New Keynesian DSGE Models: Theory, Empirical Implementation, and Specification

DISSERTATION

zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaft

eingereicht an der
Wirtschaftswissenschaftlichen Fakultät
der Universität Regensburg

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Tag der Disputation: 10.08.2012

Acknowledgements

Foremost, I am deeply grateful to my first supervisor Prof. Jürgen Jerger for giving me the opportunity to work on this exciting field of research. I am truly indebted and thankful for his guidance, patience, and continuous support during the completion of this dissertation. Further, I would sincerely like to thank my second supervisor Prof. David N. DeJong for his very helpful advice. I greatly benefited from his detailed and highly valuable comments. I also would like to show my gratitude to Prof Gabriel S. Lee for his concise comments, helpful suggestions, and his invitations to the Vienna Macroeconomics Workshop. Many thanks go to my colleagues and friends at the department of economics. It is in great part due to them that I had such an enjoyable and memorable time in Regensburg. Special thanks go to my parents and my sister. I always felt their support during the years. Finally, my deep gratitude goes to Michaela for her invaluable help, encouragement, and understanding through this timely process.

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Chapter 1

Introduction

1.1 Background

The last four decades have witnessed a fundamental change in macroeconomic modeling. This development has its origins in the 1970s when the existing conventional quantitative macroeconomic models, rooted in Keynesian economic theory, were heavily criticized on both theoretical and empirical grounds (see, for example, Lucas, 1976; Sims, 1980; Sargent, 1981). Existing mainstream macroeconomic models, including the Wharton Econometric Forecasting Model and the Brookings Model, showed a poor forecast performance, missing the economic reality of stagflation (see Galí and Gertler, 2007). As a result, the general applicability of these models for forecasting and policy analysis was questioned. In his famous critique of econometric policy evaluation, Lucas (1976) emphasized the lack of structural invariance of the current macroeconomic models making them unfit to predict the effects of alternative policies:

> "... [T]he ability to forecast the consequences of "arbitrary", unannounced sequences of policy decisions, currently claimed (at least implicitly) by the theory of economic policy, appears to be beyond the capability not only of the current-generation models, but of conceivable future models as well" (Lucas, 1976, p. 41).

A response to this critique emerged in the form of the first generation of dynamic stochastic general equilibrium (DSGE) models. The development of these models was a merit of the real business cycle (RBC) approach initiated by the seminal work of Kydland and Prescott (1982) and Long and Plosser (1983).

> "For the first time, macroeconomists had a small and coherent dynamic model of the economy, built from first principles with optimizing agents, rational expectations, and market clearing, that could generate data that resembled observed variables to a remarkable degree" (Fernández-Villaverde, 2010, p. 4).

Based on the frictionless neoclassical growth model, the RBC approach aimed to explain economic fluctuations as an optimal response of rational agents to real disturbances, particularly technology shocks (see Rebelo, 2005). From this paradigm, a distinct school of thought evolved becoming known as New Keynesian macroeconomics. Originally derived as an extension to the standard real business cycle framework, which features monetary neutrality due to the presence of flexible prices and wages, New Keynesian economics evolved into a progressive research program, accounting for the real effects of monetary policy.

> "... [New Keynesian] models integrate Keynesian elements (imperfect competition, and nominal rigidities) into a dynamic general equilibrium framework that until recently was largely associated with the Real Business Cycle (RBC) paradigm. They can be used (and are being used) to analyze the connection between money, inflation, and the business cycle, and to assess the desirability of alternative monetary policies" (Galí, 2002, p. 1).

DSGE models rapidly became a standard tool for quantitative policy analysis in macroeconomics. While, as outlined in An and Schorfheide (2007), the quantitative evaluation of the early DSGE models was typically conducted without formal statistical methods and instead relied on parameter calibration, i.e., the choice of parameter values on the basis of microeconomic evidence or long-run data properties (see Karagedikli et al., 2010), the predominance of calibration in empirical DSGE analysis decreased considerably in the 1990s when advances in computational power and the development of new econometric methods made the estimation of DSGE models more accessible.

> "There has been tremendous improvement over the last twenty years in the mathematical, probabilistic, and computational tools available to applied macroeconomists. This extended set of tools has changed

> the way researchers have approached the problem of estimating parameters, validating theories, or simply identifying regularities in the data" (Canova, 2007, p. xi).

As a result, DSGE models not only became widely used for empirical research in macroeconomics, but also for policy analysis and forecasting at policy-making institutions.

> "DSGE models are powerful tools that provide a coherent framework for policy discussion and analysis. In principle, they can help to identify sources of fluctuations; answer questions about structural changes; forecast and predict the effect of policy changes, and perform counterfactual experiments" (Tovar, 2009, p. 1).

This thesis contributes to the evolving field of applied macroeconomic research, strengthening the idea of a fruitful symbiosis between theoretical models and advanced econometric techniques.

1.2 Outline of the Thesis

The core of the dissertation consists of three chapters. Chapter 2 provides a graphical and formal representation of a basic dynamic stochastic general equilibrium economy and discusses the prerequisites needed for an empirical implementation. The aim of this chapter is to present the core features of the models used in chapter 3 and 4 of this work and to introduce the estimation techniques employed in the remainder of the thesis.

In chapter 3 we estimate a New Keynesian DSGE model on French, German, Italian, and Spanish data to check for the respective sets of parameters that are stable over time, implementing the ESS procedure ("Estimate of Set of Stable parameters") developed by Inoue and Rossi (2011). This econometric technique allows to identify the respective parameters of a DSGE model that have changed at an unknown break date. In the case of France, Germany, and Italy our results point to structural breaks after the beginning of the second stage of EMU in the mid-1990s, while the estimates for Spain show a significant break just before the start of the third stage in 1998. Specifically, we find significant changes in monetary policy behavior for France, Italy, and Spain, while we detect monetary policy to be stable over time in Germany.

The incorporation of convex adjustment costs of capital accumulation into dynamic stochastic general equilibrium models has become standard practice in the literature, since these frictions improve the ability of sticky-price models with endogenous investment to match the key features of the data considerably. In chapter 4, we use a Bayesian approach to investigate empirically how different ad-hoc specifications of adjustment costs affect the fit and the dynamics of a New Keynesian dynamic stochastic general equilibrium model with real and nominal frictions featuring several exogenous stochastic disturbances. We consider three different forms of quadratic adjustment costs: an investment adjustment cost specification and two versions of capital adjustment costs. Using both euro area and US data, we detect in part marked differences between the estimated structural parameters across the three model specifications. Further, the implementation of either investment or capital adjustment costs affects the empirical fit and the dynamics of the respective model specifications substantially. Concerning the overall empirical fit, the model specifications with capital adjustment costs outperform the model version featuring investment adjustment costs, although only the latter is able to produce data-consistent hump-shaped investment dynamics in response to exogenous shocks.

Chapter 5 concludes by summarizing the main results of this dissertation.

Chapter 2

DSGE Models: Basic Structure and Empirical Implementation

2.1 Introduction

DSGE models have become the workhorse in modern macroeconomics, receiving wide support not only among researchers, but also from policy making circles, supporting, for instance, the monetary policy decision-making process at central banks around the world (see Kremer et al., 2006; Tovar, 2009). The term DSGE thereby refers to a special class of dynamic stochastic macroeconomic models which feature a sound micro-founded general equilibrium framework, characterized by the optimizing behavior of rational agents subject to technology, budget, and institutional constraints (see Smets et al., 2010). As outlined in Fernández-Villaverde (2010), a crucial part of the recent popularity of DSGE models stems from the ability to fit these structural models to the data.

In this chapter, we present the general structure of DSGE models and discuss prerequisites needed for an empirical implementation. We focus on a standard New Keynesian model and describe basic procedures for constructing and solving this prototype model. Further, we consider three empirical methods for DSGE models. The purpose of this chapter, on the one hand, is to highlight the core features of the models used in chapter 3 and 4 of this work. On the other hand, we introduce the estimation methods employed in the remainder of the thesis.

Chapter 2 is organized as follows: Section 2.2 provides a graphical and formal presentation of a standard DSGE framework. The formal description comprises of the theoretical setup, the log-linear approximation, and the solution of a standard New Keynesian model. Section 2.3 discusses three common strategies used in the empirical analysis of DSGE models: calibration, maximum likelihood estimation, and Bayesian estimation. Technical details concerning the theoretical setup, the log-linear approximation, and the model solution appear in the appendices.

2.2 The Basic Structure of DSGE Models

2.2.1 A Graphical Exposition

In presenting a general DSGE framework, we closely follow Sbordone et al. (2010) and use a simplified diagram to illustrate the interactions among the different agents in a basic dynamic stochastic general equilibrium economy (see figure 2.1).

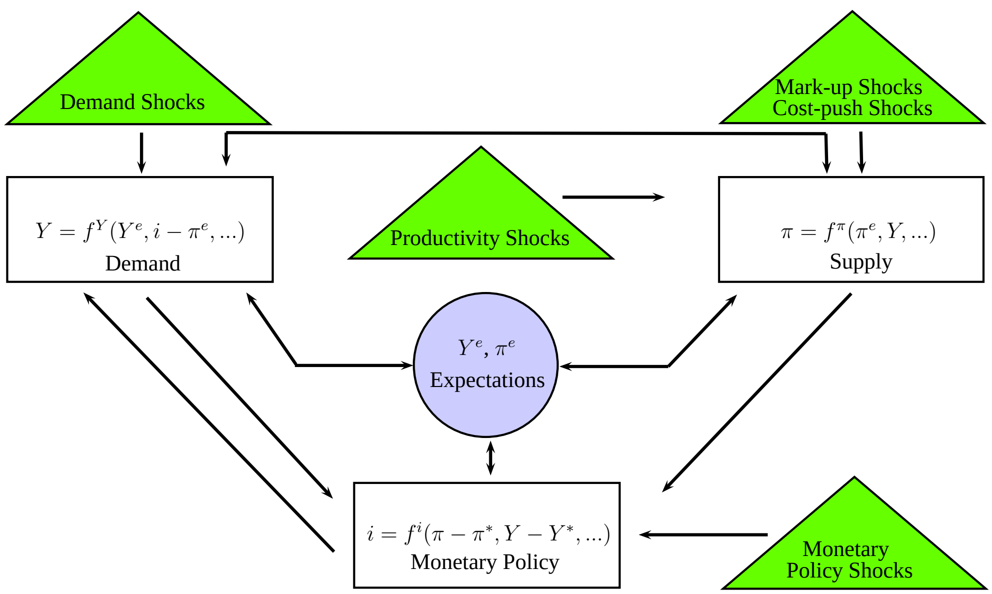

Figure 2.1: A basic DSGE framework.

The model economy can be characterized by three interrelated blocks: a demand block arising from the optimal behavior of households, a supply block describing the optimal behavior of firms, and a monetary policy equation. Each of these blocks is defined by equations derived from the underlying microeconomic structure of the model, i.e., explicit assumptions on the specific behavior of agents as well as the technological, budget, and institutional constraints in the economy.

As outlined in Sbordone et al. (2010), the demand block determines the real activity Y as a function of the expected future real activity Y^e and the real interest rate, which, according to the Fisherian equation, equals the difference between nominal interest rate r and the expected inflation rate π^e. The demand block exhibits a negative relationship between the real interest rate and real activity, since a rise in the real interest rate increases savings and lowers consumption (and investment). In contrast, the functional relation between real activity and its expected value is assumed to be positive, capturing the willingness of people to spend more in anticipation of thriving days.

The arrow pointing from the demand block to the supply block emphasizes the importance of the real activity Y emerging from demand, since it is, together with expected inflation π^e, a key input for determining the inflation rate π. The supply block captures a positive relationship between the rate of inflation and the level of real activity, implicitly expressing the pressure of factor prices on producer prices stemming from increased competition for scarce production factors. Further, the supply block accounts for a positive relation between current inflation and expected inflation.

Following Sbordone et al. (2010), the values of real activity and inflation determined by the demand and supply block enter into the monetary policy block. Monetary policy itself is often described by a central bank, which sets the short-term nominal interest rate according to a Taylor-type policy rule (see Taylor, 1993). The monetary authority adjusts the nominal interest rate r in response to deviations of current inflation π and real activity Y from their respective target values π^* and Y^*. The effects of monetary policy on real activity and inflation are stressed by the arrow running from the monetary policy block to the demand block and then on to the supply block. Thus, the policy reaction function of the monetary authority closes the model allowing for a complete description of the relationship between the key variables: output Y, inflation π, and the nominal interest rate r.

Note that the incorporation of expectations about future outcomes provides the source of (forward-looking) dynamic interactions between the three interrelated blocks. To take into account the role of expectations, figure 2.1 explicitly highlights the influence of expectations on real activity and inflation, especially emphasizing the expectations channel of monetary policy.

The stochastic nature of DSGE models originates from exogenous processes, commonly called shocks. These shocks amount to fluctuations of the model around its deterministic steady state equilibrium, a perfectly predictable path, with neither booms nor busts. Although recent micro-founded DSGE models include various types of shocks, figure 2.1 only contains some of the most common specifications (represented as triangles, with arrows pointing to the blocks that are directly affected by a specific driving process).

2.2.2 A Formal Exposition

According to Clarida et al. (2002), Ambler (2007), Blanchard and Galí (2007), and Galí (2008), New Keynesian models have become a fundamental tool for monetary policy analysis by academic economists and central banks. As outlined in Goodfriend and King (1997) and Goodfriend (2002), these models extend the neoclassical RBC setup by introducing Keynesian features like imperfect competition and sticky prices and hence provide a setting that allows monetary policy to be central to macroeconomic fluctuations.

We subsequently present the micro-foundations of a standard, closed-economy New Keynesian model in the spirit of Clarida et al. (1999), Galí (2002), Woodford (2003), or Ireland (2004). We select this standard framework for two reasons. First, it closely mimics the basic structure outlined in the previous section. Second, the model clearly outlines the core features of the more elaborated versions applied in chapter 3 and 4 of this work.

2.2.2.1 Overview

The economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$ and a monetary authority. The representative household consumes, saves, and supplies labor to the intermediate goods-producing firms. Final output is produced by a representative finished goods-producing firm acting in a perfectly competitive market. The finished goods-producing firm bundles the continuum of intermediate goods manufactured by monopolistic competitors and sells it to the household, who uses the final good for consumption. The intermediate goods-producing firms are owned by the household and each of them produces a distinct, perishable intermediate good, also indexed by $i \in [0, 1]$ during each period $t = 0, 1, 2, ...$ The assumption of monopoly power of intermediate goods-producing firms allows to introduce nominal rigidities in the form of quadratic nominal price adjustment costs. Finally, there is a monetary authority that conducts monetary policy by setting the nominal interest rate according to a Taylor-type rule.

2.2.2.2 Households

The representative household of the economy enters period t holding B_{t-1} one-period bonds. During period t the household receives $W_t l_t$ total nominal factor payments from supplying $l_t(i)$ units of labor at the nominal wage rate W_t to each intermediate goods-producing firm $i \in [0, 1]$. For all $t = 0, 1, 2, ...$, the household's choices of $l_t(i)$ must satisfy

$$l_t = \int_0^1 l_t(i)di,$$

where l_t denotes total hours worked. Further, the household receives nominal dividends from each intermediate goods producing firm $i \in [0, 1]$ aggregating to

$$D_t = \int_0^1 D_t(i)di.$$

The household uses its funds to purchase new bonds at the nominal cost B_t/r_t, where r_t denotes the gross nominal interest rate between time periods, and output for consumption purposes from the final goods sector at price P_t. We follow Woodford (2003) and assume that prices are measured in terms of a unit of account called "money", but that the economy is cashless otherwise. Therefore, the budget constraint of the representative household is given by

$$\frac{B_{t-1} + W_t l_t + D_t}{P_t} \geq c_t + \frac{B_t/r_t}{P_t}.$$

Furthermore, we impose a no-Ponzi-game condition preventing the household from excessive borrowing. Subject to these constraints, the household seeks to maximize the stream of expected utility

$$E \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right),$$

where $0 < \beta < 1$ is a discount factor and $\chi > 0$ measures the relative weight of the disutility of labor. The parameter $\sigma \geq 0$ denotes the inverse of the elasticity of intertemporal substitution for consumption while $\eta \geq 0$ is the inverse of the Frisch elasticity of labor supply.[1]

To solve this optimization problem, we form the Lagrangian

$$\begin{aligned}\max_{c_t, l_t, B_t} \Lambda &= E \sum_{t=0}^{\infty} \left[\beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right) \right. \\ & \left. -\beta^t \lambda_t \left(c_t + \frac{B_t / r_t}{P_t} - \frac{B_{t-1}}{P_t} - \frac{W_t l_t}{P_t} - \frac{D_t}{P_t} \right) \right],\end{aligned}$$

obtaining the first-order conditions

$$\Lambda_{c_t} = c_t^{-\sigma} - \lambda_t = 0,$$

$$\Lambda_{l_t} = -\chi l_t^{\eta} + \lambda_t \frac{W_t}{P_t} = 0,$$

$$\Lambda_{B_t} = \frac{\lambda_t}{P_t} - \beta r_t E_t \left(\frac{\lambda_{t+1}}{P_{t+1}} \right) = 0,$$

$$\Lambda_{\lambda_t} = c_t + \frac{B_t / r_t}{P_t} - \frac{B_{t-1}}{P_t} - \frac{W_t l_t}{P_t} - \frac{D_t}{P_t} = 0,$$

and a standard transversality condition for bonds. By rearranging the first-order conditions of the household's decision problem concerning the choice of consumption, labor supply, and bond holding we yield the following standard optimality conditions:

$$\frac{W_t}{P_t} = \chi \frac{l_t^{\eta}}{c_t^{-\sigma}}$$

and

$$c_t^{-\sigma} = \beta r_t E_t \left(\frac{c_{t+1}^{-\sigma}}{\pi_{t+1}} \right),$$

where $\pi_{t+1} = \frac{P_{t+1}}{P_t}$. While the former equation describes the intratemporal optimality condition, setting the real wage equal to marginal rate of substitution between leisure and consumption, the latter represents the Euler equation for the optimal intertemporal allocation of consumption.

[1] Note that $\sigma = 1$ implies a log utility for consumption, so that the model would be consistent with a balanced growth path if secular technical progress was introduced (see King et al., 1988; Galí, 2002).

2.2.2.3 Firms

The final good y_t is produced by a firm that acts in a perfectly competitive environment, bundling the intermediate goods $y_t(i)$ subject to the constant returns to scale technology

$$y_t \leq \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)},$$

where $\theta > 1$ represents the elasticity of substitution between intermediate goods $y_t(i)$. Profit maximization leads to the demand function for intermediate goods

$$y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t,$$

with $P_t(i)$ denoting the price of intermediate good i and

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)}.$$

Each intermediate good i is produced by a single monopolistically competitive firm according to the constant returns to scale technology

$$y_t(i) \leq z_t l_t(i),$$

where the technology shock z_t is assumed to follow the autoregressive process

$$\ln(z_t) = \rho_z \ln(z_{t-1}) + \varepsilon_{zt}$$

with $1 > \rho_z > 0$ and $\varepsilon_{zt} \sim N(0, \sigma_z^2)$. Although each firm i exerts some market power, it acts as a price taker in the factor markets. Moreover, the adjustment of the firm's nominal price $P_t(i)$ is assumed to be costly, where the cost function is convex in the size of the price adjustment. Following Rotemberg (1982), these costs are defined as

$$\frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t,$$

where $\phi_P \geq 0$ governs the size of price adjustment costs and π denotes the gross steady state rate of inflation targeted by the monetary authority. As outlined in Ireland (1997), this specification can be interpreted as the negative effects of price changes on customer-firm relationships.[2]

The typical intermediate goods-producing firm's optimization problem can be split into two steps. First, each firm wants to minimize its costs $W_t l_t(i)$ subject to the production technology $y_t(i) = z_t l_t(i)$. The Lagrangian of this problem can be written in real terms as

$$\min_{l_t(i)} \Lambda = \left(\frac{W_t}{P_t}\right) l_t(i) - \varphi_t[z_t l_t(i) - y_t(i)].$$

The first-order conditions of the firm's problem are

$$\Lambda_{l_t(i)} = W_t/P_t - \varphi_t z_t = 0$$

and

$$\Lambda_{\varphi t} = z_t l_t(i) - y_t(i) = 0,$$

where the Lagrange multiplier φ_t has the interpretation of the firm's real marginal costs. Second, since the convex adjustment costs make the firm's optimization problem dynamic (see Ireland, 2003), each firm chooses $y_t(i)$ and $P_t(i)$ to maximize its total market value

$$E \sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t]$$

subject to the demand function for intermediate goods, where λ_t measures the period t marginal utility to the representative household provided by an additional unit of profits. The firm's profits distributed to the household as dividends are defined in real terms by

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right] y_t(i) - \varphi_t y_t(i) - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t.$$

[2]Alternative specifications of nominal price rigidities in the recent New Keynesian literature include Taylor's (1980) pricing contracts and Calvo's (1983) random probability of price adjustment (for an overview see Roberts, 1995). A detailed analysis of Rotemberg (1982) and Calvo (1983) price setting mechanisms can be found in Lombardo and Vestin (2008) and Ascari et al. (2011).

The optimization problem of each firm takes the form

$$\max_{P_t(i)} E \sum_{t=0}^{\infty} \left(\beta^t \lambda_t \left\{ \left[\frac{P_t(i)}{P_t} \right]^{1-\theta} y_t - \varphi_t \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t \right.\right.$$
$$\left.\left. - \frac{\phi}{2} \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 y_t \right\} \right),$$

where $\left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t$ denotes revenues and $\varphi_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t + \frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t$ refers to costs.

The first-order condition of this problem is

$$\lambda_t \left\{ (1-\theta) \left[\frac{P_t(i)}{P_t} \right]^{-\theta} \frac{y_t}{P_t} + \varphi_t \theta \left[\frac{P_t(i)}{P_t} \right]^{-\theta-1} \frac{y_t}{P_t} - \phi_p \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right] \frac{y_t}{\pi P_{t-1}(i)} \right\}$$
$$+ \beta E_t \left\{ \lambda_{t+1} \phi_p \left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1 \right] \left[\frac{P_{t+1}(i)}{\pi P_t(i)^2} \right] y_{t+1} \right\} = 0.$$

If $\phi_P = 0$, the above expression reduces to

$$P_t(i) = \frac{\theta}{\theta - 1} \varphi_t P_t,$$

which points out, that in the case of costless price adjustment, a representative intermediate goods-producing firm sets its markup of price $P_t(i)$ over (nominal) marginal cost $\varphi_t P_t$ equal to $\theta/(\theta-1)$.

2.2.2.4 Monetary Authority

Following Clarida et al. (2000), Ireland (2000), Canova (2009), and Fernández-Villaverde et al. (2010), monetary policy can be described by a modified Taylor rule of the form:

$$\ln\left(\frac{r_t}{r}\right) = \rho_r \ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t).$$

The monetary authority gradually adjusts the short-term nominal interest rate in response to deviations of current gross inflation $\pi_t = \frac{P_t}{P_{t-1}}$ and output y_t from their steady state values, where ρ_r, ω_π, and ω_y are the parameters of the monetary policy rule.[3] The monetary policy shock υ_t follows the autoregressive process

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t},$$

where $0 < \rho_\upsilon < 1$ and $\varepsilon_{\upsilon t} \sim N(0, \sigma_\upsilon^2)$.

2.2.3 Approximating and Solving the Model

Empirical investigations using DSGE models require to find a solution to the dynamic system. Since most dynamic models do not have an exact analytical closed-form solution, a tractable approximation needs to be derived (see Aruoba et al., 2006). To reduce the computational burden, the majority of studies involving either simulation or estimation use linear approximations of the original model (see Iskrev, 2010), which then can be solved by various solution methods for linear difference models under rational expectations.[4] An extensive coverage of approximation techniques and solution methods for DSGE models can be found in Canova (2007), DeJong and Dave (2007), and McCandless (2008). In the following sections we give a short presentation of the linearized standard New Keynesian model and its solution, while a detailed description is provided in the appendices A and B.

2.2.3.1 Log-linear Approximation

As outlined in Zietz (2008), log-linearization allows to transform a system of nonlinear equations into a system that is linear in terms of the log-deviations of the underlying variables from their steady state values. To log-linearize the standard New Keynesian model we use a first-order Taylor approximation of the model around its steady state values.[5] Letting $\widehat{var}_t = log\left(\frac{var_t}{var}\right)$ denote the log-deviation

[3]Note that the gross steady state level of inflation π is determined by the inflation target of the monetary authority.

[4] Although, as outlined in Kim et al. (2008), linear approximations might be sufficiently accurate for a wide variety of purposes, using a linear approximation to the model economy can be inappropriate. Examples include welfare comparisons across policies that do not have first-order effects on the model's deterministic steady state (see Kim and Kim, 2003; An 2007; Kim et al., 2008). DeJong and Dave (2011) provide an overview of recent nonlinear solution methods and their use in empirical applications.

[5]For the ease of exposition, we assume that the inflation target is zero, implying a gross steady state inflation rate π equal to one.

of some variable var_t from its steady state var the model can be expressed as:

$$\hat{y}_t = E_t\hat{y}_{t+1} - \frac{1}{\sigma}\left(\hat{r}_t - E_t\hat{\pi}_{t+1}\right),$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\theta-1}{\phi_P}(\eta+\sigma)\hat{y}_t - \frac{\theta-1}{\phi_P}(1+\eta)\hat{z}_t,$$

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + (1-\rho_r)(\omega_\pi\hat{\pi}_t + \omega_y\hat{y}_t) + \hat{v}_t,$$

$$\hat{z}_t = \rho_z\hat{z}_{t-1} + \varepsilon_{zt},$$

and

$$\hat{v}_t = \rho_v\hat{v}_{t-1} + \varepsilon_{vt}.$$

The first equation is a so-called dynamic IS curve, capturing the features of aggregate demand outlined in section 2.2.1, whereas the properties of aggregate supply are described by the second equation, usually termed as New Keynesian Phillips curve.[6] Monetary policy is characterized by the third equation, which is a Taylor-type policy reaction function. While these equations are often referred to as "three equation New Keynesian DSGE framework" (see, for example, Schorfheide, 2008; Woodford, 2008; Christiano et al., 2010), the last two equations complete the model, describing the first-order autoregressive structure of the exogenous shocks.

2.2.3.2 Solution

Since the early work of Blanchard and Kahn (1980), several techniques for solving linear difference models under rational expectations have emerged, including the approaches of Anderson and Moore (1985), Uhlig (1999), Klein (2000), and Sims (2002). Although these procedures differ with respect to their specific methodology, they all allow the solution of the underlying model to be written in state space form, which enables the use of the Kalman filter to perform a likelihood-based analysis of DSGE models.

Employing the approach of Klein (2000) on our standard New Keynesian model

[6] As shown in appendix A, the dynamic IS curve can be derived from the Euler equation of the representative household (see section 2.2.2.2), while the New Keynesian Phillips curve is obtained from solving the monopolistically competitive firm's optimization problem under price adjustment costs (see section 2.2.2.3).

leads to a solution in state space form, characterized by a state equation

$$s_{t+1} = \Gamma_0(\mu)s_t + \Gamma_1(\mu)\boldsymbol{\varepsilon}_{t+1}$$

and an observation equation

$$f_t = \Gamma_2(\mu)s_t,$$

where

$$s_t = \left[\begin{array}{ccc} \hat{r}_{t-1} & \hat{z}_t & \hat{v}_t \end{array} \right]'$$

contains the model's state variables, including endogenous predetermined and exogenous variables[7],

$$\boldsymbol{\varepsilon}_{t+1} = \left[\begin{array}{cc} \varepsilon_{zt+1} & \varepsilon_{vt+1} \end{array} \right]'$$

consists of the serially and mutually uncorrelated innovations of the shocks, and

$$f_t = \left[\begin{array}{cc} \hat{y}_t & \hat{\pi}_t \end{array} \right]'$$

comprises the model's flow variables. The matrices $\Gamma_0(\mu), \Gamma_1(\mu)$, and $\Gamma_2(\mu)$ contain (functions of) the model's parameters μ.

2.3 Taking DSGE Models to the Data

An appealing feature of DSGE models is their applicability for empirical analysis, making them a widely used tool for empirical research in macroeconomics as well as quantitative policy analysis and forecasting at central banks all over the world (see Schorfheide, 2011). In this section, we briefly describe three common empirical strategies for taking DSGE models to the data: calibration, maximum likelihood estimation, and Bayesian estimation. For a more detailed description of empirical methods for DSGE models, we refer to Canova (2007) and DeJong and Dave (2007).

2.3.1 Calibration

Pioneered by Kydland and Prescott (1982), calibration was the most popular method for empirical analysis based on DSGE models until the late 1990s (see

[7] See appendix B for details.

Karagedikli et al., 2010). According to Kydland and Prescott (1996), "basic" calibration in the sense of an empirical methodology involves the following five steps:[8]

1. *Pose an economic question.* Such a question can either deal with policy evaluation issues or with the testing and development of theory.

2. *Use a "well-tested theory"*, i.e., an explicit set of instructions for building a mechanical imitation system to answer the question.

3. *Construct a model economy* that is appropriate to address the question.

4. *Calibrate the model economy* by choosing values for certain key parameters of the underlying preferences and technologies using evidence from other empirical studies (see also Plosser, 1989).

5. *Run the experiment.* For this, the state space representation derived in the previous section can be employed to assess theoretical implications of changes in policy or the ability of a specific model to mimic features of the real world.

Although, as outlined in Ruge-Murcia (2007), calibration is, in general, a useful tool for understanding the dynamic properties of DSGE models, the initial predominance of the calibration approach in the quantitative evaluation of DSGE models was partly due to the fact that "... macroeconomists were unsure about how to compute their models efficiently, a necessary condition to perform likelihood-based inference. Moreover, even if economists had known how to do so, most of the techniques required for estimating DSGE models using a likelihood approach did not exist" (Fernández-Villaverde and Rubio-Ramírez, 2006, p. 1). Calibration offered a solution to this problem.

2.3.2 Estimation

According to Fernández-Villaverde and Rubio-Ramírez (2006), the predominance of calibration in empirical DSGE analysis decreased considerably in the late 1990s.

[8]More advanced types of calibration are, for example, based on Bayesian Monte Carlo techniques, taking into account the degree of uncertainty in parameter values (see DeJong et al., 1996).

Advances in computer power and the development of new econometric techniques have facilitated the estimation of DSGE models, and henceforth, as described in Ruge-Murcia (2007, p. 2622), lead to several benefits compared to calibration:

- Rather than relying on (potentially inconsistent) estimates from micro-studies to calibrate the model, parameter estimates can be obtained by imposing the restrictions of the full model under consideration on the data.
- Bootstrapped confidence bands can be computed to quantify the degree of estimation uncertainty of impulse-responses.
- "... [S]tandard tools of model selection and evaluation can be readily applied."

As outlined in An and Schorfheide (2007) and Tovar (2009), the empirical literature features various econometric techniques for estimating DSGE models, including generalized method of moments (GMM) estimation of equilibrium relationships (see Christiano and Eichenbaum, 1992; Burnside et al., 1993), minimum distance estimation based on minimizing a weighted distance between structural vector autoregressive (SVAR) and DSGE model impulse-response functions (see Rotemberg and Woodford, 1997; Christiano et al., 2005), maximum likelihood (see Altug, 1989; Leeper and Sims, 1994; Ireland, 1997), and Bayesian estimation (see DeJong et al., 2000b; Schorfheide, 2000; Otrok, 2001). According to Canova (2007), a key feature distinguishing these different approaches is the amount of information processed. While limited-information procedures like GMM only exploit part of the information contained in a subset of the model's equilibrium conditions, full-information likelihood-based methods aim at estimating the entire DSGE model simultaneously. Tovar (2009, p. 14) points out, that "[i]t is for this reason that the most important strand of the literature has focused on estimation methods built around the implied likelihood function derived from the DSGE model." According to DeJong et al. (2000b), the distinction between maximum likelihood and Bayesian estimation given a specific model hinges critically on whether the data or the model parameters are interpreted as random variables.

Under the classical maximum likelihood approach to inference, "... the parameters are treated as fixed and the data are treated as unknown in the sense that their probability distribution (the likelihood) is the center of focus. The question is whether the observed data could plausibly have come from the model under a particular parameterization" (DeJong et al., 2000b, p. 210). Therefore, maximum likelihood estimation seeks to determine the parameter values that maximize the likelihood of the observed data given a specific model, where "... the uncertainty regarding the specific values estimated for the parameters is conveyed by reporting associated standard errors" (DeJong and Dave, 2007, p. 180). To test for the empirical plausibility of a specific model, formal hypothesis procedures can be applied within a maximum likelihood framework (see, for example, Ireland, 2003).

Bayesian analysis takes a different point of view, since the observed data are treated as fixed whereas the unknown parameters are regarded as random variables. According to DeJong and Dave (2007), this probabilistic interpretation of the model parameters allows for the formal incorporation of a priori information in form of prior distributions specified for the parameters. These prior distributions "... either reflect subjective opinions or summarize information derived from data sets not included in the estimation sample" (Del Negro and Schorfheide, 2008, p. 1). Therefore, Bayesian estimation is sometimes described as a bridge between estimation and calibration (see, for example, Kremer et al., 2006; Walsh, 2010), since it combines the likelihood function, formed by the structure of the model and the data, with the prior distributions (by employing Bayes's theorem) to construct a posterior distribution for the parameters of interest. Once the posterior distribution of the parameters is derived, inference like point estimation or model comparison can be performed (see Fernández-Villaverde, 2010).

2.4 Conclusion

DSGE models have become a standard tool of modern macroeconomics, capable to bridge the gap between micro-founded macroeconomic theory and the data. This appealing feature of DSGE models made them a widely used tool for empirical research in macroeconomics as well as policy analysis and forecasting in central banking (see Schorfheide, 2011).

In this chapter we describe the general structure of DSGE models and appropriate steps to take these models to the data. In particular, we consider a standard New Keynesian model and expound the basic procedure for setting up and solving such a prototype DSGE model. Finally, we briefly discuss three common strategies used in the empirical analysis of DSGE models: calibration, maximum likelihood estimation, and Bayesian estimation.

Appendices

Appendix A

Equilibrium Conditions

The appendix presents the equation system of the standard New Keynesian DSGE model.

A.1 The Economic Environment

- Households:
  The representative household chooses $\{c_t, l_t, B_t\}_{t=0}^{\infty}$ to maximize utility

  $$E\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\sigma}}{1-\sigma}-\chi\frac{l_t^{1+\eta}}{1+\eta}\right),$$

  subject to the budget constraint

  $$\frac{B_{t-1}+W_t l_t+D_t}{P_t}\geq c_t+\frac{B_t/r_t}{P_t}.$$

  Further, following Buiter and Sibert (2007), we prevent the household from excessive borrowing by imposing the no-Ponzi-game condition

  $$\lim_{t\to\infty} B_t\prod_{s=0}^{t}\frac{1}{r_s}\geq 0.$$

Accordingly, the Lagrangian can be written as follows:

$$\begin{aligned}\max_{c_t,l_t,B_t} \Lambda \quad &= \quad E\sum_{t=0}^{\infty}\left[\beta^t\left(\frac{c_t^{1-\sigma}}{1-\sigma}-\chi\frac{l_t^{1+\eta}}{1+\eta}\right)\right. \\ &\left. -\beta^t\lambda_t\left(c_t+\frac{B_t/r_t}{P_t}-\frac{B_{t-1}}{P_t}-\frac{W_tl_t}{P_t}-\frac{D_t}{P_t}\right)\right].\end{aligned}$$

The first-order conditions are obtained by setting the partial derivatives of Λ with respect to c_t, l_t, B_t, and λ_t equal to zero, yielding

$$\Lambda_{c_t} = c_t^{-\sigma} - \lambda_t = 0, \tag{2.1}$$

$$\Lambda_{l_t} = -\chi l_t^{\eta} + \lambda_t\frac{W_t}{P_t} = 0, \tag{2.2}$$

$$\Lambda_{B_t} = \frac{\lambda_t}{P_t} - \beta r_t E_t\left(\frac{\lambda_{t+1}}{P_{t+1}}\right) = 0, \tag{2.3}$$

and

$$\Lambda_{\lambda_t} = c_t + \frac{B_t/r_t}{P_t} - \frac{B_{t-1}}{P_t} - \frac{W_tl_t}{P_t} - \frac{D_t}{P_t} = 0. \tag{2.4}$$

Finally, we impose the standard transversality conditions to guarantee that bonds do not grow too quickly:

$$\lim_{t\to\infty}\beta^t\lambda_t\frac{B_t}{P_t} = 0.$$

- Finished goods-producing firms:
  The representative finished goods-producing firm seeks to maximize its profits

$$P_ty_t - \int_0^1 P_t(i)y_t(i)di$$

subject to the constant returns to scale technology

$$y_t \le \left[\int_0^1 y_t(i)^{(\theta-1)/\theta}di\right]^{\theta/(\theta-1)}.$$

Therefore, the firm's problem can be written as

$$\max_{y_t(i)} \Pi_t = P_t\left[\int_0^1 y_t(i)^{(\theta-1)/\theta}di\right]^{\theta/(\theta-1)} - \int_0^1 P_t(i)y_t(i)di,$$

which leads to the following first-order condition characterizing the demand for intermediate goods:

$$\frac{\partial \Pi_t}{\partial y_t(i)} = y_t(i) - \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t = 0.$$

By plugging this expression into the constant elasticity of scale (CES) aggregator for intermediate goods we obtain the price aggregator

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)}.$$

- Intermediate goods-producing firms:

  The typical intermediate goods-producing firm optimizes along two dimensions. First, each firm wants to minimize its costs subject to the production technology. The Lagrangian of this problem can be written in real terms as

$$\min_{l_t(i)} \Lambda = \left(\frac{W_t}{P_t}\right) l_t(i) - \varphi_t[z_t l_t(i) - y_t(i)],$$

where

$$\ln(z_t) = \rho_z \ln(z_{t-1}) + \varepsilon_{zt}. \tag{2.5}$$

Therefore, we have the first-order conditions

$$\Lambda_{l_t(i)} = \frac{W_t}{P_t} - \varphi_t z_t = 0 \tag{2.6}$$

and

$$\Lambda_{\varphi_t} = z_t l_t(i) - y_t(i) = 0. \tag{2.7}$$

Second, each intermediate goods-producing firm seeks to maximize its present discounted value of profits

$$E \sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t],$$

by choosing $\{y_t(i), P_t(i)\}_{t=0}^{\infty}$ subject to the demand for intermediate goods

$$y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t.$$

We can use the latter expression to rewrite the real value of dividends

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right] y_t(i) - \varphi_t y_t(i) - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t \tag{2.8}$$

as

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \varphi_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t. \tag{2.8$'$}$$

Therefore, the firms' intertemporal optimization problem can be written as

$$\max_{P_t(i)} E \sum_{t=0}^{\infty} \Bigg(\beta^t \lambda_t \Bigg\{ \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \varphi_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - \frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t \Bigg\} \Bigg),$$

leading to the following first-order condition:

$$\lambda_t \left\{ (1-\theta) \left[\frac{P_t(i)}{P_t}\right]^{-\theta} \frac{y_t}{P_t} + \varphi_t \theta \left[\frac{P_t(i)}{P_t}\right]^{-\theta-1} \frac{y_t}{P_t} - \phi_p \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right] \frac{y_t}{\pi P_{t-1}(i)} \right\} + \beta E_t \left\{ \lambda_{t+1} \phi_p \left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right] \left[\frac{P_{t+1}(i)}{\pi P_t(i)^2}\right] y_{t+1} \right\} = 0. \tag{2.9}$$

- The monetary authority sets the gross nominal interest rate according to the modified Taylor rule

$$\ln\left(\frac{r_t}{r}\right) = \rho_r \ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t), \tag{2.10}$$

where

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}. \tag{2.11}$$

A.2 The Nonlinear System

A.2.1 Symmetric Equilibrium

The model is characterized by the nonlinear difference equations (2.1) – (2.7), (2.8′), (2.9) – (2.11). To close the model, two additional steps are required. First, we focus on a symmetric equilibrium, where all intermediate goods-producing firms make identical decisions. This implies $P_t(i) = P_t$, $y_t(i) = y_t$, $l_t(i) = l_t$, and $D_t(i) = D_t$ for all $i \in [0,1]$ and $t = 0, 1, 2, ...$ Second, the market clearing condition for the bond market, $B_t = B_{t-1} = 0$ must hold for all $t = 0, 1, 2, ...$ By substituting these conditions into (2.1) – (2.11) we obtain:

$$c_t^{-\sigma} = \lambda_t, \tag{2.1}$$

$$\chi l_t^{\eta} = \lambda_t \frac{W_t}{P_t}, \tag{2.2}$$

$$\frac{\lambda_t}{P_t} = \beta r_t E_t \left(\frac{\lambda_{t+1}}{P_{t+1}} \right), \tag{2.3}$$

$$c_t = \frac{W_t l_t}{P_t} + \frac{D_t}{P_t}, \tag{2.4}$$

$$\ln(z_t) = \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{2.5}$$

$$\frac{W_t}{P_t} = \varphi_t z_t, \tag{2.6}$$

$$y_t = z_t l_t, \tag{2.7}$$

$$\frac{D_t}{P_t} = y_t - \varphi_t y_t - \frac{\phi_P}{2} \left[\frac{P_t}{\pi P_{t-1}} - 1 \right]^2 y_t, \tag{2.8′}$$

$$\begin{aligned} \lambda_t & \left[(1-\theta) y_t + \varphi_t \theta y_t - \phi_p \left(\frac{P_t}{\pi P_{t-1}} - 1 \right) \frac{P_t}{\pi P_{t-1}} y_t \right] \\ & = -\beta E_t \left[\lambda_{t+1} \phi_p \left(\frac{P_{t+1}}{\pi P_t} - 1 \right) \left[\frac{P_{t+1}}{\pi P_t} \right] y_{t+1} \right], \end{aligned} \tag{2.9}$$

$$\ln\left(\frac{r_t}{r}\right) = \rho_r \ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t), \tag{2.10}$$

and

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}. \tag{2.11}$$

A.2.2 Change of Variables and System Reduction

We can rewrite the nonlinear system by defining $\pi_t = \frac{P_t}{P_{t-1}}, w_t = \frac{W_t}{P_t}$, and $d_t = \frac{D_t}{P_t}$. In terms of these re-defined variables (2.1), (2.2), (2.3), (2.4), (2.5) – (2.7), (2.8′), and (2.9) – (2.11) become:

$$c_t^{-\sigma} = \lambda_t, \tag{2.1}$$

$$\chi l_t^{\eta} = \lambda_t w_t, \tag{2.2}$$

$$\lambda_t = \beta r_t E_t \left(\frac{\lambda_{t+1}}{\pi_{t+1}} \right), \tag{2.3}$$

$$c_t = w_t l_t + d_t, \tag{2.4}$$

$$\ln(z_t) = \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{2.5}$$

$$w_t = \varphi_t z_t, \tag{2.6}$$

$$y_t = z_t l_t, \tag{2.7}$$

$$d_t = y_t - \varphi_t y_t - \frac{\phi_P}{2} \left[\frac{\pi_t}{\pi} - 1 \right]^2 y_t, \tag{2.8$'$}$$

$$\begin{aligned} &\lambda_t \left[(1-\theta) y_t + \varphi_t \theta y_t - \phi_p \left(\frac{\pi_t}{\pi} - 1 \right) \left(\frac{\pi_t}{\pi} \right) y_t \right] \\ &= -\beta E_t \left[\lambda_{t+1} \phi_p \left(\frac{\pi_{t+1}}{\pi} - 1 \right) \left[\frac{\pi_{t+1}}{\pi} \right] y_{t+1} \right], \end{aligned} \tag{2.9}$$

$$\ln\left(\frac{r_t}{r}\right) = \rho_r \ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t), \tag{2.10}$$

and

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}. \tag{2.11}$$

Following King and Watson (2002), we apply a system reduction and use equation (2.1), (2.2), (2.6), (2.7), and (2.8′) to eliminate l_t, w_t, d_t, and φ_t. The reduced system can be written as

$$c_t^{-\sigma} = \beta r_t E_t \left(\frac{c_{t+1}^{-\sigma}}{\pi_{t+1}} \right), \tag{2.3$'$}$$

$$y_t = c_t + \frac{\phi_P}{2} \left[\frac{\pi_t}{\pi} - 1 \right]^2 y_t, \tag{2.4$'$}$$

$$\ln(z_t) = \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{2.5}$$

$$c_t^{-\sigma}\left[(1-\theta)y_t+\chi\left(\frac{y_t}{z_t}\right)^{1+\eta}c_t^{\sigma}\theta-\phi_p\left(\frac{\pi_t}{\pi}-1\right)\left(\frac{\pi_t}{\pi}\right)y_t\right] \\ =-\beta E_t\left[c_{t+1}^{-\sigma}\phi_p\left(\frac{\pi_{t+1}}{\pi}-1\right)\left[\frac{\pi_{t+1}}{\pi}\right]y_{t+1}\right], \tag{2.9$'$}$$

$$\ln\left(\frac{r_t}{r}\right)=\rho_r\ln\left(\frac{r_{t-1}}{r}\right)+(1-\rho_r)\left[\omega_\pi\ln\left(\frac{\pi_t}{\pi}\right)+\omega_y\ln\left(\frac{y_t}{y}\right)\right]+\ln(\upsilon_t), \tag{2.10}$$

and

$$\ln(\upsilon_t)=\rho_\upsilon\ln(\upsilon_{t-1})+\varepsilon_{\upsilon t}. \tag{2.11}$$

A.3 Steady States

In absence of the two shocks, i.e., $\varepsilon_{zt}=\varepsilon_{\upsilon t}=0$ for all $t=0,1,2...$, the economy converges to a steady state in which each of the six variables of the reduced system is constant. We use (2.5) and (2.11) to solve for

$$\begin{aligned} z &= 1, \\ \upsilon &= 1. \end{aligned}$$

By assuming that the monetary authority targets zero inflation, implying a gross steady state inflation rate π equal to one, (2.3$'$) can be used to solve for

$$r=\frac{1}{\beta}.$$

Use (2.4$'$) to solve for

$$c=y.$$

Finally use (2.4$'$), (2.5), and (2.9$'$) to solve for

$$y=\left[\frac{1}{\chi}\left(\frac{\theta-1}{\theta}\right)\right]^{\frac{1}{\eta+\sigma}}.$$

A.4 The Linearized System

The nonlinear system (2.3$'$), (2.4$'$), (2.5), (2.9$'$), (2.10), and (2.11) can be linearized by taking a log-linear approximation of the model at steady state values. For a detailed description of logarithmic approximations, we refer to Canova (2007), DeJong and Dave (2007), and Zietz (2008). Let $\widehat{var}_t \equiv log\left(\frac{var_t}{var}\right)$ denote the log-deviation of some variable var_t from its steady state var, where $log\left(\frac{var_t}{var}\right) \approx \frac{var_t - var}{var}$. A first-order Taylor approximation of equation $(2.3') - (2.11)$ at the steady state gives:

$$\hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{\sigma}\left(\hat{r}_t - E_t\hat{\pi}_{t+1}\right), \tag{2.3$'$}$$

$$\hat{y}_t = \hat{c}_t, \tag{2.4$'$}$$

$$\hat{z}_t = \rho_z\hat{z}_{t-1} + \varepsilon_{zt}, \tag{2.5}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \left(\frac{\theta - 1}{\phi_P}\right)(\sigma\hat{c}_t + \eta\hat{y}_t) - \frac{(\theta-1)(1+\eta)}{\phi_P}\hat{z}_t, \tag{2.9$'$}$$

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + (1-\rho_r)(\omega_\pi\hat{\pi}_t + \omega_y\hat{y}_t) + \hat{v}_t, \tag{2.10}$$

and

$$\hat{v}_t = \rho_v\hat{v}_{t-1} + \varepsilon_{vt}. \tag{2.11}$$

By using $(2.4')$ we can rewrite the linearized system as

$$\hat{y}_t = E_t\hat{y}_{t+1} - \frac{1}{\sigma}\left(\hat{r}_t - E_t\hat{\pi}_{t+1}\right), \tag{2.3$''$}$$

$$\hat{z}_t = \rho_z\hat{z}_{t-1} + \varepsilon_{zt}, \tag{2.5}$$

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\theta - 1}{\phi_P}(\eta + \sigma)\hat{y}_t - \frac{\theta - 1}{\phi_P}(1+\eta)\hat{z}_t, \tag{2.9$''$}$$

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + (1-\rho_r)(\omega_\pi\hat{\pi}_t + \omega_y\hat{y}_t) + \hat{v}_t, \tag{2.10}$$

and

$$\hat{v}_t = \rho_v\hat{v}_{t-1} + \varepsilon_{vt}. \tag{2.11}$$

Appendix B

Solving the Model

B.1 Klein's method

A solution of the model can be obtained by applying the approach of Klein (2000) for solving linear difference models under rational expectations. Therefore, the model is brought into the form:

$$AE_t s^0_{t+1} = Bs^0_t + C\zeta_t \tag{2.12}$$

$$\zeta_t = P\zeta_{t-1} + \varepsilon_t, \tag{2.13}$$

where A, B, and C are coefficient matrices, P contains the persistence parameters of the shocks, ζ_t consists of the model's exogenous forcing variables while the serially and mutually uncorrelated innovations are included in ε_t (see DeJong and Dave, 2007). Similar to the approach of Blanchard and Kahn (1980) s^0_t can be separated into

$$s^0_t = [s^0_{1t}\ s^0_{2t}]',$$

letting s^0_{1t} denote a vector of predetermined and s^0_{2t} a vector of non-predetermined variables, which implies that:

$$E_t s_{t+1} = [s^0_{1t+1}\ E_t s^0_{2t+1}]'.$$

The solution method relies on decoupling the system into unstable and stable portions, using a complex generalized Schur decomposition, and then solving the two components in turn. If the number of unstable generalized eigenvalues of

A and B is equal to the number of non-predetermined variables, the system is said to be saddle-path stable and a unique solution exists. In contrast to the method of Blanchard and Kahn (1980), which relies on a Jordan decomposition, Klein's procedure does not require invertibility of matrix A. The subsequent sections follow the expositions in Klein (2000), DeJong and Dave (2007), and the technical notes of Ireland (2011).[1]

B.2 Solution

Let

$$s_t^0 = \begin{bmatrix} \hat{r}_{t-1} & \hat{y}_t & \hat{\pi}_t \end{bmatrix}',$$

$$\zeta_t = \begin{bmatrix} \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$P = \begin{bmatrix} \rho_z & 0 \\ 0 & \rho_v \end{bmatrix},$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}'.$$

Then the coefficient matrices A, B, and C of the model are:

$$A = \begin{bmatrix} -\frac{1}{\sigma} & 1 & \frac{1}{\sigma} \\ 0 & 0 & -\beta \\ 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{\theta-1}{\phi_P}(\eta+\sigma) & -1 \\ \rho_r & (1-\rho_r)\omega_y & (1-\rho_r)\omega_\pi \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 0 \\ -\frac{\theta-1}{\phi_P}(1+\eta) & 0 \\ 0 & 1 \end{bmatrix}.$$

Following Klein (2000), we apply the complex generalized Schur decomposition

[1]The technical notes of Ireland (2011) are available at http://www.irelandp.com/progs/nkp.zip.

of A and B, which is given by

$$QAZ = S$$

and

$$QBZ = T$$

where Q and Z are unitary and S and T are upper triangular matrices. The generalized eigenvalues of B and A can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(B, A) = \{t_{ii}/s_{ii} | i = 1, 2, 3\}.$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues are ordered in increasing value in moving from left to right. Note that one variable in the vector s_t^0 is predetermined and two variables are non-predetermined. Given this setup, Blanchard and Kahn (1980) prove the following three propositions.

- PROPOSITION 1: If the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, then there is a unique solution to the system.

- PROPOSITION 2: If the number of eigenvalues outside the unit circle is greater than the number of non-predetermined variables, then there is no solution to the system.

- PROPOSITION 3: If the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables, then there is an infinite number of solutions.

We proceed under the case of saddle-path stability, assuming exactly two generalized eigenvalues to lie outside the unit circle and therefore allow for a unique solution. The matrices Q, Z, S, and T are portioned, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 is 1×3 and Q_2 is 2×3 and

$$\begin{aligned} Z &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \\ S &= \begin{bmatrix} S_{11} & S_{12} \\ 0_{(2\times 1)} & S_{22} \end{bmatrix}, \\ T &= \begin{bmatrix} T_{11} & T_{12} \\ 0_{(2\times 1)} & T_{22} \end{bmatrix}, \end{aligned}$$

where Z_{11}, S_{11}, and T_{11} are 1×1 and Z_{12}, S_{12}, and T_{12} are 1×2, Z_{21} is 2×1, and Z_{22}, S_{22}, and T_{22} are 2×2. To "triangularize" the system we first define the vector of auxiliary variables as

$$s_t^1 = Z^H s_t^0,$$

letting Z^H denote the conjugate transpose of matrix Z, so that

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z_{11}^H \hat{r}_{t-1} + Z_{21}^H \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} \tag{2.14}$$

is 1×1 and

$$s_{2t}^1 = Z_{12}^H \hat{r}_{t-1} + Z_{22}^H \begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} \tag{2.15}$$

is 2×1. Since Z is unitary, $Z^H Z = I$ or $Z^H = Z^{-1}$ and hence $s_t^0 = Z s_t^1$. We use this property to rewrite (2.12) as

$$AZE_t s_{t+1}^1 = BZ s_t^1 + C\zeta_t.$$

Premultiplying this equation by Q gives

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} s_{1t+1}^1 \\ s_{2t+1}^1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} C\zeta_t,$$

or in matrix partitions,

$$S_{11}E_t s^1_{1t+1} + S_{12}E_t s^1_{2t+1} = T_{11}s^1_{1t} + T_{12}s^1_{2t} + Q_1 C\zeta_t \tag{2.16}$$

and

$$S_{22}E_t s^1_{2t+1} = T_{22}s^1_{2t} + Q_2 C\zeta_t. \tag{2.17}$$

Since the generalized eigenvalues of A and B corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (2.17) can be solved forward to obtain

$$s^1_{2t} = -T_{22}^{-1}R\zeta_t,$$

where the 2×2 matrix R is given by "reshaping"[2]

$$\begin{aligned}
vec(R) &= vec\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2 C P^j = \sum_{j=0}^{\infty} vec[(S_{22}T_{22}^{-1})^j Q_2 C P^j] \\
&= \sum_{j=0}^{\infty}[P^j \otimes (S_{22}T_{22}^{-1})^j]\, vec(Q_2 C) = \sum_{j=0}^{\infty}[P \otimes (S_{22}T_{22}^{-1})]^j\, vec(Q_2 C) \\
&= [I_{(4\times 4)} - P \otimes (S_{22}T_{22}^{-1})]^{-1} vec(Q_2 C).
\end{aligned}$$

Using this result together with equation (2.15) allows to solve for

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = -(Z_{22}^H)^{-1} Z_{12}^H \hat{r}_{t-1} - (Z_{22}^H)^{-1} T_{22}^{-1} R\zeta_t. \tag{2.18}$$

Under the assumption that Z is unitary, i.e.,

$$\underbrace{\begin{bmatrix} Z_{11}^H & Z_{21}^H \\ Z_{12}^H & Z_{22}^H \end{bmatrix}}_{Z^H} \underbrace{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} I_{(1\times 1)} & 0_{(1\times 22)} \\ 0_{(2\times 1)} & I_{(2\times 2)} \end{bmatrix}}_{I},$$

we find

$$Z_{12}^H Z_{11} + Z_{22}^H Z_{21} = 0,$$

[2] As outlined in Hamilton (1994a) and DeJong and Dave (2007) the appearance of the *vec* operator accommodates the VAR specification for ζ_t. We use the relationship between *vec* operator and Kronecker product: $vec[(S_{22}T_{22}^{-1})^j Q_2 C P^j] = [(P^j)' \otimes (S_{22}T_{22}^{-1})^j]vec(Q_2 C)$. Note that $P' = P$, since P is a diagonal matrix.

$$-(Z_{22}^H)^{-1}Z_{12}^H = Z_{21}Z_{11}^{-1},$$

$$Z_{12}^H Z_{12} + Z_{22}^H Z_{22} = I,$$

and

$$(Z_{22}^H)^{-1} = Z_{22} + (Z_{22}^H)^{-1}Z_{12}^H Z_{12} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12},$$

which allows to rewrite (2.18) as

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = M_1\hat{r}_{t-1} + M_2\zeta_t, \tag{2.18$'$}$$

with

$$M_1 = Z_{21}Z_{11}^{-1}$$

and

$$M_2 = -[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}]T_{22}^{-1}R.$$

Now we can solve (2.14) for s_{1t}^1

$$s_{1t}^1 = (Z_{11}^H + Z_{21}^H Z_{21}Z_{11}^{-1})\hat{r}_{t-1} - Z_{21}^H[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}]T_{22}^{-1}R\zeta_t,$$

using

$$Z_{11}^H Z_{11} + Z_{21}^H Z_{21} = I,$$

$$Z_{11}^H + Z_{21}^H Z_{21}Z_{11}^{-1} = Z_{11}^{-1},$$

and

$$Z_{21}^H[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}] = Z_{21}^H Z_{22} - Z_{21}^H Z_{21}Z_{11}^{-1}Z_{12} = -Z_{11}^{-1}Z_{12},$$

so that

$$s_{1t}^1 = Z_{11}^{-1}\hat{r}_{t-1} + Z_{11}^{-1}Z_{12}T_{22}^{-1}R\zeta_t.$$

If we plug this expression result into equation (2.16) we get

$$\hat{r}_t = M_3\hat{r}_{t-1} + M_4\zeta_t, \tag{2.19}$$

where

$$M_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

and

$$M_4 = Z_{11} S_{11}^{-1} (T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 C + S_{12} T_{22}^{-1} RP - T_{12} T_{22}^{-1} R) - Z_{12} T_{22}^{-1} RP.$$

Hence, the model's solution can be written compactly in state space form by combining (2.13), (2.18′), and (2.19) as

$$s_{t+1} = \Gamma_0 s_t + \Gamma_1 \varepsilon_{t+1}, \tag{2.20}$$

and

$$f_t = \Gamma_2 s_t, \tag{2.21}$$

where

$$s_t = \begin{bmatrix} \hat{r}_{t-1} & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{y}_t & \hat{\pi}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}',$$

$$\Gamma_0 = \begin{bmatrix} M_3 & M_4 \\ 0_{(2\times 1)} & P \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0_{(1\times 2)} \\ I_{(2\times 2)} \end{bmatrix},$$

and

$$\Gamma_2 = \begin{bmatrix} M_1 & M_2 \end{bmatrix}.$$

Chapter 3

Testing for Parameter Stability in DSGE Models. The Cases of France, Germany, Italy, and Spain

This chapter is joint work with Jürgen Jerger. It presents an extended version of Jerger and Röhe (2012). We expand the model presented in chapter 2 by adding endogenous capital formation, capital adjustment costs, as well as preference shocks and a shock to the marginal efficiency of investment. Furthermore, we introduce real money balances into the household's utility function, to generate an explicit role for money other than that of a pure unit of account.

