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HIGHER CONGRUENCE COMPANION FORMS

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Abstract. For a rational prime $p \geq 3$ we consider p-ordinary, Hilbert modular newforms f of weight $k \geq 2$ with associated p-adic Galois representations ρ_f and $\bmod p^n$ reductions $\rho_{f,n}$. Under suitable hypotheses on the size of the image, we use deformation theory and modularity lifting to show that if the restrictions of $\rho_{f,n}$ to decomposition groups above p split then f has a companion form g modulo p^n (in the sense that $\rho_{f,n} \sim \rho_{g,n} \otimes \chi^{k-1}$).

1. Introduction

Let F be a totally real number field and let p be an odd prime. Suppose we are given a Hilbert modular newform f over F of level $\mathfrak{n}_f$, character ψ and (parallel) weight $k \geq 2$. For a prime $\mathfrak{q}$ not dividing $\mathfrak{n}_f$, let $c(\mathfrak{q}, f)$ denote the eigenvalue of the Hecke operator $T(\mathfrak{q})$ acting on f; denote by K_f the number field generated by the $c(\mathfrak{q}, f)$'s and $\psi(\mathrm{Frob}_\mathfrak{q})$'s, and by $\mathcal{O}_f$ the integer ring of K_f. Then for each prime $\wp | p$ of $\mathcal{O}_f$ one has a continuous, odd, absolutely irreducible representation $\rho_{f,\wp} : G_F \longrightarrow GL_2(\mathcal{O}_{f,\wp})$ characterized by the following property: $\rho_{f,\wp}$ is unramified outside primes dividing $p\mathfrak{n}_f$, and at a prime $\mathfrak{q} \nmid p\mathfrak{n}_f$ the characteristic polynomial of $\rho_{f,\wp}(\mathrm{Frob}_\mathfrak{q})$ is $X^2 - c(\mathfrak{q}, f)X + \psi(\mathrm{Frob}_\mathfrak{q})\mathrm{Nm}(\mathfrak{q})^{k-1}$. We denote the p-adic cyclotomic character by χ. Thus the determinant of $\rho_{f,\wp}$ is $\psi\chi^{k-1}$.

From here on we assume that the character ψ is unramified at p. Suppose that f is ordinary at p. Then, by Wiles [18], and Mazur-Wiles [14], for every prime $\mathfrak{p}|p$ we have

$$\rho_{f,\wp}|_{G_\mathfrak{p}} \sim \begin{pmatrix} \psi_{1\mathfrak{p}}\chi^{k-1} & * \\ 0 & \psi_{2\mathfrak{p}} \end{pmatrix}$$

where $\psi_{1\mathfrak{p}}$, $\psi_{2\mathfrak{p}}$ are unramified characters. In fact, with $a(\mathfrak{p}, f)$ defined to be the unit root $X^2 - c(\mathfrak{p}, f)X + \psi(\mathrm{Frob}_\mathfrak{p})\mathrm{Nm}(\mathfrak{p})^{k-1} = 0$, we have $\psi_{2\mathfrak{p}}(\mathrm{Frob}_\mathfrak{p}) = c(\mathfrak{p}, f)$. A natural question is to ask when the restriction(s) $\rho_{f,\wp}|_{G_\mathfrak{p}}$ actually split. We note that if $\rho_{f,\wp} \bmod \wp$ is absolutely irreducible then the splitting (or not) of $\rho_{f,\wp} \bmod \wp^n$ is independent of the choice of a lattice used to define $\rho_{f,\wp}$. Indeed, if for some $M \in GL_2(K_{f,\wp})$ the conjugate $M\rho_{f,\wp}M^{-1}$ is integral and stabilises the upper triangular decomposition at $\mathfrak{p}$, then M is a scalar multiple of $\left(\begin{smallmatrix} u & v \\ 0 & 1 \end{smallmatrix}\right)$ where $u \equiv 1 \mod \wp$ and $v \equiv 0 \mod \wp$. If we denote by $c_\mathfrak{p} \in H^1(G_\mathfrak{p}, \mathcal{O}_{f,\wp}(\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}}^{-1}\chi^{k-1}))$ the cohomology class for $\rho_{f,\wp}|_{G_\mathfrak{p}}$, then the cohomology class for the extension at $\mathfrak{p}$ determined by $M\rho_{f,\wp}M^{-1}$ is $uc_\mathfrak{p}$. Hence, if $\rho_{f,\wp} \bmod \wp$ is absolutely irreducible we can speak of $\rho_{f,\wp} \bmod \wp^n$ being split without any ambiguity.

Now suppose we are given a second newform g which is also ordinary at p. Fix a p-adic integer ring $\mathcal{O}$ in which $\mathcal{O}_f$ and $\mathcal{O}_g$ embed, and let π be a uniformiser. We say that g is a $\bmod \pi^n$

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weak companion form for f if $c(\mathfrak{q}, f) \equiv c(\mathfrak{q}, g)\mathrm{Nm}(\mathfrak{q})^{k-1} \mod \pi^n$ for all but finitely many primes $\mathfrak{q}$. The weight k' of g is then determined by k via the congruence $\chi^{k'-1} \equiv \chi^{1-k} \mod \pi^n$ on each decomposition group above p. Note that we do not enforce any optimality requirement on the level of g (and hence the prefix 'weak').

Classically, companion forms $\mod p$ played an important part in the weight optimisation part of Serre's Modularity Conjecture. Serre's predicted equivalence between local splitting for the residual modular representation (tame ramification) and the existence of companions was established by Gross in [8]. In much the same spirit, the main result of this paper, which we now state, proves the equivalence between splitting $\mod p^n$ and the existence of $\mod p^n$ weak companion forms.

Main Theorem. *Let F be a totally real number field, p be an odd prime unramified in F, and let f be a p-ordinary Hilbert modular newform f of squarefree level $\mathfrak{n}$, character ψ with order coprime to p and unramified at p, and weight $k \geq 2$. Let $n \geq 2$ and set $\rho_{f,n} := \rho_{f,\wp} \mod p^n$, $\overline{\rho}_f := \rho_{f,\wp} \mod \wp$, $\boldsymbol{k} := \mathcal{O}_f/\wp$. Assume the following hypotheses:*

- *Global conditions.*
  - (GC1) *$\rho_{f,n}$ takes values in $GL_2(W/p^n)$ where W is the Witt ring of $\boldsymbol{k} := \mathcal{O}_f/\wp$ (under the natural injection $W/p^n \hookrightarrow \mathcal{O}_f/\wp^n$).*
  - (GC2) *The image of $\rho_{f,\wp} \mod \wp$ contains $SL_2(\boldsymbol{k})$. Furthermore, if $p = 3$ then the image $\rho_{f,n}$ contains a transvection $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$.*
- *Local conditions.*
  - (LC1) *If $\mathfrak{p}$ is a prime dividing p then $c(\mathfrak{p}, f)^2 \not\equiv \psi(Frob_{\mathfrak{p}}) \mod \wp$.*
  - (LC2) *Let $\mathfrak{q}$ be a prime dividing the level $\mathfrak{n}$ where $\overline{\rho}_f$ is unramified. If $Nm(\mathfrak{q}) \equiv 1 \mod p$ then p divides the order of $\overline{\rho}_f(Frob_{\mathfrak{q}})$.*

Let $k' \geq 2$ be the smallest integer such that $k + k' \equiv 2 \mod (p-1)p^{n-1}$. Then f splits $\mod p^n$ if and only if it has a p-ordinary $\mod p^n$ weak companion form g of weight k' and character ψ.

The proof, given in section 3.2, relies on being able to lift $\rho_{f,n} \otimes \chi^{1-k}$ to characteristic 0 with prescribed local properties and then proving that the lift is modular by using results of Skinner and Wiles in [17] along with the existence of companion forms $\mod p$ over totally real fields due to Gee ([5, Theorem 2.1]). The construction of characteristic 0 lifts for certain classes of $\mod p^n$ representations is carried out in section 3.1. (See Theorem 3.2 for the statement.)

