On the ℓ-adic cohomology of varieties over number fields and its Galois cohomology

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If X is a smooth, projective variety over a number field k, then the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ acts on the étale cohomology groups $H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$, where $\bar{X} = X \times_k \bar{k}$ for an algebraic closure $\bar{k}$ of k. In this paper I study some properties of these G_k-modules; in particular, I am interested in the corank of the Galois cohomology groups

$$H^\nu(G_k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))).$$

It turns out that this question reduces to the situation where G_k is replaced by its decomposition groups $G_\mathfrak{p}$, or its maximal S-ramified quotient G_S for some suitable (finite) set S of places of k. In the last case, there are only very few results, while in the local case the situation is somewhat better, especially for the places $\mathfrak{p}$ of good reduction, but the general question is unsolved here, too. The aim of this article is to collect the known results and to discuss several conjectures — some well-known and some new — which would imply that the groups in question behave rather nicely.

The basic conjecture, Conjecture 1, is that $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ is finite for $i + 1 < n$ or $i + 1 > 2n$. Since the Euler-Poincaré characteristic of G_S is well-known, this amounts to predicting a very simple number for the corank of $H^1(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ in this range.

In §2 I show how this conjecture for $i + 1 < n$ is motivated by Conjecture 2, which concerns Chern class maps from algebraic K-theory to ℓ-adic cohomology and can be seen as a certain ℓ-adic analogue of a part of Beilinson's conjectures. In fact, by results of Thomason one can draw a direct connection to Beilinson's conjectures.

Another motivation for Conjecture 1 is that its function field analogue can be proved by using Deligne's proof of the Weil conjectures, see §4. Behind many results and conjectures of this paper stands the observation that the Weil conjectures and the theory of weights can also be used heavily in the number field case, and the hope that there might be an even closer analogy with the function field case than visible today. As an example for the first

statement, a vanishing theorem of Serre implies that the localization map of continuous cohomology

$$H^1(G_k, M) \to \prod_{\mathfrak{p}} H^1(G_{\mathfrak{p}}, M)$$

is injective for a torsion-free $\mathbb{Z}_\ell$-module M, which is a pure G_k-module of weight $\neq 0$.

This reduces the determination of corank $H^2(G_k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ to a purely local question, and leads to Conjecture 3 (sharpened form), saying that

$$H^i(\bar{X}, \mathbb{Q}_\ell(m))^{G_{\mathfrak{p}}}$$

vanishes except possibly for $0 \leq m \leq \frac{i}{2}$, see §5. For $\mathfrak{p} \nmid \ell$ and good reduction, this is known, again by the Weil conjectures. For $\mathfrak{p} \nmid \ell$ and bad reduction this would follow from a well-known conjecture about the monodromy filtration, and for $\mathfrak{p} \mid \ell$ and good reduction it follows from the so-called crystalline conjecture, which has partially been proved by Fontaine-Messing and Faltings. I explain why a naive form of the monodromy filtration cannot exist for $\mathfrak{p} \mid \ell$ (this was observed by Mazur, Tate, and Teitelbaum several years ago), and I discuss what the correct analogue should be, proposing some kind of crystalline conjecture for the case of bad reduction.

If one tries to investigate whether $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ at least is finite for almost all $n \in \mathbb{Z}$, one is immediately led to the question of whether a certain Iwasawa module Z is a Λ-torsion module, see §3. This would follow from Iwasawa's conjecture on the vanishing of the μ-invariant, but is unproved yet. I show that the Λ-rank of Z has to be added to Soulé's expression for the corank of $H^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ (see [**Sou4**, Theorem 1]), whose determination is closely related to our subject. In fact, this paper can be seen as a successor of [**Sou4**], trying to investigate more closely the mentioned coranks and to predict precisely the range where one expects a simple expression for them.

In §6 I briefly discuss the "unstable range" $n \leq i + 1 \leq 2n$, and in §7 I treat the case of abelian varieties, where one can at least prove the local conjectures.

This paper was initiated by the investigation of ℓ-adic Chern classes in [**J3**] and was influenced much by the work and ideas of Beilinson and Deligne, and by the study of Soulé's paper [**Sou4**]. I would like to thank W. Raskind, J. M. Fontaine, W. Messing, and especially K. Wingberg,

1. The basic conjecture.

The following notations will be valid for the whole paper:

k is a number field with algebraic closure $\bar{k}$,

$G_k = \mathrm{Gal}(\bar{k}/k)$ is the absolute Galois group of k,

$G_{\mathfrak{p}} \subset G_k$ or $G_v \subset G_k$ is a decomposition group for the prime $\mathfrak{p}$ or the place v of k (well defined up to conjugacy),

X is a smooth, projective variety of pure dimension d over k,

ℓ is a prime,

S is a finite set of places of k, containing all places above ∞ and ℓ, and all primes where X has bad reduction,

$G_S = \mathrm{Gal}(k_S/k)$ is the Galois group over k of the maximal S-ramified ($=$ unramified outside of S) extension k_S of k.

CONJECTURE 1. *Let* $\bar{X} = X \times_k \bar{k}$. *Then*

$$H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = 0 \text{ if } \quad \begin{array}{ll} a) & i+1 < n, \text{ or} \\ b) & i+1 > 2n. \end{array}$$

REMARK 1: Here $H^i(\bar{X}, \mathbb{Q}_\ell(n)) = H^i(\bar{X}, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is étale cohomology, and it follows from the smooth and proper base change theorem that the action of G_k on $H^i(\bar{X}, \mathbb{Z}_\ell(n)) = \lim_{\overleftarrow{r}} H^i(\bar{X}, \mathbb{Z}/\ell^r(n))$ is unramified outside S, i.e., factors through G_S. The sheaves $\mathbb{Z}/\ell^r(n)$ are defined as usual by $\mathbb{Z}/\ell^r(n) = \mu_{\ell^r}^{\otimes n}$, where μ_{ℓ^r} is the sheaf of ℓ^r-th roots of unity [Mi1, p. 163]. The corresponding Galois modules are denoted by the same symbols, and there is a canonical isomorphism of G_k-modules $H^i(\bar{X}, \mathbb{Z}_\ell(n)) \cong H^i(\bar{X}, \mathbb{Z}_\ell)(n)$, where the Tate twist $M(n)$ of a $\mathbb{Z}_\ell$-G_k-module is defined by $M(n) = M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(n)$, with $\mathbb{Z}_\ell(n) = \lim_{\overleftarrow{r}} \mathbb{Z}/\ell^r(n)$. The G_S-cohomology is taken via continuous cochains, and since $H^i(\bar{X}, \mathbb{Z}_\ell(n))$ is finitely generated as a $\mathbb{Z}_\ell$-module, we have

$$H^\nu(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = H^\nu(G_S, H^i(\bar{X}, \mathbb{Z}_\ell(n))) \bigotimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

compare [J1, 5.15.c]. Throughout the paper we consider profinite groups G and topological G-modules M; the cohomology will always be the continuous one. If M is discrete (e.g., for $H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \lim_{\overrightarrow{r}} H^i(\bar{X}, \mathbb{Z}/\ell^r(n)) \cong$

$H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)(n))$, this is the usual cohomology, described in [**Se1**]. Note that we write $\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$ for $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n)$. We shall use the following basic facts which follow from [**Ta3**, §2]: if $M = \lim_{\overleftarrow{r}} M_r$ for a projective system (M_r) of finite $\mathbb{Z}_\ell$-G-modules, then

$$(1) \qquad H^1(G, M) = \varprojlim_r H^1(G, M_r).$$

This may be false for H^ν, $\nu > 1$, but if G satisfies the property

$$(F_\ell) \qquad H^\nu(G, A) \text{ is finite for every finite } \mathbb{Z}_\ell\text{-}G\text{-module A,}$$

then we have $H^\nu(G, M) = \lim_{\overleftarrow{r}} H^\nu(G, M_r)$ for all ν. Moreover, if G satisfies (F_ℓ) and M is a finitely generated $\mathbb{Z}_\ell$-module, then $H^\nu(G, M)$ is finitely generated over $\mathbb{Z}_\ell$ and the canonical map

$$(2) \qquad H^\nu(G, M) \bigotimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^\nu(G, M \bigotimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is an isogeny, i.e., has finite kernel and cokernel. Similarly, if N is a cofinitely generated $\mathbb{Z}_\ell$-torsion G-module (i.e., the Pontrjagin dual N^* is a finitely generated $\mathbb{Z}_\ell$-module), then $H^\nu(G, N)$ is cofinitely generated and the canonical map

$$(3) \qquad H^\nu(G, T_\ell N) \to T_\ell H^\nu(G, N)$$

is an isogeny, where $T_\ell A = \lim_{\overleftarrow{r}} A_{\ell^r}$ is the Tate module of an abelian group A. Again this may be false without (F_ℓ).

We fix some more notation: If A is an abelian group and r is a natural number, let $A/r = A/rA$ and $A_r = \{a \in A \mid ra = 0\}$. If a group G acts on A, let A^G and A_G be the fixed and cofixed module (module of coinvariants), respectively. For an ℓ-primary torsion group A, let $\dim A$ be the corank of its maximal ℓ-divisible subgroup $\ell - \text{Div } A$; that is, $\dim A = \dim_{\mathbb{F}_\ell}(\ell - \text{Div } A)_\ell$, where $\mathbb{F}_{\ell^r}$ is the field with ℓ^r elements.

LEMMA 1. *The following statements are equivalent:*
   a) $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = 0$.
   b) $H^2(G_S, H^i(\bar{X}, \mathbb{Z}_\ell(n)))$ *is finite.*
   c) $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ *is finite.*
   d) *(if $\ell \neq 2$, or if k is totally imaginary)* $H^2(G_S, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = 0$. *where* $\tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \ell - \text{Div } H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$.

PROOF: It follows from class field theory that G_S satisfies (F_ℓ) (cf. [**Mi2**, I 4.15]). Hence the groups in b) and c) are of finite and cofinite type, respectively, and the maps

$$H^\nu(G_S, H^i(\bar{X}, \mathbb{Z}_\ell(n))) \bigotimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^\nu(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

induced by the isogeny $H^i(\bar{X}, \mathbb{Z}_\ell(n)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ and (2) are again isogenies. This shows the relation

$$\dim_{\mathbb{Q}_\ell} H(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = \text{rank}_{\mathbb{Z}_\ell} H(G_S, H^i(\bar{X}, \mathbb{Z}_\ell(n)))$$
$$= \dim H(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))),$$

and the equivalence of a), b) and c). Under the assumption in d), G_S has ℓ-cohomological dimension $cd_\ell(G_S) \leq 2$ [**Br**, 2.11]. Hence for an ℓ-divisible G_S-module N the cohomology sequence associated to $0 \to N_\ell \to N \overset{\ell}{\to} N \to 0$ shows that $H^2(G_S, N)$ is ℓ-divisible. Since an ℓ-divisible group of finite exponent is zero, the equivalence of c) and d) follows from the finiteness of $H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))/\tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$.

In the following, we shall assume $\ell \neq 2$ or k totally imaginary, for simplicity. We leave it to the reader to check that everything remains true for $\ell = 2$ up to finite groups (in §3: up to Λ-torsion modules) of exponent two, which does not matter for our purposes.

Example 1   For $X = \text{Spec}\, k$, we have a non-trivial cohomology group only for $i = 0$, $H^0(\text{Spec}\, \bar{k}, \mathbb{Z}_\ell(n)) = \mathbb{Z}_\ell(n)$, and we may take $S = S_\ell = \{v \mid \ell \cdot \infty\}$. The conjecture then claims

$$H^2(G_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = 0 \text{ for } \quad \begin{array}{ll} \text{a)} & 1 < n, \text{ or} \\ \text{b)} & 1 > n, \end{array}$$

i.e., for $n \neq 1$. This has been conjectured by Schneider [**Sch1**, p. 192]. Case a) has been proved by Soulé [**Sou1**] via higher Chern classes on algebraic K-theory; see the discussion in §2. Also for general X the case b) seems to be much more mysterious; here it contains the Leopoldt conjecture (the case $n = 0$, see [**Sch1**, §7]). Concerning $n = 1$, one has $\mathbb{Q}_\ell/\mathbb{Z}_\ell(1) = \mu_{\ell^\infty}$, the group of all ℓ-power roots of unity, and

$$H^2(G_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \cong \text{Br}_S(k)\{\ell\} \cong \mathbb{Q}_\ell/\mathbb{Z}_\ell^{\#S'-1}$$

where $S' = S\backslash\{v \mid \infty\}$, see [**Sch1**, 4.2], so this group is non-trivial in general.

Example 2 If E is an elliptic curve over k, then $H^0(\bar{E}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$ and $H^2(\bar{E}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(-1)$ are covered by the above. For the remaining case $i = 1$, we may use the Kummer sequence $0 \to \mu_{\ell^\nu} \to \mathbb{G}_m \xrightarrow{\ell^\nu} \mathbb{G}_m \to 0$ and the isomorphisms $E(\bar{k}) \cong \text{Pic}^0(\bar{E}) \subseteq \text{Pic}(\bar{E}) \cong H^1(\bar{E}, \mathbb{G}_m)$ to identify

$$H^1(\bar{E}, \mathbb{Z}/\ell^\nu(1)) = H^1(\bar{E}, \mu_{\ell^\nu}) = \text{Pic}(\bar{E})_{\ell^\nu} = E(\bar{k})_{\ell^\nu}.$$

Hence the conjecture here claims $(r = n - 1)$

$$H^2(G_S, E_{\ell^\infty}(r)) = 0 \text{ for } \quad \begin{array}{ll} \text{a)} & 1 < r, \text{ or} \\ \text{b)} & 0 > r, \end{array}$$

i.e., for $r \neq 0, 1$ (Here $E_{\ell^\infty} = \cup_r E(\bar{k})_{\ell^r}$ is the coefficient system of ℓ-power torsion points of E.) If k is imaginary quadratic and E/k has complex multiplication by k, this has been proved by K. Wingberg [**Wi2**] for primes ℓ which are regular for E. We shall see in §7 why $r = 0, 1$ definitely has to be excluded (remark 7).

It will be useful to reformulate Conjecture 1 in terms of $H^1(G_S, -)$, by using Tate's calculation of the Euler-Poincaré characteristic of G_S. For this, note that X can be regarded as a variety over $\mathbb{Q}$ via $X \to \text{Spec } k \to \text{Spec } \mathbb{Q}$, and then $X \times_{\mathbb{Q}} \mathbb{C}$ is a well-defined complex projective variety (non-connected for $k \neq \mathbb{Q}$). In particular, $X \times_{\mathbb{Q}} \mathbb{C}$ is defined over $\mathbb{R}$ and hence has a canonical involution F_∞ (the "infinite Frobenius", see [**D3**, 0.2.5]) on its Betti cohomology $H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z})$. For a ring $A \subset \mathbb{C}$ and $n \in \mathbb{Z}$, define $A(n) = A \cdot (2\pi i)^n \subset \mathbb{C}$, and let $\sigma = \bar{F}_\infty$ act on $H^i(X \times_{\mathbb{Q}} \mathbb{C}, A(n)) = H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}) \otimes_{\mathbb{Z}} A(n)$ via $F_\infty \otimes c$, where c is the complex conjugation. Let $H^i(X \times_{\mathbb{Q}} \mathbb{C}, A(n))^{\pm}$ be the eigenspace for the eigenvalue ± 1 of σ, respectively.

LEMMA 2. If $\chi(G_S, V) = \sum_{\nu=0}^3 (-1)^\nu \dim_{\mathbb{Q}_\ell} H^\nu(G_S, V)$ for a $\mathbb{Q}_\ell$-vector space V with continuous action of G_S, then

$$-\chi(G_S, V) = \sum_{v \mid \infty} \dim_{\mathbb{Q}_\ell} V - \sum_{\substack{v \mid \infty \\ v \text{ real}}} \dim_{\mathbb{Q}_\ell} V^{G_v}.$$

As a consequence,

$$\chi(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = -\dim_{\mathbb{R}} H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(n))^- =: \chi_{i,n}(X) =: \chi_{i,n};$$

in particular, these numbers are independent of ℓ and S (satisfying our assumptions) and depend only on n mod 2.