3.1 Introduction

DSGE models emerged as a standard tool of modern macroeconometrics. The attractiveness of this class of models lies in the symbiosis of theoretical macroeconomic models with the recent developments in macroeconometric analysis (see DeJong and Dave, 2007). As outlined in Fernández-Villaverde (2010), considerable advances made in both theoretical and empirical DSGE research led to a progressive discipline, reshaping our thinking about macroeconomic modeling and economic policy advise. We contribute to this area of research by employing an econometric technique, recently introduced by Inoue and Rossi (2011), to test for parameter stability in a New Keynesian model estimated for the four largest countries of the European Monetary Union (EMU): France, Germany, Italy, and Spain. Therefore, we add to a vast literature that developed around the topic of economic integration within Europe. One of the important aspects of this ongoing, gradual integration process was the introduction of a common monetary policy in the EMU. Evaluating the overall macroeconomic performance in 2008, the European Commission (2008) summarizes that the record after almost one decade of the EMU looks quite favorable. More detailed analyses of European economic integration can be grouped into four distinct strands of literature. The first looks at the implications of a common currency for other economic institutions like regulation or wage setting (see, for example, von Hagen, 1999; Cukierman and Lippi, 2001; Jerger, 2002; Fratzscher and Stracca, 2009). A second one analyzes the (change of) different transmission channels of monetary policy (Angeloni and Ehrmann, 2006; Hughes Hallett and Richter, 2009; Jarocinski, 2010). Thirdly, the availability of micro-data, especially for loans and prices, led to a large literature studying the economically convergence across countries due to monetary union (Fischer, 2012; Popov and Ongena, 2011). A fourth and relatively recent literature uses DSGE models to characterize the euro area or the economies in this region within some well-defined theoretical framework (see, for example, Smets and Wouters, 2003; Coenen and Wieland, 2005; Casares, 2007; Sahuc and Smets, 2008).

In this chapter we contribute to the last strand and add the dimensions parameter stability over time and cross country comparisons. Therefore we employ the ESS procedure ("Estimate of Set of Stable parameters") introduced by Inoue and Rossi (2011). The ESS procedure allows to pin down the subset of parameters of a model that are stable for an unknown break date. Following Inoue and Rossi (2011, p. 9), "... our analysis focuses on the situation in which there is a single, unanticipated and once for all shift in some of the parameters of the structural model at an unknown time, and in which there is an immediate convergence to a rational-expectations equilibrium after the regime change."

In the case of France, Germany, and Italy our results point to structural breaks after the beginning of the second stage of EMU in the mid-1990s, while the estimates for Spain show a significant break just before the start of the third stage in 1998. We identify significant changes in monetary policy behavior for France, Italy, and Spain, whereas monetary policy in Germany appears to be stable over time. We also find significant declines in capital and price adjustment costs in France, Italy, and Spain.

The rest of the chapter is structured as follows. Section 3.2 describes the model. Section 3.3 discusses data issues. Section 3.4 outlines the ESS procedure. Section 3.5 presents the results. Section 3.6 concludes. Technical details of the model setup, its solution, and the construction of the likelihood appear in the appendices.

3.2 The Model

3.2.1 Overview

The model we use for France, Germany, Italy, and Spain is similar to the standard closed-economy New Keynesian framework developed in Ireland (2003). The model economy features a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a monetary policy authority. During each period $t = 0, 1, 2, ...,$ the intermediate goods-producing firms owned by the household produce a distinct, perishable intermediate good, also indexed by $i \in [0, 1]$. The solution requires these firms to be treated symmetrically.

We choose a closed-economy approach, since openness complicates the modeling of a capital formation process, which is a central part of the present model (see also the discussion by DiCecio and Nelson, 2007, who apply a closed-economy model to the UK, as well as the remarks of Obstfeld, 2002, and Neiss and Nelson, 2003 concerning the choice of closed-economy models).

We next characterize the decisions taken by households and firms before looking at the behavior of the monetary authority and sketching the solution of the model.[1]

3.2.2 Households

The representative household enters period t holding M_{t-1}, B_{t-1}, and k_t units of money, one-period bonds and physical capital. In addition to this endowment, the household receives a lump sum transfer T_t from the monetary authority at

[1] Appendices C and D provide a summary of the complete model and its solution.

the beginning of period t. The household receives $W_t l_t + Q_t k_t$ total nominal factor payments from supplying $l_t(i)$ units of labor and $k_t(i)$ units of capital to each intermediate goods-producing firm $i \in [0,1]$, letting W_t and Q_t denote the nominal wage rate for labor and the nominal rental rate for capital, respectively. For all $t = 0, 1, 2, ...$, the household's choices of $l_t(i)$ and $k_t(i)$ must satisfy

$$l_t = \int_0^1 l_t(i) di,$$

where l_t denotes total hours worked[2], and

$$k_t = \int_0^1 k_t(i) di.$$

Finally, the household earns nominal dividends

$$D_t = \int_0^1 D_t(i) di$$

from the ownership of the intermediate goods-producing firms. Each source of income is measured in units of money.

The household uses its funds to purchase new bonds at the nominal cost B_t/r_t, where r_t denotes the gross nominal interest rate between time periods, or output from the final goods sector at price P_t. This good can be used for consumption c_t or investment i_t. In the latter case, quadratic capital adjustment cost given by

$$\frac{\phi_K}{2} \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 k_t$$

accrue to the household. The parameter $\phi_K \geq 0$ governs the size of these adjustment costs. The capital accumulation process is given by

$$k_{t+1} = (1 - \delta) k_t + x_t i_t, \tag{3.1}$$

with $0 < \delta < 1$ denoting the rate of depreciation and x_t representing a shock to the marginal efficiency of investment. This shock is specified as

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{3.2}$$

[2]The endowment of time is normalized to one.

with $0 < \rho_x < 1$ and $\varepsilon_{xt} \sim N(0, \sigma_x^2)$ as introduced by Greenwood et al. (1988).

The budget constraint of the representative household is given by

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t l_t + Q_t k_t + D_t}{P_t} \geq c_t + i_t + \frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t + \frac{B_t/r_t + M_t}{P_t}.$$

In addition, we impose a no-Ponzi-game condition to prevent the household from excessive borrowing. Given these constraints, the household maximizes the stream of expected utility

$$E\sum_{t=0}^{\infty}\beta^t\left\{a_t\left[\frac{\gamma}{\gamma-1}\right]\ln\left[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}\right] + \chi\ln(1-l_t)\right\},$$

where $0 < \beta < 1$ is a discount factor and $\chi > 0$ measures the relative weight of leisure in the utility function. Further, it can be shown that γ is the absolute value of the interest rate elasticity of money demand. The utility function contains two preference shocks, which are both assumed to follow an autoregressive process. In particular,

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3.3}$$

whith $0 < \rho_a < 1$ and $\varepsilon_{at} \sim N(0, \sigma_a^2)$ denotes an IS shock (McCallum and Nelson, 1999), whereas

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et} \tag{3.4}$$

represents a money demand shock with $0 < \rho_e < 1$, $e > 0$ and $\varepsilon_{et} \sim N(0, \sigma_e^2)$.

3.2.3 Firms

The final good y_t is produced by a firm, acting in a perfectly competitive market, which combines the intermediate goods $y_t(i)$ according to the constant returns to scale technology

$$y_t \leq \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)},$$

where $\theta > 1$ represents the elasticity of substitution between intermediate goods $y_t(i)$. With $P_t(i)$ denoting the price of intermediate good i, profit maximization leads to the following demand function for intermediate goods

$$y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t,$$

where

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)}.$$

Each intermediate good i is produced by a single monopolistically competitive firm according to the constant returns to scale technology

$$y_t(i) \leq k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha},$$

where $1 > \alpha > 0$ represents the elasticity of output with respect to capital. The technology shock z_t follows the autoregressive process

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt} \tag{3.5}$$

with $1 > \rho_z > 0$, $z > 0$ and $\varepsilon_{zt} \sim N(0, \sigma_z^2)$. Although each firm i enjoys some market power on its own output, it is assumed to act as a price taker in the factor markets. Furthermore, the adjustment of its nominal price $P_t(i)$ is assumed to be costly, where the cost function is convex in the size of the price adjustment. Following Rotemberg (1982), these costs are defined as

$$\frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t,$$

where $\phi_P \geq 0$ governs the size of price adjustment costs and π denotes the gross steady state rate of inflation targeted by the monetary authority. Due to these convex adjustment costs, the firm's optimization problem becomes dynamic. It chooses $l_t(i), k_t(i), y_t(i)$, and $P_t(i)$ to maximize its total market value

$$E\sum_{t=0}^{\infty} \beta^t \lambda_t \left[\frac{D_t(i)}{P_t}\right]$$

subject to the demand function for intermediate goods, where λ_t measures the period t marginal utility to the representative household provided by an additional unit of profits. The firm's profits distributed to the household as dividends, are defined in real terms by

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right] y_t(i) - \frac{W_t l_t(i) + Q_t k_t(i)}{P_t} - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t.$$

3.2.4 Monetary policy

Similar to Ireland (2001), monetary policy is represented by a generalized Taylor rule of the form

$$\ln(r_t/r) = \omega_\tau \ln(\tau_t/\tau) + \omega_\pi \ln(\pi_t/\pi) + \omega_y \ln(y_t/y) + \ln(\upsilon_t),$$

encompassing the standard Taylor (1993) rule (when $\omega_\tau = 0$), where the monetary authority changes interest rates in response to inflation and output deviations.[3] If ω_τ is non-zero, monetary policy can be considered to influence a linear combination of the interest rate r_t and the money growth rate $\tau_t = M_t/M_{t-1}$ in response to deviations of gross inflation and detrended output from their steady state values.[4] The latter specification allows for two alternative interpretations. On the one hand, the central bank responds to money growth because it wishes to protect the economy from the effects of money demand shocks, on the other hand, the monetary authority reacts since money growth is a predictor of future inflation (see Christensen and Dib, 2008) and has a predictive value beyond the other variables contained in the Taylor (1993) rule.

The monetary policy shock υ_t follows the autoregressive process

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{3.6}$$

where $0 < \rho_\upsilon < 1$ and $\varepsilon_{\upsilon t} \sim N(0, \sigma_\upsilon^2)$.

This characterization of the monetary authority does not even ask the question of optimal monetary policy. Being aware that there are a lot of alternative specifications of monetary reaction functions and that it might be doubtful to assume an identical specification of the monetary policy function for the four economies under consideration we would like to stress that we are much more interested

[3] The above Taylor rule is a simplified version of the monetary policy rule presented in chapter 2 with $\rho_r = 0$, i.e., no interest smoothing.

[4] Note that the steady state money growth rate τ is assumed to be determined by the monetary authority.

in examining the statistical relationship between short term interest rates, inflation, money growth, and the output gap in four different countries than in issues regarding the specification of monetary policy.

3.2.5 Solution and Estimation

The model is characterized by a set of nonlinear difference equations, namely the first-order conditions for the three agents' problems, the laws of motion for the five exogenous shocks, and the monetary policy rule. Two additional steps are required to close the model. First, to get from sectoral to aggregate variables, symmetric behavior within the intermediate sector is assumed, implying $P_t(i) = P_t$, $y_t(i) = y_t$, $l_t(i) = l_t$, $k_t(i) = k_t$, and $D_t(i) = D_t$ for all $i \in [0, 1]$. Second, the market clearing conditions for both the money market $M_t = M_{t-1} + T_t$ and the bond market $B_t = B_{t-1} = 0$ must hold for all $t = 0, 1, 2....$

Since the model is nonlinear, no exact analytical closed-form solution exists in general. An approximation is obtained by computing the steady state, log-linearizing the system around the steady state, and then applying the method of Blanchard and Kahn (1980) to solve linear difference models under rational expectations (see appendices C and D). The solution takes on the form of a state space representation with a state equation

$$s_{t+1} = \Gamma_0(\mu)s_t + \Gamma_1(\mu)\boldsymbol{\varepsilon}_{t+1}$$

and an observation equation

$$f_t = \Gamma_2(\mu)s_t,$$

where the vector s_t contains the model's state variables including the current capital stock, lagged real balances and the five exogenous shocks. The vector $\boldsymbol{\varepsilon}_{t+1}$ consists of the mutually as well as serially uncorrelated innovations ε_{at+1}, ε_{et+1}, ε_{xt+1}, ε_{zt+1}, ε_{vt+1} while the vector f_t comprises the model's flow variables including current values of consumption, investment, inflation and the nominal interest rate. The matrices $\Gamma_0(\mu)$, $\Gamma_1(\mu)$, and $\Gamma_2(\mu)$ contain (functions of) the parameters μ of the model. These parameters are estimated using maximum likelihood. As outlined in Canova (2007, p. 123), "... the likelihood function of a state space model can be conveniently expressed in terms of one-step-ahead forecast errors, conditional on the initial observations, and of their recursive variance, both of which can be obtained with the Kalman filter."[5] Because likelihoods can have several peaks we use multiple starting values as well as different numerical search algorithms to circumvent stalling at a local peak.[6]

3.3 Data

To estimate the structural parameters of the model we use French, German, Italian, and Spanish quarterly (seasonally adjusted) data for consumption, investment, money balances, inflation, and the interest rate.[7] While French, German, and Italian time series data run from 1980:Q1 to 2008:Q3, we decided to follow Burrriel et al. (2010) and drop the data before 1987:Q1 for Spain because the changes in the structure of the Spanish economy were too substantial in the early 1980s. Consumption and investment are measured by real personal consumption and real gross fixed capital formation in per capita terms. Real money balances are constructed by dividing the monetary aggregate M3 (again per capita) by the consumer price index, which we use to construct a measure of inflation. The interest rate is measured by the three month money market rate.[8]

Following Fagan et al. (2005), we deal with the break in the series for Germany due to re-unification by re-scaling the West German series for consumption, investment, and money prior to re-unification by the ratio of the values for West Germany and Germany at re-unification. We detrend the time series for (logs of) consumption, investment and M3 applying the Hodrick-Prescott (H-P) filter, although we are aware of the potential problem of spuriousness, as pointed out in DeJong and Dave (2007) and Canova and Ferroni (2011).[9]

Despite its relative simplicity the model contains a large number of parameters that are difficult to estimate precisely on only five time series. Hence, a number of parameters had to be fixed prior to estimation. The value of χ is set to 1.5

[5] For a detailed description of the Kalman filter we refer to appendix E.

[6] Therefore, we implement Christopher Sims' hybrid optimization algorithm "csminwel", which combines the derivative-based Broyden-Fletcher-Goldfarb-Shannon (BFGS) method with a simplex algorithm (see DeJong and Dave, 2007 and Heer and Maussner, 2009 for details). The "csminwel" program is available at http://sims.princeton.edu/yftp/optimize/.

[7] Since the model contains as many structural shocks as observable variables the problem of stochastic singularity is avoided (see Ingram, Kocherlakota and Savin, 1994).

[8] Appendix F presents the data sources.

[9] To facilitate the process of parameter estimation, we follow DeJong and Dave (2007, Chapter 11.2.5) and perform further data alignment by scaling the filtered series using their (relative) means.

which implies that the representative household's labor supply in the steady state amounts to one-third of its time. In addition, the depreciation rate δ is set to 0.025, corresponding to an annual depreciation rate of about 10 percent and θ is fixed at 6, implying a steady state markup of prices over marginal cost of 20 percent. Lastly, we set the elasticities of output with respect to capital of each country equal to their respective average capital income share, calculated from OECD data. The steady state money growth rate of each country is set equal to the average rate of inflation for the whole sample under consideration.

3.4 Estimating the Set of Stable Parameters: The ESS Procedure

In this section we outline the ESS procedure developed by Inoue and Rossi (2011), which allows to identify the subset of parameters of a model that are stable over time. They propose the following recursive procedure. First, test the joint null hypothesis that all parameters are stable, using a consistent test for structural breaks. Following Inoue and Rossi (2011), we employ Andrews' (1993) Quandt Likelihood Ratio (QLR) stability test. If the null hypothesis is not rejected, then all the parameters belong to the set of stable parameters. If it is, the p-values of the individual test statistics are calculated to test whether each of the parameters is stable. Then the parameter with the lowest p-value is eliminated from the set of stable parameters, since this is the one that is most likely to be unstable. Second, it is tested whether the remaining parameters are jointly stable. If they are, then the set of stable parameters includes those parameters; otherwise, eliminate the parameter with the second lowest p-value from the set, and continue this procedure until the joint test on the remaining parameters does not reject stability.

Two specific features of the ESS procedure have to be emphasized:

(i) The individual tests do not rely on the assumption that the other parameters are constant over time. „If the parameters that are assumed to be constant are in reality time-varying, [a "one at a time" approach] may incorrectly attribute the time variation to the wrong source" (Inoue and Rossi, 2011, p. 1186). Therefore, the individual tests allow all the other parameters to be time-varying.

(ii) The ESS approach overcomes the problem of size distortions, which arises "... in existing tests for structural breaks when used repeatedly to test structural changes in more than one subset of parameters" (Inoue and Rossi, 2011, p. 1203).[10]

3.5 Results

3.5.1 Full Sample Estimates

Here we first report the estimates for the whole sample before moving to the identification of parameter instabilities in section 3.5.2.

For each country table 3.1 presents the full sample maximum likelihood estimates of the parameters as well as the respective standard errors. The latter are computed using a parametric bootstrapping technique as in Cho and Moreno (2006) or Ireland (2007). According to Ireland (2007), this procedure simulates the estimated model for each country to generate 1000 samples of artificial data for real personal consumption, real gross fixed capital formation, real money balances, inflation, and the short term interest rate, each containing the same number of observations as the original samples of the four EMU countries, and then re-estimates the model 1000 times using these artificial data sets. For a detailed description of the parametric bootstrapping analysis we refer to Efron and Tibshirani (1993). The absolute value of the maximized log likelihood function is indicated by $|L|$.

To compare parameter estimates of the full samples across countries, we employ the Andrews and Fair (1988) Wald test. The Wald statistic can be written as

$$W = \frac{(a_i - a_j)^2}{\sigma_{a_i}^2 + \sigma_{a_j}^2},$$

where a and σ_a denote the point estimate of a parameter and the associated bootstrapped standard deviation, respectively, for country $i, j \in \{$France, Germany, Italy, Spain$\}$, $i \neq j$. The test statistic W follows a $\chi^2(1)$ distribution under the null hypothesis of $a_i = a_j$. For a detailed discussion on the use of the bootstrap

[10] For a more detailed description of the methodology, including a formal description of the algorithm and proofs, we refer to Inoue and Rossi (2011) as well as to their not-for-publication appendix; see http://econ.duke.edu/ brossi/NotforPublicationAppendixInoueRossi2009.pdf.

	France		Germany		Italy		Spain	
Parameter	Estimate	Std. Error	Estimate	Std. Error	Estimate	Std. Error	Estimate	Std. Error
β	0.9905	0.0112	0.9921	0.0008	0.9998	0.0410	0.9932	0.0277
γ	0.0152	0.0091	0.0738	0.0079	0.0067	0.0157	0.0366	0.0334
ϕ_P	10.2132	2.7778	13.9927	0.3103	46.7997	14.2531	27.0936	6.9245
ϕ_K	26.5408	4.1028	30.2681	0.4301	35.4014	6.2435	20.5672	3.4103
ω_τ	0.2009	0.0411	0.4362	0.0078	0.5647	0.1378	0.3163	0.0832
ω_π	0.9391	0.1491	1.6001	0.0020	1.0750	0.4163	0.8161	0.0901
ω_y	-0.1011	0.0842	-0.0024	0.0039	-0.1673	0.1378	-0.0711	0.0495
e	4.1884	0.0202	2.9638	0.0001	3.7456	0.9409	4.3559	0.0056
z	4214.3794	0.0001	4184.4958	0.0001	3189.9297	0.0080	1866.9879	0.0001
ρ_a	0.9678	0.0357	0.9003	0.0025	0.8587	0.0782	0.9731	0.0221
ρ_e	0.8778	0.0552	0.8796	0.0023	0.9877	0.0350	0.9360	0.0373
ρ_x	0.9615	0.0381	0.9061	0.0008	0.9873	0.0386	0.9294	0.1063
ρ_z	0.9125	0.0318	0.9162	0.0019	0.9871	0.0626	0.9210	0.0518
ρ_v	0.4826	0.0096	0.2397	0.0108	0.1425	0.3356	0.3818	0.0121
σ_a	0.0124	0.0012	0.0149	0.0010	0.0258	0.0178	0.0189	0.0020
σ_e	0.0096	0.0007	0.0145	0.0012	0.0135	0.0030	0.0102	0.0003
σ_x	0.0236	0.0201	0.0816	0.0065	0.2162	0.1450	0.0182	0.0094
σ_z	0.0090	0.0012	0.0141	0.0010	0.0334	0.0052	0.0140	0.0014
σ_v	0.0041	0.0007	0.0070	0.0005	0.0105	0.0028	0.0069	0.0008
$\lvert L\rvert$	2195.2950		2038.4201		1891.2450		1553.1251	

Table 3.1: Maximum Likelihood Estimates: Full Samples.

in hypothesis testing we refer to Cameron and Trivedi (2005).[11]

Turning to our results, we first note that the estimates for the discount factor β are below unity, but exceed 0.99 for all of the four economies.

The money demand equation derived from the household's optimization problem implies an interest elasticity for real money holdings of $-\gamma$. We find small values of this elasticity with the correct sign for all regions, although the estimates for Italy and Spain turn out to be statistically insignificant. These results are in line with a large empirical literature detecting small interest rate elasticities of (broad) money demand (see Browne et al., 2005).

Next, we turn to the estimates for the rigidity parameters. For all countries, both the adjustment cost parameters for capital ϕ_K defined in section 3.2.2 and prices ϕ_P defined in section 3.2.3 are significant. The latter is significantly higher in Italy and Spain compared to France and Germany at the 5% and 10% level, respectively. Our findings are confirmed by the results of analysis on consumer price changes conducted by Dhyne et al. (2006), identifying Italy to have the lowest incidence of price changes, whereas France shows the highest frequency of price changes among the four regions.

[11]A full set of the test statistics is available from the authors upon request.

Turning to the monetary policy reaction function, our estimates of ω_π and ω_τ are non-zero for all four countries, allowing at least for two possible interpretations of monetary policy (see section 3.2.4). Compared to France and Spain ω_π is significantly higher in Germany (at the 1% level). This result might reflect the well-documented higher pre-occupation with inflation in this country. The point estimate of ω_π for Italy is also well below the estimate for Germany, although insignificantly so. Concerning the positive estimates of ω_τ our results are consistent with the findings of Andrés et al. (2006) for the euro area. It is important to note that for each of the four countries the estimates of ω_τ and ω_π sum up to a value greater than unity. Hence, the monetary policy rule is consistent with a unique rational expectations equilibrium (see Clarida et al., 2000). For all countries the estimates of ω_y are negative. However, they are insignificant, which makes it difficult to interpret this result as a hint for the presence of an endogenous money channel.

The estimates of e and z are not interesting from an economic policy point of view; they simply allow the steady state values of real balances and output in the model to match the average values of these variables in the data (see Ireland, 1997).

The estimates of ρ_a, ρ_e, ρ_x, ρ_z, and ρ_v indicate a high persistence of the first four shocks, whereas the monetary policy shock is less persistent and even statistically insignificant for Italy. In the case of France, Germany, and Italy, the estimated standard deviations of the innovations are dominated by the ones of the investment shock, although the estimate of σ_x turns out to be insignificant for Italy. This result is consistent with the findings of Justiniano et al. (2010) for the US. Hence, the marginal efficiency of investment shock is identified as the most important driver of business cycle fluctuations. For Spain the preference shock is the most volatile followed by the marginal efficiency of investment shock.

3.5.2 Testing for Parameter Instability

For each country tables G.1 – G.4 report the parameter estimates and standard deviations in both sub-samples, while tables G.5 – G.8 show the p-values of the QLR test on individual parameters as well as the p-values at each step of the ESS procedure. The set of stable parameters at the 10% significance level is denoted by $\mathcal{S}$. To structure the following discussion, it is useful to divide the parameters into three groups:

(i) private sector parameters: $\beta, \gamma, \phi_P, \phi_K$;

(ii) monetary policy parameters: $\omega_\tau, \omega_\pi, \omega_y$;

(iii) shock parameters: $e, z, \rho_a, \rho_e, \rho_x, \rho_z, \rho_\upsilon, \sigma_a, \sigma_e, \sigma_x, \sigma_z$, and σ_υ.

In the case of France, the QLR stability test indicates a significant break in 1994:Q3. Concerning the private sector parameters, table G.5 reports instabilities of γ and ϕ_P. The estimates of γ are lower in both sub-samples than in the full sample, the estimate for the 1980:Q1 to 1994:Q2 period is insignificant, however. Table G.1 shows a sharp decline of the price rigidity parameter ϕ_P. Further, we find significant changes in the monetary policy parameters ω_y and ω_π, both increasing in absolute values. Concerning the shock parameters, the ESS procedure identifies only the technology shock to be stable with respect to both persistence and volatility. The direction of change in the persistence of the remaining shocks is ambiguous, while we find an overall decline in the volatilities $\sigma_a, \sigma_e, \sigma_x$, and σ_υ.

For Germany we locate a break in 1994:Q2.[12] As reported in table G.6, the set of stable parameters $\mathcal{S}$ contains $(\sigma_x, \sigma_e, \omega_\tau, \omega_\pi, \rho_e, \rho_a, \rho_z, \omega_y, \rho_x)$. Most interestingly, we find monetary policy to be constant over time. This result suggests no discernible difference between the monetary policy conducted in the 1980:Q2 to 1994:Q1 period by the German Bundesbank and the 1994:Q2 to 2008:Q3 period, although the latter is affected by the inception of EMU and the monetary policy strategy of the ECB. Further, we find instabilities in all of the private sector parameters, as well as the persistence of the monetary policy shock and the volatilities of the preference shock a_t, the technology shock z_t and the monetary policy shock υ_t. Concerning the direction of change, only the volatility of the monetary policy shock increases, while the volatilities of the other shocks decline or stay constant over time.

We detect a significant break in 1994:Q4 for Italy. With respect to the private sector parameters, table G.7 shows instabilities of γ, ϕ_P, and ϕ_K. According to

[12] We cannot rule out a test bias due to the treatment of re-unification outlined in section 3.

table G.3 the interest elasticity of money demand turns out to increase over time, while we find a significant decline in capital and price adjustment costs after the break. Concerning the monetary policy parameters, ω_y appears to be stable over time, whereas ω_τ and ω_π both change significantly. More specifically, table G.3 presents a sharp decline of ω_τ and a substantial increase of ω_π in the 1994:Q4 to 2008:Q3 period. With exception of ρ_x, we find the persistence parameters to be unstable. While ρ_a and ρ_v increase, ρ_e and ρ_z turn out to decrease after the break. Regarding the volatilities of the five shocks, the ESS procedure identifies σ_a and σ_e to be stable, whereas σ_v, σ_x and σ_z decrease over time.

Turning to Spain, we find a significant break in 1998:Q1. Moreover, we detect instabilities in the private sector parameters (γ, ϕ_P, ϕ_K), the monetary policy parameters $(\omega_\tau, \omega_\pi)$ and the shock parameters $(e, z, \rho_z, \rho_v, \sigma_a, \sigma_e, \sigma_x, \sigma_z$, and $\sigma_v)$. While ω_τ decreases, ω_π is significantly higher after the break (see table G.4). Furthermore, we observe a sharp decline in capital and price adjustment costs. Regarding the persistence of the technology shock and the money policy shock, table G.4 shows a decrease in both, while the latter declines sharply after the break. With the exception of the money demand shock, we also find a decrease in the volatilities of the shocks a_t, x_t, z_t, and v_t.

3.6 Conclusions

Despite some skepticism voiced in the literature DSGE models became a cornerstone of modern macroeconometrics leading to a high acceptance both in academia and central banking (see Tovar, 2009). Being firmly rooted in microeconomic foundations, this class of models is able to identify structural characteristics of economies that are not easily recovered from a necessarily parsimonious set of macroeconomic time series. Apart from their frequent use as a tool for the description and evaluation of monetary policy, DSGE models enable cross-country comparisons of such characteristics without having to resort to micro-data (see Smets and Wouters, 2005).

In this chapter, we apply a New Keynesian model to French, German, Italian, and Spanish data and formally test for parameter stability over time. Parameter instabilities are detected by the ESS procedure developed by Inoue and Rossi (2011). This procedure allows to identify the parameters of the model that have changed at an unknown break date. In the cases of France, Germany, and Italy our results point to structural breaks in the mid-1990s after the beginning of the second stage of EMU, while the estimates for Spain show a significant break just before the start of the third stage of EMU in 1998. An interesting result is that France, Italy, and Spain show significant changes in monetary policy behavior after the break dates, while monetary policy in Germany is found to be stable over time. Furthermore, France, Italy, and Spain exhibit a significant decline in capital and price adjustment costs after the break. Moreover, we find at least four out of the five shocks to be either constant or declining after the break date for all economies under consideration.

On a methodological level, we demonstrate that the use of DSGE models is able to sheds some light on the process of economic integration in Europe by allowing to look at the stability of structural and policy parameters both across countries and across time. This process yields numerous explanations for changes of allegedly "deep" parameters questioning the full compliance with the well-known Lucas (1976) critique. However, as set out in Inoue and Rossi (2011, p. 1195), "... the definition of structural parameters (in the sense of the Lucas critique) is that these parameters are policy invariant, not necessarily time invariant." Therefore, future research faces an important challenge in developing techniques able to identify the specific factors responsible for parameter instabilities, allowing to assess the applicability of the respective DSGE setting for policy analysis and forecasting.

Appendices

Appendix C

Equilibrium Conditions

The appendix contains a detailed description of the estimated DSGE model. The exposition is based on the technical notes of Ireland (2003).[1]

C.1 The Economic Environment

- Households:
  The representative household chooses $\{c_t, l_t, M_t, B_t, k_{t+1}, i_t\}_{t=0}^{\infty}$ to maximize utility

$$E \sum_{t=0}^{\infty} \beta^t \{a_t[\gamma/(\gamma-1)] \ln[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}] + \chi \ln(1-l_t)\},$$

  subject to the budget constraint

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t l_t + Q_t k_t + D_t}{P_t} \geq c_t + i_t + \frac{\phi_k}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t + \frac{B_t/r_t + M_t}{P_t},$$

  and the law of motion for capital

$$k_{t+1} = (1-\delta)k_t + x_t i_t. \tag{3.1}$$

[1] The technical notes are available at http://www.irelandp.com/progs/endogenous.zip.

Further, following Buiter and Sibert (2007), we prevent the household from excessive debts by imposing the no-Ponzi-game condition:

$$\lim_{t\to\infty} B_t \prod_{s=0}^{t} \frac{1}{r_s} \geq 0.$$

Accordingly the Lagrangian can be written as follows:

$$\begin{aligned}\Lambda \;=\; & E\sum_{t=0}^{\infty}\Bigg(\beta^t\Bigg\{a_t\,[\gamma/(\gamma-1)]\ln\left[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}\left(\frac{M_t}{P_t}\right)^{(\gamma-1)/\gamma}\right]+\chi\ln(1-l_t)\Bigg\} \\ & -\beta^t\lambda_t\Bigg\{c_t+\left[\frac{k_{t+1}-(1-\delta)k_t}{x_t}\right]+\frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t}-1\right)^2 k_t+\frac{B_t/r_t+M_t}{P_t} \\ & -\left(\frac{M_{t-1}+T_t+B_{t-1}+W_t l_t+Q_t k_t+D_t}{P_t}\right)\Bigg\}\Bigg).\end{aligned}$$

The first-order conditions are obtained by setting the partial derivatives of Λ with respect to $c_t, l_t, M_t, B_t, k_{t+1}$, and λ_t equal to zero, yielding

$$\Lambda_{c_t} = a_t - \lambda_t c_t^{1/\gamma}\left[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}\left(\frac{M_t}{P_t}\right)^{(\gamma-1)/\gamma}\right] = 0, \tag{3.7}$$

$$\Lambda_{l_t} = \chi - \lambda_t\left(\frac{W_t}{P_t}\right)(1-l_t) = 0, \tag{3.8}$$

$$\begin{aligned}\Lambda_{M_t} = & \left(\frac{M_t}{P_t}\right)^{1/\gamma}\left[c_t^{(\gamma-1)/\gamma}+e_t^{1/\gamma}\left(\frac{M_t}{P_t}\right)^{(\gamma-1)\gamma}\right]\left[\lambda_t-\beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right)\right] \\ & - a_t e_t^{1/\gamma} = 0,\end{aligned} \tag{3.9}$$

$$\Lambda_{B_t} = \lambda_t - \beta r_t E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right) = 0, \tag{3.10}$$

$$
\begin{aligned}
\Lambda_{k_{t+1}} = \lambda_t \left[\frac{1}{x_t} + \phi_K \left(\frac{k_{t+1}}{k_t} - 1 \right) \right] \\
- \left\{ \beta E_t \left[\lambda_{t+1} \left(\frac{Q_{t+1}}{P_{t+1}} + \frac{1-\delta}{x_{t+1}} \right) \right] \right. \\
- \left(\frac{\beta \phi_K}{2} \right) E_t \left[\lambda_{t+1} \left(\frac{k_{t+2}}{k_{t+1}} - 1 \right)^2 \right] \\
\left. + \beta \phi_K E_t \left[\lambda_{t+1} \left(\frac{k_{t+2}}{k_{t+1}} - 1 \right) \left(\frac{k_{t+2}}{k_{t+1}} \right) \right] \right\} = 0,
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
\Lambda_{\lambda_t} = c_t + \left[\frac{k_{t+1} - (1-\delta) k_t}{x_t} \right] + \frac{\phi_K}{2} \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 k_t + \frac{B_t / r_t + M_t}{P_t} \\
- \left(\frac{M_{t-1} + T_t + B_{t-1} + W_t l_t + Q_t k_t + D_t}{P_t} \right) = 0.
\end{aligned}
\tag{3.12}
$$

Note that we can rewrite (3.9) by using (3.7) and (3.10) to obtain

$$
c_t e_t - \left(\frac{M_t}{P_t} \right) \left(1 - \frac{1}{r_t} \right)^{\gamma} = 0. \tag{3.9$'$}
$$

Finally, we impose the standard transversality conditions to guarantee that money, bonds and capital do not grow too quickly:

$$
\begin{aligned}
\lim_{t \to \infty} \beta^t \lambda_t \frac{M_t}{P_t} = 0, \\
\lim_{t \to \infty} \beta^t \lambda_t \frac{B_t}{P_t} = 0, \\
\lim_{t \to \infty} \beta^t \lambda_t k_{t+1} = 0.
\end{aligned}
$$

- Finished goods-producing firms:
  The representative finished goods-producing firm seeks to maximize its profits

$$
P_t y_t - \int_0^1 P_t(i) y_t(i) di
$$

subject to the constant returns to scale technology

$$
y_t \leq \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)}.
$$

Therefore, the firm's optimization problem can be written as

$$\max_{y_t(i)} \Pi_t = P_t \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} - \int_0^1 P_t(i) y_t(i) di,$$

which leads to the following first-order condition characterizing the demand for intermediate goods:

$$\frac{\partial \Pi_t}{\partial y_t(i)} = y_t(i) - \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t = 0.$$

By plugging this expression into the constant elasticity of substitution aggregator of intermediate goods we obtain the price aggregator

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{1/(1-\theta)}.$$

- Intermediate goods-producing firms:
  Each intermediate goods-producing firm seeks to maximize its present discounted value of profits

$$E \sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t],$$

by choosing $\{l_t(i), k_t(i), y_t(i), P_t(i)\}_{t=0}^{\infty}$ subject to the Cobb-Douglas technology constraint

$$y_t(i) \leq k_t(i)^{\alpha} [z_t l_t(i)]^{1-\alpha}$$

and the above demand for intermediate goods

$$y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t.$$

We can use the latter expression to rewrite the real value of dividends

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t} \right] y_t(i) - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t} \right] - \frac{\phi_P}{2} \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 y_t$$

as

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t}\right] - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t. \tag{3.13}$$

Therefore, the Lagrangian for the firms' intertemporal optimization problem can be written as:

$$\begin{aligned} \Lambda &= E\sum_{t=0}^{\infty}\Bigg(\beta^t\lambda_t\Bigg\{\left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t}\right] - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t\Bigg\} \\ &\quad -\beta^t\xi_t\Bigg\{\left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha}\Bigg\}\Bigg). \end{aligned}$$

Setting the partial derivatives of Λ with respect to $l_t(i), k_t(i), P_t(i)$, and ξ_t equal to zero leads to the first order conditions:

$$\Lambda_{l_t(i)} = \frac{\lambda_t W_t l_t(i)}{P_t} - (1-\alpha)\xi_t k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha} = 0, \tag{3.14}$$

$$\Lambda_{k_t(i)} = \frac{\lambda_t Q_t k_t(i)}{P_t} - \alpha\xi_t k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha} = 0, \tag{3.15}$$

$$\begin{aligned} \Lambda_{P_t(i)} &= \phi_P\lambda_t\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]\left[\frac{P_t}{\pi P_{t-1}(i)}\right] \\ &\quad -(1-\theta)\lambda_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta} - \theta\xi_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta-1} \\ &\quad -\beta\phi_P E_t\left\{\lambda_{t+1}\left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)P_t}{\pi P_t(i)^2}\right]\left(\frac{y_{t+1}}{y_t}\right)\right\} = 0, \end{aligned} \tag{3.16}$$

and

$$\Lambda_{\xi_t} = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - k_t(i)^{\alpha}[z_t l_t(i)]^{1-\alpha} = 0. \tag{3.17}$$

- The monetary authority sets the gross nominal interest rate according to the generalized Taylor rule:

$$\ln\left(\frac{r_t}{r}\right) = \omega_\tau \ln\left(\frac{\tau_t}{\tau}\right) + \omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right) + \ln(\upsilon_t). \tag{3.18}$$

C.2 The Nonlinear System

C.2.1 Symmetric Equilibrium

The dynamic system is described by the nonlinear difference equations (3.1) – (3.8), (3.9′), (3.10) – (3.18). To close the model, we complete the following two steps. First, we consider a symmetric equilibrium where all intermediate goods-producing firms make identical decisions. This assumption implies $P_t(i) = P_t$, $y_t(i) = y_t$, $l_t(i) = l_t$, $k_t(i) = k_t$, and $D_t(i) = D_t$ for $t = 0, 1, 2...$ and all $i \in [0, 1]$. Second, the market clearing condition for both the bond market, $B_t = B_{t-1} = 0$, and the money market, $M_t = M_{t-1} + T_t$, must hold for all $t = 0, 1, 2....$ By substituting these conditions into (3.1) – (3.18) and defining the average product of labor as $n_t = y_t/l_t$ and the money growth rate as $\tau_t = \frac{M_t}{M_{t-1}}$ we get:

$$k_{t+1} = (1-\delta)k_t + x_t i_t, \tag{3.1}$$

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{3.2}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3.3}$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3.4}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{3.5}$$

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{3.6}$$

$$a_t = \lambda_t c_t^{1/\gamma} \left[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma} \left(\frac{M_t}{P_t} \right)^{(\gamma-1)/\gamma} \right], \tag{3.7}$$

$$\chi = \lambda_t \left(\frac{W_t}{P_t} \right) (1 - l_t), \tag{3.8}$$

$$c_t e_t = \left(\frac{M_t}{P_t} \right) \left(1 - \frac{1}{r_t} \right)^{\gamma}, \tag{3.9′}$$

$$\lambda_t = \beta r_t E_t \left(\lambda_{t+1} \frac{P_t}{P_{t+1}} \right), \tag{3.10}$$

$$\lambda_t \left[\frac{1}{x_t} + \phi_K \left(\frac{k_{t+1}}{k_t} - 1\right)\right] = \beta E_t \left[\lambda_{t+1} \left(\frac{Q_{t+1}}{P_{t+1}} + \frac{1-\delta}{x_{t+1}}\right)\right] - \left(\frac{\beta\phi_K}{2}\right) E_t \left[\lambda_{t+1} \left(\frac{k_{t+2}}{k_{t+1}} - 1\right)^2\right] + \beta\phi_K E_t \left[\lambda_{t+1} \left(\frac{k_{t+2}}{k_{t+1}} - 1\right) \left(\frac{k_{t+2}}{k_{t+1}}\right)\right], \tag{3.11}$$

$$c_t + \left[\frac{k_{t+1} - (1-\delta)k_t}{x_t}\right] + \frac{\phi_K}{2} \left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t = \left(\frac{W_t l_t + Q_t k_t + D_t}{P_t}\right), \tag{3.12}$$

$$\frac{D_t}{P_t} = y_t - \frac{W_t l_t + Q_t k_t}{P_t} - \frac{\phi_P}{2} \left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 y_t, \tag{3.13}$$

$$\lambda_t \left(\frac{W_t}{P_t}\right) l_t = (1-\alpha)\xi_t k_t^\alpha [z_t l_t]^{1-\alpha}, \tag{3.14}$$

$$\lambda_t \left(\frac{Q_t}{P_t}\right) k_t = \alpha \xi_t k_t^\alpha [z_t l_t]^{1-\alpha}, \tag{3.15}$$

$$\phi_P \lambda_t \left[\frac{P_t}{\pi P_{t-1}} - 1\right] \left[\frac{P_t}{\pi P_{t-1}}\right] = (1-\theta)\lambda_t + \theta\xi_t + \beta\phi_P E_t \left[\lambda_{t+1} \left(\frac{P_{t+1}}{\pi P_t} - 1\right) \left(\frac{P_{t+1}}{\pi P_t}\right) \left(\frac{y_{t+1}}{y_t}\right)\right], \tag{3.16}$$

$$y_t = k_t^\alpha [z_t l_t]^{1-\alpha}, \tag{3.17}$$

$$\ln\left(\frac{r_t}{r}\right) = \omega_\tau \ln\left(\frac{\tau_t}{\tau}\right) + \omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right) + \ln(\upsilon_t), \tag{3.18}$$

$$n_t = \frac{y_t}{l_t}, \tag{3.19}$$

and

$$\tau_t = \frac{M_t}{M_{t-1}}. \tag{3.20}$$

Note that we can rewrite (3.12) by using (3.13) to obtain:

$$y_t = c_t + i_t + \frac{\phi_K}{2} \left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t + \frac{\phi_P}{2} \left[\frac{P_t}{\pi P_{t-1}} - 1\right]^2 y_t. \tag{3.12$'$}$$

Further, (3.17) can be used to rewrite (3.14) and (3.15) as

$$\lambda_t \left(\frac{W_t}{P_t}\right) l_t = (1-\alpha)\xi_t y_t \tag{3.14$'$}$$

and

$$\lambda_t \left(\frac{Q_t}{P_t}\right) k_t = \alpha \xi_t y_t. \tag{3.15$'$}$$

C.2.2 Change of Variables

We can rewrite the nonlinear system by defining $\pi_t = \frac{P_t}{P_{t-1}}$, $m_t = \frac{M_t}{P_t}$, $w_t = \frac{W_t}{P_t}$, $q_t = \frac{Q_t}{P_t}$, and $d_t = \frac{D_t}{P_t}$. With these re-defined variables, (3.1)–(3.8), (3.9′), (3.10), (3.11), (3.12′), (3.13), (3.14′), (3.15′), (3.16) – (3.20) become:

$$k_{t+1} = (1-\delta)k_t + x_t i_t, \tag{3.1}$$

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{3.2}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{3.3}$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{3.4}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{3.5}$$

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{3.6}$$

$$a_t = \lambda_t c_t^{1/\gamma}\left[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma} m_t^{(\gamma-1)/\gamma}\right], \tag{3.7}$$

$$\chi = \lambda_t w_t (1-l_t), \tag{3.8}$$

$$c_t e_t = m_t \left(1 - \frac{1}{r_t}\right)^{\gamma}, \tag{3.9$'$}$$

$$\lambda_t = \beta r_t E_t \left(\frac{\lambda_{t+1}}{\pi_{t+1}}\right), \tag{3.10}$$

$$\begin{aligned}
\lambda_t \left[\frac{1}{x_t} + \phi_K \left(\frac{k_{t+1}}{k_t} - 1\right)\right] &= \beta E_t \left[\lambda_{t+1}\left(q_{t+1} + \frac{1-\delta}{x_{t+1}}\right)\right] \\
&\quad - \left(\frac{\beta\phi_K}{2}\right) E_t \left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)^2\right] \\
&\quad + \beta\phi_K E_t \left[\lambda_{t+1}\left(\frac{k_{t+2}}{k_{t+1}} - 1\right)\left(\frac{k_{t+2}}{k_{t+1}}\right)\right],
\end{aligned} \tag{3.11}$$

$$y_t = c_t + i_t + \frac{\phi_K}{2}\left(\frac{k_{t+1}}{k_t} - 1\right)^2 k_t + \frac{\phi_P}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{3.12$'$}$$

$$d_t = y_t - w_t l_t - q_t k_t - \left(\frac{\phi_P}{2}\right)\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t, \tag{3.13}$$

$$\lambda_t w_t l_t = (1-\alpha)\xi_t y_t, \tag{3.14$'$}$$

$$\lambda_t q_t k_t = \alpha \xi_t y_t, \tag{3.15$'$}$$

$$\begin{aligned} \phi_P \lambda_t \left(\frac{\pi_t}{\pi} - 1\right)\left(\frac{\pi_t}{\pi}\right) &= (1-\theta)\lambda_t + \theta\xi_t \\ &+ \beta\phi_P E_t\left[\lambda_{t+1}\left(\frac{\pi_{t+1}}{\pi} - 1\right)\left(\frac{\pi_{t+1}}{\pi}\right)\left(\frac{y_{t+1}}{y_t}\right)\right], \end{aligned} \tag{3.16}$$

$$y_t = k_t^{\alpha}[z_t l_t]^{1-\alpha}, \tag{3.17}$$

$$\ln\left(\frac{r_t}{r}\right) = \omega_\tau \ln\left(\frac{\tau_t}{\tau}\right) + \omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right) + \ln(\upsilon_t), \tag{3.18}$$

$$n_t = \frac{y_t}{l_t}, \tag{3.19}$$

and

$$\tau_t = \left(\frac{m_t}{m_{t-1}}\right)\pi_t. \tag{3.20}$$

C.3 Steady States

In absence of the five shocks, i.e., $\varepsilon_{xt} = \varepsilon_{at} = \varepsilon_{et} = \varepsilon_{zt} = \varepsilon_{vt} = 0$ for all $t = 0, 1, 2, ...$, the economy converges to a steady state, where each of the 20 variables is constant. We use (3.2), (3.3), (3.4), (3.5), and (3.6) to solve for

$$\begin{aligned} x &= 1, \\ a &= 1, \\ e &= e, \\ z &= z, \\ \upsilon &= 1. \end{aligned}$$

Assuming that the steady state money growth rate τ is determined by policy, (3.10) and (3.20) can be used to solve for

$$\pi = \tau$$

and

$$r = \frac{\pi}{\beta}.$$

Next, (3.11) and (3.16) can be used to solve for

$$q = \frac{1}{\beta} - 1 + \delta$$

and

$$\xi = \left[\frac{(\theta - 1)}{\theta}\right] \lambda.$$

Equations (3.7) and (3.9′) can be used to solve for

$$c = \left[1 + e\left(\frac{r}{r-1}\right)^{\gamma - 1}\right]^{-1} \left(\frac{1}{\lambda}\right)$$

and

$$m = e\left(\frac{r}{r-1}\right)^{\gamma} c.$$

Use (3.1), (3.12′), (3.15′), and (3.16) to solve for

$$y = \left[1 - \delta\left(\frac{\alpha}{q}\right)\left(\frac{\theta - 1}{\theta}\right)\right]^{-1}.$$

Use (3.15′) and (3.16) to solve for

$$k = \left(\frac{\alpha}{q}\right)\left(\frac{\theta - 1}{\theta}\right) y.$$

Equations (3.1), (3.13), (3.14′),(3.17), and (3.19) can be used to solve for

$$i = \delta k,$$

$$l = \frac{1}{z}\left(\frac{y}{k^{\alpha}}\right)^{1/(1-\alpha)},$$

$$w = (1 - \alpha)\left(\frac{\theta - 1}{\theta}\right)\left(\frac{y}{l}\right),$$

$$d = y - wl - qk,$$

and

$$n = \frac{y}{l}.$$

Finally, (3.8), (3.14′), and (3.16) can be used to solve for

$$\lambda = \frac{\chi + (1-\alpha)\left[1 + e\left(\frac{r}{r-1}\right)^{\gamma-1}\right]^{-1}\left[\left(\frac{\theta}{\theta-1}\right) - \delta\left(\frac{\alpha}{q}\right)\right]^{-1}}{(1-\alpha)z\left(\frac{\theta-1}{\theta}\right)^{1/(1-\alpha)}\left(\frac{\alpha}{q}\right)^{\alpha/(1-\alpha)}}.$$

C.4 The Linearized System

To linearize the nonlinear system (3.1) – (3.20), we perform a log-linear approximation of the model at steady state values.[2] Let $\widehat{var}_t \equiv log\left(\frac{var_t}{var}\right)$ denote the log-deviation of some variable var_t from its steady state var, where $log\left(\frac{var_t}{var}\right) \approx \frac{var_t - var}{var}$. A first-order Taylor approximation of equation (3.1) – (3.8), (3.9′), (3.10), (3.11), (3.12′), (3.13), (3.14′), (3.15′), (3.16) – (3.20) at the steady state gives:

$$k\hat{k}_{t+1} = (1-\delta)k\hat{k}_t + i\hat{x}_t + i\hat{i}_t, \tag{3.1}$$

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_{xt}, \tag{3.2}$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{3.3}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{3.4}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{3.5}$$

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \tag{3.6}$$

$$\gamma r\hat{a}_t = \gamma r\hat{\lambda}_t + r\left[1 + (\gamma-1)\lambda c\right]\hat{c}_t + (r-1)\lambda m\hat{e}_t + (\gamma-1)(r-1)m\hat{m}_t, \tag{3.7}$$

$$\lambda wl\hat{l}_t = \chi\hat{\lambda}_t + \chi\hat{w}_t, \tag{3.8}$$

$$(r-1)\hat{c}_t + (r-1)\hat{e}_t = (r-1)\hat{m}_t + \gamma\hat{r}_t, \tag{3.9′}$$

$$\hat{\lambda}_t = \hat{r}_t + E_t\hat{\lambda}_{t+1} - E_t\pi_{t+1}, \tag{3.10}$$

$$\hat{\lambda}_t - \hat{x}_t - \phi_k\hat{k}_t = E_t\hat{\lambda}_{t+1} + \beta qE_t\hat{q}_{t+1} - \beta(1-\delta)E_t\hat{x}_{t+1} + \beta\phi_K E_t\hat{k}_{t+2} - (1+\beta)\phi_K\hat{k}_{t+1}, \tag{3.11}$$

$$y\hat{y}_t = c\hat{c}_t + i\hat{i}_t, \tag{3.12′}$$

$$d\hat{d}_t = y\hat{y}_t - wl\hat{w}_t - wl\hat{l}_t - qk\hat{q}_t - qk\hat{k}_t, \tag{3.13}$$

[2] Canova (2007), DeJong and Dave (2007), and Zietz (2008) provide a detailed description of logarithmic approximations.

$$\hat{\lambda}_t + \hat{w}_t + \hat{l}_t = \hat{\xi}_t + \hat{y}_t, \tag{3.14$'$}$$

$$\hat{\lambda}_t + \hat{q}_t + \hat{k}_t = \hat{\xi}_t + \hat{y}_t, \tag{3.15$'$}$$

$$\phi_P \hat{\pi}_t = (1-\theta)\hat{\lambda}_t + (\theta - 1)\hat{\xi}_t + \beta\phi_P E_t \hat{\pi}_{t+1}, \tag{3.16}$$

$$\hat{y}_t = \alpha\hat{k}_t + (1-\alpha)\hat{z}_t + (1-\alpha)\hat{l}_t, \tag{3.17}$$

$$\hat{r}_t = \omega_\tau \hat{\tau}_t + \omega_\pi \hat{\pi}_t + \omega_y \hat{y}_t + \hat{v}_t, \tag{3.18}$$

$$\hat{n}_t = \hat{y}_t - \hat{l}_t, \tag{3.19}$$

and

$$\hat{\tau}_t = \hat{m}_t - \hat{m}_{t-1} + \hat{\pi}_t. \tag{3.20}$$

To facilitate the model's solution we follow Ireland (2003) and use (3.20) to rewrite (3.7) and (3.9$'$) as

$$\begin{aligned}\gamma r\hat{a}_t = {} & \gamma r\hat{\lambda}_t + r[1 + (\gamma-1)\lambda c]\hat{c}_t + (r-1)\lambda m\hat{e}_t \\ & + (\gamma-1)(r-1)\lambda m\hat{\tau}_t + (\gamma-1)(r-1)\lambda m\hat{m}_{t-1} - (\gamma-1)(r-1)\lambda m\hat{\pi}_t\end{aligned} \tag{3.7$'$}$$

and

$$(r-1)\hat{e}_t + (r-1)\hat{c}_t = (r-1)\hat{\tau}_t + (r-1)\hat{m}_{t-1} - (r-1)\hat{\pi}_t + \gamma\hat{r}_t. \tag{3.9$''$}$$

Further, we make use of (3.1) and (3.2) to rewrite (3.11) as

$$\begin{aligned} & \hat{\lambda}_t - \left\{1 + \beta\left[\delta\phi_K - (1-\delta)\right]\rho_x\right\}\hat{x}_t - \phi_K\hat{k}_t \\ = \; & E_t\hat{\lambda}_{t+1} + \beta q E_t\hat{q}_{t+1} + \phi_K\left[\beta(1-\delta) - (1+\beta)\right]\hat{k}_{t+1} \\ & + \beta\delta\phi_K E_t\hat{i}_{t+1}.\end{aligned} \tag{3.11$'$}$$

Appendix D

Solving the Model

To solve the linear difference model under rational expectations described by equations (3.1)–(3.6), (3.7′), (3.8), (3.9″), (3.10), (3.11′), (3.12′), (3.13),(3.14′),(3.15′),(3.16)–(3.20), we apply the method proposed by Blanchard and Kahn (1980). The subsequent sections follow the expositions in Blanchard and Kahn (1980), Farmer (1999), the technical notes of Ireland (2003), and DeJong and Dave (2007).