The Main Theorem, in practice, is not useful for checking when a given newform fails to split $\mod p^n$ because we have very little control over the level of the weak companion form. However, as we show in Section 4, in the situation when the dimension of the tangent space $\mathbf{t}_{\mathcal{D}}$ (associated to a deformation condition $\mathcal{D}$ of $\overline{\rho}$) is 0, we can use higher companion forms to computationally verify a conjecture of Greenberg connecting local splitting with complex multiplication. We conclude by giving examples in support of this conjecture.

2. Toolkit

The method we use for obtaining a fine structure on deformations of a $\mod p^n$ representation is an adaptation of the more familiar $\mod p$ case and has two key components: the existence of sufficiently well behaved local deformations; and, for the existence of characteristic 0 liftings, being able to place local constraints so that the dual Selmer group vanishes. Naturally, both of these present difficulties in the general $\mod p^n$ case. In this section, we discuss the tools that will enable us to manage the difficulties for certain classes of $\mod p^n$ representations.

Throughout this section p is an odd prime, $\boldsymbol{k}$ is a finite field of characteristic p and W is the Witt ring of $\boldsymbol{k}$.

2.1. **Deformations and substantial deformation conditions.** In the main, we follow Mazur's treatment of deformations and deformation conditions in [13]. Given a residual representation, a deformation condition is simply a collection of liftings satisfying some additional properties (closure under projections, a Mayer-Vietoris property etc). The fundamental consequence then is the existence of a (uni)versal deformation.

We expand on this: Suppose we are given a 'nice' profinite group Γ, and a continuous representation $\overline{\rho} : \Gamma \longrightarrow GL_2(\boldsymbol{k})$. If $\mathcal{D}$ is a deformation condition for $\overline{\rho}$ then there is a complete local Noetherian W-algebra R with residue field $\boldsymbol{k}$ and a lifting $\rho : \Gamma \longrightarrow GL_2(R)$ in $\mathcal{D}$ with the following property: If $\rho' : \Gamma \longrightarrow GL_2(A)$ is a lifting of $\overline{\rho}$ in $\mathcal{D}$ then there is a morphism $R \longrightarrow A$ which gives, on composition with ρ, a representation strictly equivalent to ρ'. In addition, we require that the morphism above is unique when A is the ring of dual numbers $\boldsymbol{k}[\epsilon]/(\epsilon^2)$. If the projective image of $\overline{\rho}$ has trivial centralizer then R, together with ρ, represents the functor that assigns type $\mathcal{D}$ deformations to a coefficient ring. We shall use the natural identification of the tangent space $\mathbf{t}_{\mathcal{D}}$ with a subspace of $H^1(\Gamma, \mathrm{ad}\overline{\rho})$ (and as a subspace of $H^1(\Gamma, \mathrm{ad}^0\overline{\rho})$ when considering deformations with a fixed determinant). The (uni)versal deformation ring R then has a presentation $W[[T_1, \dots, T_n]]/J$ where $n = \dim_{\boldsymbol{k}} \mathbf{t}_{\mathcal{D}}$. We will be particularly interested in smooth deformation conditions (so the ideal of relations J will be (0)).

As hinted in the beginning of this section, the method we use for constructing smooth global deformation conditions depends upon being able to find local (uni)versal deformation rings smooth in a large number of variables. It will be convenient to make the following definition:

Definition 2.1. Let F be a local field and let $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$ be a residual representation.

(a) We call a deformation condition for $\overline{\rho}$ with fixed determinant *substantial* if it is smooth and its tangent space $\mathbf{t}$ satisfies the inequality

$$\dim_{\boldsymbol{k}} \mathbf{t} \geq \dim_{\boldsymbol{k}} H^0(G_F, \mathrm{ad}^0\overline{\rho}) + [F : \mathbb{Q}_p]\delta$$

where δ is 1 when F has residue characteristic p and 0 otherwise.

(b) A deformation $\rho : G_F \longrightarrow GL_2(A)$ of $\overline{\rho}$ is substantial if it is part of a substantial deformation condition.

We now give examples of substantial deformation conditions. From here on, for the rest of the section, F is a finite extension of $\mathbb{Q}_l$ for some prime l. As in the definition above, let $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$ be a residual representation.

Example 2.2. Assume that the residue characteristic of F is different from p. Suppose that the order of $\overline{\rho}(I_F)$ is co-prime to p, and let $d : G_F \longrightarrow W^\times$ be a character lifting $\det\overline{\rho}$. The collection of liftings of $\overline{\rho}$ which factor through $G_F/(I_F \cap \ker\overline{\rho})$ and have determinant d is a substantial deformation condition. The tangent space has dimension $\dim_{\boldsymbol{k}} H^0(G_F, \mathrm{ad}^0\overline{\rho})$.

Example 2.3. Suppose that

$$\overline{\rho} \sim \begin{pmatrix} \overline{\chi} & * \\ 0 & 1 \end{pmatrix} \overline{\varepsilon}$$

for some character $\overline{\varepsilon} : G_F \longrightarrow \boldsymbol{k}^\times$. Moreover, assume that if $\overline{\rho}$ is semi-simple then $\overline{\chi}$ is non-trivial. Fix a character $\varepsilon : G_F \longrightarrow W^\times$ lifting $\overline{\varepsilon}$. Then the collection of liftings strictly equivalent to

$$\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix} \varepsilon$$

is a substantial deformation condition. Note that $\overline{\rho}$ is equivalent to a representation of the form considered above only if p divides the order of $\overline{\rho}(I_F)$. (See Example 3.3 of [11].)

Example 2.4. We now assume that the residue characteristic of F is p. Suppose we are given an integer $k \geq 2$ and a representation $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$ such that

$$\overline{\rho} = \begin{pmatrix} \overline{\chi}^{k-1}\overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \end{pmatrix}$$

where $\overline{\psi}_1, \overline{\psi}_2$ are unramified characters. Let ψ be the Teichmüller lift of $\overline{\psi}_1\overline{\psi}_2$. If A is a coefficient ring, we shall call a lifting $\rho_A : G_F \longrightarrow GL_2(A)$ of $\overline{\rho}$ a *$\overline{\psi}_2$-good weight k lifting with character ψ* if ρ_A is strictly equivalent to a representation of the form

$$\begin{pmatrix} \widetilde{\psi_1}\chi^{k-1} & * \\ 0 & \widetilde{\psi_2} \end{pmatrix}$$

for some unramified characters $\widetilde{\psi_1}, \widetilde{\psi_2} : G_F \longrightarrow A^\times$ lifting $\overline{\psi}_1, \overline{\psi}_2$ and $\widetilde{\psi_1}\widetilde{\psi_2} = \psi$.

We then have the following property of weight k liftings (proof immediate, Example 3.4 in [11]):

Proposition 2.5. *Let $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$ be as above in Example 2.4, and further assume that $\overline{\chi}^{k-1}\psi_1 \neq \overline{\chi}\psi_2$. Then the deformation condition consisting of weight k liftings of $\overline{\rho}$ is a smooth deformation condition. The dimension of its tangent space is equal to $[F : \mathbb{Q}_p] + dim_{\boldsymbol{k}} H^0(G_F, ad^0\overline{\rho})$.*

We conclude by determining all substantial deformation conditions for residual representations of a particular shape. For the remainder of this section, we assume F has residue field of order q with $p \nmid q$ and let $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$ be an unramified representation with $\overline{\rho}(\mathrm{Frob}) = \begin{pmatrix} q\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Of course, any lift of $\overline{\rho}$ necessarily factors through the maximal tamely ramified extension F^{tr} of F. We fix a generator τ of the tame inertia, a lift σ of Frobenius to $\mathrm{Gal}(F^{\mathrm{tr}}/F)$. One then has the following description:

$$\mathrm{Gal}(F^{\mathrm{tr}}/F) = \langle \sigma, \tau | \sigma\tau\sigma^{-1} = \tau^q \rangle.$$

First define polynomials $h_n(T) \in \mathbb{Z}[T], n \geq 1$, by the recursion $h_{n+2} = Th_{n+1} - h_n$ and initial values $h_1 := 1, h_2 := T$. The following properties of h_n are easily verified by induction:

- $h_n(2) = n$
- If M is a 2 by 2 matrix over any commutative ring with trace t and determinant 1 then, $M^n = h_n(t)M - h_{n-1}(t)I$
- $h_n^2 - Th_nh_{n-1} + h_{n-1}^2 = 1$

We then have the following proposition.

