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ANALYTIC VARIETIES WITH FINITE VOLUME AMOEBAS ARE ALGEBRAIC

FARID MADANI AND MOUNIR NISSE

Abstract. In this paper, we study the amoeba volume of a given $k-$dimensional generic analytic variety V of the complex algebraic torus $(\mathbb{C}^*)^n$. When $n \geq 2k$, we show that V is algebraic if and only if the volume of its amoeba is finite. Moreover, in this case, we establish a comparison theorem for the volume of the amoeba and the coamoeba. Examples and applications to the $k-$linear spaces will be given.

1. Introduction

Some fundamental questions concerning the complex logarithm lead us to study certain mathematical objects called amoebas and coamoebas, which are natural projections of complex varieties. They have strong relations to several other areas of mathematics such as real algebraic geometry, tropical geometry, complex analysis, mirror symmetry, algebraic statistics and several other areas. Amoebas degenerate to piecewise-linear objects called *tropical varieties*, (see [M1-02], [M2-04], [M3-00], [FPT-00], [NS-11], [PR-04] and [PS-04]). The behavior of an amoeba at the infinity is called *the logarithmic limit set* of the variety, and its analogous for coamoebas is called *the phase limit set* of the variety. For a k-dimensional complex algebraic variety, the phase limit set contains an arrangement of k-dimensional flat tori, which plays a crucial role in the geometry and the topology of both amoeba and coamoeba. Moreover, these objects are used as an intermediate link between complex algebraic geometry and tropical geometry. In this paper, we underline that the geometry of the amoeba affects the algebraic structure of the variety.

It was shown by Passare and Rullgård [PR-04] that the area of complex algebraic plane curve amoebas are finite. In [MN-11], we proved that the volume of the amoeba of a generic k-dimensional algebraic variety of the complex algebraic torus $(\mathbb{C}^*)^n$ with $n \geq 2k$, is finite. There now arises the reverse question: let $\mathcal{I}$ be an ideal in the ring of holomorphic functions on $\mathbb{C}^n$ and W be the set of zeros of $\mathcal{I}$, which we assume generic of dimension $k \leq \frac{n}{2}$, where the genericity in our context means that the Jacobian of logarithmic map restricted to our variety is of maximal rank. If we suppose that the

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volume of the amoeba of $V = W \cap (\mathbb{C}^*)^n$ is finite, then, is V algebraic? Theorem 1.1 gives an affirmative answer to this question.
The *Main theorem* of this paper is the following:

Theorem 1.1. *Let V be a generic k-dimensional analytic variety in $(\mathbb{C}^*)^n$ with $n \geq 2k$. The following assertions are equivalent:*
(i) The variety V is algebraic;
(ii) The volume of $\mathscr{A}(V)$ is finite.

This paper is organized as follows. In Section 3, we prove a comparison theorem, asserting that up to a rational number, the amoeba volume is bounded above and below by the coamoeba volume. In Section 4, using the geometry and the combinatorial structure of the logarithmic limit and the phase limit sets, we prove Theorem 1.1 in the special case of curves. The method used to prove this case is actually the crucial step to show the main result in general. In Section 5, we show Theorem 1.1 for varieties of higher dimensions. In Section 6, we give some examples of plane and spatial complex curves, underlying the importance of the finiteness of their amoeba areas. Finally, in Section 7, we give an application of the comparison theorem to the $k-$dimensional affine linear spaces in $(\mathbb{C}^*)^{2k}$. We compute the amoeba volumes of $k-$dimensional real affine linear spaces in $(\mathbb{C}^*)^{2k}$.

2. Preliminaries

Let W be a complex variety of $\mathbb{C}^n$ defined by an ideal $\mathcal{I}$ of holomorphic functions on $\mathbb{C}^n$. We say that a subvariety V of the complex algebraic torus $(\mathbb{C}^*)^n$ is analytic if there exists a complex variety W as above such that $V = W \cap (\mathbb{C}^*)^n$. All the analytic varieties considered in this paper are defined as above. The *amoeba* $\mathscr{A}$ of V is by definition (see M. Gelfand, M.M. Kapranov and A.V. Zelevinsky [GKZ-94]) the image of V under the map :

$$\begin{array}{rccl} \mathrm{Log} : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & (z_1, \ldots, z_n) & \longmapsto & (\log|z_1|, \ldots, \log|z_n|). \end{array}$$

It is well known that the amoeba of a variety of codimension one is closed and its complement components in $\mathbb{R}^n$ are convex (see [FPT-00]). A. Henriques gives an analogous definition for the convexity of amoeba complements of higher codimension varieties as follows:

Definition 2.1. *A subset $A \subset \mathbb{R}^n$ is called l-convex if for all oriented affine $(l+1)$-plane $L \subset \mathbb{R}^n$, the induced homomorphism $H_l(L \cap A) \longrightarrow H_l(A)$ does not send non-zero elements of $\tilde{H}_l^+(L \cap A)$ to zero, where $\tilde{H}_l(X)$ (resp. $\tilde{H}_l^+(X)$) denotes the reduced homology groups associated to the corresponding augmented complexes (resp. elements of $H_l(L \cap A)$ such that their image in $\tilde{H}_l(L \setminus p) \sim \mathbb{Z}$ are non-negative for all $p \in L \cap A$).*

Also, he proves that if V is a variety of codimension l, L is an l-plane of rational slope and c is a non-zero $(l-1)$-cycle in $H_{l-1}(L \setminus \mathscr{A})$, then its image in $H_{l-1}(\mathbb{R}^n \setminus \mathscr{A})$ is non-zero and $\mathbb{R}^n \setminus \mathscr{A}$ is $(l-1)$-convex (see [H-03] Theorem 4.1).