D.1 Blanchard and Kahn's Method

A solution to the linear difference model under rational expectations can be obtained by making use of the approach developed by Blanchard and Kahn (1980). Therefore, a vector s_t^0 is defined which can be separated into

$$s_t^0 = [s_{1t}^0 \; s_{2t}^0]',$$

letting s_{1t}^0 denote a $n \times 1$ vector of predetermined and s_{2t}^0 a $m \times 1$ vector of non-predetermined variables, which implies that:

$$E_t s_{t+1} = [s_{1t+1}^0 \; E_t s_{2t+1}^0]'.$$

Next, to apply Blanchard and Kahn's (1980) procedure, the model is written as

$$E_t s_{t+1}^0 = A s_t^0 + B\zeta_t \tag{3.21}$$

with

$$\zeta_t = P\zeta_{t-1} + \varepsilon_t, \tag{3.22}$$

where A and B are $(n+m) \times (n+m)$ and $(n+m) \times k$ coefficient matrices, P is $k \times k$ matrix containing the persistence parameters of the shocks, ζ_t is a $k \times 1$ vector consisting of the model's exogenous forcing variables while the serially and mutually uncorrelated innovations are included in the $k \times 1$ vector ε_t. The solution method relies on decoupling (3.21) into unstable and stable portions, using a Jordan decomposition, and then solving the two components in turn. If the number of unstable eigenvalues (with absolute value greater than one) of matrix A is equal to the number of non-predetermined variables, the system is said to be saddle-path stable and a unique solution exists (see Blanchard and Kahn, 1980).[1]

D.2 System Reduction

The solution of the system comprising (3.21) and (3.22) can be simplified by applying a system reduction first. According to King and Watson (2002, p. 2) the idea of a system reduction is to isolate a "... reduced-dimension, nonsingular dynamic system in a subset of variables of the full vector of endogenous variables. Once the rational expectations solution to this smaller system is obtained, it is easy to also calculate the solution for the remaining variables as these are governed by dynamic identities."

Let

$$f_t^0 = \begin{bmatrix} \hat{y}_t & \hat{c}_t & \hat{i}_t & \hat{l}_t & \hat{n}_t & \hat{\tau}_t & \hat{w}_t & \hat{q}_t & \hat{d}_t & \hat{r}_t \end{bmatrix}',$$

$$s_t^0 = \begin{bmatrix} \hat{k}_t & \hat{m}_{t-1} & \hat{\pi}_t & \hat{\lambda}_t & \hat{\xi}_t \end{bmatrix}',$$

and

$$\zeta_t = \begin{bmatrix} \hat{a}_t & \hat{e}_t & \hat{x}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}'.$$

Then, the linearized equilibrium conditions (3.1), (3.10), (3.11′), (3.16), and (3.20) can be written in matrix form as

$$\Omega_0 E_t s_{t+1}^0 + \Omega_1 E_t f_{t+1}^0 = \Omega_2 s_t^0 + \Omega_3 f_t^0 + \Omega_4 \zeta_t \tag{3.23}$$

[1] A detailed presentation of the Blanchard-Kahn (1980) conditions is given in appendix $B.2$.

with

$$\Omega_0 = \begin{bmatrix} k & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \phi_K[\beta(1-\delta)-(1+\beta)] & 0 & 0 & 1 & 0 \\ 0 & 0 & \beta\phi_P & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\delta\phi_K & 0 & 0 & 0 & 0 & \beta q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_2 = \begin{bmatrix} (1-\delta)k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -\phi_K & 0 & 0 & 1 & 0 \\ 0 & 0 & \phi_P & \theta-1 & -(\theta-1) \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix},$$

$$\Omega_3 = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Omega_4 = \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1-\beta\left[\delta\phi_K-(1-\delta)\right]\rho_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Likewise, (3.7′), (3.8), (3.9″), (3.12′), (3.13), (3.14′), (3.15′), (3.17), (3.18), and (3.19) can be written as

$$\Omega_5 f_t^0 = \Omega_6 s_t^0 + \Omega_7 \zeta_t \tag{3.24}$$

with

$$\Omega_5 = \begin{bmatrix} y & -c & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r\left[1+(\gamma-1)\lambda c\right] & 0 & 0 & 0 & (\gamma-1)(r-1)\lambda m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda wl & 0 & 0 & -\chi & 0 & 0 & 0 \\ 0 & r-1 & 0 & 0 & 0 & -(r-1) & 0 & 0 & 0 & -\gamma \\ y & 0 & 0 & -wl & 0 & 0 & -wl & -qk & -d & 0 \\ 1 & 0 & 0 & -(1-\alpha) & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \omega_y & 0 & 0 & 0 & 0 & \omega_\tau & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -(\gamma-1)(r-1)\lambda m & (\gamma-1)(r-1)\lambda m & -\gamma r & 0 \\ 0 & 0 & 0 & \chi & 0 \\ 0 & r-1 & -(r-1) & 0 & 0 \\ qk & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -\omega_\pi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Omega_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \gamma r & -(r-1)\lambda m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -(r-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, (3.2) – (3.6) can directly be written as

$$\zeta_t = P\zeta_{t-1} + \varepsilon_t \tag{3.22}$$

with

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & \rho_v \end{bmatrix}.$$

To cast (3.23) and (3.24) in the form of (3.21), we start by iterating forward equation (3.22) j periods which implies:

$$P^j \zeta_t = E_t \zeta_{t+j}.\text{[2]} \tag{3.22$'$}$$

Rewriting (3.24) as

$$f_t^0 = \Omega_5^{-1}\Omega_6 s_t^0 + \Omega_5^{-1}\Omega_7 \zeta_t \tag{3.24$'$}$$

and substituting this expression together with (3.22′) into (3.23) yields

$$\begin{aligned}
\Omega_0 E_t s_{t+1}^0 + \Omega_1 E_t\left(\Omega_5^{-1}\Omega_6 s_{t+1}^0 + \Omega_5^{-1}\Omega_7\zeta_{t+1}\right) &= \Omega_2 s_t^0 + \Omega_3\left(\Omega_5^{-1}\Omega_6 s_t^0 + \Omega_5^{-1}\Omega_7\zeta_t\right) + \Omega_4\zeta_t \\
\left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right) E_t s_{t+1}^0 + \Omega_1\Omega_5^{-1}\Omega_7 E_t\zeta_{t+1} &= \left(\Omega_2 + \Omega_3\Omega_5^{-1}\Omega_6\right) s_t^0 + \left(\Omega_4 + \Omega_3\Omega_5^{-1}\Omega_7\right)\zeta_t \\
\left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right) E_t s_{t+1}^0 + \Omega_1\Omega_5^{-1}\Omega_7 P\zeta_t &= \left(\Omega_2 + \Omega_3\Omega_5^{-1}\Omega_6\right) s_t^0 + \left(\Omega_4 + \Omega_3\Omega_5^{-1}\Omega_7\right)\zeta_t \\
\left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right) E_t s_{t+1}^0 &= \left(\Omega_2 + \Omega_3\Omega_5^{-1}\Omega_6\right) s_t^0 \\
&\quad + \left(\Omega_4 + \Omega_3\Omega_5^{-1}\Omega_7 - \Omega_1\Omega_5^{-1}\Omega_7 P\right)\zeta_t.
\end{aligned}$$

If $\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6$ is nonsingular, we can rewrite the expression above as

$$\begin{aligned}
E_t s_{t+1}^0 = {}& \left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right)^{-1}\left(\Omega_2 + \Omega_3\Omega_5^{-1}\Omega_6\right) s_t^0 \\
& + \left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right)^{-1}\left(\Omega_4 + \Omega_3\Omega_5^{-1}\Omega_7 - \Omega_1\Omega_5^{-1}\Omega_7 P\right)\zeta_t,
\end{aligned}$$

which is in the same form as (3.21) with

$$A = \left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right)^{-1}\left(\Omega_2 + \Omega_3\Omega_5^{-1}\Omega_6\right)$$

and

$$B = \left(\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6\right)^{-1}\left(\Omega_4 + \Omega_3\Omega_5^{-1}\Omega_7 - \Omega_1\Omega_5^{-1}\Omega_7 P\right)$$

so that

$$E_t s_{t+1}^0 = A s_t^0 + B\zeta_t \tag{3.21}$$

or

$$E_t\begin{bmatrix} \hat{k}_{t+1} \\ \hat{m}_t \\ \hat{\pi}_{t+1} \\ \hat{\lambda}_{t+1} \\ \hat{\xi}_{t+1} \end{bmatrix} = A\begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \\ \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix} + B\begin{bmatrix} \hat{a}_t \\ \hat{e}_t \\ \hat{x}_t \\ \hat{z}_t \\ \hat{v}_t \end{bmatrix}.$$

[2]To derive (3.22′) we use the Law of Iterated Expectations: $E_t[E_{t+1}(\cdot)] = E_t(\cdot)$.

It should be emphasized, that the transformation of the linearized model into (3.21) and (3.22) hinges critically on the invertibility of $\Omega_0 + \Omega_1\Omega_5^{-1}\Omega_6$.

D.3 Solution

Blanchard and Kahn's (1980) solution strategy relies on simplifying the model

$$E_t s_{t+1}^0 = A s_t^0 + B\zeta_t \tag{3.21}$$

with

$$\zeta_t = P\zeta_{t-1} + \varepsilon_t \tag{3.22}$$

by transforming it into canonical form. As outlined in DeJong and Dave (2007), the method begins with a Jordan decomposition of A such that

$$A = M^{-1}NM,$$

where the diagonal elements of N, consisting of the eigenvalues of A, are ordered in increasing absolute value and the columns of M^{-1} are the eigenvectors of A. We proceed under the case of saddle-path stability and assume that $m = 3$, i.e., the number of eigenvalues outside the unit circle equals the number non-predetermined variables in s_t^0 and therefore allow for a unique solution. Thus, N can be written as

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix},$$

where the eigenvalues of the 2×2 matrix N_1 are on or inside the unit circle and the eigenvalues of the 3×3 matrix N_2 lie outside the unit circle. M and B are decomposed accordingly, so that

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where M_{11} is a 2×2, M_{12} is a 2×3, M_{21} is a 3×2, M_{22} is a 3×3, B_1 is a 2×5, and B_2 is a 3×5 matrix. Now (3.21) can be rewritten as

$$E_t s^0_{t+1} = M^{-1} N M s^0_t + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \zeta_t. \tag{3.21$'$}$$

Pre-multiplying (3.21′) by M gives

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s^0_{t+1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^0_t + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \zeta_t$$

or in terms of matrix partitions

$$E_t s^1_{1t+1} = N_1 s^1_{1t} + Q_1 \zeta_t \tag{3.25}$$

and

$$E_t s^1_{2t+1} = N_2 s^1_{2t} + Q_2 \zeta_t, \tag{3.26}$$

where

$$s^1_{1t} = M_{11} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{12} \begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix}, \tag{3.27}$$

$$s^1_{2t} = M_{21} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{22} \begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix}, \tag{3.28}$$

$$Q_1 = M_{11} B_1 + M_{12} B_2,$$

and

$$Q_2 = M_{21} B_1 + M_{22} B_2.$$

Since the eigenvalues of N_2 all lie outside the unit circle, (3.26) can be solved forward. Using the result of this forward iteration together with (3.22′) we obtain

$$s^1_{2t} = -N_2^{-1} R \zeta_t, \tag{3.29}$$

where the 3×5 matrix R is obtained by "reshaping"[3]

$$\begin{aligned}
vec(R) &= vec\sum_{j=0}^{\infty} N_2^{-j} Q_2 P^j = \sum_{j=0}^{\infty} vec\left(N_2^{-j} Q_2 P^j\right) \\
&= \sum_{j=0}^{\infty} \left[P^j \otimes \left(N_2^{-1}\right)^j\right] vec(Q_2) = \sum_{j=0}^{\infty} \left(P \otimes N_2^{-1}\right)^j vec(Q_2) \\
&= \left[I_{(15\times 15)} - P \otimes N_2^{-1}\right]^{-1} vec(Q_2).
\end{aligned}$$

Substituting (3.29) into (3.28) gives

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_2 \zeta_t, \tag{3.30}$$

where

$$S_1 = -M_{22}^{-1} M_{21}$$

and

$$S_2 = -M_{22}^{-1} N_2^{-1} R.$$

We can next substitute (3.30) into (3.27) to solve for

$$s_{1t}^1 = (M_{11} + M_{12} S_1) \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{12} S_2 \zeta_t. \tag{3.31}$$

Substituting (3.31) into (3.25) gives

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{m}_t \end{bmatrix} = S_3 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_4 \zeta_t, \tag{3.32}$$

where

$$S_3 = (M_{11} + M_{12} S_1)^{-1} N_1 (M_{11} + M_{12} S_1)$$

[3]According Hamilton (1994a) and DeJong and Dave (2007), the appearance of the *vec* operator accommodates the VAR specification for ζ_t. In particular, we use the relationship between *vec* operator and Kronecker product: $vec\left(N_2^{-j} Q_2 P^j\right) = \left[(P^j)' \otimes N_2^{-j}\right] vec(Q_2)$. Note that $P' = P$, since P is a diagonal matrix.

and

$$S_4 = (M_{11} + M_{12}S_1)^{-1}(Q_1 + N_1M_{12}S_2 - M_{12}S_2P).$$

Finally, by using (3.30), (3.24′) can be written as

$$\begin{aligned} f_t^0 &= \Omega_5^{-1}\Omega_6 s_t^0 + \Omega_5^{-1}\Omega_7\zeta_t \\ &= \Omega_5^{-1}\Omega_6 \begin{bmatrix} I_{(2\times 2)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + \Omega_5^{-1}\Omega_6 \begin{bmatrix} 0_{(2\times 5)} \\ S_2 \end{bmatrix} \zeta_t + \Omega_5^{-1}\Omega_7\zeta_t, \end{aligned}$$

or simply

$$f_t^0 = S_5 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_6\zeta_t, \tag{3.33}$$

where

$$S_5 = \Omega_5^{-1}\Omega_6 \begin{bmatrix} I_{(2\times 2)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = \Omega_5^{-1}\Omega_6 \begin{bmatrix} 0_{(2\times 5)} \\ S_2 \end{bmatrix} + \Omega_5^{-1}\Omega_7.$$

Hence, the model's solution can be written compactly in state space form by combining (3.22), (3.30), (3.32), and (3.33) as

$$s_{t+1} = \Gamma_0 s_t + \Gamma_1 \varepsilon_{t+1} \tag{3.34}$$

and

$$f_t = \Gamma_2 s_t, \tag{3.35}$$

where

$$s_t = \begin{bmatrix} \hat{k}_t & \hat{m}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{x}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{y}_t & \hat{c}_t & \hat{i}_t & \hat{l}_t & \hat{n}_t & \hat{\tau}_t & \hat{w}_t & \hat{q}_t & \hat{d}_t & \hat{r}_t & \hat{\pi}_t & \hat{\lambda}_t & \hat{\xi}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{xt} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}',$$

$$\Gamma_0 = \begin{bmatrix} S_3 & S_4 \\ 0_{(5\times 2)} & P \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0_{(2\times 5)} \\ I_{(5\times 5)} \end{bmatrix},$$

and

$$\Gamma_2 = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

Appendix E

Estimation

E.1 Empirical State Space Model

Since the model is estimated using an observed sample X including consumption, investment, money, inflation, and interest rates, we can define a sequence of observations $\{X_t\}_{t=1}^{T}$ with a measured data vector

$$X_t = \begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix}.$$

To distinguish the theoretical model from the empirical model, we rewrite (3.34) and (3.35) as

$$s_{t+1} = \Psi_0 s_t + \Psi_1 \varepsilon_{t+1} \tag{3.36}$$

and

$$X_t = \Psi_2 s_t, \tag{3.37}$$

where $\Gamma_o = \Psi_0$, $\Gamma_1 = \Psi_1$ and Ψ_2 is formed from the rows $(\cdot)$ of Γ_0 and Γ_2 as

$$\Psi_2 = \begin{bmatrix} \Gamma_2(2) \\ \Gamma_2(3) \\ \Gamma_0(2) \\ \Gamma_2(11) \\ \Gamma_2(10) \end{bmatrix}.$$

Given the empirical state space model, the Kalman filter can be used to estimate the model parameters with maximum likelihood and to draw inferences about the unobserved components of the state vector s_t exploiting the information contained in the five observable series (see Ireland, 2004).

E.2 Kalman Filter

As stated, for example, in Ruge-Murcia (2007) the maximum likelihood estimation of a DSGE model in state space form calls for the construction and evaluation of the likelihood function

$$\mathcal{L}(\mu|X) = p(X|\mu) = \prod_{t=1}^{T} p(X_t|\mu),$$

letting X denote the T observations of a vector of observable variables X_t given the model's parameters μ. According to Hamilton (1994a) the Kalman filter can be used to calculate the likelihood function for such a state space system. More precisely, as outlined in Canova (2007, p. 123), "... the likelihood function of a state space model can be conveniently expressed in terms of one-step-ahead forecast errors, conditional on the initial observations, and of their recursive variance, both of which can be obtained with the Kalman filter." While a general and detailed treatment of the Kalman filter can be found in Harvey (1993), Hamilton (1994a), and Hamilton (1994b), we only give a brief exemplary exposition of the Kalman filter by applying the recursive algorithm originally developed by Kalman (1960) and Kalman and Bucy (1961) to the empirical state space model formed by state equation (3.36) and observation equation (3.37):

$$s_{t+1} = \Psi_0 s_t + \Psi_1 \varepsilon_{t+1}, \tag{3.36}$$

$$X_t = \Psi_2 s_t. \tag{3.37}$$

Note that s_t is a vector of possibly unknown state variables, X_t denotes a vector of observed variables, Ψ_0, Ψ_1, and Ψ_2 depend on the structural parameters of the model and the vector ε_{t+1} comprises the serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{at+1} & \varepsilon_{et+1} & \varepsilon_{xt+1} & \varepsilon_{zt+1} & \varepsilon_{vt+1} \end{bmatrix}',$$

which are assumed to be normally distributed with zero mean and diagonal covariance matrix

$$\Sigma_\varepsilon = E\varepsilon_{t+1}\varepsilon'_{t+1} = \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_v^2 \end{bmatrix}.$$

E.2.1 Kalman Filter Recursion

In order to outline the Kalman filter recursion, we follow the expositions of Hamilton (1994a) and Lütkepohl (2005). Let

$$\begin{aligned} s_{t|j} &= E(s_t|X_1, ..., X_j), \\ \Sigma_s(t|j) &= E(s_t - s_{t|j})(s_t - s_{t|j})', \\ X_{t|j} &= E(X_t|X_1, ..., X_j), \\ \Sigma_X(t|j) &= E(X_t - X_{t|j})(X_t - X_{t|j})'. \end{aligned}$$

Further, the initial state s_0 and the conditional distribution of s given X are assumed to be normally distributed with $s_0 \sim \mathcal{N}(\mu_{s0}, \Sigma_0)$ and $(s|X) \sim \mathcal{N}(\mu_s, \Sigma)$, respectively. Given the previous conditions, the normality assumption implies

$$\begin{aligned} (s_t|X_1, ..., X_{t-1}) &\sim \mathcal{N}(s_{t|t-1}, \Sigma_s(t|t-1)) \quad \text{for } t = 2, ..., T, \\ (s_t|X_1, ..., X_t) &\sim \mathcal{N}(s_{t|t}, \Sigma_s(t|t)) \quad \text{for } t = 1, ..., T, \\ (X_t|X_1, ..., X_{t-1}) &\sim \mathcal{N}(X_{t|t-1}, \Sigma_X(t|t-1)) \quad \text{for } t = 2, ..., T. \end{aligned}$$

According to in Lütkepohl (2005) the conditional means and covariance matrices can be obtained by the subsequent Kalman filter recursions:

- Initialization:

$$s_{0|0} = \mu_{s0}, \Sigma_s(0|0) = \Sigma_0.$$

- Prediction step ($1 \leq t \leq T$):

$$\begin{aligned}
s_{t|t-1} &= \Psi_0 s_{t-1|t-1}, \\
\Sigma_s(t|t-1) &= \Psi_0 \Sigma_s(t-1|t-1)\Psi_0' + \Psi_1 \Sigma_\varepsilon \Psi_1', \\
X_{t|t-1} &= \Psi_2 s_{t|t-1}, \\
\Sigma_X(t|t-1) &= \Psi_2 \Sigma_s(t|t-1)\Psi_2', \\
u_t &= X_t - X_{t|t-1}.
\end{aligned}$$

- Correction step ($1 \leq t \leq T$):

$$\begin{aligned}
s_{t|t} &= s_{t|t-1} + \Upsilon_t u_t, \\
\Sigma_s(t|t) &= \Sigma_s(t|t-1) - \Upsilon_t \Sigma_X(t|t-1)\Upsilon_t',
\end{aligned}$$

  where the Kalman gain Υ_t is defined as

$$\Upsilon_t = s_{t|t-1}\Psi_2' \Sigma_X(t|t-1)^{-1} .$$

As outlined in Lütkepohl (2005) the recursions proceed by performing the prediction step for $t = 1$. Then, the correction step is performed for $t = 1$. Next, the prediction and correction steps are repeated for $t = 2$ and so on.

E.2.2 Log Likelihood Function

The observation vector estimation errors $\{u_t\}_{t=1}^T$ can be used to form the Gaussian log likelihood function for $\{X_t\}_{t=1}^T$:

$$\begin{aligned}
\ln \mathcal{L}(\mu|X) &= \sum_{t=1}^{T} \ln p(X_t|\mu) \\
&= -\frac{5T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln|\Sigma_X(t|t-1)| - \frac{1}{2}\sum_{t=1}^{T} u_t' \Sigma_X(t|t-1)^{-1} u_t.
\end{aligned}$$

Appendix F

Data sources

All data have been retrieved from Thomson Reuters Datastream. The original sources are detailed below.

- France:

  Real personal consumption: EUROSTAT

  Gross fixed capital formation: EUROSTAT

  Money balances (M3): Banque de France

  Consumer price index: OECD

  Interest rate (Pibor): OECD

  Population: National Institute for Statistics and Economic Studies (INSEE)

- Germany:

  Real personal consumption: Federal Statistical Office

  Gross fixed capital formation: Federal Statistics Office

  Money balances (M3): Deutsche Bundesbank

  Consumer price index: OECD

  Interest rate (Fibor): OECD

  Population: Federal Statistics Office

- Italy:

  Real personal consumption: Oxford Economics

  Gross fixed capital formation: Oxford Economics

  Money balances (M3): Oxford Economics

Consumer price index: Oxford Economics

Interest rate (three-month money market rate): Oxford Economics

Population: Oxford Economics

- Spain:

  Real personal consumption: EUROSTAT

  Gross fixed capital formation: EUROSTAT

  Money balances (M3): Banco de España

  Consumer price index: OECD

  Interest rate (three-month money market rate): OECD

  Population: EUROSTAT

Appendix G

Tables

	1980:Q1 - 1994:Q2		1994:Q3 - 2008:Q3	
Parameter	Estimate	Std. Error	Estimate	Std. Error
β	0.9906	0.0013	0.9913	0.0024
γ	0.0000	0.0007	0.0043	0.0014
ϕ_P	10.3880	0.5796	3.2691	0.3101
ϕ_K	30.0492	0.5400	28.8285	2.1778
ω_τ	0.2980	0.0081	0.2792	0.0188
ω_π	1.1974	0.0095	1.4680	0.0807
ω_y	-0.0075	0.0115	-0.1417	0.0605
e	4.4410	0.0006	4.3587	0.0115
z	4185.6183	0.0001	4181.1612	0.0001
ρ_a	0.8963	0.0065	0.8507	0.0137
ρ_e	0.9000	0.0071	0.8132	0.0128
ρ_x	0.9011	0.0078	0.9817	0.0067
ρ_z	0.8995	0.0188	0.9222	0.0061
ρ_υ	0.4999	0.0076	0.1976	0.0249
σ_a	0.0202	0.0004	0.0082	0.0002
σ_e	0.0096	0.0001	0.0089	0.0001
σ_x	0.0554	0.0069	0.0324	0.0021
σ_z	0.0080	0.0003	0.0082	0.0002
σ_υ	0.0057	0.0001	0.0044	0.0003

Table G.1: Maximum Likelihood Estimates: France.

	1980:Q1 - 1994:Q1		1994:Q2 - 2008:Q3	
Parameter	Estimate	Std. Error	Estimate	Std. Error
β	0.9917	0.0001	0.9926	0.0001
γ	0.0757	0.0001	0.0731	0.0001
ϕ_P	13.9735	0.0128	14.0138	0.0143
ϕ_K	29.9619	0.1973	30.5119	0.1615
ω_τ	0.4368	0.0007	0.4353	0.0008
ω_π	1.5998	0.0007	1.6005	0.0005
ω_y	-0.0025	0.0008	-0.0023	0.0030
e	2.9640	0.0001	2.9633	0.0001
z	4195.9727	0.0001	4158.7916	0.0001
ρ_a	0.9000	0.0007	0.9001	0.0006
ρ_e	0.8795	0.0007	0.8800	0.0006
ρ_x	0.9061	0.0006	0.9061	0.0005
ρ_z	0.9162	0.0006	0.9162	0.0004
ρ_v	0.2400	0.0002	0.2388	0.0004
σ_a	0.0186	0.0011	0.0105	0.0009
σ_e	0.0150	0.0009	0.0134	0.0006
σ_x	0.0850	0.0036	0.0777	0.0031
σ_z	0.0170	0.0008	0.0113	0.0007
σ_v	0.0063	0.0001	0.0075	0.0001

Table G.2: Maximum Likelihood Estimates: Germany.

	1980:Q1 - 1994:Q3		1994:Q4 - 2008:Q3	
Parameter	Estimate	Std. Error	Estimate	Std. Error
β	0.9992	0.0100	0.9975	0.0036
γ	0.0054	0.0057	0.0350	0.0057
ϕ_P	64.6013	4.7897	31.7841	1.8189
ϕ_K	33.1134	0.8700	14.5576	1.4126
ω_τ	0.6722	0.0287	0.0538	0.0096
ω_π	0.8736	0.0287	1.6598	0.2561
ω_y	-0.1286	0.0302	-0.1627	0.0832
e	3.9728	0.0855	3.2327	0.5237
z	3343.6115	0.0006	3336.9033	0.0013
ρ_a	0.8379	0.0108	0.9935	0.0142
ρ_e	0.9929	0.0093	0.9093	0.0119
ρ_x	0.9952	0.0085	0.9891	0.0077
ρ_z	0.9953	0.0195	0.8519	0.0062
ρ_υ	0.0899	0.0662	0.5551	0.0022
σ_a	0.0291	0.0076	0.0119	0.0005
σ_e	0.0153	0.0021	0.0110	0.0001
σ_x	0.2501	0.0364	0.0192	0.0017
σ_z	0.0533	0.0017	0.0101	0.0003
σ_υ	0.0128	0.0015	0.0034	0.0006

Table G.3: Maximum Likelihood Estimates: Italy.

	1987:Q1 - 1997:Q4		1998:Q1 - 2008:Q3	
Parameter	Estimate	Std. Error	Estimate	Std. Error
β	0.9929	0.0067	0.9957	0.0020
γ	0.0189	0.0069	0.0518	0.0075
ϕ_P	66.9756	3.5472	2.7164	0.3119
ϕ_K	26.8170	0.7038	7.4710	0.4382
ω_τ	0.4707	0.0313	0.2367	0.0171
ω_π	0.6868	0.0339	1.2448	0.0480
ω_y	-0.0646	0.0098	-0.1006	0.0140
e	4.6627	0.0034	4.1651	0.0035
z	1932.4221	0.0001	1771.8852	0.0001
ρ_a	0.9542	0.0123	0.9411	0.0147
ρ_e	0.9440	0.0098	0.9648	0.0071
ρ_x	0.9625	0.0142	0.9903	0.0122
ρ_z	0.9477	0.0122	0.7833	0.0173
ρ_v	0.4565	0.0027	0.0333	0.0025
σ_a	0.0235	0.0006	0.0079	0.0002
σ_e	0.0084	0.0001	0.0107	0.0002
σ_x	0.0389	0.0086	0.0083	0.0004
σ_z	0.0227	0.0008	0.0073	0.0002
σ_v	0.0071	0.0003	0.0054	0.0002

Table G.4: Maximum Likelihood Estimates: Spain.

Model Parameters	Individual p-value	ESS p-value
z	0	0
σ_a	0	0
ρ_υ	0	0
ϕ_P	0	0
ρ_x	0	0
e	0	0
ρ_e	0	0
σ_υ	0	0
σ_e	0	0
ω_π	0.0503	0
σ_x	0.0723	0
ρ_a	0.1106	0
γ	0.2181	0
ω_y	0.5459	0
ρ_z	1	1
ω_τ	1	1
σ_z	1	1
ϕ_K	1	1
β	1	1

Set of stable parameters (90% probability level):
$\mathcal{S} = \{\rho_z, \omega_\tau, \sigma_z, \phi_K, \beta\}$

Table G.5: The table shows the p-values of Andrews'(1993) QLR test on individual parameters for France. In addition the set of stable parameters is reported as well as the p-values at each step of Inoue and Rossi's (2011) ESS procedure.

Model Parameters	Individual p-value	ESS p-value
z	0	0
γ	0	0
e	0	0
β	0	0
σ_υ	0	0
σ_a	0	0
σ_z	0	0
ρ_υ	0.1515	0
ϕ_K	0.5622	0
ϕ_P	0.6054	0
σ_x	1	0.5929
σ_e	1	0.4773
ω_τ	1	1
ω_π	1	1
ρ_e	1	1
ρ_a	1	1
ρ_z	1	1
ω_y	1	1
ρ_x	1	1

Set of stable parameters (90% probability level):
$\mathcal{S} = \{\sigma_x, \sigma_e, \omega_\tau, \omega_\pi, \rho_e, \rho_a, \rho_z, \omega_y, \rho_x\}$

Table G.6: The table shows the p-values of Andrews'(1993) QLR test on individual parameters for Germany. In addition the set of stable parameters is reported as well as the p-values at each step of Inoue and Rossi's (2011) ESS procedure.

Model Parameters	Individual p-value	ESS p-value
z	0	0
σ_z	0	0
ω_τ	0	0
ϕ_K	0	0
ρ_a	0	0
ρ_v	0	0
ρ_z	0	0
ϕ_P	0	0
σ_x	0	0
σ_v	0	0
ρ_e	0	0
γ	0.0180	0
ω_π	0.1498	0
σ_a	0.4892	0.6207
σ_e	0.6370	0.7320
e	1	1
ρ_x	1	1
ω_y	1	1
β	1	1

Set of stable parameters (90% probability level):
$\mathcal{S} = \{\sigma_a, \sigma_e, e, \rho_x, \omega_y, \beta\}$

Table G.7: The table shows the p-values of Andrews'(1993) QLR test on individual parameters for Italy. In addition the set of stable parameters is reported as well as the p-values at each step of Inoue and Rossi's (2011) ESS procedure.

Model Parameters	Individual p-value	ESS p-value
z	0	0
ρ_υ	0	0
e	0	0
σ_a	0	0
ϕ_K	0	0
σ_z	0	0
ϕ_P	0	0
σ_e	0	0
ω_π	0	0
ρ_z	0	0
ω_τ	0	0
σ_υ	0	0
σ_x	0.0249	0
γ	0.0066	0
ω_y	0.5959	0.6288
ρ_e	0.8332	1
ρ_x	1	1
ρ_a	1	1
β	1	1

Set of stable parameters (90% probability level):
$\mathcal{S} = \{\omega_y, \rho_e, \rho_x, \rho_a, \beta\}$

Table G.8: The table shows the p-values of Andrews'(1993) QLR test on individual parameters for Spain. In addition the set of stable parameters is reported as well as the p-values at each step of Inoue and Rossi's (2011) ESS procedure.

Chapter 4

Comparing Quadratic Costs of Capital Accumulation: An Empirical Assessment

4.1 Introduction

Over the last three decades DSGE models have become the paradigm for monetary policy and business cycle analysis, both in academic and policy making circles (see Canova and Ferroni, 2011). The origin of this class of models dates back to the work of Kydland and Prescott (1982) and Long and Plosser (1983), who develop a small scale frictionless neoclassical framework in which utility-maximizing rational agents operate subject to budget constraints, technological restrictions, and Hicks-neutral technology shocks. As outlined in Shea (1998), this so-called real business cycle (RBC) model was seminal in several ways. First, the Schumpeterian idea that random changes in productivity (technology shocks) can generate business cycle fluctuations was reintroduced. Second, these business cycle fluctuations were seen as optimal responses of rational agents to erratic changes in technology, leaving no need for government policy interventions. Third, business cycles were explained in a dynamic stochastic general equilibrium framework in which optimal behavior of agents can explicitly be derived from microeconomic first principles by specifying preferences, technologies, budget and resource constraints, and the institutional environment.

Since the pioneering contribution of Kydland and Prescott (1982) and Long and Plosser (1983), the DSGE research program broadened considerably, generating a wide set of extensions of the basic RBC model, including New Keynesian features like monopolistic competition and nominal rigidities (see Woodford, 2003). Also the initial view of technology shocks as the ultimate source of business cycle fluctuations has soon been challenged by a number of studies, leading to the incorporation of a variety of economic disturbances into mainstream DSGE models (see Danthine and Donaldson, 1993 and Galí and Rabanal, 2004 for an overview).

A prominent candidate shock is the marginal efficiency of investment shock. Even though already Keynes (1936, p. 313) assumed that the phenomenon of the business cycle is "... mainly due to the way in which the marginal efficiency of capital fluctuates ...", Greenwood et al. (1988) were the first to provide a theoretical and quantitative analysis of the marginal efficiency of investment shock in a single-shock dynamic stochastic general equilibrium framework of the RBC type. Employing a model with variable capacity utilization, Greenwood et al. (1988) conclude that their calibrated RBC model is able to match the observed cyclical fluctuations in US data as well as DSGE models purely driven by technology shocks.

Based on these investigations, DeJong et al. (2000a) choose a multiple-shock approach to analyze the relative importance of a total factor productivity shock and a marginal efficiency of investment shock in explaining US business cycle fluctuations. Using Bayesian estimation techniques, DeJong et al. (2000a, p. 328) find both shocks to play an important role in driving aggregate fluctuations with the total factor productivity shock having "... a greater initial impact on output and investment ...", whereas the marginal efficiency of investment shock shows "... a more lasting impact."

Also, Justiniano et al. (2010) adopt this empirical approach and estimate a multiple-shock DSGE model with New Keynesian features by employing Bayesian inference methods. The estimated medium-scale model contains a host of nominal and real frictions, like imperfect competition, sticky prices, habit formation in consumption, variable capacity utilization, and investment adjustment costs as well as several shocks including a marginal efficiency of investment shock. As a result, Justiniano et al. (2010, p. 144) find, "... that investment shocks – shocks to the marginal efficiency of investment – are the main drivers of movements in hours, output and investment over the [US] cycle ...", whereas the incorporation of frictions such as convex investment adjustment costs plays "... a crucial role in turning investment shocks into a viable driving force of fluctuations" (Justiniano et al., 2010, p. 133).

The inclusion of adjustment costs into New Keynesian models to constrain physical capital accumulation has become standard practice in the recent DSGE literature, since, as outlined in Ireland (2003) and Smets and Wouters (2007), these frictions improve the ability of sticky-price models with endogenous investment to match the key features of the data substantially. A familiar way to model these real rigidities is to assume that capital owners are subject to quadratic adjustment costs (see, for example, Kim, 1998; Ireland, 2003; Zanetti, 2007 and Cogley and Yagihashi, 2010). As described in Groth and Kahn (2010), these convex adjustment costs specifications can be classified into either costs to adjusting the level of capital (capital adjustment costs) or costs to changing the level of investment (investment adjustment costs). In particular, the latter specification becomes widely used in the context of monetary policy analysis, since, as argued by Christiano et al. (2005), Smets and Wouters (2007), and Christiano et al. (2010, p. 49), adjustment costs as a function of the change in investment are able to "... reproduce VAR-based evidence that investment has a humped-shaped response to a monetary policy shock."[1]

The purpose of this chapter is to investigate empirically how these different specifications of quadratic adjustment costs affect the fit and the dynamics of a DSGE model with real and nominal frictions featuring several exogenous stochastic disturbances. Specifically, a preference shock, a monetary policy shock, and a shock to the marginal efficiency of investment compete with a standard RBC technology shock in driving aggregate fluctuations. We consider three different specifications of quadratic costs of capital accumulation. Thus, each variant of adjustment costs defines a distinct version of the underlying New Keynesian model. Following DeJong et al. (2000a) and Justiniano et al. (2010), we use a Bayesian approach to estimate and compare the different model versions for both the euro area and the US.

The main results of the analysis are as follows. First, we find in part marked differences between the estimated structural parameters across the three model specifications. In particular, the estimates of persistence and volatility of the marginal efficiency of investment shock vary with the choice of investment or cap-

[1] Groth and Khan (2010) outline further arguments made in favor of investment adjustment costs in the recent DSGE literature.

ital adjustment costs. The use of Monte Carlo filtering techniques allows us to take a closer look at the causes of these differences. Second, the implementation of either investment or capital adjustment costs affects the dynamics of the respective model specifications substantially. Our results confirm the findings of Christiano et al. (2005) and Smets and Wouters (2007, p. 589), who point out that "... modeling capital adjustment costs as a function of the change in investment rather than its level introduces additional dynamics in the [model's] investment equation, which is useful in capturing the hump-shaped response of investment to various shocks." Third, despite the ability of investment adjustment to generate hump-shaped investment responses, the posterior odds comparison, which evaluates the relative empirical fit of a DSGE model, provides decisive evidence in favor of the model specifications featuring capital adjustment costs. To further evaluate where the model specifications fail to match the data, we compare the model versions' implied characteristics to the actual data using standard moment criteria. Consistent with the results from the posterior odds analysis, we obtain a better fit of the model versions with capital adjustment costs than with the specification including investment adjustment costs. Our findings appear to be qualitatively robust across both data sets. We conclude that using estimated DSGE models with quadratic costs of capital accumulation for policy analysis should be done with caution, since the results could be influenced by the choice of either investment or capital adjustment costs both having in common the lack of an explicit microfoundation.

The remainder of the chapter is organized as follows: Section 4.2 presents the theoretical setup of the three model specifications. Section 4.3 describes the solution of the model versions and the estimation technique applied. Section 4.4 introduces the concepts of Monte Carlo filtering and regionalized sensitivity analysis. Section 4.5 explains the data and the priors used. Section 4.6 exposes the use of Bayesian posterior odds analysis for model evaluation. Section 4.7 discusses the results obtained from the Bayesian estimation, the MCF analysis, the posterior odds comparison, the moment analysis, the impulse response analysis, and the variance decomposition. Section 4.8 concludes. Technical details concerning the theoretical setup, the model solution, and the construction of the likelihood appear in the appendices.

4.2 The Model

4.2.1 Overview

The model economy is a simple cashless closed-economy New Keynesian model featuring a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a monetary authority. We consider three versions of the model that differ only with respect to the particular specification of capital accumulation costs. The representative household consumes, saves, and supplies labor and capital services to the intermediate goods-producing firms. Moreover, the household is assumed to face convex adjustment costs of capital accumulation. We consider three different quadratic specifications frequently used in the recent DSGE literature. The final output is produced by a representative finished goods-producing firm acting in a perfectly competitive market. The finished goods-producing firm bundles the continuum of intermediate goods manufactured by monopolistic competitors and sells it to the household who divides the final good between consumption an investment. The intermediate goods-producing firms are owned by the household and each of them produces a distinct, perishable intermediate good, also indexed by $i \in [0, 1]$ during each period $t = 0, 1, 2, ...$ The assumption of monopoly power of intermediate goods-producing firms allows nominal rigidities to arise in the form of quadratic nominal price adjustment costs. Finally, there is a monetary authority, which conducts monetary policy by setting the nominal interest rate according to a Taylor-type rule. We next characterize the decisions taken by households and firms before looking at the behavior of the monetary authority and sketching the solution of the model.[2]

4.2.2 Households

The representative household enters period t holding B_{t-1} nominal one-period bonds and k_t units of physical capital. During period t the household receives $W_t l_t + Q_t k_t$ total nominal factor payments from supplying $l_t(i)$ units of labor and $k_t(i)$ units of capital to each intermediate goods-producing firm $i \in [0, 1]$. W_t and Q_t denote the nominal wage rate for labor and the nominal rental rate for capital,

[2] Appendices H and I provide a summary of the complete model and its solution.

respectively. For all $t = 0, 1, 2, ...$, the household's choices of $l_t(i)$ and $k_t(i)$ must satisfy

$$l_t = \int_0^1 l_t(i)di,$$

where l_t denotes total hours worked,[3] and

$$k_t = \int_0^1 k_t(i)di.$$

Further, the household receives nominal dividends from each intermediate goods producing firm $i \in [0, 1]$ aggregating to

$$D_t = \int_0^1 D_t(i)di.$$

The household uses its funds to purchase new bonds at the nominal cost B_t/r_t, where r_t denotes the gross nominal interest rate between time periods, and output from the final goods sector at price P_t. Following Woodford (2003), we assume that prices are measured in terms of a unit of account called "money", but the economy is cashless otherwise. The final good can be used for consumption c_t or investment i_t. In the latter case convex (quadratic) adjustment costs accrue to the household measured in terms of the finished good. As outlined in Kim (2000, p. 335) quadratic costs are justified on the basis that "... it is easier to absorb new capacity into the firm at a slow rate." We consider three different specifications of quadratic adjustment costs S indexed by $j \in \{1, 2, 3\}$, which are frequently used in the recent DSGE literature, namely:

i)

$$S_1(i_{t-1}, i_t) = g_1(i_{t-1}, i_t)i_t = \frac{\phi}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2 i_t,$$

employed by Schmitt-Grohé and Uribe (2005), Del Negro and Schorfheide (2008), and Fernández-Villaverde (2010)

ii)

$$S_2(i_t, k_t) = g_2(i_t, k_t)k_t = \frac{\phi}{2}\left(\frac{i_t}{k_t} - \delta\right)^2 k_t,$$

[3] The endowment of time is normalized to one. Therefore, $1 - l_t$ denotes leisure.

as in Ireland (2003), Dellas (2006), Christensen and Dib (2008), and Cogley and Yagihashi (2010)

iii)

$$S_3(i_t, k_t) = g_3(i_t, k_t)i_t = \frac{\phi}{2}\left(\frac{i_t}{k_t} - \delta\right)^2 i_t,$$

similar in spirit to the specifications used in Kim (1998), Kim (2000), and Groth and Khan (2010).

Letting the function $g_j(\cdot,\cdot)$ parameterize the adjustment costs, it holds that $g_j(\cdot,\cdot) = g_j'(\cdot,\cdot) = 0$ and $g_j''(\cdot,\cdot) > 0$ in the steady state. Therefore, as described in Christiano et al. (2005), the adjustment costs will only depend on the second-order derivative. The parameter $\phi \geq 0$ governs the size of these adjustment costs. Note that each adjustment cost specification S_j defines a distinct version $\mathcal{M}_j$ of the model.[4]

The capital accumulation process is given by

$$k_{t+1} = (1 - \delta)k_t + x_t i_t,$$

with $0 < \delta < 1$ denoting the rate of depreciation and x_t representing a shock to the marginal efficiency of investment introduced by Greenwood et al. (1988). The shock is specified as

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{4.1}$$

with $0 < \rho_x < 1$ and $\varepsilon_{xt} \sim N(0, \sigma_x^2)$.

The budget constraint of the representative household is given by

$$\frac{B_{t-1} + W_t l_t + Q_t k_t + D_t}{P_t} \geq c_t + i_t + S_j(\cdot,\cdot) + \frac{B_t/r_t}{P_t}.$$

Moreover, we impose a no-Ponzi-game condition to prevent the household to make excessive debts. Facing these constraints, the household maximizes the stream of expected utility

$$E\sum_{t=0}^{\infty}\beta^t a_t\left[\log(c_t - hc_{t-1}) - \chi\frac{l_t^{1+\eta}}{1+\eta}\right],$$

[4]Since g_2 equals g_3, we a priori expect $\mathcal{M}_2$ and $\mathcal{M}_3$ to share the same dynamics, whereas differences presumably occur in the estimated magnitude of the parameter ϕ.

where $0 < \beta < 1$ is a discount factor, $0 \leq h < 1$ is the parameter that controls the "degree" of habit persistence, $\chi > 0$ measures the relative weight of leisure, and $\eta \geq 0$ denotes the inverse of the Frisch labor supply elasticity. The expected utility function is subject to an intertemporal preference shock, which is assumed to follow the autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{4.2}$$

where $0 < \rho_a < 1$ and $\varepsilon_{at} \sim N(0, \sigma_a^2)$. Following Primiceri et al. (2006), we refer to a_t as a "discount factor shock", affecting both the marginal utility of consumption and the marginal disutility of hours worked.

4.2.3 Firms

The final good y_t is produced by a firm in a perfectly competitive environment, bundling together the differentiated intermediate goods $y_t(i)$ according to the constant returns to scale technology

$$y_t \leq \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)},$$

where $\theta > 1$ represents the elasticity of substitution between intermediate goods $y_t(i)$. Letting $P_t(i)$ denoting the price of intermediate good i, the following demand function for intermediate goods is obtained from profit maximization:

$$y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t,$$

with

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)}.$$

Each intermediate good i is produced by a single monopolistically competitive firm having access to the constant returns to scale technology

$$y_t(i) \leq k_t(i)^\alpha [z_t l_t(i)]^{1-\alpha},$$

where $1 > \alpha > 0$ represents the elasticity of output with respect to capital. The technology shock z_t follows the autoregressive process

$$\ln(z_t) = (1 - \rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{4.3}$$

with $1 > \rho_z > 0$, $z > 0$, and $\varepsilon_{zt} \sim N(0, \sigma_z^2)$. While each firm i exerts some market power, it acts as a price taker in the factor markets. Furthermore, the adjustment of the firm's nominal price $P_t(i)$ is assumed to be costly, where the cost function is convex in the size of the price adjustment. According to Rotemberg (1982), these costs are defined as

$$\frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t,$$

where $\phi_P \geq 0$ governs the size of price adjustment costs and π denotes the gross steady state rate of inflation targeted by the monetary authority. Following Ireland (1997), this specification accounts for the negative effects of price changes on customer-firm relationships. As a consequence of these convex adjustment costs, the firm's optimization problem becomes dynamic. In accordance with Ireland (2003), each firm chooses $l_t(i), k_t(i), y_t(i)$, and $P_t(i)$ to maximize its total market value

$$E\sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t]$$

subject to the demand function for intermediate goods, where λ_t measures the period t marginal utility to the representative household provided by an additional unit of profits. The firm's profits are distributed to the household as dividends, which are defined in real terms by

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right] y_t(i) - \frac{W_t l_t(i) + Q_t k_t(i)}{P_t} - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t.$$

4.2.4 Monetary Authority

Following Clarida et al. (2000), Ireland (2000), Canova (2009), and Fernández-Villaverde et al. (2010), monetary policy is conducted through setting the short-term nominal gross interest rate r_t according to a modified Taylor rule (see Taylor, 1993):

$$\ln\left(\frac{r_t}{r}\right) = \rho_r \ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t).$$

The monetary authority gradually adjusts the nominal interest rate in response to deviations of current gross inflation $\pi_t = \frac{P_t}{P_{t-1}}$ and output y_t from their steady state values, where ρ_r, ω_π and ω_y are the parameters of the monetary policy rule.[5] The monetary policy shock υ_t follows the autoregressive process

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{4.4}$$

where $0 < \rho_\upsilon < 1$ and $\varepsilon_{\upsilon t} \sim N(0, \sigma_\upsilon^2)$.

4.3 Solution and Estimation

Each version $\mathcal{M}_j$ of the model is characterized by a set of nonlinear difference equations, encompassing the first-order conditions for the three agents' problems, the laws of motion for the four exogenous shocks, and the monetary policy rule. To close the model, we complete the following two steps. First, we assume symmetric behavior within the intermediate sector to get from sectoral to aggregate variables, which implies $P_t(i) = P_t$, $y_t(i) = y_t$, $l_t(i) = l_t$, $k_t(i) = k_t$, and $D_t(i) = D_t$, for $t = 0, 1, 2...$ and all $i \in [0, 1]$. Second, for all $t = 0, 1, 2...$, the market clearing conditions for the bond market $B_t = B_{t-1} = 0$ must hold.

The empirical implementation requires additional preparation of the underlying model. Since the model is nonlinear, no exact analytical closed-form solution can be derived in general. Therefore, an approximate solution is obtained by computing the steady state, log-linearizing the system around the steady state, and then applying a complex generalized Schur decomposition to solve the linear difference model under rational expectations (see appendices H and I).[6] The

[5] Note that the gross steady state level of inflation π is assumed to be determined by the monetary authority.

[6] As outlined in Adjemian et al. (2011), the core of the DYNARE solution algorithm for computing the solution of a linear rational expectations model is predicated on a complex generalized Schur decomposition as presented in Klein (2000).

solution can be written in state space form with a state equation

$$s_{t+1} = \Gamma_0(\mu)s_t + \Gamma_1(\mu)\boldsymbol{\varepsilon}_{t+1}$$

and an observation equation

$$f_t = \Gamma_2(\mu)s_t,$$

where the vector s_t contains the model's state variables, the vector $\boldsymbol{\varepsilon}_{t+1}$ consists of the serially and mutually uncorrelated innovations[7], and the vector f_t comprises the model's flow variables. The matrices $\Gamma_0(\mu), \Gamma_1(\mu)$, and $\Gamma_2(\mu)$ contain (functions of) the model's parameters μ_i, $i = 1, 2, \ldots, k$. These parameters are estimated using Bayesian methods. In contrast to classical statistical inference, where the parameters of interest are considered as fixed, but unknown quantities, the Bayesian approach assigns a probabilistic interpretation to the model's parameters under consideration (see Robert, 2001; DeJong and Dave, 2007). Hence, as stated in An and Schorfheide (2007), the Bayesian framework allows to incorporate information about the model's parameters that is not contained in the estimation sample through re-weighting the likelihood function of a model by a prior density. Therefore, according to DeJong and Dave (2007), the likelihood function has to be formed at first, providing the foundation for both classical and Bayesian approaches to statistical inference. As outlined in Canova (2007, p. 123), "... the likelihood function of a state space model can be conveniently expressed in terms of one-step-ahead forecast errors, conditional on the initial observations, and of their recursive variance, both of which can be obtained with the Kalman filter."[8]

If

$$\mathcal{L}(\mu|X) \equiv p(X|\mu) = \prod_{t=1}^{T} p(X_t|\mu)$$

represents the likelihood function of the model, letting X denote the T observations of an $m \times 1$ vector of observable variables X_t and μ define a $k \times 1$ vector of the model's parameters, then, according to the Bayes theorem, for any specification

[7]Specifically, we assume $\boldsymbol{\varepsilon}_t \sim N(0, \sum_{\varepsilon}(\mu))$.

[8]See appendix J for a detailed presentation of the Kalman filter.

of the prior distribution $p(\mu)$[9], the posterior distribution of the model is given by

$$\begin{aligned} p(\mu|X) &= \frac{p(\mu)p(X|\mu)}{p(X)} \\ &= \frac{p(\mu)\mathcal{L}(\mu|X)}{p(X)} \\ &\propto p(\mu)\mathcal{L}(\mu|X), \end{aligned}$$

where the unnormalized posterior density $p(\mu)\mathcal{L}(\mu|X)$ is referred to as the posterior kernel $\mathcal{K}$.[10] Since, as described in Canova (2009), an analytical computation of the posterior is practically impossible when μ has many dimensions, we use a Markov Chain Monte-Carlo (MCMC) method to obtain draws from the unknown posterior distribution. "[This] method involves simulating from a complex and generally multivariate target distribution ... indirectly, by generating a Markov chain with the target density as its stationary density" (Brooks and Gelman, 1998, p. 434).

Specifically, we employ a Random-Walk Metropolis (RWM)-algorithm which was first applied to deliver draws from the posterior distribution of DSGE model parameters by Schorfheide (2000) and Otrok (2001). According to An and Schorfheide (2007) and Fernández-Villaverde (2010) the RWM-algorithm can be characterized by the following steps:

1. Employ a numerical optimization routine to maximize the log posterior kernel $\ln \mathcal{K}(\mu|X) = \ln p(\mu) + \ln \mathcal{L}(\mu|X)$ with respect to μ. Let $\tilde{\mu}$ be the posterior mode.

2. To initialize the procedure draw μ^0 from the symmetric jumping distribution $N(\tilde{\mu}, c\tilde{\sum})$, where $\tilde{\sum}$ defines the inverse of the Hessian computed at the posterior mode $\tilde{\mu}$ and c denotes a scale factor.[11] Alternatively, specify a starting value directly.

3. For $t = 1, \ldots, n_{sim}$ draw a proposal μ^* from the jumping distribution $J(\mu^*|\mu^{t-1}) = N(\mu^{t-1}, c\tilde{\sum})$. The jump from μ^{t-1} is accepted ($\mu^t = \mu^*$) if

[9] Subsequently, we assume that the prior distribution can be factored as $p(\mu) = \prod_{i=1}^{k} p(\mu_i)$.

[10] As stated in Hamilton (1994a) and DeJong and Dave (2007), $p(X)$, which assigns probabilities to specific values of X, can be regarded as a constant from the perspective of the distribution of μ.

[11] According to Otrok (2001), a suitable jumping distribution should be both easy to simulate from and symmetric. These requirements are satisfied by the multivariate normal distribution.

ϑ_t drawn from the uniform distribution $U(0,1)$ satisfies $\vartheta_t \leq min\{r,1\}$, with $r = \frac{p(\mu^*|X)}{p(\mu^{t-1}|X)} = \frac{\mathcal{K}(\mu^*|X)}{\mathcal{K}(\mu^{t-1}|X)}$, and rejected otherwise ($\mu^t = \mu^{t-1}$).