PROOF: This is proved as Proposition 2 in [Sou4], cf. also [Sch1, 4.6]. Since it is important for the sequel, we sketch the argument. Let Λ be a G_S-equivariant lattice in V (i.e., a $\mathbf{Z}_\ell$-submodule with $\Lambda \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \cong V$), and let $A = V/\Lambda$. Then $\dim_{\mathbf{Q}_\ell} H^\nu(G_S, V) = \dim H^\nu(G_S, A) = \dim_{\mathbf{F}_\ell} H^\nu(G_S, A)_\ell$ $- \dim_{\mathbf{F}_\ell} H^\nu(G_S, A)/\ell$, so the cohomology sequence for $0 \to A_\ell \to A \xrightarrow{\ell} A \to 0$ shows

$$\chi(G_S, V) = \chi(G_S, A_\ell) - \dim_{\mathbf{F}_\ell} H^2(G_S, A)/\ell,$$

where $\chi(G_S, A_\ell) = \sum_{\nu=0}^{3}(-1)^\nu \dim_{\mathbf{F}_\ell} H^\nu(G_S, A_\ell) = -\sum_{v|\infty} \dim_{\mathbf{F}_\ell} A_\ell + \sum_{\substack{v|\infty \\ v \text{ real}}} \dim_{\mathbf{F}_\ell} H^0(G_v, A_\ell)$ by Tate [Ta1]. Since $H^2(G_S, A)/\ell \cong \oplus_{v|\infty} H^2(G_v, A)$, one easily deduces as in [Sou4]

$$-\chi(G_S, V) = \sum_{v|\infty} \dim A - \sum_{\substack{v|\infty \\ v \text{ real}}} \dim A^{G_v},$$

i.e., the first formula. Since $H^i(X \times_{\mathbf{Q}} \bar{Q}, \mathbf{Q}_\ell) \cong \mathrm{Ind}_{G_k}^{G_{\mathbf{Q}}} H^i(\bar{X}, \mathbf{Q}_\ell)$, where $\mathrm{Ind}_{G_k}^{G_{\mathbf{Q}}}$ denotes induction from G_k to $G_{\mathbf{Q}}$, this formula implies

$$- \chi(G_S, H^i(\bar{X}, \mathbf{Q}_\ell(n)))$$
$$= \dim_{\mathbf{Q}_\ell} H^i(X \times_{\mathbf{Q}} \bar{Q}, \mathbf{Q}_\ell(n)) - \dim_{\mathbf{Q}_\ell} H^i(X \times_{\mathbf{Q}} \bar{Q}, \mathbf{Q}_\ell(n))^{G_\infty},$$

where $G_\infty \subset G_{\mathbf{Q}}$ is a decomposition group at ∞. The choice of G_∞ corresponds to the choice of an embedding $\tau : \bar{Q} \hookrightarrow \mathbf{C}$, and via the canonical comparison isomorphism

$$H^i_{\text{ét}}(X \times_{\mathbf{Q}} \bar{Q}, \mathbf{Q}_\ell) \xrightarrow{\tau} H^i_{\text{ét}}(X \times_{\mathbf{Q}} \mathbf{C}, \mathbf{Q}_\ell) \to H^i(X \times_{\mathbf{Q}} \mathbf{C}, \mathbf{Z}) \underset{\mathbf{Z}}{\otimes} \mathbf{Q}_\ell$$

the action of F_∞ corresponds to the action of the non-trivial element in G_∞. Finally, the Tate twist (± 1) on both sides changes the action of G_∞ and σ by a sign (i.e., by the non-trivial characters $G_\infty \to \{\pm 1\}$, $\langle \sigma \rangle \to \{\pm 1\}$, respectively), which shows

$$\dim_{\mathbf{Q}_\ell} H^i(X \times_{\mathbf{Q}} \bar{Q}, \mathbf{Q}_\ell(n))^{G_\infty} = \dim_{\mathbf{R}} H^i(X \times_{\mathbf{Q}} \mathbf{C}, \mathbf{R}(n))^+.$$

COROLLARY 1. a) $\dim_{\mathbf{Q}_\ell} H^1(G_S, H^i(\bar{X}, \mathbf{Q}_\ell(n))) \geq -\chi_{i,n}(X)$, with equality if and only if $H^i(\bar{X}, \mathbf{Q}_\ell(n))^{G_S} = 0 = H^2(G_S, H^i(\bar{X}, \mathbf{Q}_\ell(n)))$.

b) $\chi(G_S, H^i(\bar{X}, \mathbf{Q}_\ell(n))) + \chi(G_S, H^i(\bar{X}, \mathbf{Q}_\ell(i + 1 - n))) = -[k : \mathbf{Q}] \cdot \dim_{\mathbf{Q}_\ell} H^i(\bar{X}, \mathbf{Q}_\ell)$.

The fixed module $H^i(\bar{X}, \mathbf{Q}_\ell(n))^{G_k}$ is related to Tate's conjecture for $i = 2n$, but it always vanishes for $i \neq 2n$:

LEMMA 3. *If* $i \neq 2n$, *then* $H^i(\bar{X}, \mathbb{Q}_\ell(n))^{G_k} = 0 = H^i(\bar{X}, \mathbb{Q}_\ell(n))_{G_k}$ *and* $H^i(\bar{X}, \mathbb{Q}_\ell(n))^{G_\mathfrak{p}} = 0 = H^i(\bar{X}, \mathbb{Q}_\ell(n))_{G_\mathfrak{p}}$ *for every* $\mathfrak{p} \notin S$.

PROOF: Let $\mathcal{X}$ be a smooth and proper model of X over $\mathcal{O}_S$, the ring of S-integers in k, and let $\mathcal{X}_\mathfrak{p} = \mathcal{X} \times_{\mathcal{O}_S} \kappa(\mathfrak{p})$ be its fiber at $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_S = \operatorname{Spec} \mathcal{O} \backslash S$, $\kappa(\mathfrak{p})$ denoting the residue field of $\mathfrak{p}$. By the smooth and proper base change theorem (see [**Mi1**, VI 4.2]), there is a canonical isomorphism (note that $\mathfrak{p} \nmid \ell$)

$$H^i(\bar{X}, \mathbb{Q}_\ell(n)) \cong H^i(\bar{\mathcal{X}_\mathfrak{p}}, \mathbb{Q}_\ell(n)),$$

where $\bar{\mathcal{X}_\mathfrak{p}} = \mathcal{X}_\mathfrak{p} \times_{\kappa(\mathfrak{p})} \overline{\kappa(\mathfrak{p})}$ for an algebraic closure $\overline{\kappa(\mathfrak{p})}$ of $\kappa(\mathfrak{p})$. This is an isomorphism of $G_\mathfrak{p}$-modules via $G_\mathfrak{p} \twoheadrightarrow \operatorname{Gal}(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p}))$, and by Deligne's proof of the Weil conjectures, $H^i(\bar{\mathcal{X}_\mathfrak{p}}, \mathbb{Q}_\ell)$ is pure of weight i; that is, the eigenvalues of the geometric Frobenius $\operatorname{Fr}_\mathfrak{p} \in \operatorname{Gal}(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p}))$ are algebraic integers whose archimedan absolute values are equal to $(N\mathfrak{p})^{\frac{i}{2}}$, where $N\mathfrak{p}$ is the cardinality of $\kappa(\mathfrak{p})$ [**D4**, 3.3.9]. Hence $H^i(\bar{\mathcal{X}_\mathfrak{p}}, \mathbb{Q}_\ell)(n)$ is of weight $i - 2n$ ($\operatorname{Fr}_\mathfrak{p}$ acts on $\mathbb{Q}_\ell(1)$ via multiplication by $(N\mathfrak{p})^{-1}$). There can only be a non-zero fixed or cofixed module, if the eigenvalue 1 appears, that is, if the weight is zero.

2. Connections with algebraic K-theory.

Beilinson's conjectures [**Be1**] concern the diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
\dfrac{H^{\nu-1}(X, \mathbb{C})^+}{H^{\nu-1}(X, \mathbb{R}(n))^+ + F^n} \\
\downarrow
\end{array}
$$

(4) $\quad K_{2n-\nu}(X/\mathbb{Z})^{(n)} \subseteq K_{2n-\nu}(X)^{(n)} \overset{r}{\longrightarrow} H^\nu_{\mathcal{D}}(X \times_\mathbb{Q} \mathbb{R}, \mathbb{R}(n))$

$$
\begin{array}{ccc}
0 & \underset{\text{for } \nu \neq 2n}{=} & H^\nu(X, \mathbb{R}(n))^+ \cap F^n \\
& & \downarrow \\
& & 0
\end{array}
$$

Here $H^\nu_{\mathcal{D}}$ denotes the Deligne cohomology, which sits in a short sequence as indicated, $H^*(X, -)$ stands for the Betti cohomology $H^*(X \times_\mathbb{Q} \mathbb{C}, -)$,

$F^{\cdot}$ is the Hodge filtration on $H^*(X, \mathbb{C})$, and $K_m(X)^{(n)}$ is the subspace of $K_m(X) \otimes \mathbb{Q}$ on which the Adams operators ψ^k operate via multiplication by k^n. The "regulator map" r is defined by Chern characters on Quillen's higher K-groups $K_m(X)$ with values in the Deligne cohomology, and factorizes as indicated for $\nu \neq 2n$, since then $F^n \cap \overline{F^n} = H^{n,n}$ is zero. Beilinson thinks that the definition

$$K_m(X/\mathbb{Z})^{(n)} = \mathrm{Im}\big(K_m(\mathcal{X})^{(n)} \to K_m(X)^{(n)}\big),$$

$\mathcal{X}$ a flat, proper model for X over $\mathbb{Z}$, does not depend on the choice of $\mathcal{X}$ and then conjectures, among other things, the following, where we set $i = \nu - 1$ for convenience.

CONJECTURE (Beilinson). *For $i < 2n - 2$ the regulator map*

$$r : K_{2n-i-1}(X/\mathbb{Z})^{(n)} \bigotimes \mathbb{R} \to H^i(X, \mathbb{C})^+/(H^i(X, \mathbb{R}(n))^+ + F^n)$$

is an isomorphism, and the dimension of these vector spaces is the order of zero of the L-function $L(V, s)$ for $V = \{H^i(\bar{X}, \mathbb{Q}_\ell)\}$ at $s = m := i + 1 - n$.

REMARK 2: The L-function of the strictly compatible system of ℓ-adic representations V is defined by the Euler product

$$L(V, s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(V, s) = \prod_{\mathfrak{p}} \frac{1}{\det(1 - \mathrm{Fr}_{\mathfrak{p}}(N\mathfrak{p})^{-s} \mid V^{I_{\mathfrak{p}}})}.$$

Here $I_{\mathfrak{p}}$ is an inertia group at the prime $\mathfrak{p}$, $\mathrm{Fr}_{\mathfrak{p}}$ is a geometric Frobenius at $\mathfrak{p}$, and it is conjectured that

$$\det(1 - \mathrm{Fr}_{\mathfrak{p}} T \mid V^{I_{\mathfrak{p}}}) = \det(1 - \mathrm{Fr}_{\mathfrak{p}} T \mid H^i(\bar{X}, \mathbb{Q}_\ell)^{I_{\mathfrak{p}}}), \mathfrak{p} \nmid \ell,$$

lies in $\mathbb{Z}[T]$, is independent of ℓ (with $\mathfrak{p} \nmid \ell$), and that $L(V, s)$ is convergent for $\mathrm{Re}(s) \geq \frac{i}{2} + 1$. By Deligne's proof of the Weil conjectures [D4], all this is known if one omits the factors where X has bad reduction. For the consideration of $m = i + 1 - n < \frac{i}{2}$, one has to assume meromorphic continuation of $L(V, s)$. The conjectured analytic behavior of $L(V, s)$ (see [D3]) is roughly described by the following picture.

functional equation $s \leftrightarrow i + 1 - s$
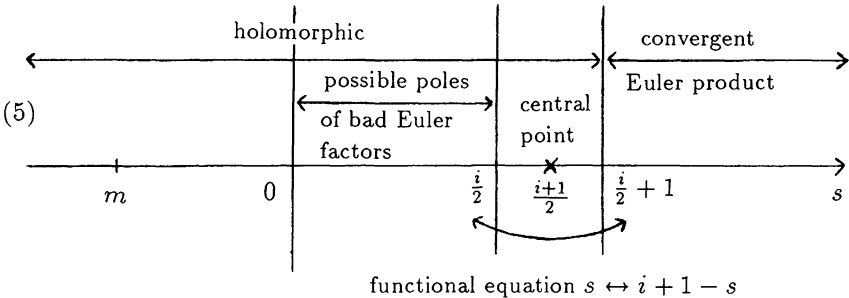
The equality

(6) $\quad \operatorname*{ord}_{s=m} L(V,s) = \dim_{\mathbf{R}} H^i(X,\mathbf{C})^+/(H^i(X,\mathbf{R}(n))^+ + F^n), m < \dfrac{i}{2},$

follows from the shape of the Γ-factors and the conjectured functional equation, see [**Be1**, 3.3] or [**J2**, 4.14.a].

REMARK 3: The domain $m < 0$ ($\Leftrightarrow i+1 < n$) can be considered as the "stable range" for various reasons:

a) One has $F^n H^i(X,\mathbf{C}) = 0$ for the Hodge filtration.
b) One expects equality $K_{2n-i-1}(X/\mathbf{Z})^{(n)} = K_{2n-i-1+}(X)^{(n)}$ by conjectures on the K-theory of schemes over finite fields (cf. [**Be1**, 2.4.2]).
c) The local Euler factors are expected to be holomorphic here (see also the discussion in §4).

Via a) and b), Beilinson's conjecture predicts

(7) $\quad K_{2n-i-1}(X)^{(n)} \otimes \mathbf{R} \xrightarrow{\;r\;}_{\sim} H^i(X,\mathbf{C})^+/H^i(X,\mathbf{R}(n))^+$ for $m < 0$.

Since $H^i(X,\mathbf{C}) = H^i(X,\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$ is an induced module for $\langle \sigma \rangle$, we have $\dim_{\mathbf{R}} H^i(X,\mathbf{C})^+ = \dim_{\mathbf{R}} H^i(X,\mathbf{R})$. Hence the dimension of the right-hand side of (7) is $-\chi_{i,n}(X)$. On the other hand, c) would give

(8) $\quad \operatorname*{ord}_{s=m} L(V,s) = \operatorname*{ord}_{s=m} L_S(V,s)$ for $m < 0,$

where $L_S(V,s) = \prod_{\mathfrak{p} \notin S} \det(1 - \mathrm{Fr}_{\mathfrak{p}}(N\mathfrak{p})^{-s} \mid H^i(\bar{X},\mathbf{Q}_\ell))^{-1}$ is the partial L-function.

We now consider an ℓ-adic analogue of the above. Let $H^*_{\mathrm{cont}}(X,\mathbf{Z}_\ell(n))$ be the continuous étale cohomology of X (see [**DF**] and [**J1**]). Then there is a Hochschild-Serre spectral sequence

(9) $\quad E_2^{p,q} = H^p(G_k, H^q(\bar{X},\mathbf{Z}_\ell(n))) \Rightarrow H^{p+q}_{\mathrm{cont}}(X,\mathbf{Z}_\ell(n)),$

see [**J1**, 3.4]. There are also Chern characters on $K_*(X)$ with values in $H^*_{\mathrm{cont}}(X,\mathbf{Q}_\ell(*)) = H^*_{\mathrm{cont}}(X,\mathbf{Z}_\ell(*)) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ (see [**Sou3**]), and, letting $H^\nu_{\mathrm{cont}}(X,\mathbf{Q}_\ell(n))_0 = \mathrm{Ker}(H^\nu_{\mathrm{cont}}(X,\mathbf{Q}_\ell(n)) \xrightarrow{\mathrm{res}} H^\nu(\bar{X},\mathbf{Q}_\ell(n)))$, (9) defines a

diagram

$$0$$
$$\downarrow$$

$$H^{\nu}_{\text{cont}}(X, \mathbb{Q}_{\ell}(n))_0 \xrightarrow{\ \pi\ } H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Q}_{\ell}(n)))$$

$$K_{2n-\nu}(X)^{(n)} \xrightarrow{\ r\ } H^{\nu}_{\text{cont}}(X, \mathbb{Q}_{\ell}(n))$$

$$\text{res} \downarrow$$

$$H^{\nu}(X, \mathbb{Q}_{\ell}(n))^{G_k} = 0 \text{ for } \nu \neq 2n$$

which should be considered as the ℓ-adic analogue of (4). Since $cd_{\ell}(G_k) \leq 2$, π is surjective. One can also show that res is surjective and that $\text{Ker}\,\pi = H^2(G_k, H^{\nu-2}(\bar{X}, \mathbb{Q}_{\ell}(n)))$, but this will not be needed in what follows.

CONJECTURE 2. *The regulator map*

$$r : K_{2n-i-1}(X)^{(n)} \bigotimes \mathbb{Q}_{\ell} \to H^1(G_k, H^i(\bar{X}, \mathbb{Q}_{\ell}(n)))$$

is an isomorphism for $i + 1 < n$ *(* $\Longleftrightarrow m := i + 1 - n < 0$ *).*

The analogy with Beilinson's conjecture becomes quite clear by the formulae

$$H^1(G_k, H^i(\bar{X}, \mathbb{Q}_{\ell}(n))) = \text{Ext}^1_{G_k}(\mathbb{Q}_{\ell}, H^i(\bar{X}, \mathbb{Q}_{\ell}(n))),$$
$$H^i(X, \mathbb{C})^+ / (H^i(X, \mathbb{R}(n))^+ + F^n) = \text{Ext}^1_{MH_{\mathbb{R}}}(\mathbb{R}, H^i(X, \mathbb{R}(n))),$$

where $MH_{\mathbb{R}}$ is the category of real Hodge structures over $\mathbb{R}$ (i.e., with "infinite Frobenius F_{∞}"), see [J2, 4.13]. In fact, the philosophy behind both conjectures is that the motivic cohomology $H^{i+1}_{\mathcal{M}} = K_{2n-i-1}(X)^{(n)}$ describes extensions of the trivial motive by the motive $H^i(X)(n)$, and that the regulator maps r are obtained by passing to the associated realizations, see [Be2, 5,10], [D5], and [J2, §4]. In contrast to the Hodge realization, we have a group $H^2(G_k, H^{\nu-2}(\bar{X}, \mathbb{Q}_{\ell}(n))) = \text{Ext}^2_{G_k}(\mathbb{Q}_{\ell}, H^{\nu-2}(\bar{X}, \mathbb{Q}_{\ell}(n)))$ for the ℓ-adic realizations, but comparison with Beilinson's conjecture and the general philosophy that there are "no non-trivial motivic 2-extensions over $\mathbb{Q}$" up to torsion (see [D5]) suggests neglecting this group for the regulator map.

All this is discussed in more detail in [**J3**], together with the function field analogue of Conjecture 2, which would follow from several well-known conjectures on K-theory and ℓ-adic cohomology of varieties over finite fields. In these notes we restrict ourselves to the connection with Conjecture 1. We use the following fact, which has also been observed by P. Schneider and W. Raskind.

LEMMA 4. *If $i \neq 2n - 2$, then the inflation map*

$$H^1(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) \overset{\inf}{\to} H^1(G_k, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$$

is an isomorphism and, consequently, $H^1(G_k, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$ is finite-dimensional.