Let V be an analytic variety of the complex algebraic torus. We denote by $\mathscr{L}^\infty(V)$ its logarithmic limit set which is the boundary of the closure of

$r(\mathscr{A}(V))$ in the $n-$dimenstional ball B^n, where r is the map defined by (see Bergman [B-71]):

$$\begin{array}{rcrcl} r & : & \mathbb{R}^n & \longrightarrow & B^n \\ & & x & \longmapsto & r(x) = \frac{x}{1+|x|}. \end{array}$$

If V is an algebraic variety of dimension k, then its logarithmic limit set is a finite rational spherical polyhedron of dimension $k-1$ (i.e., a finite union of finite intersections of closed hemispheres and can be described in terms of a finite number of inequalities with integral coefficients). More precisely, we have the following structure theorem (see [B-71] and [BG-84]):

Theorem 2.1 (Bergman, Bieri-Groves)**.** *The logarithmic limit set* $\mathscr{L}^\infty(V)$ *of an algebraic variety* V *in* $(\mathbb{C}^*)^n$ *is a finite union of rational spherical polyhedrons. The maximal dimension of a polyhedron* P *in this union is such that* $\dim_{\mathbb{R}} P = \dim_{\mathbb{R}} \mathscr{L}^\infty(V) = \dim_{\mathbb{C}} V - 1$.

The argument map is the map defined as follows:

$$\begin{array}{rcrcl} \mathrm{Arg} & : & (\mathbb{C}^*)^n & \longrightarrow & (S^1)^n \\ & & (z_1,\ldots,z_n) & \longmapsto & (\arg(z_1),\ldots,\arg(z_n)). \end{array}$$

where $\arg(z_j) = \frac{z_j}{|z_j|}$. The *coamoeba* of V, denoted by $co\mathscr{A}$, is its image under the argument map (defined for the first time by Passare on 2004). On 2009, Sottile and the second author [NS-11] defined the *phase limit set* of V, $\mathscr{P}^\infty(V)$, as the set of accumulation points of arguments of sequences in V with unbounded logarithm. If V is an algebraic variety of dimension k, $\mathscr{P}^\infty(V)$ contains an arrangement of k-dimensional real subtori.

3. Comparison between Amoeba and Coamoeba volumes

For a given map f, we denote by $\mathrm{Jac}(f)$ the Jacobian matrix of f and by $\mathrm{J}(f)$ the determinant of $\mathrm{Jac}(f)$ when it exists.

Proposition 3.1. *Let* V *be a* k*-dimensional complex submanifold in* $(\mathbb{C}^*)^n$*. The maps* Log *and* Arg *are well defined on* V *and*

$$\partial \mathrm{Log} = \frac{1}{\mathrm{Arg}}\partial \mathrm{Arg}\,, \quad \bar{\partial}\mathrm{Log} = \frac{-1}{\mathrm{Arg}}\bar{\partial}\mathrm{Arg} \tag{1}$$

Proof. We denote by $\{z_j\}_{1\leq j\leq n}$ the complex coordinates on $\mathbb{C}^n$ and by $\{t_j\}_{1\leq j\leq k}$ the complex coordinates on V given by a local chart (Ω, f) (i.e. $\forall z\in\Omega,\ t_j = f_j(z)$), where Ω is an open set of V and f is a holomorphic function from an open set of $\mathbb{C}^n$ to $\mathbb{C}^k$. Since V is a complex submanifold of $(\mathbb{C}^*)^n$, the injection map $\imath : V \hookrightarrow (\mathbb{C}^*)^n$ is holomorphic. By definition, for any $z\in V$ we have $\imath(z) = e^{Log z}\mathrm{Arg}\, z$. Since $\imath$ is holomorphic, $\bar{\partial}\imath(z) = 0$ for any $z\in V$ (i.e. $\forall j\leq k,\ \partial_{\bar{t}_j}\imath(z) = 0$). It implies that for any $j = 1,\ldots,k$ and $z\in\Omega$ we have

$$\partial_{\bar{t}_j}\mathrm{Log}\,(z) = -\frac{1}{\mathrm{Arg}\,(z)}\partial_{\bar{t}_j}\mathrm{Arg}\,(z), \quad \partial_{t_j}\mathrm{Log}\,(z) = \frac{1}{\mathrm{Arg}\,(z)}\partial_{t_j}\mathrm{Arg}\,(z),$$

where the second equality holds by conjugating the first one. The statement of the proposition follows. □

An immediate consequence of Proposition 3.1 is the following:

Corollary 3.1. *Let V be a k-dimensional complex submanifold in $(\mathbb{C}^*)^n$. The set of critical points of the maps* Log *and* Arg *are the same and the map* $\operatorname{Log} \circ \operatorname{Arg}^{-1}$ *conserves locally the volumes (i.e. for any regular value $\theta \in co\mathscr{A}(V)$, we have $|\mathrm{J}(\operatorname{Log} \circ \operatorname{Arg}^{-1})(\theta)| = 1$).*

Proof. Let

$$S = \{z \in V \mid \text{rank Jac} \operatorname{Log}_z < \min(n, 2k)\}$$

be the set of critical points of Log. Using Proposition 3.1, we conclude that the Jacobian matrices of Log and Arg have the same rank at any point in V. Hence, the set of critical points of Arg is also S.
We know that there exists an open set $U \subset V \setminus S$ such that the map $\operatorname{Arg} : U \to co\mathscr{A}(V) \setminus \operatorname{Arg}(S)$ is a diffeomorphism. Using Proposition 3.1, it follows that $|\mathrm{J}(\operatorname{Log} \circ \operatorname{Arg}^{-1})(\theta)| = 1$ for any $\theta \in co\mathscr{A}(V) \setminus \operatorname{Arg} S$. □

As an application of corollary 3.1, we have the following comparison theorem.

Theorem 3.1. *Let V be a generic analytic variety of $(\mathbb{C}^*)^n$ of dimension $k \leq \frac{n}{2}$. Let $\mathscr{A}$, $co\mathscr{A}$ be the amoeba and coamoeba of V respectively. We suppose that* $\operatorname{Log} : V \to \mathscr{A}$ *and* $\operatorname{Arg} : V \to co\mathscr{A}$ *are locally finite coverings. We define the following two rational numbers*

$$p = \frac{\min\limits_{\theta \in co\mathscr{A} \setminus \operatorname{Arg} S} \#\operatorname{Arg}^{-1}\{\theta\}}{\max\limits_{y \in \mathscr{A} \setminus \operatorname{Log} S} \#\operatorname{Log}^{-1}\{y\}}, \quad P = \frac{\max\limits_{\theta \in co\mathscr{A} \setminus \operatorname{Arg} S} \#\operatorname{Arg}^{-1}\{\theta\}}{\min\limits_{y \in \mathscr{A} \setminus \operatorname{Log} S} \#\operatorname{Log}^{-1}\{y\}}.$$

Then,

$$p\, vol(co\mathscr{A}) \leq vol(\mathscr{A}) \leq P\, vol(co\mathscr{A}).$$

In particular, the volume of $\mathscr{A}$ is finite.