Proposition 2.6. *Let $\overline{\rho}$ be as above with $q\alpha \neq \alpha^{-1}$. Denote by $\hat{\alpha}$ the Teichmuller lift of α. Let R, resp. $\rho : Gal(F^{tr}/F) \to GL_2(R)$, be the versal deformation ring, resp. the versal representation, for liftings of $\overline{\rho}$ with determinant χ.*

(i) *Suppose $\alpha^2 \neq 1$ and $q^2\alpha^2 \neq 1$. If $q \equiv 1 \mod p$ then $R \cong W(k)[[S,T]]/\langle (1+T)^q - (1+T)\rangle$ and*

$$\rho(\sigma) = \begin{pmatrix} q\hat{\alpha}(1+S) & 0 \\ 0 & (\hat{\alpha}(1+S))^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 1+T & 0 \\ 0 & (1+T)^{-1} \end{pmatrix}.$$

If $q \not\equiv 1 \mod p$ then $R \cong W[[S]]$ and $\rho(\sigma) = \begin{pmatrix} q\hat{\alpha}(1+S) & 0 \\ 0 & (\hat{\alpha}(1+S))^{-1} \end{pmatrix}$. In any case, a deformation condition is substantial if and only if it is unramified.

(ii) *If $\alpha^2 = 1$ and $q^2 \not\equiv 1 \mod p$ then $R \cong W(k)[[S,T]]/(ST)$ and*

$$\rho(\sigma) = \hat{\alpha} \begin{pmatrix} q(1+S) & 0 \\ 0 & (1+S)^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}.$$

A deformation condition for $\overline{\rho}$ with determinant χ is substantial if and only if it is either unramified or of the type considered in Example 2.3.

(iii) *If $\alpha^2 = 1$ and $q \equiv -1 \mod p$, then $R := W(k)[[S,T_1,T_2]]/J$ where*

$$J := \langle T_1\big(q(1+S)^2 - h_q(2\sqrt{1+T_1T_2})\big), T_2\big(1 - q(1+S)^2 h_q(2\sqrt{1+T_1T_2})\big)\rangle, \quad \text{and}$$

$$\rho(\sigma) = \hat{\alpha} \begin{pmatrix} q(1+S) & 0 \\ 0 & (1+S)^{-1} \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} \sqrt{1+T_1T_2} & T_1 \\ T_2 & \sqrt{1+T_1T_2} \end{pmatrix}.$$

The only ramified substantial deformation condition for $\overline{\rho}$ is the of the type given Example 2.3: it corresponds to the quotient $W(k)[[S,T_1,T_2]]/(S,T_2)$.

Proof. Let A be a coefficient ring with maximal ideal $\mathfrak{m}_A$, and let be ρ_A a lifting of $\overline{\rho}$ with determinant χ. By Hensel's Lemma, we can assume that $\rho_A(\sigma)$ is diagonal. Let

$$\rho_A(\sigma) = \begin{pmatrix} q\hat{\alpha}(1+s) & 0 \\ 0 & (\hat{\alpha}(1+s))^{-1} \end{pmatrix}, \quad \rho_A(\tau) = \begin{pmatrix} a & t_1 \\ t_2 & d \end{pmatrix} \tag{2.1}$$

with $s, t_1, t_2, a-1, d-1 \in \mathfrak{m}_A$ and $ad - t_1t_2 = 1$. Since $\sigma\tau\sigma^{-1} = \tau^q$, we have

$$\begin{pmatrix} a & t_1 q(\hat{\alpha}(1+s))^2 \\ t_2 q^{-1}(\hat{\alpha}(1+s))^{-2} & d \end{pmatrix} = \begin{pmatrix} ah_q(t) - h_{q-1}(t) & t_1 h_q(t) \\ t_2 h_q(t) & dh_q(t) - h_{q-1}(t) \end{pmatrix} \tag{2.2}$$

where $t = a + d$ is the trace. Note that $t \equiv 2 \mod \mathfrak{m}_A$ and so $h_q(t) \equiv q \mod \mathfrak{m}_A$.

If $\alpha^2 \neq 1$ then $q(\hat{\alpha}(1+s))^2 - h_q(t)$ is a unit and we get $t_1 = 0$. Similarly, if $q^2\alpha^2 \neq 1$ then $t_2 = 0$. The claims made in part (i) of the proposition are now immediate.

We now continue our analysis of ρ_A under the assumption that $\alpha^2 = 1$ and $q \not\equiv \pm 1 \mod p$. For ease of notation, we shall in fact assume that $\hat{\alpha} = 1$. Since $1 - h_q(t)$ is a unit, taking the difference of the diagonal entries on both sides of (2.2) gives $a - d = 0$ and so $a = d = \sqrt{1 + t_1 t_2}$. Comparison of the off-diagonal entries of (2.2) (followed by multiplication) produces $t_1 t_2 (1 - h_q(t)^2) = 0$.

Suppose now that $q \not\equiv -1 \mod p$. Then $t_1 t_2 = 0$ and so $t = 2, h_q(t) = q$. We can now simplify the two relations from the off-diagonal entries to get $t_1 = t_1(1+s)^2$ and $t_2 = t_2 q^2 (1+s)^2$, and finally deduce that $st_1 = 0, t_2 = 0$. Part (ii) of the proposition now follows easily.

Finally, we consider the case $q \equiv -1 \mod p$. The presentation for R and ρ follows from the presentation of an arbitrary lift along with the fact that $\dim H^1(G_F, \mathrm{ad}^0 \overline{\rho}) = 3$. We now indicate how to determine the substantial deformation conditions. Take A to be characteristic 0 (and $\hat{\alpha} = 1$). In the presentation (2.1) the trace of $\rho_A(\tau)$ is $2\sqrt{1 + t_1 t_2}$. If $t_1 t_2 \neq 0$ then $\rho_A(\tau)$ has distinct eigenvalues - contradicting the fact that ρ_A is twist equivalent to $\left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)$ over its field of fractions. Hence we must have $t_2 = 0$ and $s = 0$. □

2.2. **Subgroups of $GL_2(W/p^n)$.** We now derive some properties of certain subgroups of $GL_2(W/p^n)$ which will be of relevance in constructing global deformations. Let's recall that p is an odd prime, and that $\boldsymbol{k}$ is the residue field W/p. We denote by ad^0 the the vector space of 2×2-matrices over $\boldsymbol{k}$ with $GL_2(W/p^n)$ acting by conjugation, and by $\mathrm{ad}^0(i)$ its twist by the i-th power of the determinant. For convenience, we record the following useful identity

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a + cx & b - 2ax - cx^2 \\ c & -a - cx \end{pmatrix}. \tag{2.3}$$

It is well known—see Lemma 2.48 of [3], for instance—that $H^1(SL_2(\boldsymbol{k}), \mathrm{ad}^0) = 0$ except when $\boldsymbol{k} = \mathbb{F}_5$. We now state and proof (for completeness) the following result in the exceptional case.