PROOF: More generally, let M be a finitely generated, torsion-free $\mathbb{Z}_\ell$-module with continuous action of G_S. Then there is an exact sequence

$$0 \to H^1(G_S, M) \overset{\inf}{\to} H^1(G_k, M) \to H^1(G_{k_S}, M)^{G_S}$$

(by [**J1**, §2], or by (1) and the well-known case of finite coefficients). By assumption, G_{k_S} acts trivially on M. For every $\mathfrak{p} \notin S$ choose a decomposition group $G_\mathfrak{p} \subseteq G_k$ for $\mathfrak{p}$ and let $I_\mathfrak{p} \subset G_\mathfrak{p}$ be the inertia group. Then G_{k_S} is the smallest closed normal subgroup of G_k containing the $I_\mathfrak{p}$, $\mathfrak{p} \notin S$, and hence the restriction

$$H^1(G_{k_S}, M)^{G_S} = \operatorname{Hom}_{\mathrm{cont}, G_S}(G_{k_S}^{\mathrm{ab}}, M) \hookrightarrow \prod_{\mathfrak{p} \notin S} \operatorname{Hom}_{\mathrm{cont}, G_\mathfrak{p}}(I_\mathfrak{p}^{\mathrm{ab}}, M)$$

is injective. Since $\mathfrak{p} \nmid \ell$ for $\mathfrak{p} \notin S$, there are isomorphisms

$$\operatorname{Hom}_{\mathrm{cont}, G_\mathfrak{p}}(I_\mathfrak{p}^{\mathrm{ab}}, M) \cong \operatorname{Hom}_{\mathrm{cont}, G_\mathfrak{p}}(\mathbb{Z}_\ell(1), M) \cong M(-1)^{G_\mathfrak{p}}$$

by local class field theory. Hence the inflation is an isomorphism, if $M(-1)^{G_\mathfrak{p}} = 0$ for $\mathfrak{p} \notin S$. For $M = H^1(\bar{X}, \mathbb{Z}_\ell(n))/$ torsion, this is the case for $i \neq 2(n-1)$ by Lemma 3.

COROLLARY 2. *If Beilinson's conjecture is true, then Conjecture 1 a) is equivalent to the equality of dimensions in Conjecture 2.*

PROOF: As explained in Remark 3, Beilinson's conjecture implies

$$(11) \qquad \dim_\mathbb{Q} K_{2n-i-1}(X)^{(n)} = -\chi_{i,n}(X) \text{ for } i+1 < n.$$

Hence the claim is clear from Corollary 1 a) and Lemma 3.

Example 3 Consider $X = \operatorname{Spec} k$: It follows from results of Borel and Soulé that $K_{2n}(\mathcal{O}_k)$ is finite for $n > 0$ and that the ℓ-adic Chern class

$$c_{1,n} : K_{2n-1}(\mathcal{O}_k) \otimes \mathbb{Z}_\ell \to H^1_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_k[\tfrac{1}{\ell}]), \mathbb{Z}_\ell(n))$$

has finite kernel and cokernel for $n > 1$, where $\mathcal{O}_k$ is the ring of integers in k (see [**Sou1**] and [**Sou2**]). Now $H^1_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_k[\tfrac{1}{\ell}]), \mathbb{Z}_\ell(n)) = H^1(G_S, \mathbb{Z}_\ell(n))$ for $S = S_\ell$ (cf. (16) below), and $K_r(\mathcal{O}_k) \otimes \mathbb{Q} \xrightarrow{\sim} K_r(k) \otimes \mathbb{Q}$ for $r \geq 2$, as follows from the localization sequence and the finiteness of $K_m(F)$ for a finite field F and $m \neq 0$. Finally, $c_{1,n}$ vanishes on $K_{2n-1}(\mathcal{O}_k)^{(\nu)}$ for $\nu \neq n$ by standard properties of Chern classes. Putting all this together, one obtains the equality $K_{2n-1}(k) \otimes \mathbb{Q} = K_{2n-1}(k)^{(n)}$ for all $n \geq 1$ and the bijectivity of

$$r : K_{2n-1}(k)^{(n)} \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^1(G_k, \mathbb{Q}_\ell(n)), \quad n > 1,$$

using Lemma 4. Hence Conjecture 2 is true for $X = \operatorname{Spec} k$. Since, by Borel, Beilinson's conjectures are also true for $X = \operatorname{Spec} k$, we may deduce Conjecture 1 a) from this. The argument in [**Sou1**] is different: there it is proved that the Chern classes

$$c_{2,n} : K_{2n-2}(\mathcal{O}_k) \otimes \mathbb{Z}_\ell \to H^2_{\text{ét}}(\operatorname{Spec} \mathcal{O}_k[\tfrac{1}{\ell}], \mathbb{Z}_\ell(n))$$

have finite cokernels for $n > 1$.

Example 4 Let k be an imaginary quadratic field and let $X = E$ be an elliptic curve with complex multiplication by k. Assume that ℓ splits in k, $\ell = \mathfrak{p}\mathfrak{p}^*$, and that E has good reduction at ℓ. In [**Sou5**] Soulé considers a map

$$r' : \lim_{\overleftarrow{r}} K_{2n-2}(E, \mathbb{Z}/\ell^r) \to H^1(G_k, H^1(\bar{E}, \mathbb{Z}_\ell(n))), \quad n > 1.$$

Here $K_{2n-2}(E, \mathbb{Z}/\ell^r)$ denotes K-theory with coefficients, sitting in an exact sequence

$$0 \to K_{2n-2}(E)/\ell^r \to K_{2n-2}(E, \mathbb{Z}/\ell^r) \to K_{2n-3}(E)_{\ell^r} \to 0,$$

and the composition of r' with $K_{2n-2}(E) \to \lim_{\overleftarrow{r}} K_{2n-2}(E, \mathbb{Z}/\ell^r)$ coincides with the ℓ-adic Chern character up to some factorials after tensoring

with $\mathbb{Q}_\ell$. Soulé finds that for a regular prime ℓ the composition of r' with the restriction

$$\alpha_{\mathfrak{p}} : H^1(G_k, H^1(\bar{E}, \mathbb{Z}_\ell(n))) \to H^1(G_{\mathfrak{p}}, H^1(\bar{E}, \mathbb{Z}_\ell(n))), \quad n > 1$$

to a decomposition group $G_{\mathfrak{p}}$ at $\mathfrak{p}$ has a finite cokernel. On the other hand, it follows from the results of K. Wingberg mentioned in Example 2 that $\alpha_{\mathfrak{p}}$ has a finite kernel for regular ℓ. Hence r' has a finite cokernel, in support of Conjecture 2.

In concordance with a conjecture of Quillen, Thomason [**Th**] has shown that for $\mu > N := \frac{8}{3}(d+2)(d+3)(d+4) - 15$ the map

$$K_\mu(X, \mathbb{Z}/\ell^r) \to K_\mu^{\text{ét}}(X, \mathbb{Z}/\ell^r)$$

is a naturally split surjection, where $K_\mu^{\text{ét}}$ is the étale (or topological) K-theory. This implies that the Chern class

$$c_{\nu,n} : \varprojlim_r K_{2n-\nu}(X, \mathbb{Z}/\ell^r) \to \varprojlim_r H_{\text{ét}}^\nu(X, \mathbb{Z}/\ell^r(n))$$

has cokernel of finite exponent for $2n > \nu + n$. On the other hand, the spectral sequence (9) induces a surjective map

$$\varprojlim_r H_{\text{ét}}^\nu(X, \mathbb{Z}/\ell^r(n))_0 \xrightarrow{\pi} H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Z}_\ell(n))),$$

where $H_{\text{ét}}^\nu(X, \mathbb{Z}/\ell^r(n))_0 = \text{Ker}(H^\nu(X, \mathbb{Z}/\ell^r(n)) \xrightarrow{\text{res}} H^\nu(\bar{X}, \mathbb{Z}/\ell^r(n)))$. In fact, we have seen that (9) gives a surjective map

$$H_{\text{cont}}^\nu(X, \mathbb{Z}_\ell(n))_0 \xrightarrow{\pi} H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Z}_\ell(n))),$$

and this map factors through $\varprojlim_r H_{\text{ét}}^\nu(X, \mathbb{Z}/\ell^r(n))_0$, since $H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Z}_\ell(n))) = \varprojlim_r H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Z}/\ell^r(n)))$ is ℓ-complete, see [**J1**, 4.4 and 5.16]. Altogether we obtain a surjective map

$$\mathbb{Q}_\ell \bigotimes_{\mathbb{Z}_\ell} \varprojlim_r K_{2n-i-1}(X, \mathbb{Z}/\ell^r) \twoheadrightarrow H^1(G_k, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$$

for $2n > i + 1 + N$ (Note that $H^\nu(\bar{X}, \mathbb{Q}(n))^{G_k} = 0$ for $\nu \neq 2n$). By the exact sequences

$$0 \to K_\mu(X)^\wedge \to \varprojlim_r K_\mu(X, \mathbb{Z}/\ell^r) \to T_\ell K_{\mu-1}(X) \to 0,$$

where $K_\mu(X)^\wedge = \varprojlim_r K_\mu(X)/\ell^r$ is the ℓ-completion of $K_\mu(X)$, this would imply the surjectivity of the regulator map

$$(12) \qquad r : K_{2n-i-1}(X)^{(n)} \bigotimes \mathbb{Q}_\ell \to H^1(G_k, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$$

for $2n > i + 1 + N$, if $T_\ell K_{2n-i-2}(X) = 0$, i.e., if $K_{2n-i-2}(X)$ has no ℓ-divisible torsion subgroup. Here we have used two facts: for an abelian group A, a finitely generated $\mathbb{Z}_\ell$-module B and a homomorphism $f : A \to B$, the induced map $A \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is surjective if and only if the induced map $\hat{A} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is surjective (applied to $A = K_{2n-i-1}(X)$ and $B = H^1(G_k, H^i(\bar{X}, \mathbb{Z}_\ell(n))/H^i(\bar{X}, \mathbb{Z}_\ell(n))_{\text{tor}}))$, and the Chern character $ch_{i+1,n} : K_{2n-i-1}(X) \otimes \mathbb{Q} \to H^1(G_k, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$ vanishes on $K_{2n-i-1}(X)^{(r)}$ for $r \neq n$. Via the localization sequence

$$\cdots \to K_\mu(\mathcal{X}) \to K_\mu(X) \to \bigoplus_{\mathfrak{p} \notin S} K_{\mu-1}(\mathcal{X}_\mathfrak{p}) \to K_{\mu-1}(\mathcal{X}) \to \cdots,$$

where $\mathcal{X}$ is a smooth model of X over the ring of S-integers $\mathcal{O}_S$ and $\mathcal{X}_\mathfrak{p} = \mathcal{X} \times_{\mathcal{O}_S} \kappa(\mathfrak{p})$ is the fiber at $\mathfrak{p}$, the vanishing of $T_\ell K_\mu(X)$ for all μ would follow from Bass's conjecture that $K_\mu(Y)$ is finitely generated for a regular scheme Y of finite type over $\mathbb{Z}$.

In any case, if (12) is surjective, Beilinson's conjecture would also imply its injectivity. In fact, we would get $-\chi_{i,n}(X) = \dim_{\mathbb{Q}} K_{2n-i-1}(X)^{(n)} \geq \dim_{\mathbb{Q}_\ell} H^1(G_k, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$ and hence equality of dimensions by Corollary 1 a).

3. Connections with Iwasawa theory.

Let $M = H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ and define

$$d_{i,n} = \dim H^1(G_S, M(n)), \quad d'_{i,n} = \dim H^1(G_k, M(n)),$$
$$r_{i,n} = \dim H^2(G_S, M(n)), \quad r'_{i,n} = \dim H^2(G_k, M(n)).$$

LEMMA 5. For $i \neq 2n - 2$ these numbers are finite and independent of S, and one has $d_{i,n} = d'_{i,n}$, $r_{i,n} \geq r'_{i,n}$.

PROOF: For any ℓ-torsion G_S-module N and any set of places $T \supseteq S$, one has an exact sequence

$$(13) \quad 0 \to H^1(G_S, N) \overset{\text{inf}}{\to} H^1(G_T, N) \to \bigoplus_{\mathfrak{p} \in T \setminus S} N(-1)^{G_\mathfrak{p}}$$

$$\to H^2(G_S, N) \overset{\text{inf}}{\to} H^2(G_T, N) \to \bigoplus_{\mathfrak{p} \in T \setminus S} N(-1)_{G_\mathfrak{p}} \to 0,$$

compare [**Sou4**, p. 117]. If N is of cofinite type and if we assume

(14) $\qquad N(-1)^{G_\mathfrak{p}}$ and $N(-1)_{G_\mathfrak{p}}$ are finite for $\mathfrak{p} \in T \backslash S$

(when $N = H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$) this is the case for $i \neq 2(n-1)$ by the Weil conjectures), then we immediately get

$$\dim H^\nu(G_S, N) = \dim H^\nu(G_T, N), \quad \nu \geq 0, \quad \text{if } T \text{ is finite.}$$

If T is the set of all places, then $G_T = G_k$, and we still get

$$\dim H^1(G_S, N) = \dim H^1(G_k, N),$$
$$\dim H^2(G_S, N) \geq \dim H^2(G_k, N),$$

in view of the fact that $H^2(G_S, N)$ is cofinitely generated.

LEMMA 6. *Let* $f_S : \mathcal{X}_S \to \operatorname{Spec}\mathcal{O}_S$ *be a smooth and proper model of* X *over* $\mathcal{O}_S$, *the ring of* S-*integers in* k. *If* $\nu \neq 2n-1, 2n, 2n+1$, *then*

$$\dim H^\nu(\mathcal{X}_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = d_{\nu-1,n} + r_{\nu-2,n}$$
$$= -\chi_{\nu-1,n} + r_{\nu-1,n} + r_{\nu-2,n},$$
$$\dim H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = d'_{\nu-1,n} + r'_{\nu-2,n}$$
$$= -\chi_{\nu-1,n} + r_{\nu-1,n} + r'_{\nu-2,n}.$$

PROOF: The first equality follows from the Leray spectral sequence

(15) $\qquad E_2^{p,q} = H^p(\operatorname{Spec}\mathcal{O}_S, R^q f_{S*}\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \Rightarrow H^{p+q}(\mathcal{X}_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)).$

Namely, by base change one has an isomorphism of sheaves

$$R^q f_{S*}\mathbb{Q}_\ell/\mathbb{Z}_\ell(n) \cong j_* R^q f_*\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$$

for the morphisms $f : X \to \operatorname{Spec} k$ and $j : \operatorname{Spec} k \hookrightarrow \operatorname{Spec}\mathcal{O}_S$, where $R^q f_*\mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$ can be identified with the G_k-module $H^q(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. Since

(16) $\qquad H^p(\operatorname{Spec}\mathcal{O}_S, j_*N) = H^p(G_S, N)$

for an ℓ-torsion G_S-module N (regarded as a sheaf on $\operatorname{Spec} k$ on the left-hand side), cf. [**Mi2**, II.2.9], we can rewrite (15) as

(17) $\qquad E_2^{p,q} = H^p(G_S, H^q(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \Rightarrow H^{p+q}(\mathcal{X}_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)).$

This implies the desired equality, since all groups occurring in (17) are cofinitely generated and $H^q(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_k}$ is finite for $q \neq 2n$. For the latter reason, we have by Lemma 2

(18)
$$-\chi_{i,n} = d_{i,n} - r_{i,n} \text{ for } i \neq 2n,$$

which shows the second equality.

For the third equality, we use the Hochschild-Serre spectral sequence

(19)
$$E_2^{p,q} = H^p(G_k, H^q(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \Rightarrow H^{p+q}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)).$$

Since these groups in general are not cofinitely generated, we have to be a little bit careful with adding up dimensions in an exact sequence. Set $H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_0 = \text{Ker}(H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \overset{\text{res}}{\to} H^\nu(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$. Then (19) gives an exact sequence

(20)
$$H^{\nu-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))^{G_k} \to H^2(G_k, H^{\nu-2}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$
$$\to H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_0 \to H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \to 0,$$

since $cd_\ell(G_k) \leq 2$. Now the second group in (20) is the direct sum of its maximal ℓ-divisible subgroup and a group of finite exponent (since $H^2(G_k, N)$ is ℓ-divisible for an ℓ-divisible module N), and the first group is finite for $\nu - 1 \neq 2n$. From this we easily deduce

$$\dim H^2(G_k, H^{\nu-2}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) + \dim H^1(G_k, H^{\nu-1}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$
$$= \dim H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))_0 \qquad \text{for } \nu \neq 2n+1$$
$$= \dim H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \qquad \text{if in addition } \nu \neq 2n,$$

and the last claimed equality again follows with (18). $\qquad\square$

Let $k_{\ell c} = k(\mu_{\ell^\infty})$ be the ℓ-cyclotomic extension of k. Then $\text{Gal}(k_{\ell c}/k) \cong \Delta \times \Gamma$ with $\Gamma \cong \mathbb{Z}_\ell$ and $\#\Delta \mid (\ell - 1)$. Let $H_S = \text{Gal}(k_S/k_{\ell c})$ and define

$$Y := Y_i := H^1(H_S, M)^*, \qquad \tilde{Y} := \tilde{Y}_i := H^1(H_S, \tilde{M})^*,$$
$$Z := Z_i := H^1(H_S, M)^*, \qquad \tilde{Z} := \tilde{Z}_i := H^1(H_S, \tilde{M})^*,$$

where $\tilde{M} = \ell - \text{Div}(M)$ is the maximal divisible subgroup of M. These are compact Λ-modules, where $\Lambda = \mathbb{Z}_\ell[[\Gamma]]$ is the completed group ring. Let $\chi : \text{Gal}(k_{\ell c}/k) \to \mathbb{Z}_\ell^\times$ be the ℓ-cyclotomic character (defined by $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for $\zeta \in \mu_{\ell^\infty}$ and $\sigma \in \text{Gal}(k_{\ell c}/k))$ and let $e_n = (\#\Delta)^{-1} \sum_{\tau \in \Delta} \chi^{-n}(\tau)\tau \in \mathbb{Z}_\ell[\Delta]$ be the idempotent projecting to the part where Δ acts via χ^n ($n \in \mathbb{Z}$). Of course, e_n only depends on $n \bmod \#\Delta$.

LEMMA 7. a) *For each* $n \in \mathbb{Z}$, Ye_n *and* Ze_n *are noetherian* Λ-*modules, and their ranks are independent of* S.

    b) $\mathrm{rank}_\Lambda Ye_n - \mathrm{rank}_\Lambda Ze_n = -\chi_{i,n}(X)$.

    c) $\tilde{Z}e_n$ *is a free* Λ-*module for all* $n \in \mathbb{Z}$.