We have the following result that the authors had already proven in [MN-11] using another method:

Corollary 3.2. *The amoeba of a k-dimensional generic complex algebraic variety in $(\mathbb{C}^*)^n$, with $2k \leq n$, has finite volume.*

Proof. If V is a generic complex algebraic variety in $(\mathbb{C}^*)^n$ of dimension $k \leq \frac{n}{2}$, then $\operatorname{Log} : V \to \mathscr{A}$ and $\operatorname{Arg} : V \to co\mathscr{A}$ are locally finite coverings. Therefore, the statement follows from Theorem 3.1, since the volume of the coamoeba is always finite. □

Proof of Theorem 3.1. We have $V = V_{reg} \cup V_{sing}$, where V_{reg} is the regular part of V, which is a k-dimensional complex submanifold in $\mathbb{C}^n$ and V_{sing} is the singular part of V, which is an analytic subset of pure dimension less or equal to $k-1$. By Sard's theorem, the $2k$-measure of $\operatorname{Log}(V_{sing})$ and $\operatorname{Arg}(V_{sing})$ are zero. Hence, without loss of generality, we may assume that $V = V_{reg}$ is a k-dimensional complex submanifold endowed with the induced metric $\imath^*\mathcal{E}_{2n}$, where $\mathcal{E}_{2n}$ is the standard Euclidean metric on $\mathbb{C}^n$ and $\imath : V \to \mathbb{C}^n$ is the injection map. Let

$$S = \{z \in V \mid \text{rank Jac} \operatorname{Log}_z < 2k\}$$

be the set of critical points of Log. Using Proposition 3.1, we conclude that the Jacobian matrices of Log and Arg have the same rank at any point in V.

Hence, the set of critical points of Arg is also S. Using again Sard's theorem, the $2k$-measure of $\mathrm{Log}\, S$ and $\mathrm{Arg}\, S$ is zero. It yields $vol(\mathscr{A}) = vol(\mathscr{A}\setminus \mathrm{Log}\, S)$ and $vol(co\mathscr{A}) = vol(co\mathscr{A} \setminus \mathrm{Arg}\, S)$. The sets $\mathscr{A} \setminus \mathrm{Log}\, S$ and $co\mathscr{A} \setminus \mathrm{Arg}\, S$ are $2k$-dimensional real immersed submanifolds of $\mathbb{R}^n$ and $\mathbb{T}^n$ respectively. They are endowed with the induced metric $\imath^*\mathcal{E}_n$. Let U_1 and U_2 be two open sets in $V \setminus S$ such that $\mathrm{Log} : U_1 \to \mathscr{A} \setminus \mathrm{Log}\, S$ and $\mathrm{Arg} : U_2 \to co\mathscr{A} \setminus \mathrm{Arg}\, S$ are diffeomorphisms. By construction, we have the following identities:

$$vol(\mathscr{A}) = vol(\mathscr{A} \setminus \mathrm{Log}\, S, \imath^*\mathcal{E}_n) = vol(U_1, \mathrm{Log}^*\mathcal{E}_n),$$
$$vol(co\mathscr{A}) = vol(co\mathscr{A} \setminus \mathrm{Arg}\, S, \imath^*\mathcal{E}_n) = vol(U_2, \mathrm{Arg}^*\mathcal{E}_n).$$

Following the construction in [MN-11] (see Section 3) and the identities above, the volume of $\mathscr{A}$ and $co\mathscr{A}$ are given by

$$vol(\mathscr{A}) = vol(U_1, \mathrm{Log}^*\mathcal{E}_n) = \int_{U_1} \left|\psi_{2k}\right|_{\mathrm{Log}^*\mathcal{E}_n} \mathrm{d}v, \tag{2}$$

$$vol(co\mathscr{A}) = vol(U_2, \mathrm{Arg}^*\mathcal{E}_n) = \int_{U_2} \left|\psi_{2k}\right|_{\mathrm{Arg}^*\mathcal{E}_n} \mathrm{d}v, \tag{3}$$

where $\mathrm{d}v$ is the restriction of the volume form of $\mathbb{C}^n$ to V and ψ_{2k} is the $2k-$vector field in $\Lambda^{2k}TV$, such that $\mathrm{d}v(\psi_{2k}) = 1$. In local complex coordinates $\{t_j\}_{1\leq j\leq k}$, the volume form $\mathrm{d}v$ and ψ_{2k} are given by

$$\mathrm{d}v = i^k \mathrm{d}t \wedge \mathrm{d}\bar{t}, \qquad \psi_{2k} = (-i)^k \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial \bar{t}},$$

where $\mathrm{d}t = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_k$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_k}$. It yields that

$$\left|\psi_{2k}\right|^2_{\mathrm{Log}^*\mathcal{E}_n} = \left|\frac{\partial \mathrm{Log}}{\partial t} \wedge \frac{\partial \mathrm{Log}}{\partial \bar{t}}\right|^2_{\mathcal{E}_n} = \sum_{I\subset\{1,\dots,n\}, \#I=2k} \left|\mathrm{J}(\mathrm{Log}_I)\right|^2,$$

$$\left|\psi_{2k}\right|^2_{\mathrm{Arg}^*\mathcal{E}_n} = \left|\frac{\partial \mathrm{Arg}}{\partial t} \wedge \frac{\partial \mathrm{Arg}}{\partial \bar{t}}\right|^2_{\mathcal{E}_n} = \sum_{I\subset\{1,\dots,n\}, \#I=2k} \left|\mathrm{J}(\mathrm{Arg}_I)\right|^2,$$

where $\mathrm{Log}_I(z) = (\log|z_{i_1}|, \dots, \log|z_{i_{2k}}|)$ for all $z \in V$ and $I = \{i_1, \dots, i_{2k}\}$ and $\mathrm{J}(\mathrm{Log}_I)$ is the Jacobian determinant of Log_I with respect to $\{t_j, \bar{t}_j\}_{1\leq j\leq k}$. We have from Proposition 3.1, $\left|\mathrm{J}(\mathrm{Log}_I)\right| = \left|\mathrm{J}(\mathrm{Arg}_I)\right|$ for any $I \subset \{1, \dots, n\}$ of cardinality $2k$. We deduce that

$$\left|\psi_{2k}\right|_{\mathrm{Log}^*\mathcal{E}_n} = \left|\psi_{2k}\right|_{\mathrm{Arg}^*\mathcal{E}_n}.$$