4. Approximate the posterior expected value of a function $g(\mu)$ by $\frac{1}{n_{sim}} \sum_{t=1}^{n_{sim}} g(\mu^t)$, where the Ergodic theorem implies that $\lim_{n_{sim}\to\infty} \frac{1}{n_{sim}} \sum_{t=1}^{n_{sim}} g(\mu^t) \xrightarrow{a.s.} E[g(\mu)]$ (for details see Canova, 2007).

Following DeJong et al. (2000b), different specifications of $g(\mu)$ can be considered. If, for example, $g(\mu)$ defines the identity function, then the sequence of accepted drawings $\{\mu^t\}$ can be used to approximate the posterior mean of μ. Alternatively, $g(\mu)$ "... might be an indicator for a small interval, in which case the function of interest is the (average) value of the posterior on that interval" (see DeJong et al., 2000b, p. 213).

For a more detailed description of MCMC methods we refer to Gelman et al. (2003) and Geweke (2005). The estimation procedure is implemented using DYNARE, which is a public domain toolbox for the simulation and estimation of DSGE models. To maximize the posterior kernel we employ Christopher Sim's hybrid optimization algorithm "csminwel", which combines the derivative-based Broyden-Fletcher-Goldfarb-Shannon (BFGS) method with a simplex algorithm.[12] As proposed by Roberts et al. (1997), the scale factor c is chosen to ensure an acceptance rate around 25%. We simulate five chains[13] of 100000 draws each, and discarded the first 50% as burn-in to eliminate any dependence of the chain from its starting values. The stationarity of the chains was monitored using the convergence checks proposed by Brooks and Gelman (1998).

4.4 Monte Carlo Filtering and Regionalized Sensitivity Analysis

4.4.1 Methodology

According to Ratto (2008), a number of issues regularly arises concerning the estimation and evaluation process of DSGE models, especially when these models

[12]For a detailed presentation of simplex and BFGS methods we refer to DeJong and Dave (2007) and Heer and Maussner (2009).

[13]Note that only one chain is used for inference, whereas the remaining chains are employed to diagnose MCMC convergence.

feature a complex structure and a rich parametrization:

i) Which portion of the prior parameter space violates the Blanchard-Kahn (1980) conditions, leading to indeterminacy or instability of the model?[14]

ii) Which of the structural parameters mostly drive the fit of a particular observed times series?

iii) Are there any conflicts or trade-off's between the fit of one observed time series versus another?

As shown in Ratto (2008), Monte Carlo filtering (MCF) and regionalized (or generalized) sensitivity analysis (RSA) can be used to answer these questions. Subsequently, we give a brief general description of both techniques before we focus on the application of these tools in a DSGE context. A detailed treatment of MCF and RSA can be found in Saltelli et al. (2004) and Saltelli et al. (2008).

According to Young et al. (1996) and Saltelli (2002), MCF denotes a process of rejecting sets of model simulations that do not fulfill certain pre-specified characteristics. More precisely, following Giglioli et al. (2004, p. 279), MCF is performed by "... mapping the space of the input factors into one or more model outputs, censoring the model output set into acceptable/non-acceptable (or behavioral or non-behavioral), and mapping back the acceptable (or behavioral) set into the space of input factors." Hence, two tasks are required for an MCF exercise (see Saltelli et al., 2008, p. 154):

- "[A] qualitative definition of the system behavior (a set of constraints: thresholds, ceilings, time bounds etc. based on available information on the system);

- [A] binary classification of model outputs based on the specified behavior definition (qualifies a simulation as behavioral, B, if the model output lies within constraints, non-behavioral, $\bar{B}$, otherwise)."

RSA, originally developed in the context of environmental models by Spear and Hornberger (1980), goes one step further by using the output of a MCF experiment for sensitivity analysis purposes (see Giglioli et al., 2004; Pappenberger et al., 2008). As in MCF, for a given vector of input factors, model output is

[14] A detailed description of the Blanchard-Kahn (1980) conditions is given in appendix I.

categorized (filtered) into either behavioral B or non-behavioral $\bar{B}$. "The $[B-\bar{B}]$ categorization is mapped back onto the [input factors], each of which is thus also partitioned into [an acceptable and a non-acceptable sub-sample]" (Saltelli et al., 2008, p. 185). Then, for each input factor, the sets of model inputs being classified as part of the behavioral subset B are compared with the sets of model inputs in subset $\bar{B}$, qualified as non-behavioral. Specifically, following Saltelli (2002), a statistical hypothesis test is applied to check, whether the two subsets are samples from the same distribution. An input factor is regarded as key factor in driving the model behavior, when the generated sample distributions are significantly different (see Ratto, 2008).

4.4.2 Mapping Stability

As outlined in Riggi and Tancioni (2010), the identification of the stability domain of the model under consideration is a fundamental issue in Bayesian DSGE model estimation, because it allows to initialize the estimation within the portion of the parameter space that satisfies the Blanchard-Kahn (1980) conditions for a unique stable solution. Since for most DSGE models an analytical derivation of the stability properties is very difficult if not impossible, RSA provides a valuable tool to detect the stability region of a model (see Ratto, 2008). Following Saltelli et al. (2004) and Ratto (2008), the steps for the analysis are as follows:

- Initially, N Monte Carlo runs are performed, sampling the model parameters from their prior distributions and propagating them through the model.[15] Therefore, each Monte Carlo run is associated with a specific vector of values of the input parameters.

- Depending on whether or not Blanchard-Kahn (1980) conditions are satisfied, the model output is is categorized into either behavioral or non-behavioral and then mapped back onto the input parameters, each of which is thus also partitioned into an acceptable and a non-acceptable sub-sample. Target stable behavior is classified as B, unacceptable behavior, i.e., instability or indeterminacy, is classified as $\bar{B}$. Hence, given the total number N

[15] In the applications presented below, the samples are generated using Sobol' quasi-Monte Carlo (LP_τ) sequences, which allow better efficiency properties compared to traditional pseudo-random Monte Carlo sampling (see Ratto, 2008; Saltelli et al., 2008). For a detailed description of quasi-Monte Carlo techniques we refer to Judd (1998), Sobol' (1998), and Saltelli et al. (2008).

of Monte Carlo runs, two subsets are obtained: $(\mu_i|B)$ of n elements and $(\mu_i|\bar{B})$ of $\bar{n}$ elements, where $n+\bar{n}=N$. According to Saltelli et al. (2008, p. 185), "[i]n general, the two sub-samples will come from different unknown probability density functions $f_n(\mu_i|B)$ and $f_{\bar{n}}(\mu_i|\bar{B})$."

- To detect the parameters that are mostly responsible for driving the DSGE model into the target stable behavior, the distributions $f_n(\mu_i|B)$ and $f_{\bar{n}}(\mu_i|\bar{B})$ are compared for each parameter independently, using a nonparametric test statistic of the Kolmogorov-Smirnov type. Since we are dealing with empirical distributions, the Smirnov two-sample test (two-sided version) is performed (see Conover, 1999; Saltelli et al., 2004), to compare the null hypothesis that the distributions $f_n(\mu_i|B)$ and $f_{\bar{n}}(\mu_i|\bar{B})$ are identical against the alternative hypothesis that the distributions are different:

$$\begin{aligned} H_0 &: f_n(\mu_i|B) = f_{\bar{n}}(\mu_i|\bar{B}) \\ H_1 &: f_n(\mu_i|B) \neq f_{\bar{n}}(\mu_i|\bar{B}). \end{aligned}$$

  The test statistic $d_{n,\bar{n}}$ is defined as the greatest absolute vertical distance between the cumulative probability functions $F_n(\mu_i|B)$ and $F_{\bar{n}}(\mu_i|\bar{B})$, i.e.,

$$d_{n,\bar{n}}(\mu_i) = \sup |F_n(\mu_i|B) - F_{\bar{n}}(\mu_i|\bar{B})|.$$

  Hence, the test is able to answer the following question (see Saltelli et al., 2004, p. 154): "At what significance level α does the computed values of $d_{n,\bar{n}}$ determine the rejection of H_0?" According to Ratto (2008, p. 118), the smaller α (or, equivalently, the larger $d_{n,\bar{n}}$), "... the more important is the parameter in driving the behavior of the DSGE model".

4.4.3 Mapping the Fit

While the estimation procedure outlined in section 4.3 selects values for the structural parameters that allow an overall optimal fit of the model with respect to a multivariate data set, MCF techniques can be used to map the fit of each singular time series in complex multivariate systems, i.e., MCF can detect "... which parameter values would be selected if one single observed series at a time were to be fitted" (Ratto, 2008, p. 125).

According to Ratto (2008), the MCF procedure is initialized, sampling the model parameters μ_i, $i = 1, 2, \ldots, k$, from their respective posterior distributions. Each Monte Carlo run is associated with a specific vector of values of the model parameters. Let $X_t = [\tilde{x}_{1t}, ..., \tilde{x}_{mt}]'$ encompass all observed times series $\tilde{x}_j$ with $j = 1, 2, \ldots, m$. Then, for each observed time series a binary classification of the structural parameters into either B or $\bar{B}$ is performed, where, based on the root mean squared error (RMSE) of the one-step-ahead model prediction, B labels the parameter values that produce the smallest 10 percent RMSEs. This procedure leads to m distinct filtering rules, each rule identifying the structural parameters that provide optimal fit for a specific observed series. Hence, given the total number m of observed time series $\tilde{x}_j$, we obtain m distinct empirical distributions $f_{\tilde{x}_j}(\mu_i|B)$ for each parameter μ_i. Therefore, MCF enables to identify the presence of trade-offs or conflicts for structural parameters when taking the DSGE model to the data. Following Ratto (2008, p. 126), trade-off's for a given parameter are detected, if the following two conditions are fulfilled:

- "[A]t least two distributions $f_{\tilde{x}_j}(\mu_i|B)$ are significantly different from the posterior distribution
- and such distributions are significantly different to each other."

If only the first condition holds, there are likely to be conflicts between the prior distribution and the likelihood.

4.5 Data and Priors

To estimate the structural parameters of the model specifications we use quarterly (seasonally adjusted) euro area and US data from 1980:Q1 to 2006:Q4.[16] Following Canova and Ferroni (2012), we decide to stop at 2006 to avoid complications stemming from the recent financial crisis. The euro area data come from the Area Wide Model (AWM) database (see Fagan et al. 2005) and the Euro Area Real-Time Database (RTDB), whereas the US data are taken from the FRED database of the Federal Reserve Bank of St. Louis.[17] We treat four variables as directly

[16]We verified that our main results are robust to extending the US data back to 1948 (see table L.1).

[17]Appendix K presents the data sources.

observed: real consumption, real investment, gross domestic product (GDP) price inflation, and the short-run interest rate.[18] The series for consumption and investment are expressed in per capita terms.[19] We detrend all time series applying the Hodrick-Prescott (H-P) filter, although being aware of the potential problem of spuriousness, as discussed in DeJong and Dave (2007) and Canova and Ferroni (2011).[20] By detrending inflation and the short-run interest rate, we follow Coenen and Wieland (2005), Juillard et al. (2006), Casares (2007), Canova and Ferroni (2012) and eliminate the downward trend in both series that occurs over the sample.[21]

Tables 4.1 and 4.2 present the prior distributions of the parameters, which are selected according to the following rule:

- Beta distributions are chosen for parameters that must lie in an interval $[0, 1]$ (ρ_r, ρ_a, ρ_z, ρ_x, ρ_v, h and α);
- gamma distributions are used for parameters that must be positive (ϕ_p, ϕ, $\ln(z)$);
- inverse gamma distribution are selected for the standard deviation of the shocks (σ_a, σ_z, σ_x and σ_v);
- and normal distributions are picked for all other parameters (ω_π and ω_y).

We follow Sahuc and Smets (2008) and assume a prior mean of 0.75 for ρ_r, ρ_a, ρ_z, and ρ_x, setting each standard deviations to 0.15. Regarding ρ_v and h we chose prior means of 0.5 and standard deviations of 0.1 and 0.2, respectively. With respect to α, we follow Smets and Wouters (2007) and set a prior with a mean of 0.3 and a standard deviation of 0.05. For the parameter ϕ_P, we assume a prior mean of 50 and a standard deviation of 10. Regarding the adjustment cost parameter ϕ, the prior mean is set to 4 ($\mathcal{M}_1$), 30 ($\mathcal{M}_2$), and 1500 ($\mathcal{M}_3$) with a prior standard deviation equal to one-fourth of the respective mean. Concerning

[18]Note that the model contains as many structural shocks as observable variables, so that the problem of stochastic singularity is avoided (see Ingram et al., 1994).

[19]Concerning the euro area time series on population, quarterly data is interpolated from the annual series using cubic spline interpolation.

[20] To facilitate the process of parameter estimation, we follow the procedure suggested in DeJong and Dave (2007, Chapter 11.2.5) and perform further data alignment by scaling the filtered series using their (relative) means.

[21]The use of original (not detrended) or quadratically detrended series of inflation and short-run interest rates did not alter our results substantially.

$\ln(z)$, we chose a prior mean of 8.13 for the euro area and 9.18 for the US, both with a standard deviation of 1.[22] For σ_a, σ_z, σ_x, and σ_v we follow Canova (2009) and assume priors with a mean of 0.01 and a standard deviation of 0.5. The prior distributions of the reaction coefficients ω_π and ω_y are centered on the prior mean values 1.3 and 0.125 with standard deviations equal to 0.3 and 0.2, respectively.

Since the model contains several parameters that are difficult to estimate precisely, we decide to fix them prior to estimation (i.e. impose dogmatic priors). The discount factor β is set equal to 0.99 following standard practice. We set η to a standard intermediate value of 1.35 (see Fernández-Villaverde, 2010), while χ is calibrated to assure that the representative household's labor supply in the steady state amounts to one-third of its time. In addition, the depreciation rate δ is set to 0.025, corresponding to an annual depreciation rate of about 10 percent and θ is fixed at 6, implying a steady state markup of prices over marginal cost of 20 percent. Finally, the respective steady state inflation rate is set equal to the average rate of inflation for the whole sample under consideration.

To identify which portion of the prior parameter space violates the Blanchard-Kahn (1980) conditions, leading to indeterminacy or instability of the three model specifications, we employ the RSA procedure based on a sample of size 2048 for both regions.[23] The RSA analysis shows that for all model specifications and both regions 85.1 percent of the prior space is stable, while 14.9 percent of the prior domain gives indeterminacy. Hence, the RSA results suggest a well-defined prior space, ensuring that the subsequent posterior update is not affected by a considerable portion of violations of Blanchard-Kahn (1980) conditions (see Ratto et al., 2009). Figures $L.1 - L.6$ show, that indeterminacy is essentially driven by the monetary policy parameter ω_π. The Smirnov test statistic $d_{n,\bar{n}}(\omega_\pi)$ rejects the null hypothesis based on a significance level $\alpha = 1\%$. The cumulative distributions for stable behavior are shifted to the right, indicating that the model specification are most likely to have a unique stable solution if $\omega_\pi > 1$, i.e., when the Taylor principle is satisfied.

[22] We set the prior means of $\ln(z)$ so that the steady state values of c_t and i_t in the model match the respective average values of consumption and investment in the data.

[23] The RSA is performed using the Sensitivity Analysis Toolbox for DYNARE, a collection of MATLAB routines developed by Marco Ratto (2009).

4.6 Posterior Odds Comparison

According to Geweke (1999), Bayesian inference provides a framework to assess the empirical performance of several competing models by comparing the different specifications through their posterior odds ratio. Moreover, as shown by Fernández-Villaverde and Rubio-Ramírez (2004), asymptotically, the best model under the Kullback-Leibler distance will have the highest posterior probability, with the former measure having a complete axiomatic foundation that justifies why it is precisely the criterion a rational agent should use to choose between models (even if these models are misspecified and/or nonnested).[24]

To derive the posterior odds ratio, we follow DeJong and Dave (2007) and rewrite the posterior to emphasize that the probability assigned to a given value of μ is conditional not only on the observations X, but also on the specific versions $\mathcal{M}_j$ of the model.[25] For a given model specification $\mathcal{M}_j$, the posterior can be written as

$$p(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j) = \frac{p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)}{p(X|\mathcal{M}_j)},$$

letting the notation $\mu_{\mathcal{M}_j}$ accentuate the potential for μ to be specific to a particular version of the model $\mathcal{M}_j$. Integrating this expression over $\mu_{\mathcal{M}_j}$ gives

$$\underbrace{\int p(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)d\mu_{\mathcal{M}_j}}_{=1} = \int \frac{p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)}{p(X|\mathcal{M}_j)}d\mu_{\mathcal{M}_j},$$

which can be rewritten to form an expression for the marginal likelihood associated with model specification $\mathcal{M}_j$:

$$p(X|\mathcal{M}_j) = \int p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)d\mu_{\mathcal{M}_j}.$$

The marginal likelihood is the probability that the model specification assigns to having observed the data.

As stated in DeJong and Dave (2007), just as the Bayes theorem can be used to derive the conditional probability associated with $\mu_{\mathcal{M}_j}$, it can also be applied

[24] As stated in Canova (2007), the Kullback-Leibler distance measures the discrepancy between the model distribution and the true distribution of the data.

[25] For the sake of consistency with section 4.2, we refer to specific versions of the model rather than to specific models.

to calculate the conditional probability with respect to a particular version of the model $\mathcal{M}_j$:

$$\begin{aligned} p(\mathcal{M}_j|X) &= \frac{p(\mathcal{M}_j)p(X|\mathcal{M}_j)}{p(X)} \\ &= \frac{p(\mathcal{M}_j)\left[\int p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)d\mu_{\mathcal{M}_j}\right]}{p(X)}, \end{aligned}$$

where $p(\mathcal{M}_j)$ denotes the prior probability assigned to a particular version of the model $\mathcal{M}_j$. Taking the ratio of conditional probabilities for two model specifications $\mathcal{M}_j$ and $\mathcal{M}_{j'}$ with $j, j' \in \{1, 2, 3\}$ gives the posterior odds ratio:

$$\begin{aligned} PO_{\mathcal{M}_j,\mathcal{M}_{j'}} &= \frac{p(\mathcal{M}_j|X)}{p(\mathcal{M}_{j'}|X)} \\ &= \frac{p(\mathcal{M}_j)[\int p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)d\mu_{\mathcal{M}_j}]}{p(\mathcal{M}_{j'})[\int p(\mu_{\mathcal{M}_{j'}}|\mathcal{M}_{j'})\mathcal{L}(\mu_{\mathcal{M}_{j'}}|X,\mathcal{M}_{j'})d\mu_{\mathcal{M}_{j'}}]}, \end{aligned}$$

where

$$\frac{p(\mathcal{M}_j)}{p(\mathcal{M}_{j'})}$$

is called the prior odds ratio and

$$\frac{[\int p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X,\mathcal{M}_j)d\mu_{\mathcal{M}_j}]}{[\int p(\mu_{\mathcal{M}_{j'}}|\mathcal{M}_{j'})\mathcal{L}(\mu_{\mathcal{M}_{j'}}|X,\mathcal{M}_{j'})d\mu_{\mathcal{M}_{j'}}]}$$

is referred to as the Bayes factor. As stated in Kass and Raftery (1995, p. 777), the Bayes factor represents "a summary of the evidence by the data in favor of one scientific theory, represented by a statistical model, as opposed to another." Note that the Bayes factor corresponds to the posterior odds ratio if the prior odds ratio is set to unity, i.e., either model specification is equally probable a priori. To interpret the Bayes factor, Jeffreys (1961) suggests the following rule of thumb:

i) a Bayes factor between 1 and 3 provides very slight evidence,

ii) a Bayes factor between 3 and 10 provides slight evidence,

iii) a Bayes factor between 10 and 32 provides strong evidence,

iv) a Bayes factor between 32 and 100 provides very strong evidence and

v) a Bayes factor above 100 provides decisive evidence

in favor of specification $\mathcal{M}_j$ against $\mathcal{M}_{j'}$.

Further, relaxing the assumption of a unity prior odds ratio, posterior odds can be used to compare a set of model specifications by computing their posterior probabilities. Following Fernández-Villaverde and Rubío-Ramirez (2004) and An and Schorfheide (2007), the posterior probability of $\mathcal{M}_1$ against $\mathcal{M}_2$ and $\mathcal{M}_3$ is given by

$$p_{\text{post}_1} = \frac{1}{PO_{\mathcal{M}_1,\mathcal{M}_1} + PO_{\mathcal{M}_2,\mathcal{M}_1} + PO_{\mathcal{M}_3,\mathcal{M}_1}}.$$[26]

As outlined in An and Schorfheide (2007) and Canova (2007), the practical difficulty in implementing posterior odds comparisons is the computation of the marginal likelihood because the integral

$$p(X|\mathcal{M}_j) = \int p(\mu_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu_{\mathcal{M}_j}|X, \mathcal{M}_j)d\mu_{\mathcal{M}_j}$$

typically has no analytical solution. Since there are several approaches to compute marginal likelihoods, usually relying on approximations or simulation-based methods, we decide to choose Geweke's (1999) modified harmonic mean estimator to calculate the marginal likelihood based on the output of the RWM-algorithm.[27] The theoretical foundation of this approach is the harmonic mean identity, which implies that the reciprocal of the integrated likelihood is equal to the posterior harmonic mean of the likelihood (see Raftery et al., 2007). Following Canova (2006) and An and Schorfheide (2007), the modified harmonic mean estimator can be described as follows: For each version of the model $\mathcal{M}_j$, the marginal likelihood $p(X|\mathcal{M}_j)$ is approximated using

$$\left[\frac{1}{n_{sim}} \sum_{t=1}^{n_{sim}} \frac{f(\mu^t_{\mathcal{M}_j})}{p(\mu^t_{\mathcal{M}_j}|\mathcal{M}_j)\mathcal{L}(\mu^t_{\mathcal{M}_j}|X, \mathcal{M}_j)}\right]^{-1},$$

where every single vector $\mu^t_{\mathcal{M}_j}$ comes from the RWM iterations, denoting the draw t of the parameters μ of model specification $\mathcal{M}_j$ and f is a truncated normal dis-

[26]The computation of p_{post_2} and p_{post_3} proceeds in the same manner.

[27]While DYNARE allows to choose between a Laplace-approximation method and the modified harmonic mean estimator, we prefer the latter, motivated by the results of Adolfson et al. (2007), who find the modified harmonic mean estimator to be numerically stable in the RWM case.

tribution, which, to make the numerical approximation efficient, should be chosen so that the summands are of equal magnitude. For a more detailed discussion of the modified harmonic mean estimator we refer to Geweke (1999).

4.7 Results

4.7.1 Bayesian Estimation

Table 4.1 and 4.2 present the parameter estimates of the three model specifications across the two data sets. Each table lists posterior means, posterior standard deviations, and highest posterior density (HPD) intervals containing the 90 percent highest posterior density.[28] The respective prior and posterior distributions of the parameters are graphed in figures $L.7 - L.12$.

For both regions, figures $L.7 - L.12$ show the data to be reasonably informative about the shock parameters, since their prior and posterior distributions appear to be relatively distinct. The posterior means of ρ_a, ρ_z, and ρ_v indicate the three shocks to be quite persistent across all model specifications $\mathcal{M}_1 - \mathcal{M}_3$, whereas the monetary policy shock appears to be less persistent than the preference and the technology shock. Contrary to these results, ρ_x differs substantially in magnitude between $\mathcal{M}_1$ on the one hand and $\mathcal{M}_2$ and $\mathcal{M}_3$ on the other hand, since, for both regions, the respective HPD intervals do not overlap. While the latter specifications exhibit a high persistence of the marginal efficiency of investment shock, the estimated mean of the persistence parameter is considerably lower under $\mathcal{M}_1$. Except for ρ_z, we find the persistence parameters of the shocks to be relatively higher in the euro area. For example, the posterior mean of the autoregressive parameter ρ_a is almost two times higher in the euro area than in the US (across all model specifications), which is broadly consistent with the findings reported in Smets and Wouters (2005).

Further differences between $\mathcal{M}_1$ and the specifications $\mathcal{M}_2/\mathcal{M}_3$ appear when turning to the estimated standard deviations of the innovations. Except for σ_v, the standard deviations of the innovations appear to be higher under model specification $\mathcal{M}_1$ in both the euro area and the US. For both regions, the posterior

[28] For a detailed description of HPD intervals and their construction we refer to Gill and King (2003), whereas the computation of posterior standard deviations is outlined in DeJong and Dave (2007).

means of σ_a and σ_z increase by at least one respective posterior standard deviation under $\mathcal{M}_1$, while in the case of σ_x the HPD intervals of $\mathcal{M}_2$ and $\mathcal{M}_3$ do not even overlap with the intervals of $\mathcal{M}_1$. Despite these differences, $\mathcal{M}_1 - \mathcal{M}_3$ coincide in detecting the shock to the marginal efficiency of investment to be the most volatile followed by the technology and the preference shock, whereas the monetary policy shock features the lowest volatility. Our estimates are in line with the results of Adolfson et al. (2008) and Gelain et al. (2012) for closed economy DSGE models of the euro area featuring adjustment costs of the form S_1, both finding the marginal efficiency of investment shock to be relatively low in persistence but high in volatility. For the US, our results coincide with the findings of Ireland (2003) and Justiniano et al. (2010), who both obtain relatively large estimates of σ_x, implementing adjustment cost specifications of the form S_2 and S_1, respectively. Overall, we find the standard deviations of the innovations (σ_a, σ_z, σ_x, and σ_v) to be relatively higher in the US than in the euro area.

Compared to the results of Smets and Wouters (2003) and Adjemian et al. (2007) for the euro area, we obtain smaller values of the degree of backward-looking behavior of consumption across all model specifications, whereas our estimates for the US, which appear to be higher than in the euro area, are in line with the findings of Christiano et al. (2005), Smets and Wouters (2005), and Ireland (2007). Further, we find the posterior mean of h under $\mathcal{M}_1$ to exceed the estimated means under $\mathcal{M}_2$ and $\mathcal{M}_3$ by more than one posterior standard deviation in case of the euro area, while these differences become less apparent for the US.

Regarding the price adjustment cost parameter ϕ_P, we find the posterior means of $\mathcal{M}_2$ and $\mathcal{M}_3$ to increase by at least one posterior standard deviation under $\mathcal{M}_1$, both in the euro area and the US. Concerning its magnitude, our estimates of the price adjustment cost parameter are consistent with the results of Gerali et al. (2010) for the euro area and Ireland (2001) for the US. Moreover, the degree of price stickiness seems to be higher in the US than in the euro area, although this result is in line with the findings of Smets and Wouters (2005) it is at odds with the micro-evidence on price stickiness presented in Álvarez et al. (2006).

Turning first to the results for the euro area, the data point to lower adjustment costs S_1 than were embodied in the prior, since we find the posterior distribution for ϕ to be left-shifted relative to the prior distribution with a posterior mean lying more than one posterior standard deviation below its prior mean. Concerning adjustment cost specifications S_2 and S_3, the respective prior and posterior means of ϕ appear to be similar in size, whereas the posterior distributions are more tightly distributed than the priors.

In case of the US, we find the data to be quite informative about the adjustment costs parameter across all specifications $S_1 - S_3$, since the posterior distributions for ϕ are left-shifted relative to the prior distributions with the respective posterior means lying partly more than one posterior standard deviation below their prior means. For S_1, our results are in line with Levin et al. (2005) and Zubairy (2010), whereas our findings for S_2 are consistent with the estimates of Ireland (2003). By broadening the scope of the analysis across data sets, we perceive adjustment costs to be higher in the euro area than in the US.

The estimates of α, measuring share of capital in the production function for intermediate goods, are almost exactly 0.3 across all model specifications for both regions. As shown in figures $L.7 - L.12$ the respective posterior distributions of α are centered very close to the prior, but are much more tightly distributed, both which is due to the data alignment described in section 4.5.

Turning to the monetary policy reaction function estimated for the euro area, we find the respective posterior means of ω_π and ω_y to be approximately equal across the model specifications $\mathcal{M}_1 - \mathcal{M}_3$, whereas the posterior mean of the interest-rate-smoothing parameter ρ_r under $\mathcal{M}_1$ exceeds the respective estimates under $\mathcal{M}_2$ and $\mathcal{M}_3$ by more than two posterior standard deviations. The estimated means of ω_π are in line with results for the euro area documented by Smets and Wouters (2003), Adolfson et al. (2008), and Sahuc and Smets (2008), while we find smaller values for the posterior mean of ρ_r. Like in Smets and Wouters (2003) and Adolfson et al. (2008), we obtain small estimates of the posterior mean of ω_y, albeit our results all show a negative sign.

In case of the US a similar pattern occurs, since the respective posterior means of ω_π and ω_y are approximately equal across the model specifications $\mathcal{M}_1 - \mathcal{M}_3$, whereas the posterior mean of the interest-rate-smoothing parameter ρ_r under $\mathcal{M}_1$ exceeds the respective estimates under $\mathcal{M}_2$ and $\mathcal{M}_3$ by more than one posterior standard deviation. In accordance with the results for the euro area, we find small values of the posterior mean of ω_y, although the estimates turn out to be positive for the US. The posterior means of ω_π and ω_y are consistent with the estimates provided in Smets and Wouters (2007) and Sahuc and Smets (2008), while we find smaller values for the posterior mean of ρ_r. By comparing the estimated policy rules across regions, it turns out that the interest rate persistence is higher and the response to inflation is stronger in the US.

Finally, we find the estimates of $\ln(z)$ to be almost equal across all model specifications within each region. Note that $\ln(z)$ has no impact on the dynamics of the model specifications, since it only serves to determine steady states (see section 4.5). As displayed in figures $L.7 - L.12$ prior and posterior distribution are almost indistinguishable, indicating that the data are uninformative regarding the location of $\ln(z)$.

		$\mathcal{M}_1$						$\mathcal{M}_2$					
		prior		posterior				prior		posterior			
	Type	Mean	Std.	Mean	Std.	HPD Interval		Mean	Std.	Mean	Std.	HPD Interval	
ρ_r	Beta	0.7500	0.1500	0.5436	0.0602	0.4536	0.6399	0.7500	0.1500	0.3887	0.0753	0.2683	0.5028
ρ_a	Beta	0.7500	0.1500	0.6846	0.0706	0.5660	0.7994	0.7500	0.1500	0.7316	0.0687	0.6368	0.8256
ρ_z	Beta	0.7500	0.1500	0.9553	0.0252	0.9183	0.9920	0.7500	0.1500	0.9618	0.0217	0.9347	0.9906
ρ_x	Beta	0.7500	0.1500	0.1886	0.0677	0.0805	0.2911	0.7500	0.1500	0.8166	0.0479	0.7384	0.8939
ρ_v	Beta	0.5000	0.1000	0.4462	0.0536	0.3643	0.5323	0.5000	0.1000	0.4010	0.0532	0.3286	0.4756
h	Beta	0.5000	0.2000	0.2857	0.0769	0.1593	0.4120	0.5000	0.2000	0.1743	0.0801	0.0749	0.2682
α	Beta	0.3000	0.0500	0.3001	0.0006	0.2988	0.3014	0.3000	0.0500	0.3001	0.0009	0.2986	0.3016
ϕ_P	Gamma	50.0000	10.0000	41.2423	9.9792	29.0034	53.7091	50.0000	10.0000	26.3339	14.7360	17.8193	34.4437
ϕ	Gamma	4.0000	1.0000	3.1538	0.6462	2.0259	4.2973	30.0000	7.5000	31.3504	4.9778	23.4267	39.2136
ω_π	Normal	1.3000	0.3000	1.7634	0.2137	1.4256	2.1145	1.3000	0.3000	1.7021	0.1902	1.3921	2.0199
ω_y	Normal	0.1250	0.2000	-0.0447	0.0273	-0.1012	0.0067	0.1250	0.2000	-0.0400	0.0255	-0.0906	0.0110
$\ln(z)$	Gamma	8.1300	1.0000	8.1267	0.9981	6.4052	9.6600	8.1300	1.0000	8.1369	1.0060	6.4460	9.7428
σ_a	Inverse gamma	0.0100	0.5000	0.0054	0.0007	0.0043	0.0065	0.0100	0.5000	0.0043	0.0007	0.0034	0.0052
σ_z	Inverse gamma	0.0100	0.5000	0.0094	0.0011	0.0077	0.0110	0.0100	0.5000	0.0079	0.0010	0.0068	0.0090
σ_x	Inverse gamma	0.0100	0.5000	0.0309	0.0065	0.0194	0.0428	0.0100	0.5000	0.0087	0.0015	0.0063	0.0111
σ_v	Inverse gamma	0.0100	0.5000	0.0020	0.0002	0.0016	0.0023	0.0100	0.5000	0.0024	0.0003	0.0018	0.0029

		$\mathcal{M}_3$					
		prior		posterior			
	Type	Mean	Std.	Mean	Std.	HPD Interval	
ρ_r	Beta	0.7500	0.1500	0.4046	0.0739	0.2866	0.5208
ρ_a	Beta	0.7500	0.1500	0.7310	0.0719	0.6332	0.8305
ρ_z	Beta	0.7500	0.1500	0.9631	0.0219	0.9365	0.9922
ρ_x	Beta	0.7500	0.1500	0.8208	0.0460	0.7438	0.8966
ρ_v	Beta	0.5000	0.1000	0.4076	0.0538	0.3306	0.4802
h	Beta	0.5000	0.2000	0.1842	0.0825	0.0829	0.282
α	Beta	0.3000	0.0500	0.3002	0.0010	0.2985	0.3018
ϕ_P	Gamma	50.0000	10.0000	27.4259	13.7310	18.647	36.313
ϕ	Gamma	1500.0000	375.0000	1368.5365	281.3410	1005.659	1730.2949
ω_π	Normal	1.3000	0.3000	1.7171	0.1987	1.3984	2.0413
ω_y	Normal	0.1250	0.2000	-0.0371	0.0285	-0.0853	0.0132
$\ln(z)$	Gamma	8.1300	1.0000	8.1245	1.0065	6.4623	9.7284
σ_a	Inverse gamma	0.0100	0.5000	0.0045	0.0007	0.0035	0.0054
σ_z	Inverse gamma	0.0100	0.5000	0.0080	0.0010	0.0069	0.0091
σ_x	Inverse gamma	0.0100	0.5000	0.0095	0.0021	0.0068	0.0122
σ_v	Inverse gamma	0.0100	0.5000	0.0023	0.0003	0.0018	0.0028

Table 4.1: Estimates: Euro area.

		$\mathcal{M}_1$						$\mathcal{M}_2$					
		prior		posterior				prior		posterior			
	Type	Mean	Std.	Mean	Std.	HPD Interval		Mean	Std.	Mean	Std.	HPD Interval	
ρ_r	Beta	0.7500	0.1500	0.6338	0.0431	0.5632	0.7020	0.7500	0.1500	0.5597	0.0499	0.4749	0.6487
ρ_a	Beta	0.7500	0.1500	0.3640	0.1157	0.1820	0.5398	0.7500	0.1500	0.4031	0.1234	0.2147	0.5872
ρ_z	Beta	0.7500	0.1500	0.9706	0.0132	0.9462	0.9967	0.7500	0.1500	0.9635	0.0177	0.9363	0.9910
ρ_x	Beta	0.7500	0.1500	0.1872	0.0598	0.0885	0.2856	0.7500	0.1500	0.7543	0.0588	0.6566	0.8461
ρ_v	Beta	0.5000	0.1000	0.2623	0.0540	0.1666	0.3539	0.5000	0.1000	0.2724	0.0547	0.1811	0.3650
h	Beta	0.5000	0.2000	0.5371	0.0656	0.4390	0.6410	0.5000	0.2000	0.4715	0.0708	0.3635	0.5869
α	Beta	0.3000	0.0500	0.2997	0.0010	0.2978	0.3015	0.3000	0.0500	0.3000	0.0008	0.2985	0.3015
ϕ_P	Gamma	50.0000	10.0000	70.1532	6.7694	54.2108	85.3110	50.0000	10.0000	55.3468	7.6791	40.4427	69.3412
ϕ	Gamma	4.0000	1.0000	2.3902	0.8028	1.2067	3.5746	30.0000	7.5000	20.2893	10.1310	12.0551	28.0795
ω_π	Normal	1.3000	0.3000	2.1442	0.2112	1.7932	2.4765	1.3000	0.3000	2.1798	0.2064	1.8402	2.5082
ω_y	Normal	0.1250	0.2000	0.0124	0.0300	-0.0336	0.0601	0.1250	0.2000	0.0152	0.0297	-0.0322	0.0616
$\ln(z)$	Gamma	9.1800	1.0000	9.1809	0.9935	7.5237	10.8273	9.1800	1.0000	9.1917	1.0055	7.5210	10.8158
σ_a	Inverse gamma	0.0100	0.5000	0.0089	0.0009	0.0071	0.0106	0.0100	0.5000	0.0074	0.0009	0.0060	0.0089
σ_z	Inverse gamma	0.0100	0.5000	0.0123	0.0010	0.0102	0.0143	0.0100	0.5000	0.0113	0.0009	0.0097	0.0129
σ_x	Inverse gamma	0.0100	0.5000	0.0856	0.0279	0.0427	0.1266	0.0100	0.5000	0.0211	0.0097	0.0130	0.0289
σ_v	Inverse gamma	0.0100	0.5000	0.0023	0.0002	0.0020	0.0026	0.0100	0.5000	0.0025	0.0002	0.0021	0.0029

		$\mathcal{M}_3$					
		prior		posterior			
	Type	Mean	Std.	Mean	Std.	HPD Interval	
ρ_r	Beta	0.7500	0.1500	0.5713	0.0486	0.4883	0.6584
ρ_a	Beta	0.7500	0.1500	0.395	0.1231	0.2095	0.5835
ρ_z	Beta	0.7500	0.1500	0.9633	0.0183	0.9366	0.9919
ρ_x	Beta	0.7500	0.1500	0.7566	0.0594	0.6634	0.8514
ρ_v	Beta	0.5000	0.1000	0.2746	0.0548	0.1818	0.3690
h	Beta	0.5000	0.2000	0.4841	0.0707	0.3725	0.5944
α	Beta	0.3000	0.0500	0.3000	0.0007	0.2985	0.3016
ϕ_P	Gamma	50.0000	10.0000	57.6791	7.5121	43.0400	72.3395
ϕ	Gamma	1500.0000	375.0000	944.4802	539.1763	549.9274	1335.0799
ω_π	Normal	1.3000	0.3000	2.1752	0.2043	1.8281	2.5026
ω_y	Normal	0.1250	0.2000	0.0182	0.0297	-0.0297	0.0652
$\ln(z)$	Gamma	9.1800	1.0000	9.1926	1.0051	7.5516	10.8149
σ_a	Inverse gamma	0.0100	0.5000	0.0076	0.0009	0.0061	0.0092
σ_z	Inverse gamma	0.0100	0.5000	0.0114	0.0009	0.0098	0.0130
σ_x	Inverse gamma	0.0100	0.5000	0.0243	0.0130	0.0145	0.0340
σ_v	Inverse gamma	0.0100	0.5000	0.0024	0.0002	0.0021	0.0028

Table 4.2: Estimates: US.

4.7.2 Monte Carlo Filtering Analysis

To identify the presence of trade-offs or conflicts for the structural parameters when taking the three model specifications to the data, we apply the MCF procedure described in section 4.4.3 combined with a Smirnov two-sample test.[29] The Smirnov test answers the question: "At what significance level α can the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution be rejected?"[30] Figures $L.13 - L.36$ present the cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and the respective cumulative posterior probability distributions (base), while the tables $L.2 - L.7$ contain the p-values (in percent) of the Smirnov two-sample test.

In section 4.7.1, we found that the most remarkable differences between specification $\mathcal{M}_1$ on the one hand and $\mathcal{M}_2$ and $\mathcal{M}_3$ on the other hand occur with respect to the parameters ρ_x and σ_x, capturing the persistence and volatility of the marginal efficiency of investment shock. We detected the posterior means of ρ_x under specification $\mathcal{M}_2$ and $\mathcal{M}_3$ to be considerably higher than under $\mathcal{M}_1$, whereas the estimates of σ_x turned out to be markedly lower. In the remainder of this section we will use the MCF analysis to identify the causes of these differences.

Concerning ρ_x, the three specifications differ markedly with regard to the cumulative empirical probability distributions $F_r(\rho_x|B)$. We find no significant differences between $F_r(\rho_x|B)$ and the cumulative posterior distribution (euro area) or even support for smaller values of the persistence parameter (US) in case of $\mathcal{M}_1$, while observations of r clearly support larger values of ρ_x, i.e., a more persistent shock to the marginal efficiency of investment, in specifications $\mathcal{M}_2$ and $\mathcal{M}_3$.

This provides an explanation for the relative higher estimates of the marginal efficiency of investment shock's persistence under specification $\mathcal{M}_2$ and $\mathcal{M}_3$ compared to $\mathcal{M}_1$.

Turning to σ_x, the model versions differ especially regarding the relative location of $F_i(\sigma_x|B)$ and $F_\pi(\sigma_x|B)$ with respect to the particular cumulative posterior

[29] We perform the MCF analysis using the Sensitivity Analysis Toolbox for DYNARE, a collection of MATLAB routines developed by Marco Ratto (2009). The MCF procedure is based on a sample of size 6000.

[30] Recall from section 4.4.3, that B labels the parameter values producing the smallest 10 percent RMSEs.

distributions. For the euro area, the observations for i significantly prefer larger values of σ_x under $\mathcal{M}_1$, whereas in case of $\mathcal{M}_2$ and $\mathcal{M}_3$, i data support smaller values than the ones implied by the posterior distribution. With regard to the US, the preferences of the observations are qualitatively consistent across all model specifications, although the p-values show that the preferences of i and π observations for lower values of σ_x are considerably stronger under $\mathcal{M}_2$ and $\mathcal{M}_3$ than under $\mathcal{M}_1$. Hence, the MCF results give a clue why specifications $\mathcal{M}_2$ and $\mathcal{M}_3$ exhibit a lower volatility of the marginal efficiency of investment shock than $\mathcal{M}_1$.

4.7.3 Model Evaluation

4.7.3.1 Bayesian Model Comparison

To assess the overall time series fit of the three model specifications under consideration, we list the respective prior probabilities, marginal log likelihoods and posterior probabilities in table 4.3. By assigning equal probabilities to the model specifications a priori, the exponentiated differences of marginal log likelihoods can be interpreted as posterior odds.[31]

	Euro Area		
	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$
Prior probability	1/3	1/3	1/3
Marginal log likelihood	1790.2238	1800.6993	1800.6304
Posterior probability	0.0000	0.5172	0.4828

	US		
	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$
Prior probability	1/3	1/3	1/3
Marginal log likelihood	1606.2136	1617.3322	1616.0600
Posterior probability	0.0000	0.7811	0.2189

Table 4.3: Prior probabilities, marginal log likelihoods and posterior probabilities across model specifications and regions.

[31] The posterior odds of model specification $\mathcal{M}_j$ versus $\mathcal{M}_{j'}$ with $j, j' \in \{1, 2, 3\}$ can be obtained by multiplying the prior odds, which are unity in the case of equal prior probabilities, with the Bayes factor $\exp\left[\ln p(\mathcal{M}_j|X) - \ln p(\mathcal{M}_{j'}|X)\right]$.

For both data sets we find $\mathcal{M}_2$ to perform best in explaining the data showing the highest posterior probability, whereas specification $\mathcal{M}_1$, gaining the lowest marginal log likelihood, turns out to have a zero posterior probability. More specifically, the posterior odds provide very slight evidence in favor of $\mathcal{M}_2$ against $\mathcal{M}_3$ and decisive evidence in favor of $\mathcal{M}_2$ against $\mathcal{M}_1$ for the euro area. In case of the US, Jeffreys' (1961) rule of thumb indicates slight evidence in favor of $\mathcal{M}_2$ against $\mathcal{M}_3$ and decisive evidence in favor of $\mathcal{M}_2$ against $\mathcal{M}_1$.

As discussed in Rabanal and Rubio-Ramírez (2008), policymakers are often interested in how theoretical models compare with more densely parameterized and less restrictive reference models. Hence, in line with Schorfheide (2000), Smets and Wouters (2003), and Juillard et al. (2008), we broaden the scope of the analysis by comparing the three model specifications with Bayesian vector autoregressive (BVAR) models at various lag lengths (lags 1 to 4).[32] Concerning the prior distributions of the BVARs we follow Lubik and Schorfheide (2005) and use a modified version of the so-called Minnesota prior, which originally dates back to Litterman (1980) and Doan et al. (1984).[33] The prior is made of two components, with a dummy observation prior, constructed according to Sims' version of the Minnesota prior (see Doan et al.), being augmented by Jeffrey's improper prior. For a detailed description of the BVAR setup we refer to Lubik and Schorfheide (2005).[34]

Table 4.4 displays prior probabilities, marginal log likelihoods and posterior probabilities of the three model specifications and the four BVARs.[35] For both data sets our three model specifications do not compare favorably to the best fitting BVAR. In contrast to large-scale DSGE models as developed by Smets and Wouters (2007) or Ratto et al. (2009), which are able to outperform BVAR models, our deliberately stylized specifications lack of features like wage and price indexation or the usage of a large set of shocks. Although to some extent criticized

[32]Pre-samples are used to initialize the BVARs.

[33]Since, as outlined in Lütkepohl (2005), the Minnesota prior was primarily suggested for certain non-stationary processes, we slightly modify the prior, following the approach of Lubik and Schorfheide (2005).

[34]As outlined in Del Negro and Schorfheide (2011), the Minnesota prior depends on several hyperparameters. According to Smets and Wouters (2007) and Ratto et al. (2009), we set the prior decay and tightness parameters to 0.5 and 3, respectively. Further, the parameter determining the weight on own-persistence (sum-of-coefficients on own lags) is set at 2 and the parameter determining the degree of co-persistence is set at 5.

[35]The marginal likelihoods of the BVARs are computed using the MATLAB codes provided by Christopher Sims, which are available at http://sims.princeton.edu/yftp/VARtools/.

Euro Area							
	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	BVAR(1)	BVAR(2)	BVAR(3)	BVAR(4)
Prior probability	1/7	1/7	1/7	1/7	1/7	1/7	1/7
Marginal log likelihood	1790.2238	1800.6993	1800.6304	1807.1802	1813.9321	1808.1124	1803.7778
Posterior probability	0.0000	0.0000	0.0000	0.0012	0.9958	0.0030	0.0000

US							
	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	BVAR(1)	BVAR(2)	BVAR(3)	BVAR(4)
Prior probability	1/7	1/7	1/7	1/7	1/7	1/7	1/7
Marginal log likelihood	1606.2136	1617.3322	1616.0600	1636.0099	1647.9345	1656.7848	1660.5414
Posterior probability	0.0000	0.0000	0.0000	0.0000	0.0000	0.0228	0.9772

Table 4.4: Cross-validation: BVARs and model specifications.

(see, for example, Chari et al., 2009), we would expect that the implementation of these features would improve the macroeconomic fit of our model specifications considerably.

4.7.3.2 Standard Moment Criteria

To understand why model specification $\mathcal{M}_1$ fits the data worse than $\mathcal{M}_2$ and $\mathcal{M}_3$, we present some selected unconditional second moments of the data and compare them with their theoretical counterparts implied by the different model versions. Tables L.8 and L.9 report standard deviations and first-order autocorrelations of consumption, investment, inflation, and the nominal interest rate derived from the data and the moments obtained from the three model specifications. Following Schmitt-Grohé and Uribe (2008) and Gabriel et al. (2012), the model implied statistics were computed by simulating the model specifications at the respective posterior means obtained from estimation. As outlined in Rabanal (2007), likelihood-based methods try to fit all second moments of the data, so this selection is just illustrative of where the model specifications fail.

Turning to the standard deviations first, we find the data to exhibit a higher volatility in consumption, investment, and the nominal interest rate for the US than for the euro area. Although the three model specifications are able to match the empirical observation that investment is more volatile than consumption, all specifications generate too much volatility of consumption compared to the data. In line with the Bayesian model comparison, we detect marked differences between $\mathcal{M}_1$ on the one hand and $\mathcal{M}_2$ and $\mathcal{M}_3$ on the other hand in fitting the data. Specifically, $\mathcal{M}_2$ and $\mathcal{M}_3$ outperform $\mathcal{M}_1$ in matching the volatility of consumption, investment and inflation in the data for both regions. Most noticeably, specification $\mathcal{M}_1$ performs relatively poorly in capturing the standard deviation of investment, since the volatility implied by the model specification departs considerably from its counterpart in the data. Model version $\mathcal{M}_1$ is able to outperform the two other specifications only in capturing the volatility of the nominal interest rate in the data with respect to the euro area.

Concerning first-order autocorrelations all model specifications appear to match the serial correlations of the data well. The most notable discrepancies between model specification and data can be found for the euro area, since actual inflation turns out to have a much lower first-order autocorrelation than implied by the model specifications.

4.7.3.3 Impulse Response Analysis

To gain insight into the dynamic properties of the three model specifications, we compare impulse response functions (IRFs) of output, consumption, investment, hours, inflation, and the nominal interest rate with respect to the four structural shocks (see figures $L.37 - L.44$).[36] Following Levin et al. (2005), all impulse responses are computed by simulating the model specifications at the posterior means reported in table 4.1 and 4.2.

First of all, the respective model specifications show similar qualitative responses across the data sets. However, to some extent we find marked differences between the IRFs of specification $\mathcal{M}_1$ on the one hand and $\mathcal{M}_2$ and $\mathcal{M}_3$ on the other hand within the data sets.

In accordance with the results of Smets and Wouters (2003) and Peersman and Straub (2006), a positive preference shock crowds out investment, but has a positive impact on consumption and output. Further, the increase in capacity necessary to satisfy demand causes a rise in hours worked. Figures L.37 and L.41 show that the increase of overall demand puts upward pressure on inflation. Consequently, the movements in output and inflation lead to a rise in the nominal interest rate under the estimated monetary policy rule. While the responses of output, consumption, hours, inflation, and the interest rate turn out to be quite

[36] The impulse responses are the log-deviations from the steady state to a one-standard deviation innovation. A detailed description of the computation of IRFs can be found in Hamilton (1994a), Lütkepohl (2005), and Canova (2007).

similar across the three model specifications, we find marked differences with respect to investment. In contrast to $\mathcal{M}_2$ and $\mathcal{M}_3$, investment declines in a hump-shaped pattern under $\mathcal{M}_1$.

Figures L.38 and L.42 report the impulse responses to the marginal efficiency of investment shock. We find the dynamics to be fairly similar across the model specifications. In accordance with the results of Peersman and Straub (2006) and Justiniano et al. (2010), output, investment, and hours rise after a positive impulse. With respect to consumption, we find an initially negative response, followed by a rise after a few quarters. Inflation and the nominal interest rate both rise in response to a positive marginal efficiency of investment shock. Justiniano et al. (2009) therefore attest the marginal efficiency of investment shock to have the typical features of a demand shock, which moves quantities and prices in the same direction, leading to a tightening of monetary policy.

Although the shape of the IRFs differs to some extend considerably between $\mathcal{M}_1$ and the model versions $\mathcal{M}_2/\mathcal{M}_3$, all model specifications predict a rise in output, consumption, and investment after a positive technology shock (see figures L.39 and L.43). Once again, we find hump-shaped investment dynamics under $\mathcal{M}_1$. Further, consistent with empirical evidence presented in Galí (1999), the presence of nominal price rigidities causes a decrease in hours worked in response to a positive technology shock. In all model specifications inflation falls contemporaneously following a positive technology shock, because the increase in productivity lowers real marginal costs. Likewise, the nominal interest rate falls in the three model specifications, since, according to the estimated Taylor-type rules, monetary policy responds more strongly to inflation than to output.

Finally, figures L.40 and L.44 depict the effects of an unpredicted monetary policy disturbance. The shock leads to a rise in the nominal short-term interest rate, which induces a fall in output, consumption, investment, hours, and inflation. As discussed in Christiano et al. (2005), the investment adjustment cost specification under $\mathcal{M}_1$ generates a hump-shaped response of aggregate investment to the monetary policy shock, and therefore improves the ability of the model specification to reproduce VAR-based evidence.

4.7.3.4 Variance Decomposition

To shed further light on the dynamics of our three model specifications, we examine the relative importance of the four structural shocks for the fluctuations of output, consumption, investment, hours worked, inflation, and the nominal interest rate.[37] To that effect, we compute the fraction of the forecast error variance of the six variables attributable to each type of shock .[38] Tables $L.10 - L.15$ present the forecast error variance decompositions across model specifications and regions at the one-year, three-year, and infinite horizon. [39]

Turning to the euro area first, we find the technology shock to represent the dominant force of movements in output and consumption for all three model specifications at all horizons. However, the role of the four shocks in explaining the fluctuations of investment, hours worked, inflation, and the nominal interest rate differ to some extend considerably between $\mathcal{M}_1$ on the one hand and $\mathcal{M}_2$ and $\mathcal{M}_3$ on the other hand at the different horizons. The marginal efficiency of investment shock, for instance, accounts for the largest part of the conditional variance in forecasting investment in the short run (up to one year) under $\mathcal{M}_2$ and $\mathcal{M}_3$, while we find the technology shock to be relatively more important in driving the short term movements in investment under $\mathcal{M}_1$. Further, we find that hours worked are mainly driven by the technology shock over the medium (three-year horizon) to the long run (infinite horizon) under $\mathcal{M}_1$, whereas the monetary policy shock explains more than 60 percent of the forecast error variance of hours worked under $\mathcal{M}_2$ and $\mathcal{M}_3$ at all horizons. We partly attribute this high contribution of the monetary policy shock to the relatively lower estimates of habit persistence under $\mathcal{M}_2$ and $\mathcal{M}_3$, since, following Bouakez et al. (2005), a higher degree of habit formation induces agents to adjust their labor supply more gradually in response to monetary policy shocks. Although the monetary policy shock is also the most important driving force behind short, medium and long run fluctuations of inflation across all model specifications, the policy shock turns out to be relatively more important under $\mathcal{M}_2$ and $\mathcal{M}_3$ than under $\mathcal{M}_1$. According to all model specifications, the preference shock represents a dominant source

[37]Following Rabanal (2009), the variance decomposition is performed evaluating the model (specification) parameters at their posterior means.

[38]For a detailed description of the computation of forecast error variance decompositions, we refer to Hamilton (1994a), Lütkepohl (2005), or Canova (2007).

[39]Note that as the horizon increases, the conditional variance of the forecast error of a given variable converges to the unconditional variance of that variable (see Bouakez et al., 2005).

of movements in the nominal interest at all horizons, while the technology shock appears to be relatively more important for explaining fluctuations in the nominal interest rate under $\mathcal{M}_1$ than under $\mathcal{M}_2$ and $\mathcal{M}_3$.

By analyzing the forecast error variances decomposition for the US, we also find the technology shock to explain most of the fluctuations in output and consumption across all model specifications at all horizons. Although the three model specifications likewise coincide in attributing most of the short run movements in investment to the marginal efficiency shock of investment, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ clearly differ at higher horizons, since the technology shock explains most of the investment forecast error variance under $\mathcal{M}_1$, while the investment fluctuations under $\mathcal{M}_2$ and $\mathcal{M}_3$ continue to be mainly driven by the investment shock (even though the contribution of the technology shock increases over the medium to the long run). The case is similar for hours worked, as we find the technology shock to play a more important role under $\mathcal{M}_1$ than under $\mathcal{M}_2$ and $\mathcal{M}_3$ at all horizons, albeit the marginal efficiency shock of investment represents the dominant source of movements in hours worked across all three specifications. Focusing on the sources of inflation fluctuation, it turns out that the marginal efficiency of investment shock and the monetary policy shock are about equally important under $\mathcal{M}_1$, both constituting the two most important driving forces behind short and medium run fluctuations, while the technology shock accounts for most of the inflation forecast error variance in the long run. Under $\mathcal{M}_2$ and $\mathcal{M}_3$, in contrast, the monetary policy shock accounts for most of the forecast error variance of inflation at all horizons, although the investment shock and the technology shock play a supporting role in driving inflation fluctuations. According to all model specifications, the marginal efficiency of investment shock represents an important source of movements in the nominal interest rate at all horizons. Furthermore, the contribution of the technology shock to the forecast error variance of the nominal interest rate increases across all model specifications over the medium to the long run. Moreover, we find the monetary policy shock to be relatively more important than the preference shock in explaining the fluctuations of the nominal interest rate under $\mathcal{M}_1$, while the reverse holds under specification $\mathcal{M}_2$ and $\mathcal{M}_3$.