Lemma 2.7. *$H^1(GL_2(\mathbb{F}_5), ad^0(i)) = 0$ if $i = 0, 1$ or 3.*

Proof. Let $B \supset U$ be the subgroups of $GL_2(\mathbb{F}_5)$ consisting of matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ respectively. We then need to verify that $H^1(B, \mathrm{ad}^0(i)) \cong H^1(U, \mathrm{ad}^0(i))^{B/U} = (0)$.

Let $\sigma := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \tau := \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. From (2.3) it follows that $(\sigma - 1)\mathrm{ad}^0$ is the subspace of upper triangular matrices in ad^0. Thus if $0 \neq \xi \in H^1(U, \mathrm{ad}^0(i))$ then we can assume that $\xi(\sigma) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and ξ is fixed by B/U if and only if $(\tau * \xi)(\sigma) - \xi(\sigma)$ is upper triangular. Now $(\tau * \xi)(\sigma) = \tau \xi(\sigma^2) \tau^{-1} = \left(\begin{smallmatrix} 1 & 2 \\ -1 & 1 \end{smallmatrix}\right)$ and so $\xi \in H^1(U, \mathrm{ad}^0(i))^{B/U}$ if and only if $1 + 3^i = 0$. □

Proposition 2.8. *Let G be a subgroup of $GL_2(W/p^n)$. Suppose the mod p reduction of G contains $SL_2(\boldsymbol{k})$. Furthermore, assume that if $p = 3$ then G contains a transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then the following statements hold.*

(a) *G contains $SL_2(W/p^n)$.*
(b) *Suppose that $p \geq 5$. If $\boldsymbol{k} = \mathbb{F}_5$ assume further that $G \mod 5 = GL_2(\mathbb{F}_5)$. Then $H^1(G, ad^0(i)) = 0$ for $i = 0, 1$.*
(c) *The restriction map $H^1\left(G, ad^0(i)\right) \longrightarrow H^1\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), ad^0(i)\right)$ is an injection (for all $p \geq 3$).*

Proof. Part (a). The claim is likely to be familiar. Certainly the case $\boldsymbol{k} = \mathbb{F}_p$ is well known and can be found in Serre's book [16].

We shall only verify that if G is a subgroup of $SL_2(W/p^n)$ whose mod p^{n-1} reduction is $SL_2(W/p^{n-1})$ then $G = SL_2(W/p^n)$. The kernel of the reduction map $G \longrightarrow SL_2(W/p^{n-1})$

consists of matrices of the form $I + p^{n-1}A$ with A an element of some additive subgroup of ad^0 stable under the action of G. Consequently either $G = SL_2(W/p^n)$ or else the reduction map $G \longrightarrow SL_2(W/p^{n-1})$ is an isomorphism.

We will now discount the second possibility. So suppose that $G \longrightarrow SL_2(W/p^{n-1})$ is an isomorphism. Since G contains a transvection when $p = 3$ we must have $p \geq 3$. Now let $g \in G$ be the pre-image of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(W/p^{n-1})$. So g must have order p^{n-1}. If we write g as $\left(I + p^{n-1}\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, a simple calculation using (2.3) shows that $g^p = \left(\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right)$ and so g has order p^n—a contradiction.

Part (b). The hypothesis implies that $H^1\left(G \mod p, \mathrm{ad}^0(i)\right) = 0$. So let's assume that $n \geq 2$ and that $H^1\left(G \mod p^{n-1}, \mathrm{ad}^0(i)\right) = 0$, and suppose that $0 \neq \xi \in H^1(G, \mathrm{ad}^0(i))$. Then the restriction ξ to $H := \ker\left(G \longrightarrow G \mod p^{n-1}\right)$ is a group homomorphism compatible with the action of $SL_2(W/p^{n-1})$. It follows from part (a) that H is in fact $\ker\left(SL_2(W/p^n) \longrightarrow SL_2(W/p^{n-1})\right)$. Since H is naturally identified with ad^0, it follows that $\xi|_H$ is an isomorphism. Let's denote by W^ν the ring $W/p^{n-1} \oplus \boldsymbol{k}\epsilon$ where $\epsilon^2 = p\epsilon = 0$. (Or equivalently $W^\nu \cong W[\epsilon]/(p^{n-1}, \epsilon^2, p\epsilon)$.) We then see that the homomorphism $SL_2(W/p^n) \longrightarrow SL_2(W^\nu)$ given by

$$g \longrightarrow (I + \epsilon\xi(g))\,(g \mod p^{n-1})$$

is an isomorphism. To finish off, we proceed as in part (a): The transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in SL_2(W^\nu)$ has order p^{n-1} while its pre-image in $SL_2(W/p^n)$, a matrix of the form $(I + p^{n-1})\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right))\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, has order p^n.

Part (c). Suppose $0 \neq \xi \in H^1(G, \mathrm{ad}^0(i))$ restricts to a trivial cohomology class in $H^1\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \mathrm{ad}^0(i)\right)$. Then the restriction of ξ to $\left(\begin{smallmatrix} 1 & p^{n-1} \\ 0 & 1 \end{smallmatrix}\right)$ is trivial. Set $N := \ker\left(G \longrightarrow G \mod p^{n-1}\right)$. Then $\xi|_N$ has a non-trivial kernel, and hence $\xi|_N$ is trivial. Thus ξ is a non-zero element of $H^1(G \mod p^{n-1}, \mathrm{ad}^0(i))$. We are thus reduced to the case when $n = 1$. Now $H^1(SL_2(\boldsymbol{k}), \mathrm{ad}^0) = 0$ except when $\boldsymbol{k} = \mathbb{F}_5$, so we are reduced to the case when G is a subgroup of $GL_2(\mathbb{F}_5)$ containing $SL_2(\mathbb{F}_5)$. But in this case $(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ is the Sylow 5-subgroup of G, and hence if $\xi|_{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)} = 0$ then $\xi = 0$. □

3. Constructing characteristic 0 lifts of mod p^n Galois representations

We can now formulate precise conditions under which a given mod p^n Galois representation can be lifted to characteristic 0, and use the lifts constructed to prove the existence of weak companion forms.

3.1. Deformations of mod p^n representations to $W(k)$.

We now suppose we are given a totally real number field F and continuous odd representations $\overline{\rho} : G_F \longrightarrow GL_2(\boldsymbol{k})$, $\rho_n : G_F \longrightarrow GL_2(W/p^n), n \geq 2$, with $\overline{\rho} = \rho_n \mod p$. We shall also assume that the $\overline{\rho}, \rho_n$ satisfy the following.

Hypothesis A. *The image of $\overline{\rho}$ contains $SL_2(\boldsymbol{k})$. Furthermore, if $p = 3$ then the image of ρ_n contains the transvection $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

Fix a character $\epsilon : G_F \longrightarrow W^\times$ lifting the determinant of ρ_n. We wish to consider global deformation conditions $\mathcal{D}$ for $\overline{\rho}$ with determinant ϵ such that ρ_n is a deformation of type $\mathcal{D}$. We shall abbreviate this and call $\mathcal{D}$ a deformation condition for ρ_n. Except for a change in choice of lettering for primes of F we keep the notation of [11]. Thus $\mathcal{D}_\mathfrak{q}$ is the local component at a prime q, $\mathbf{t}_{\mathcal{D}_\mathfrak{q}}$ is the tangent space there, and $\mathbf{t}^\perp_{\mathcal{D}_\mathfrak{q}} \subseteq H^1(G_{F_\mathfrak{q}}, \mathrm{ad}^0\overline{\rho}(1))$ is the orthogonal complement of $\mathbf{t}_{\mathcal{D}_\mathfrak{q}}$ under the pairing induced by

$$\mathrm{ad}^0\overline{\rho} \times \mathrm{ad}^0\overline{\rho}(1) \xrightarrow{\text{trace}} \boldsymbol{k}(1).$$