It remains to compare between the volume of $\mathscr{A}$ and $co\mathscr{A}$, given by (2), (3). Since $\mathrm{Log} : V \to \mathscr{A}$ and $\mathrm{Arg} : V \to co\mathscr{A}$ are locally finite coverings,

$$\frac{1}{M_1}\int_{V\setminus S} \left|\psi_{2k}\right|_{\mathrm{Log}^*\mathcal{E}_n} \mathrm{d}v \leq vol(\mathscr{A}) \leq \frac{1}{m_1}\int_{V\setminus S} \left|\psi_{2k}\right|_{\mathrm{Log}^*\mathcal{E}_n} \mathrm{d}v,$$
$$\frac{1}{M_2}\int_{V\setminus S} \left|\psi_{2k}\right|_{\mathrm{Arg}^*\mathcal{E}_n} \mathrm{d}v \leq vol(co\mathscr{A}) \leq \frac{1}{m_2}\int_{V\setminus S} \left|\psi_{2k}\right|_{\mathrm{Arg}^*\mathcal{E}_n} \mathrm{d}v,$$

where $M_1 = \max_{y\in\mathscr{A}\setminus\mathrm{Log}\, S} \#\mathrm{Log}^{-1}\{y\}$, $m_1 = \min_{y\in\mathscr{A}\setminus\mathrm{Log}\, S} \#\mathrm{Log}^{-1}\{y\}$, $M_2 = \max_{\theta\in\mathscr{A}\setminus\mathrm{Arg}\, S} \#\mathrm{Arg}^{-1}\{\theta\}$ and $m_2 = \min_{\theta\in\mathscr{A}\setminus\mathrm{Arg}\, S} \#\mathrm{Arg}^{-1}\{\theta\}$. We conclude

that
$$\frac{m_2}{M_1} vol(co\mathscr{A}) \leq vol(\mathscr{A}) \leq \frac{M_2}{m_1} vol(co\mathscr{A}).$$
□

4. Analytic curves with finite area amoebas are algebraic

The purpose of this section is to prove the following:

Theorem 4.1. *Let $\mathscr{C} \subset (\mathbb{C}^*)^n$ be a generic analytic curve defined by an ideal of holomorphic functions $\mathcal{I}(\mathscr{C})$ with $n \geq 2$. The curve $\mathscr{C}$ is algebraic if and only if the area of its amoeba is finite*

In order to prove Theorem 4.1, we start by proving a series of three lemmas.

Lemma 4.1. *Let $\mathscr{C}$ be a curve as in Theorem 4.1, and assume that the area of its amoeba is finite. Then, the logarithmic limit set $\mathscr{L}^\infty(\mathscr{C})$ has the following properties:*

(i) *Each point $s \in \mathscr{L}^\infty(\mathscr{C})$ has a rational slope (i.e., there exists $\lambda \in \mathbb{R}$ such that $\lambda \cdot \overrightarrow{Os} \in \mathbb{Z}^n$);*
(ii) *The logarithmic limit set $\mathscr{L}^\infty(\mathscr{C})$ is a finite set.*

Proof. (i) Assume on the contrary that $\mathscr{L}^\infty(\mathscr{C})$ contains a point s, such that the vector $\overrightarrow{Os}$ has an irrational slope. Let $s = (u_1, \dots, u_n)$ and
$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R}^n \\ x &\longmapsto (u_1 x + a_1, \dots, u_n x + a_n) \end{aligned}$$
be the parametrization of the straight line D_s in $\mathbb{R}^n$ of direction s, and asymptotic to the amoeba $\mathscr{A}(\mathscr{C})$. Under this assumption, the phase limit set $\mathscr{P}^\infty(\mathscr{C})$ contains a subset of dimension at least two. Indeed, there exists an affine line in the universal covering of the real torus $(S^1)^n$, parametrized by $y \mapsto (u_1 y_1 + b_1, \dots, u_n y_n + b_n)$, such that the closure of its projection in $(S^1)^n$ is a 2-dimensional torus T_s contained in the phase limit set $\mathscr{P}^\infty(\mathscr{C})$. This implies that there exists a regular open subset $U \subset co\mathscr{A}(\mathscr{C})$, which is covered infinitely many times under the argument map. Namely, if we denote by Arg (resp. Log) the restriction of the argument (resp. logarithmic) map to the curve $\mathscr{C}$, i.e., $\mathrm{Arg} = \mathrm{Arg}_{|\mathscr{C}}$ (resp. $\mathrm{Log} = \mathrm{Log}_{|\mathscr{C}}$), then $\mathrm{Arg}^{-1}(U) = \cup_{i=1}^\infty V_i$, where V_i are open, regular and disjoint sets in $\mathscr{C}$. In fact, let U be a regular open subset of $co\mathscr{A}(\mathscr{C})$ contained in T_s. Let $\tilde{V}_1$ be a connected component of $\mathrm{Arg}^{-1}(U)$ and $V_1 = \mathrm{Log}(\tilde{V}_1)$. Let I_2 be a segment in $D_s \setminus \overline{V}_1$, such that $\mathrm{Arg}(\mathrm{Log}^{-1}(I_2))$ intersects U. Indeed, I_2 exists because the immersed circle of slope s in the real torus $(S^1)^n$ is dense in T_s. Let $\tilde{V}_2$ be the connected component of $\mathrm{Arg}(\mathrm{Log}^{-1}(U))$, such that $\mathrm{Log}(\tilde{V}_2)$ contains I_2 and $V_2 = \mathrm{Log}(\tilde{V}_2)$. By construction, $V_1 \cap V_2$ is empty. We do the same thing to V_2, and we obtain an infinite sequence of open subsets $V_1, V_2, \dots$ in the amoeba, such that $V_i \cap V_j = \phi$ for each $i \neq j$ and the area of V_i is equal to the area of U for each i, since the map $\mathrm{Log} \circ \mathrm{Arg}^{-1}$ conserves the volume (see Corollary 3.1). This is a contradiction with the finiteness of the area of the amoeba. Hence, s cannot be irrational.