Overall, also the forecast error variance decomposition reveals to some extent remarkable differences between specification $\mathcal{M}_1$ and model versions $\mathcal{M}_2$ and $\mathcal{M}_3$. In addition, it appears that the technology shock explains most of the fluctuations in output and consumption in both regions. However, we find the marginal efficiency of investment shock to play a more important role in business cycle fluctuations in the US than in the euro area. According to tables $L.10 - L.15$, the preference shock represents a dominant source of movements in the nominal interest in the euro area, whereas the same holds for the shock to the marginal efficiency of investment in case of the US. Finally, we identify the monetary policy shock to be much more important in explaining the forecast error variance of hours worked and inflation in the euro area than in the US.

4.8 Conclusions

In this chapter we use a Bayesian approach to analyze the impact of different specifications of quadratic adjustment costs on the fit and the dynamics of a New Keynesian model using both euro area and US data. The underlying dynamic stochastic general equilibrium model features real and nominal frictions as well as several exogenous stochastic disturbances, namely a preference shock, a technology shock, a shock to the marginal efficiency of investment, and a monetary policy shock. We consider three different specifications of quadratic adjustment costs: an investment adjustment cost specification and two forms of capital adjustment costs. Therefore, each variant of quadratic adjustment costs defines a distinct version of our New Keynesian DSGE model.

We find to some extent noticeable differences between the estimated structural parameters across the three model versions. The estimates of persistence and volatility of the marginal efficiency of investment shock turn out to vary substantially with the choice of investment or capital adjustment costs.

Moreover, the implementation of either investment or capital adjustment costs affects the dynamics of the respective model specifications considerably. Our results are in line with the findings of Christiano et al. (2005) and Smets and Wouters (2007), who show that sticky-price models incorporating investment adjustment costs are able to produce data-consistent hump-shaped investment dynamics. However, despite the ability of investment adjustment costs to generate hump-shaped investment responses to various shocks, the Bayesian posterior odds comparison indicates that the two model specifications with capital adjustment costs outperform the model version incorporating investment adjustment costs in terms of overall empirical fit.

To further evaluate where the model specifications fail to match the data, we compare the model version's implied characteristics with the actual data using standard moment criteria. In line with the results from the posterior odds analysis, the model versions with capital adjustment costs provide a better fit to the data than the specification including investment adjustment costs. Our findings appear to be qualitatively robust across both data sets.

Therefore, using estimated DSGE models with quadratic costs of capital accumulation for policy analysis should be done with caution, since the results could be affected by the choice of either investment or capital adjustment costs both being modeling shortcuts. Hence, our results give further evidence to encourage the efforts of a sound microfoundation of adjustment costs for capital accumulation as recently put forward by Wang and Wen (2012).

Appendices

Appendix H

Equilibrium Conditions

The appendix outlines the complete system of equations of the three DSGE model specifications.

H.1 The Economic Environment

- Households:
  The representative household chooses $\{c_t, l_t, B_t, k_{t+1}, i_t\}_{t=0}^{\infty}$ to maximize utility

  $$E\sum_{t=0}^{\infty}\beta^t a_t\left[\log(c_t - hc_{t-1}) - \chi\frac{l_t^{1+\eta}}{1+\eta}\right],$$

  subject to the budget constraint

  $$\frac{B_{t-1} + W_t l_t + Q_t k_t + D_t}{P_t} \geq c_t + i_t + S_j(\cdot,\cdot) + \frac{B_t/r_t}{P_t}$$

  and the law of motion for capital

  $$k_{t+1} = (1-\delta)k_t + x_t i_t.$$

  According to Buiter and Sibert (2007), we prevent the household to make excessive debts by imposing the no-Ponzi-game condition

  $$\lim_{t\to\infty} B_t \prod_{s=0}^{t}\frac{1}{r_s} \geq 0.$$

  .

Hence, the Lagrangian can be written as follows:

$$\Lambda = E\sum_{t=0}^{\infty}\Bigg\{\beta^t a_t\Bigg[\log(c_t - hc_{t-1}) - \chi\frac{l_t^{1+\eta}}{1+\eta}\Bigg]$$
$$- \beta^t\lambda_t\Bigg[c_t + i_t + S_j(\cdot,\cdot) + \frac{B_t/r_t}{P_t} - \frac{B_{t-1}}{P_t} - \frac{W_t l_t}{P_t} - \frac{Q_t k_t}{P_t} - \frac{D_t}{P_t}\Bigg]$$
$$- \beta^t\psi_t[k_{t+1} - (1-\delta)k_t - x_t i_t]\Bigg\}.$$

The first-order conditions are obtained by setting the partial derivatives of Λ with respect to $c_t, l_t, B_t, k_{t+1}, i_t, \lambda_t$, and ψ_t equal to zero, yielding

$$\Lambda_{c_t} = a_t(c_t - hc_{t-1})^{-1} - h\beta E_t[a_{t+1}(c_{t+1} - hc_t)^{-1}] - \lambda_t = 0, \tag{4.5}$$

$$\Lambda_{l_t} = \lambda_t\frac{W_t}{P_t} - a_t\chi l_t^{\eta} = 0, \tag{4.6}$$

$$\Lambda_{B_t} = r_t\beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right) - \lambda_t = 0, \tag{4.7}$$

and, concerning the different specifications of $S_j(\cdot,\cdot)$,

– $S_1(i_{t-1}, i_t)$:

$$\Lambda_{k_{t+1}} = \psi_t - \beta E_t\left(\lambda_{t+1}\frac{Q_{t+1}}{P_{t+1}}\right) - \beta E_t[\psi_{t+1}(1-\delta)] = 0, \tag{4.8}$$

$$\begin{aligned}\Lambda_{i_t} = \lambda_t + \lambda_t\Bigg[\phi\left(\frac{i_t}{i_{t-1}} - 1\right)\left(\frac{i_t}{i_{t-1}}\right) + \frac{\phi}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\Bigg]\\ - \psi_t x_t - \beta E_t\Bigg\{\lambda_{t+1}\Bigg[\phi\left(\frac{i_{t+1}}{i_t} - 1\right)\left(\frac{i_{t+1}}{i_t}\right)^2\Bigg]\Bigg\} = 0,\end{aligned} \tag{4.9}$$

– $S_2(i_t, k_t)$:

$$\begin{aligned}\Lambda_{k_{t+1}} = \psi_t + \beta E_t\Bigg\{\lambda_{t+1}\Bigg[\frac{\phi}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)^2\Bigg]\Bigg\}\\ - \beta E_t\Bigg\{\lambda_{t+1}\Bigg[\phi\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{i_{t+1}}{k_{t+1}}\right)\Bigg]\Bigg\}\\ - \beta E_t\left(\lambda_{t+1}\frac{Q_{t+1}}{P_{t+1}}\right) - \beta E_t[\psi_{t+1}(1-\delta)] = 0,\end{aligned} \tag{4.8$'$}$$

$$\Lambda_{i_t} = \psi_t x_t - \lambda_t - \lambda_t\left[\phi\left(\frac{i_t}{k_t} - \delta\right)\right] = 0, \tag{4.9$'$}$$

– $S_3(i_t, k_t)$:

$$\begin{aligned}\Lambda_{k_{t+1}} = \psi_t - \beta E_t\left\{\lambda_{t+1}\left[\phi\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{i_{t+1}}{k_{t+1}}\right)^2\right]\right\}\\ - \beta E_t\left(\lambda_{t+1}\frac{Q_{t+1}}{P_{t+1}}\right) - \beta E_t[\psi_{t+1}(1-\delta)] = 0,\end{aligned} \tag{4.8$''$}$$

$$\begin{aligned}\Lambda_{i_t} = \psi_t x_t - \lambda_t - \lambda_t\phi\left(\frac{i_t}{k_t} - \delta\right)\left(\frac{i_t}{k_t}\right)\\ - \lambda_t\frac{\phi}{2}\left(\frac{i_t}{k_t} - \delta\right)^2 = 0,\end{aligned} \tag{4.9$''$}$$

as well as

$$\Lambda_{\lambda_t} = c_t + i_t + S(\cdot,\cdot) + \frac{B_t/r_t}{P_t} - \frac{B_{t-1}}{P_t} - \frac{W_t l_t}{P_t} - \frac{Q_t k_t}{P_t} - \frac{D_t}{P_t} = 0, \tag{4.10}$$

and

$$\Lambda_{\psi_t} = k_{t+1} - (1-\delta)k_t - x_t i_t = 0. \tag{4.11}$$

Finally, we impose the standard transversality conditions to guarantee that bonds and capital do not grow too quickly:

$$\begin{aligned}\lim_{t\to\infty} \beta^t \lambda_t \frac{B_t}{P_t} = 0,\\ \lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0.\end{aligned}$$

- Finished goods-producing firms:
  The representative finished goods-producing wants to maximize its profits

$$P_t y_t - \int_0^1 P_t(i) y_t(i) di$$

subject to the constant returns to scale technology

$$y_t \le \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)}.$$

Hence, the firm's optimization problem can be written as

$$\max_{y_t(i)} \Pi_t = P_t \left[\int_0^1 y_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} - \int_0^1 P_t(i) y_t(i) di,$$

which leads to the following first-order condition characterizing the demand for intermediate goods:

$$\frac{\partial \Pi_t}{\partial y_t(i)} = y_t(i) - \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t = 0.$$

By substituting this expression into the constant elasticity of substitution aggregator of intermediate goods, we derive the price aggregator

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di \right]^{1/(1-\theta)}.$$

- Intermediate goods-producing firms:
  Each intermediate goods-producing firm maximizes its present discounted value of profits

$$E \sum_{t=0}^{\infty} \beta^t \lambda_t [D_t(i)/P_t],$$

by choosing $\{l_t(i), k_t(i), y_t(i), P_t(i)\}_{t=0}^{\infty}$ subject to the Cobb-Douglas technology constraint

$$y_t(i) \leq k_t(i)^{\alpha} [z_t l_t(i)]^{1-\alpha}$$

and the demand for intermediate goods outlined above

$$y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta} y_t.$$

We can use the latter expression to rewrite the real value of dividends

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t} \right] y_t(i) - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t} \right] - \frac{\phi_P}{2} \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1 \right]^2 y_t$$

as

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t}\right] - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t. \tag{4.12}$$

Accordingly, the Lagrangian for the firms' intertemporal optimization problem can be written as:

$$\begin{aligned}\Lambda \;=\; & E\sum_{t=0}^{\infty}\Bigg(\beta^t\lambda_t\Big\{\left[\frac{P_t(i)}{P_t}\right]^{1-\theta} y_t - \left[\frac{W_t l_t(i) + Q_t k_t(i)}{P_t}\right] - \frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 y_t\Big\} \\ & -\beta^t\xi_t\Big\{\left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - k_t(i)^\alpha[z_t l_t(i)]^{1-\alpha}\Big\}\Bigg).\end{aligned}$$

By setting the partial derivatives of Λ with respect to $l_t(i), k_t(i), P_t(i)$, and ξ_t equal to zero we have the first-order conditions:

$$\Lambda_{l_t(i)} = \frac{\lambda_t W_t l_t(i)}{P_t} - (1-\alpha)\xi_t k_t(i)^\alpha[z_t l_t(i)]^{1-\alpha} = 0, \tag{4.13}$$

$$\Lambda_{k_t(i)} = \frac{\lambda_t Q_t k_t(i)}{P_t} - \alpha\xi_t k_t(i)^\alpha[z_t l_t(i)]^{1-\alpha} = 0, \tag{4.14}$$

$$\begin{aligned}\Lambda_{P_t(i)} = {} & \phi_P\lambda_t\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]\left[\frac{P_t}{\pi P_{t-1}(i)}\right] \\ & - (1-\theta)\lambda_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta} - \theta\xi_t\left[\frac{P_t(i)}{P_t}\right]^{-\theta-1} \\ & - \beta\phi_P E_t\Big\{\lambda_{t+1}\left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)P_t}{\pi P_t(i)^2}\right]\left(\frac{y_{t+1}}{y_t}\right)\Big\} = 0,\end{aligned} \tag{4.15}$$

and

$$\Lambda_{\xi_t} = \left[\frac{P_t(i)}{P_t}\right]^{-\theta} y_t - k_t(i)^\alpha[z_t l_t(i)]^{1-\alpha} = 0. \tag{4.16}$$

- The monetary authority sets the gross nominal interest rate according to the Taylor-type rule:

$$\ln\left(\frac{r_t}{r}\right) = \rho_r\ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi\ln\left(\frac{\pi_t}{\pi}\right) + \omega_y\ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t). \tag{4.17}$$

H.2 The Nonlinear System

H.2.1 Symmetric Equilibrium

Each version $\mathcal{M}_j$ of the model is characterized by a nonlinear system of 17 equations, namely:

- $\mathcal{M}_1 : (4.1) - (4.7), (4.8), (4.9), (4.10) - (4.17)$,

- $\mathcal{M}_2 : (4.1) - (4.7), (4.8'), (4.9'), (4.10) - (4.17)$, and

- $\mathcal{M}_3 : (4.1) - (4.7), (4.8''), (4.9''), (4.10) - (4.17)$.

The model is closed through two additional steps. First, we focus on a symmetric equilibrium where all intermediate goods-producing firms make identical decisions. This assumption implies $P_t(i) = P_t$, $y_t(i) = y_t$, $l_t(i) = l_t$, $k_t(i) = k_t$, and $D_t(i) = D_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ Second, the market clearing condition for the bond market $B_t = B_{t-1} = 0$ must hold for all $t = 0, 1, 2, ...$ By substituting these conditions into $(4.1) - (4.17)$ and defining the average product of labor as $n_t = y_t/l_t$ for each version $\mathcal{M}_j$ of the underlying model we obtain:

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{4.1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{4.2}$$

$$\ln(z_t) = (1 - \rho_z) \ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{4.3}$$

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{4.4}$$

$$a_t(c_t - hc_{t-1})^{-1} - h\beta E_t[a_{t+1}(c_{t+1} - hc_t)^{-1}] = \lambda_t, \tag{4.5}$$

$$\lambda_t \frac{W_t}{P_t} = a_t \chi l_t^\eta, \tag{4.6}$$

$$\lambda_t = r_t \beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right), \tag{4.7}$$

$$\psi_t = \beta E_t\left(\lambda_{t+1}\frac{Q_{t+1}}{P_{t+1}}\right) + \beta E_t[\psi_{t+1}(1 - \delta)], \tag{4.8}$$

$$\psi_t x_t = \lambda_t + \lambda_t\left[\phi\left(\frac{i_t}{i_{t-1}} - 1\right)\left(\frac{i_t}{i_{t-1}}\right) + \frac{\phi}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] \\ - \beta E_t\left\{\lambda_{t+1}\left[\phi\left(\frac{i_{t+1}}{i_t} - 1\right)\left(\frac{i_{t+1}}{i_t}\right)^2\right]\right\}, \tag{4.9}$$

$$\beta E_t[\psi_{t+1}(1-\delta)] = \psi_t + \beta E_t\left\{\lambda_{t+1}\left[\frac{\phi}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)^2\right]\right\} \\ - \beta E_t\left\{\lambda_{t+1}\left[\phi\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]\right\} \\ - \beta E_t\left(\lambda_{t+1}\frac{Q_{t+1}}{P_{t+1}}\right), \tag{4.8$'$}$$

$$\psi_t x_t = \lambda_t + \lambda_t\left[\phi\left(\frac{i_t}{k_t} - \delta\right)\right], \tag{4.9$'$}$$

$$\psi_t = \beta E_t\left\{\lambda_{t+1}\left[\phi\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{i_{t+1}}{k_{t+1}}\right)^2\right]\right\} \\ + \beta E_t\left(\lambda_{t+1}\frac{Q_{t+1}}{P_{t+1}}\right) + \beta E_t[\psi_{t+1}(1-\delta)], \tag{4.8$''$}$$

$$\psi_t x_t = \lambda_t + \lambda_t\phi\left(\frac{i_t}{k_t} - \delta\right)\left(\frac{i_t}{k_t}\right) \\ + \lambda_t\frac{\phi}{2}\left(\frac{i_t}{k_t} - \delta\right)^2, \tag{4.9$''$}$$

$$y_t = c_t + i_t + S_j(\cdot,\cdot) + \frac{\phi_P}{2}\left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 y_t, \tag{4.10}$$

$$k_{t+1} = (1-\delta)k_t + x_t i_t, \tag{4.11}$$

$$\frac{D_t}{P_t} = y_t - \frac{W_t l_t + Q_t k_t}{P_t} - \frac{\phi_P}{2}\left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 y_t, \tag{4.12}$$

$$\lambda_t\frac{W_t}{P_t}l_t = (1-\alpha)\xi_t y_t, \tag{4.13}$$

$$\lambda_t\frac{Q_t}{P_t}k_t = \alpha\xi_t y_t, \tag{4.14}$$

$$\phi_P\lambda_t\left(\frac{P_t}{\pi P_{t-1}} - 1\right)\left(\frac{P_t}{\pi P_{t-1}}\right) = (1-\theta)\lambda_t + \theta\xi_t \\ + \beta\phi_P E_t\left[\lambda_{t+1}\left(\frac{P_{t+1}}{\pi P_t} - 1\right)\left(\frac{P_{t+1}}{\pi P_t}\right)\left(\frac{y_{t+1}}{y_t}\right)\right], \tag{4.15}$$

$$y_t = k_t^{\alpha}[z_t l_t]^{1-\alpha}, \tag{4.16}$$

$$\ln\left(\frac{r_t}{r}\right) = \rho_r \ln\left(\frac{r_{t-1}}{r}\right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi}\right) + \omega_y \ln\left(\frac{y_t}{y}\right)\right] + \ln(\upsilon_t), \tag{4.17}$$

and

$$n_t = \frac{y_t}{l_t}. \tag{4.18}$$

H.2.2 Change of Variables

Let $\pi_t = \frac{P_t}{P_{t-1}}, w_t = \frac{W_t}{P_t}, q_t = \frac{Q_t}{P_t}$, and $d_t = \frac{D_t}{P_t}$. Using these re-defined variables the nonlinear system encompassing (4.1) – (4.18) becomes:

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{4.1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{4.2}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{4.3}$$

$$\ln(\upsilon_t) = \rho_\upsilon \ln(\upsilon_{t-1}) + \varepsilon_{\upsilon t}, \tag{4.4}$$

$$a_t(c_t - hc_{t-1})^{-1} - h\beta E_t[a_{t+1}(c_{t+1} - hc_t)^{-1}] = \lambda_t, \tag{4.5}$$

$$\lambda_t w_t = a_t \chi l_t^{\eta}, \tag{4.6}$$

$$\lambda_t = r_t \beta E_t\left(\frac{\lambda_{t+1}}{\pi_{t+1}}\right), \tag{4.7}$$

$$\psi_t = \beta E_t\Big(\lambda_{t+1} q_{t+1}\Big) + \beta E_t[\psi_{t+1}(1-\delta)], \tag{4.8}$$

$$\begin{aligned}\psi_t x_t = \lambda_t + \lambda_t\left[\phi\left(\frac{i_t}{i_{t-1}} - 1\right)\left(\frac{i_t}{i_{t-1}}\right) + \frac{\phi}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] \\ - \beta E_t\left\{\lambda_{t+1}\left[\phi\left(\frac{i_{t+1}}{i_t} - 1\right)\left(\frac{i_{t+1}}{i_t}\right)^2\right]\right\},\end{aligned} \tag{4.9}$$

$$\begin{aligned}\beta E_t[\psi_{t+1}(1-\delta)] = \psi_t + \beta E_t\left\{\lambda_{t+1}\left[\frac{\phi}{2}\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)^2\right]\right\} \\ - \beta E_t\left\{\lambda_{t+1}\left[\phi\left(\frac{i_{t+1}}{k_{t+1}} - \delta\right)\left(\frac{i_{t+1}}{k_{t+1}}\right)\right]\right\} \\ - \beta E_t\Big(\lambda_{t+1} q_{t+1}\Big),\end{aligned} \tag{4.8$'$}$$

$$\psi_t x_t = \lambda_t + \lambda_t \left[\phi\left(\frac{i_t}{k_t} - \delta \right) \right], \tag{4.9$'$}$$

$$\begin{aligned} \psi_t = \beta E_t \left\{ \lambda_{t+1} \left[\phi\left(\frac{i_{t+1}}{k_{t+1}} - \delta \right) \left(\frac{i_{t+1}}{k_{t+1}} \right)^2 \right] \right\} \\ + \beta E_t \left(\lambda_{t+1} q_{t+1} \right) + \beta E_t [\psi_{t+1}(1-\delta)], \end{aligned} \tag{4.8$''$}$$

$$\begin{aligned} \psi_t x_t = \lambda_t + \lambda_t \phi \left(\frac{i_t}{k_t} - \delta \right) \left(\frac{i_t}{k_t} \right) \\ + \lambda_t \frac{\phi}{2} \left(\frac{i_t}{k_t} - \delta \right)^2, \end{aligned} \tag{4.9$''$}$$

$$y_t = c_t + i_t + S_j(\cdot,\cdot) + \frac{\phi_P}{2} \left(\frac{\pi_t}{\pi} - 1 \right)^2 y_t, \tag{4.10}$$

$$k_{t+1} = (1-\delta)k_t + x_t i_t, \tag{4.11}$$

$$d_t = y_t - w_t l_t - q_t k_t - \frac{\phi_P}{2} \left(\frac{\pi_t}{\pi} - 1 \right)^2 y_t, \tag{4.12}$$

$$\lambda_t w_t l_t = (1-\alpha)\xi_t y_t, \tag{4.13}$$

$$\lambda_t q_t k_t = \alpha \xi_t y_t, \tag{4.14}$$

$$\begin{aligned} \phi_P \lambda_t \left(\frac{\pi_t}{\pi} - 1 \right) \left(\frac{\pi_t}{\pi} \right) = (1-\theta)\lambda_t + \theta \xi_t \\ + \beta \phi_P E_t \left[\lambda_{t+1} \left(\frac{\pi_{t+1}}{\pi} - 1 \right) \left(\frac{\pi_{t+1}}{\pi} \right) \left(\frac{y_{t+1}}{y_t} \right) \right], \end{aligned} \tag{4.15}$$

$$y_t = k_t^\alpha [z_t l_t]^{1-\alpha}, \tag{4.16}$$

$$\ln\left(\frac{r_t}{r} \right) = \rho_r \ln\left(\frac{r_{t-1}}{r} \right) + (1-\rho_r)\left[\omega_\pi \ln\left(\frac{\pi_t}{\pi} \right) + \omega_y \ln\left(\frac{y_t}{y} \right) \right] + \ln(\upsilon_t), \tag{4.17}$$

and

$$n_t = \frac{y_t}{l_t}. \tag{4.18}$$

H.3 Steady States

In absence of the four shocks, i.e., $\varepsilon_{xt} = \varepsilon_{at} = \varepsilon_{zt} = \varepsilon_{vt} = 0$ for all $t = 0, 1, 2, ...$, the economy converges to a steady state, where each of the 18 variables is constant. Due to the absence of adjustment costs of capital accumulation in the steady state, the determination of the model's steady state values is independent from a specific form of adjustment costs. Therefore, the subsequent steady state computation can be undertaken using just one out of the three adjustment costs specifications under consideration. We use (4.1), (4.2), (4.3) and (4.4) to solve for

$$\begin{aligned} x &= 1, \\ a &= 1, \\ z &= z, \\ v &= 1. \end{aligned}$$

Assuming that steady state (gross) inflation target π is determined by policy, (4.7) can be used to solve for

$$r = \frac{\pi}{\beta}.$$

Next, (4.8) and (4.9) can be used to solve for

$$q = \frac{1}{\beta} - 1 + \delta$$

and

$$\psi = \lambda.$$

Use (4.15) to solve for

$$\xi = \left(\frac{\theta - 1}{\theta}\right)\lambda.$$

Equation (4.5) can be used to solve for

$$c = \left(\frac{1 - h\beta}{1 - h}\right)\left(\frac{1}{\lambda}\right).$$

Use (4.10)-(4.16), and (4.18) to solve for

$$y = \left[1 - \delta\left(\frac{\alpha}{q}\right)\left(\frac{\theta - 1}{\theta}\right)\right]^{-1} c,$$

$$k = \left(\frac{\alpha}{q}\right)\left(\frac{\theta - 1}{\theta}\right)y,$$

$$i = \delta k,$$

$$l = \left(\frac{1}{z}\right)\left(\frac{y}{k^{\alpha}}\right)^{\frac{1}{1-\alpha}},$$

$$w = (1-\alpha)\left(\frac{\theta-1}{\theta}\right)\left(\frac{y}{l}\right),$$

$$d = y - wl - qk,$$

and

$$n = \frac{y}{l}.$$

Finally, use (4.6) to solve for

$$\lambda = \frac{\chi\left\{\frac{(1-\alpha)\left(\frac{\theta-1}{\theta}\right)\left[1-\delta\left(\frac{\alpha}{q}\right)\left(\frac{\theta-1}{\theta}\right)\right]^{-1}\left(\frac{1-h\beta}{1-h}\right)}{\chi}\right\}^{\frac{\eta}{1+\eta}}}{(1-\alpha)z\left(\frac{\alpha}{q}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\theta-1}{\theta}\right)^{\frac{1}{1-\alpha}}}.$$

H.4 The Linearized System

The nonlinear system (4.1) – (4.18) can be linearized by taking a log-linear approximation of the model at steady state values. For a detailed description of logarithmic approximations we refer to Canova (2007), DeJong and Dave (2007), and Zietz (2008). Let $\widehat{var}_t \equiv log\left(\frac{var_t}{var}\right)$ denote the log-deviation of some variable var_t from its steady state var, where $log\left(\frac{var_t}{var}\right) \approx \frac{var_t - var}{var}$. A first-order Taylor approximation of equation (4.1) – (4.18) at the steady state gives:

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_{xt}, \tag{4.1}$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{4.2}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{4.3}$$

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \tag{4.4}$$

$$(1-h)(1-h\beta)\hat{\lambda}_t = (1-h)(1-h\beta\rho_a)\hat{a}_t + h\hat{c}_{t-1} - (1+h^2\beta)\hat{c}_t + h\beta E_t\hat{c}_{t+1}, \tag{4.5}$$

$$\hat{\lambda}_t + \hat{w}_t = \hat{a}_t + \eta\hat{l}_t, \tag{4.6}$$

$$\hat{\lambda}_t = \hat{r}_t + E_t\hat{\lambda}_{t+1} - E_t\hat{\pi}_{t+1}, \tag{4.7}$$

$$\hat{\psi}_t = \beta q E_t\hat{\lambda}_{t+1} + \beta q E_t\hat{q}_{t+1} + \beta(1-\delta)E_t\hat{\psi}_{t+1}, \tag{4.8}$$

$$\hat{\psi}_t + \hat{x}_t = \hat{\lambda}_t - \phi \hat{i}_{t-1} + (1+\beta)\phi \hat{i}_t - \beta\phi E_t \hat{i}_{t+1}, \tag{4.9}$$

$$\beta(1-\delta)E_t\hat{\psi}_{t+1} = \hat{\psi}_t - \beta\phi\left(\frac{i}{k}\right)^2 E_t\hat{i}_{t+1} + \beta\phi\left(\frac{i}{k}\right)^2 \hat{k}_{t+1} - \beta q E_t\hat{\lambda}_{t+1} - \beta q E_t \hat{q}_{t+1}, \tag{4.8$'$}$$

$$\hat{\psi}_t + \hat{x}_t = \hat{\lambda}_t + \phi\left(\frac{i}{k}\right)\hat{i}_t - \phi\left(\frac{i}{k}\right)\hat{k}_t, \tag{4.9$'$}$$

$$\begin{aligned}\hat{\psi}_t = \beta\phi\left(\frac{i}{k}\right)\left(\frac{i}{k}\right)^2 E_t\hat{i}_{t+1} - \beta\phi\left(\frac{i}{k}\right)^3 \hat{k}_{t+1} \\ + \beta q E_t\hat{q}_{t+1} + \beta q E_t\hat{\lambda}_{t+1} + \beta(1-\delta)E_t\hat{\psi}_{t+1},\end{aligned} \tag{4.8$''$}$$

$$\hat{\psi}_t + \hat{x}_t = \hat{\lambda}_t + \phi\left(\frac{i}{k}\right)^2\hat{i}_t - \phi\left(\frac{i}{k}\right)^2\hat{k}_t, \tag{4.9$''$}$$

$$y\hat{y}_t = c\hat{c}_t + i\hat{i}_t, \tag{4.10}$$

$$k\hat{k}_{t+1} = (1-\delta)k\hat{k}_t + i\hat{x}_t + i\hat{i}_t, \tag{4.11}$$

$$d\hat{d}_t = y\hat{y}_t - wl\hat{w}_t - wl\hat{l}_t - qk\hat{q}_t - qk\hat{k}_t, \tag{4.12}$$

$$\hat{\lambda}_t + \hat{w}_t + \hat{l}_t = \hat{\xi}_t + \hat{y}_t, \tag{4.13}$$

$$\hat{\lambda}_t + \hat{q}_t + \hat{k}_t = \hat{\xi}_t + \hat{y}_t, \tag{4.14}$$

$$\phi_P\hat{\pi}_t = (1-\theta)\hat{\lambda}_t + (\theta-1)\hat{\xi}_t + \beta\phi_P E_t\hat{\pi}_{t+1}, \tag{4.15}$$

$$\hat{y}_t = \alpha\hat{k}_t + (1-\alpha)\hat{z}_t + (1-\alpha)\hat{l}_t, \tag{4.16}$$

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + (1-\rho_r)(\omega_\pi\hat{\pi}_t + \omega_y\hat{y}_t) + \hat{v}_t, \tag{4.17}$$

and

$$\hat{n}_t = \hat{y}_t - \hat{l}_t. \tag{4.18}$$

Appendix I

Solving the Model

According to Adjemian et al. (2011), the core of the DYNARE solution algorithm for computing the solution of a linear rational expectations model is based on a complex generalized Schur decomposition as presented in Klein (2000). Hence, we give a brief description of how to solve the three model specifications under consideration using this solution method. For a more detailed presentation of the complex generalized Schur decomposition we refer to Golub and Van Loan (1996).

I.1 Klein's method

Each model version $\mathcal{M}_j$ can be solved following the approach of Klein (2000) outlined in appendix B.2. Therefore, each of the model's specifications is brought into the form:

$$AE_t s_{t+1}^0 = Bs_t^0 + C\zeta_t \tag{4.19}$$

$$\zeta_t = P\zeta_{t-1} + \varepsilon_t, \tag{4.20}$$

where A, B, and C are coefficient matrices, P contains the persistence parameters of the shocks, ζ_t consists of the model's exogenous forcing variables, while the serially and mutually uncorrelated innovations are included in ε_t. The vector s_t^0 can be separated into

$$s_t^0 = [s_{1t}^0 \; s_{2t}^0]',$$

letting s_{1t}^0 denote a vector of predetermined and s_{2t}^0 a vector of non-predetermined variables, which implies that:

$$E_t s_{t+1} = [s^0_{1t+1} \; E_t s^0_{2t+1}]'.$$

The solution technique relies on decoupling the system into unstable and stable portions, using a complex generalized Schur decomposition, and then solving the two components in turn. If, as set out in Blanchard and Kahn (1980), the number of generalized eigenvalues with modulus larger than unity equals the number of non-predetermined variables, a unique solution exits and system is said to be saddle-path stable (see DeJong and Dave, 2007).[1] Since the versions $\mathcal{M}_2$ and $\mathcal{M}_3$, of the model differ with respect to the number of predetermined variables from $\mathcal{M}_1$, we need to subdivide the description of the solution method. The subsequent sections follow the expositions in Klein (2000), DeJong and Dave (2007), and the technical notes of Ireland (2011).[2]

I.2 Solving $\mathcal{M}_2$ and $\mathcal{M}_3$

Let

$$s^0_t = \begin{bmatrix} \hat{c}_{t-1} & \hat{k}_t & \hat{r}_{t-1} & \hat{\lambda}_t & \hat{w}_t & \hat{l}_t & \hat{\pi}_t & \hat{\psi}_t & \hat{q}_t & \hat{i}_t & \hat{y}_t & \hat{d}_t & \hat{\xi}_t & \hat{n}_t & \hat{c}_t \end{bmatrix}',$$

$$\zeta_t = \begin{bmatrix} \hat{x}_t & \hat{a}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$P = \begin{bmatrix} \rho_x & 0 & 0 & 0 \\ 0 & \rho_a & 0 & 0 \\ 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & \rho_v \end{bmatrix},$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{xt} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}'.$$

[1] A detailed presentation of the Blanchard-Kahn (1980) conditions is given in appendix $B.2$.

[2] The technical notes of Ireland (2011) are available at http://www.irelandp.com/progs/nkp.zip.

Then the coefficient matrices A, B, and C for the different versions of the model are:

- $\mathcal{M}_2$:

$$A = \begin{bmatrix}
-(1+h^2\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h\beta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta\phi\left(\frac{i}{k}\right)^2 & 0 & \beta q & 0 & 0 & 0 & \beta(1-\delta) & \beta q & \beta\phi\left(\frac{i}{k}\right)^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta\phi_P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$B = \begin{bmatrix}
-h & 0 & 0 & (1-h)(1-h\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \phi\left(\frac{i}{k}\right) & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -\phi\left(\frac{i}{k}\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & y & 0 & 0 & 0 & 0 \\
0 & (1-\delta)k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & -qk & 0 & 0 & -wl & -wl & 0 & 0 & -qk & 0 & y & -d & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -(1-\theta) & 0 & 0 & \phi_P & 0 & 0 & 0 & 0 & 0 & -(\theta-1) & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & (1-\alpha) & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_r & 0 & 0 & 0 & (1-\rho_r)\omega_\pi & 0 & 0 & 0 & (1-\rho_r)\omega_y & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & -(1-h)(1-h\beta\rho_a) & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\alpha) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- $\mathcal{M}_3$:

$$A = \begin{bmatrix} -(1+h^2\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta\phi\left(\frac{i}{k}\right)^3 & 0 & \beta q & 0 & 0 & 0 & \beta(1-\delta) & \beta q & \beta\phi\left(\frac{i}{k}\right)^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta\phi_P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -h & 0 & 0 & (1-h)(1-h\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \phi\left(\frac{i}{k}\right)^2 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -\phi\left(\frac{i}{k}\right)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & y & 0 & 0 & 0 & 0 \\ 0 & (1-\delta)k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & -qk & 0 & 0 & -wl & -wl & 0 & 0 & -qk & 0 & y & -d & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -(1-\theta) & 0 & 0 & \phi_P & 0 & 0 & 0 & 0 & 0 & -(\theta-1) & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & (1-\alpha) & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_r & 0 & 0 & 0 & (1-\rho_r)\omega_\pi & 0 & 0 & 0 & (1-\rho_r)\omega_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$C=\begin{bmatrix}0 & -(1-h)(1-h\beta\rho_a) & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & (1-\alpha) & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Following Klein (2000), we apply the complex generalized Schur decomposition of A and B, which is given by

$$QAZ = S$$

and

$$QBZ = T,$$

where Q and Z are unitary and S and T are upper triangular matrices. The generalized eigenvalues of B and A can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(B, A) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 15\}.$$

The matrices $Q, Z, S,$ and T can always be arranged so that the generalized eigenvalues are ordered in increasing value in moving from left to right. Note that the vector s_t^0 comprises of three predetermined and twelve non-predetermined variables.

We proceed under the case of saddle-path stability, assuming exactly twelve generalized eigenvalues to lie outside the unit circle, and therefore allow for a unique solution. The matrices $Q, Z, S,$ and T are portioned, so that

$$Q=\begin{bmatrix}Q_1\\ Q_2\end{bmatrix},$$

where Q_1 is 3×15 and Q_2 is 12×15 and

$$\begin{aligned}
Z &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \\
S &= \begin{bmatrix} S_{11} & S_{12} \\ 0_{(12\times 3)} & S_{22} \end{bmatrix}, \\
T &= \begin{bmatrix} T_{11} & T_{12} \\ 0_{(12\times 3)} & T_{22} \end{bmatrix},
\end{aligned}$$

where Z_{11}, S_{11}, and T_{11} are 3×3 and Z_{12}, S_{12}, and T_{12} are 3×12, Z_{21} is 12×3, and Z_{22}, S_{22}, and T_{22} are 12×12. To "triangularize" the system we first define the vector of auxiliary variables as

$$s_t^1 = Z^H s_t^0,$$

letting Z^H denote the conjugate transpose of matrix Z, so that

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z_{11}^H \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} + Z_{21}^H \begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{\imath}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \end{bmatrix} \tag{4.21}$$

is 3×1 and

$$s_{2t}^1 = Z_{12}^H \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} + Z_{22}^H \begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{i}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \end{bmatrix} \tag{4.22}$$

is 12×1.

Since Z is unitary, $Z^H Z = I$ or $Z^H = Z^{-1}$ and hence $s_t^0 = Z s_t^1$. We use this property to rewrite (4.19) as

$$AZE_t s_{t+1}^1 = BZ s_t^1 + C\zeta_t.$$

Premultiplying this equation by Q gives

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} s_{1t+1}^1 \\ s_{2t+1}^1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} C\zeta_t,$$

or in matrix partitions,

$$S_{11} E_t s_{1t+1}^1 + S_{12} E_t s_{2t+1}^1 = T_{11} s_{1t}^1 + T_{12} s_{2t}^1 + Q_1 C\zeta_t \tag{4.23}$$

and

$$S_{22} E_t s_{2t+1}^1 = T_{22} s_{2t}^1 + Q_2 C\zeta_t. \tag{4.24}$$

Since the generalized eigenvalues corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (4.24) can be solved forward to obtain

$$s_{2t}^1 = -T_{22}^{-1} R\zeta_t,$$

where the 12×4 matrix R is given by "reshaping"[3]

$$\begin{aligned} vec(R) &= vec\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2 C P^j = \sum_{j=0}^{\infty} vec[(S_{22}T_{22}^{-1})^j Q_2 C P^j] \\ &= \sum_{j=0}^{\infty}[P^j \otimes (S_{22}T_{22}^{-1})^j] vec(Q_2 C) = \sum_{j=0}^{\infty}[P \otimes (S_{22}T_{22}^{-1})]^j vec(Q_2 C) \\ &= [I_{(48\times 48)} - P \otimes (S_{22}T_{22}^{-1})]^{-1} vec(Q_2 C). \end{aligned}$$

Using this result together with equation (4.22) allows to solve for

$$\begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{i}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \end{bmatrix} = -(Z_{22}^H)^{-1} Z_{12}^H \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} - (Z_{22}^H)^{-1} T_{22}^{-1} R \zeta_t. \tag{4.25}$$

Under the assumption that Z is unitary, i.e.,

$$\underbrace{\begin{bmatrix} Z_{11}^H & Z_{21}^H \\ Z_{12}^H & Z_{22}^H \end{bmatrix}}_{Z^H} \underbrace{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} I_{(3\times 3)} & 0_{(3\times 12)} \\ 0_{(12\times 3)} & I_{(12\times 12)} \end{bmatrix}}_{I},$$

we find

$$Z_{12}^H Z_{11} + Z_{22}^H Z_{21} = 0,$$

[3]Following Hamilton (1994a) and DeJong and Dave (2007), the appearance of the *vec* operator accommodates the VAR specification for ζ_t. We use the relationship between *vec* operator and Kronecker product: $vec[(S_{22}T_{22}^{-1})^j Q_2 C P^j] = [(P^j)' \otimes (S_{22}T_{22}^{-1})^j] vec(Q_2 C)$. Note that $P' = P$, since P is a diagonal matrix.

$$-(Z_{22}^H)^{-1} Z_{12}^H = Z_{21} Z_{11}^{-1},$$

$$Z_{12}^H Z_{12} + Z_{22}^H Z_{22} = I,$$

and

$$(Z_{22}^H)^{-1} = Z_{22} + (Z_{22}^H)^{-1} Z_{12}^H Z_{12} = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12},$$

which allows to rewrite (4.25) as

$$\begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{i}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \end{bmatrix} = M_1 \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} + M_2 \zeta_t, \tag{4.25$'$}$$

with

$$M_1 = Z_{21} Z_{11}^{-1}$$

and

$$M_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Now we can solve (4.21) for s_{1t}^1

$$s_{1t}^1 = (Z_{11}^H + Z_{21}^H Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} - Z_{21}^H [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R \zeta_t,$$

using

$$Z_{11}^H Z_{11} + Z_{21}^H Z_{21} = I,$$

$$Z_{11}^H + Z_{21}^H Z_{21} Z_{11}^{-1} = Z_{11}^{-1},$$

and

$$Z_{21}^H[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}] = Z_{21}^H Z_{22} - Z_{21}^H Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12},$$

so that

$$s_{1t}^1 = Z_{11}^{-1} \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R \zeta_t.$$

If we plug this result into equation (4.23), we get

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_{t+1} \\ \hat{r}_t \end{bmatrix} = M_3 \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \end{bmatrix} + M_4 \zeta_t, \tag{4.26}$$

where

$$M_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

and

$$M_4 = Z_{11} S_{11}^{-1}(T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 C + S_{12} T_{22}^{-1} RP - T_{12} T_{22}^{-1} R) - Z_{12} T_{22}^{-1} RP.$$

Hence, the model's solution can be written compactly in state space form by combining (4.20), (4.25′) and (4.26) as

$$s_{t+1} = \Gamma_0 s_t + \Gamma_1 \varepsilon_{t+1}, \tag{4.27}$$

and

$$f_t = \Gamma_2 s_t, \tag{4.28}$$

where

$$s_t = \begin{bmatrix} \hat{c}_{t-1} & \hat{k}_t & \hat{r}_{t-1} & \hat{x}_t & \hat{a}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{\lambda}_t & \hat{w}_t & \hat{l}_t & \hat{\pi}_t & \hat{\psi}_t & \hat{q}_t & \hat{i}_t & \hat{y}_t & \hat{d}_t & \hat{\xi}_t & \hat{n}_t & \hat{c}_t \end{bmatrix}',$$

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{xt} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}',$$

$$\Gamma_0 = \begin{bmatrix} M_3 & M_4 \\ 0_{(4\times 3)} & P \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 0_{(3\times 4)} \\ I_{(4\times 4)} \end{bmatrix},$$

and

$$\Gamma_2 = \begin{bmatrix} M_1 & M_2 \end{bmatrix}.$$

I.3 Solving $\mathcal{M}_1$

Let

$$s_t^0 = \begin{bmatrix} \hat{c}_{t-1} & \hat{k}_t & \hat{r}_{t-1} & \hat{i}_{t-1} & \hat{\lambda}_t & \hat{w}_t & \hat{l}_t & \hat{\pi}_t & \hat{\psi}_t & \hat{q}_t & \hat{y}_t & \hat{d}_t & \hat{\xi}_t & \hat{n}_t & \hat{c}_t & \hat{i}_t \end{bmatrix}',$$

$$\zeta_t = \begin{bmatrix} \hat{x}_t & \hat{a}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$P = \begin{bmatrix} \rho_x & 0 & 0 & 0 \\ 0 & \rho_a & 0 & 0 \\ 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & \rho_v \end{bmatrix},$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{xt} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}'.$$

Then the coefficient matrices A, B and C for version $\mathcal{M}_1$ of the model are:

$$A = \begin{bmatrix}
-(1+h^2\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h\beta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \beta q & 0 & 0 & 0 & \beta(1-\delta) & \beta q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1+\beta)\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta\phi \\
c & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & k & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\phi_P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$
B = \begin{bmatrix}
-h & 0 & 0 & 0 & (1-h)(1-h\beta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \phi & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y & 0 & 0 & 0 & 0 & 0 \\
0 & (1-\delta)k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -qk & 0 & 0 & 0 & -wl & -wl & 0 & 0 & -qk & y & -d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(1-\theta) & 0 & 0 & \phi_P & 0 & 0 & 0 & 0 & -(\theta-1) & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 & (1-\alpha) & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_r & 0 & 0 & 0 & 0 & (1-\rho_r)\omega_\pi & 0 & 0 & (1-\rho_r)\omega_y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

and

$$
C = \begin{bmatrix}
0 & -(1-h)(1-h\beta\rho_a) & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & (1-\alpha) & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Just as in the case of $\mathcal{M}_2$ and $\mathcal{M}_3$ we follow the approach of Klein (2000), and perform a complex generalized Schur decomposition of A and B, which is given by

$$QAZ = S$$

and

$$QBZ = T,$$

letting Q and Z denote unitary and S and T upper triangular matrices. The generalized eigenvalues of B and A can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(B, A) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 16\}.$$

Again, the matrices Q, Z, S, and T can be arranged so that the generalized eigenvalues appear in ascending order. Note that there are four predetermined variables and twelve non-predetermined variables in the vector s_t^0. We proceed under the case of saddle-path stability and assume that there are exactly twelve generalized eigenvalues that lie outside the unit circle, and therefore allow for a unique solution. The matrices Q, Z, S, and T are portioned, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 is 4×16 and Q_2 is 12×16 and

$$\begin{aligned} Z &= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \\ S &= \begin{bmatrix} S_{11} & S_{12} \\ 0_{(12\times 4)} & S_{22} \end{bmatrix}, \\ T &= \begin{bmatrix} T_{11} & T_{12} \\ 0_{(12\times 4)} & T_{22} \end{bmatrix}, \end{aligned}$$

where Z_{11}, S_{11}, and T_{11} are 4×4 and Z_{12}, S_{12}, and T_{12} are 4×12, Z_{21} is 12×4, and Z_{22}, S_{22}, and T_{22} are 12×12. To "triangularize" the system we first define the vector of auxiliary variables as

$$s_t^1 = Z^H s_t^0,$$

letting Z^H denote the conjugate transpose of matrix Z, so that

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z_{11}^H \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \\ \hat{i}_{t-1} \end{bmatrix} + Z_{21}^H \begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \\ \hat{i}_t \end{bmatrix} \tag{4.29}$$

is 4×1 and

$$s_{2t}^1 = Z_{12}^H \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \\ \hat{i}_{t-1} \end{bmatrix} + Z_{22}^H \begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \\ \hat{i}_t \end{bmatrix} \tag{4.30}$$

is 12×1.

Since Z is unitary, $Z^H Z = I$ or $Z^H = Z^{-1}$ and hence $s_t^0 = Z s_t^1$. We use this property to rewrite (4.19) as

$$AZE_t s_{t+1}^1 = BZs_t^1 + C\zeta_t.$$

Premultiplying this equation by Q gives

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} s^1_{1t+1} \\ s^1_{2t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s^1_{1t} \\ s^1_{2t} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} C\zeta_t,$$

or in terms of the matrix partitions,

$$S_{11}E_t s^1_{1t+1} + S_{12}E_t s^1_{2t+1} = T_{11}s^1_{1t} + T_{12}s^1_{2t} + Q_1 C\zeta_t \tag{4.31}$$

and

$$S_{22}E_t s^1_{2t+1} = T_{22}s^1_{2t} + Q_2 C\zeta_t. \tag{4.32}$$

Since the generalized eigenvalues corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (4.32) can be solved forward to obtain

$$s^1_{2t} = -T_{22}^{-1} R\zeta_t,$$

where the 12×4 matrix R is given by "reshaping"

$$\begin{aligned} vec(R) &= vec\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2 C P^j = \sum_{j=0}^{\infty} vec[(S_{22}T_{22}^{-1})^j Q_2 C P^j] \\ &= \sum_{j=0}^{\infty}[P^j \otimes (S_{22}T_{22}^{-1})^j]\, vec(Q_2 C) = \sum_{j=0}^{\infty}[P \otimes (S_{22}T_{22}^{-1})]^j\, vec(Q_2 C) \\ &= [I_{(48\times 48)} - P \otimes (S_{22}T_{22}^{-1})]^{-1}\, vec(Q_2 C). \end{aligned}$$

Using this result together with equation (4.30) allows to solve for

$$\begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \\ \hat{i}_t \end{bmatrix} = -(Z_{22}^H)^{-1} Z_{12}^H \begin{bmatrix} c_{t-1} \\ k_t \\ r_{t-1} \\ i_{t-1} \end{bmatrix} - (Z_{22}^H)^{-1} T_{22}^{-1} R \zeta_t. \tag{4.33}$$

Under the assumption that Z is unitary, i.e.,

$$\underbrace{\begin{bmatrix} Z_{11}^H & Z_{21}^H \\ Z_{12}^H & Z_{22}^H \end{bmatrix}}_{Z^H} \underbrace{\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}}_{Z} = \underbrace{\begin{bmatrix} I_{(4\times4)} & 0_{(4\times12)} \\ 0_{(12\times4)} & I_{(12\times12)} \end{bmatrix}}_{I},$$

we find

$$Z_{12}^H Z_{11} + Z_{22}^H Z_{21} = 0,$$

$$-(Z_{22}^H)^{-1} Z_{12}^H = Z_{21} Z_{11}^{-1},$$

$$Z_{12}^H Z_{12} + Z_{22}^H Z_{22} = I,$$

and

$$(Z_{22}^H)^{-1} = Z_{22} + (Z_{22}^H)^{-1} Z_{12}^H Z_{12} = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12},$$

which allows to rewrite (4.33) as

$$\begin{bmatrix} \hat{\lambda}_t \\ \hat{w}_t \\ \hat{l}_t \\ \hat{\pi}_t \\ \hat{\psi}_t \\ \hat{q}_t \\ \hat{y}_t \\ \hat{d}_t \\ \hat{\xi}_t \\ \hat{n}_t \\ \hat{c}_t \\ \hat{i}_t \end{bmatrix} = \tilde{M}_1 \begin{bmatrix} c_{t-1} \\ k_t \\ r_{t-1} \\ i_{t-1} \end{bmatrix} + \tilde{M}_2 \zeta_t, \tag{4.33'}$$

with

$$\tilde{M}_1 = Z_{21} Z_{11}^{-1}$$

and

$$\tilde{M}_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Now we can solve (4.29) for s_{1t}^1

$$s_{1t}^1 = (Z_{11}^H + Z_{21}^H Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \\ \hat{i}_{t-1} \end{bmatrix} - Z_{21}^H [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R \zeta_t,$$

making use of

$$Z_{11}^H Z_{11} + Z_{21}^H Z_{21} = I,$$

$$Z_{11}^H + Z_{21}^H Z_{21} Z_{11}^{-1} = Z_{11}^{-1},$$

and

$$Z_{21}^H [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] = Z_{21}^H Z_{22} - Z_{21}^H Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12},$$

so that

$$s_{1t}^1 = Z_{11}^{-1} \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \\ \hat{i}_{t-1} \end{bmatrix} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R \zeta_t.$$

If we plug this result into equation (4.31) we get

$$\begin{bmatrix} \hat{c}_t \\ \hat{k}_{t+1} \\ \hat{r}_t \\ \hat{i}_t \end{bmatrix} = \tilde{M}_3 \begin{bmatrix} \hat{c}_{t-1} \\ \hat{k}_t \\ \hat{r}_{t-1} \\ \hat{i}_{t-1} \end{bmatrix} + \tilde{M}_4 \zeta_t, \tag{4.34}$$

where

$$\tilde{M}_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

and

$$\tilde{M}_4 = Z_{11} S_{11}^{-1} (T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 C + S_{12} T_{22}^{-1} R P - T_{12} T_{22}^{-1} R) - Z_{12} T_{22}^{-1} R P.$$

The model's solution can be written compactly in state space form by combining (4.20), (4.33′), and (4.34) as

$$\tilde{s}_{t+1} = \tilde{\Gamma}_0 \tilde{s}_t + \tilde{\Gamma}_1 \varepsilon_{t+1}, \tag{4.35}$$

and

$$\tilde{f}_t = \tilde{\Gamma}_2 \tilde{s}_t, \tag{4.36}$$

where

$$\tilde{s}_t = \begin{bmatrix} \hat{c}_{t-1} & \hat{k}_t & \hat{r}_{t-1} & \hat{i}_{t-1} & \hat{x}_t & \hat{a}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$\tilde{f}_t = \begin{bmatrix} \hat{\lambda}_t & \hat{w}_t & \hat{l}_t & \hat{\pi}_t & \hat{\psi}_t & \hat{q}_t & \hat{y}_t & \hat{d}_t & \hat{\xi}_t & \hat{n}_t & \hat{c}_t & \hat{i}_t \end{bmatrix}',$$

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{xt} & \varepsilon_{at} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}',$$

$$\tilde{\Gamma}_0 = \begin{bmatrix} \tilde{M}_3 & \tilde{M}_4 \\ 0_{(4\times4)} & P \end{bmatrix},$$

$$\tilde{\Gamma}_1 = \begin{bmatrix} 0_{(4\times 4)} \\ I_{(4\times 4)} \end{bmatrix},$$

and

$$\tilde{\Gamma}_2 = \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 \end{bmatrix}.$$

Appendix J

Estimation

J.1 Empirical State Space Model

Since the model is estimated using an observed sample X including consumption, investment, inflation and interest rates, we can define a sequence of observations $\{X_t\}_{t=1}^{T}$ with a measured data vector

$$X_t = \begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix}.$$

To distinguish the theoretical model from the empirical model, we rewrite (4.27) and (4.28) as

$$s_{t+1} = \Psi_0 s_t + \Psi_1 \varepsilon_{t+1}, \tag{4.37}$$

and

$$X_t = \Psi_2 s_t, \tag{4.38}$$

where $\Gamma_o = \Psi_0$, $\Gamma_1 = \Psi_1$, and Ψ_2 is formed from the rows $(\cdot)$ of Γ_0 and Γ_2 as

$$\Psi_2 = \begin{bmatrix} \Gamma_0(1) \\ \Gamma_2(7) \\ \Gamma_2(4) \\ \Gamma_0(3) \end{bmatrix},$$

while (4.35) and (4.36) can be expressed as

$$\tilde{s}_{t+1} = \tilde{\Psi}_0 \tilde{s}_t + \tilde{\Psi}_1 \varepsilon_{t+1}, \tag{4.39}$$

and

$$X_t = \tilde{\Psi}_2 \tilde{s}_t, \tag{4.40}$$

where $\tilde{\Gamma}_o = \tilde{\Psi}_0$, $\tilde{\Gamma}_1 = \tilde{\Psi}_1$, and $\tilde{\Psi}_2$ is formed from the rows $(\cdot)$ of $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_2$ as

$$\tilde{\Psi}_2 = \begin{bmatrix} \tilde{\Gamma}_0(1) \\ \tilde{\Gamma}_0(4) \\ \tilde{\Gamma}_2(4) \\ \tilde{\Gamma}_0(3) \end{bmatrix}.$$

J.2 Kalman Filter

Bayesian estimation of a DSGE model in state space form requires the construction and evaluation of the likelihood function

$$\mathcal{L}(\mu|X) = p(X|\mu) = \prod_{t=1}^{T} p(X_t|\mu),$$

where X denotes the T observations of a vector of observable variables X_t and μ a $k \times 1$ vector of the model's parameters. Following Hamilton (1994a), the Kalman filter provides a way to calculate the likelihood function for a state space system. As outlined in Canova (2007, p. 123), "... the likelihood function of a state space model can be conveniently expressed in terms of one-step-ahead forecast errors, conditional on the initial observations, and of their recursive variance, both of which can be obtained with the Kalman filter." Since a general and detailed treatment of the Kalman filter can be found in Harvey (1993), Hamilton (1994a) and Hamilton (1994b), we give a brief exemplary exposition of the Kalman filter by applying the recursive algorithm originally developed by Kalman (1960) and Kalman and Bucy (1961) to the empirical state space model formed by state equation (4.37) and observation equation (4.38)[1]:

$$s_{t+1} = \Psi_0 s_t + \Psi_1 \varepsilon_{t+1}, \tag{4.37}$$

$$X_t = \Psi_2 s_t. \tag{4.38}$$

Recall that s_t is a vector of possibly unknown state variables, X_t denotes a vector of observed variables, Ψ_0, Ψ_1, and Ψ_2 depend on the structural parameters of the model and the vector ε_{t+1} comprises the serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{xt+1} & \varepsilon_{at+1} & \varepsilon_{zt+1} & \varepsilon_{vt+1} \end{bmatrix}',$$

which are assumed to be normally distributed with zero mean and diagonal covariance matrix

$$\Sigma_\varepsilon = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_x^2 & 0 & 0 & 0 \\ 0 & \sigma_a^2 & 0 & 0 \\ 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & \sigma_v^2 \end{bmatrix}.$$

J.2.1 Kalman Filter Recursion

To analyze the Kalman filter recursion, we follow the expositions of Hamilton (1994a) and Lütkepohl (2005). Let

$$\begin{aligned} s_{t|j} &= E(s_t|X_1, ..., X_j), \\ \Sigma_s(t|j) &= E(s_t - s_{t|j})(s_t - s_{t|j})', \\ X_{t|j} &= E(X_t|X_1, ..., X_j), \\ \Sigma_X(t|j) &= E(X_t - X_{t|j})(X_t - X_{t|j})'. \end{aligned}$$

Further, the initial state s_0 and the conditional distribution of s given X are assumed to be normally distributed with $s_0 \sim \mathcal{N}(\mu_{s0}, \Sigma_0)$ and $(s|X) \sim \mathcal{N}(\mu_s, \Sigma)$, respectively. Given the previous conditions, the normality assumption implies

$$\begin{aligned} (s_t|X_1, ..., X_{t-1}) &\sim \mathcal{N}(s_{t|t-1}, \Sigma_s(t|t-1)) && \text{for } t = 2, ..., T, \\ (s_t|X_1, ..., X_t) &\sim \mathcal{N}(s_{t|t}, \Sigma_s(t|t)) && \text{for } t = 1, ..., T, \\ (X_t|X_1, ..., X_{t-1}) &\sim \mathcal{N}(X_{t|t-1}, \Sigma_X(t|t-1)) && \text{for } t = 2, ..., T. \end{aligned}$$

[1] The Kalman filter recursion for the state equation (4.39) and observation equation (4.40) would proceed in exactly the same way.