PROOF: a) Let $H'_S = \mathrm{Gal}(k_S/k_\infty)$, where k_∞/k is the unique $\mathbb{Z}_\ell$-extension contained in $k_{\ell c}/k$. Then we have canonical isomorphisms

$$(21) \quad H^r(H'_S, M(n)) \cong H^r(H_S, M(n))^\Delta \cong (H^r(H_S, M)e_{-n})(n), \quad r \geq 0,$$

since $\ell \nmid \#\Delta$, so the dual of this module is noetherian if and only if $H^r(H_S, M)^* e_n$ is. Now the Hochschild-Serre spectral sequence for $1 \to H'_S \to G_S \to \Gamma \to 1$ gives exact sequences

$$(22) \quad 0 \to H^1(\Gamma, M(n)^{H'_S}) \to H^1(G_S, M(n)) \to H^1(H'_S, M(n))^\Gamma \to 0,$$

$$(23) \quad 0 \to H^1(\Gamma, H^1(H'_S(M(n)))) \to H^2(G_S, M(n)) \to H^2(H'_S, M(n))^\Gamma \to 0,$$

and an isomorphism

$$(24) \qquad H^1(\Gamma, H^2(H'_S, M(n))) \cong H^3(G_S, M(n)) = 0,$$

since $cd_\ell \Gamma = 1$, $cd_\ell G_S \leq 2$ and $cd_\ell H'_S \leq 2$. Since $H^r(G_S, M(n))$ is cofinitely generated for all r and n, the same is true for $H^r(H'_S, M(n))^\Gamma$. Hence $(H^r(H'_S, M(n))^*)_\Gamma$ is finitely generated over $\mathbb{Z}_\ell$. This implies that $H^r(H'_S, M(n))^*$ is finitely generated over Λ. For the second claim let T be a finite set of places containing S. By applying (13) to each intermediate layer $k \subseteq k_r \subseteq k_\infty$ and passing to the limit over r, we get an exact sequence

$$(25) \quad 0 \to H^1(H'_S, M(n)) \to H^1(H'_T, M(n)) \to \bigoplus_{\mathfrak{p} \in T \backslash S(k_\infty)} M(n-1)$$
$$\to H^2(H'_S, M(n)) \to H^2(H'_T, M(n)) \to 0,$$

since $\varprojlim_r M(n-1)_{\mathrm{Gal}(k_\infty/k_r)} = 0$ (the limit is taken via norms). The set $T \backslash S(k_\infty)$ of places in k_∞ above $T \backslash S$ is finite, since all primes $\mathfrak{p}$ decompose into only finitely many primes in the cyclotomic extension. Therefore $\left(\bigoplus_{\mathfrak{p} \in T \backslash S(k_\infty)} M(n-1) \right)^*$ is a Λ-torsion module (in fact, finitely generated over $\mathbb{Z}_\ell$), and the claim follows, in view of (21).

    For b), we use the fact that

$$(26) \qquad \mathrm{rank}_\Lambda W = \mathrm{rank}_{\mathbb{Z}_\ell} W_\Gamma - \mathrm{rank}_{\mathbb{Z}_\ell} W^\Gamma$$

for a noetherian Λ-module W (see, e.g., [**Wi1** (1.4)]). Hence by (22) and (23) we get

$$\text{rank}_\Lambda Y e_n - \text{rank}_\Lambda Z e_n = \text{rank}_\Lambda H^1(H'_S, M(n))^* - \text{rank}_\Lambda H^2(H'_S, M(n))^*$$
$$= \dim H^1(G_S, M(n)) - \dim H^2(G_S, M(n)) - \dim H^1(\Gamma, M(n)^{H'_S}) = -\chi_{i,n},$$

since $\dim H^1(\Gamma, M(n)^{H'_S}) = \dim M(n)^{G_S}$ by the exact sequence

$$0 \longrightarrow N^\Gamma \longrightarrow N \xrightarrow{\gamma-1} N \longrightarrow N_\Gamma \longrightarrow 0$$

$$\parallel \qquad\qquad\qquad\qquad\qquad \parallel$$

$$M(n)^{G_S} \qquad\qquad\qquad\qquad H^1(\Gamma, N)$$

for $N = M(n)^{H'_S}$ and γ a topological generator of Γ.

For c), we use the criterion (see [**Wi1** (1.2)])

(27) $\qquad\qquad W$ free $\iff W^\Gamma = 0$ and W_Γ is torsion-free

for the Λ-module $W = H^2(H'_S, \tilde{M}(n))^*$. Namely, $H^2(H'_S, \tilde{M}(n))_\Gamma \cong H^1(\Gamma, H^2(H'_S, \tilde{M}(n))) = 0$ by (24), and $H^2(H'_S, \tilde{M}(n))^\Gamma$ is ℓ-divisible as quotient of the ℓ-divisible group $H^2(G_S, \tilde{M}(n))$. $\qquad\qquad\square$

Iwasawa's conjecture "$\mu = 0$" for all number fields K with $k \subseteq K \subseteq k_S$ would imply $cd_\ell(H_S) = 1$ (see [**Sch3**, 3.8]) and in particular the vanishing of $Z = H^2(H_S, M)^*$. In any case, we have

LEMMA 8. Let $G_{S,r} = \text{Gal}(k_S/k_r)$, where $k_r = k(\mu_{\ell^r}) \cap k_\infty$. Then the following statement are equivalent for $n \in \mathbf{Z}$:

  a) $Z e_n$ is a Λ-torsion module.
  b) $H^2(H_S, \tilde{M})e_{-n} = 0$.
  c) $\dim H^2(G_{S,r}, M(n))$ is bounded for $r \geq 1$.
  c') For all $n' \in \mathbf{Z}$ with $n' \equiv n \bmod \#\Delta$, $\dim H^2(G_{S,r}, M(n'))$ is bounded for $r \geq 1$.
  d) $H^2(G_S, M(n))$ is finite for one $n' \in \mathbf{Z}$ with $n' \equiv n \bmod \#\Delta$.
  d') $H^2(G_S, M(n'))$ is finite for almost all $n' \in \mathbf{Z}$ with $n' \equiv n \bmod \#\Delta$.

PROOF: Since $\text{rank}_\Lambda Z e_n = \text{rank}_\Lambda \tilde{Z} e_n$, and $\tilde{Z} e_n \cong H^2(H'_S, \tilde{M}(n))^*(n)$ is Λ-free, the equivalence of a) and b) is clear. The equivalence with c) and d) follows from Lemma 1 and the exact sequences

(28) $\quad 0 \to H^1(\Gamma_r, H^1(H'_S, \tilde{M}(n'))) \to H^2(G_{S,r}, \tilde{M}(n'))$
$$\to H^2(H'_S, \tilde{M}(n'))^{\Gamma_r} \to 0,$$

$\Gamma_r = \mathrm{Gal}(k_\infty/k_r)$. Indeed, $\dim(\Lambda^*)^{\Gamma_r} = [k_r : k]$, and for a noetherian Λ-module W the numbers

$$\dim H^1(\Gamma_r, W^*(\nu))$$

are bounded for $r \geq 0$, $\nu \in \mathbb{Z}$, and zero for almost all twists $\nu \in \mathbb{Z}$, cf. [Sou4, Lemma 2]. Note that $H^r(H'_S, \tilde{M}(n'))(\nu) = H^r(H'_S, \tilde{M}(n' + \nu))$ for $\nu \equiv 0 \bmod \#\Delta$. The equivalence with c$'$) and d$'$) is now clear, since a) and b) only depend on $n \bmod \#\Delta$. $\qquad\square$

In general, we obtain from (23)

$$\text{(29)} \quad \begin{aligned} r_{i,n} = \dim H^2(G_S, M(n)) &= \mathrm{rank}_\Lambda\, Z_i e_n + \mathrm{rank}_{\mathbb{Z}_\ell}(Y_i e_n(-n))^\Gamma \\ &= \mathrm{rank}_\Lambda\, Z_i e_n \quad \text{for almost all } n \in \mathbb{Z}, \end{aligned}$$

and, via Lemma 6 and Corollary 4 below,

$$\text{(30)} \quad \dim H^\nu(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = -\chi_{\nu-1,n} + \mathrm{rank}_\Lambda\, Z_{\nu-1} e_n$$

$$\text{for almost all } n \in \mathbb{Z}.$$

These are the correct versions of Proposition 3 and Theorem 1 ii) in [Sou4], as long as we do not know the vanishing of $\tilde{Z}$. The gap in *loc. cit.* 1.5 is that one cannot conclude $cd_p H = 1$ from $cd_p H_p = 1$ without any extra argument, but the reasoning there still gives the following interesting result:

LEMMA 9. *Let K be a finite extension of k over which $\tilde{M}(n)_\ell$ becomes a trivial Galois module, let $K_\infty = K \cdot k_\infty$ be the cyclotomic $\mathbb{Z}_\ell$-extension of K, and let $\mathcal{M}$ be the maximal abelian ℓ-extension of K_∞ which is unramified outside ℓ. If the μ-invariant of the noetherian $\mathbb{Z}_\ell[[\mathrm{Gal}(K_\infty/K)]]$-module $\mathrm{Gal}(\mathcal{M}/K_\infty)$ is zero, then $H^2(H_S, \tilde{M})e_{-n} = 0$.*

PROOF: It follows from the assumption on the μ-invariant that $H'_S(K)(\ell)$, the maximal pro-ℓ-factor group of $H'_S(K) = \mathrm{Gal}(k_S/K_\infty)$, is pro-$\ell$-free (see [Wi1, 5.4 and 5.6]). Now by definition G_K lies in the kernel of the composite of the two homomorphisms

$$G_k \xrightarrow{\rho} \mathrm{Aut}(\tilde{M}(n)) \xrightarrow{\pi} \mathrm{Aut}(\tilde{M}(n)_\ell)$$

given by the action of G_k on $\tilde{M}(n)$ and the restriction to $\tilde{M}(n)_\ell$. Since $\mathrm{Ker}\,\pi$ is a pro-ℓ-group, G_K and consequently $H'_S(K)$ act on $\tilde{M}(n)$ via their pro-ℓ-quotients. Hence we can consider the inflation

$$H^2(H'_S(K)(\ell), \tilde{M}(n)) \xrightarrow{\mathrm{inf}} H^2(H'_S(K), \tilde{M}(n)),$$

and this map is an isomorphism (see [**Neu**, Cor. 1]). Furthermore the corestriction

$$H^2(H'_S(K), \tilde{M}(n)) \overset{\text{cor}}{\twoheadrightarrow} H^2(H'_S, \tilde{M}(n))$$

is surjective, since $cd_\ell(H'_S) \leq 2$ (see [**Se1**, I 3.3]). It follows that $cd_\ell(H'_S(K)(\ell)) \leq 1$ implies $H^2(H'_S, \tilde{M}(n)) = 0$, hence the result, by (21). $\square$

The conjectured vanishing of $\tilde{Z}$, which would follow both from Iwasawa's "$\mu = 0$" conjecture or from Conjecture 1, may be more easily accessible than either of these. For example, in the case $X = \text{Spec } k$ one can in fact prove $H^2(H_S, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ (which is also called the weak Leopoldt conjecture) without proving anything about $cd_\ell H_S$ or Conjecture 1, see [**Sch1**, 4.7].

4. Global results.

Conjecture 1 is in part motivated by the following result.

THEOREM 1. *The analogue of Conjecture 1 is true in the function field case.*

This follows from the following, more general, fact.

THEOREM 2. *Let U be a smooth curve over a finite field F_q and let F be a smooth (= twisted-constant) $\mathbb{Q}_\ell$-sheaf on U ($\ell \neq p = \text{char } \mathsf{F}_q$) which is mixed [D4, 1.2.2]. Then $H^2(U, F) = 0$ if*

    a) $F^\vee(2)$ *is entire [D4, 3.3.2], or*

    b) *the weights in F are ≥ 0.*

(Here $F^\vee$ is the dual of F, and $F^\vee(n)$ is its n-fold Tate twist.)

COROLLARY 3. *Let F be smooth of weight i (see [D4, 1.2.2]: for each closed point $x \in U$ the eigenvalues α of the geometric Frobenius Fr_x on the stalk $F_{\bar{x}}$ are algebraic numbers with absolute value $|\alpha| = (Nx)^{i/2}$ for every complex embedding). Assume that F is positive (or effective), by which we mean the following: F is entire (i.e., the α are algebraic integers), and $F^\vee(-i)$ is entire. Then $H^2(U, F(n)) = 0$ if*

    a) $i + 1 < n$, *or*

    b) $i + 1 > 2n$.

PROOF OF COROLLARY 3: If $F^\vee(-i)$ is entire, then $F(n)^\vee(2) = F^\vee(-i)(i+2-n)$ is entire for $i+2-n \leq 0$, since $\mathbf{Q}_\ell(\nu)$ is entire for $\nu \leq 0$. If the weight of F is i, then the weight of $F(n)$ is $i - 2n$, which is ≥ 0 for $i + 1 > 2n$.

COROLLARY 3 $\Rightarrow$ THEOREM 1: Let X be a smooth, proper variety over the function field $k = \mathbf{F}_q(U)$ of U and assume that X has a smooth, proper model $f : \mathcal{X} \to U$ (i.e., good reduction over U). By smooth and proper base change, $F = R^i f_* \mathbf{Q}_\ell$ is a twisted-constant sheaf on U, and for each $x \in U$ there is an isomorphism $F_{\bar{x}} \cong H^i(\bar{\mathcal{X}}_x, \mathbf{Q}_\ell)$, where $\mathcal{X}_x = \mathcal{X} \times_U \operatorname{Spec} \kappa(x)$ is the fiber of f at x and $\bar{\mathcal{X}}_x = \mathcal{X}_x \times_{\kappa(x)} \overline{\kappa(x)}$ for a separable closure $\overline{\kappa(x)}$ of the residue field $\kappa(x)$ of x. If U is affine, and $\eta = \operatorname{Spec} k$ is the generic point of U, then there are canonical isomorphisms

$$H^r(U, F(n)) \cong H^r(\pi_1(U, \bar{\eta}), F(n)_{\bar{\eta}}) \cong H^r(\pi_1(U, \bar{\eta}), H^i(\bar{X}, \mathbf{Q}_\ell(n))),$$

for all $r \geq 0$, where $\bar{\eta}$ is the geometric point over η given by $\operatorname{Spec} \bar{k}$, see [Mi2, II 2.9]. In this situation, U is the analogue of $\operatorname{Spec} \mathcal{O}_S$; note that by definition $G_S = \pi_1(\operatorname{Spec} \mathcal{O}_S, \bar{\eta})$ in the number field situation.

To apply Corollary 3, we have to show that F is positive. But by Deligne's proof of the Weil conjectures, $H^i(\bar{\mathcal{X}}_x, \mathbf{Q}_\ell)$ is entire of weight i for $x \in U$ closed, and by Poincaré duality and hard Lefschetz [D4, 4.1.1]

$$H^i(\bar{\mathcal{X}}_x, \mathbf{Q}_\ell)^\vee(-i) \cong H^{2d-i}(\bar{\mathcal{X}}_x, \mathbf{Q}_\ell)(d-i) \cong H^i(\bar{\mathcal{X}}_x, \mathbf{Q}_\ell)$$

is again entire ($d = \dim X$).

PROOF OF THEOREM 2: If the weights in F are $\geq r$, then the weights in $F^\vee$ are $\leq -r$. By Deligne's fundamental result [D4, 3.3.4], the weights w' occurring in $H^1_c(\bar{U}, F^\vee(1))$, the cohomology with compact support of $\bar{U} = U \times_{\mathbf{F}_q} \bar{\mathbf{F}}$, satisfy $w' \leq -r + 1 - 2 = -r - 1$ (note that $\mathbf{Q}_\ell(1)$ has weight -2), and if $F^\vee(2)$ is entire, we have $w' - 2 \geq 0$, see [D4, 3.3.3]. By Poincaré duality, we have

$$H^1(\bar{U}, F) \cong H^1_c(\bar{U}, F^\vee(1))^\vee.$$

Hence for the weights w occurring in this mixed $G_{\mathbf{F}_q}$-module, where $G_{\mathbf{F}_q} = \operatorname{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q)$, we have $w \geq r + 1$, and $w \leq -2$ under assumption a). Similarly, $H^2(\bar{U}, F) \cong H^0_c(\bar{U}, F^\vee(1))^\vee$ has weights $\geq r + 2$, and ≤ -2 under assumption a). We apply this to the exact sequence

$$(31) \qquad 0 \to \underbrace{H^1(\bar{U}, F)}_{\substack{G_{\mathbf{F}_q} \\ w \leq -2, \text{ case a)} \\ w \geq 1, \text{ case b)}}} \to H^2(U, F) \to \underbrace{H^2(\bar{U}, F)^{G_{\mathbf{F}_q}}}_{\substack{w \leq -2, \text{ case a)} \\ w \geq 2, \text{ case b)}}} \to 0$$

obtained from the Hochschild-Serre spectral sequence for $\bar{\mathbf{F}}_q/\mathbf{F}_q$. We conclude that $H^2(U, F) = 0$ in both cases, since a mixed $G_{\mathbf{F}_q}$-module can have a non-zero fixed or cofixed module only if the weight zero occurs.

REMARK 4: The bounds in Theorem 2 seem to be sharp, and I expect the same of the bounds in Conjecture 1, except for an improvement by hard Lefschetz: $H^i(\bar{X}, \mathbb{Q}_\ell) \cong H^{2d-i}(\bar{X}, \mathbb{Q}_\ell(d-i))$, which would imply $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = 0$ for $n > d+1$.

Question 1   Is the analogue of Theorem 2 true in the number field case (U being replaced by $\operatorname{Spec}\mathcal{O}_S$ for a finite set S of primes containing all primes above ℓ)?

By the same arguments as above, a positive answer to this question would imply Conjecture 1. So far, one does not know any number field analogue of Deligne's theorem that $H^\nu_c(\bar{U}, F)$ is mixed of weights $\leq \nu + r$, if F is mixed of weights $\leq r$, or, what is more or less the same, the analogous result for $H^\nu(\bar{Y}, G)$, where Y is the smooth compactification of U. According to the general philosophy of Iwasawa theory, the analogue of $H^\nu(\bar{Y}, G)$ should be a certain Iwasawa module over k_∞ — for example, the Galois group $\operatorname{Gal}(L/k_\infty)$ of the maximal abelian unramified ℓ-extension L of k_∞ for $\nu = 1$ and $G = \mathbb{Q}_\ell(1)$. But no integrality results exist in this setting and perhaps cannot be expected in a "naive" way, cf. [**Wa2**]. Is it possible that, nevertheless, there exists some theory of weights for these Iwasawa modules?