(ii) If $\mathscr{L}^\infty(\mathscr{C})$ is not finite, then it contains an accumulation point; because S^{n-1} is compact. Let $s \in \mathscr{L}^\infty(\mathscr{C})$ be an accumulation point. Namely, there exists a sequence of points $s_m \in \mathscr{L}^\infty(\mathscr{C})$ such that $\lim_{m\to+\infty} s_m = s$, where each s_m has a rational slope and up to a multiplication by a real number, $\overrightarrow{Os_m} = (u_1^{(m)}, \dots, u_n^{(m)})$, with $u_j^{(m)} = \frac{a_j^{(m)}}{b_j^{(m)}}$, where $a_j^{(m)}$ and $b_j^{(m)}$ are integers. The sequence $\{s_m\}$ converges and is not stationary. This means that there exists $1 \leq j \leq n$ such that the sequence $\{b_j^{(m)}\}$ is unbounded. The circles corresponding to that slopes, have length greater or equal to $2\pi b_j^{(m)}$. Hence, if U is a regular subset of measure different from zero in the coamoeba, then it is covered infinitely many times under the argument map since the $b_j^{(m)}$ tend to infinity. This contradicts the fact that the area of the amoeba is finite. So, $\mathscr{L}^\infty(\mathscr{C})$ is finite.

□

Lemma 4.2. *Let $\mathscr{C} \subset (\mathbb{C}^*)^n$ be a generic analytic curve, such that the area of its amoeba is finite and let s be a point in $\mathscr{L}^\infty(\mathscr{C})$. The number of ends $\mathscr{E}$ of the amoeba $\mathscr{A}(\mathscr{C})$, such that $\overline{r(\mathscr{E})} \cap S^{n-1} = \{s\}$, is finite.*

Proof. Assume on the contrary that the number of ends $\mathscr{E}$ of the amoeba $\mathscr{A}(\mathscr{C})$, such that $\overline{r(\mathscr{E})} \cap S^{n-1} = \{s\}$, is infinite. Hence, two cases can hold. The first case: the number of corresponding circles of s in $(S^1)^n$ is finite. Then, there exists at least a circle C covered by the argument mapping infinitely many times. Take a regular open set $U \subset co\mathscr{A}(\mathscr{C})$ containing a segment in C. Then, the number of connected components of $\mathrm{Arg}^{-1}(U)$ is infinite and $\mathrm{Log}(\mathrm{Arg}^{-1}(U))$ has also an infinite number of connected components with the same area as U. This is in contradiction with the assumption.
The second case: the number of corresponding circles of s in $(S^1)^n$ is infinite. Then, there exists a circle C which is an accumulation circle with the same slope s, because the real torus is compact. Using the same reasoning as above, we obtain a contradiction.
This implies that the number of ends of the amoeba with the same slope is finite.

□

Lemma 4.3. *Let $\mathscr{C}$ be a generic curve in $(\mathbb{C}^*)^n$. If S^{n-2} is a subsphere of $S^{n-1} = \partial B^n$, invariant under the involution $-\mathrm{id}$, then $\mathscr{L}^\infty(\mathscr{C})$ intersects the interior of each connected component of $S^{n-1} \setminus S^{n-2}$.*

Proof. Using Henriques theorem [H-03], we know that the complement components of amoebas are $l-$convex, if $\mathrm{codim} V = l+1$. Lemma 4.3 is a consequence of this fact. If $\mathscr{C}$ is generic (it is not contained in a complex algebraic torus of smaller dimension), then the intersection of the closed half spaces in $\mathbb{R}^n$ bounded by the hyperplanes normal to the directions $s \in \mathscr{L}^\infty(\mathscr{C})$ is compact. □

Let $\mathscr{C} \subset (\mathbb{C}^*)^n$ be a generic analytic curve with defining ideal $\mathcal{I}(\mathscr{C})$ such that the area of its amoeba is finite. Let $s \in \mathscr{L}^\infty(\mathscr{C})$ with slope $(u_1, \dots, u_n)$, and D_s be a straight line in $\mathbb{R}^n$ directed by s and asymptotic to the amoeba $\mathscr{A}$.

We denote by $\mathscr{H}(D_s)$ the holomorphic cylinder which is the lifting of D_s and asymptotic to the end of $\mathscr{C}$ corresponding to s.

Lemma 4.4. *Let $\mathscr{C} \subset (\mathbb{C}^*)^n$ be a generic analytic curve as above. Then, the ideal $\mathcal{I}(\mathscr{C})$ is generated by a set of holomorphic functions $\{f_1, \dots, f_q\}$ such that for any $j = 1, \dots, q$, we have $f_j = h_j g_j$, where h_j are holomorphic functions without zeros in a neighborhood of $\mathscr{H}(D_s)$, and g_j are entire functions where the terms of their power series have powers in the closed half space $\{\alpha \in \mathbb{R}^n \mid \sum_{i=1}^n \alpha_i u_i \leq 0\}$.*

Proof. Without loss of generality, by Lemma 4.2, we may assume that the amoeba contains only one end corresponding to $s \in \mathscr{L}^\infty(\mathscr{C})$. If we do not have such a decomposition of the f_j's, then s is an accumulation point and this cannot happen because the area of the amoeba is finite by assumption. So, there exist holomorphic functions h_j, g_j and $U(s)$ a neighborhood of a holomorphic cylinder on which the h_j's do not vanish. Namely, the holomorphic cylinder is the lifting in $(\mathbb{C}^*)^n$ via the logarithmic map of a straight line in $\mathbb{R}^n$ of slope $(u_1, \dots, u_n)$. This holomorphic cylinder is such that the closure in B^n of the retraction by r (defined in Section 2) of the image under the logarithmic map of one of its ends, intersects the boundary of B^n precisely on s. The functions g_j's are entire and their power series expansion is of the form $\sum a_\alpha z^\alpha$ with $\sum_{i=1}^n u_i \alpha_i \leq 0$. Without loss of generality, we may assume that $\mathscr{C}$ is an irreducible curve. Hence, $\mathscr{C}$ is contained in the zero locus of the g_j's. In other words, our curve $\mathscr{C}$ is contained in the curve with defining ideal $\mathcal{I}_s$ spanned by the g_j's. □