As outlined in Lütkepohl (2005), the conditional means and covariance matrices can be obtained by the following Kalman filter recursions:

- Initialization:

$$s_{0|0} = \mu_{s0}, \Sigma_s(0|0) = \Sigma_0.$$

- Prediction step ($1 \leq t \leq T$):

$$\begin{aligned}
s_{t|t-1} &= \Psi_0 s_{t-1|t-1}, \\
\Sigma_s(t|t-1) &= \Psi_0 \Sigma_s(t-1|t-1)\Psi_0' + \Psi_1 \Sigma_\varepsilon \Psi_1', \\
X_{t|t-1} &= \Psi_2 s_{t|t-1}, \\
\Sigma_X(t|t-1) &= \Psi_2 \Sigma_s(t|t-1)\Psi_2', \\
u_t &= X_t - X_{t|t-1}.
\end{aligned}$$

- Correction step ($1 \leq t \leq T$):

$$\begin{aligned}
s_{t|t} &= s_{t|t-1} + \Upsilon_t u_t, \\
\Sigma_s(t|t) &= \Sigma_s(t|t-1) - \Upsilon_t \Sigma_X(t|t-1)\Upsilon_t',
\end{aligned}$$

  where the Kalman gain Υ_t is defined as

$$\Upsilon_t = s_{t|t-1}\Psi_2' \Sigma_X(t|t-1)^{-1} .$$

As described in Lütkepohl (2005), the recursions proceed by performing the prediction step for $t = 1$. Then, the correction step is performed for $t = 1$. Next, the prediction and correction steps are repeated for $t = 2$ and so on.

J.2.2 Log Likelihood Function

The observation vector estimation errors $\{u_t\}_{t=1}^T$ can be used to form the Gaussian log likelihood function for $\{X_t\}_{t=1}^T$ as

$$
\begin{aligned}
\ln \mathcal{L}(\mu|X) &= \sum_{t=1}^{T} \ln p(X_t|\mu) \\
&= -\frac{4T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln|\Sigma_X(t|t-1)| - \frac{1}{2}\sum_{t=1}^{T} u_t'\Sigma_X(t|t-1)^{-1}u_t.
\end{aligned}
$$

Appendix K

Data Sources

- Euro area:
  Real personal consumption: AWM database
  Gross fixed capital formation: AWM database
  Consumer price index: AWM database
  Interest rate (short term): AWM database
  Population: RTDB

- US:
  Real personal consumption: FRED database
  Gross fixed capital formation: FRED database
  Consumer price index: FRED database
  Interest rate (three-month money market rate): FRED database
  Population: FRED database

Appendix L

Figures and Tables

		$\mathcal{M}_1$						$\mathcal{M}_2$					
		prior		posterior				prior		posterior			
	Type	Mean	Std.	Mean	Std.	HPD Interval		Mean	Std.	Mean	Std.	HPD Interval	
ρ_r	Beta	0.7500	0.1500	0.7483	0.0341	0.6973	0.7999	0.7500	0.1500	0.5650	0.0623	0.4346	0.7289
ρ_a	Beta	0.7500	0.1500	0.1311	0.0485	0.0519	0.2099	0.7500	0.1500	0.3882	0.0819	0.1206	0.5824
ρ_z	Beta	0.7500	0.1500	0.9419	0.0201	0.9085	0.9750	0.7500	0.1500	0.9207	0.0161	0.8893	0.9504
ρ_x	Beta	0.7500	0.1500	0.1466	0.0464	0.0714	0.2169	0.7500	0.1500	0.7162	0.0443	0.6367	0.7902
ρ_v	Beta	0.5000	0.1000	0.3207	0.0481	0.2419	0.3994	0.5000	0.1000	0.2830	0.0348	0.2104	0.3555
h	Beta	0.5000	0.2000	0.5962	0.0408	0.5292	0.6639	0.5000	0.2000	0.2109	0.0561	0.0168	0.4759
α	Beta	0.3000	0.0500	0.3000	0.0005	0.2989	0.3011	0.3000	0.0500	0.3000	0.0004	0.2993	0.3007
ϕ_P	Gamma	50.0000	10.0000	76.3238	8.4470	61.8581	91.1359	50.0000	10.0000	33.0100	3.9714	17.6470	53.2806
ϕ	Gamma	4.0000	1.0000	1.5240	0.2981	0.9092	2.1257	30.0000	7.5000	8.3811	0.8444	4.1573	14.8177
ω_π	Normal	1.3000	0.3000	1.6976	0.2158	1.3411	2.0448	1.3000	0.3000	1.9626	0.2026	1.6069	2.3371
ω_y	Normal	0.1250	0.2000	-0.0121	0.0147	-0.0399	0.0165	0.1250	0.2000	-0.0491	0.0195	-0.0919	-0.0069
$\ln(z)$	Gamma	9.1800	1.0000	9.1908	0.9940	7.5527	10.8146	9.1800	1.0000	9.1959	0.9940	7.5372	10.8149
σ_a	Inverse gamma	0.0100	0.5000	0.0164	0.0017	0.0135	0.0193	0.0100	0.5000	0.0081	0.0005	0.0056	0.0118
σ_z	Inverse gamma	0.0100	0.5000	0.0213	0.0016	0.0184	0.0240	0.0100	0.5000	0.0163	0.0008	0.0145	0.0182
σ_x	Inverse gamma	0.0100	0.5000	0.0750	0.0145	0.0443	0.1038	0.0100	0.5000	0.0110	0.0009	0.0058	0.0191
σ_v	Inverse gamma	0.0100	0.5000	0.0022	0.0002	0.0020	0.0025	0.0100	0.5000	0.0035	0.0005	0.0023	0.0044

		$\mathcal{M}_3$					
		prior		posterior			
	Type	Mean	Std.	Mean	Std.	HPD Interval	
ρ_r	Beta	0.7500	0.1500	0.5719	0.0631	0.4549	0.7093
ρ_a	Beta	0.7500	0.1500	0.3788	0.0862	0.1446	0.5826
ρ_z	Beta	0.7500	0.1500	0.9208	0.0162	0.8912	0.9490
ρ_x	Beta	0.7500	0.1500	0.7118	0.0450	0.6380	0.7910
ρ_v	Beta	0.5000	0.1000	0.2861	0.0350	0.2167	0.3575
h	Beta	0.5000	0.2000	0.2192	0.0623	0.0282	0.4370
α	Beta	0.3000	0.0500	0.3000	0.0004	0.2993	0.3007
ϕ_P	Gamma	50.0000	10.0000	32.8950	4.1066	19.1493	48.8868
ϕ	Gamma	1500.0000	375.0000	322.7626	34.7720	177.7162	506.7580
ω_π	Normal	1.3000	0.3000	1.9543	0.2046	1.6084	2.3121
ω_y	Normal	0.1250	0.2000	-0.0487	0.0197	-0.0879	-0.0103
$\ln(z)$	Gamma	9.1800	1.0000	9.180	0.9940	7.5715	10.8237
σ_a	Inverse gamma	0.0100	0.5000	0.0080	0.0005	0.0058	0.0108
σ_z	Inverse gamma	0.0100	0.5000	0.0164	0.0008	0.0145	0.0181
σ_x	Inverse gamma	0.0100	0.5000	0.0105	0.0010	0.0061	0.0164
σ_v	Inverse gamma	0.0100	0.5000	0.0034	0.0005	0.0024	0.0043

Table L.1: Estimates: US, 1948:Q1–2006:Q4.

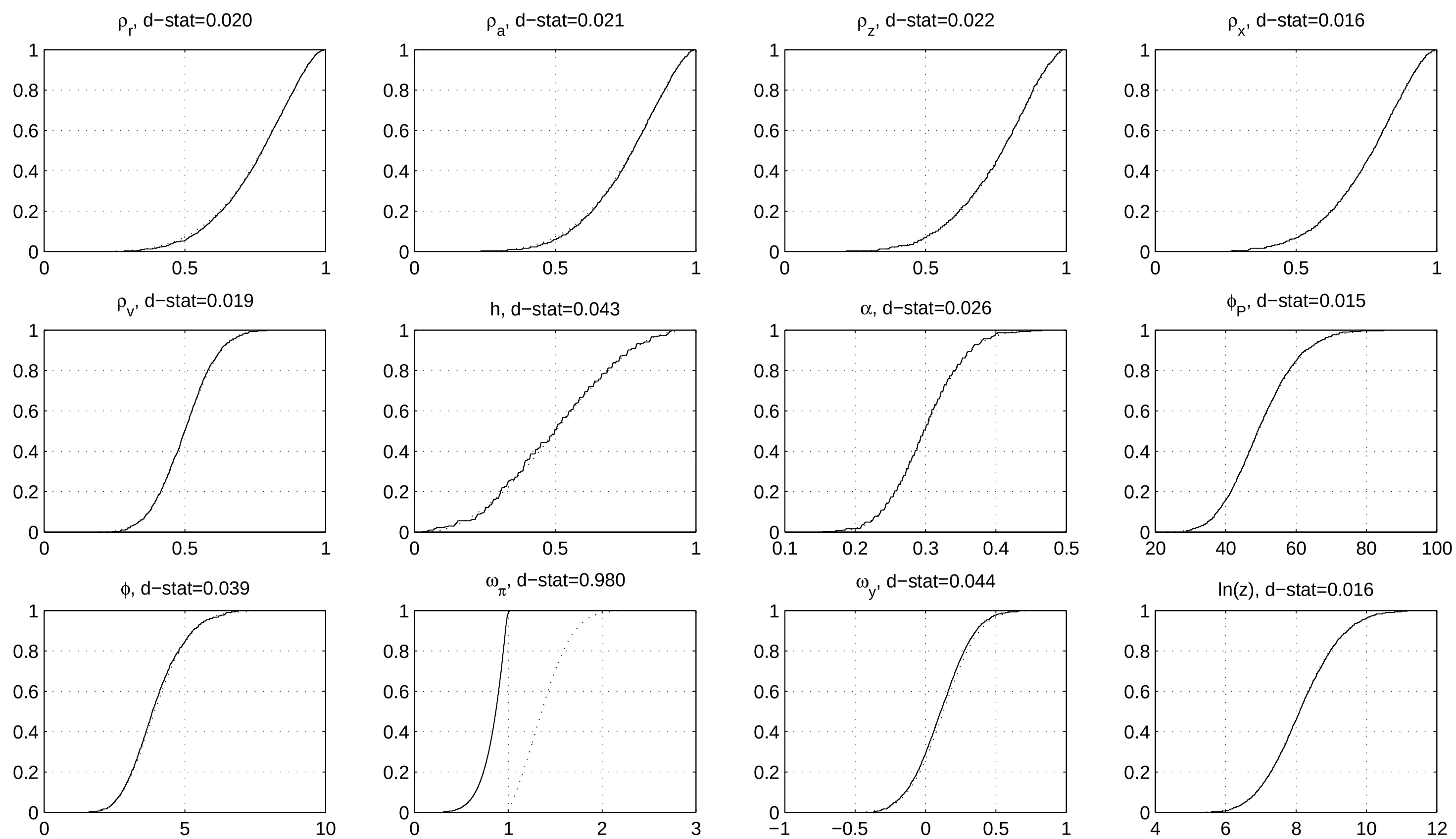

Figure L.1: Euro area: $\mathcal{M}_1$. Smirnov two-sample test for stability analysis: $F_n(\mu_i|B)$ (dotted lines), $F_{\bar{n}}(\mu_i|\bar{B})$ (solid lines) and Smirnov test statistic (d-stat).

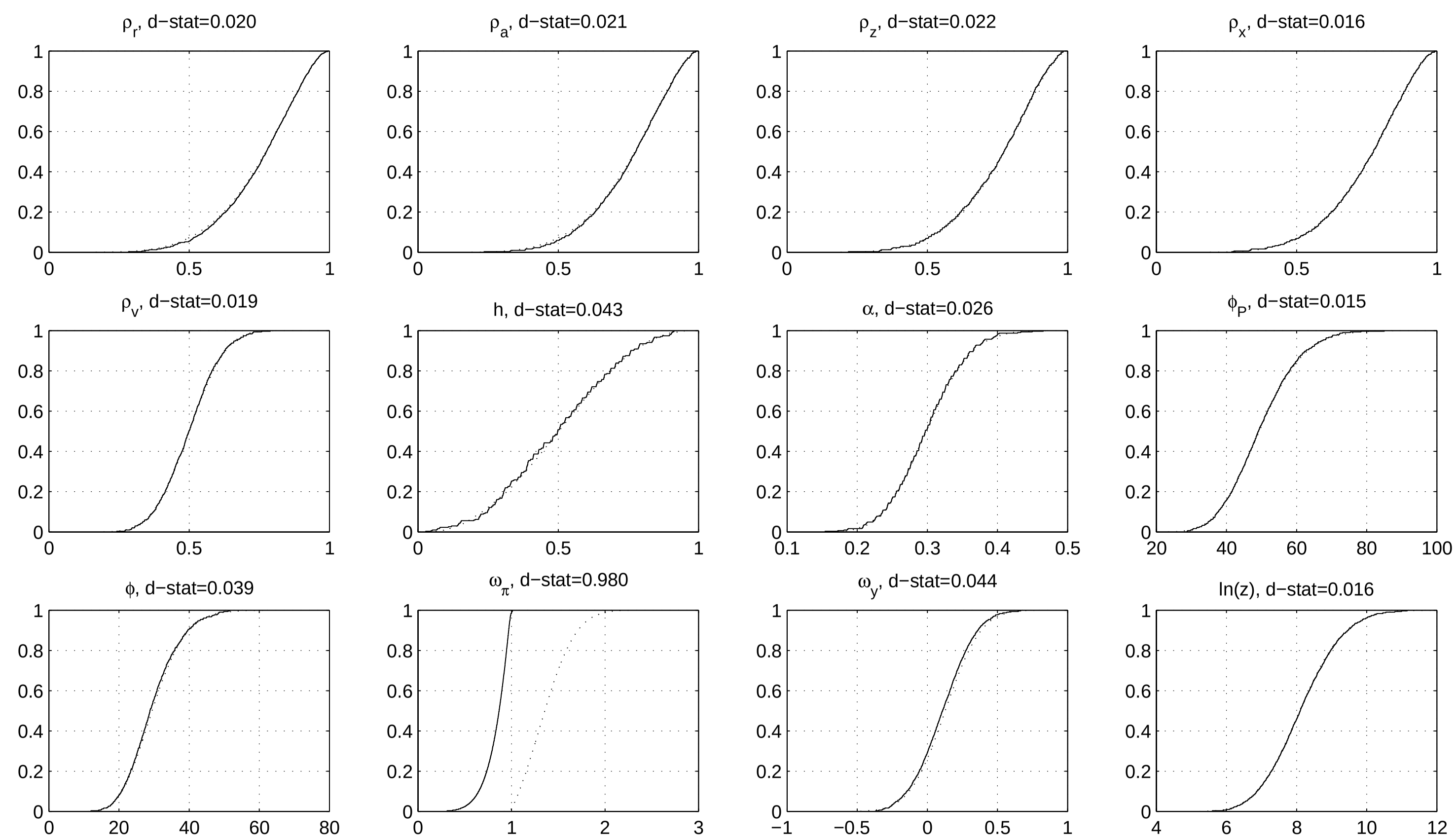

Figure L.2: Euro area: $\mathcal{M}_2$. Smirnov two-sample test for stability analysis: $F_n(\mu_i|B)$ (dotted lines), $F_{\bar{n}}(\mu_i|\bar{B})$ (solid lines) and Smirnov test statistic (d-stat).

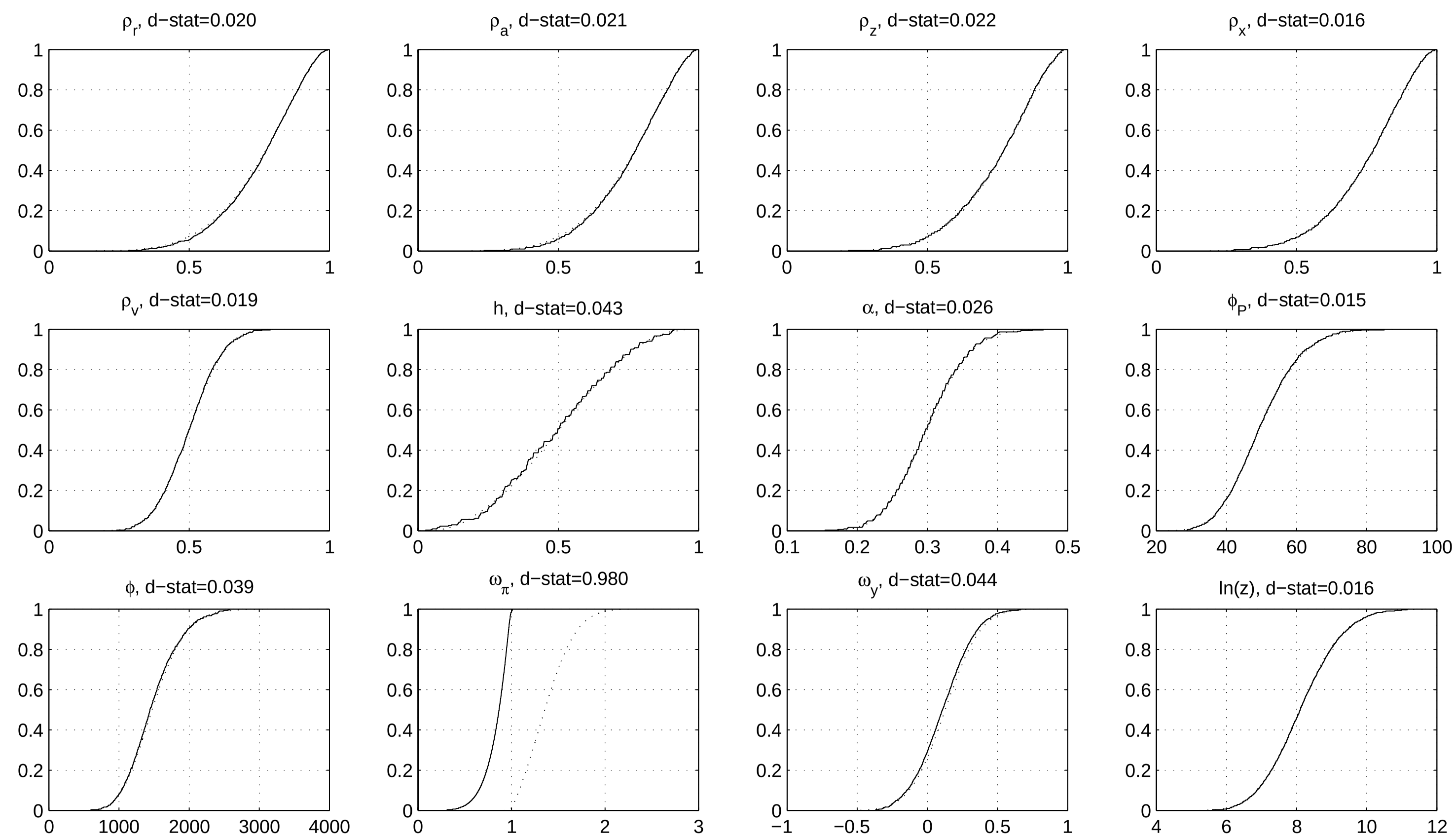

Figure L.3: Euro area: $\mathcal{M}_3$. Smirnov two-sample test for stability analysis: $F_n(\mu_i|B)$ (dotted lines), $F_{\bar{n}}(\mu_i|\bar{B})$ (solid lines) and Smirnov test statistic (d-stat).

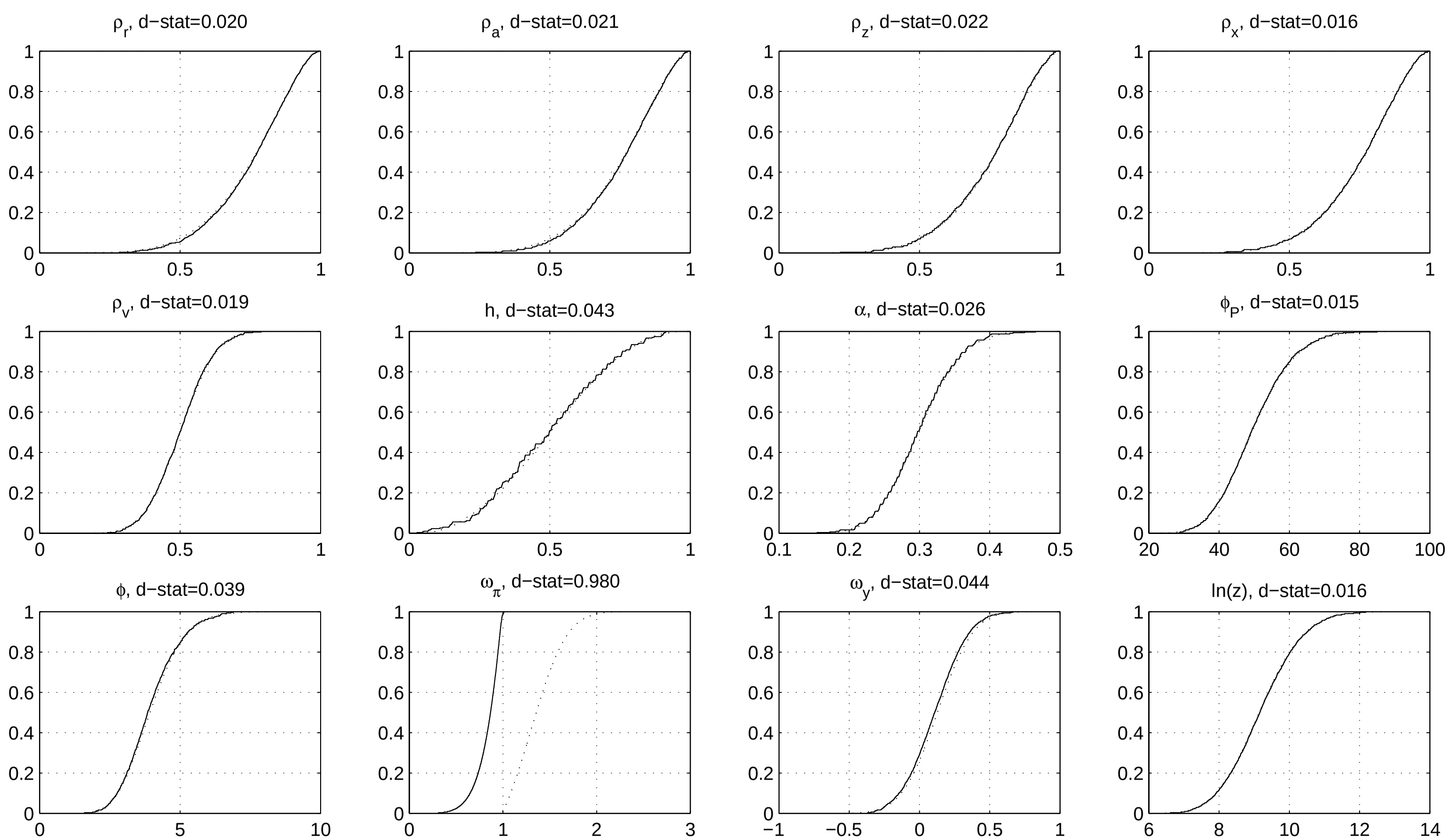

Figure L.4: US: $\mathcal{M}_1$. Smirnov two-sample test for stability analysis: $F_n(\mu_i|B)$ (dotted lines), $F_{\bar{n}}(\mu_i|\bar{B})$ (solid lines) and Smirnov test statistic (d-stat).

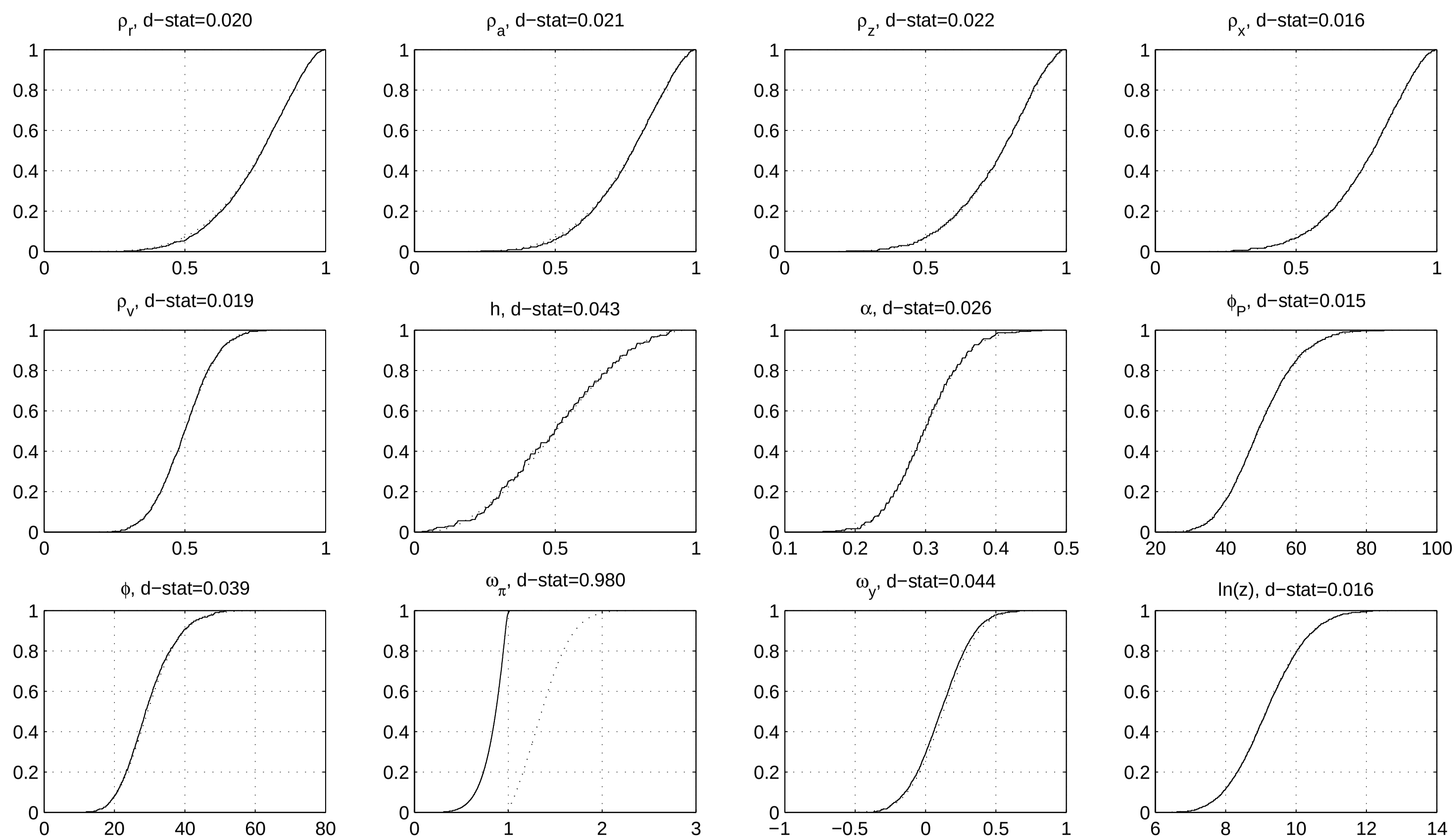

Figure L.5: US: $\mathcal{M}_2$. Smirnov two-sample test for stability analysis: $F_n(\mu_i|B)$ (dotted lines), $F_{\bar{n}}(\mu_i|\bar{B})$ (solid lines) and Smirnov test statistic (d-stat).

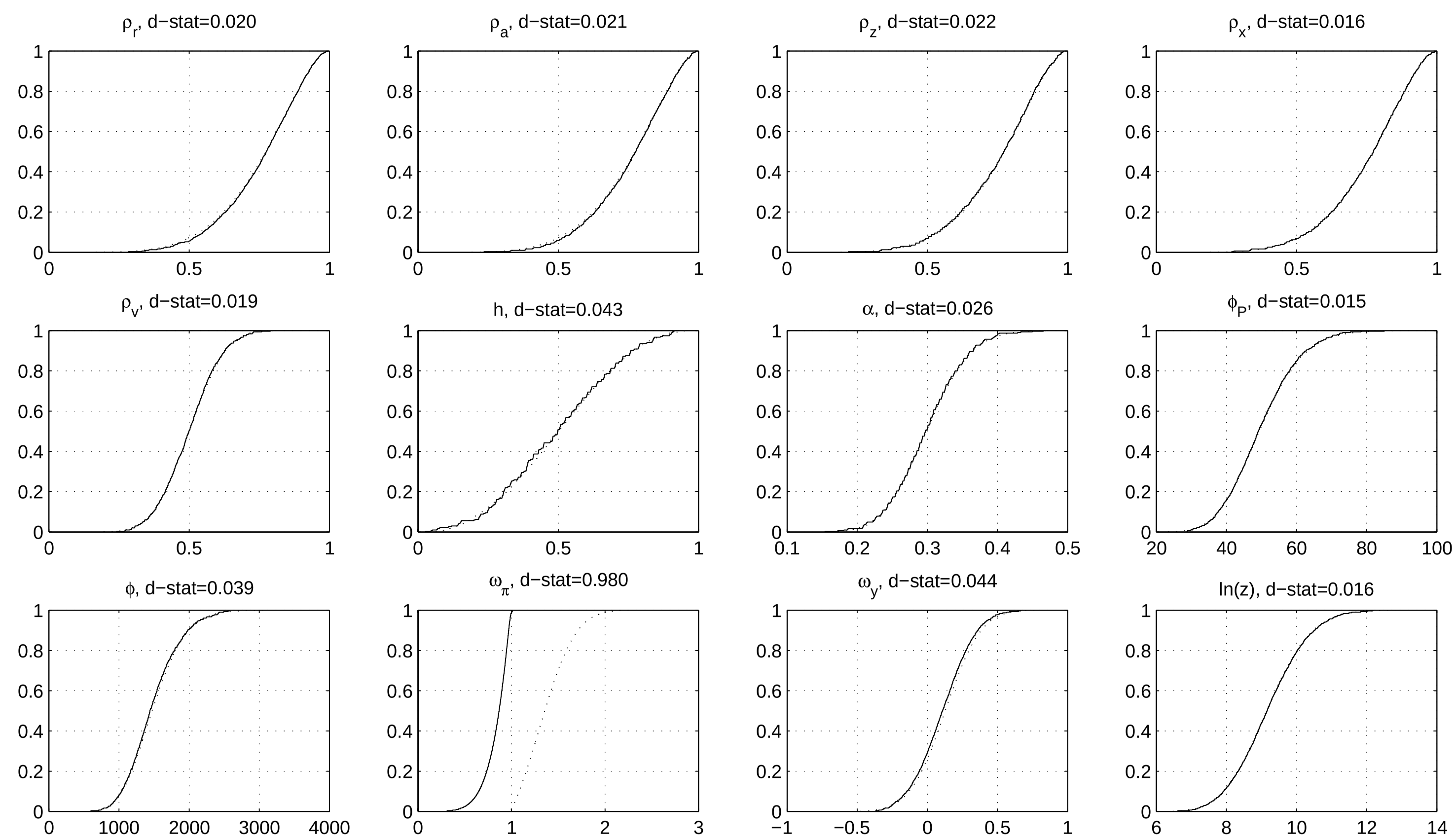

Figure L.6: US: $\mathcal{M}_3$. Smirnov two-sample test for stability analysis: $F_n(\mu_i|B)$ (dotted lines), $F_{\bar{n}}(\mu_i|\bar{B})$ (solid lines) and Smirnov test statistic (d-stat).

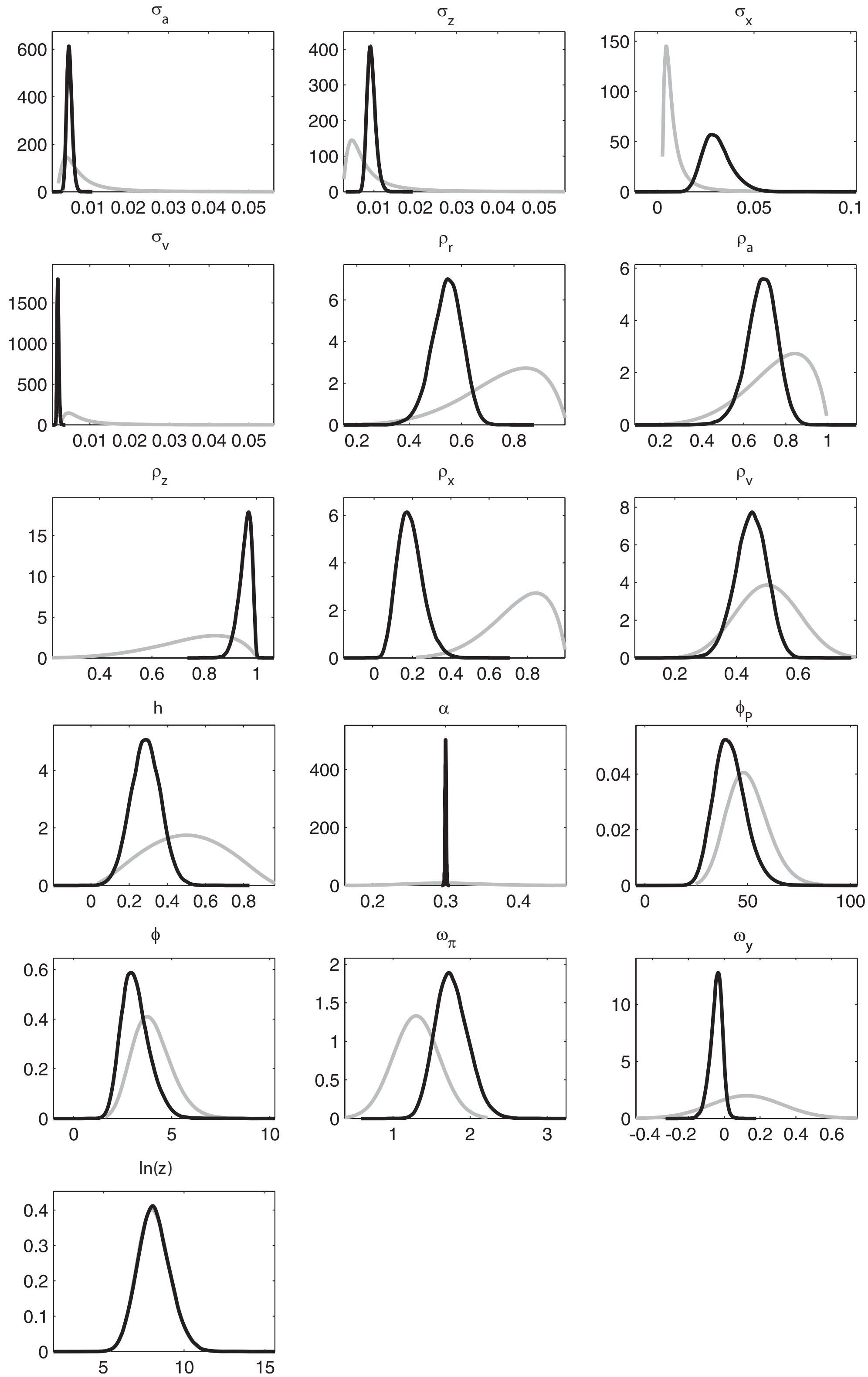

Figure L.7: Euro area: $\mathcal{M}_1$. Prior distributions (gray lines) and posterior distributions (black lines) of the estimated parameters.

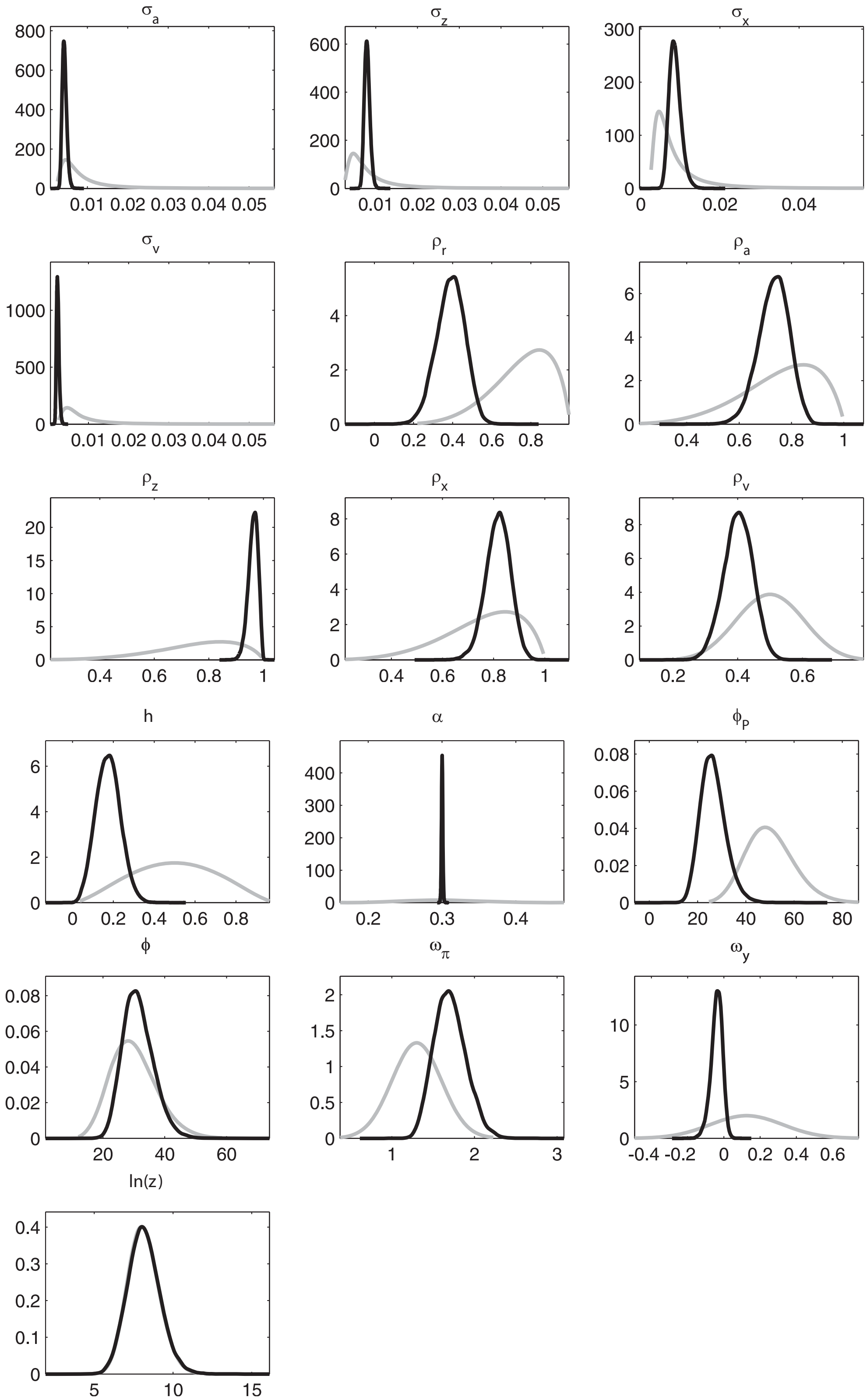

Figure L.8: Euro area: $\mathcal{M}_2$. Prior distributions (gray lines) and posterior distributions (black lines) of the estimated parameters.

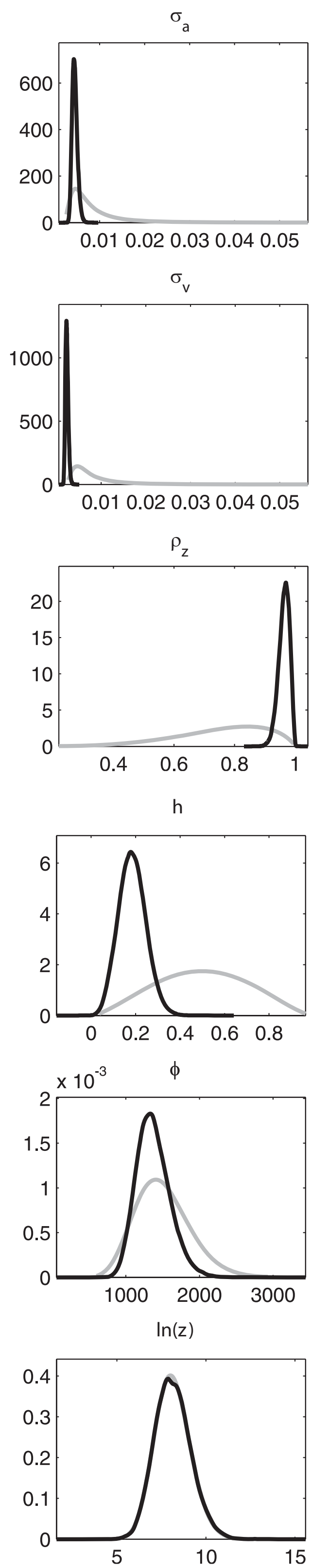

Figure L.9: Euro area: $\mathcal{M}_3$. Prior distributions (gray lines) and posterior distributions (black lines) of the estimated parameters.

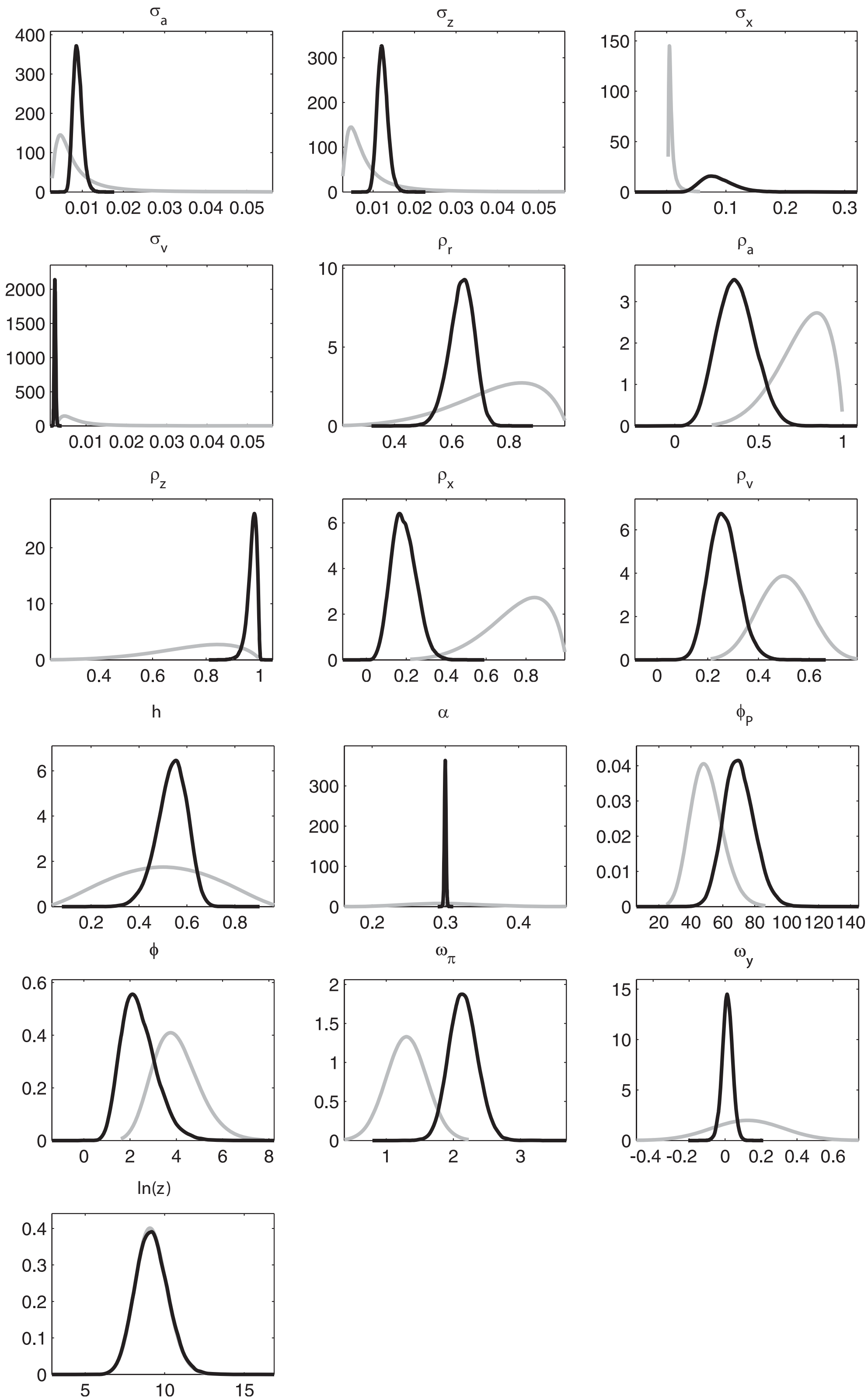

Figure L.10: US: $\mathcal{M}_1$. Prior distributions (gray lines) and posterior distributions (black lines) of the estimated parameters.

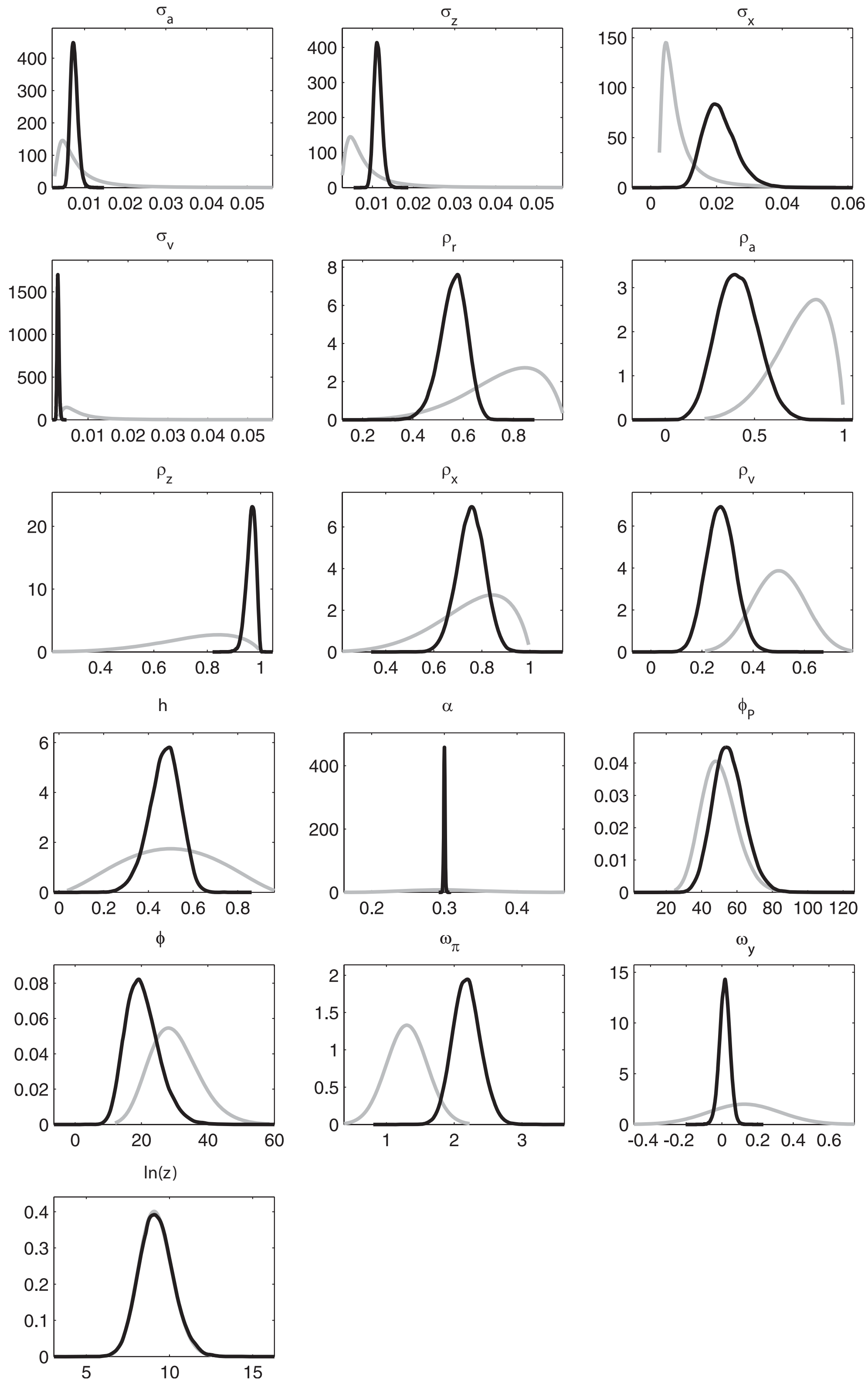

Figure L.11: US: $\mathcal{M}_2$. Prior distributions (gray lines) and posterior distributions (black lines) of the estimated parameters.

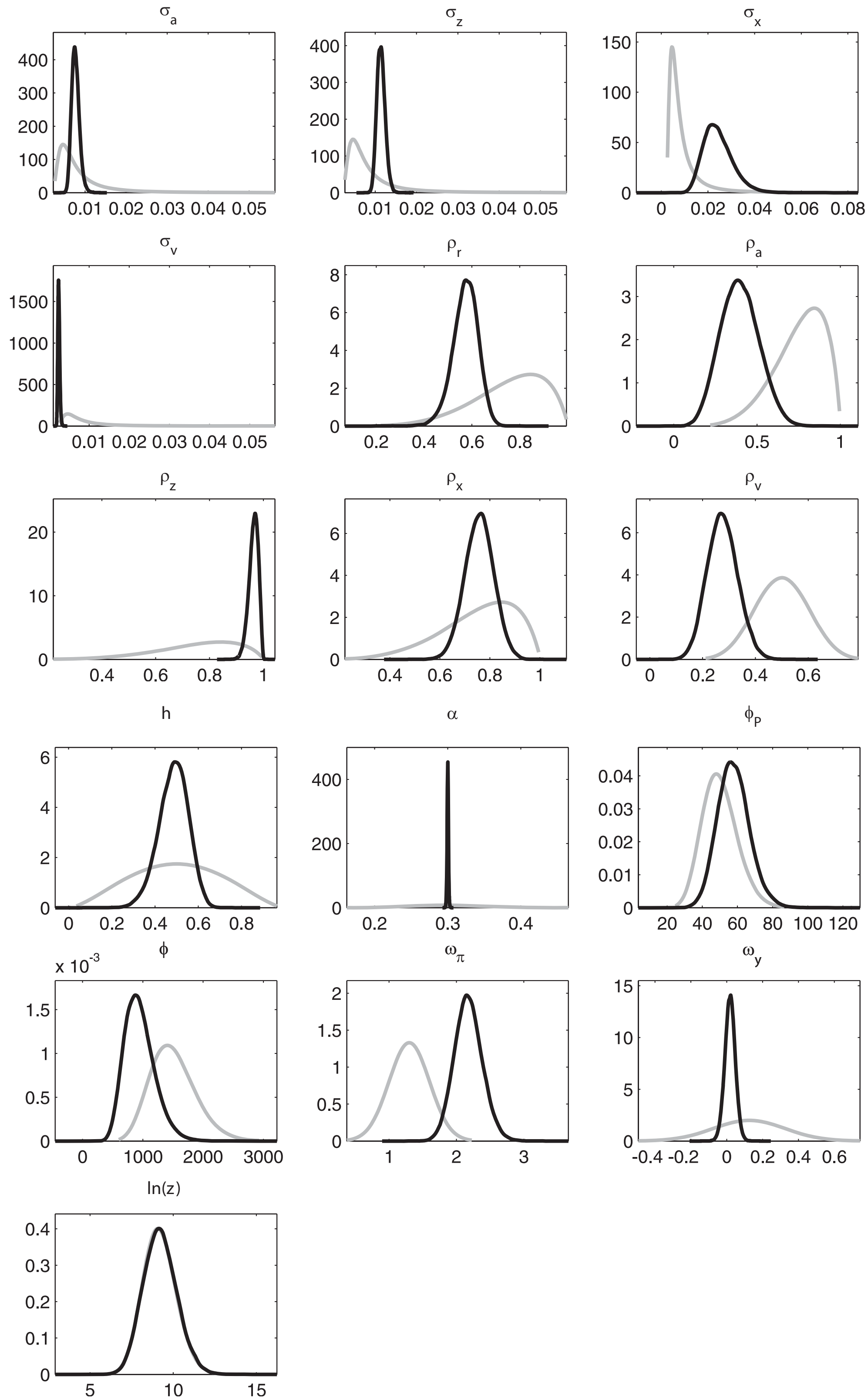

Figure L.12: US: $\mathcal{M}_3$. Prior distributions (gray lines) and posterior distributions of the estimated parameters.