The tangent space for $\mathcal{D}$ is the Selmer group $H^1_{\{\mathbf{t}_{\mathcal{D}_\mathfrak{q}}\}}(F, \mathrm{ad}^0\overline{\rho})$; the dual Selmer group $H^1_{\{\mathbf{t}^\perp_{\mathcal{D}_\mathfrak{q}}\}}(F, \mathrm{ad}^0\overline{\rho}(1))$ is determined by the local conditions $\mathbf{t}^\perp_{\mathcal{D}_\mathfrak{q}}$. (See for instance [15, Definition 8.6.19].) We also set

$$\delta(\mathcal{D}) := \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}_{\mathcal{D}_\mathfrak{q}}\}}(F, \mathrm{ad}^0\overline{\rho}) - \dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}^\perp_{\mathcal{D}_\mathfrak{q}}\}}(F, \mathrm{ad}^0\overline{\rho}(1)).$$

Proposition 3.1. *Suppose we are given a deformation condition $\mathcal{D}$ for ρ_n with determinant ϵ. Let S be a fixed finite set of primes of F including primes where $\mathcal{D}$ is ramified and all the infinite primes. If $\delta(\mathcal{D}) \geq 0$ we can find a deformation condition $\mathcal{E}$ for ρ_n with determinant ϵ such that:*

- *The local conditions $\mathcal{E}_\mathfrak{q}$ and $\mathcal{D}_\mathfrak{q}$ are the same at primes $\mathfrak{q} \in S$;*
- *$\mathcal{E}_\mathfrak{q}$ is a substantial deformation condition for $\mathfrak{q} \notin S$; and,*
- *$H^1_{\{\mathbf{t}^\perp_{\mathcal{E}_\mathfrak{q}}\}}(F, ad^0\overline{\rho}(1)) = (0)$.*

Proof. Let K be the splitting field of ρ_n adjoined p^n-th roots of unity. We claim that we can find elements $g, h \in \mathrm{Gal}(K/F)$ such that

(R1) $\rho_n(g) \sim \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\chi(g) = -1 \mod p^n$;
(R2) $\rho_n(h) \sim a\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\chi(h) = 1 \mod p^n$.

For R1, we can take g to be complex conjugation. For R2, by considering $\epsilon = \chi(\epsilon\chi^{-1})$ or otherwise, we can write $\epsilon = \chi\epsilon_0\epsilon_1^2$ where ϵ_0 is a finite order character of order co-prime to p. Our assumptions on the size of $\overline{\rho}$ and ρ_n (when $p = 3$) along with Proposition 2.8 imply that the image of the twist of $\rho_n \otimes \epsilon_1^{-1}$ contains $SL_2(W/p^n)$. Thus we can find $h_1 \in \mathrm{Gal}(K/F)$ such that $\rho_n(h_1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\epsilon_1(h_1)$ and we get $\epsilon_0(h_1)\chi(h_1) = 1$. We can then take h to be $h_1^{p^k-1}$ where p^k is the cardinality of $\boldsymbol{k}$.

We first adjust $\mathcal{D}$ and define a deformation condition $\mathcal{E}_0$ for ρ_n with determinant ϵ as follows. We make no change if $p \geq 5$ and the projective image of $\overline{\rho}$ strictly contains $PSL_2(\mathbb{F}_5)$; so $\mathcal{E}_0$ is $\mathcal{D}$. Now for the remaining cases: Suppose that either $p = 3$ or the projective image of $\overline{\rho}$ is A_5 (so $\boldsymbol{k}$ is necessarily $\mathbb{F}_5$). Using the Chebotarev Density Theorem and R2 above, we can find a prime $\mathfrak{q}_0 \notin S$ with $\mathfrak{q}_0 \equiv 1 \mod p^n$ and $\rho_n(\mathrm{Frob}_{\mathfrak{q}_0}) = a\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\mathcal{E}_0$ be the deformation condition of $\overline{\rho}$ with determinant ϵ characterized by the following local conditions:

- at primes $\mathfrak{q} \neq \mathfrak{q}_0$, $\mathcal{E}_{0\mathfrak{q}} = \mathcal{D}_\mathfrak{q}$;
- at $\mathfrak{q}_0$, $\mathcal{E}_{0\mathfrak{q}_0}$ consists of deformations of the form

$$\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}\epsilon'$$

where $\epsilon' : G_{v_0} \longrightarrow W^\times$ is unramified and $\epsilon|_{G_{\mathfrak{q}_0}} = \chi\epsilon'^2$.

By our choice of $\mathfrak{q}_0$, $\mathcal{E}_0$ is a deformation condition for ρ_n. Further, $\mathcal{E}_{0\mathfrak{q}_0}$ is a substantial deformation and all non-zero cohomology classes in $\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}_0}}, \mathbf{t}^\perp_{\mathcal{E}_{0\mathfrak{q}_0}}$ are ramified.

We claim that the restriction maps

$$H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\overline{\rho}) \longrightarrow H^1(G_K, \mathrm{ad}^0\overline{\rho}) \quad \text{and} \quad H^1_{\{\mathbf{t}^\perp_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\overline{\rho}(1)) \longrightarrow H^1(G_K, \mathrm{ad}^0\overline{\rho}(1))$$

are injective. When $p \geq 5$ and the projective image of $\overline{\rho}$ strictly contains A_5 an easy calculation using Proposition 2.8 shows that $H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\overline{\rho})$ and $H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\overline{\rho}(1))$ are trivial, and so the injectivity follows. In the case when $p = 3$ or the projective image of $\overline{\rho}$ is A_5, we argue as follows: If $\xi \in \ker\left(H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\overline{\rho}) \longrightarrow H^1(G_K, \mathrm{ad}^0\overline{\rho})\right)$, then ξ is naturally an element of $H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\overline{\rho})$. Thus ξ is unramified at $\mathfrak{q}_0$ and so the restriction of ξ to the decomposition group at $\mathfrak{q}_0$ must be trivial. Using Proposition 2.8 it follows that $\xi \in H^1(\mathrm{Gal}(K/F), \mathrm{ad}^0\overline{\rho})$ is trivial. A similar argument works for $\mathrm{ad}^0\overline{\rho}(1)$.

The proof is now standard: If the dual Selmer group for $\mathcal{E}_0$ is non-trivial then we can find

$$0 \neq \xi \in H^1_{\{\mathbf{t}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\overline{\rho}), \quad 0 \neq \psi \in H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{0\mathfrak{q}}}\}}(F, \mathrm{ad}^0\overline{\rho}(1)).$$

Take $g \in \mathrm{Gal}(K/L)$ as in R1, consider pairs $(M_1, N_1), (M_2, N_2)$ where $\{\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)\} = N_1 \subset M_1 = \mathrm{ad}^0\overline{\rho}$, $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} = N_2 \subset M_2 = \mathrm{ad}^0\overline{\rho}(1)$ and apply Proposition 2.2 of [11]. One can then find a prime $\mathfrak{r} \notin S \cup \{\mathfrak{q}_0\}$ lifting g such that the restrictions of ξ, ψ to $G_{\mathfrak{r}}$ are not in $H^1(G_{\mathfrak{r}}, N_1), H^1(G_{\mathfrak{r}}, N_2)$.

Now take $\mathcal{E}_1$ to be the deformation condition with determinant ϵ as follows: $\mathcal{E}_1$ and $\mathcal{E}_0$ differ only at $\mathfrak{r}$, and at $\mathfrak{r}$, the local component consists of deformations of the form $\left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)(\epsilon/\chi)^{1/2}$ considered in Example 2.3. Here, $(\epsilon/\chi)^{1/2}$ is the unramified character determined by taking the square-root of $\epsilon(\mathrm{Frob}_{\mathfrak{r}})\chi^{-1}(\mathrm{Frob}_{\mathfrak{r}})$. Since $\mathrm{Frob}_{\mathfrak{r}}$ lifts g we have $\chi(\mathrm{Frob}_{\mathfrak{r}}) \equiv -1 \mod p^n$, and consequently $\mathcal{E}_1$ is a substantial deformation condition for ρ_n. The rest is identical to the proof of Proposition 4.2, [11]: The dual Selmer group for $\mathcal{E}_1$ has dimension one less than that of the dual Selmer group for $\mathcal{E}_0$. (Of course $\delta(\mathcal{E}_1) = \delta(\mathcal{E}_0) = \delta(\mathcal{D})$.) $\square$

We can now prove a general result for lifting a $\mod p^n$ representation to characteristic 0.