Proof of Theorem 4.1. If the area of the amoeba of $\mathscr{C}$ is finite, then using Lemma 4.3, and doing the same thing for each vertex in $\mathscr{L}^\infty(\mathscr{C})$ as in Lemma 4.4 we obtain polynomials $p_1, \dots, p_q$ such that $\mathscr{C}$ is contained in $\mathscr{C}_r$, where $\mathscr{C}_r$ is the algebraic curve defined by the ideal generated by $p_1, \dots, p_q$. Hence, $\mathscr{C}$ is an irreducible component of $\mathscr{C}_r$. This means that $\mathscr{C}$ is algebraic. If the curve is algebraic, then using Corollary 3.2, we deduce that the area of the amoeba of $\mathscr{C}$ is finite. □

Corollary 4.1. *Let $\mathscr{C}$ be a generic analytic curve in $(\mathbb{C}^*)^n$. Then $\mathscr{L}^\infty(\mathscr{C})$ is the union of a finite number of isolated points with rational slopes and a finite number of geodesic arcs with rational end slopes. In particular, if $\mathscr{C}$ is not algebraic, then the number of arcs in $\mathscr{L}^\infty(\mathscr{C})$ is different from zero.*

This is a consequence of the proof of Lemma 4.1. The arcs are geodesic (i.e., contained in some circle invariant under the involution $x \mapsto -x$) because the contrary means that the phase limit set contains a flat torus of dimension at least three and this contradicts the fact that the dimension of the coamoeba is equal to two.

Corollary 4.2. *Let f be an entire function in two variables. There exist a holomorphic function h which does not vanish in the complex algebraic torus and a polynomial p, such that $f = hp$ if and only if the area of the amoeba of the holomorphic curve in $(\mathbb{C}^*)^2$ defined by f is finite.*

5. Proof of the Main theorem

In this section, we generalize the result of Section 4 to k-dimensional analytic varieties in the complex algebraic torus $(\mathbb{C}^*)^n$ with $n \geq 2k$. The following result generalizes Lemma 4.1 to higher dimensions.

Proposition 5.1. *Let V be a $k-$dimensional generic analytic variety in $(\mathbb{C}^*)^n$, with $n \geq 2k$. Assume that the volume of its amoeba is finite. Then its logarithmic limit set $\mathscr{L}^\infty(V)$ is a finite rational spherical polyhedron of dimension $k-1$.*

Proof. If $\mathscr{L}^\infty(V)$ is not a rational spherical polyhedron and v is a vertex of $\mathscr{L}^\infty(V)$ with irrational slope u, then the phase limit set $\mathscr{P}^\infty(V)$ contains a torus of dimension at least $(k-1)+2 = k+1$, which is the closure of an immersed circle of slope u. Using the fact that the map $\mathrm{Log} \circ \mathrm{Arg}^{-1}$ preserves the volumes (see Corollary 3.1), which is already used in the curve, we deduce that the volume of $\mathscr{A}(V)$ is infinite. Indeed, if U is a regular subset of the coamoeba $co\mathscr{A}(V)$ with nonvanishing volume, then it is covered via the argument map infinitely many times. This is in contradiction with the assumption on the volume of the amoeba $\mathscr{A}(V)$. Therefore, $\mathscr{L}^\infty(V)$ is rational.
Now, suppose that $\dim(\mathscr{L}^\infty(V)) > k-1$. Then, it contains a point of irrational slope in some $l-$cell with $l \geq k$. Hence, its phase limit set contains a torus of dimension at least $l+1 \geq k+1$. By the same argument as in the proof of Lemma 4.1, we obtain a contradiction with the assumption on the volume of the amoeba.
It remains to show that $\mathscr{L}^\infty(V)$ is finite. Otherwise, it contains an accumulation $(k-1)$-cell, which means that $\mathscr{P}^\infty(V)$ contains a torus of dimension at least $k+1$ (here $k+1 = (k-1)+1+1$, where $k-1$ is the dimension of the cells, to which we add one dimension for the circle corresponding to the direction s and one dimension for the accumulation direction). Using the same reasoning as in Lemma 4.1 (ii), we get a contradiction with the assumption on the amoeba volume. $\square$

Definition 5.1. *Let s be a vertex of $\mathscr{L}^\infty(V)$. A polyhedron P is in the direction of s, if for any point $x \in P$, there exists a vector v_x in P with starting point x and slope s.*

The following lemmas are the analogous of Lemma 4.2 and Lemma 4.4 in higher dimension and their proofs are similar.

Lemma 5.1. *Let $V \subset (\mathbb{C}^*)^n$ be a generic k-dimensional analytic variety with defining ideal $\mathcal{I}(V)$, such that $n \geq 2k$ and the volume of its amoeba is finite. Let $s \in Vert(\mathscr{L}^\infty(V))$ be a vertex and Σ_s be the open sub-complex of $\mathscr{L}^\infty(V)$ with only one vertex s. Then, there is a finite number of polyhedrons P of dimension k, asymptotic to the amoeba $\mathscr{A}(V)$ in the direction of s and such that $\overline{r(P)} \cap S^{n-1} \subset \Sigma_s$.*

Proof. If the number of polyhedrons P of dimension k, asymptotic to the amoeba $\mathscr{A}(V)$ in direction s and such that $\overline{r(P)} \cap S^{n-1} \subset \Sigma_s$, is infinite,

then there exists a sequence of parallel polyhedrons $\{P_m\}$ satisfying the same property. Hence, there are two possibilities:
(i) the number of their corresponding k-dimensional real tori in $(S^1)^n$ is finite, which means that at least one of them is covered under the argument map infinitely many times. This contradicts the fact that the volume of the amoeba is finite.
(ii) the number of their corresponding k-dimensional real tori in $(S^1)^n$ is infinite, which means that they contain at least one accumulation k-torus (of course parallel to all of them). This is a contradiction with the assumption on the volume of the amoeba. $\square$

Let $V \subset (\mathbb{C}^*)^n$ be a generic k-dimensional analytic variety with defining ideal $\mathcal{I}(V)$, such that the volume of its amoeba is finite. Let s be a vertex of $\mathscr{L}^\infty(V)$ with slope $(u_1, \dots, u_n)$, and $\mathscr{P}_s$ be the finite set of polyhedrons in the direction of s and asymptotic to the amoeba $\mathscr{A}(V)$. We denote by $\mathscr{H}(\mathscr{P}_s)$ the union of the holomorphic cylinders which are the lifting of the polyhedrons in $\mathscr{P}_s$ and asymptotic to the ends of V corresponding to s.