We return to number fields and to the notations at the beginning of §1. It turns out that the numbers $r'_{i,n} = \dim H^2(G_k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ are much more accessible than the numbers $r_{i,n} = \dim H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$.

THEOREM 3. *Let P be a set of primes of k of density 1.*

a) *The localization map (induced by the restrictions to the decomposition groups $G_\mathfrak{p}$)*

$$\alpha_1^{i,n} : H^1(G_k, H^i(\bar{X}, \mathbb{Z}_\ell(n))) \to \prod_{\mathfrak{p} \in P} H^1(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Z}_\ell(n)))$$

has a finite kernel for $i \neq 2n$.

b) *Let $\tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n)) = H^i(\bar{X}, \mathbb{Z}_\ell(n))/H^i(\bar{X}, \mathbb{Z}_\ell(n))_{\mathrm{tors}}$, where A_{tors} is the torsion subgroup of an abelian group A. Then*

$$\tilde{\alpha}_1^{i,n} : H^1(G_k, \tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n))) \to \prod_{\mathfrak{p} \in P} H^1(G_\mathfrak{p}, \tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n)))$$

is injective for $i \neq 2n$.

c) *The localization map*

$$\alpha_2^{i,n} : H^2(G_k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \to \bigoplus_{\text{all } \mathfrak{p}} H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

has a finite kernel and cokernel for $i \neq 2(n-1)$.

d) *Let $\tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) = \ell - \mathrm{Div}\, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. Then*

$$\tilde{\alpha}_2^{i,n} : H^2(G_k, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \to \bigoplus_{\mathfrak{p} \in S} H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

is an isomorphism for $i \neq 2(n-1)$.

PROOF: This follows directly from results of Serre: For a profinite group G and a topological G-module M let

$$H^1_*(G, M) = \mathrm{Ker}\left(H^1(G, M) \overset{\text{res}}{\to} \prod_C H^1(C, M)\right),$$

where C runs over all (pro-) cyclic subgroups of G (and the cohomology is the continuous one, as always). If $\mathcal{G} = \mathrm{Im}(G \to \mathrm{Aut}(M))$, then

$$H^1_*(\mathcal{G}, M) \overset{\text{inf}}{\underset{\sim}{\to}} H^1_*(G, M),$$

[**Se2**, Prop. 6]. We now consider $M = H^i(\bar{X}, \mathbb{Z}_\ell(n))$, $i \neq 2n$, and $G = G_k$. Then, by the Weil conjectures proved by Deligne, for a Frobenius $\mathrm{Fr}_\mathfrak{p} \in \mathcal{G}$ at $\mathfrak{p} \notin S$ its eigenvalues on M are of weight $i - 2n \neq 0$ and thus satisfy the condition in Lemma 2 of [**Se3**]. Hence the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ satisfies the hypothesis of Theorem 1 in [**Se3**], and we get

$$H^r(\mathfrak{g}, M \otimes \mathbb{Q}_p) = 0 \text{ for all } r \geq 0.$$

Hence

$$H^r(\mathcal{G}, M) \text{ is finite for all } r \geq 0$$

by results of Lazard, which show that $H^r(\mathcal{G}, M)$ is a finitely generated $\mathbb{Z}_\ell$-module for all r and that there is an injection $H^r(\mathcal{G}, M) \otimes \mathbb{Q}_\ell \hookrightarrow H^r(\mathfrak{g}, M \otimes \mathbb{Q}_\ell)$, cf. [**Se3**, Cor. to Lemma 3]. Moreover, since $(M \otimes \mathbb{Q}_p)^{\mathrm{Fr}_\mathfrak{p}} = 0$, we have

$$H^1_*(\mathcal{G}, \tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n))) = 0 \qquad (i \neq 2n),$$

see [**Se2**, Theorem 1]. This implies b), since the kernel of the map in b) is contained in $H^1_*(\mathcal{G}, \tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n)))$ ([**Se2**, Prop. 7 and 8] — the proof also works for a set of primes of density 1, cf. [**Se4**, II Notes].

For a) we may assume without restriction that $S \cap P = \emptyset$. Then $\tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n))^{G_\mathfrak{p}} = 0$ for all $\mathfrak{p} \in P$ ($i \neq 2n$). Thus the claim easily follows via the exact sequence

$$0 \to H^i(\bar{X}, \mathbb{Z}_\ell(n))_{\mathrm{tors}} \to H^i(\bar{X}, \mathbb{Z}_\ell(n)) \to \tilde{H}^i(\bar{X}, \mathbb{Z}_\ell(n)) \to 0,$$

and the fact that the localization map for the finite module $H^i(\bar{X}, \mathbb{Z}_\ell(n))_{\mathrm{tors}}$ has a finite kernel by the theorem of Tate and Poitou [**Ta1**, 3.1].

When P is the set of all primes, the cited global duality theorem also gives an exact sequence

$$(32) \quad 0 \to \left(\mathrm{Ker}\,\alpha_1^{2d-i,d+1-n}\right)^* \to H^2(G_k, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$
$$\xrightarrow{\alpha_2^{i,n}} \bigoplus_{\mathrm{all}\ \mathfrak{p}} H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \to H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))_{G_k} \to 0,$$

by passing to the limit over the corresponding exact sequences for $H^i(\bar{X}, \mathbb{Z}/\ell^r(n))$. Here we have used the fact that

$$(33) \quad \mathrm{Hom}\big(H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\big) \cong H^{2d-i}(\bar{X}, \mathbb{Z}_\ell(d+1-n))$$

by Poincaré duality, and that

$$(34) \quad H^0(G_k, \mathrm{Hom}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)))^* \cong \mathrm{Hom}_{G_k}(A(-1), \mathbb{Q}_\ell/\mathbb{Z}_\ell)^* \cong A(-1)_{G_k}$$

for the ℓ-torsion G_k-module $A = H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. Now $\mathrm{Ker}\,\alpha_1^{2d-i,d+1-n}$ is finite for $i \neq 2(n-1)$ by a), and $H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))_{G_k}$ is finite for $i \neq 2(n-1)$ by the Weil conjectures, which shows c).

If we replace $H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ by $\tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ in (32), then $\mathrm{Ker}\,\alpha_1^{2d-i,d+1-n}$ is replaced by $\mathrm{Ker}\,\tilde{\alpha}_1^{2d-i,d+1-n}$, since (33) induces an isomorphism

$$(35) \quad \mathrm{Hom}\big(\tilde{H}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)), \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)\big) \cong \tilde{H}^{2d-i}(\bar{X}, \mathbb{Z}_\ell(d+1-n)).$$

Furthermore, the last group in (32) is replaced by $\tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))_{G_k}$, which is zero for $i \neq 2(n-1)$, as a finite and ℓ-divisible group. Finally, by Tate's local duality theorem [**Ta1**, 2.1] and (34) we get

$$(36) \quad H^2(G_\mathfrak{p}, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \cong \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))_{G_\mathfrak{p}}$$

for all primes $\mathfrak{p}$. For $\mathfrak{p} \notin S$ this is zero for $i \neq 2(n-1)$ by the Weil conjectures and the same arguments as above (good reduction, $\mathfrak{p} \nmid \ell$). Thus d) follows from b).

COROLLARY 4. $H^2(G_k, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) = 0$ (equivalently, $r'_{i,n} = 0$) for almost all $n \in \mathbb{Z}$.

PROOF: This follows from Theorem 3 d) and (36), since obviously $\tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))_{G_\mathfrak{p}} \neq 0$ only for finitely many n (at most $\dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell)$ many for fixed $\mathfrak{p}$).

For the Tate module $T_\ell A$ of an abelian variety A, the injectivity of the localization map $H^1(G_k, T_\ell A) \to \prod_\mathfrak{p} H^1(G_\mathfrak{p}, T_\ell A)$ has been shown by Wake, by a different method (see [**Wak**] and also [**Mi2**, I p. 109 ff.]). Conjecturally, $\operatorname{Ker} \alpha_1^{i,n}$ and $\operatorname{Ker} \alpha_2^{i,n}$ are finite for all $n \in \mathbb{Z}$. Namely, Grothendieck and Serre have conjectured that the action of G_k on $H^i(\bar{X}, \mathbb{Q}_\ell)$ is semi-simple, and one has

LEMMA 10. If the action of G_k on $H^i(\bar{X}, \mathbb{Q}_\ell)$ is semi-simple, then the localization maps $\alpha_1^{i,n}$ and $\alpha_2^{i,n}$ have finite kernels for all $n \in \mathbb{Z}$.

PROOF: We have only to consider α_1, since then the case of α_2 follows by the global duality theorem. Let $M = H^i(\bar{X}, \mathbb{Z}_\ell(n))$ and

$$M^\delta = \bigcup_{K/k} M^{G_K},$$

where K runs over all finite extensions of k. By the assumed semi-simplicity, the exact sequence

$$0 \to M^\delta \underset{r}{\overset{i}{\rightleftarrows}} M \to M/M^\delta \to 0$$

has a rational retraction, i.e., a map r as indicated with $r \circ i = \ell^\nu$ for some $\nu \geq 0$, and one has $(M/M^\delta)^\delta = 0$. By the existence of r, it suffices to show that α_1 has a finite kernel for M^δ and for M/M^δ (the kernels are always finitely generated over $\mathbb{Z}_\ell$). By passing to a finite extension of k and using the Hochschild-Serre spectral sequence, one easily reduces to the case where G_k acts trivially on M^δ. But α_1 is even injective for the module $\mathbb{Z}_\ell$, since G_k is generated by the $G_\mathfrak{p}$ for $\mathfrak{p} \in P$. On the other hand, let $\mathcal{G} = \operatorname{Im}(G_k \overset{\rho}{\to} \operatorname{Aut}(M/M^\delta))$ and let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}$. Then

$((M/M^\delta) \otimes \mathbb{Q}_\ell)^{\mathfrak{g}} = ((M/M^\delta) \otimes \mathbb{Q}_\ell)^\delta$ (see [L, V 2.4.10], and this space is zero as remarked above. Since $\mathfrak{g}$ is semi-simple by assumption this implies $H^r(\mathfrak{g}, (M/M^\delta) \otimes \mathbb{Q}_\ell) = 0$ for all $r \geq 0$, see [ChE, Theorem 24.1], and we may proceed as in the proof of Theorem 3 a).

5. The local case.

We now investigate the local groups $H^2(G_\mathfrak{p}, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$.

LEMMA 11. a) Let $m = i + 1 - n$; then

$$\dim H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) = \dim_{\mathbb{Q}_\ell} H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$$
$$= \dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell(n-1))_{G_\mathfrak{p}} = \dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell(m))^{G_\mathfrak{p}}.$$

b) If $\mathfrak{p} \nmid \ell$, then $\dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell(m))^{G_\mathfrak{p}} \leq -\operatorname{ord}_{s=m} L_\mathfrak{p}(H^i(\bar{X}, \mathbb{Q}_\ell), s)$, with equality if $\operatorname{Fr}_\mathfrak{p}$ acts semi-simply on $H^i(\bar{X}, \mathbb{Q}_\ell)^{I_\mathfrak{p}}$.

PROOF: a) The group $G_\mathfrak{p}$ satisfies the property (F_ℓ) and hence all conclusions of Remark 1. This shows the first equality. The second one follows from (36), and the third one from the isomorphisms

$$(37) \quad H^i(\bar{X}, \mathbb{Q}_\ell(n-1))^\vee \cong H^{2d-i}(\bar{X}, \mathbb{Q}_\ell(d+1-n)) \cong H^i(\bar{X}, \mathbb{Q}_\ell(i+1-n))$$

obtained from Poincaré duality and hard Lefschetz.

b) follows from the relation

$$-\operatorname*{ord}_{s=m} L(H^i(\bar{X}, \mathbb{Q}_\ell), s) = \operatorname*{ord}_{s=m} \det\left(1 - \operatorname{Fr}_\mathfrak{p}(N\mathfrak{p})^{-s} \mid H^i(\bar{X}, \mathbb{Q}_\ell)^{I_\mathfrak{p}}\right)$$
$$= \operatorname*{ord}_{s=0} \det\left(1 - \operatorname{Fr}_\mathfrak{p}(N\mathfrak{p})^{-s} \mid H^i(\bar{X}, \mathbb{Q}_\ell)^{I_\mathfrak{p}}(m)\right)$$

(note that $\operatorname{Fr}_\mathfrak{p}$ acts on $\mathbb{Q}_\ell(m)$ via multiplication by $(N\mathfrak{p})^{-m}$). Hence this number is the multiplicity of the eigenvalue 1 for $\operatorname{Fr}_\mathfrak{p}$ on $H^i(\bar{X}, \mathbb{Q}_\ell(m))^{I_\mathfrak{p}}$, while $H^i(\bar{X}, \mathbb{Q}_\ell(m))^{G_\mathfrak{p}} = \left(H^i(\bar{X}, \mathbb{Q}_\ell(m))^{I_\mathfrak{p}}\right)^{\langle \operatorname{Fr}_\mathfrak{p} \rangle}$ is the space of eigenvectors for this eigenvalue. $\square$

By Lemma 5 and Theorem 3, or directly by the exact sequence of Tate's duality theorem (compare (32))

$$H^2(G_S, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) \to \bigoplus_{\mathfrak{p} \in S} H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$$

$$\to H^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n-1))_{G_S} \to 0$$

and the known case of $H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ for $\mathfrak{p} \notin S$ we see that Conjecture 1 implies the local

CONJECTURE 3. $H^2(G_\mathfrak{p}, H^k(\bar{X}, \mathbb{Q}_\ell(n))) = 0$ if

   a) $i + 1 < n$, or

   b) $i + 1 > 2n$.

(Equivalent formulation by Lemma 11: $H^i(\bar{X}, \mathbb{Q}_\ell(m))^{G_\mathfrak{p}} \neq 0$ at most for $0 \leq m \leq \frac{i+1}{2}$.)

The investigation of this divides into two cases.

Case $\mathfrak{p} \nmid \ell$: The case of good reduction has been treated above. Similarly, one has

LEMMA 12. If X has potentially good reduction at $\mathfrak{p}$, then $H^2(G_\mathfrak{p}, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))) = 0$ for $i \neq 2(n - 1)$.

PROOF: Since $cd_\ell(G_\mathfrak{p}) \leq 2$, the corestriction is surjective on H^2, so we may pass to some finite extension, where X has good reduction and where the result follows from the Weil conjecture.

In general, we may pass to some open subgroup $G'_\mathfrak{p}$ of $G_\mathfrak{p}$ such that the action of its ramification group on $V = H^i(\bar{X}, \mathbb{Q}_\ell)$ becomes trivial and the action of its inertia group $I'_\mathfrak{p}$ becomes unipotent (see [**SGA 7 I**]). Let $M_\bullet$ be the associated monodromy filtration, cf. [**D4**, 1.6.1]. This is a finite, ascending, $G_\mathfrak{p}$-equivariant filtration on V such that $I'_\mathfrak{p}$ acts trivially on $\mathrm{Gr}_r^M V = M_r V / M_{r-1} V$ for all $r \in \mathbb{Z}$. There is the well-known

MONODROMY CONJECTURE (see [**D1**] and [**RZ**, Introduction]). *One has*

$$0 = M_{-i-1}V \subseteq M_{-i}V \subseteq \cdots \subseteq M_{i-1}V \subseteq M_i V = V,$$

*and $\mathrm{Gr}_r^M V$ is pure of weight $i+r$ (with respect to the action of the Frobenius in $G'_\mathfrak{p}/I'_\mathfrak{p}$; this does not depend on the choice of $G'_\mathfrak{p}$ and can be reformulated purely in terms of $G_\mathfrak{p}$, cf. [**D4**, 1.7]).*

The function field analogue was proved by Deligne [**D4**, 1.8.4]. For number fields the conjecture is known for abelian varieties; for further results and discussions see [**RZ**].

By definition of $M_\bullet$ one has

(38) $$V^{I_\mathfrak{p}} \subseteq M_0 V,$$

hence the Monodromy Conjecture would imply that this $G_\mathfrak{p}/I_\mathfrak{p}$-representation is mixed with weights between 0 and i. In particular, this would imply the following, well-known conjecture (cf. [**D3**, p. 319]):

CONJECTURE (on the bad factors of the L-function). $L_\mathfrak{p}(H^i(\bar{X}, \mathbb{Q}_\ell), s)$ has poles at most for $0 \le \mathrm{Re}(s) \le \frac{i}{2}$.

This conjecture has been proved in many cases; by Lemma 11 b) it implies Conjecture 3 for $\mathfrak{p} \nmid \ell$ (in fact, we only need the case $s = m \in \mathbb{Z}$).

Case $\mathfrak{p} \mid \ell = p$: Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Faltings [**Fa1**] has proved the Hodge-Tate decomposition

$$(39) \qquad H^i(\bar{X}, \mathbb{Q}_p) \bigotimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{r+t=i} H^t(X, \Omega_X^r) \bigotimes_k \mathbb{C}_p(-r),$$

depending on an embedding $\bar{k} \hookrightarrow \mathbb{C}_p$ inducing $\mathfrak{p}$. This is an isomorphism of topological $G_\mathfrak{p}$-modules, and Tate has proved

$$\mathbb{C}_p(\nu)^{G_\mathfrak{p}} = \begin{cases} k_\mathfrak{p} \text{ (completion of } k \text{ at } \mathfrak{p}), & \nu = 0, \\ 0, & \nu \ne 0 \end{cases}$$

see [**Ta2**, Theorem 2].

