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A note on the vanishing of certain local cohomology modules

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Abstract

For a finite module M over a local, equicharacteristic ring (R, m), we show that the well-known formula $\mathrm{cd}(m, M) = \dim M$ becomes trivial if ones uses Matlis duals of local cohomology modules together with spectral sequences. We also prove a new, ring-theoretic vanishing criterion for local cohomology modules.

1 Introduction

Let R be a noetherian ring, I an ideal of R and M an R-module; one denotes the n-th local cohomology module of M with respect to I by $H_I^n(M)$ and the cohomological dimension of I on M by

$$\mathrm{cd}(I, M) := \sup\{l | H_I^l(M) \neq 0\}.$$

From now on assume that (R, m) is local and M is finitely generated. Grothendieck's Vanishing Theorem (VT) says that $\mathrm{cd}(I, M) \leq \dim M$ and Grothendieck's Non-Vanishing Theorem (NVT) says $H_m^{\dim M}(M) \neq 0$. Both are well-known theorems with various proofs, see e. g. [1, Theorem 6.1.2], [2, Theorem 2.7] (a version for sheaves) for VT and [1, Theorem 6.1.4], [1, Theorem 7.3.2] for NVT. The case $I = m$ of VT and NVT *together* say that the cohomological dimension is the Krull dimension:

$$\mathrm{cd}(m, M) = \dim M. \qquad (*)$$

The first aim of this paper is to show that, using Matlis duals of local cohomology modules, formula $(*)$ become almost trivial once one knows:

(A) The fact that local cohomology can be written as the direct limit of Koszul cohomologies; it is an easy exercise to check that immediate consequences of this are

- (A_1) the base-change formula ${}_R H_{IS}^i(N) = H_I^i({}_R N)$ (S/R a noetherian algebra, N an S-module, I an ideal of R and $i \in \mathbb{N}$)
- (A_2) the formula

$$H^j_{(X_1,\ldots,X_i)}(k[[X_1,\ldots,X_i]]) = \begin{cases} 0, & \text{if } j > i \\ E_{k[[X_1,\ldots,X_i]]}(k) = k[X_1^{-1},\ldots,X_i^{-1}], & \text{if } j = i \end{cases}$$

(k a field, $X_1, \ldots, X_i$ indeterminates)

(A_3) the fact that each local cohomology functor of the form $H^j_{(x_1,\ldots,x_i)R}$ is zero for $j > i$; in particular, $H^i_{(x_1,\ldots,x_i)}R$ is right exact.

(B) Some Matlis duality theory and some spectral sequence theory. Both serve as *technical tools.*

Our method works *only in the equicharacteristic case.*

The second aim is to prove theorem 3.1, which is a new (sufficient) criterion for the vanishing of local cohomology modules, which is of a ring-theoretic nature; the idea which is used in its proof is, to the best of our knowledge, completely new in this context.

2 (Non-)Vanishing Theorem

Everything in this paper is based on the following easy

Lemma 2.1. *Let (R,m) be a noetherian local complete ring containing a field k, M an R-module and $x_1,\ldots,x_i \in R$. Then*

$$H^i_{\underline{x}R}(M) \neq 0 \iff \dim(R_0) = i \text{ and } \operatorname{Hom}_{R_0}(M, R_0) \neq 0$$

where $R_0 := k[[x_1,\ldots,x_i]]$ as a subring of R and $\underline{x} := x_1,\ldots,x_i$.

Proof. $\Rightarrow$: Assume $\dim(R_0) < i$. Write $R_0 = k[[X_1,\ldots,X_i]]/I$ where $X_1,\ldots,X_i$ are indeterminates and I is a non-zero ideal of $k[[X_1,\ldots,X_i]] =: S$. Then

$$H^i_{\underline{x}R_0}(R_0) \overset{(A_1),(A_3)}{=} H^i_{\underline{X}S}(S) \otimes_S (S/I) = 0$$

as every $0 \neq f \in I$ operates injectively on S and hence ((B)) surjectively on $H^i_{\underline{X}S}(S)(\overset{(A_2)}{\cong} E_S(k))$. In particular,