Lemma 5.2. *Let $V \subset (\mathbb{C}^*)^n$ be a generic k-dimensional analytic variety such that $n \geq 2k$ and with finite volume amoeba. Then, the ideal $\mathcal{I}(V)$ is generated by a set of holomorphic functions $\{f_1, \dots, f_q\}$ such that for any $j = 1, \dots, q$, we have $f_j = h_j g_j$, where h_j are holomorphic functions without zeros in a neighborhood of $\mathscr{H}(\mathscr{P}_s)$, and g_j are entire functions where the terms of their power series have powers in the closed half space $\{\alpha \in \mathbb{R}^n \mid \sum_{i=1}^n \alpha_i u_i \leq 0\}$.*

Proof. The proof of this lemma is similar to the proof of Lemma 4.4. If we have not such decomposition, then the $\mathscr{L}^\infty(V)$ is at least of dimension k, and this contradict the fact that the volume of the amoeba is finite. $\square$

End of the proof of Theorem 1.1 . The implication (i) $\Rightarrow$ (ii) is a consequence of Corollary 3.2. Without loss of generality, we may assume that V is irreducible. So (ii) $\Rightarrow$ (i) is a consequence of Proposition 5.1, Lemma 5.1, Lemma 5.2 and Lemma 4.3. $\square$

6. Examples

1. Let $\mathscr{C}$ be the complex plane curve in $(\mathbb{C}^*)^2$ given by the zeros of the holomorphic function $f(z_1, z_2) = z_2 - e^{z_1}$. Since the complex rank of $\mathrm{Jac}(f)$ is one, $\mathscr{C}$ is a Riemann surface. A parametrization of $\mathscr{C}$ is given by $t \in \mathbb{C}^* \mapsto (t, e^t) \in (\mathbb{C}^*)^2$. The amoeba $\mathscr{A}(\mathscr{C})$ is the set of points in $\mathbb{R}^2$ delimited by the graphs of the two functions $x \mapsto \pm e^x$ (see Figure 1). The set of critical points of Log restricted to $\mathscr{C}$ is $S = \{(x, e^x) \in (\mathbb{R}^*)^2\}$. The map Log is 2-sheets covering over its regular values. However, the map Arg is not a locally finite covering (i.e., $\#\mathrm{Arg}^{-1}\theta = +\infty$ for any regular value $\theta \in co\mathscr{A}$). Since the closure of the coamoeba is compact, its area is always finite. But, the amoeba has an infinite area. This means that the assumption on Theorem 3.1 is necessary.
Moreover, we can check that the logarithmic limit set of $\mathscr{C}$ has 1-dimensional

connected component and one isolated point of rational slope. The phase limit set of $\mathscr{C}$ is the whole torus.

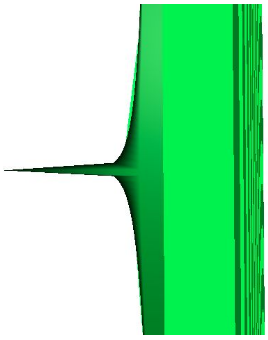

FIGURE 1. The amoeba of the plane holomorphic curve given by the parametrization $g(t) = (t, e^t)$.

2. Let $\mathscr{C}$ be the complex plane curve in $(\mathbb{C}^*)^2$ parametrized by

$$\begin{aligned} \rho : D &\longrightarrow (\mathbb{C}^*)^2 \\ t &\longmapsto (\cos t, \sin t), \end{aligned}$$

where $D = \{t = a + ib \,|\, (a, b) \in (]0, 2\pi[\setminus\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}) \times \mathbb{R}\}$ is the fundamental domain. We can check that $\mathscr{C}$ is contained in the algebraic curve with defining polynomial $p(x, y) = x^2 + y^2 - 1$. On the other hand, we know that the last curve is irreducible. Hence, $\mathscr{C}$ is algebraic and defined by the same polynomial. The critical points of Log are $\rho((]0, 2\pi[\setminus\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}) \times \{0\})$. The logarithmic limit set consists of three points with coordinates $(-1, 0)$, $(0, -1)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. The phase limit set consists of three pairs of circles with distinct slopes given by $-\infty$, 0 and 1.

3. Let $\mathscr{C}$ be the complex plane curve in $(\mathbb{C}^*)^2$ parametrized by

$$\begin{aligned} \rho : D &\longrightarrow (\mathbb{C}^*)^2 \\ t &\longmapsto (t, e^t, t + 1). \end{aligned}$$

The logarithmic limit set of $\mathscr{C}$ is the union of two points and an arc in the sphere S^2. Its phase limit set is an arrangement of two circles and a 2-dimensional flat torus. The curve $\mathscr{C}$ is not algebraic and the area of its amoeba (see Figure 2) is infinite.

7. AMOEBAS OF k-DIMENSIONAL AFFINE LINEAR SPACES

Let k and s be two positives integers and $\mathscr{P}(k) \subset (\mathbb{C}^*)^{k+s}$ be the affine linear space of dimension k given by the parametrization

$$\begin{array}{cccc} \rho & : & (\mathbb{C}^*)^k & \longrightarrow & (\mathbb{C}^*)^{k+s} \\ & & (t_1, \ldots, t_k) & \longmapsto & (t_1, \ldots, t_k, f_1(t_1, \ldots, t_k), \ldots, f_s(t_1, \ldots, t_k)), \end{array}$$

such that $f_j(t_1, \ldots, t_k) = b_j + \sum_{i=1}^k a_{ji} t_i$, where a_{ji} and b_j are complex numbers for $i = 1, \ldots, k$ and $j = 1, \ldots, s$. First of all, we may assume

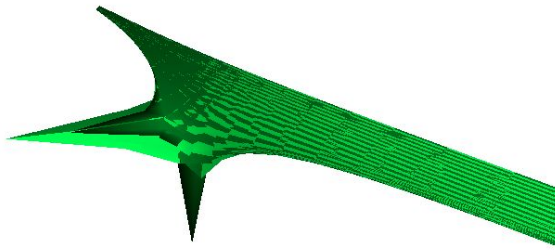

FIGURE 2. The amoeba of the spatial holomorphic curve given by the parametrization $g(t) = (t, e^t, t+1)$.

that $f_1(t_1, \dots, t_k) = 1 + \sum_{i=1}^{k} t_i$. Moreover, we assume that the affine linear spaces are in general position.