COROLLARY 5 (compare [**Sou4**, Theorem 2 iii]). *For $\mathfrak{p} \mid p$ one has*

$$\left(H^i(\bar{X}, \mathbb{Q}_p(m)) \bigotimes_{\mathbb{Q}_p} \mathbb{C}_p \right)^{G_\mathfrak{p}} = 0 \text{ for} \quad \begin{array}{l} a) \ m < 0 \ (\iff i+1 < n), \text{ or} \\ b') \ m > i \ (\iff n < 1). \end{array}$$

In particular, the same vanishing holds for $H^i(\bar{X}, \mathbb{Q}_p(m))^{G_\mathfrak{p}} \subseteq (H^i(\bar{X}, \mathbb{Q}_p(m)) \otimes \mathbb{C}_p)^{G_\mathfrak{p}}$, *and, by Lemma 11,* $H^2(G_\mathfrak{p}, \tilde{H}^i(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)))$ *vanishes for the indicated values of* $n = i + 1 - m$.

To obtain a better bound than b'), we consider the finer, crystalline theory. First assume that X has good reduction at $\mathfrak{p}$. Let $\mathcal{X}(\mathfrak{p})$ be a smooth and proper model of X over $\mathcal{O}_\mathfrak{p}$, the ring of integers in $k_\mathfrak{p}$, and let $\mathcal{X}_\mathfrak{p} = \mathcal{X}(\mathfrak{p}) \times_{\mathcal{O}_\mathfrak{p}} \kappa(\mathfrak{p})$ be the special fiber, where $\kappa(\mathfrak{p})$ is the residue field of $\mathcal{O}_\mathfrak{p}$. Let $k_\mathfrak{p}^0$ be the maximal unramified extension of $\mathbb{Q}_p$ in $k_\mathfrak{p}$ and let $\mathcal{O}_\mathfrak{p}^0$ be its ring of integers (note that $\mathcal{O}_\mathfrak{p}^0 = W(\kappa(\mathfrak{p}))$ is the ring of Witt vectors for $\kappa(\mathfrak{p})$). Then one has the

CRYSTALLINE CONJECTURE (Fontaine [**Fo2**, A.11]). *There are canonical $G_\mathfrak{p}$-isomorphisms*

$$H^i(\bar{X}, \mathbb{Q}_p(m)) \cong \left\{ v \in H_{\mathrm{cris}}^i(\mathcal{X}_\mathfrak{p}/\mathcal{O}_\mathfrak{p}^0) \bigotimes_{\mathcal{O}_\mathfrak{p}^0} B_{\mathrm{cris}} \mid \phi v = p^m v \right\} \cap F^m$$

for all $i \geq 0$ and $m \in \mathbf{Z}$.

Here H^i_{cris} denotes the crystalline cohomology, and $B_{\text{cris}} = B_{\text{cris}}(k_{\mathfrak{p}})$ is a certain ring containing $k^0_{\mathfrak{p}}$ equipped with three compatible structures:

- an action of $G_{\mathfrak{p}}$
- a decreasing filtration $F^{\bullet}$ on $B_{\text{cris}} \otimes_{k^0_{\mathfrak{p}}} k_{\mathfrak{p}}$
- a Frobenius endomorphism ϕ inducing the absolute Frobenius σ on $k^0_{\mathfrak{p}}$.

The Frobenius ϕ on $H^i_{\text{cris}} \otimes B_{\text{cris}}$ is the tensor product of ϕ on B_{cris} and the crystalline Frobenius ϕ on H^i_{cris}; this makes sense, since both are σ-linear. The filtration $F^{\bullet}$ in the Crystalline Conjecture is naturally defined on $(H^i_{\text{cris}} \otimes B_{\text{cris}}) \otimes_{k^0_{\mathfrak{p}}} k_{\mathfrak{p}} \cong (H^i_{\text{cris}} \otimes_{\mathcal{O}^0_{\mathfrak{p}}} k_{\mathfrak{p}}) \otimes_{k_{\mathfrak{p}}} (B_{\text{cris}} \otimes_{k^0_{\mathfrak{p}}} k_{\mathfrak{p}})$, as the tensor product of the filtration $F^{\bullet}$ on $B_{\text{cris}} \otimes_{k^0_{\mathfrak{p}}} k_{\mathfrak{p}}$ and the Hodge filtration, via the canonical isomorphism with the de Rham cohomology

$$H^i_{\text{cris}}(\mathcal{X}_{\mathfrak{p}}/\mathcal{O}^0_{\mathfrak{p}}) \bigotimes_{\mathcal{O}^0_{\mathfrak{p}}} k_{\mathfrak{p}} \cong H^i_{DR}(X \times_k k_{\mathfrak{p}}/k_{\mathfrak{p}}),$$

see [**BO**].

Let $\phi_{\mathfrak{p}} = \phi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]}$ be the Frobenius with respect to $\kappa(\mathfrak{p})$. Then $\phi_{\mathfrak{p}}$ acts $k^0_{\mathfrak{p}}$-linearly on $H^i_{\text{cris}}(\mathcal{X}_{\mathfrak{p}}/\mathcal{O}^0_{\mathfrak{p}}) \otimes_{\mathcal{O}^0_{\mathfrak{p}}} k^0_{\mathfrak{p}}$, and it is known that the eigenvalues of $\phi_{\mathfrak{p}}$ on this space are the same as the eigenvalues of the (geometric) Frobenius $\text{Fr}_{\mathfrak{p}}$ on $H^i_{\text{ét}}(\bar{\mathcal{X}}_{\mathfrak{p}}, \mathbf{Q}_{\ell})$ for $\ell \neq p$, where $\bar{\mathcal{X}}_{\mathfrak{p}} = \mathcal{X}_{\mathfrak{p}} \times_{\kappa(\mathfrak{p})} \overline{\kappa(\mathfrak{p})}$ for an algebraic closure $\overline{\kappa(\mathfrak{p})}$ of $\kappa(\mathfrak{p})$, see [**KM**]. On the other hand, one has

$$(F^{\nu} B_{\text{cris}})^{G_{\mathfrak{p}}} = \begin{cases} k^0_{\mathfrak{p}}, & \nu \leq 0, \\ 0, & \nu > 0. \end{cases}$$

In particular, the Crystalline Conjecture would imply

$$(40) \qquad H^i(\bar{X}, \mathbf{Q}_p(m))^{G_{\mathfrak{p}}} \subseteq \left\{ v \in H^i_{\text{cris}}(\mathcal{X}_{\mathfrak{p}}/\mathcal{O}^0_{\mathfrak{p}}) \bigotimes k^0_{\mathfrak{p}} \mid \phi_{\mathfrak{p}} v = (N\mathfrak{p})^m v \right\}.$$

Combining this with the (proved!) Weil conjectures, we would obtain

$$H^i(\bar{X}, \mathbf{Q}_{\mathfrak{p}}(m))^{G_{\mathfrak{p}}} = 0 \text{ for } i \neq 2m,$$

as in the ℓ-adic case of good reduction.

If $k_{\mathfrak{p}} = k^0_{\mathfrak{p}}$, the Crystalline Conjecture has been proved by Fontaine and Messing (for $p > \dim X$, see [**FM**]) and Faltings (in general, [**Fa2**]). Hence we have

COROLLARY 6. *If $\mathfrak{p}$ is unramified in $k/\mathbb{Q}$ and X has good reduction at $\mathfrak{p} \mid p$, then $H^i(\bar{X}, \mathbb{Q}_p(m))^{G_{\mathfrak{p}}} = 0$ for $i \neq 2m$.*

For arbitrary X there does not yet exist a "crystalline conjecture". I think that some analogue of the Monodromy Conjecture should hold. There is an obvious guess: that there is an $G_{\mathfrak{p}}$-equivariant filtration

$$0 = M_{-i-1}V_p \subseteq M_{-i}V_p \subseteq \cdots \subseteq M_{i-1}V_p \subseteq M_i V_p = V_p \text{ of } V_p = H^i(\bar{X}, \mathbb{Q}_p)$$

such that $\mathrm{Gr}_r^M V_p$ is potentially crystalline [**Fo2**, 5.6], and over a finite extension of $k_{\mathfrak{p}}$ the polynomial of the "crystalline" Frobenius on the associated filtered module [**Fo2**, 5.1] is the same as the polynomial of the Frobenius on $\mathrm{Gr}_r^M V_\ell$ for the corresponding ℓ-adic representations $V_\ell = H^i(\bar{X}, \mathbb{Q}_\ell)$ for $\ell \neq p$. However, this guess is false, as can be seen in the following counterexample, due to Mazur, Tate and Teitelbaum.

Let $\omega_4(z) = (\eta(z)\eta(5z))^4 = q - 4q^2 + 2q^3 - 5q^5 - \ldots, q = e^{2\pi i z}$, where $\eta(z)$ is the Dedekind η-function. This is a new form of weight 4 of "Haupttyp" on $\Gamma_0(5)$ and the associated compatible system $V = \{V_\ell\}$ of two-dimensional ℓ-adic representations of $G_{\mathbb{Q}}$ [**D2**] has the following properties [**MTT**, I §12, II §15]:

i) the image of G_5 in $Gl(V_5) \cong Gl_2(\mathbb{Q}_5)$ is open,

ii) for $\ell \neq 5$, $V_\ell^{I_5}$ is one-dimensional, and the Frobenius Fr_5 acts on it by multiplication by -5.

For ii) we have combined the facts that $a_5 = -5$ in the q-series $\omega_4 = \sum_{n \geq 1} a_n q^n$ and that the L-function of V equals the L-function of ω_4 at *all* places (see [**C**]; indeed, the assertion needed for ii) was already proved by Langlands in his Antwerp article and, independently, by Deligne in his letter to Piatetski-Shapiro).

Consequently, V_ℓ has a non-trivial monodromy filtration for the decomposition group G_5 for all $\ell \neq 5$, while V_5 has no non-trivial G_5-filtration at all; the same holds over any finite extension of $\mathbb{Q}$. (The ℓ-adic Monodromy Conjecture is true here, by the way; since $\det V_\ell \cong \mathbb{Q}_\ell(-3)$, Fr_5 must act by multiplication with -25 on the one-dimensional space $V_{\ell_{I_5}}$ for $\ell \neq 5$.) To see the connection with the subject of this paper, note that $\{V_\ell\}$ is a direct factor of $\{H^3(\bar{X}, \mathbb{Q}_\ell)\}$ for a certain three-dimensional, smooth projective variety X over $\mathbb{Q}$ [**D2**].

This example is useful for further observations. One can also show:

iii) V is unramified outside 5 (i.e., each V_ℓ is unramified outside 5 and ℓ; this follows from the construction in [**D2**]),

iv) the image of $G_{\mathbb{Q}(\mu_5)}$ in $Gl(V_5)$ is a pro-5-group [**Ma2**].

We deduce from this:

LEMMA 13. *Let k be a finite extension of $\mathbb{Q}$, and let $\Lambda_\ell \subseteq V_\ell$, for each ℓ, be a G_k-invariant lattice (i.e., $\Lambda_\ell \cong \mathbb{Z}_\ell^2$ as a $\mathbb{Z}_\ell$-module, with $\Lambda_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = V_\ell$).*

a) $V_\ell(\nu)^{G_\mathfrak{p}} = \mathbb{Q}_\ell$, *if $\nu = 1$, $\ell \neq 5$, $\mathfrak{p} \mid 5$, and the degree $f(\mathfrak{p} \mid 5)$ of $\mathfrak{p}$ over 5 is even; $V_\ell(\nu)^{G_\mathfrak{p}} = 0$, otherwise. In particular, $H^2(G_k, V_\ell/\Lambda_\ell(\nu)) = 0$ for $\nu \neq 3$, and $H^2(G_k, V_5/\Lambda_5(3)) = 0$.*

b) *If $\mathbb{Q}(\mu_3) \subseteq k$, then $\dim H^2(G_k, V_\ell/\Lambda_\ell(3)) > 0$ for $\ell \neq 5$.*

c) *Let $k = \mathbb{Q}(\mu_3)$. If $S \supseteq S_5$, then $H^2(G_S, V_5(\nu)) = 0$ for all $\nu \in \mathbb{Z}$. If $\ell \neq 5$ and $S \supseteq S_5 \cup S_\ell$, then $\dim_{\mathbb{Q}_\ell} H^2(G_S, V_\ell(3)) \geq 1$. In particular, $\dim_{\mathbb{Q}_5} H^1(G_k, V_5(3)) = 2$, but $\dim_{\mathbb{Q}_\ell} H^1(G_k, V_\ell(3)) > 2$ for $\ell \neq 5$.*

PROOF: a) The case $\mathfrak{p} \nmid 5$, $\mathfrak{p} \nmid \ell$ is clear by good reduction and the Weil conjectures. For $\mathfrak{p} \nmid 5$ and $\mathfrak{p} \mid \ell$ the $G_\mathfrak{p}$-representation V_ℓ is crystalline, since this is the case for $k = \mathbb{Q}$ by Fontaine-Messing-Faltings, and the associated admissible filtered module is the base extension of that for $k = \mathbb{Q}$ (cf. [**Fo1**, 7.3.2]). Hence we may argue using the Weil conjectures as in Corollary 6. For $\mathfrak{p} \mid 5$ we have $V_5(\nu)^{G_\mathfrak{p}} = 0$ for all $\nu \in \mathbb{Z}$, since the image of $G_\mathfrak{p}$ in $\mathrm{Aut}(\Lambda_5) \cong Gl_2(\mathbb{Z}_5)$ is open by i). For $\mathfrak{p} \mid 5 \neq \ell$ we have $V_\ell(\nu)^{G_\mathfrak{p}} = \{v \in V_\ell^{I_\mathfrak{p}} \mid \mathrm{Fr}_\mathfrak{p}\, v = (N\mathfrak{p})^\nu \cdot v\}$; by ii) this is non-zero if and only if $\nu = 1$ and $\mathrm{Fr}_\mathfrak{p}$ is an even power of Fr_5. The other assertions of a) now follow from the analogues of Lemma 11 a) and Theorem 3 d) for V_ℓ; recall that V_ℓ is a direct factor of $H^3(\bar{X}, \mathbb{Q}_\ell)$, moreover, (37) can be replaced by the isomorphism $V_\ell^\vee \cong V_\ell(3)$ (note that $(R^1 f_* \mathbb{Q}_\ell)^\vee \cong R^1 f_* \mathbb{Q}_\ell(1)$ in [**D4**, 3.7] by Poincaré duality).

b) follows similarly from the local results in a), since $f(\mathfrak{p} \mid 5) = 2$ for the prime $\mathfrak{p} \mid 5$ in $\mathbb{Q}(\mu_3)$.

For the first assertion of c) note that the action of $G_{S_5}(\mathbb{Q}(\mu_{15}))$ factors through its maximal pro-5-quotient $\tilde{G}$ by iii), and that this quotient is pro-5-free by [**Br**, 3.3], since there is only one prime above 5 in $\mathbb{Q}(\mu_{15})$ and the class number of $\mathbb{Q}(\mu_{15})$ is 1 [**Wa1**, Tables]. We obtain the vanishing of $H^2(G_{S_5}(\mathbb{Q}(\mu_{15})), V_5/\Lambda_5(\nu))$, since the inflation

$$H^2(\tilde{G}, V_5, \Lambda_5(\nu)) \overset{\mathrm{inf}}{\underset{\sim}{\to}} H^2(G_{S_5}(\mathbb{Q}(\mu_{15})), V_5/\Lambda_5(\nu))$$

is an isomorphism (cf. [**Neu**, Cor. 1]). By the surjectivity of the corestriction on H^2, valid since $cd_5(G_{S_5}(\mathbb{Q}(\mu_3)) \leq 2$, we get the same vanishing for

G_{S_5} over $\mathbb{Q}(\mu_3)$, hence for G_S with $S \supset S_5$, cf. (13). The other results are clear from a), b) and the analogues for V_ℓ of Lemmas 1 to 5; we have $\chi(G_S, V_\ell) = -[k : \mathbb{Q}]$ by Lemma 2, since the Hodge structure $V_{\mathbb{R}}$ belonging to V in $H^3(X(\mathbb{C}), \mathbb{R})$ is of type $\{(0,3), (3,0)\}$ [**D3**, §7]. $\qquad\square$

From this example we learn that for a compatible system $V = \{V_\ell\}$ of ℓ-adic representations the dimension of its Galois cohomology can get smaller for the primes ℓ where V has bad reduction. In particular, the numbers $d_{i,n}$, $r_{i,n}$ and $r'_{i,n}$ in §3 may depend on ℓ. I would expect that they are the same for all ℓ where X has good reduction. For $r'_{i,n}$ this would follow from the Monodromy Conjecture and the Crystalline Conjecture, via Theorem 3. For $r_{i,n}$ (and hence $d_{i,n}$), compare §6.

I think that there should exist a p-adic analogue of the monodromy filtration on the "crystalline side", in the following sense. Consider the following category $MF_{k_\mathfrak{p}}^{f,N}$: objects are weakly admissible filtered modules D over $k_\mathfrak{p}$ (see [**Fo2**, 5.1]) together with a nilpotent homomorphism of $k_\mathfrak{p}^0$-modules $N : D \to D$ satisfying $\phi^{-1}N\phi = pN$ (*no* compatibility with the filtration $F^\bullet$).[1] This implies that ϕ respects the monodromy filtration $M_\bullet$ associated to N. Morphisms between these objects are morphisms of filtered modules which respect N. There should exist a category $\mathrm{Rep}_{\mathrm{cris},N}(k_\mathfrak{p})$ of $\mathbb{Q}_p$-representations of $G_\mathfrak{p} = \mathrm{Gal}(\bar{k}_\mathfrak{p}/k_\mathfrak{p})$, containing the crystalline ones, and an equivalence of tensor categories between $MF_{k_\mathfrak{p}}^{f,N}$ and $\mathrm{Rep}_{\mathrm{cris},N}(k_\mathfrak{p})$. Under this equivalence of categories, the crystalline representations should correspond to those objects in $MF_{k_\mathfrak{p}}^{f,N}$ for which $N = 0$ (note that this subcategory can be identified with $MF_{k_\mathfrak{p}}^{f}$, the category of weakly admissible filtered modules over $k_\mathfrak{p}$). $H^i(\bar{X}, \mathbb{Q}_p)$ should be in $\mathrm{Rep}_{\mathrm{cris},N}(k_\mathfrak{p})$ potentially, i.e., over a finite extension of $k_\mathfrak{p}$. If $H^i(\bar{X}, \mathbb{Q}_p)$ is in $\mathrm{Rep}_{\mathrm{cris},N}(k_\mathfrak{p})$, and D is the associated object in $MF_{k_\mathfrak{p}}^{f,N}$, the characteristic polynomial of $\phi_\mathfrak{p}$ on $\mathrm{Gr}_r^M D$ should be the same as the characteristic polynomial of $\mathrm{Fr}_\mathfrak{p}$ on $\mathrm{Gr}_r^M V_\ell$ for $V_\ell = H^i(\bar{X}, \mathbb{Q}_\ell)$, $\ell \neq p$.