$$H^i_{\underline{x}R}(M) \overset{(A_3)}{=} M \otimes_{R_0} H^i_{\underline{x}R_0}(R_0) = 0,$$

contradiction. Therefore, $\dim(R_0) = i$, $R_0 \cong k[[X_1,\ldots,X_i]]$ with indeterminates $X_1,\ldots,X_i$ and one has

$$\begin{aligned} 0 &\overset{(B)}{\neq} \operatorname{Hom}_{R_0}(H^i_{\underline{x}R}(M), E_{R_0}(k)) \\ &\overset{(A_3)}{=} \operatorname{Hom}_{R_0}(M \otimes_{R_0} H^i_{\underline{x}R_0}(R_0), E_{R_0}(k)) \\ &= \operatorname{Hom}_{R_0}(M, \operatorname{Hom}_{R_0}(H^i_{\underline{x}R_0}(R_0), E_{R_0}(k))) \\ &\overset{(A_2),(B)}{=} \operatorname{Hom}_{R_0}(M, R_0) \end{aligned}$$

$\Leftarrow$: Again, $R_0 \cong k[[X_1,\ldots,X_i]]$ with indeterminates $X_1,\ldots,X_i$; now,

$$0 \neq \operatorname{Hom}_{R_0}(M, R_0) = \operatorname{Hom}_{R_0}(H^i_{\underline{x}R}(M), E_{R_0}(k))$$

follows like above. □

Theorem 2.2. *(i) If R is a noetherian ring containing a field, $\underline{x} = x_1, \dots, x_i \in R$ and M is an R-module (not necessarily finitely generated) such that* $\dim_R(M) < i$*, then $H^i_{\underline{x}R}(M) = 0$.*

(ii) If (R, m) is a noetherian local ring containing a field and $\underline{x} = x_1, \dots, x_i$ is part of a system of parameters of a finitely generated R-module M then $H^i_{\underline{x}R}(M) \neq 0$; in particular, $H^{\dim_R(M)}_m(M) \neq 0$.

(iii) If (R, m) is a noetherian local ring containing a field and M is a finitely generated R-module then $\mathrm{cd}(m, M) = \dim_R(M)$.

Proof. (i) By localizing and completing we may assume that R is local and complete. Set $R_0 := k[[x_1, \dots, x_i]]$ as a subring of R like in lemma 2.1; we may assume that $\dim(R_0) = i$, i. e. $R_0 \cong k[[X_1, \dots, X_i]]$, where $X_1, \dots, X_i$ are indeterminates. Due to dimension reasons it is clear that $\mathrm{Hom}_{R_0}(M, R_0) = 0$ and the claim follows from lemma 2.1.

(ii) We may assume that R is complete ($\hat{R}/R$ is faithfully flat); by base-change, we may replace R by $R/\mathrm{Ann}_R(M)$; set $d := \dim(R)$. We choose $x_{i+1}, \dots, x_d \in R$ such that $x_1, \dots, x_d$ is a system of parameters of M. Then $R_0 := k[[x_1, \dots, x_d]] \subseteq R$ is a regular d-dimensional subring of R and, because M is module-finite over R_0, $\mathrm{Hom}_{R_0}(M, R_0) \neq 0$; lemma 2.1 implies $H^d_{(x_1,\dots,x_d)R}(M) \neq 0$. Now a formal spectral sequence argument (namely for the spectral sequence of composed functors $E_2^{p,q} = H^p_{(x_{i+1},\dots,x_d)R}(H^q_{(x_1,\dots,x_i)R}(M)) \Rightarrow H^{p+q}_{(x_1,\dots,x_d)R}(M)$; note that $H^p_{(x_{i+1},\dots,x_d)R} = 0$ for each $p > d - i$ and that $H^q_{(x_1,\dots,x_i)R} = 0$ for each $q > i$, by (A_3)) shows

$$0 \neq H^d_{(x_1,\dots,x_d)R}(M) = H^{d-i}_{(x_{i+1},\dots,x_d)R}(H^i_{(x_1,\dots,x_i)R}(M))$$