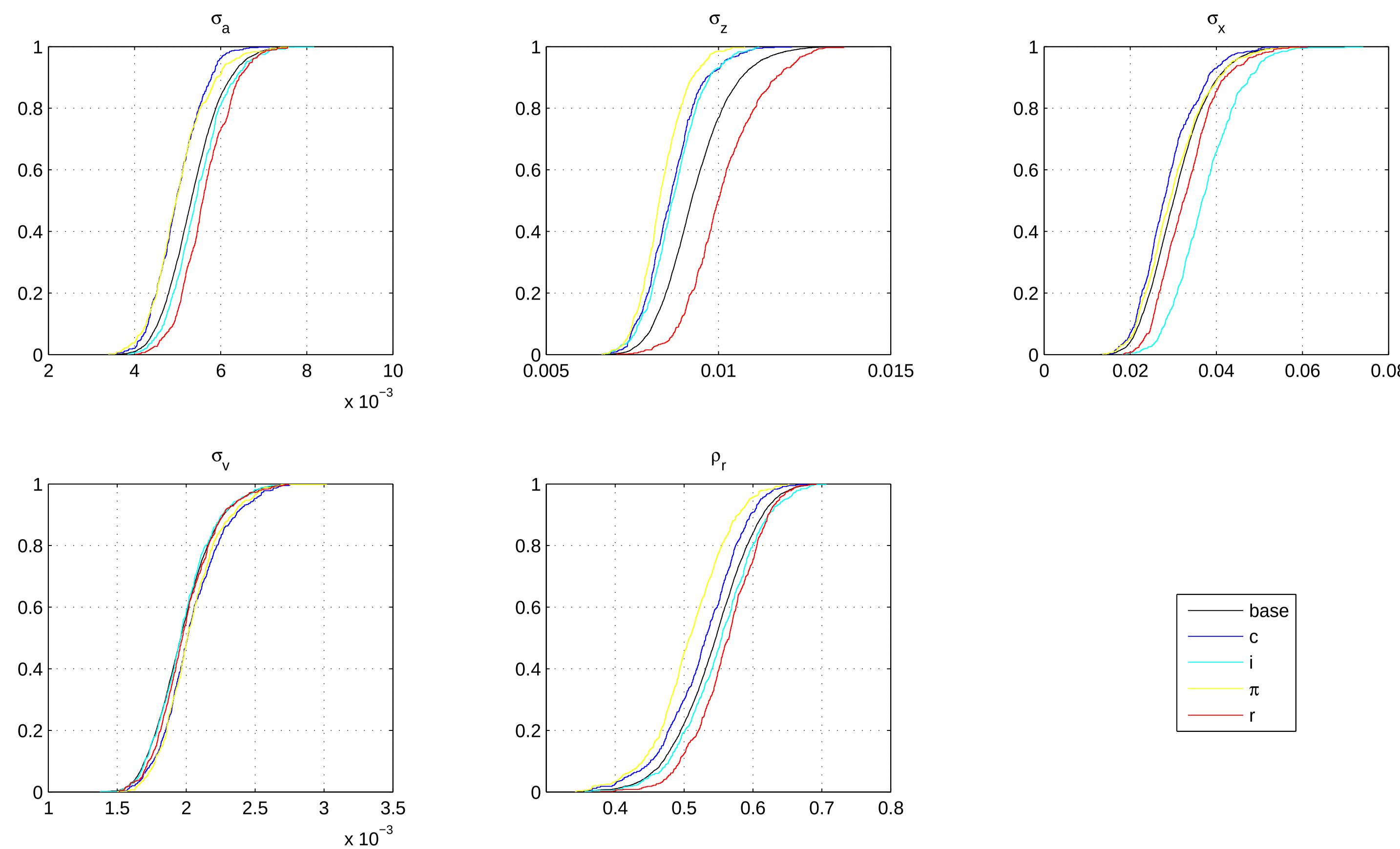

Figure L.13: Euro area: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

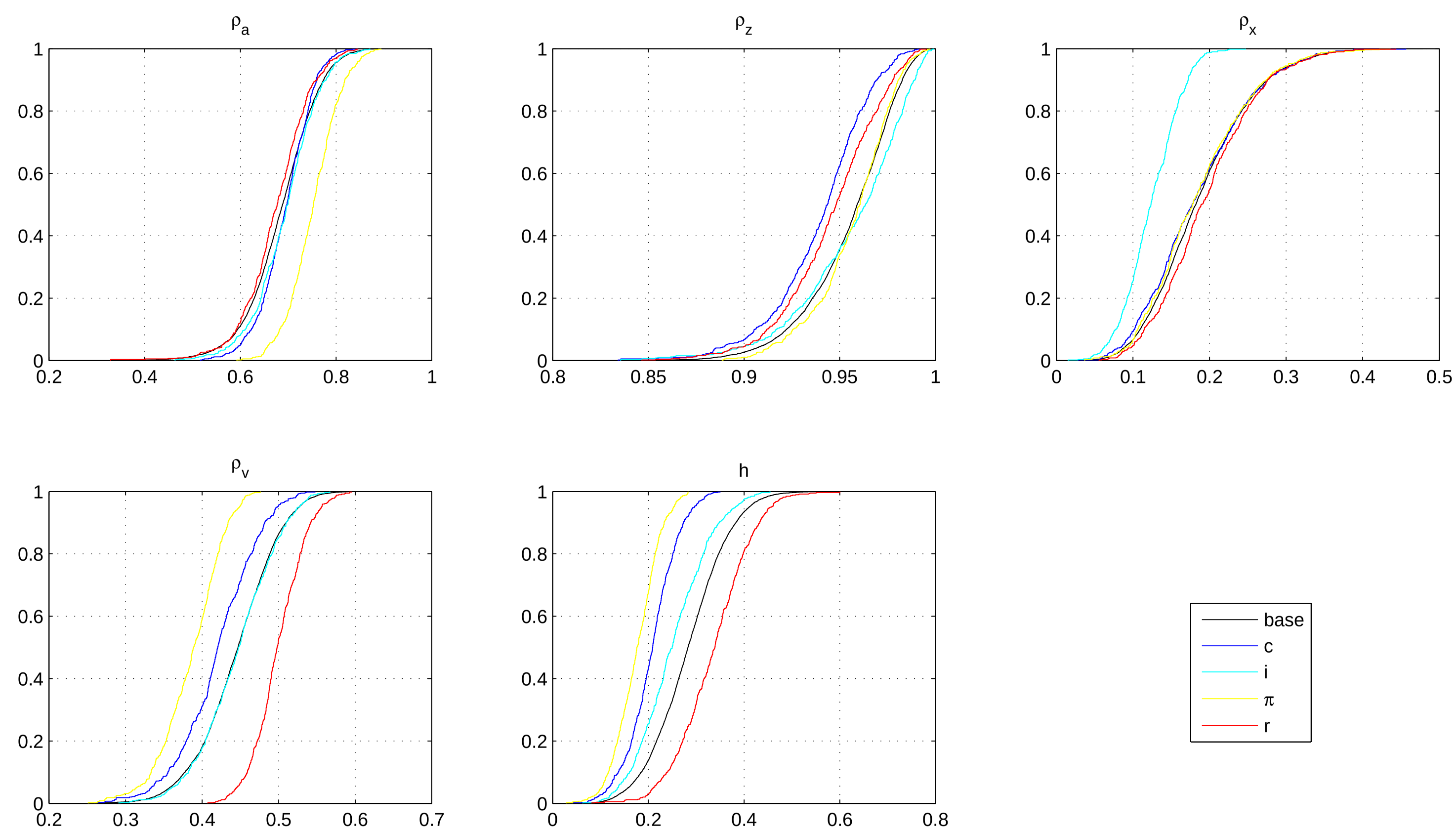

Figure L.14: Euro area: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

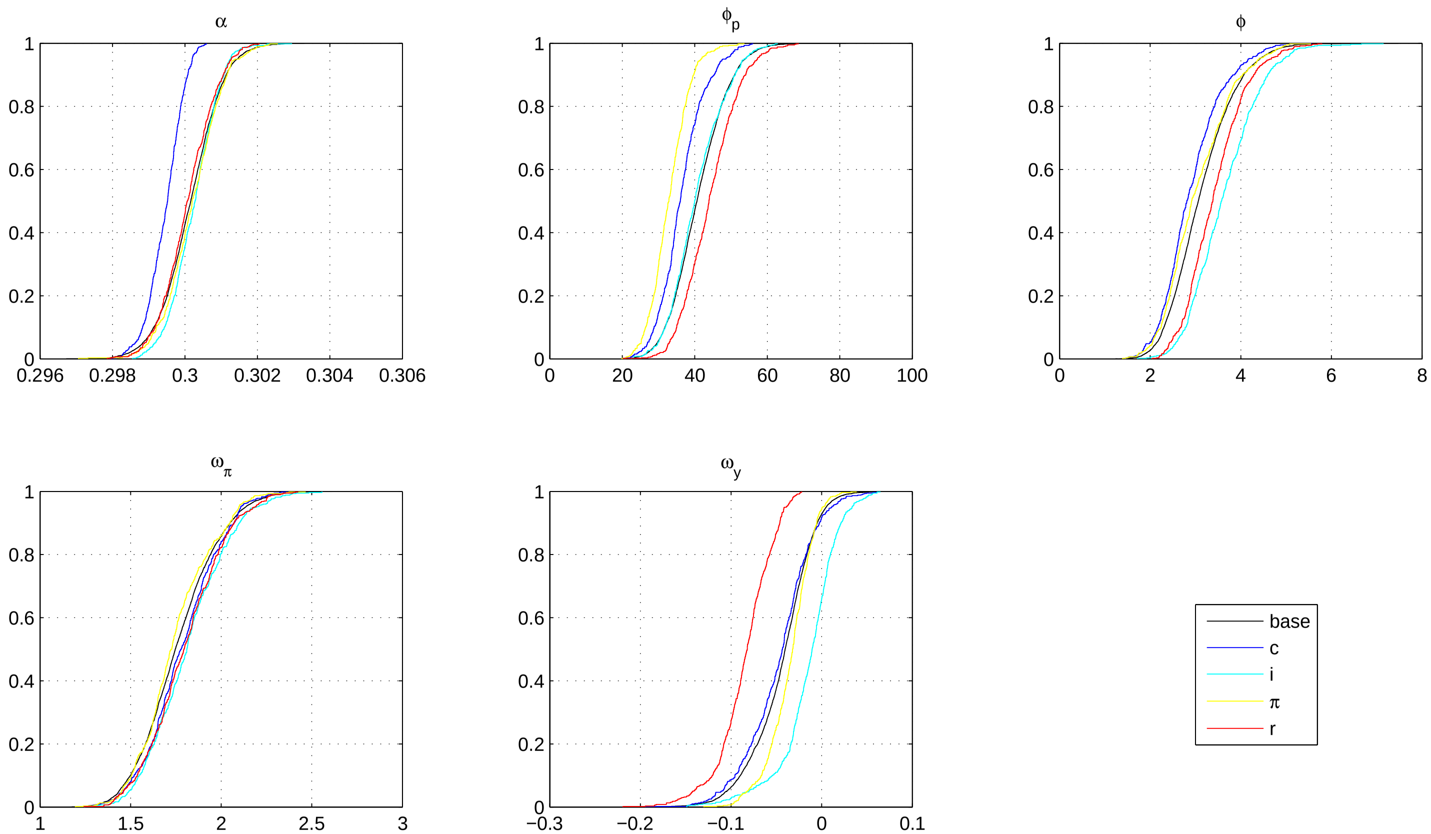

Figure L.15: Euro area: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

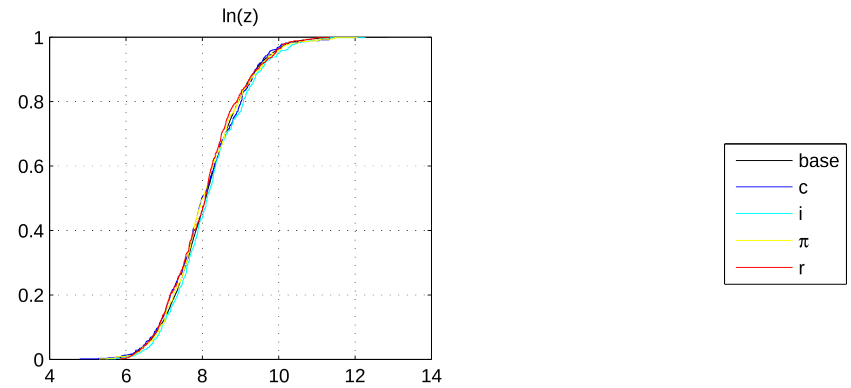

Figure L.16: Euro area: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

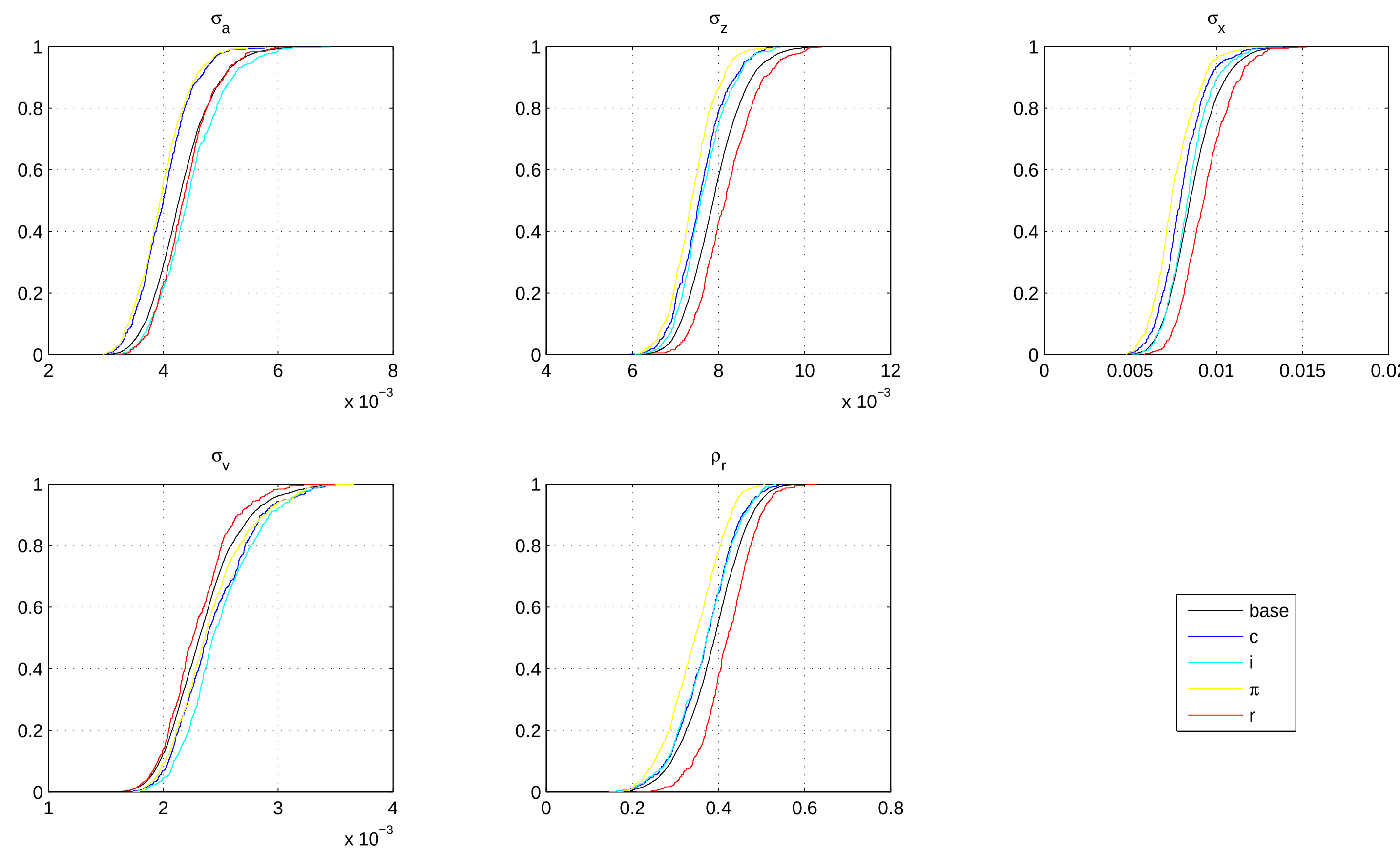

Figure L.17: Euro area: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

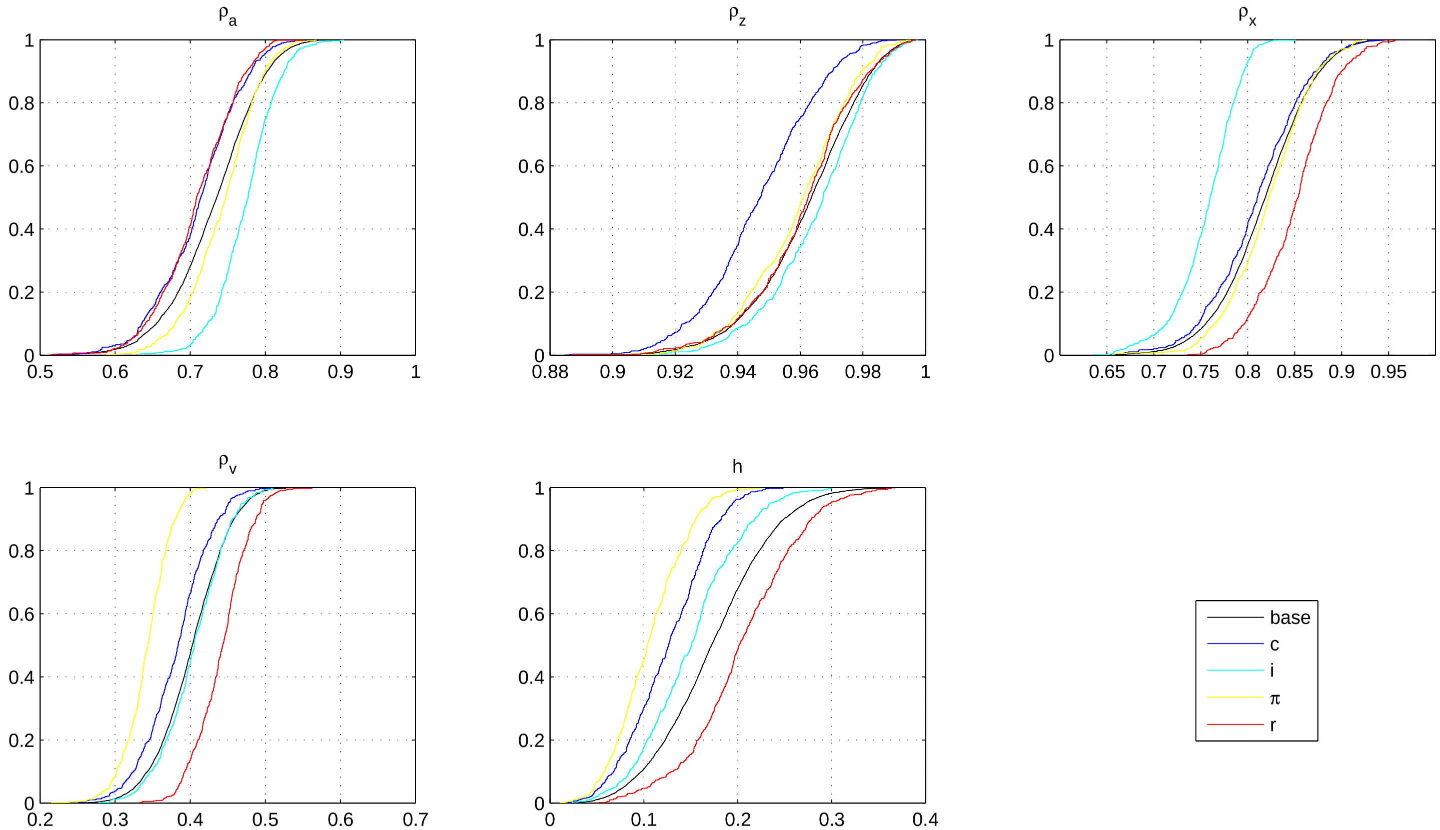

Figure L.18: Euro area: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

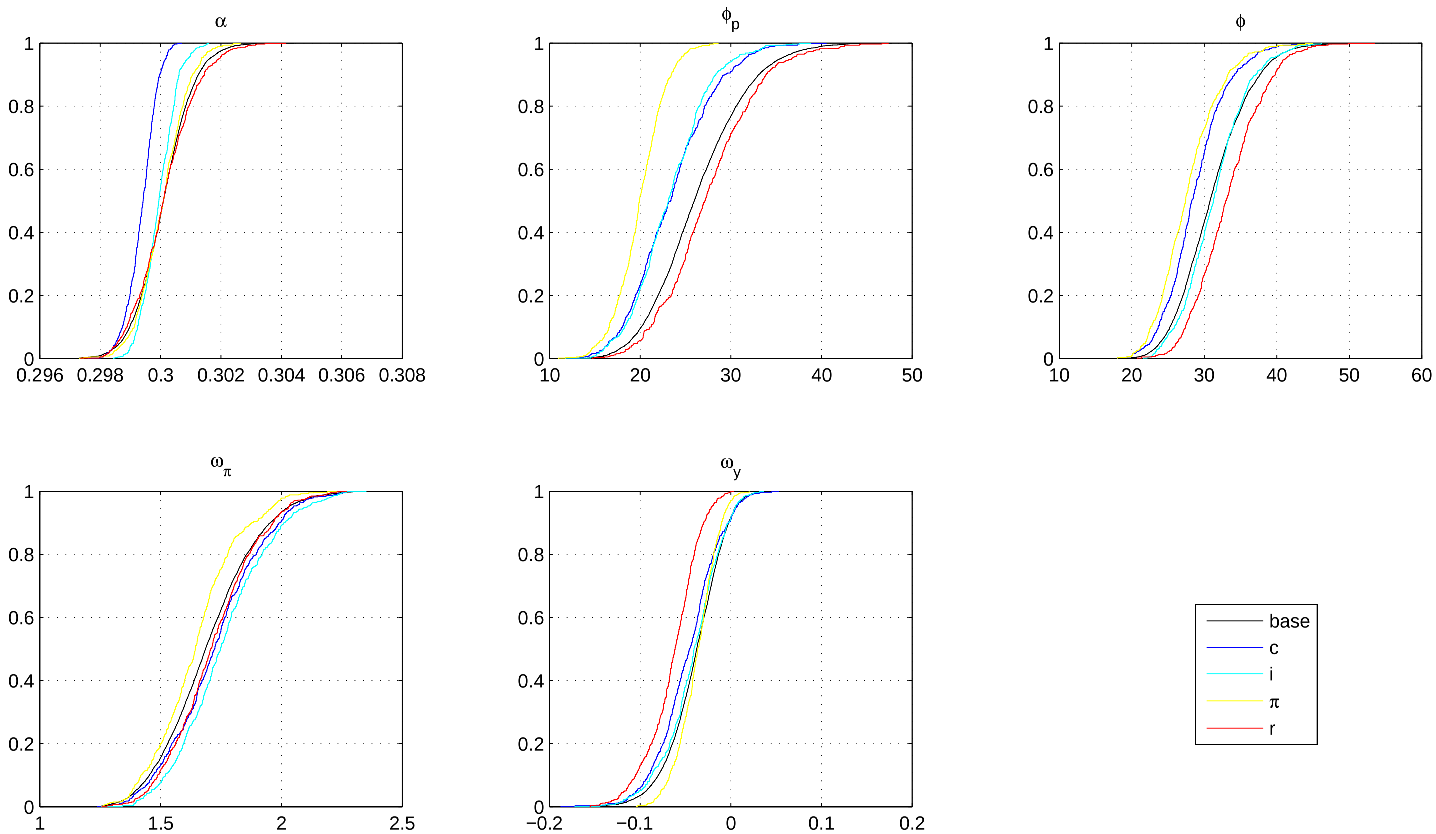

Figure L.19: Euro area: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

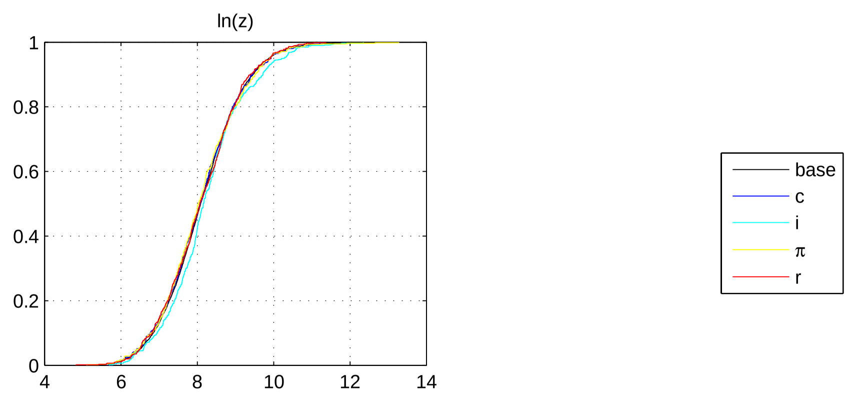

Figure L.20: Euro area: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

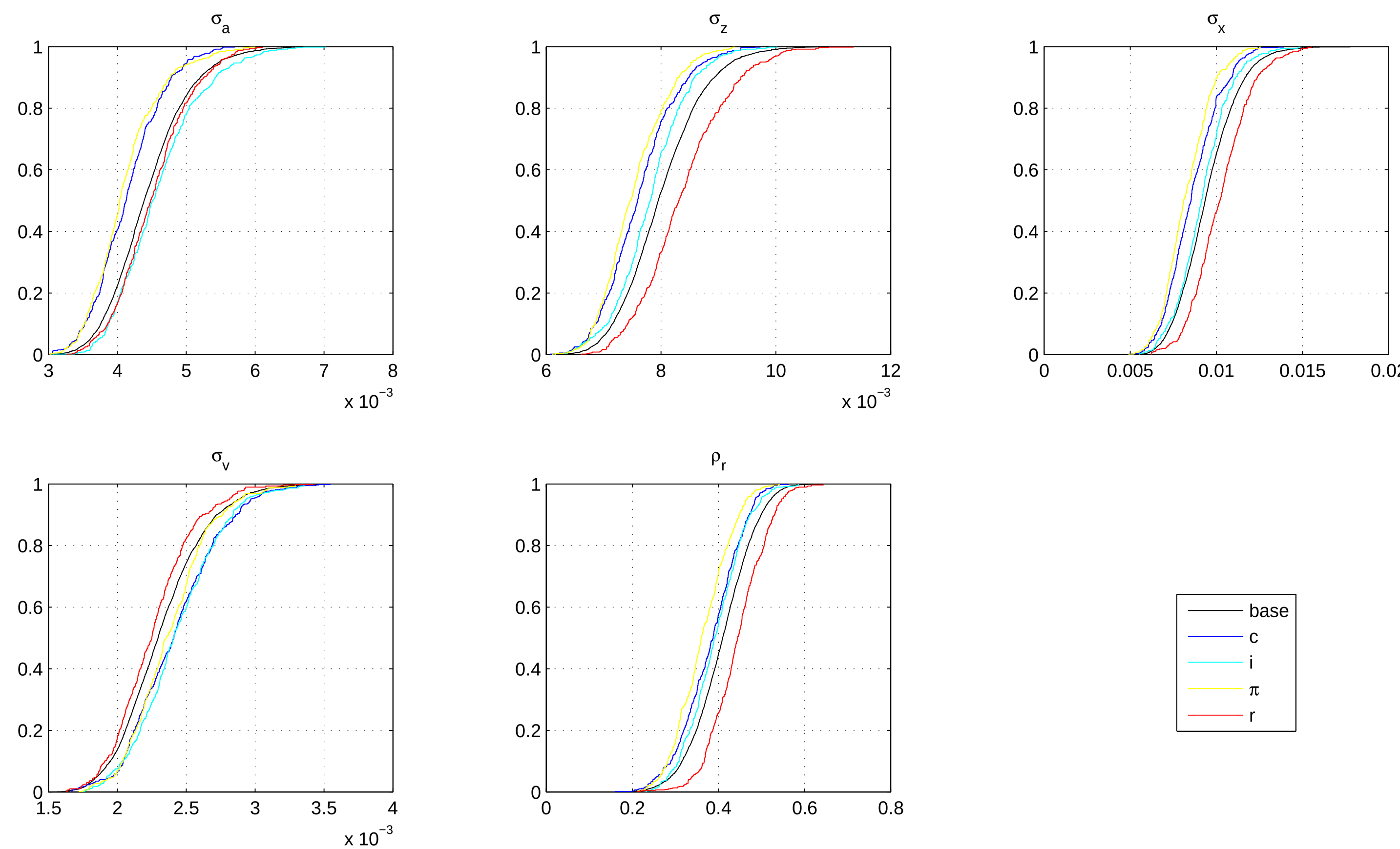

Figure L.21: Euro area: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

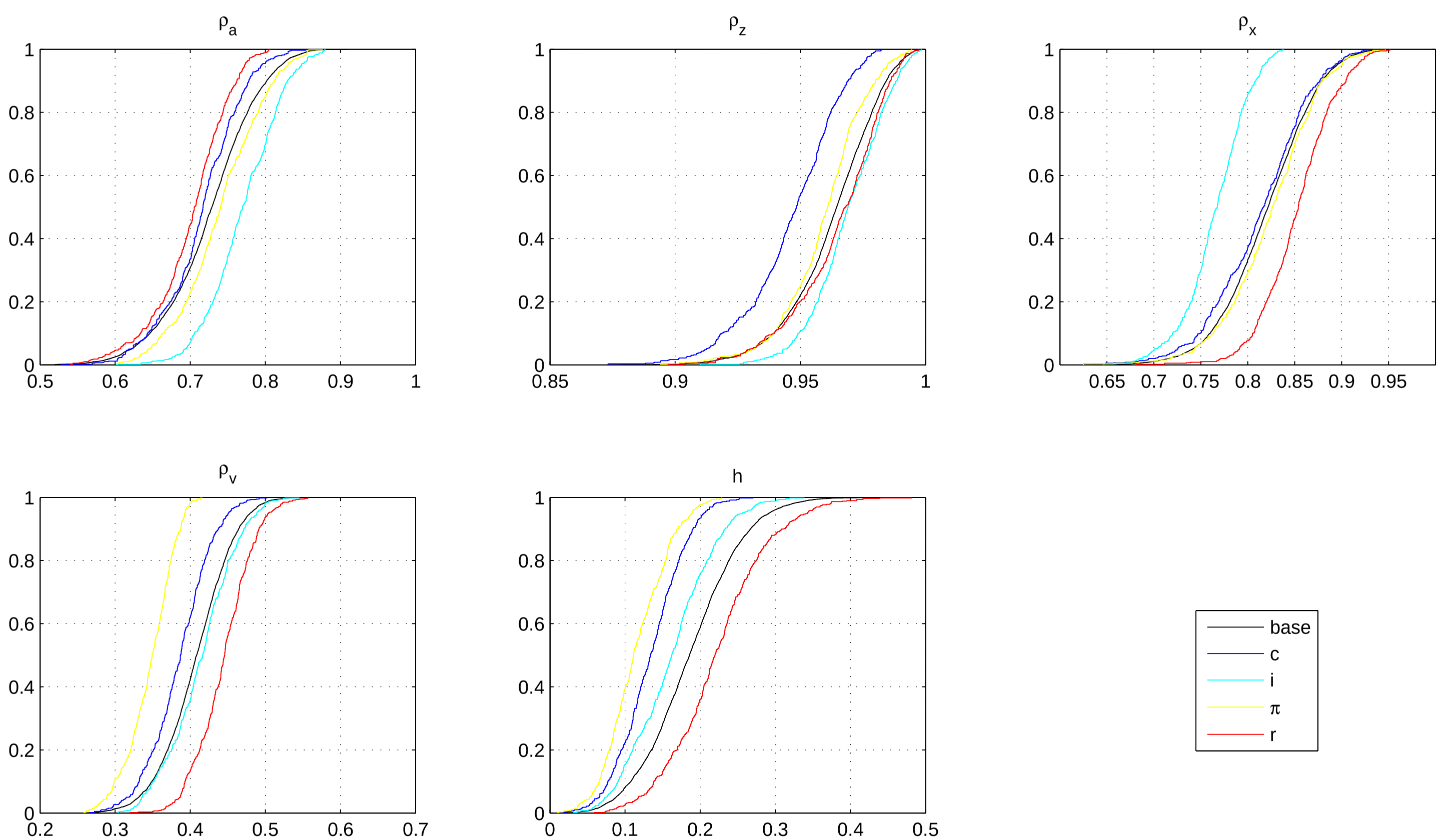

Figure L.22: Euro area: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

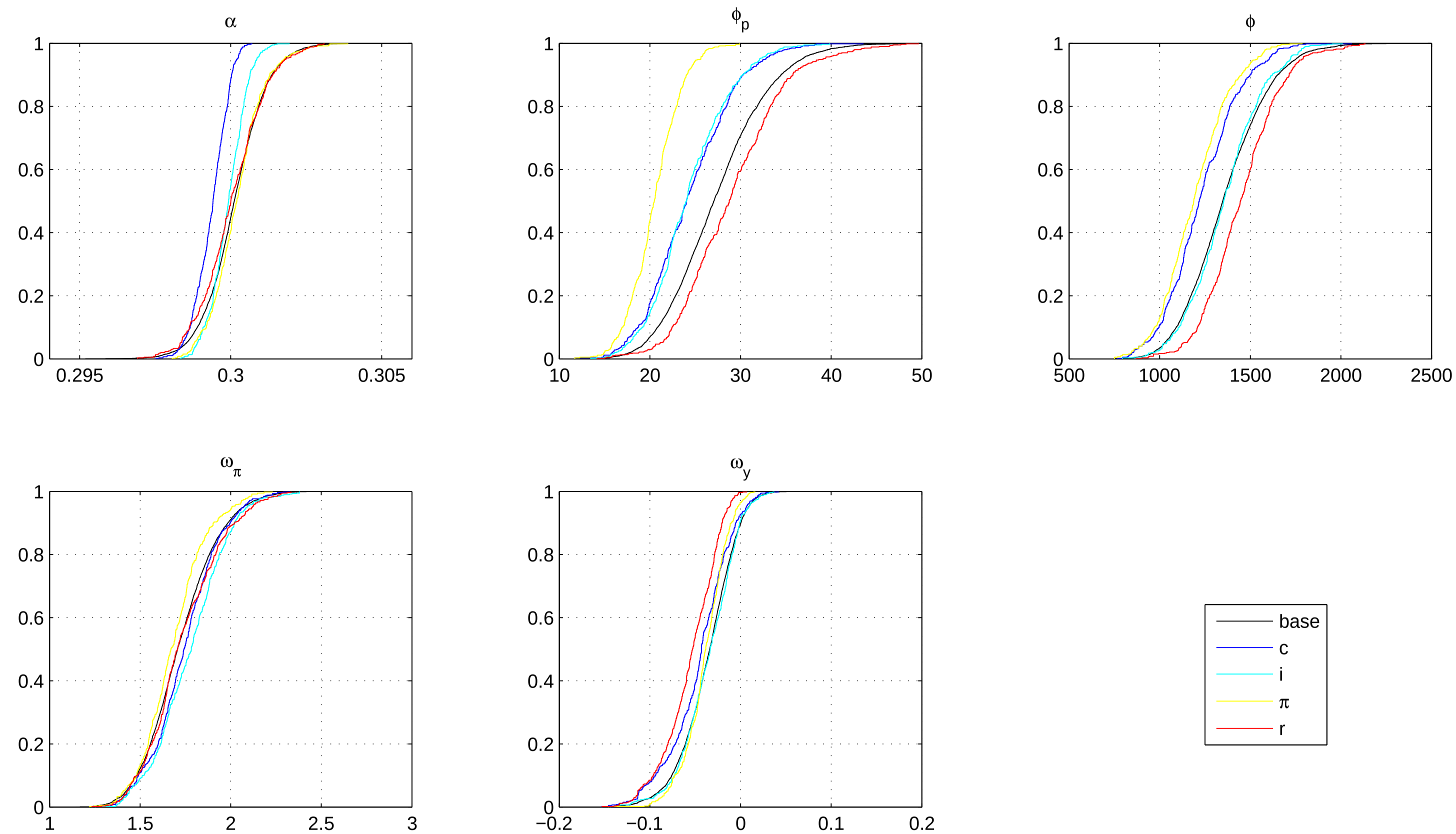

Figure L.23: Euro area: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

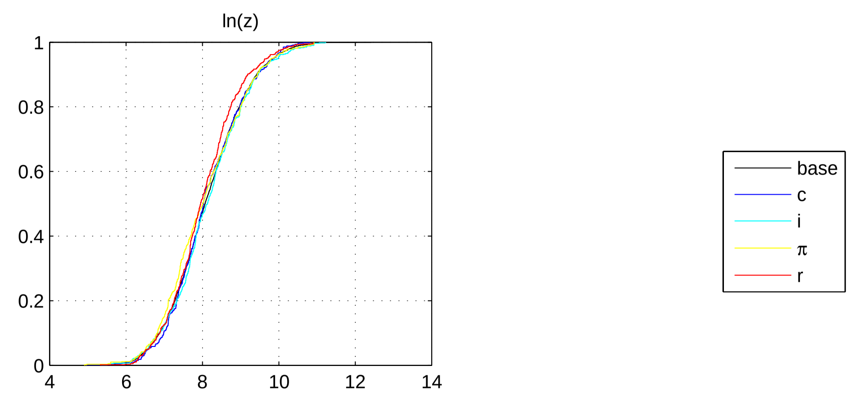

Figure L.24: Euro area: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

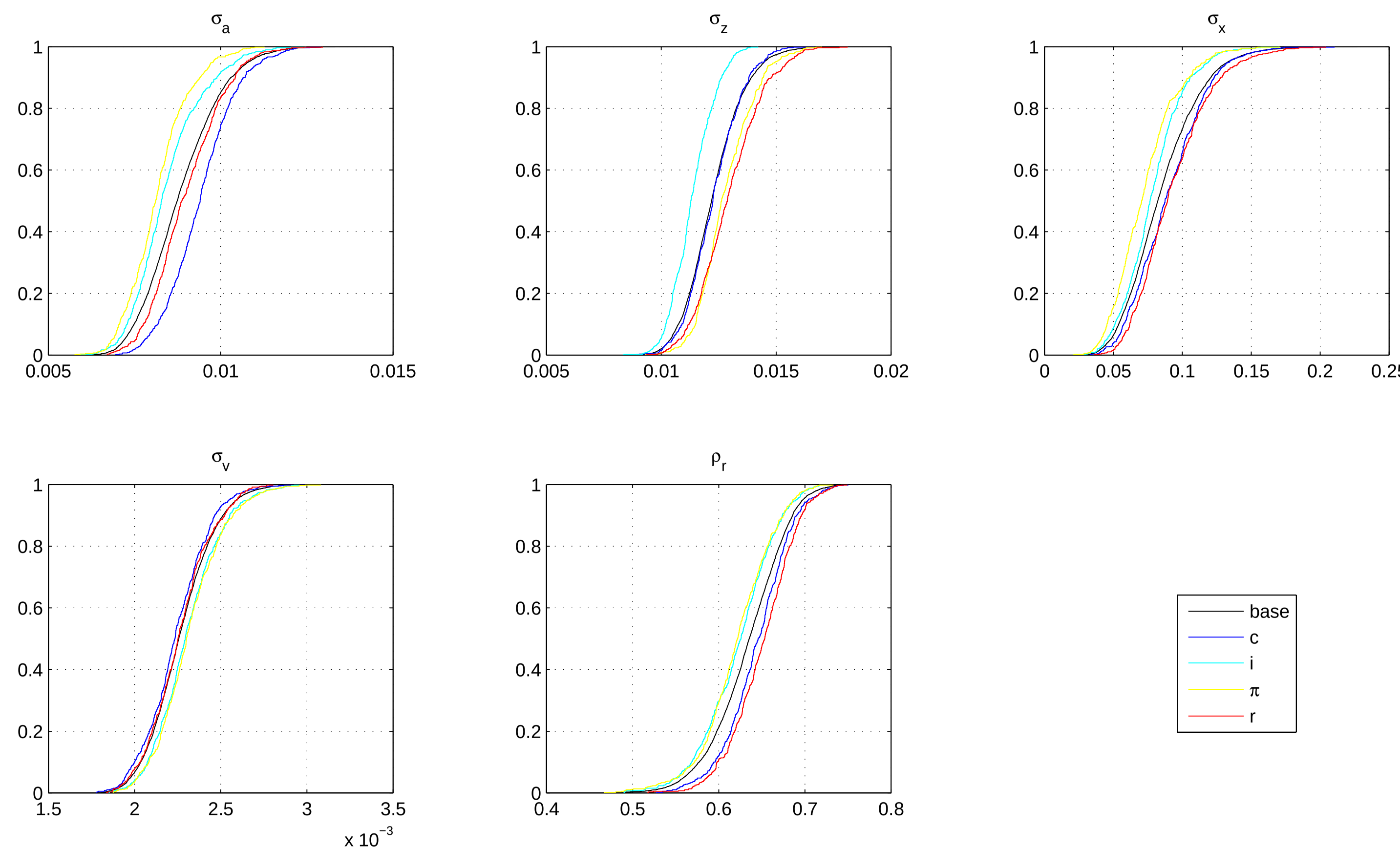

Figure L.25: US: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

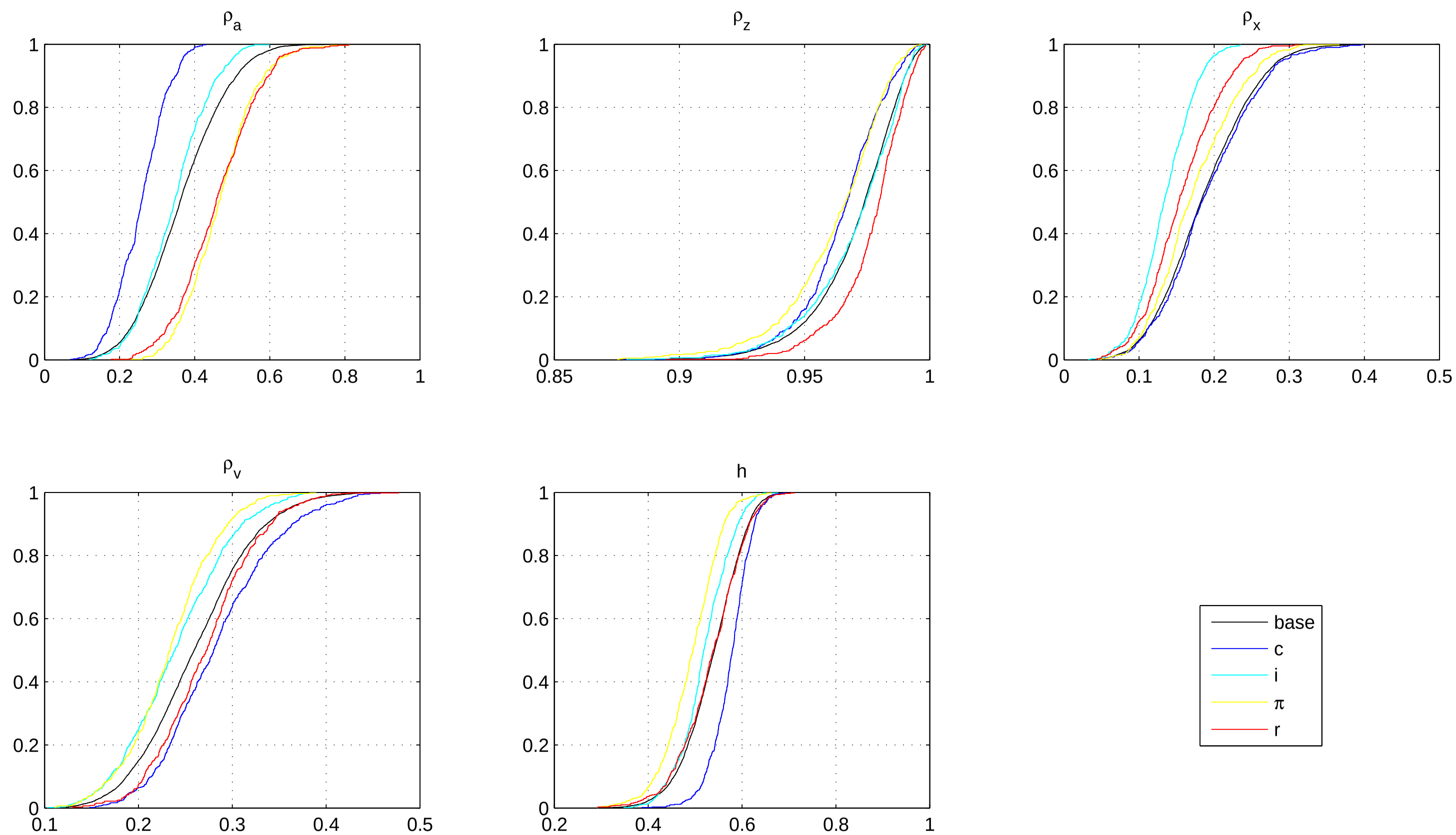

Figure L.26: US: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

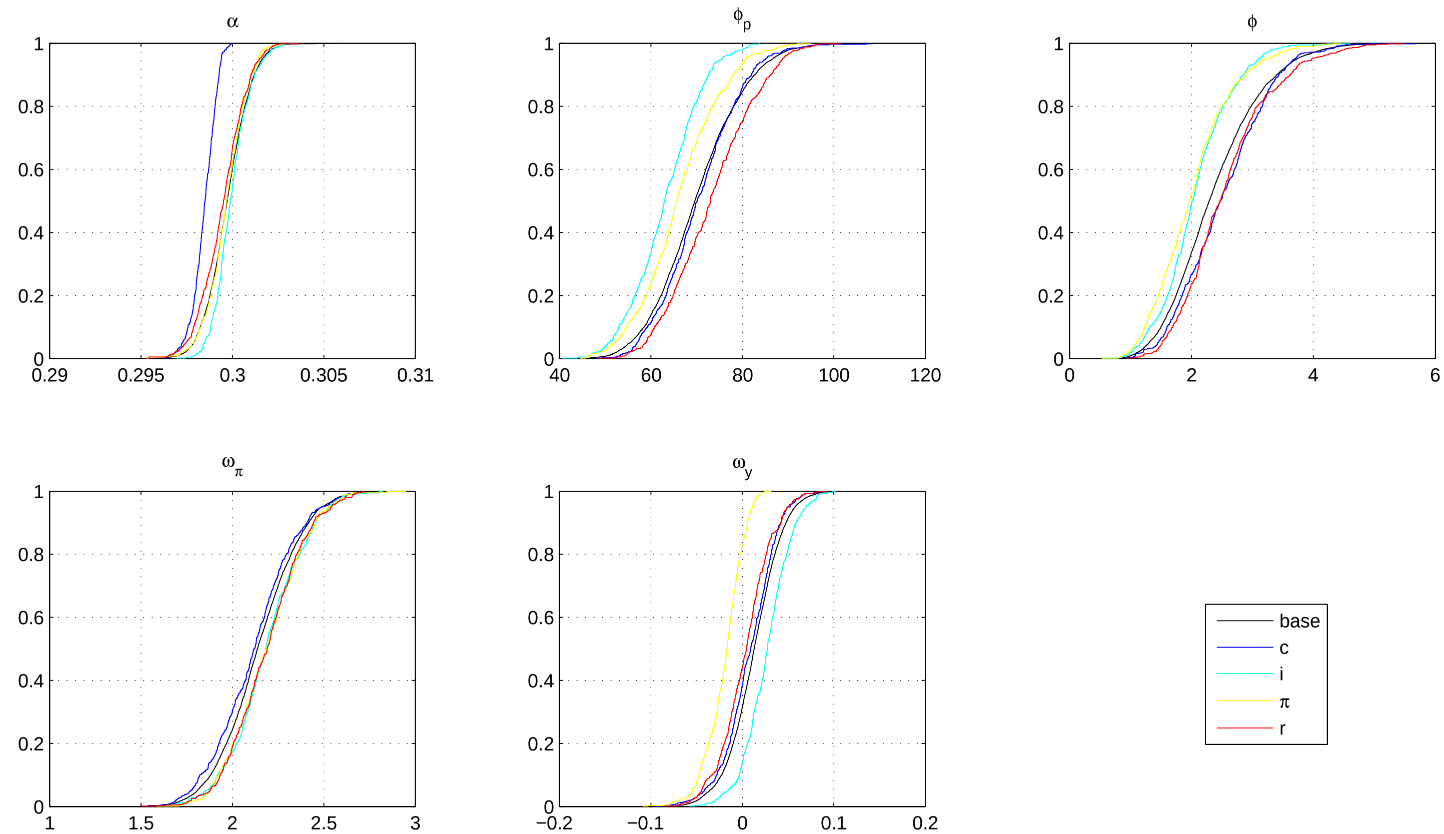

Figure L.27: US: $\mathcal{M}_1$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

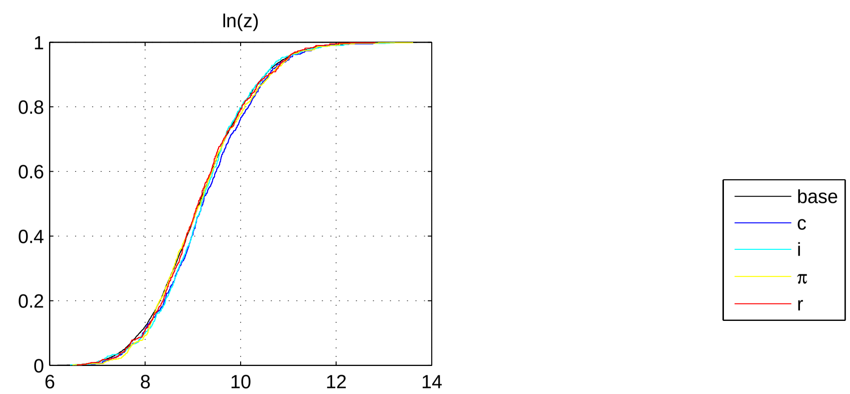

Figure L.28: US: $\mathcal{M}_1$.Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

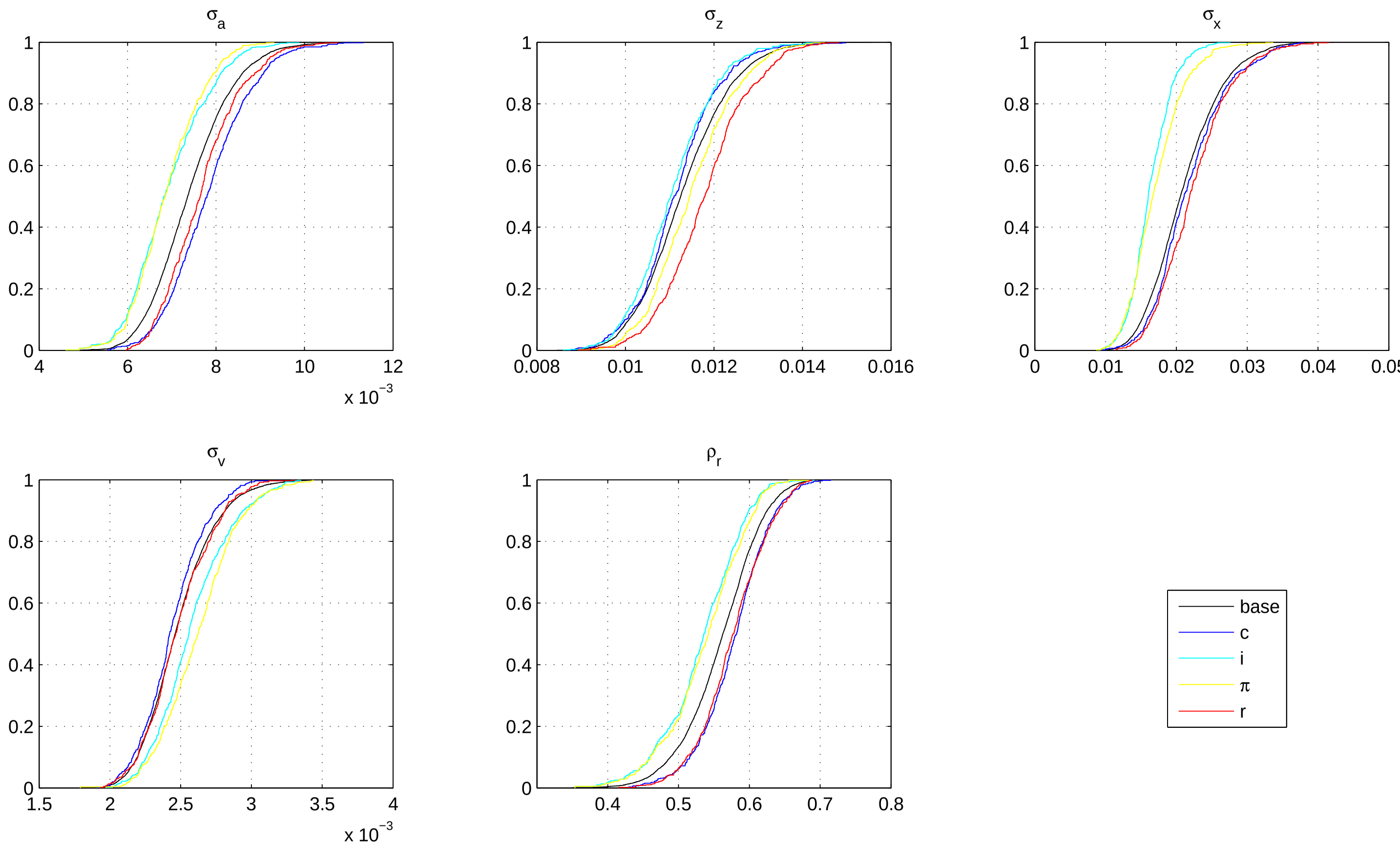

Figure L.29: US: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

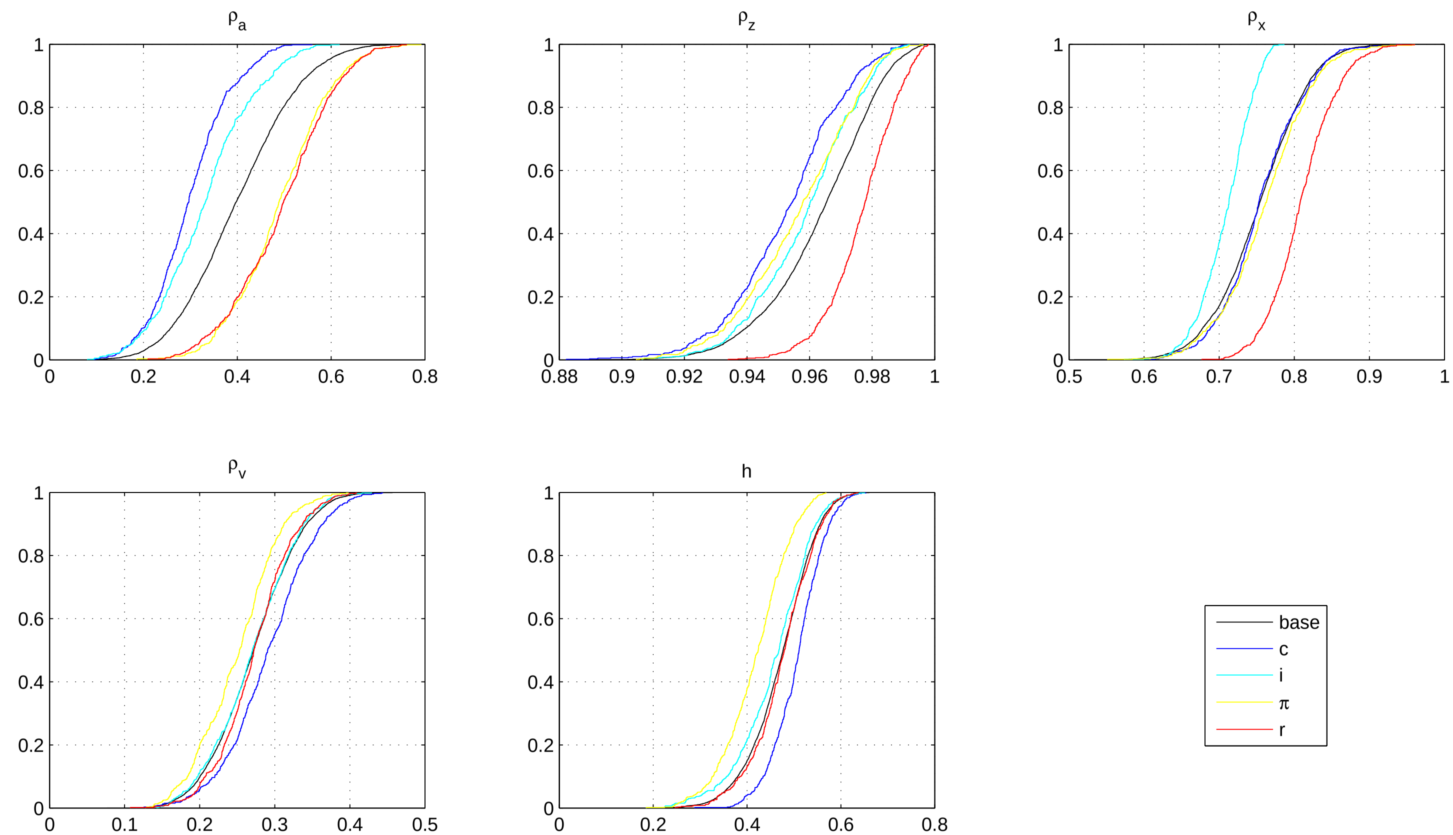

Figure L.30: US: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

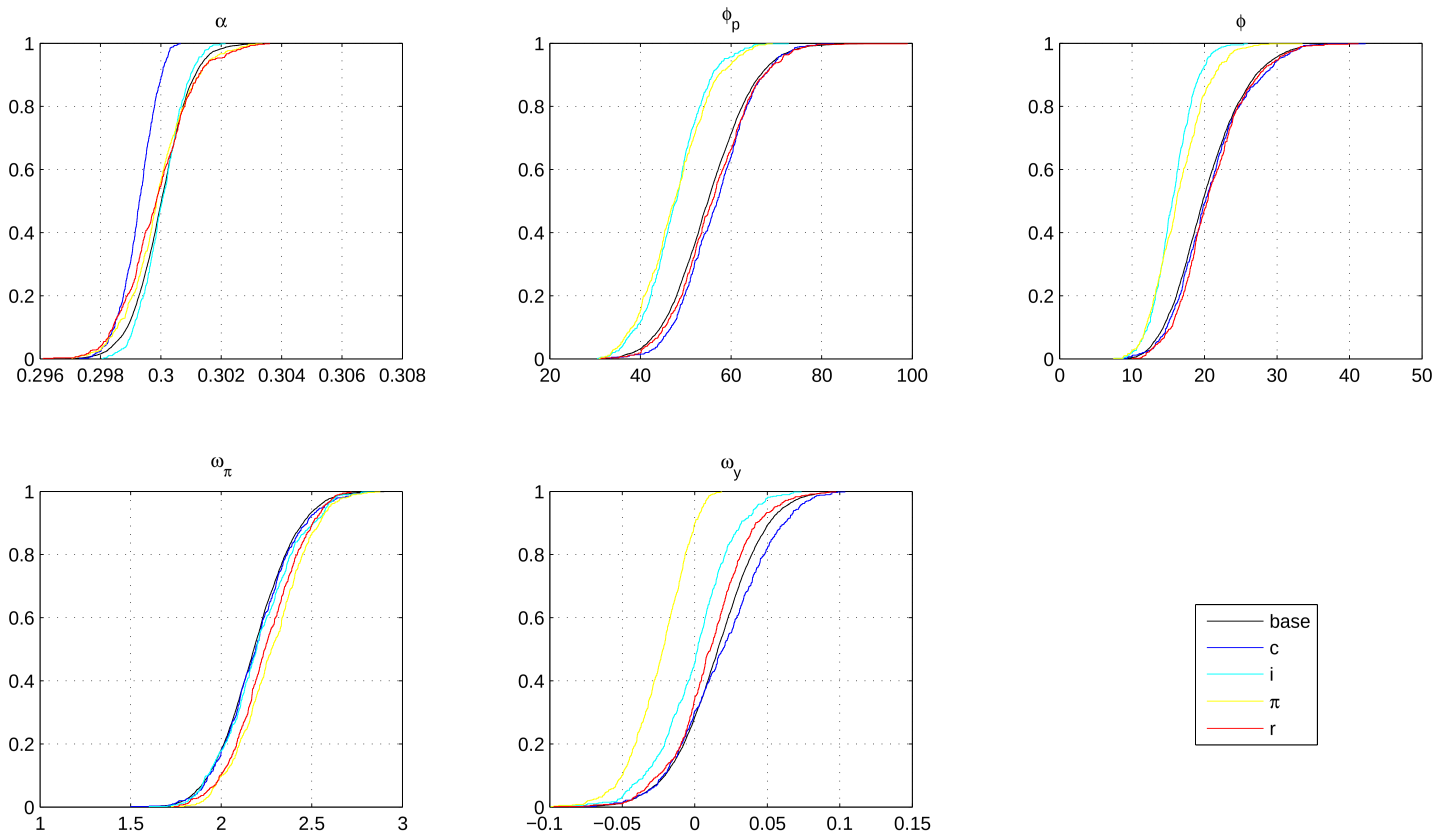

Figure L.31: US: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

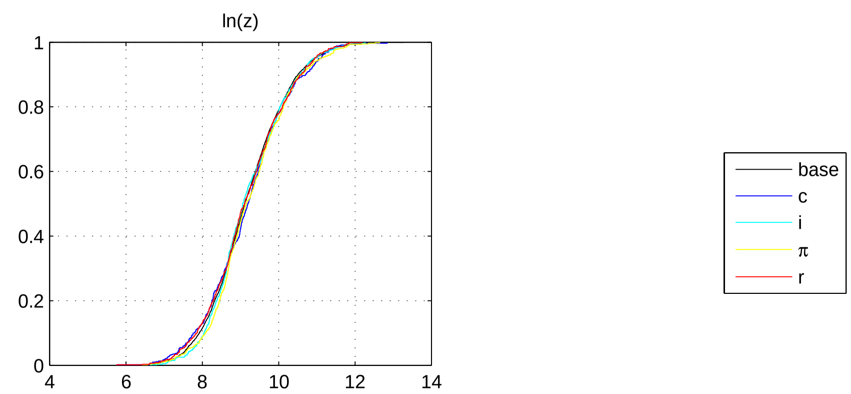

Figure L.32: US: $\mathcal{M}_2$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