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[1]**Added in proof:** In a letter to the author (November 1987), J.-M. Fontaine observed a wrong normalization of ϕ in the first version of this paper and pointed out that the modules in $MF_{k_\mathfrak{p}}^{f,N}$ should just be weakly N-admissible: in the notation of [**Fo1**] this means postulating $t_H(D') \leq t_N(D')$ only for submodules $D' \subseteq D$ respected by N. By developing a formalism quite similar to the one for his Crystalline Conjecture, Fontaine then gives a definition for $\mathrm{Rep}_{\mathrm{cris},N}(k_\mathfrak{p})$ and the functor to $MF_{k_\mathfrak{p}}^{f,N}$. This leads to a precise formulation of the p-adic Monodromy Conjecture, and Fontaine proves it for abelian varieties.

This would imply

(41) $H^i(\bar{X}, \mathbf{Q}_p(m))^{G_\mathfrak{p}} = \mathrm{Hom}_{G_\mathfrak{p}}(\mathbf{Q}_p, H^i(\bar{X}, \mathbf{Q}_p(m)))$

$\cong \mathrm{Hom}_{MF^{f,N}_{k_\mathfrak{p}}}(k^0_\mathfrak{p}, D(m)) \cong \{v \in D \mid \phi v = p^m v, Nv = 0\} \cap F^m.$

We see that the dimension of this space can be smaller than the dimension of

(42) $$\{v \in D \mid \phi v = p^m v, Nv = 0\},$$

which should equal the dimension of

(43) $\{v \in H^i(\bar{X}, \mathbf{Q}_\ell) \mid \mathrm{Fr}_\mathfrak{p}\, v = (N\mathfrak{p})^m v, Nv = 0\}$

$$\cong H^i(\bar{X}, \mathbf{Q}_\ell(m))^{G_\mathfrak{p}} \quad \ell \neq p.$$

This would explain the phenomena discussed in Lemma 13.

The compatibility of N with $F^\bullet$ decides whether the mondromy filtration "lifts to characteristic zero", i.e., to V_p (compare the relation between filtrations and lifting problems for p-divisible groups [Gr]). Consider the example of the two-dimensional $G_{\mathbf{Q}_5}$-representation V_5 associated to ω_4. For the associated object D of $MF^{f,N}_{\mathbf{Q}_5}$ one expects a filtration

$$0 = M_{-2}D \subsetneq M_{-1}D = \mathrm{Im}\, N = \mathrm{Ker}\, N = M_0 D \subsetneq M_1 D = D$$

such that ϕ acts via -5 on $\mathrm{Gr}^M_{-1} D$ and via -25 on $\mathrm{Gr}^M_1 D$. If N is compatible with $F^\bullet$, then $\mathrm{Ker}\, N$ and $\mathrm{Coker}\, N$ are weakly admissible; this implies $D = F^0 = F^1 \supsetneq F^2 \supsetneq F^3 = 0$. The associated 5-adic representation would be a non-trivial extension of $\mathbf{Q}_5(-2)(\chi)$ by $\mathbf{Q}_5(-1)(\chi)$, where (χ) denotes the twist by the character belonging to the unramified extension of degree 2 of $\mathbf{Q}_5$. However, the Hodge filtration on D should satisfy

$$D = F_0 \supsetneq F^1 = F^2 = F^3 \supsetneq F^4 = 0$$

(recall that $V_{\mathbf{R}}$ is of type $\{(0,3),(3,0)\}$). Then Newton polygon and Hodge polygon (see [Fo1, 4.3]) look like:

(44)
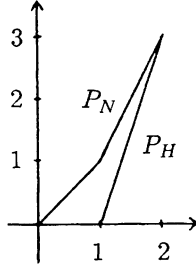
It follows from [**Fo1**, 4.3.3] that D is weakly admissible if and only if ϕ does not respect F^1D. It is then also admissible, since the length is $4 < p = 5$ [**FL**, Theorem 8.4]. Moreover, we see from (44) that there are no proper weakly admissible submodules, therefore the associated crystalline $G_{\mathbb{Q}_5}$-representation $\tilde{V}_5$ is irreducible as V_5 is. Nevertheless, $\tilde{V}_5$ should not be isomorphic to V_5, since it corresponds to the module D with $N = 0$.[2]

REMARK 5: The discussion above suggests that one should — for all $\mathfrak{p}$ and ℓ — also expect $H^2(G_\mathfrak{p}, H^i(\bar{X}, \mathbb{Q}_\ell(n))) = 0$ for $i + 1 = 2n$ (that is, $H^i(\bar{X}, \mathbb{Q}_\ell(m))^{G_\mathfrak{p}} = 0$ for $m = \frac{i+1}{2}$), thus sharpening Conjecture 3. Also, everything could be formulated for varieties over $k_\mathfrak{p}$, not mentioning a global field, and the conjectures should still be true.

6. The case $n \leq i + 1 \leq 2n$.

Let V be an ℓ-adic representation of G_S, then the kernels of the localization maps

(45)
$$\alpha_{1,S}(V) : H^1(G_S, V) \longrightarrow \bigoplus_{\mathfrak{p} \in S} H^1(G_\mathfrak{p}, V)$$
$$\alpha_{2,S}(V^\vee(1)) : H^2(G_S, V^\vee(1)) \longrightarrow \bigoplus_{\mathfrak{p} \in S} H^2(G_\mathfrak{p}, V^\vee(1))$$

are $\mathbb{Q}_\ell$-dual to each other by Tate's duality theorem (here $V^\vee$ is the $\mathbb{Q}_\ell$-dual of V).

Question 2   Let V be pure of weight w. Is $\alpha_{1,S}(V)$ injective for $w \neq -1$ (or, equivalently, is $\alpha_{2,S}(V)$ injective for $w \neq -1$)?

REMARK 6: Both G_S and the $G_\mathfrak{p}$ satisfy the condition (F_ℓ) of Remark 1. Hence, if $\Lambda \subset V$ is a G_S-equivariant lattice, then we have $\dim_{\mathbb{Q}_\ell} \operatorname{Ker} \alpha_{r,S}(V)$

---

[2] **Added in proof:** In view of the previous footnote, it is possible that V_5 corresponds to the module $D = \mathbb{Q}_5 e_1 \oplus \mathbb{Q}_5 e_2$ with $\phi e_1 = -5e_1$, $\phi e_2 = -25e_2$, $Ne_1 = 0$, $Ne_2 = e_1$, $F^0 D = D$, $F^1 D = F^3 D = \mathbb{Q}_5 e_2$, and $F^4 D = 0$, which is weakly N-admissible but not weakly admissible.

$= \dim \operatorname{Ker} \alpha_{r,S}(V/\Lambda) = \operatorname{rank}_{\mathbb{Z}_\ell} \operatorname{Ker} \alpha_{r,S}(\Lambda)$, so we may equivalently ask for the finiteness of the last two kernels. Also, we may replace Λ or V/Λ by isogenous modules. In the commutative diagram

$$
\begin{array}{ccc}
H^2(G_S, V/\Lambda) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S}(G_{\mathfrak{p}}, V/\Lambda) \\
{\scriptstyle \inf} \downarrow & & \downarrow \\
H^2(G_k, V/\Lambda) & \longrightarrow & \bigoplus_{\text{all } \mathfrak{p}} H^2(G_{\mathfrak{p}}, V/\Lambda),
\end{array}
$$

the bottom map is injective for V pure of weight $w \neq -2$ by the arguments of Theorem 3. Hence for $w \neq -2$ we have $\dim \operatorname{Ker} \alpha_{2,S}(V/\Lambda) = \dim \operatorname{Ker}(H^2(G_S, V/\Lambda) \overset{\inf}{\to} H^2(G_k, V/\Lambda))$ and may reformulate Question 2 in terms of the above inflation map. It is more complicated to describe what this implies for the H^2-inflation for V or Λ.

THEOREM 4. *The function field analogue of Question 2 has a positive answer for $w \neq 0, -1, -2$. More precisely, let U be a smooth, affine curve over a finite field $\mathbf{F}_q$ and let V be a $\mathbb{Q}_\ell$-representation of $\pi_1(U, \bar{\eta})$ of weight w (here $\bar{\eta}$ is a geometric point over the generic point $\eta = \operatorname{Spec} k$ of U). Let Y be the smooth, projective compactification of U, and for each $x \in Y \backslash U$ let $G_x \subseteq G_k = \pi(\eta, \bar{\eta})$ be a decomposition group at x. Then the localization map*

$$
\alpha_1 : H^1(\pi_1(U, \bar{\eta}), V) \to \bigoplus_{x \in Y \backslash U} H^1(G_x, V)
$$

is injective for $w \neq -1$, except possibly when $w = 0$ and $V^{G_k} \neq 0$, and

$$
\alpha_2 : H^2(\pi_1(U, \bar{\eta}), V) \to \bigoplus_{x \in Y \backslash U} H^2(G_x, V)
$$

is injective for $w \neq -1$, except possibly when $w = -2$ and $V(-1)_{G_k} \neq 0$. If $w = -1$, then α_1 and α_2 are the zero maps.

PROOF: Let F be the smooth $\mathbb{Q}_\ell$-sheaf on U corresponding to V, and let $j : U \hookrightarrow Y$ be the inclusion. Then the localization maps α_r can be identified with the maps α_r in the relative cohomology sequence

$$
(46) \quad \cdots \to H^r(Y, j_! F) \overset{\beta_r}{\to} H^r(U, F) \overset{\alpha_r}{\to}
$$
$$
\bigoplus_{x \in Y \backslash U} H^{r+1}_x(Y, j_! F) \to H^{r+1}(Y, j_! F) \to \ldots,
$$

compare [**Mi2**, 4.13.c]. It remains to consider the maps

$$\beta_r : H_c^r(U,F) = H^r(Y, j_! F) \to H^r(U,F).$$

Now β_r factors through $H^r(Y, j_* F)$, and the Hochschild-Serre spectral sequence gives an exact sequence

$$0 \to H^{r-1}(\bar{Y}, j_* F)_\Gamma \to H^r(Y, j_* F) \to H^r(\bar{Y}, j_* F)^\Gamma \to 0,$$

$\Gamma = \mathrm{Gal}(\bar{\mathbf{F}}_q / \mathbf{F}_q)$ and $\bar{Y} = Y \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q$. Moreover, Deligne has shown that $H^r(\bar{Y}, j_* F)$ is pure of weight $r + w$, see [**D4**, 3.2.3]. This implies the vanishing of $H^r(Y, j_* F)$ for $r + w \neq 0$ and $r - 1 + w \neq 0$.

For $r = 1$ and $w = 0$ we get the isomorphism $H^1(Y, j_* F) \cong H^0(\bar{Y}, j_* F)_\Gamma$; the latter group has the same dimension as

$$H^0(\bar{Y}, j_* F)^\Gamma = H^0(U, F) = V^{G_k}.$$

For $r = 2$ and $w = -2$ we have $H^2(Y, j_* F) \cong H^2(\bar{Y}, j_* F)^\Gamma \cong H^0(\bar{Y}, j_* F^\vee(1))_\Gamma{}^\vee$, see *loc. cit.*; this has the same dimension as

$$H^0(\bar{Y}, j_* F^\vee(1))^\Gamma = H^0(U, F^\vee(1)) = \mathrm{Hom}_{G_k}(V, \mathbf{Q}_\ell(1)).$$

Finally, for $w = -1$ and $x \in S$, we obtain

$$H^0(G_x, V) = (V^{I_x})^{\mathrm{Gal}(\bar{\mathbf{F}}_q / \kappa(x))} = 0$$

($I_x \subset G_x$ the inertia group), since the stalk $F_x = V^{I_x}$ has weights ≤ -1 [**D4**, 1.8.1]. By local duality, we also have

$$H^2(G_k, V) \cong H^0(G_k, V^\vee(1))^\vee = 0.$$

This implies $H^1(G_x, V) = 0$, since

$$0 = \chi(G_x, N) := \sum_{\nu=0}^{3} (-1)^\nu \dim_{\mathbf{Q}_\ell} H^\nu(G_x, N)$$

for every $\mathbf{Q}_\ell$-representation N of G_x (this follows, e.g., from [**Ta1**, 2.2] as in Lemma 2).

REMARK 7: If V is a semi-simple $\pi_1(U, \bar{\eta})$-representation, we get the injectivity of α_r also for $w = 0$ and $w = -2$: If $V \cong \mathbf{Q}_\ell^r \oplus V_1$ with $V_1^{G_k} = 0$, we

have $\operatorname{Ker}\alpha_1(V_1) = 0$ by the theorem, and $\operatorname{Ker}\alpha_1(Q_\ell) = 0$, since the class group of k is finite. The case of α_2 follows by duality.

The weight of $H^i(\bar{X}, Q_\ell(n))$ is $i - 2n$; hence we ask for the injectivity of $\alpha_{r,S}^{i,n} := \alpha_{r,S}(H^i(\bar{X}, Q_\ell(n)))$ for $i \neq 2n - 1$. Conjecture 1 implies that $\alpha_{2,S}^{i,n}$ is injective for a) $i + 1 < n$ or b) $i + 1 > 2n$, hence that $\alpha_{1,S}^{i,n}$ is injective for a) $n < 0$ or b) $i + 1 < 2n$, by using (37). Case b) would imply that we could consider the regulator maps

$$ r : K_{2n-i-1}(X)^{(n)} \bigotimes Q_\ell \to H^1(G_S, H^i(\bar{X}, Q_\ell(n))), \quad 2n - i - 1 > 0, $$

locally. Conversely, a positive answer for Question 2 would imply that $r_{i,n} = r'_{i,n}$ for $i \neq 2n - 1$, $i \neq 2n - 2$, hence that $r_{i,n}$ could be computed in purely local terms except for these values of i and n. If $i = 2n - 2$, then $r'_{i,n}$ can be infinite and $r_{i,n}$ may depend on S, cf. Example 1. If $i = 2n - 1$, then $r_{i,n}$ is of "global nature", and α_2 is expected to be zero, cf. Remark 5 and Remark 8. Similar statements hold for the numbers $d_{i,n}$, by the formula

$$ \chi_{i,n} = \dim_{Q_\ell} H^i(\bar{X}, Q_\ell)^{G_k} - d_{i,n} + r_{i,n}, $$

in which $H^i(\bar{X}, Q_\ell(n))^{G_k}$ vanishes for $i \neq 2n$ and is related to Tate's conjecture for $i = 2n$.

I conclude this chapter with some speculation on the regulator maps for $n \leq i + 1 \leq 2n$. I think that they are still injective, at least if one restricts to $K_{2n-i-1}(X/Z)^{(n)} \otimes Q_\ell$, and that the image of the last group in $H^1(G_S, H^i(\bar{X}, Q_\ell(n)))$ can be described by local conditions. (For $i + 1 = 2n$, the above regulator maps are defined on $K_0(X)_0^{(n)} \otimes Q_\ell$, where $K_0(X)_0^{(n)} = \operatorname{Ker}(K_0(X)^{(n)} \to H^{2n}(\bar{X}, Q_\ell(n))^{G_k})$, and we consider this whole space, since $K_0(X/Z)^{(n)} = K_0(X)^{(n)}$). The nature of these local conditions is suggested by work of R. Greenberg (on the Selmer group of a compatible system of ℓ-adic representations), S. Bloch (on Tamagawa numbers of motives), and P. Schneider (on a p-adic, local version of Beilinson's conjectures, aiming at relations between p-adic regulators and p-adic L-functions). The subspace in question should be defined via the localization map

$$ H^1(G_S, H^i(\bar{X}, Q_\ell(n))) \to \bigoplus_{\mathfrak{p} \in S} H^1(G_\mathfrak{p}, H^i(\bar{X}, Q_\ell(n))), $$

as the pre-image of a space $\oplus_{\mathfrak{p} \in S} A_{\mathfrak{p}}$, where for good reduction at ℓ the spaces $A_{\mathfrak{p}} \subseteq H^1(G_{\mathfrak{p}}, H^i(\bar{X}, \mathbb{Q}_\ell(n)))$ are defined as follows

$$A_{\mathfrak{p}} = 0 \text{ for } \mathfrak{p} \nmid \ell$$
$$A_{\mathfrak{p}} = H^i_{DR}(X \times k_{\mathfrak{p}}/k_{\mathfrak{p}})/(\phi - p^n) F^n H^i_{DR}(X \times k_{\mathfrak{p}}/k_{\mathfrak{p}}) \text{ for } \mathfrak{p} \mid \ell.$$

The embedding of the last group in $H^1(G_{\mathfrak{p}}, -)$ is obtained via the connecting morphism of the $G_{\mathfrak{p}}$-cohomology sequence associated to the exact sequence

$$(47) \quad 0 \to H^i(\bar{X}, \mathbb{Q}_p(n)) \to F^n\big(H^i_{DR}(X \times k_{\mathfrak{p}}/k_{\mathfrak{p}}) \otimes B_{\mathrm{cris}}\big)$$
$$\xrightarrow{\phi - p^n} H^i_{DR}(X \times k_{\mathfrak{p}}/k_{\mathfrak{p}}) \otimes B_{\mathrm{cris}} \to 0$$

derived from the Crystalline Conjecture (and proved for $k_{\mathfrak{p}} = k^0_{\mathfrak{p}}$ and $p > \dim X$ by Fontaine and Messing). The idea to consider these spaces for $\mathfrak{p} \mid p$ is due to K. Kato.