Theorem 3.2. *Let $\mathcal{D}$ be a deformation condition for ρ_n with determinant ϵ, and let S be a fixed finite set of primes of F including primes where $\mathcal{D}$ is ramified and all the infinite primes. Suppose that each local component is substantial. We can then find a deformation condition $\mathcal{E}$ for ρ_n with determinant ϵ such that:*

- *The local conditions $\mathcal{E}_{\mathfrak{q}}$ and $\mathcal{D}_{\mathfrak{q}}$ are the same at primes $\mathfrak{q} \in S$;*
- *Each local component is a substantial deformation condition;*
- *The dual Selmer group $H^1_{\{\mathbf{t}^{\perp}_{\mathcal{E}_{\mathfrak{q}}}\}}(F, ad^0\overline{\rho}(1))$ is trivial.*

$\mathcal{E}$ is a smooth deformation condition and the universal deformation ring is a power series ring over W in $\delta(\mathcal{D})$ variables. In particular, there is a representation $\rho : G_F \longrightarrow GL_2(W)$ of type $\mathcal{E}$ lifting ρ_n.

Proof. The only verification required is to check that $\delta(\mathcal{D}) \geq 0$ and that $\dim_{\boldsymbol{k}} H^1_{\{\mathbf{t}_{\mathcal{E}_{\mathfrak{q}}}\}}(F, \mathrm{ad}^0\overline{\rho}) = \delta(\mathcal{E}) = \delta(\mathcal{D})$. This is done using Wiles' formula (cf [15, Theorem 8.6.20]). $\square$

3.2. **Modular characteristic 0 lifts and proof of Main Theorem.** We now look at the question of producing characteristic zero liftings which are modular. Given a $\mod p^n$ Galois representation $\rho_n : G_F \longrightarrow GL_2(W/p^n)$ with $\rho_n \mod p$ modular, when can we guarantee the existence of a modular form f with $\rho_{f,p} \mod p^n \sim \rho_n$? Our answer is a modest attempt using Theorem 3.2 to produce a characteristic 0 lift and then invoking results of Skinner and Wiles [17] to prove that it is modular.

For the rest of this section, F is a totally real field and $\psi : G_F \longrightarrow W^\times$ is a finite order character of G_F unramified at primes dividing p.

Proposition 3.3. *Let $\rho_n : G_F \longrightarrow GL_2(W/p^n)$ be a continuous odd representation satisfying 3.1. Suppose $\epsilon := \psi\chi^a, a \geq 1$ lifts the determinant of ρ_n. Assume that:*

(i) *At a prime $\mathfrak{q} \nmid p$ where ρ_n is ramified, the restriction $\rho_n|_{G_{\mathfrak{q}}}$ is substantial there and that a substantial deformation condition $\mathcal{D}_{\mathfrak{q}}$ is specified for ρ_n.*

(ii) *At a prime* $\mathfrak{p}$ *dividing* p,

$$\rho_n|_{G_\mathfrak{p}} \sim \begin{pmatrix} \chi^a\psi_{1\mathfrak{p}} & * \\ 0 & \psi_{2\mathfrak{p}} \end{pmatrix}$$

where $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ *are unramified,* $\chi^a\psi_{1\mathfrak{p}} \not\equiv \psi_{2\mathfrak{p}} \mod p$ *and* $\chi^a\psi_{1\mathfrak{p}} \not\equiv \chi\psi_{2\mathfrak{p}} \mod p$.

(iii) *There is an ordinary, parallel weight at least 2, modular form which is a* $(\psi_{2\mathfrak{p}} \mod p)$*- good lift of* $\rho_n \mod p$.

There is then a modular form f *such that its associated* p*-adic representation* $\rho_{f,p} : G_F \longrightarrow GL_2(W)$ *lifts* ρ_n*, has determinant* $\psi\chi^a$*, is of of type* $\mathcal{D}_\mathfrak{q}$ *at primes* $\mathfrak{q} \nmid p$ *where* ρ_n *is ramified, and*

$$\rho_{f,p}|_{G_\mathfrak{p}} \sim \begin{pmatrix} \psi'_{1\mathfrak{p}}\chi^a & * \\ 0 & \psi'_{2\mathfrak{p}} \end{pmatrix}$$

at primes $\mathfrak{p}|p$ *with* $\psi'_{2\mathfrak{p}}$ *an unramified lift of* $\psi_{2p} \mod p$.

Proof. At a prime $\mathfrak{p}|p$ take $\mathcal{D}_\mathfrak{p}$ to be the class of deformations of the form

$$\begin{pmatrix} \psi'_{1\mathfrak{p}}\chi^a & * \\ 0 & \psi'_{2\mathfrak{p}} \end{pmatrix}$$

where $\psi'_{1\mathfrak{p}}$ (resp. $\psi'_{2\mathfrak{p}}$) is an unramified lifting of $\psi_{1\mathfrak{p}} \mod p$ (resp. $\psi_{2\mathfrak{p}} \mod p$), and $\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}} = \psi$. This is a substantial deformation for ρ_n at $\mathfrak{p}$ by Proposition 2.5. By Theorem 3.2, there is a smooth deformation condition $\mathcal{E}$ for ρ_n which agrees with $\mathcal{D}_\mathfrak{p}$ at primes above p and primes where ρ_n is ramified. Thus there is continuous representation $\rho : G_F \longrightarrow GL_2(W)$ with $\rho \mod p^n = \rho_n$, unramified outside finitely many primes, determinant $\psi\chi^a$ and $\rho|_{I_\mathfrak{p}} \sim \left(\begin{smallmatrix} \chi^a & * \\ 0 & 1 \end{smallmatrix}\right)$ at primes $\mathfrak{p}|p$. The proposition now follows from Skinner-Wiles [17]. □

Proof of Main Theorem. Let's recall the set up: We are given a Hilbert modular newform f of weight $k \geq 2$ character ψ which is ordinary at p and whose reduction $\mod p^n$ gives $\rho_{f,n} : G_F \longrightarrow GL_2(W/p^n)$. For each prime $\mathfrak{p}$ of F over p, let $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ be the unramified characters such that

$$\rho_f|_{G_\mathfrak{p}} \sim \begin{pmatrix} \psi_{1\mathfrak{p}}\chi^{k-1} & * \\ 0 & \psi_{2\mathfrak{p}} \end{pmatrix}.$$

As $\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}} = \psi$ and $\psi_{2\mathfrak{p}}(\mathrm{Frob}_\mathfrak{p}) = c(\mathfrak{p}, f)$, hypothesis LC1 ensures $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ are distinct modulo $\wp$. From this, one deduces easily that if f has a weak companion form $\mod p^n$ then $\rho_{f,n}$ splits at p. We now show that 'split at p' implies the existence of a weak companion form.

Let $\rho_n := \rho_{f,n} \otimes \chi^{1-k}$, and set $\overline{\rho} := \rho_n \mod p$. Recall that $k' \geq 2$ is the smallest integer satisfying the congruence $k + k' \equiv 2 \mod (p-1)p^{n-1}$. Define a global deformation condition $\mathcal{D}$ for $\overline{\rho} \otimes \chi^{1-k} \mod p$ by the following requirements:

(a) Deformations are unramified outside primes dividing $p\mathfrak{n}$ and have determinant $\psi\chi^{k'-1}$.