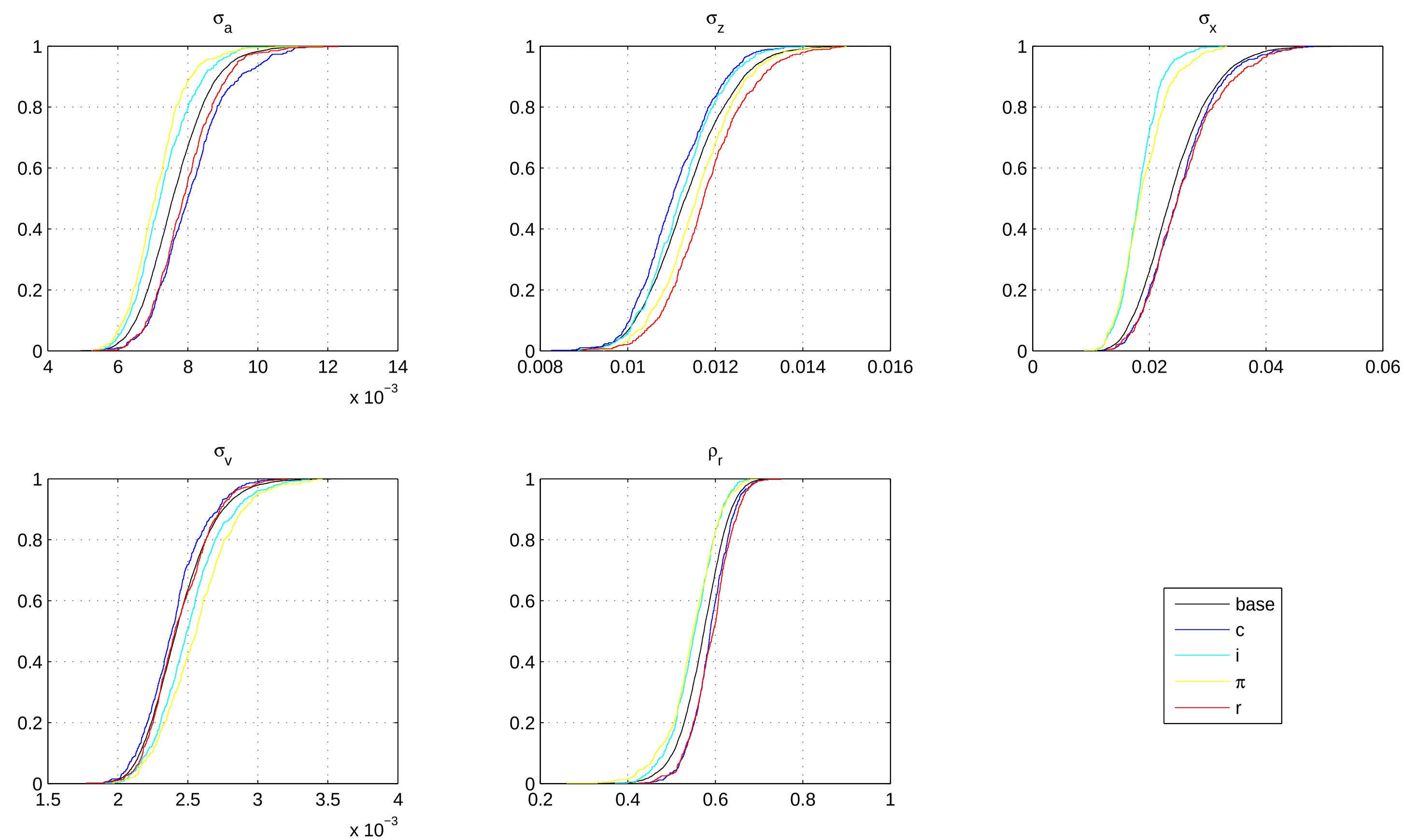

Figure L.33: US: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

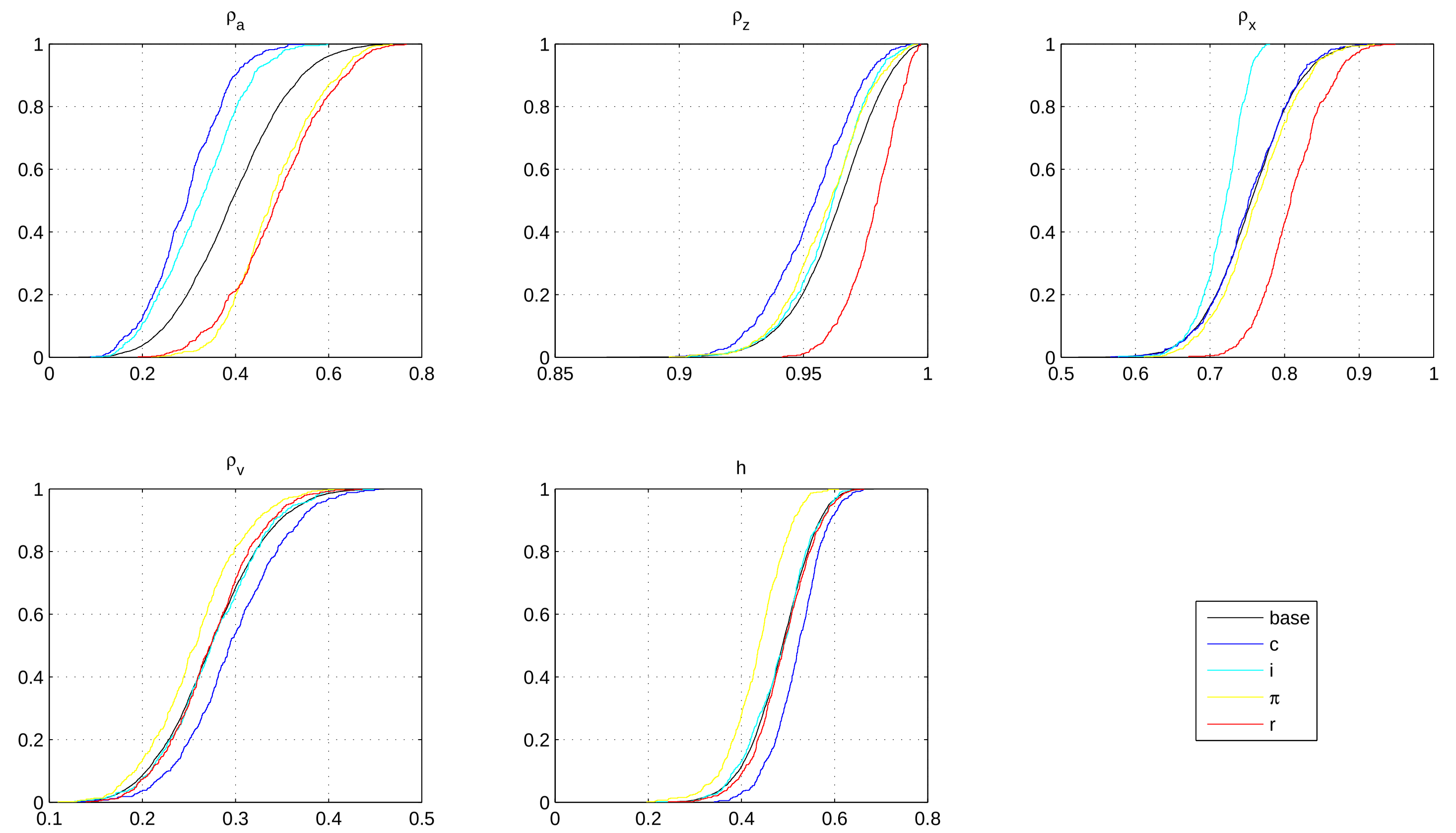

Figure L.34: US: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

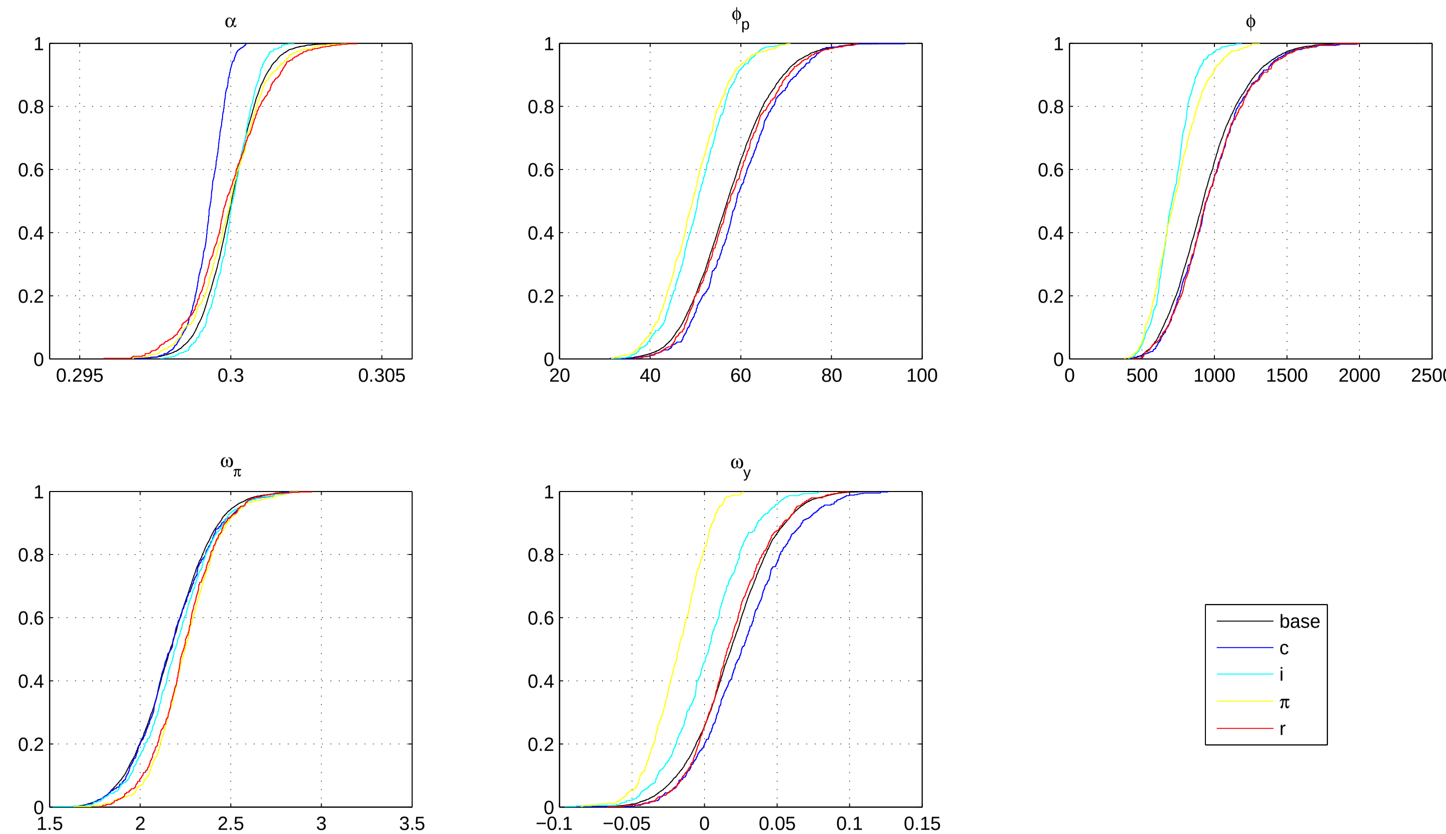

Figure L.35: US $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

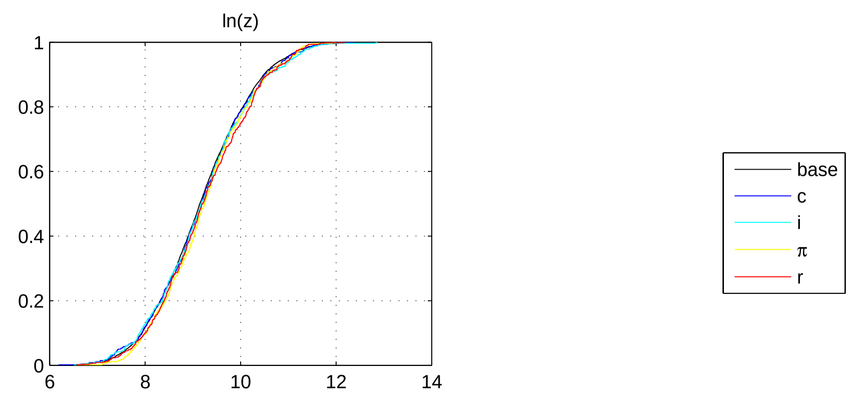

Figure L.36: US: $\mathcal{M}_3$. Cumulative empirical probability distributions of the filtered samples corresponding to the best fit for each observed time series (c, i, π, r) and cumulative posterior probability distributions (base).

	c	i	π	r
σ_a	4.230E-28	0.113	4.490E-29	5.490E-21
σ_z	3.000E-54	3.030E-38	1.120E-114	2.000E-59
σ_x	7.190E-10	1.290E-76	0.756	4.860E-09
σ_v	1.150E-06	71.700	9.110E-08	5.350
ρ_r	2.840E-05	0.168	6.560E-39	1.770E-11
ρ_a	1.350E-04	0.289	4.550E-103	0.028
ρ_z	5.380E-50	2.280E-06	3.000	1.760E-19
ρ_x	3.390	3.560E-128	8.040	0.185
ρ_v	4.100E-26	92.800	2.860E-149	8.000E-146
h	8.280E-128	6.870E-20	8.700E-240	5.030E-45
α	5.050E-115	0.003	13.900	1.160
ϕ_P	9.220E-48	16.200	7.710E-134	1.920E-19
ϕ	1.220E-12	2.350E-46	0.002	7.080E-22
ω_π	0.348	1.210E-06	0.197	0.004
ω_y	1.360	3.950E-89	2.490E-09	6.910E-148
$\ln(z)$	15.000	9.360	26.100	28.500

Table L.2: Euro area: $\mathcal{M}_1$. Cells contain the p-values (in percent) of the Smirnov two-sample test. Question answered by the test: "At what significance level α is the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution rejected?" Gray cells indicate p-values$<0.1\%$.

	c	i	π	r
σ_a	7.390E-32	4.110E-07	4.390E-47	0.070
σ_z	1.970E-29	1.040E-18	2.340E-71	3.010E-16
σ_x	1.440E-20	0.006	1.860E-61	2.140E-25
σ_v	3.300E-07	1.020E-20	0.083	0.011
ρ_r	2.220E-05	1.660E-05	1.150E-34	1.900E-21
ρ_a	1.280E-14	2.670E-73	4.820E-05	8.900E-18
ρ_z	5.160E-71	1.350E-04	0.055	0.564
ρ_x	4.450E-02	6.680E-201	1.160	1.500E-57
ρ_v	2.480E-21	15.300	5.040E-215	2.490E-85
h	9.310E-77	4.510E-22	6.580E-158	2.980E-29
α	6.710E-128	6.410E-18	10.400	22.700
ϕ_P	1.820E-32	6.860E-40	1.620E-220	2.340E-05
ϕ	8.590E-30	3.560	1.300E-57	4.870E-19
ω_π	0.017	1.370E-10	2.150E-10	0.271
ω_y	1.690E-07	14.700	0.012	1.330E-70
$\ln(z)$	78.700	0.516	30.500	71.000

Table L.3: Euro area: $\mathcal{M}_2$. Cells contain the p-values (in percent) of the Smirnov two-sample test. Question answered by the test: "At what significance level α is the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution rejected?" Gray cells indicate p-values$<0.1\%$.

	c	i	π	r
σ_a	9.490E-28	5.300E-05	3.030E-49	0.197
σ_z	1.060E-32	3.240E-08	1.730E-57	1.520E-22
σ_x	2.080E-28	0.003	1.560E-49	1.020E-20
σ_v	1.450E-11	2.090E-15	4.180E-05	0.002
ρ_r	3.160E-10	1.360E-05	2.930E-42	7.710E-27
ρ_a	2.240E-07	1.730E-59	0.002	8.180E-25
ρ_z	9.010E-75	2.150E-10	3.360E-10	6.720E-05
ρ_x	0.051	4.120E-162	3.390	1.810E-53
ρ_v	4.990E-26	0.142	5.160E-209	3.400E-75
h	2.770E-89	2.590E-18	2.040E-158	1.060E-32
α	9.040E-113	1.440E-20	1.260	0.298
ϕ_P	2.960E-31	3.630E-39	2.520E-226	3.240E-08
ϕ	5.830E-39	28.500	9.820E-59	3.230E-21
ω_π	0.005	3.940E-11	4.700E-06	5.480
ω_y	5.480E-09	23.500	6.720E-06	3.180E-33
$\ln(z)$	28.000	31.500	0.247	0.133

Table L.4: Euro area: $\mathcal{M}_3$. Cells contain the p-values (in percent) of the Smirnov two-sample test. Question answered by the test: "At what significance level α is the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution rejected?" Gray cells indicate p-values$< 0.1\%$.

	c	i	π	r
σ_a	5.540E-37	6.960E-18	3.040E-46	0.016
σ_z	63.000	2.610E-60	5.770E-19	1.920E-24
σ_x	2.340E-04	4.310E-09	1.450E-24	1.790E-07
σ_v	0.317	0.001	1.420E-06	50.200
ρ_r	4.240E-06	2.950E-06	3.030E-11	4.120E-16
ρ_a	5.010E-134	3.450E-05	2.100E-93	4.330E-68
ρ_z	7.080E-22	25.700	5.710E-17	2.770E-17
ρ_x	39.300	1.180E-96	7.440E-06	9.900E-25
ρ_v	8.680E-10	2.830E-11	8.590E-30	4.190E-04
h	2.570E-56	1.350E-13	4.120E-50	28.000
α	3.840E-188	6.110E-08	20.400	0.006
ϕ_P	13.900	4.140E-58	3.480E-19	2.770E-13
ϕ	3.790E-05	6.550E-21	4.990E-26	4.110E-07
ω_π	0.532	3.070E-04	1.470E-03	1.780E-04
ω_y	0.006	5.100E-30	2.260E-155	6.680E-11
$\ln(z)$	6.290	11.100	49.500	86.200

Table L.5: US: $\mathcal{M}_1$. Cells contain the p-values (in percent) of the Smirnov two-sample test. Question answered by the test: "At what significance level α is the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution rejected?" Gray cells indicate p-values$< 0.1\%$.

	c	i	π	r
σ_a	2.050E-20	2.520E-32	2.760E-36	1.340E-08
σ_z	0.010	7.390E-05	9.590E-04	4.980E-29
σ_x	0.431	5.520E-124	2.920E-70	1.190E-08
σ_v	0.162	6.520E-15	5.880E-35	66.600
ρ_r	4.250E-13	1.830E-23	2.700E-14	6.860E-08
ρ_a	2.920E-101	2.070E-41	5.740E-69	1.060E-71
ρ_z	4.860E-44	4.920E-10	1.170E-13	4.070E-76
ρ_x	15.600	1.030E-99	0.548	1.340E-112
ρ_v	7.880E-12	89.500	8.720E-14	1.650
h	5.610E-31	7.26E-02	1.500E-57	17.500
α	1.910E-100	2.920	0.025	2.280E-08
ϕ_P	5.460E-04	2.300E-88	2.340E-71	5.480
ϕ	13.900	3.370E-121	9.150E-65	0.050
ω_π	60.100	3.070	9.270E-22	1.620E-09
ω_y	0.002	3.960E-33	2.130E-220	6.220E-04
$\ln(z)$	21.100	63.700	12.000	93.200

Table L.6: US: $\mathcal{M}_2$. Cells contain the p-values (in percent) of the Smirnov two-sample test. Question answered by the test: "At what significance level α is the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution rejected?" Gray cells indicate p-values$<$ 0.1%.

	c	i	π	r
σ_a	1.170E-16	2.770E-17	1.130E-40	1.890E-10
σ_z	1.430E-08	0.050	2.000E-07	2.230E-20
σ_x	0.002	3.060E-139	7.890E-88	0.001
σ_v	0.001	4.920E-10	4.980E-29	61.500
ρ_r	1.450E-11	1.380E-14	2.810E-18	4.120E-16
ρ_a	1.530E-92	1.400E-42	4.390E-67	1.600E-63
ρ_z	5.610E-31	1.350E-04	2.810E-04	8.920E-93
ρ_x	50.200	2.620E-105	0.756	2.160E-104
ρ_v	1.270E-16	74.500	7.880E-12	13.100
h	2.720E-30	28.000	5.400E-57	7.690
α	4.720E-111	0.898	1.990	9.790E-05
ϕ_P	2.580E-05	2.770E-66	1.130E-90	16.900
ϕ	1.650	1.280E-137	8.570E-83	1.440
ω_π	78.000	0.337	1.900E-21	9.690E-20
ω_y	1.490E-06	4.080E-31	4.540E-199	12.500
$\ln(z)$	83.200	70.300	2.440	10.800

Table L.7: US: $\mathcal{M}_3$. Cells contain the p-values (in percent) of the Smirnov two-sample test. Question answered by the test: "At what significance level α is the null hypothesis that $f_{\tilde{x}_j}(\mu_i|B)$ equals the respective posterior distribution rejected?" Gray cells indicate p-values$<$ 0.1%.

	Standard Deviations (Percent)			
	Consumption	Investment	Inflation	Interest Rate
Data	0.80	2.37	0.22	0.25
$\mathcal{M}_1$	2.59	6.35	0.32	0.23
$\mathcal{M}_2$	2.35	3.10	0.28	0.21
$\mathcal{M}_3$	2.43	3.10	0.27	0.21
	Autocorrelations (First-order)			
	Consumption	Investment	Inflation	Interest Rate
Data	0.84	0.87	0.21	0.87
$\mathcal{M}_1$	0.98	0.98	0.59	0.81
$\mathcal{M}_2$	0.98	0.92	0.50	0.82
$\mathcal{M}_3$	0.98	0.92	0.51	0.82

Table L.8: Euro area: Selected second moments.

	Standard Deviations (Percent)			
	Consumption	Investment	Inflation	Interest Rate
Data	1.05	6.32	0.22	0.34
$\mathcal{M}_1$	4.43	10.79	0.30	0.39
$\mathcal{M}_2$	3.45	6.69	0.28	0.36
$\mathcal{M}_3$	3.46	6.60	0.28	0.37
	Autocorrelations (First-order)			
	Consumption	Investment	Inflation	Interest Rate
Data	0.88	0.82	0.45	0.83
$\mathcal{M}_1$	0.99	0.94	0.53	0.81
$\mathcal{M}_2$	0.99	0.84	0.51	0.81
$\mathcal{M}_3$	0.99	0.84	0.53	0.82

Table L.9: US: Selected second moments.

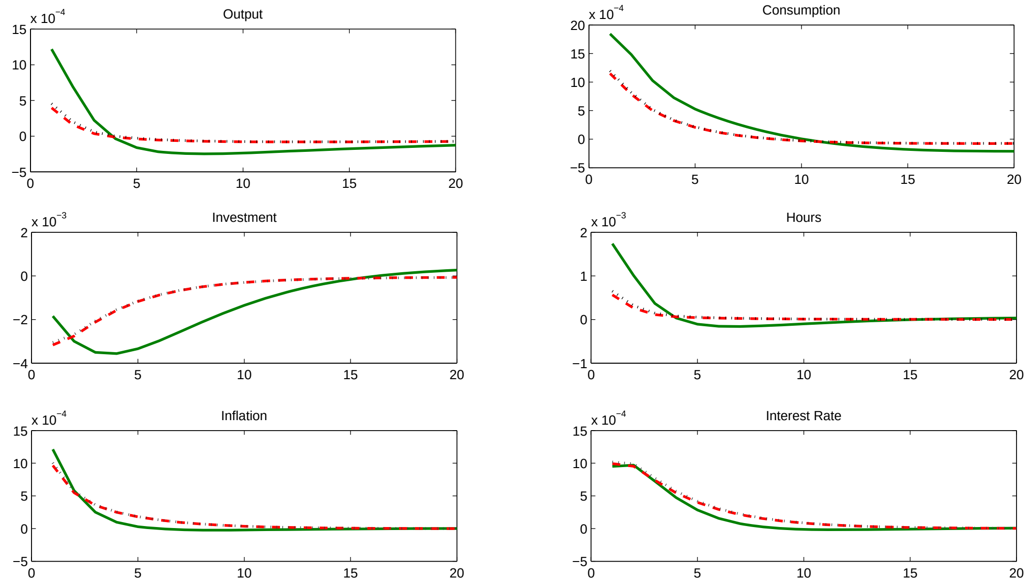

Figure L.37: Euro area: Impulse responses (log-deviations from the steady state) to a one standard deviation preference shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

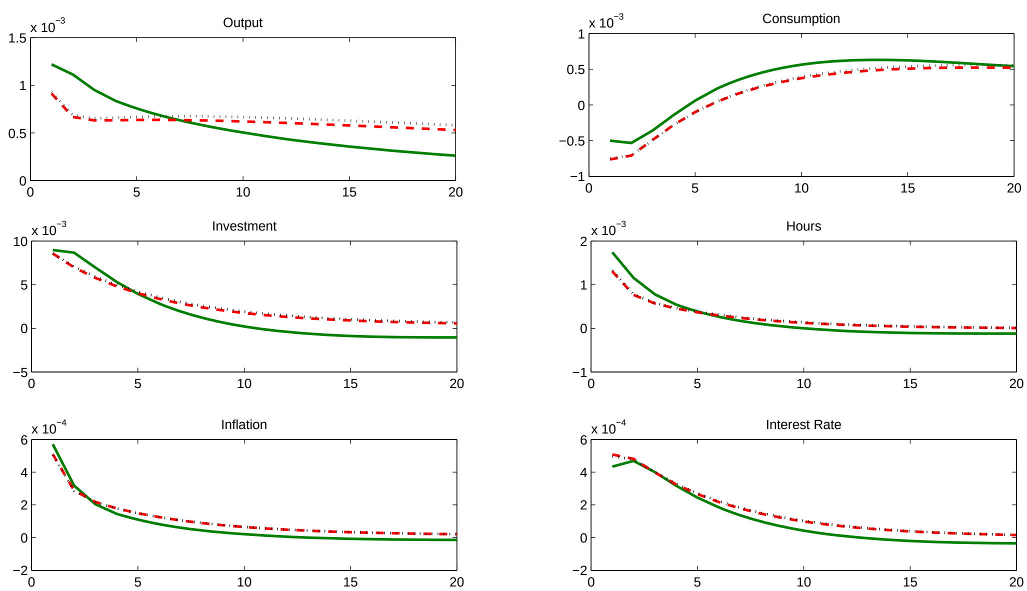

Figure L.38: Euro area: Impulse responses (log-deviations from the steady state) to a one standard deviation marginal efficiency of investment shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

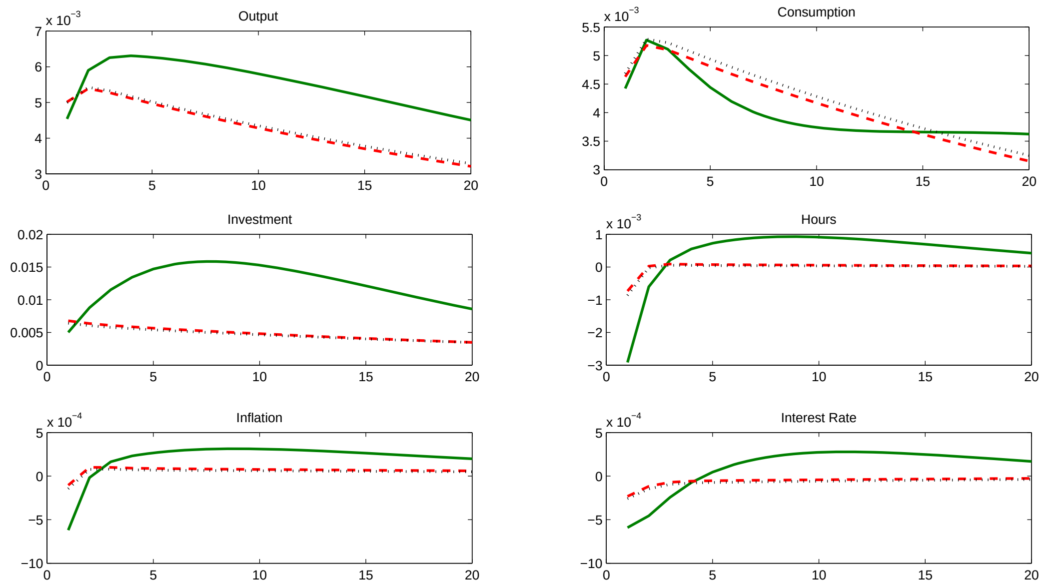

Figure L.39: Euro area: Impulse responses (log-deviations from the steady state) to a one standard deviation technology shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

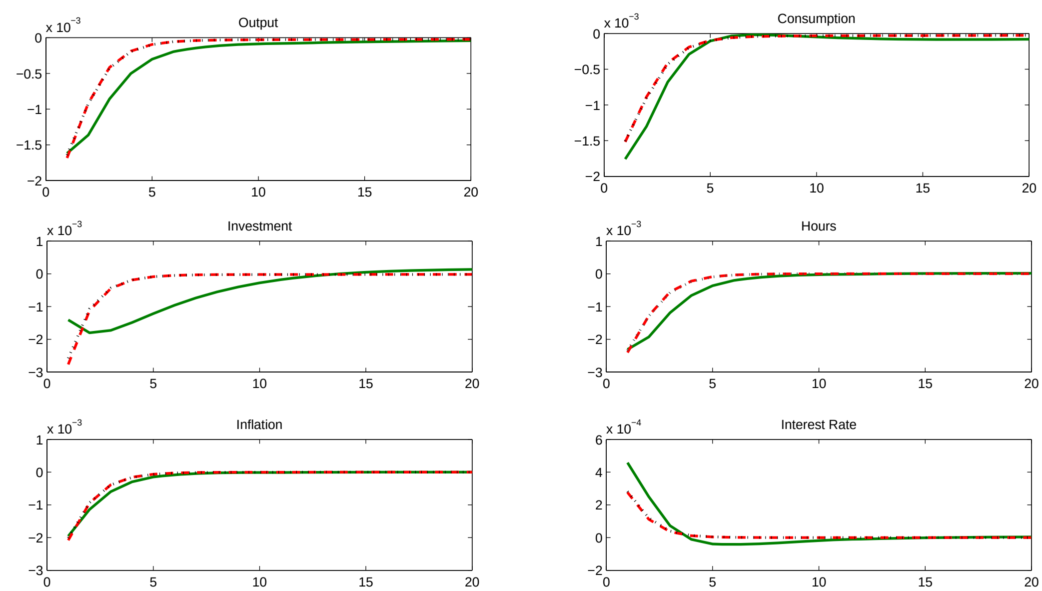

Figure L.40: Euro area: Impulse responses (log-deviations from the steady state) to a one standard deviation monetary policy shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

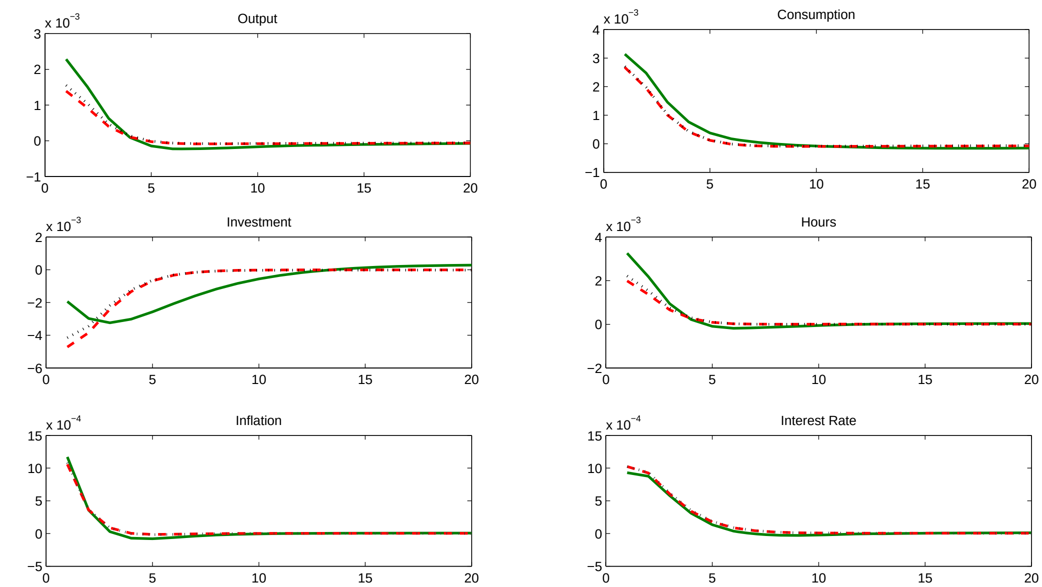

Figure L.41: US: Impulse responses (log-deviations from the steady state) to a one standard deviation preference shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

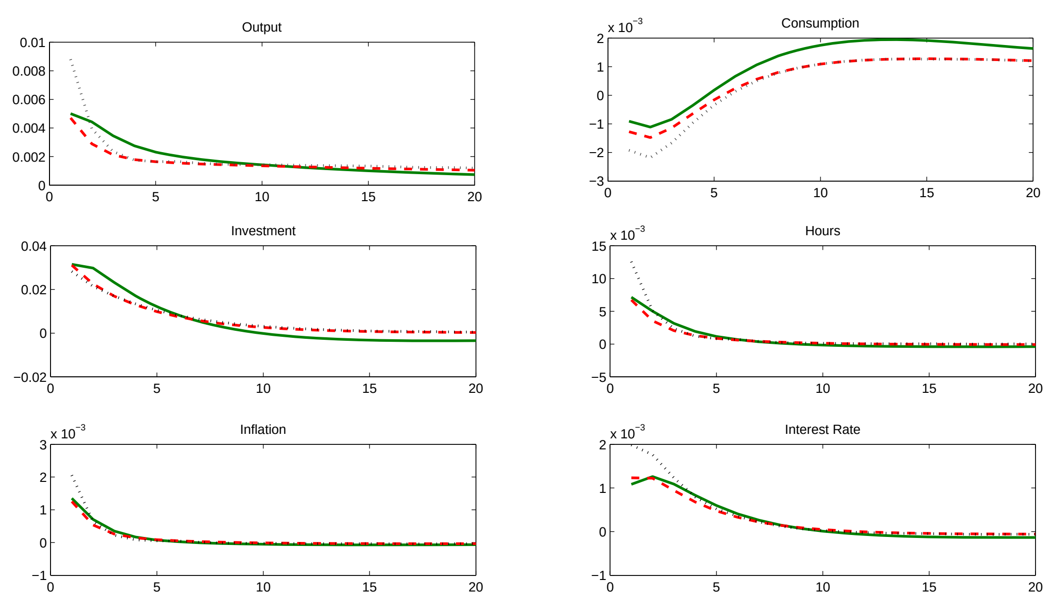

Figure L.42: US: Impulse responses (log-deviations from the steady state) to a one standard deviation marginal efficiency of investment shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

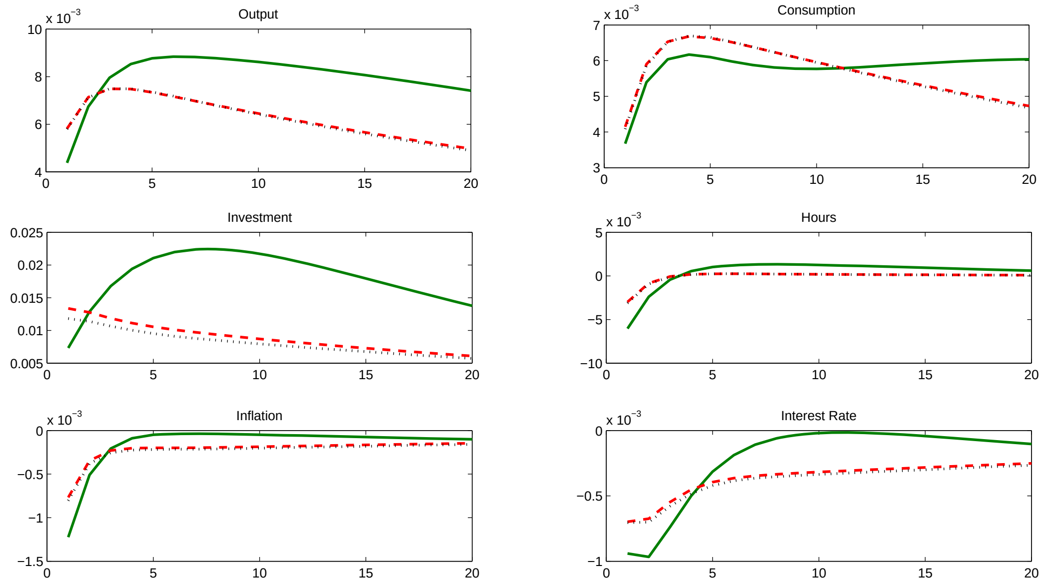

Figure L.43: US: Impulse responses (log-deviations from the steady state) to a one standard deviation technology shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

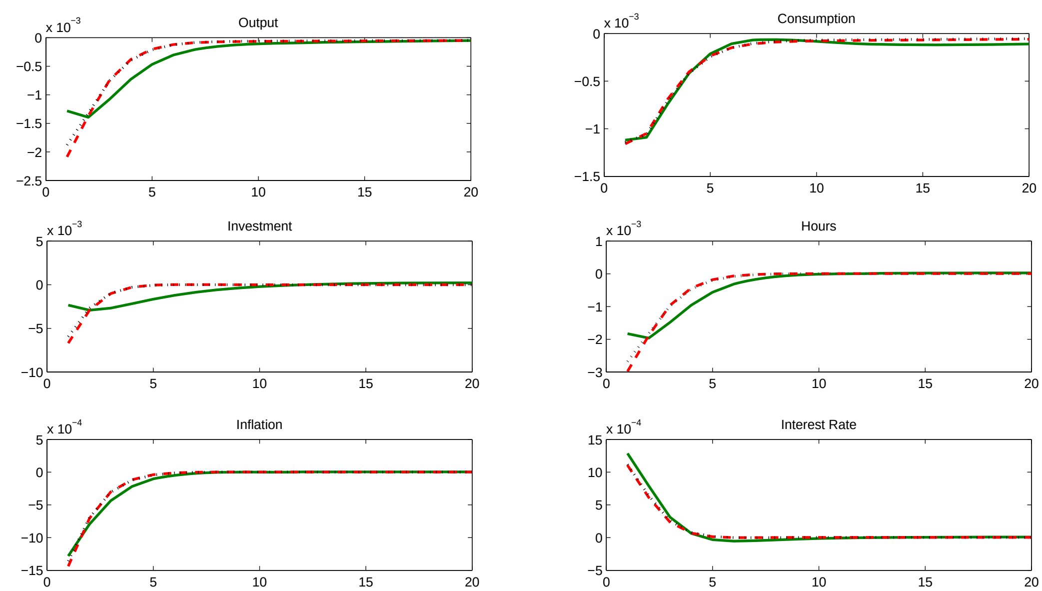

Figure L.44: US: Impulse responses (log-deviations from the steady state) to a one standard deviation monetary policy shock for 20 quarters; $\mathcal{M}_1$ (solid lines), $\mathcal{M}_2$ (dashed lines), $\mathcal{M}_3$ (dotted lines).

Variable	Fraction of the Variance Due to			
	Preference Shock	Investment Shock	Technology Shock	Monetary Policy Shock
		One-Year Horizon		
Output	1.33	2.89	91.88	3.91
Consumption	6.6	0.63	87.98	4.79
Investment	5.33	33.38	59.83	1.47
Hours	13.5	17.11	31.12	38.27
Inflation	22.07	5.69	5.54	66.7
Interest Rate	62.51	16.01	15.11	6.38
		Three-Year Horizon		
Output	0.54	1.6	96.49	1.36
Consumption	3.28	0.99	93.49	2.23
Investment	2.84	10	86.64	0.52
Hours	11.37	14.73	42.04	31.86
Inflation	20.24	5.49	12.88	61.39
Interest Rate	56.32	16.53	21.49	5.66
		Infinite Horizon		
Output	0.29	0.90	98.17	0.64
Consumption	1.29	1.20	96.71	0.80
Investment	1.92	7.03	90.70	0.35
Hours	10.09	13.62	48.22	28.07
Inflation	18.76	5.12	19.27	56.85
Interest Rate	49.89	15.07	30.03	5.01

Table L.10: Euro area: $\mathcal{M}_1$. Forecast error variance decomposition.

Variable	Fraction of the Variance Due to			
	Preference Shock	Investment Shock	Technology Shock	Monetary Policy Shock
		One-Year Horizon		
Output	0.15	1.79	94.53	3.53
Consumption	2.18	1.32	93.44	3.06
Investment	6.63	48.40	42.53	2.44
Hours	3.26	23.51	4.77	68.46
Inflation	19.31	5.71	0.54	74.44
Interest Rate	75.10	20.52	2.11	2.27
		Three-Year Horizon		
Output	0.08	1.85	96.63	1.45
Consumption	0.92	0.81	97.02	1.26
Investment	4.39	36.69	57.50	1.43
Hours	3.18	25.75	4.84	66.22
Inflation	19.71	6.52	1.20	72.57
Interest Rate	73.14	22.66	2.23	1.97
		Infinite Horizon		
Output	0.07	2.55	96.68	0.70
Consumption	0.47	2.15	96.79	0.59
Investment	2.91	25.18	70.97	0.94
Hours	3.18	25.78	4.99	66.05
Inflation	19.39	6.58	2.65	71.38
Interest Rate	72.72	22.76	2.57	1.95

Table L.11: Euro area: $\mathcal{M}_2$. Forecast error variance decomposition.

Variable	Fraction of the Variance Due to			
	Preference Shock	Investment Shock	Technology Shock	Monetary Policy Shock
		One-Year Horizon		
Output	0.21	1.89	94.53	3.37
Consumption	2.26	1.27	93.51	2.96
Investment	6.37	51.62	39.84	2.18
Hours	4.40	24.37	6.22	65.01
Inflation	21.47	5.86	0.55	72.12
Interest Rate	75.80	19.27	2.65	2.28
		Three-Year Horizon		
Output	0.10	2.01	96.52	1.37
Consumption	0.94	0.80	97.05	1.20
Investment	4.23	39.92	54.58	1.28
Hours	4.29	26.82	6.09	62.79
Inflation	21.90	6.71	0.97	70.42
Interest Rate	73.63	21.42	2.98	1.97
		Infinite Horizon		
Output	0.08	2.9	96.37	0.65
Consumption	0.47	2.34	96.63	0.56
Investment	2.74	27.19	69.25	0.82
Hours	4.29	26.89	6.14	62.68
Inflation	21.63	6.81	2.04	69.53
Interest Rate	72.93	21.44	3.69	1.95

Table L.12: Euro area: $\mathcal{M}_3$. Forecast error variance decomposition.

Variable	Fraction of the Variance Due to			
	Preference Shock	Investment Shock	Technology Shock	Monetary Policy Shock
		One-Year Horizon		
Output	2.78	22.63	72.62	1.97
Consumption	13.17	2.23	82.40	2.20
Investment	0.86	74.47	23.97	0.69
Hours	10.03	55.59	27.64	6.74
Inflation	17.91	29.53	21.96	30.60
Interest Rate	17.54	39.75	22.53	20.18
		Three-Year Horizon		
Output	0.88	9.44	89.03	0.65
Consumption	4.37	4.28	90.61	0.74
Investment	0.62	38.95	60.04	0.40
Hours	9.23	52.10	32.25	6.42
Inflation	17.93	29.54	22.00	30.53
Interest Rate	16.55	42.29	22.24	18.91
		Infinite Horizon		
Output	0.31	3.75	95.70	0.23
Consumption	0.98	3.48	95.36	0.18
Investment	0.42	27.77	71.53	0.27
Hours	8.65	50.30	35.04	6.02
Inflation	16.27	27.66	28.37	27.70
Interest Rate	13.93	37.67	32.50	15.90

Table L.13: US: $\mathcal{M}_1$. Forecast error variance decomposition.

Variable	Fraction of the Variance Due to			
	Preference Shock	Investment Shock	Technology Shock	Monetary Policy Shock
		One-Year Horizon		
Output	1.18	15.01	80.87	2.94
Consumption	7.60	3.38	87.12	1.91
Investment	1.67	73.46	22.86	2.02
Hours	6.80	66.83	10.95	15.43
Inflation	18.81	28.17	12.23	40.79
Interest Rate	24.48	43.60	15.06	16.86
		Three-Year Horizon		
Output	0.47	8.48	89.89	1.16
Consumption	2.59	2.40	94.33	0.67
Investment	1.26	61.00	36.25	1.50
Hours	6.68	67.06	11.10	15.16
Inflation	17.99	27.10	15.90	39.01
Interest Rate	21.83	41.90	21.50	14.77
		Infinite Horizon		
Output	0.22	6.22	93.02	0.54
Consumption	1.04	4.52	94.16	0.27
Investment	1.01	48.94	48.85	1.20
Hours	6.65	66.96	11.29	15.10
Inflation	16.39	25.15	22.91	35.55
Interest Rate	18.48	36.21	32.81	12.50

Table L.14: US: $\mathcal{M}_2$. Forecast error variance decomposition.

	Fraction of the Variance Due to			
Variable	Preference Shock	Investment Shock	Technology Shock	Monetary Policy Shock
		One-Year Horizon		
Output	1.47	16.49	79.38	2.65
Consumption	7.65	3.12	87.38	1.85
Investment	1.30	77.72	19.39	1.59
Hours	7.67	66.41	13.18	12.73
Inflation	19.36	28.54	14.96	37.14
Interest Rate	23.73	42.21	17.34	16.72
		Three-Year Horizon		
Output	0.58	9.54	88.83	1.05
Consumption	2.60	2.44	94.31	0.65
Investment	1.00	66.27	31.53	1.20
Hours	7.54	66.82	13.10	12.55
Inflation	18.32	27.15	19.38	35.15
Interest Rate	20.85	40.05	24.68	14.42
		Infinite Horizon		
Output	0.27	7.52	91.72	0.48
Consumption	1.06	5.34	93.33	0.27
Investment	0.81	53.88	44.34	0.97
Hours	7.52	66.78	13.18	12.52
Inflation	16.45	24.98	26.99	31.58
Interest Rate	17.47	34.43	36.02	12.08

Table L.15: US: $\mathcal{M}_3$. Forecast error variance decomposition.

Chapter 5

Conclusion

During the last three decades there has been a remarkable progress in the development of DSGE models. Starting with the influential work of Kydland and Prescott (1982) and Long and Plosser (1983), DSGE models rapidly attracted increasing interest among the profession thus becoming a centrepiece of modern macroeconomics. Among the vast body of literature that evolved in recent years two main categories can be identified:

i) Research, primarily dealing with the empirical implementation of DSGE models and

ii) literature with a particular focus on the exact specification of the underlying theoretical model.

Contributing to both categories, this thesis builds on the remarkable progress achieved in the DSGE research program so far, but also points out important problems and challenges that need to be addressed in the future.

Chapter 2 starts by presenting the general setup of DSGE models as well as techniques that enable their empirical implementation. For the sake of clarity, we focus on a standard New Keynesian model and expound the structure and solution for this prototype model. Further, we briefly introduce three common strategies used in the empirical analysis of DSGE models: calibration, maximum likelihood estimation, and Bayesian estimation. The objective of this chapter is to lay out the core features of the models used in the subsequent chapters and to introduce the estimation techniques employed.

In chapter 3, we apply an extended version of the standard New Keynesian model to French, German, Italian, and Spanish data and test for parameter stability over time. The model is estimated employing a maximum likelihood approach. Parameter instabilities are identified by the ESS procedure of Inoue and Rossi (2011). This procedure aims at detecting the parameters of a specific model that have changed at an unknown break date, overcoming the drawbacks known from "one at a time approaches" by allowing all parameters to be time-varying but, at the same time, avoids size distortions. For France, Germany, and Italy we find structural breaks in the mid-1990s after the beginning of the second stage of EMU, while the estimates for Spain show a significant break just before the start of the third stage of EMU in 1998. Concerning monetary policy behavior, France, Italy, and Spain show significant changes after the break dates, whereas monetary policy in Germany turns out to be stable over time. Moreover, France, Italy, and Spain exhibit a significant decline in capital and price adjustment costs after the break. Further, we find at least four out of the five shocks to be either constant or declining after the break date for all economies under consideration. The identification of changes in both policy and structural parameters might let the DSGE framework appear to be vulnerable to the Lucas (1976) critique. However, as outlined in Inoue and Rossi (2011, p. 1195), "... the definition of structural parameters (in the sense of the Lucas critique) is that these parameters are policy invariant, not necessarily time invariant." Therefore, future research faces an important challenge in developing techniques able to extract the factors responsible for parameter instabilities, allowing to assess the applicability of the respective DSGE setting for policy analysis and forecasting.

In chapter 4 we use a Bayesian approach to analyze the impact of distinct ad hoc specifications of adjustment costs on the fit and the dynamics of a New Keynesian model using both euro area and US data. Particularly, we consider three different theoretical specifications of quadratic adjustment costs of capital accumulation frequently used in the literature: an investment adjustment cost specification and two forms of capital adjustment costs. Our findings suggest that caution should be exercised when using estimated DSGE models with quadratic costs of capital accumulation for policy analysis, since the results could be affected by the choice of either investment or capital adjustment costs both being modeling shortcuts. Hence, our analysis provides further evidence for encouraging the efforts of a rigorous microfoundation of adjustment costs for capital accumulation.

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