(iii) Follows from (i) and (ii). □

3 A Ring-theoretic Vanishing Criterion

Theorem 3.1. *Let (R, m) be a noetherian local complete domain containing a field and $\underline{x} = x_1, \dots, x_i$ a sequence in R. Then the implication*

$$H^i_{\underline{x}R}(R) \neq 0 \Rightarrow \dim(R_0) = i \text{ and } R \cap Q(R_0) = R_0$$

holds, where $R_0 := k[[x_1, \dots, x_i]] \subseteq R$, $Q(R_0)$ denotes the quotient field of R_0 and the intersection is taken inside $Q(R)$.

Proof. By lemma 2.1, $R_0 \cong k[[X_1, \dots, X_i]]$, $X_1, \dots, X_i$ indeterminates, $\dim(R_0) = i$.

Let $r \in R, r_0 \in R_0$ such that $r_0 \cdot r \in R_0$. We have to show that $r \in R_0$: by lemma 2.1, $\mathrm{Hom}_{R_0}(R, R_0) \neq 0$ and so we can choose $\varphi \in \mathrm{Hom}_{R_0}(R, R_0)$ such that $\varphi(1_R) \neq 0$ (namely by composing a $\varphi' \in \mathrm{Hom}_{R_0}(R, R_0)$ that has $\varphi(r') \neq 0$ (for some $r' \in R$) with the multiplication map $R \xrightarrow{r'} R$). Set $r_0' := r_0 r$. One has

$$r_0 \varphi(r) = \varphi(r_0') = r_0' \varphi(1_R)$$

and then

$$\varphi(1_R) r = \varphi(1_R) \frac{r_0'}{r_0} = \varphi(r) \in R_0$$

On the other hand, we have

$$r_0'^2 = r_0^2 r^2$$

and thus

$$r_0^2 \varphi(r^2) = r_0'^2 \varphi(1_R)$$

and

$$\varphi(1_R) r^2 = \varphi(1_R) \frac{r_0'^2}{r_0^2} = \varphi(r^2) \in R_0 \quad .$$

Continuing in the same way, one sees that, for every $l \geq 1$, one has

$$\varphi(1_R) r^l \in R_0 \quad .$$

But this implies that the R_0-module

$$\varphi(1_R) \cdot < 1, r, r^2, \cdots >_{R_0}$$

is finitely generated ($< 1, r, r^2, \cdots >_{R_0}$ stands for the R_0-submodule of R generated by $1, r, r^2, \ldots$). But, as R is a domain,

$$< 1, r, r^2, \cdots >_{R_0}$$

is then finitely generated, too, i. e. r is necessarily contained in R_0. □

Remarks 3.2. *(i) $H^i_{\underline{x}R}(R) \neq 0$ (and thus $R \cap Q(R_0) = R_0$) are clear if $\underline{x}$ is an R-regular sequence; but the condition $\underline{x}$ being a regular sequence is not necessary as the following example shows: $H^2_{(y_1 y_2, y_1 y_3)}(k[[y_1, y_2, y_3]])$ is non-zero (and thus $R \cap Q(R_0) = R_0$) though $y_1 y_2, y_1 y_3$ is not a regular sequence (k a field, y_1, y_2, y_3 indeterminates).*

(ii) In the situation of theorem 3.1 without the assumption $H^i_{\underline{x}R}(R) \neq 0$ the condition $R \cap Q(R_0) = R_0$ does not hold in general: e. g. for $R_0 = k[[y_1 y_2, y_1 y_2^2]] \subseteq k[[y_1, y_2]] = R$ (k a field, y_1, y_2 indeterminates) one has $y_2 \in (R \cap Q(R_0)) \setminus R_0$.

Remark 3.3. *If R is regular, the implication from theorem 3.1 is an equivalence for $i = 1$; while this is easy to see, the case $i = 2$ seems already unclear.*

Question 3.4. *Under what conditions can the implication from theorem 3.1 be reversed?*

References

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