7. The case $i = 1$: abelian varieties.

The Kummer sequences $0 \to \mu_{\ell^r} \to \mathbb{G}_m \xrightarrow{\ell^r} \mathbb{G}_m \to 0$ induce an isomorphism

$$H^1(\bar{X}, \mathbb{Z}_\ell(1)) \cong T_\ell H^1(\bar{X}, \mathbb{G}_m) = T_\ell A,$$

where $A = \underline{\mathrm{Pic}}^0_{X/k}$ is the Picard variety of X/k. (Conversely, if we start with an abelian variety A, then we get such an isomorphism for $X = A'$, the dual abelian variety). By [SGA 7 I, exp. IX, §6 f.], the local conjectures are true in this case for every prime $\mathfrak{p}$: There is a filtration of $G_{\mathfrak{p}}$-submodules

$$(48) \qquad\qquad 0 \subseteq T_\ell(A)^{\mathrm{et}} \subseteq T_\ell(A)^{\mathrm{ef}} \subseteq T_\ell(A)$$

(et = essentially toric, ef = essentially finite), coinciding with the monodromy filtration after tensoring with $\mathbb{Q}_\ell$ for $\mathfrak{p} \nmid \ell$ (for the last property see [SGA 7 1, exp. I, §6]), and a semi-abelian variety $A^{\natural 0}$ over $k_{\mathfrak{p}}$ (the connected Raynaud scheme) with the following properties: There is an exact sequence of group schemes

$$0 \to T \to A^{\natural 0} \to B \to 0$$

such that T is a torus, B is an abelian variety with potentially good reduction, and one has canonical $G_{\mathfrak{p}}$-isomorphisms

$$(49) \qquad\qquad T_\ell(T) \cong T_\ell(A)^{\mathrm{et}},$$

$$(50) \qquad\qquad T_\ell(A^{\mathfrak{t}0}) \cong T_\ell(A)^{\mathrm{ef}},$$

(*all* primes ℓ). Moreover, there is an ismorphism

$$(51) \qquad T_\ell(A)/T_\ell(A)^{\mathrm{ef}} \cong \mathrm{Hom}(T_\ell(A')^{\mathrm{et}}, \mathbb{Z}_\ell(1)),$$

where A' is the dual abelian variety. Hence for $\mathfrak{p} \nmid \ell$, $T_\ell(A)^{\mathrm{et}}$, $T_\ell(A)^{\mathrm{ef}}/T_\ell(A)^{\mathrm{et}} \cong T_\ell(B)$ and $T_\ell(A)/T_\ell(A)^{\mathrm{ef}}$ are pure of weight -2, -1 and 0, respectively. For $\mathfrak{p} \mid \ell = p$, it follows from the work of Fontaine [**Fo1**, 3.3.5], [**Fo2**, 6.2] that $T_p(T)$, $T_p(B)$ and $\mathrm{Hom}(T_p(T'), \mathbb{Z}_p(1))$ are potentially crystalline (here T' is the maximal subtorus of $(A')^{\mathfrak{t}0}$), and that over a suitable extension of $k_{\mathfrak{p}}$ the characteristic polynomial of the crystalline Frobenius on the associated admissible filtered modules is the same as the Frobenius polynomial on the ℓ-adic counterparts, $\ell \neq p$. By using the isogeny $T_p(A)/T_p(A)^{\mathrm{ef}} \to T_p(A)^{\mathrm{et}}(-1)$ coming from [**SGA 7 1**, exp. IX, §9], it is even possible to define a p-adic monodromy operator $N = N_p : V_p(A) \to V_p(A)(-1)$ such that the associated monodromy filtration coincides with (48) after tensoring with $\mathbb{Q}_p$. Thus, in this case the "naive" form of the p-adic monodromy is present, but this shall not be needed in the following.

THEOREM 5. *Let* $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

    a) $H^1(\bar{X}, \mathbb{Q}_\ell(r+1))^{G_{\mathfrak{p}}} \cong V_\ell(A)(r)^{G_{\mathfrak{p}}} = 0$ *for* $r \neq -1$.

    b) $H^1(\bar{X}, \mathbb{Q}_\ell)^{G_{\mathfrak{p}}} \cong V_\ell(A)(-1)^{G_{\mathfrak{p}}} \cong X_*(T)_{k_{\mathfrak{p}}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$,

where $X_*(T)_{k_{\mathfrak{p}}} = \mathrm{Hom}_{k_{\mathfrak{p}}}(\mathbb{G}_m, T)$ *is the group of cocharacters over* $k_{\mathfrak{p}}$ *of* T *(and* T *is the torus associated to* A *over* $k_{\mathfrak{p}}$ *as above). If* A *has semi-stable reduction at* $\mathfrak{p}$*, then* $\mathrm{rank}\, X_*(T)_{k_{\mathfrak{p}}} = \dim T_s^0$*, where* T_s *is the maximal torus of* $\mathcal{A}_s^0$*, the connected component of the special fibre* A_s *of the Néron model* $\mathcal{A}$ *of* A *(over* $\mathcal{O}_k$ *or* $\mathcal{O}_{\mathfrak{p}}$*), and* $T_s^0 \subset T_s$ *is the maximal split torus.*

PROOF: For $\mathfrak{p} \nmid \ell$ one has $T_\ell(A)(r)^{G_{\mathfrak{p}}} \subseteq T_\ell(A)(r)^{I_{\mathfrak{p}}} \subseteq T_\ell(A)^{\mathrm{ef}}(r)$ (coincidence of (48) with the monodromy filtration). Now $T_\ell(B)(r)^{G_{\mathfrak{p}}} = 0$ for all $r \in \mathbb{Z}$ and $T_\ell(T)(r)^{G_{\mathfrak{p}}} = 0$ for $r \neq -1$ by questions of weights, hence we get a) and

$$T_\ell(A)(-1)^{G_{\mathfrak{p}}} = T_\ell(A)^{\mathrm{ef}}(-1)^{G_{\mathfrak{p}}} = T_\ell(T)(-1)^{G_{\mathfrak{p}}}$$
$$= \mathrm{Hom}_{G_{\mathfrak{p}}}(T_\ell \mathbb{G}_m, T_\ell T) = \mathrm{Hom}_{k_{\mathfrak{p}}}(\mathbb{G}_m, T) \bigotimes_{\mathbb{Z}} \mathbb{Z}_\ell$$

by the theory of tori.

For $\mathfrak{p} \mid \ell = p$, we use the Hodge-Tate decomposition

$$H^1(\bar{X}, \mathbb{Q}_p) \underset{\mathbb{Q}_p}{\bigotimes} \mathbb{C}_p \cong H^1(X, \mathcal{O}_X) \underset{k}{\bigotimes} \mathbb{C}_p \oplus H^0(X, \Omega_X^1) \underset{k}{\bigotimes} \mathbb{C}_p(-1)$$

as in Corollary 5 to deduce $H^1(\bar{X}, \mathbb{Q}_p)(r+1)^{G_\mathfrak{p}} = 0$ for $r \neq 0, -1$. For $r = 0$ we have $T_p(A)^{G_\mathfrak{p}} = T_p(A(k_\mathfrak{p})) = 0$, since the p-torsion in $A(k_\mathfrak{p})$ is finite. For $r = -1$ we first observe that $(T_p(A)/T_p(A)^{\text{ef}}(-1))^{G_\mathfrak{p}} = 0$, since $G_\mathfrak{p}$ acts on $T_p(A)/T_p(A)^{\text{ef}}$ via a finite quotient by (51). Furthermore, $T_p(B)(-1)^{G_\mathfrak{p}} = 0$ by results of Tate: let $L/k_\mathfrak{p}$ be a finite extension such that B has good reduction over L, that is, extends to an abelian scheme $\mathcal{B}$ over $\mathcal{O}_L$. Let $\mathcal{B}(p)$ be the associated p-divisible group over $\mathcal{O}_L$. Then we have

$$\begin{aligned}
T_p(B)(-1)^{G_L} &= \operatorname{Hom}_{G_L}(T_p \mathbb{G}_m, T_p B) \\
&= \operatorname{Hom}_{\mathcal{O}_L}(\mathbb{G}_m(p), \mathcal{B}(p)) \quad [\text{Ta2, Theorem 3}] \\
&= \operatorname{Hom}_{\mathcal{O}_L}(\mathcal{B}(p)^D, \mathbb{Q}_p/\mathbb{Z}_p) = 0.
\end{aligned}$$

Here $\mathcal{B}(p)^D = \operatorname{Hom}(\mathcal{B}(p), \mathbb{G}_m)$ is the Cartier dual of $\mathcal{B}(p)$, which is connected, since it is the p-divisible group of the dual abelian scheme $\mathcal{B}'$. We conclude as above

$$T_p(A)(-1)^{G_\mathfrak{p}} = T_p(T)(-1)^{G_\mathfrak{p}} = \operatorname{Hom}_{k_\mathfrak{p}}(\mathbb{G}_m, T) \underset{\mathbb{Z}}{\bigotimes} \mathbb{Z}_p.$$

If A has semi-stable reduction, then $X_*(T) \cong X_*(T_s)$ (isomorphism of $G_\mathfrak{p}$-modules via $G_\mathfrak{p} \twoheadrightarrow \operatorname{Gal}(\overline{\kappa(\mathfrak{p})}/\kappa(\mathfrak{p})) =: \Gamma$) by construction of T, and rank $X_*(T_s)^\Gamma = \dim T_s^0$ by definition.

COROLLARY 7. a) $H^2(G_k, H^1(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r+1))) = H^2(G_k, A_{\ell^\infty}(r)) = 0 = H^2(G_\mathfrak{p}, A_{\ell^\infty}(r))$ for $r \neq 1$.

b) Let S_0 be the finite set of places, where A does not have potentially good reduction. Then

$$H^2(G_k, A_{\ell^\infty}(1)) \cong \bigoplus_{\mathfrak{p} \in S_0} H^2(G_\mathfrak{p}, A_{\ell^\infty}(1))$$

$$\cong \bigoplus_{\mathfrak{p} \in S_0} A_{\ell^\infty}(k_\mathfrak{p})_{G_\mathfrak{p}} \cong \bigoplus_{\mathfrak{p} \in S_0} X_*(T'_\mathfrak{p})^*_{k_\mathfrak{p}},$$

where $T'_\mathfrak{p}$ is formed as above, but for the dual abelian variety A' (i.e., $T'_\mathfrak{p}$ is the maximal torus of the Raynaud group scheme $(A' \times_k k_\mathfrak{p})^{\natural 0}$).

PROOF: This is clear from Theorem 4, Theorem 3 and the local duality

$$(52) \qquad H^2(G_\mathfrak{p}, A_{\ell^\infty}(1)) \cong (T_\ell A'(-1)^{G_\mathfrak{p}})^*$$

coming from the perfect Weil pairings $A_{\ell^r}(\bar{k}) \times A'_{\ell^r}(\bar{k}) \to \mu_{\ell^r}$.

REMARK 8: From the above, we see that $H^2(G_k, A_{\ell^\infty}(1))$ may well be non-zero. Hence the same is true for the group $H^2(G_S, A_{\ell^\infty}(1))$ mapping onto it (cf. Lemma 5). Next we explain why $H^2(G_S, A_{\ell^\infty})$ can be non-zero (compare Example 2): The Kummer sequences $0 \to A_{\ell^\nu} \to A \xrightarrow{\ell^\nu} A \to 0$ induce an exact sequence

$$0 \to A(k) \bigotimes \mathbb{Z}_\ell \to H^1(G_k, T_\ell A) \to T_\ell H^1(k, A) \to 0,$$

where we have used the fact that $A(k)$ is a finitely generated group by the theorem of Mordell-Weil. We obtain

$$\dim_{\mathbb{Q}_\ell} H^1(G_S, V_\ell A) = \dim_{\mathbb{Q}_\ell} H^1(G_k, V_\ell A) \geq \operatorname{rank}_{\mathbb{Z}} A(k).$$

On the other hand, we compute by Lemma 2

$$\chi(G_S, V_\ell A) = \chi_{1,1}(X) = -[k : \mathbb{Q}] \cdot \dim A,$$

since the $\mathbb{R}$-Hodge structure $H^1(X \times_{\mathbb{Q}} \mathbb{R}, \mathbb{R})$ is of type $\{(0,1),(1,0)\}$ and thus an induced module for F_∞. Hence $\dim_{\mathbb{Q}_\ell} H^2(G_S, V_\ell A) \geq \operatorname{rank}_{\mathbb{Z}} A(k) - [k : \mathbb{Q}] \cdot \dim A$, and this number can be strictly positive (there are elliptic curves over $\mathbb{Q}$ with Mordell-Weil rank > 1).

This also shows that $\alpha_{2,S}(V_\ell(A))$ can have a non-zero kernel, since its target is zero by Corollary 5 a); compare the exception in Question 2 ($V_\ell(A)$ has weight -1).

As explained, these counterexamples concern the case $(i, n) = (1, 1)$, but they may also apply to the case $i = 2n - 1$, since $H^{2n-1}(\bar{X}, \mathbb{Z}_\ell(n))$ may contain the Tate module of an abelian variety, namely the "abelian part" of the n-th intermediate Jacobian, compare [Bl].

REMARK 9: Consider the exact sequence

$$\cdots \to \bigoplus_{\mathfrak{p} \notin S} A_{\ell^\infty}(r-1)^{G_\mathfrak{p}} \xrightarrow{\delta} H^2(G_S, A_{\ell^\infty}(r)) \xrightarrow{\inf} H^2(G_k, A_{\ell^\infty}(r)) \to 0$$

(cf. (13)). Surprisingly, the finite groups $A_{\ell^\infty}(r-1)^{G_\mathfrak{p}}$ ($\mathfrak{p} \notin S$) can add up to an infinitely divisible group via δ for $r = 0$ (can one describe this "adding up" in a precise way?). Conjecture 1 claims that this cannot happen for $r \neq 0, 1$, and Question 2 asks whether this can happen for $r = 1$ (compare Remark 6).

In the end, we remark that the Iwasawa theoretic question about $Z_1 = H^2(H_S, H^1(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell))^*$ (cf. §3) is related to Mazur's conjecture [**Ma1**, p. 184]. In fact, let A be an abelian variety over k, let ℓ be a prime of good reduction for A, and let k_∞/k be the cyclotomic $\mathbb{Z}_\ell$-extension, $H'_S = \mathrm{Gal}(k_S/k_\infty)$ etc. as in §3. Then Mazur's conjecture, as generalized to supersingular primes by Schneider [**Sch3**, p. 348], says that $H^2_{fppf}(\mathcal{O}_\infty, \mathcal{A}(\ell)) = 0$, where $\mathcal{O}_\infty$ is the ring of integers in k_∞, $\mathcal{A}$ is the Néron model of A, and $\mathcal{A}(\ell)$ is the associated ℓ-divisible group, regarded as a sheaf for the $fppf$-topology.

LEMMA 14. *Assume that the conjecture of Mazur-Schneider is true for A and ℓ, and let S be a set of places of k containing all places above $\ell \cdot \infty$, and all primes where A has bad reduction. Then $H^2(H'_S, A_{\ell\infty}) = 0$ (i.e., by (21) and Lemma 8, $H^2(G_S, A_{\ell\infty}(r)) = 0$ for almost all $r \equiv 0 \bmod [k(\mu_\ell) : k]$).*

PROOF: Let $\mathcal{O}_{S,\infty}$ be the ring of S-integers in k_∞. It suffices to show the surjectivity of the restriction

$$(53) \qquad H^2_{fppf}(\mathcal{O}_\infty, \mathcal{A}(\ell)) \to H^2_{fppf}(\mathcal{O}_{S,\infty}, \mathcal{A}(\ell)),$$

since there are canonical isomorphisms

$$H^2_{fppf}(\mathcal{O}_{S,\infty}, \mathcal{A}(\ell) \cong H^2_{\text{ét}}(\mathcal{O}_{S,\infty}, \mathcal{A}(\ell)) \cong H^2(H'_S, A_{\ell\infty}).$$

Indeed, the first one holds, since $\mathcal{A}(\ell)$ is étale over $\mathcal{O}_{S,\infty}$, and the second one follows by the same arguments as (16). For the surjectivity of (53), it suffices to show

$$(54) \qquad H^3_{fppf, \mathfrak{P}}(\mathcal{O}_\infty, \mathcal{A}(\ell)) = 0 \text{ for } \mathfrak{P} \in \mathrm{Spec}\,\mathcal{O}_\infty \backslash \mathrm{Spec}\,\mathcal{O}_{S,\infty},$$

and for this we may pass to the Henselization $\tilde{\mathcal{O}}_{\infty,\mathfrak{P}}$ of $\mathcal{O}_\infty$ at $\mathfrak{P}$.

If $\mathfrak{P} \mid \ell$, then the claim follows from [**Sch2**, Lemma 5], while for $\mathfrak{P} \nmid \ell$ we may again pass to étale cohomology and use the exact sequence

$$\cdots \to H^{\nu-1}(\tilde{k}_{\infty,\mathfrak{P}}, A_{\ell\infty}) \to H^\nu_{\text{ét},\mathfrak{p}}(\tilde{\mathcal{O}}_{\infty,\mathfrak{P}}, \mathcal{A}(\ell)) \to H^\nu_{\text{ét}}(\tilde{\mathcal{O}}_{\infty,\mathfrak{P}}, \mathcal{A}(\ell)) \to \cdots,$$

where $\tilde{k}_{\infty,\mathfrak{P}}$ is the field of fractions of $\tilde{\mathcal{O}}_{\infty,\mathfrak{P}}$, and $H^\nu_{\text{ét}}(\tilde{\mathcal{O}}_{\infty,\mathfrak{P}}, \mathcal{A}(\ell)) \cong H^\nu(\kappa(\mathfrak{P}), \mathcal{A}(\ell) \times_{\mathcal{O}_\infty} \kappa(\mathfrak{P}))$ for the residue field $\kappa(\mathfrak{P})$ of $\mathfrak{P}$ [**Mil**, III 3.11]. But $cd_\ell(\kappa(\mathfrak{P})) = 0$ since $\kappa(\mathfrak{P})$ contains all ℓ-power roots of unity, and $cd_\ell(\tilde{k}_{\infty,\mathfrak{P}}) \le 1$ for the same reason.

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