_S$-modules one has to proceed as in [SGA 5, VII 3.2].

(6.13) **Proposition.** *Let* S *be a scheme and* l *a prime invertible on* S. *Let* E *be a locally free* $\mathcal{O}_S$*-module of rank* m, *and* $P=\mathbb{P}_S(Q)\xrightarrow{p}S$ *be the associated projective fibre bundle. Let* $\mathcal{O}_P(1)$ *be the canonical invertible sheaf on* P *and* $\xi=\xi_E$ *be the image of its class* $[\mathcal{O}_P(1)]$ *in* Pic(P) *under the morphism*

$$\delta_P: \text{Pic}(P)\to H^2_{\text{cont}}(P,\mathbb{Z}_l(1))$$

defined in (3.26). *Then via* p^* *the bi-graded ring* $\bigoplus_{i,j} H^i_{\text{cont}}(P,\mathbb{Z}_l(j))$ *is a free module over* $\bigoplus_{r,s} H^r_{\text{cont}}(S,\mathbb{Z}_l(s))$ *with basis* $1,\xi,\ldots,\xi^{m-1}$ *i.e., the map*

$$\bigoplus_{\nu=0}^{m-1} H^{i-2\nu}_{\text{cont}}(S,\mathbb{Z}_l(j-\nu))\to H^i_{\text{cont}}(P,\mathbb{Z}_l(j))$$
$$(x_0,\ldots,x_{m-1})\mapsto\sum_{\nu=0}^{m-1} p^*x_\nu\cup\xi^\nu$$

are isomorphisms for every i *and* j.

Proof. By (6.8), this follows, by passing to $\varprojlim$ and $\varprojlim^1$, from the corresponding result with $\mathbb{Z}_l(j)$ replaced by the finite coefficients $\mathbb{Z}/l^n(j)$ for any n, which holds by [SGA 5, VII 2.2.4].

We finish this section with two properties of the cycle map.

(6.14) **Lemma.** *Let X be a smooth variety over a field k, $l \neq \mathrm{char}(k)$.*

i) *If two cycles Z_1 and Z_2 on X are linearly equivalent, then $\mathrm{cl}^X(Z_1) = \mathrm{cl}^X(Z_2)$. Therefore the cycle classes induce homomorphisms for $i \geqq 1$*

$$\mathrm{cl}^X = \mathrm{cl}^{X,i} : CH^i(X) \to H^{2i}_{\mathrm{cont}}(X, \mathbb{Z}_l(i)),$$

where $CH^i(X)$ is the Chow group of cycles of codimension i modulo linear equivalence.

ii) *If Z_1 and Z_2 are two cycles on X and $Z_1 \cdot Z_2$ is their intersection product, then* $\mathrm{cl}^X(Z_1 \cdot Z_2) = \mathrm{cl}^X(Z_1) \cup \mathrm{cl}^X(Z_2)$.

Proof. i) By assumption, there are a cycle Z on $\mathbb{P}^1_k \times X$ and two points t_1, t_2 of $\mathbb{P}^1_k$ such that for the induced morphisms $\varphi_i : X \to \mathbb{P}^1_k \times X$, $i = 1, 2$ one has $\varphi_1^*(Z) = Z_1$ and $\varphi_2^*(Z) = Z_2$. Choosing an automorphism α of $\mathbb{P}^1_k$ with $\alpha(t_1) = t_2$ we have $(\alpha \times \mathrm{id}_x) \circ \varphi_1 = \varphi_2$ and therefore

$$\mathrm{cl}(Z_2) = \varphi_2^*(\mathrm{cl}(Z)) = \varphi_1^*((\alpha \times \mathrm{id}_x)^* \mathrm{cl}(Z)) = \varphi_1^*(\mathrm{cl}(Z)) = \mathrm{cl}(Z_1),$$

since $(\alpha \times \mathrm{id}_x)^*$ acts trivially on $H^*(\mathbb{P}^1_k \times X)$ by (6.12): it fixes $H^*(X)$ and the canonical ample class which comes from $\mathbb{P}^1_k$.

ii) Once the cupproduct is defined, this follows as in (3.25) by passing to classes with support and then to finite coefficients, where the statement holds by [SGA $4\frac{1}{2}$, (cycle) 2.3.9]. Note that by i) and Chow's moving lemma we may assume that Z_1 and Z_2 intersect properly.