(b) At a prime $\mathfrak{p} \mid p$, the local condition $\mathcal{D}_{\mathfrak{p}}$ consists of deformations of the form

$$\begin{pmatrix} \psi'_{2\mathfrak{p}}\chi^{k'-1} & * \\ 0 & \psi'_{1\mathfrak{p}} \end{pmatrix}$$

where $\psi'_{1\mathfrak{p}}$ (resp. $\psi'_{2\mathfrak{p}}$) is an unramified lifting of $\psi_{1\mathfrak{p}} \mod p$ (resp. $\psi_{2\mathfrak{p}} \mod p$), and $\psi_{1\mathfrak{p}}\psi_{2\mathfrak{p}} = \psi$.

(c) Let $\mathfrak{q}$ be a prime dividing $\mathfrak{n}$, the level of f. We need to distinguish two cases:
   (i) If $\mathfrak{q}$ does not divide the conductor of ψ then $\overline{\rho}|_{G_{\mathfrak{q}}} \sim \left(\begin{smallmatrix} \overline{\chi} & * \\ 0 & 1 \end{smallmatrix}\right)\bar{\epsilon}$ for some character $\bar{\epsilon}$. Further, hypothesis LC2 ensures that if $\overline{\rho}|_{G_{\mathfrak{q}}}$ is semisimple then $\overline{\chi} \neq 1$. We then take $\mathcal{D}_{\mathfrak{q}}$ to be local liftings with determinant $\psi\chi^{k'-1}$ of the type considered in Example 2.3.
   (ii) If $\mathfrak{q}$ divides the conductor of ψ then $\rho_f(I_{\mathfrak{q}}), \overline{\rho}(I_{\mathfrak{q}})$ are finite and have the same order. In this case we take $\mathcal{D}_{\mathfrak{q}}$ as in Example 2.2 i.e. lifts with detereminant $\psi\chi^{k'-1}$ which factor through $G_{\mathfrak{q}}/(I_{\mathfrak{q}} \cap \ker\overline{\rho})$.

It then follows that ρ_n is a deformation of type $\mathcal{D}$ and that at each prime $\mathfrak{q} \nmid p$ where ρ_n is ramified the local deformation condition $\mathcal{D}_{\mathfrak{q}}$ is substantial there. As p is unramified in F, the distinctness of $\psi_{1\mathfrak{p}}, \psi_{2\mathfrak{p}}$ modulo p implies that $\overline{\rho}$ satisfies hypothesis (ii) of Proposition 3.3. From the existence of $\mod p$ companion forms, ([5, Theorem 2.1]), it follows that $\overline{\rho}$ has an ordinary modular lift which is $(\psi_{1\mathfrak{p}} \mod p)_{\mathfrak{p}|p}$-good. The existence of a $\mod p^n$ weak companion form g for f of weight k' character ψ now follows from Proposition 3.3. $\square$

4. Checking local splitting: A computational approach

The lifting result of the previous section is not suitable for computational purposes in general because, except in the case when dual Selmer group was already trivial, we had no control of the level. There is, however, one case when we do have absolute control. We now describe this situation and go on to verify examples of local splitting.

4.1. **A special case.** Suppose $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\boldsymbol{k})$ is absolutely irreducible and $\mathcal{D}$ is a deformation condition for $\overline{\rho}$ such that its tangent space is 0 dimensional. Then the universal deformation ring $R_{\mathcal{D}}$ is a quotient of $W(\mathbf{k})$. If we also knew that there is a characteristic 0 lift of type $\mathcal{D}$, then we must have $R_{\mathcal{D}} \simeq W$. Consequently any $\mod p^n$ representation of type $\mathcal{D}$ lifts to characteristic 0.

The question now is: How can one check if the tangent space is 0 dimensional? Observe that we must necessarily have exactly one characteristic 0 lift of type $\mathcal{D}$. This alone might not be enough though. For instance, $R_{\mathcal{D}}$ might be $W[X]/(X^2)$.

To proceed further, and with the examples we have in mind, we shall assume that $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\boldsymbol{k})$ is an absolutely irreducible representation with determinant $\overline{\chi}$ such that

- $\overline{\rho}|_{G_p} \sim \left(\begin{smallmatrix} \overline{\chi}\psi^{-1} & * \\ 0 & \psi \end{smallmatrix}\right)$, with ψ unramified and $\psi \neq \psi^{-1}$,
- if $q \nmid p$ then $\#\overline{\rho}(I_q)|p$.

By Lemma 3.24 of [3], $\overline{\rho}|_{G_L}$ is absolutely irreducible where $L = \mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$.

Let N be the Artin conductor of $\overline{\rho}$. For an integer $k \geq 2$, let $S(k, N, \overline{\rho})$ be the (possibly empty) set of newforms of level N with $\rho_f \mod p \simeq \overline{\rho}$.

With notation as in Theorem 3.42 of [3], we then have an isomorphism $R_{\emptyset} \xrightarrow{\sim} \mathbb{T}_{\emptyset}$, where $R_{\emptyset}$ is the universal deformation ring for minimally ramified ordinary lifts and $\mathbb{T}_{\emptyset}$ is the reduced Hecke algebra generated by the Fourier coefficients of newforms in $S(2, N, \overline{\rho})$. In particular, the dimension of the tangent space in the minimally ramified case is 0 if and only if $\#S(2, N, \overline{\rho}) = 1$.

For $n \geq 1$ set $k_n := (p-1)p^{n-1} - (p-1) + 2$ and define a deformation condition $\mathcal{D}_{k_n}$ for $\overline{\rho}$ as follows: A lift $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(A)$ is a deformation of type $\mathcal{D}_{k_n}$ if

- $\det \rho = \chi^{k_n - 1}$ and ρ is unramified outside primes dividing N,
- at primes $q|N$, $\rho|_{G_q} \sim \left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)$ up to twist, and
- at p, $\rho|_{G_p} \sim \begin{pmatrix} \tilde{\psi}^{-1}\chi^{k_n-1} & * \\ 0 & \tilde{\psi} \end{pmatrix}$, where $\tilde{\psi}$ is an unramified lift of ψ.

Note that for $n = 1$ the universal deformation ring $R_{\mathcal{D}_{k_n}}$ is $R_{\emptyset} \simeq \mathbb{T}_{\emptyset}$. Clearly, the type $\mathcal{D}_{k_n}$ deformations to $\boldsymbol{k}[\epsilon]/(\epsilon^2)$ are in bijection with type $\mathcal{D}_2$ deformations. Hence if the tangent space of $\mathcal{D}_2$ has dimension 0 then so does $\mathcal{D}_{k_n}$. We conclude that $R_{\mathcal{D}_{k_n}} \simeq W$, corresponding to a unique newform in $S(k_n, N, \overline{\rho})$.

Proposition 4.1. *Let f be a newform of weight $k \geq 2$, level N, trivial character and ordinary at p, such that*

- *$\overline{\rho}_f$ is absolutely irreducible,*
- *the conductor of $\overline{\rho}_f$ is N,*
- *$\overline{\rho}|_{G_p} \sim \left(\begin{smallmatrix} * & * \\ 0 & \psi \end{smallmatrix}\right)$ with ψ unramified and $\psi^2 \neq 1$,*
- *if $q \nmid p$ then $\#\overline{\rho}_f(I_q)|p$.*

Assume that $p-1|k$ and that f has exactly one companion form $\mod p$. Then ρ_f splits $\mod p^n$ iff f has a companion form $\mod p^n$.

Proof. The proposition is immediate from the preceding discussion by taking $\overline{\rho}$ to be $\overline{\rho}_f \otimes \overline{\chi}^{1-k}$. $\square$

4.2. **Greenberg's conjecture.** In this section we give some examples of the existence (or non-existence) of higher companion forms. We shall restrict ourselves to the setting of classical elliptic modular forms as we only give examples in this case.