Definition 7.1. *Let $V \subset (\mathbb{C}^*)^n$ be an algebraic subvariety and* conj *be the involution on $\mathbb{C}^n$ given by conjugation on each coordinate. We say the variety V is real if it is invariant under* conj*, i.e.,* $\mathrm{conj(V)} = \mathrm{V}$*. The real part of V denoted by $\mathbb{R}V$ is the set of points in V fixed by* conj*.*

In this section, we assume $n = 2k + m$, where m is a nonnegative integer (i.e., $s = k + m$). The goal of this section is to prove the following theorem:

Theorem 7.1. *Let $\mathscr{P}(k)$ be a generic linear space in $(\mathbb{C}^*)^{2k+m}$ and θ be a regular value of the argument map. Then, the cardinality of $\mathrm{Arg}^{-1}(y)$ is equal to one.*

Before we give the proof of this theorem, let us start by looking at the case $k = 1$.

Remark 7.1. Without loss of generality, we may assume that $\mathscr{P}(1)$ is the line in $(\mathbb{C}^*)^{2+m}$ given by the parametrization:

$$\begin{array}{cccc} \rho & : & \mathbb{C}^* & \longrightarrow & (\mathbb{C}^*)^{2+m} \\ & & t_1 & \longmapsto & (t_1, f_1(t_1), \dots, f_{1+m}(t_1)), \end{array}$$

such that $f_1(t_1) = 1 + t_1$ and $f_j(t_1) = b_j + a_{j1}t_1$, where a_{j1} and b_j are complex numbers for $j = 2, \dots, 1+m$. Let $y = (y_1, \dots, y_{2+m})$ be a regular value of the logarithmic map. The number of points $t_1 \in \mathbb{C}^*$ with $|t_1| = e^{y_1}$ and $|f_1(t_1)| = e^{y_2}$ is at most two (because the intersection of two circles cannot exceed two points; the first circle has the center at the origin and radius e^{y_2} and the second circle has the center at $(1, 0)$ and radius e^{y_1}). It is clear that if y is regular, then this number is equal two. Otherwise, the circles are tangent and then y is critical. Indeed, if we make a small perturbation of t_1 in some direction, the point goes out of the amoeba, which means that y is in the boundary of the amoeba. Moreover, we can check that these two points are conjugate. So, if the number is equal two, then $|f_j(t_1)|$ should be equal to

$|f_j(\bar{t}_1)|$ for any $j = 1, \dots, 1+m$. This means that $|b_j + a_{j1}t_1| = |b_j + a_{j1}\bar{t}_1|$, or equivalently $\cos(\arg(a_{j1}) + \arg(t_1) - \arg(b_j)) = \cos(\arg(a_{j1}) + 2\pi - \arg(t_1) - \arg(b_j))$. Hence, $\arg(a_{j1}) - \arg(b_j) = 0 \mod (\pi)$, which is equivalent to $\frac{a_{j1}}{b_j} \in \mathbb{R}^*$ for any j and then the curve is real. Otherwise, the cardinality of $\operatorname{Log}^{-1}(y)$ is equal to one.

Proof of Theorem 7.1. Let $\theta = (\theta_1, \dots, \theta_k, \psi_1, \dots, \psi_{k+m})$ be a regular value of the argument map. If $\rho(t) \in \mathscr{P}(k)$ belongs to the inverse image by the argument map of θ, then $\arg(t_j) = \theta_j$ for each $j = 1, \dots, k$ and $\arg(f_l(t)) = \psi_l$ for each $l = 1, \dots, k+m$. The k-plane $\mathscr{P}(k)$ is parametrized by ρ as above. For each l, we can always view f_l, b_l and the $a_{lj}t_j$'s as vectors in the plane $\mathbb{C}$, such that their arguments are in the increasing order. If we put them in the plane with this order, we obtain a convex polygon. We can check that if there exist $t \neq t'$ with $\rho(t)$ and $\rho(t')$ in $\mathscr{P}(k)$ and $\operatorname{Arg}(\rho(t)) = \operatorname{Arg}(\rho(t')) = \theta$, then the inverse image of θ by Arg has dimension strictly greater than zero. Therefore, θ is not regular. In fact, if $|t_j| \neq |t'_j|$ for some j, then for any $(\lambda|t_j| + (1-\lambda)|t'_j|)e^{\theta_j}$, with $\lambda \in [0,1]$, there exist $t_s = \mu_s e^{\theta_s}$ with $\mu_s \in \mathbb{R}_+$, $s \in \{1, \dots, k\} \setminus \{j\}$ and $\operatorname{Arg}(f_l) = \psi_l$ for each $l = 1, \dots, k+m$. This means that the dimension of $\operatorname{Arg}^{-1}(\theta)$ is at least one. We conclude that the cardinality of $\operatorname{Arg}^{-1}(\theta)$ is equal to one for any regular point θ. $\square$

In the case of k-dimensional real linear spaces of $(\mathbb{C}^*)^{2k}$, the second author, and Passare [NP-11] prove that each regular value of the logarithmic map is covered by 2^k points and the volume of the coamoeba is equal to π^{2k}. Hence, the numbers p and P of Theorem 3.1 are equal to $\frac{1}{2^k}$ in this case.

Corollary 7.1. *Let $\mathscr{A}(k)$ be the amoeba of a k-dimensional real linear space of $(\mathbb{C}^*)^{2k}$. Then*

$$vol(\mathscr{A}) = \frac{\pi^{2k}}{2^k}.$$

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