(6.15) *Remark.* a) It is not true in general, that the cycle map factorizes through *algebraic* equivalence, unless k is algebraically closed. Indeed by the theorem of Mordell-Weil the morphism

$$\mathrm{cl}^{X,1} = \delta_X : \mathrm{Pic}(X) \to H^2_{\mathrm{cont}}(X, \mathbb{Z}_l(1))$$

is *injective* for k of finite type over the prime field and X projective over k: by the Kummer sequence (3.27)

$$0 \to \mathbb{Z}_l(1) \to (G_m, l) \to \underline{G_m} \to 0$$

and the functoriality of (3.1), the kernel of the above map is l-Div(Pic(X)) for any scheme X. If moreover $H^2(X, G_m)$ is finite, the induced map $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{H}^2_{\mathrm{cont}}(X, \mathbb{Z}_l(1))$ is an isomorphism. It would be very interesting to get similar results on $\mathrm{cl}^{X,i}$ for $i \geqq 2$.

b) It follows from the definitions that the cycle map

$$\overline{\mathrm{cl}^X} : CH^i(X) \to H^{2i}(\bar{X}, \mathbb{Z}_l(i))^{G_k}$$

considered by Tate in [15] is the composition of cl^X and the restriction map

$$H^{2i}_{\mathrm{cont}}(X, \mathbb{Z}_l(i)) \xrightarrow{\mathrm{res}} H^{2i}(\bar{X}, \mathbb{Z}_l(i))^{G_k}$$

where $\bar{X} = X \times_k \bar{k}$ for an algebraic closure $\bar{k}$ of k. Tate conjectures that for a finitely generated field k and X/k smooth projective $\overline{\mathrm{cl}^X}$ becomes surjective after tensoring with $\mathbb{Q}_l$ [15, (7)]. A necessary condition for this to be true therefore is that res is surjective up to torsion or, equivalently, that all differentials

$$d_r^{0,2i}: E_r^{0,2i} \to E_r^{r,2i-r+1}$$

in the spectral sequence of (3.5b) have torsion images. In a following paper we shall in fact show this property for all differentials $d_r^{p,q}$, $p, q \geqq 0$.

c) Let $CH^i(X)_0$ be the kernel of cl^X, i.e., the subgroup generated by cycles homologous to zero (perhaps depending on l). Then the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & CH^i(X)_0 & \longrightarrow & CH^i(X) & \longrightarrow & CH^i(X)/CH^i(X)_0 & \longrightarrow 0 \\
& \downarrow & & \downarrow \mathrm{cl}^X & \searrow & \overline{\mathrm{cl}^X} \downarrow & \\
0 \longrightarrow & F^1 H^{2i}_{\mathrm{cont}}(X, \mathbb{Z}_l(i)) & \longrightarrow & H^{2i}_{\mathrm{cont}}(X, \mathbb{Z}_l(i)) & \xrightarrow{\mathrm{res}} & H^{2i}(X, \mathbb{Z}_l(i))^{G_k} = E_2^{0,2i} & \\
& \downarrow & & & & & \\
& \multicolumn{3}{l}{E_2^{1,2i-1} = H^1_{\mathrm{cont}}(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)))} & & &
\end{array}$$

induced by the Hochschild-Serre spectral sequence (3.5b) defines an "Abel-Jacobi map"

$$\mathrm{cl}_0^X: CH^i(X)_0 \to H^1_{\mathrm{cont}}(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))),$$

and it can be shown that for a cycle Z on X homologous to zero

$$\mathrm{cl}_0^X(Z) \in H^1_{\mathrm{cont}}(G_k, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i))) = \mathrm{Ext}^1_{G_k}(\mathbb{Z}_l, H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)))$$

is the class of the extension of continuous G_k-modules

$$\begin{array}{ccccccc}
0 \to & H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)) & \to & H^{2i-1}(\bar{U}, \mathbb{Z}_l(k)) & \to & H^{2i}_{|\bar{Z}|}(\bar{X}, \mathbb{Z}_l(i)) & \to H^{2i}(\bar{X}, \mathbb{Z}_l(i)) \\
& \| & & \uparrow & & \uparrow & \\
0 \to & H^{2i-1}(\bar{X}, \mathbb{Z}_l(i)) & \to & E & \to & \mathbb{Z}_l \cdot \mathrm{cl}_{\bar{Z}}^{\bar{X}}(\bar{Z}) & \to \quad 0
\end{array}$$

obtained by pull-back, where $U = X \setminus |Z|$. This will be discussed in another paper.

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Received December 31, 1986