Recall that a newform f is said to have **complex multiplication**, or just CM, by a quadratic character $\phi : G_{\mathbb{Q}} \longrightarrow \{\pm 1\}$ if $T(q)f = \phi(\mathrm{Frob}_q)c(q, f)f$ for almost all primes q. We will also refer to CM by the corresponding quadratic extension. It is well known that a modular form has CM if and only if its associated p-adic representation is induced from an algebraic Hecke character.

Let $\mathcal{O}$ be a p-adic integer ring with residue field $\boldsymbol{k}$. Suppose the newform f is p-ordinary and $\rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O})$. Then, as indicated in the introduction, $\rho_f|G_p$ can be assumed to be upper triangular with an unramified lower diagonal entry and this leads to the natural question of determining when ρ_f splits at p.

There is a well-known conjectural connection between ρ_f to be split at p and f to have CM. The antecedents are sketchy, but Hida [9] calls it Greenberg's local non-semisimplicity conjecture; we will simply refer to it as Greenberg's conjecture which asserts:

If f is ordinary at p and ρ_f splits at p then f has complex multiplication.

This is satisfactorily known for modular forms over $\mathbb{Q}$ of weight 2. For higher weights, the question remains largely unresolved although some interesting results involving Hida families are shown in Ghate [6] which also has a survey of results for weight 2. The analogous problem for Λ-adic modular forms was resolved in Ghate-Vatsal [7] by using deformation theory but similar methods appear not to bear fruit in the classical case. Emerton [4] shows how this conjecture would follow from a p-adic version of the variational Hodge conjecture. Through the main theorem and proposition 4.1, higher congruence companions offer a slightly different perspective to the question of ρ_f splitting at p.

To describe this further, let N be the level of the p-ordinary newform f. Assume that f has trivial character and has weight $p-1$. For each positive integer n we set $k_n := p^{n-1}(p-1)-(p-1)+2$. We then proceed as follows:

(a) Check that f has a companion form $\mod p$. Check congruences to make sure that the residual representation $\rho_f \mod \pi$ is absolutely irreducible and that $c(p,f) \not\equiv \pm 1 \mod \pi$. We can therefore write
$$\rho_f \mod \pi = \begin{pmatrix} \overline{\chi}^{k-1}\overline{\psi} & 0 \\ 0 & \overline{\psi}^{-1} \end{pmatrix}$$
with $\overline{\psi}^{-1} \neq \overline{\psi}$.

(b) In order to be able to check fewer cases, ensure that ρ_f is minimally ramified i.e. the Artin conductor of $\rho_f \mod \pi$ is N.

(c) Set $\overline{\rho} := \rho_f \otimes \chi^{1-k} \mod \pi$. For each $n \geq 1$ let $\mathcal{D}_{k_n}$ be the weight k_n, trivial character deformation condition as described in section 4.1. Check if the tangent spaces can be taken to be 0 dimensional. Thus we have to check if f has precisely one companion form of type $\mathcal{D}_{k_1}$. We then apply proposition 4.1 to deduce that ρ_f splits $\mod p^n$ if and only if f has a companion form $\mod p^n$ *i.e.* there is a newform g of level N, trivial character, weight k_n such that $f \equiv g \otimes \chi^{k-1} \mod p^n$

We check Greenberg's conjecture explicitly for two known non-CM forms of weight 4. The computations were done on MAGMA. In both cases $p=5$. We note that in these examples, taking N to be the level of f, one may check its companionship with a form g "*by hand*" by simply verifying the congruences $c(f,m) \equiv m^3 c(g,m) \mod 5^n$ for $(m,5N)=1$ up to the Sturm bound.

Example 4.2. Let f be the newform of weight 4, level 21 and trivial character with the following Fourier expansion:

$$g = q - 3q^2 - 3q^3 + q^4 - 18q^5 + 9q^6 + 7q^7 + 21q^8 + 9q^9 + 54q^{10} - 36q^{11} - 3q^{12} - 34q^{13} - 21q^{14} + 54q^{15} - 71q^{16} + 42q^{17} - 27q^{18} - 124q^{19} - 18q^{20} - 21q^{21} + 108q^{22} - 63q^{24} + 199q^{25} + 102q^{26} - 27q^{27} + 7q^{28} + 102q^{29} - 162q^{30} - 160q^{31} + 45q^{32} + 108q^{33} - 126q^{34} - 126q^{35} + 9q^{36} + 398q^{37} + 372q^{38} + 102q^{39} - 378q^{40} - 318q^{41} + 63q^{42} - 268q^{43} - 36q^{44} - 162q^{45} + 240q^{47} + 213q^{48} + 49q^{49} - 597q^{50} + \cdots$$

MAGMA outputs modulo 5, a unique companion form g weight 2, level 21 and trivial character with the following Fourier expansion:

$$f = q - q^2 + q^3 - q^4 - 2q^5 - q^6 - q^7 + 3q^8 + q^9 + 2q^{10} + 4q^{11} - q^{12} - 2q^{13} + q^{14} - 2q^{15} - q^{16} - 6q^{17} - q^{18} + 4q^{19} + 2q^{20} - q^{21} - 4q^{22} + 3q^{24} - q^{25} + 2q^{26} + q^{27} + q^{28} - 2q^{29} + 2q^{30} - 5q^{32} + 4q^{33} + 6q^{34} + 2q^{35} - q^{36} + 6q^{37} - 4q^{38} - 2q^{39} - 6q^{40} + 2q^{41} + q^{42} - 4q^{43} - 4q^{44} - 2q^{45} - q^{48} + q^{49} + q^{50} + \cdots$$

Clearly there are no companions of weight 2 and level 3 or 7. Modulo 5^2, f has no companion forms of weight 18, level dividing 21 and trivial character. Thus f does not split $\mod 5^2$.

Example 4.3. Let f be the newform of weight 4, level 57 and trivial character with Fourier expansion

$$f = q - q^2 + 3q^3 - 7q^4 - 12q^5 - 3q^6 - 20q^7 + 15q^8 + 9q^9 + 12q^{10} - 4q^{11} - 21q^{12} - 76q^{13} + 20q^{14} - 36q^{15} + 41q^{16} + 22q^{17} - 9q^{18} - 19q^{19} + 84q^{20} - 60q^{21} + 4q^{22} + 82q^{23} + 45q^{24} + 19q^{25} + 76q^{26} + 27q^{27} + 140q^{28} + 242q^{29} + 36q^{30} - 126q^{31} - 161q^{32} - 12q^{33} - 22q^{34} + 240q^{35} - 63q^{36} - 180q^{37} + 19q^{38} - 228q^{39} - 180q^{40} - 390q^{41} + 60q^{42} + 308q^{43} + 28q^{44} - 108q^{45} - 82q^{46} - 522q^{47} + 123q^{48} + 57q^{49} - 19q^{50} + \cdots$$

It has a unique mod 5 companion form g of weight 2, level 57 and trivial character with Fourier expansion

$$g = q - 2q^2 - q^3 + 2q^4 - 3q^5 + 2q^6 - 5q^7 + q^9 + 6q^{10} + q^{11} - 2q^{12} + 2q^{13} + 10q^{14} + 3q^{15} - 4q^{16} - q^{17} - 2q^{18} - q^{19} - 6q^{20} + 5q^{21} - 2q^{22} - 4q^{23} + 4q^{25} - 4q^{26} - q^{27} - 10q^{28} - 2q^{29} - 6q^{30} - 6q^{31} + 8q^{32} - q^{33} + 2q^{34} + 15q^{35} + 2q^{36} + 2q^{38} - 2q^{39} - 10q^{42} - q^{43} + 2q^{44} - 3q^{45} + 8q^{46} - 9q^{47} + 4q^{48} + 18q^{49} - 8q^{50} + \cdots$$

and no other companions of level dividing 57. Modulo 5^2, f has no companion forms of weight 18, level dividing 57 and trivial character.

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